```
> with(DifferentialGeometry) : with(LieAlgebras) : with(Library) :
> StructureEquation := [[x, y] = λ·y, [x, z] = μ·z];
```

$$StructureEquation := [[x, y] = \lambda\, y, [x, z] = \mu\, z] \quad (1)$$

```
> L := LieAlgebraData(StructureEquation, [x, y, z], Alg1);
```

$$L := [[e1, e2] = \lambda\, e2, [e1, e3] = \mu\, e3] \quad (2)$$

```
> DGsetup(L);
```

$$\text{Lie algebra: Alg1} \quad (3)$$

To find: Conjugacy Classes of the subalgebras of this Lie Algebra.
Step 1: Find all the ideals of g.

```
> MultiplicationTable("LieTable");
```

$$\left[\begin{array}{ccccc} & `|` & e1 & e2 & e3 \\ & --- & ---- & ---- & ---- \\ e1 & `|` & 0 & \lambda\, e2 & \mu\, e3 \\ e2 & `|` & -\lambda\, e2 & 0 & 0 \\ e3 & `|` & -\mu\, e3 & 0 & 0 \end{array}\right] \quad (4)$$

```
> A := DerivedAlgebra( );
```

$$A := [\lambda\, e2, \mu\, e3] \quad (5)$$

```
Alg1 > C := Center( );
```

$$C := [\ ] \quad (6)$$

```
Alg1 > B := [e2, e3]
```

$$B := [e2, e3] \quad (7)$$

```
Alg1 > for i from 1 to 2 do LieBracket(B[1], B[i]) end do;
```

$$0\ e1$$
$$0\ e1 \quad (8)$$

```
> for i from 1 to 2 do LieBracket(B[2], B[i]) end do;
```

$$0\ e1$$
$$0\ e1 \quad (9)$$

here, center of derived algebra is equal to derived algebra. Thus common eigenvectors are y and z. So <y> and <z> are 1-dimensional ideals.

```
Alg1 > h1 := [e2] : m1 := [e1, e3] :
Alg1 > L2 := QuotientAlgebra(h1, m1, Alg2);
```

$$L2 := [[e1, e2] = \mu\, e2] \quad (10)$$

Here e1=e1+<e4> and e3=e3+<e4>. From Lemma 2.2.1, <e3> is the only one dimensional ideal of g/<e2>.

```
Alg1 > h2 := [e3] : m2 := [e1, e2] :
Alg1 > QuotientAlgebra(h2, m2);
```

$$[[e1, e2] = \lambda\, e2] \quad (11)$$

From Lemma 2.2.1, <e2> is the only one dimensional ideal of g/<e3>. Thus <e2,e3> is the only 2-dimensional ideal of g.
Thus the proper ideals of g are <y>,<z> and <y,z>. So the one dimensional subalgebras are of the form

<y>,<z>,<y+kz> and <x+ly+mz>.

Alg1 > $X := [e2 + k \cdot e3]$

$$X := [e2 + k\, e3] \tag{12}$$

Alg1 > $X1 := SubalgebraNormalizer(X);$

$$X1 := [e3, e2] \tag{13}$$

Alg1 > $X2 := SubalgebraNormalizer(X1);$

$$X2 := [e3, e2, e1] \tag{14}$$

Here, we use the series of normalizers

$\langle \boldsymbol{X} \rangle \subseteq \langle \boldsymbol{X, e3} \rangle \subseteq \langle \boldsymbol{X, e1, e3} \rangle$ *.The adjoint group thus factorizes as* $\mathrm{e}^{ad\,\langle \boldsymbol{e1} \rangle}\mathrm{e}^{ad\,\langle \boldsymbol{e3} \rangle}\mathrm{e}^{ad\,\langle \boldsymbol{X} \rangle}$.

Alg1 > $LieBracket(e3, e2 + k \cdot e3);$

$$0\, e1 \tag{15}$$

Alg1 > $g := (x, y) \mapsto y + LieBracket(x, y)$

$$g := (x, y) \rightarrow y + DifferentialGeometry{:-}LieBracket(x, y) \tag{16}$$

Alg1 > $g(e1, e2 + k \cdot e3)$

$$e2 + k\, e3 + \lambda\, e2 + k\, \mu\, e3 \tag{17}$$

So, $\langle y + kz \rangle \sim \langle \mathrm{e}^{t\lambda} y + \mathrm{e}^{t\mu} kz \rangle = \langle y + \mathrm{e}^{t(\mu - \lambda)} kz \rangle$.

