> with(DifferentialGeometry) : with(LieAlgebras) : *with*(*Library*) :

> $DGsetup([x, t, u], N);$

frame name: N **(1)**

N > $\Gamma := evalDG([t * D\_x + D\_u, x * D\_x + 3\,t * D\_t - 2\,u * D\_u, D\_t, D\_x])$

$$\Gamma := [t\,D\_x + D\_u, x\,D\_x + 3\,t\,D\_t - 2\,u\,D\_u, D\_t, D\_x] \quad \textbf{(2)}$$

N > $L := LieAlgebraData(\Gamma, Alg1)$

$$L := [[e1, e2] = -2\,e1, [e1, e3] = -e4, [e2, e3] = -3\,e3, [e2, e4] = -e4] \quad \textbf{(3)}$$

N > $DGsetup(L)$

Lie algebra: Alg1 **(4)**

Alg1 > *MultiplicationTable*("LieTable");

$$\left[\begin{array}{cccccc} & `|` & e1 & e2 & e3 & e4 \\ & --- & ---- & ---- & ---- & ---- \\ e1 & `|` & 0 & -2\,e1 & -e4 & 0 \\ e2 & `|` & 2\,e1 & 0 & -3\,e3 & -e4 \\ e3 & `|` & e4 & 3\,e3 & 0 & 0 \\ e4 & `|` & 0 & e4 & 0 & 0 \end{array}\right] \quad \textbf{(5)}$$

> $A := DerivedAlgebra(\,);$

$$A := [-2\,e1, -e4, -3\,e3] \quad \textbf{(6)}$$

Alg1 > $C := Center(\,);$

$$C := [\,] \quad \textbf{(7)}$$

Alg1 > $B := [e1, e3, e4]$

$$B := [e1, e3, e4] \quad \textbf{(8)}$$

> **for** i **from** 1 **to** 3 **do** $LieBracket(B[1], B[i])$ **end do**;

$$0\,e1$$
$$-e4$$
$$0\,e1 \quad \textbf{(9)}$$

Alg1 > **for** i **from** 1 **to** 3 **do** $LieBracket(B[2], B[i])$ **end do**;

$$e4$$
$$0\,e1$$
$$0\,e1 \quad \textbf{(10)}$$

Alg1 > **for** i **from** 1 **to** 3 **do** $LieBracket(B[3], B[i])$ **end do**;

$$0\,e1$$
$$0\,e1$$
$$0\,e1 \quad \textbf{(11)}$$

Since Z(g')=<e4>. <e4> is the only one dimensional ideal.

Quotient Algebra:-

Since we have found a 1 dimentional Ideal of g i.e.

$\langle e4 \rangle$, *so let* $I = \langle e4 \rangle$, **then** *Quotient Algebra is defined as* $\frac{g}{I} := \langle e1 + \langle e4 \rangle, e2 + \langle e4 \rangle, e3 + \langle e4 \rangle \rangle$ *which is constructed with the help of complementary bases.* As [e2,e1]=2e1 and [e2,e3]=-3e3, where λ=2 and μ=-3. Thus by lemma 2.2.2 one dimensional ideals of g/<e4> are <e1> and <e3>. Thus 2-dimensional ideals of g are <e1,e4> and <e3,e4>.

Alg1 > $h1 := [e4] : m1 := [e1, e2, e3] :$

Alg1 > $QuotientAlgebra(h1, m1);$

$$L2 := [[e1, e2] = -2\, e1, [e2, e3] = -3\, e3] \tag{12}$$

For 3-dimensional ideals of g:

Alg1 > $h2 := [e1, e4] : m2 := [e2, e3] :$

Alg1 > $QuotientAlgebra(h2, m2);$

$$[[e1, e2] = -3\, e2] \tag{13}$$

Alg1 > $h3 := [e3, e4] : m3 := [e1, e2] :$

Alg1 > $QuotientAlgebra(h3, m3);$

$$[[e1, e2] = -2\, e1] \tag{14}$$

hence,
all the ideals of g are $\langle\ \rangle$, $\langle \boldsymbol{e4} \rangle$, $\langle \boldsymbol{e1, e4} \rangle$, $\langle \boldsymbol{e3, e4} \rangle$, $\langle \boldsymbol{e1, e3, e4} \rangle$ **and** $\langle \boldsymbol{e1, e2, e3, e4} \rangle$.
Therefore, the 1-dimensional subalgebras of g are
$\langle \boldsymbol{e4} \rangle$, $\langle \boldsymbol{e1 + ae4} \rangle$, $\langle \boldsymbol{e3 + be4} \rangle$, $\langle \boldsymbol{e1 + ce3 + de4} \rangle$ **and** $\langle \boldsymbol{e2 + fe1 + ge3 + he4} \rangle$.

