

Analytical results for tests

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Abstract

We summarise the analytical results used in the tests for `scikit-mcquad`.

Following Numerical Recipes, we calculate the standard error in the Monte-Carlo integration of the function f as:

$$\text{Err}_N(f) = \sqrt{\frac{\langle f^2 \rangle - \langle f \rangle^2}{N}} \quad (1)$$

where N is the number of points and $\langle g \rangle = \int_{\Omega} g(x) dx$, where Ω is the volume being sampled during the integration and x denotes all the variables of integration.

1 Constant function

Let $f(x) = 1$. Then, $\langle f \rangle = \Omega$, where Ω is the volume of integration. $\langle f^2 \rangle = 1$, such that $\text{Err}_N = 0$ for all $N > 0$.

2 Product function

Let $f(x) = \prod_i^d x_i$, where d is the dimensionality of the integration. Thus, if $d = 2$, $f(x, y) = xy$. We consider the d -dimensional hypercube with upper and lower limit b and a , respectively, such that each $a \leq x_i \leq b$.

$$\langle f \rangle = \int \cdots \int_a^b \prod_i x_i dx_i = \left(\frac{b^2 - a^2}{2} \right)^d \quad (2)$$

$$\langle f^2 \rangle = \int \cdots \int_a^b \prod_i x_i^2 dx_i = \left(\frac{b^3 - a^3}{3} \right)^d \quad (3)$$

If $a = 0$ and $b = 1$, we have $\langle f \rangle = 1/2^d$ and $\langle f^2 \rangle = 1/3^d$. Then, $\text{Err}_N(f) = \frac{\sqrt{1/3^d - 1/4^d}}{N}$.

3 Gaussian

Let $f(x) = \prod_i^d \exp(-\beta^2 x_i^2)$. Again, we consider the hypercube such that $a \leq x_i \leq b$ for each i . Then:

$$\langle f \rangle = \frac{\sqrt{\pi}}{2\beta} (\text{erf}(\beta b) - \text{erf}(\beta a)) \quad (4)$$

$$\langle f^2 \rangle = \frac{\sqrt{\pi/2}}{2\beta} (\text{erf}(\sqrt{2}\beta b) - \text{erf}(\sqrt{2}\beta a)) \quad (5)$$