Consequently, if k≠0, we have $\langle y + kz \rangle \ \sim\ \langle y + \epsilon^2 z \rangle$, *where* $\epsilon^2 = 1$.

Alg1 > $X := [e1 + \lambda \cdot e2 + \mu \cdot e3]$

$$X := [e1 + \lambda\, e2 + \mu\, e3] \tag{18}$$

Alg1 > $X1 := SubalgebraNormalizer(X);$

$$X1 := \left[\frac{e1}{\mu} + \frac{\lambda\, e2}{\mu} + e3\right] \tag{19}$$

The adjoint group thus factorizes as $\mathrm{e}^{ad\,\langle \boldsymbol{e2} \rangle}\mathrm{e}^{ad\,\langle \boldsymbol{e3} \rangle}\mathrm{e}^{ad\,\langle \boldsymbol{X} \rangle}$.

Alg1 > $LieBracket(e2, e1 + \lambda \cdot e2 + \mu \cdot e3);$

$$-\lambda\, e2 \tag{20}$$

Alg1 > $LieBracket(e3, e1 + \lambda \cdot e2 + \mu \cdot e3);$

$$-\mu\, e3 \tag{21}$$

Alg1 > $g(e2, e1 + \lambda \cdot e2 + \mu \cdot e3)$

$$e1 + \lambda\, e2 + \mu\, e3 - \lambda\, e2 \tag{22}$$

$\langle x + \lambda y + \mu z \rangle \sim \langle x + \mu z \rangle$.

Alg1 > $g(e3, e1 + \mu \cdot e3)$

$$e1 + \mu\, e3 - \mu\, e3 \tag{23}$$

$\langle x + \mu z \rangle \sim \langle x \rangle$. Consequently, representatives of conjugacy classes of 1 dimensional subalgebras of g are $\langle x \rangle$, $\langle y \rangle$, $\langle z \rangle$, $\langle y + z \rangle$ **and** $\langle y - z \rangle$.

2-dimensional subalgebras of g:

Alg1 > $P := [[e1], [e2], [e3], [e2+e3], [e2-e3]]$

$$P := [[e1], [e2], [e3], [e2+e3], [e2-e3]] \quad \textbf{(24)}$$

Alg1 > **for** i **from** 1 **to** 5 **do** $SubalgebraNormalizer(P[i])$ **end do**;

$$[e1]$$
$$[e3, e2, e1]$$
$$[e3, e2, e1]$$
$$[e3, e2]$$
$$[e3, e2] \quad \textbf{(25)}$$

Alg1 > $H := [[e2], [e3], [e2], [e2]] : M := [[e1, e3], [e1, e2], [e3], [e3]] :$

Alg1 > **for** i **from** 1 **to** 4 **do** $QuotientAlgebra(H[i], M[i])$ **end do**;

$$[[e1, e2] = \mu\, e2]$$
$$[[e1, e2] = \lambda\, e2]$$
$$[\ ]$$
$$[\ ] \quad \textbf{(26)}$$

So, 1-dimensional subalgebras of $\frac{N\langle y\rangle}{\langle y\rangle}$ *are* $< x' >$ **and** $\langle z'\rangle$.

Thus 2-dimensional subalgebras obtained by extending $\langle y\rangle$ *are* $< x, y >$ **and** $\langle z, y\rangle$.
Thus 2-dimensional subalgebras obtained by extending $\langle z\rangle$ *are* $< x, z >$ **and** $\langle y, z\rangle$.
Thus 2-dimensional subalgebra obtained by extending $\langle y+z\rangle$ *is* $\langle y+z, z\rangle = \langle y, z\rangle$.
Similarly the 2-dimensional algebra with base $\langle y-z\rangle$ *is* $\langle y-z, z\rangle = \langle y, z\rangle$.
Therefore, the representatives of conjugacy classes of 2-dimensional subalgebras of g are
$\langle x, y\rangle$, $\langle x, z\rangle$ **and** $\langle y, z\rangle$.