>

Subalgebras:
Here we use this formula

$$g := (x, y) \mapsto y + s \cdot LieBracket(x, y) + \frac{1}{2!} \cdot s^2 \cdot LieBracket(x, LieBracket(x, y)) + \frac{1}{3!} \cdot s^3 \cdot LieBracket(x, LieBracket(x, LieBracket(x, y)))$$

But we ignore the heigher terms of this formula.

Alg1 > $g := (x, y) \mapsto y + s \cdot LieBracket(x, y)$

$$g := (x, y) \rightarrow y + s\, DifferentialGeometry{:-}LieBracket(x, y) \tag{15}$$

Alg1 > $X := [e2 + f \cdot e1 + g \cdot e3 + h \cdot e4] :$

Alg1 > $X1 := SubalgebraNormalizer(X);$

$$X1 := \left[e4, \frac{1}{3} \frac{e2}{a} + e3, e1\right] \tag{16}$$

The adjoint group thus factorizes as $\mathrm{e}^{ad \langle \boldsymbol{e4} \rangle} \mathrm{e}^{ad \langle \boldsymbol{e3} \rangle} \mathrm{e}^{ad \langle \boldsymbol{e1} \rangle} \mathrm{e}^{ad \langle \boldsymbol{X} \rangle}$,
and <X,e1,e3,e4> is a basis of g with X normalizing the ideal <e1,e3,e4>.

Alg1 > $X := [e1 + c \cdot e3 + d \cdot e4]$

$$X := [e1 + c\, e3 + d\, e4] \tag{17}$$

Alg1 > $X1 := SubalgebraNormalizer(X);$

$$X1 := \left[e4, \frac{e1}{c} + e3\right] \tag{18}$$

Alg1 > $X2 := SubalgebraNormalizer(X1);$

$$X2 := [e4, e3, e1] \tag{19}$$

Alg1 > $X3 := SubalgebraNormalizer(X2);$

$$X3 := [e4, e3, e2, e1] \tag{20}$$

The adjoint group thus factorizes as $e^{ad\langle e2\rangle}e^{ad\langle e3\rangle}e^{ad\langle e4\rangle}e^{ad\langle X\rangle}$.

Alg1 > $LieBracket(e4, e1 + c\cdot e3 + d\cdot e4)$

$$0\ e1 \tag{21}$$

So, X and e4 commute, we only need to compute conjugates under $e^{ad\langle e2\rangle}e^{ad\langle e3\rangle}$.

Alg1 > $g(e3, e1 + c\cdot e3 + d\cdot e4)$

$$e1 + c\ e3 + d\ e4 + s\ e4 \tag{22}$$

So, <X>∼<e1+ce3>.

Alg1 > $g(e2, e1 + c\cdot e3)$

$$e1 + c\ e3 + s\ (2\ e1 - 3\ c\ e3) \tag{23}$$

So, <X>∼<e1+ce3>∼<$e^{2s}e1 + ce^{-3s}e3$ >∼ $\langle e1 + ce^{-5s}e3\rangle = \langle e1 + \epsilon e3\rangle$, *where* $\epsilon = +-1$.

Alg1 > $X := [e3 + b\cdot e4]$

$$X := [e3 + b\ e4] \tag{24}$$

Alg1 > $X1 := SubalgebraNormalizer(X);$

$$X1 := [e4, e3, 2\ b\ e1 + e2] \tag{25}$$

The adjoint group thus factorizes as $e^{ad\langle e4\rangle}e^{ad\langle e2\rangle}e^{ad\langle e1\rangle}e^{ad\langle X\rangle}$.

Alg1 > $g(e1, e3 + b\cdot e4)$

$$e3 + b\ e4 - s\ e4 \tag{26}$$

So, <X>∼<e3>.

Alg1 > $X := [e1 + a\cdot e4]$

$$X := [e1 + a\ e4] \tag{27}$$

Alg1 > $X1 := SubalgebraNormalizer(X);$

$$X1 := \left[\frac{1}{3}\ \frac{e2}{a} + e3, e4, e1\right] \tag{28}$$

Alg1 > $g(e2, e1 + a\cdot e4)$

$$e1 + a\ e4 + s\ (2\ e1 - a\ e4) \tag{29}$$

So, <X>∼<e1>. Also, <X>∼<e2>.Thus representatives of for conjugacy classes of 1-dimensional subalgebras are <e4>,<e1>,<e3>,<e1+e3>,<e1-e3> and <e2>.

2-dimensional subalgebras:

Alg1 > $P := [[e1], [e2], [e3], [e4], [e1 + e3], [e1 - e3]]$

$$P := [[e1], [e2], [e3], [e4], [e1 + e3], [e1 - e3]] \tag{30}$$

Alg1 > **for** i **from** 1 **to** 6 **do** $SubalgebraNormalizer(P[i])$ **end do**;

$$[e4, e2, e1]$$
$$[e2]$$
$$[e4, e3, e2]$$
$$[e4, e3, e2, e1]$$
$$[e4, e1 + e3]$$
$$[e4, -e1 + e3] \tag{31}$$

Alg1 > $h1 := [e4] : m1 := [e2, e1, e3] :$

Alg1 > $QuotientAlgebra(h1, m1)$

$$[[e1, e2] = 2\ e2, [e1, e3] = -3\ e3] \tag{32}$$

By Lemma 4.2, the conjugacy classes of 1-dimensional subalgebras of N<e4>/<e4> as <e2'>,<e1'>,

<e3'>,<e1'+e3'>,<e1'-e3'>.
Hence, the 2-dimensional subalgebras obtained by extending <e4> are <e4,e2>,<e4,e1>,<e4,e3>,<e4, e1+e3> and <e4,e1-e3>.
Since, N<e1>=<e4,e2,e1> and N<e1>/<e1>=<e2',e4'>, by Lemma 4.1 the 1-dimensional subspaces of N<e1>/<e1> are <e2'> and <e4'>. Thus the 2-dimensional subalgebras obtained by extending <e1> are <e1,e2> and <e1,e4>.
Also, N<e3>/<e3>=<e4',e2'>, by Lemma 4.1 the 1-dimensional subspaces are <e4'> and <e2'>. Thus the 2-dimensional subalgebras obtained by extending <e3> are <e3,e4> and <e2,e3>.
Hence, representatives for classes of 2 dimensional subalgebras of g are <e4,e2>, <e4,e1>, <e4,e3>, <e4,e1+e3>, <e4,e1-e3>, <e2,e3> and <e1,e4>.
3-dimensional subalgebras:

Alg1 > $Q := [[e1, e2], [e4, e1], [e3, e2], [e4, e3], [e4, e2], [e4, e1 + e3], [e4, e1 - e3]]$

$$Q := [[e1, e2], [e4, e1], [e3, e2], [e4, e3], [e4, e2], [e4, e1 + e3], [e4, e1 - e3]] \quad (33)$$

Alg1 > **for** i **from** 1 **to** 7 **do** $SubalgebraNormalizer(Q[i])$ **end do**;

$$[e2, e1]$$
$$[e4, e3, e2, e1]$$
$$[e3, e2]$$
$$[e4, e3, e2, e1]$$
$$[e2, e4]$$
$$[e4, e3, e1]$$
$$[e4, e3, e1] \quad (34)$$

Alg1 > $h1 := [e4, e1] : m1 := [e2, e3] :$

Alg1 > $QuotientAlgebra(h1, m1)$

$$[[e1, e2] = -3\, e2] \quad (35)$$

Alg1 > $h2 := [e4, e3] : m2 := [e1, e2] :$

Alg1 > $QuotientAlgebra(h2, m2)$

$$[[e1, e2] = -2\, e1] \quad (36)$$

By using Lemma 4.1, 3-dimensional subalgebras obtained by extending <e4,e1> are <e4,e1,e2> and <e4,e1,e3>.
Similarly for <e4,e3>, extentions are <e4,e3,e1> and <e4,e3,e2>.
Hence, representatives for classes of 3-dimensional subalgebras are <e4,e1,e2>,<e4,e1,e3> and <e4, e3,e2>.