Periodic Functions and Fourier Series

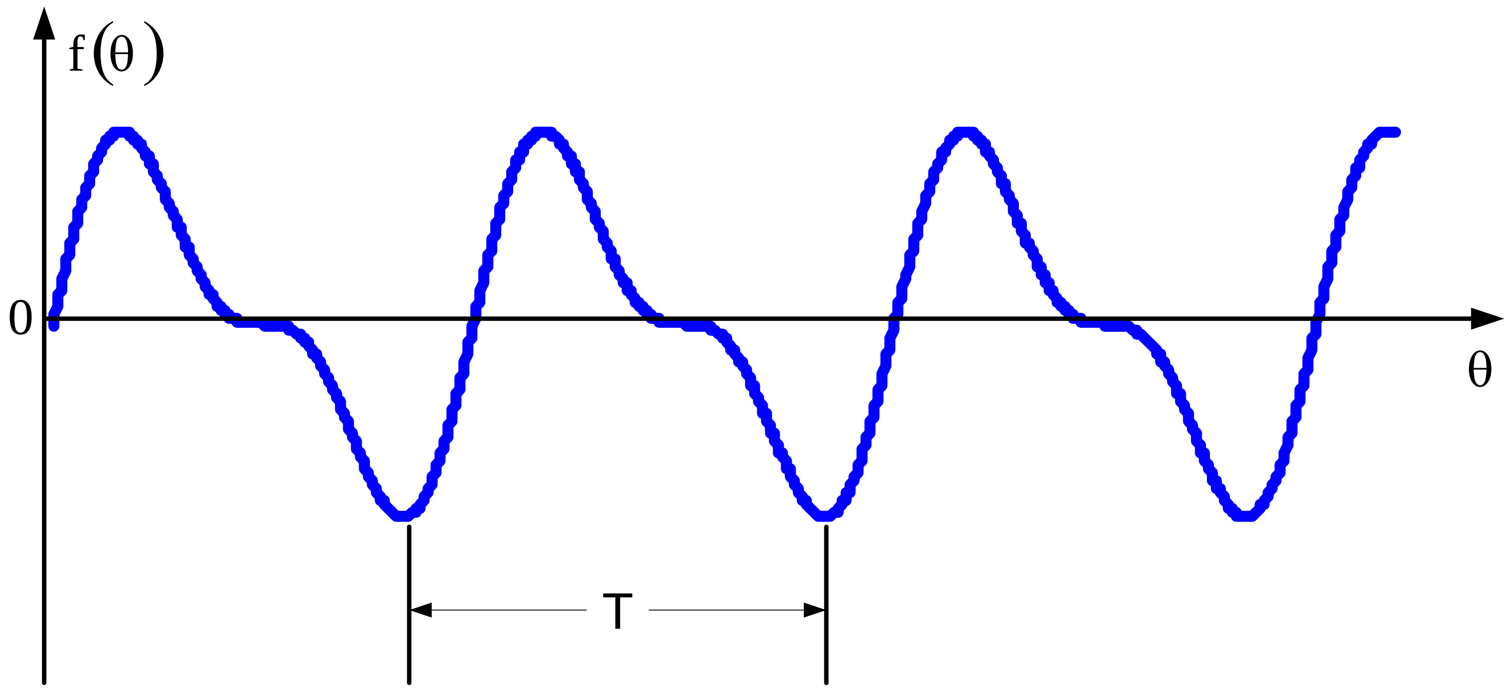
Periodic Functions

A function $f(\theta)$ is periodic

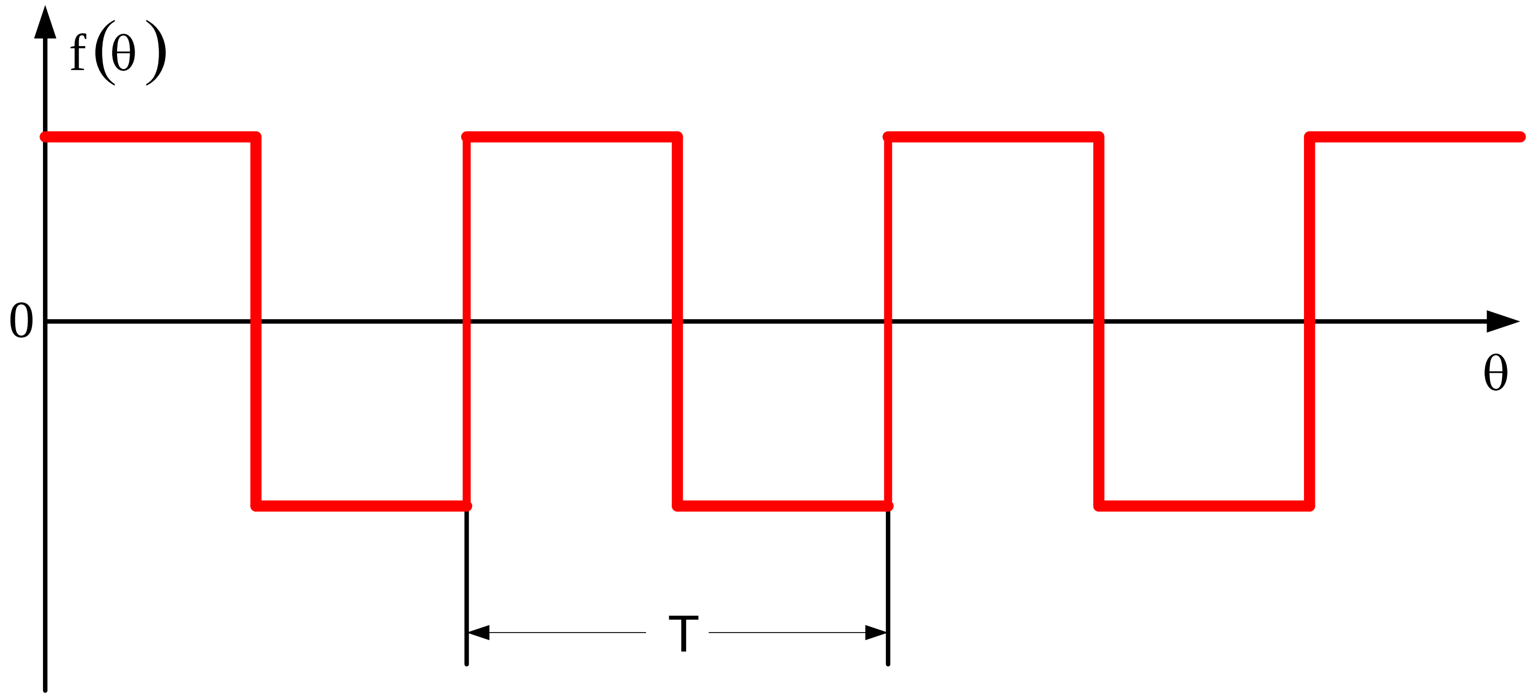
if it is defined for all real θ

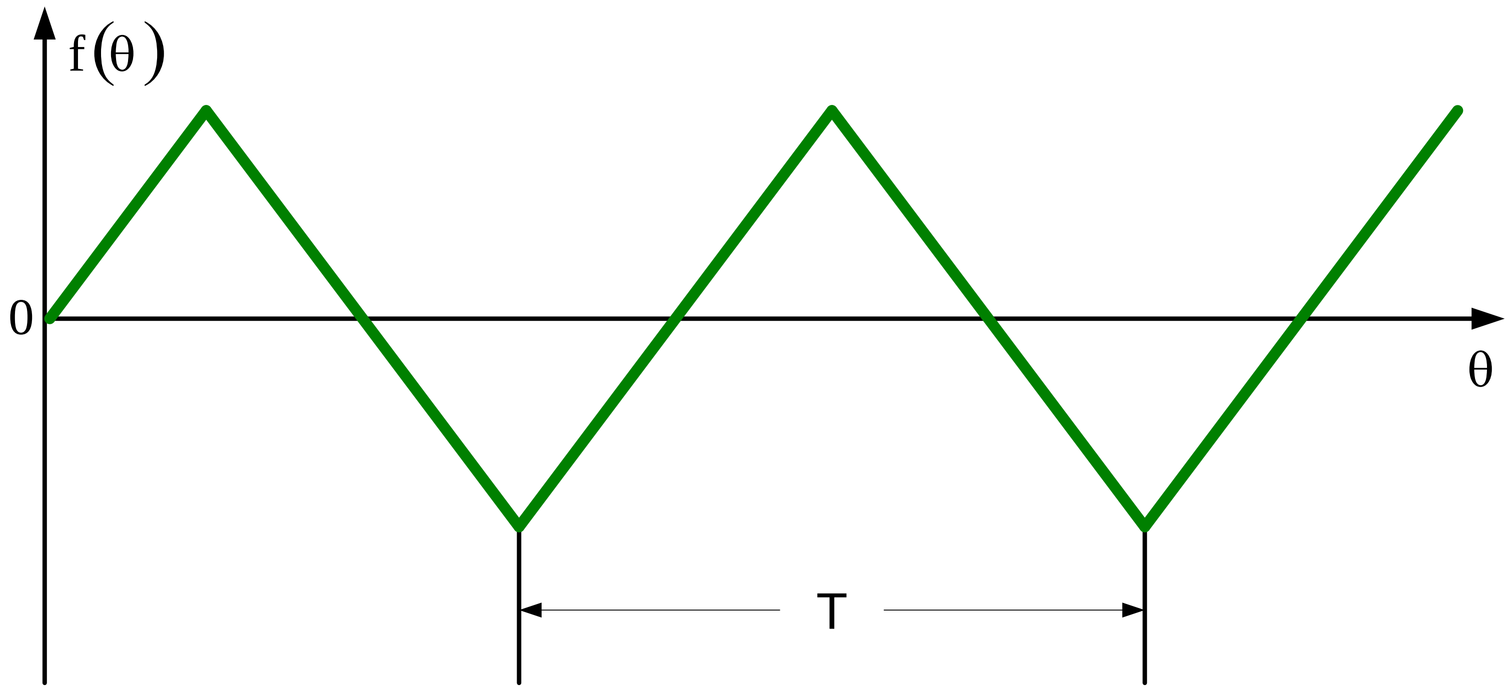
and if there is some positive number,

T such that $f(\theta+T)=f(\theta)$.



f(θ)
0
θ
T


f(θ)
0
θ
T



f(θ)
0
θ
T

Fourier Series

$f(\theta)$ be a periodic function with period 2π

The function can be represented by
a trigonometric series as:

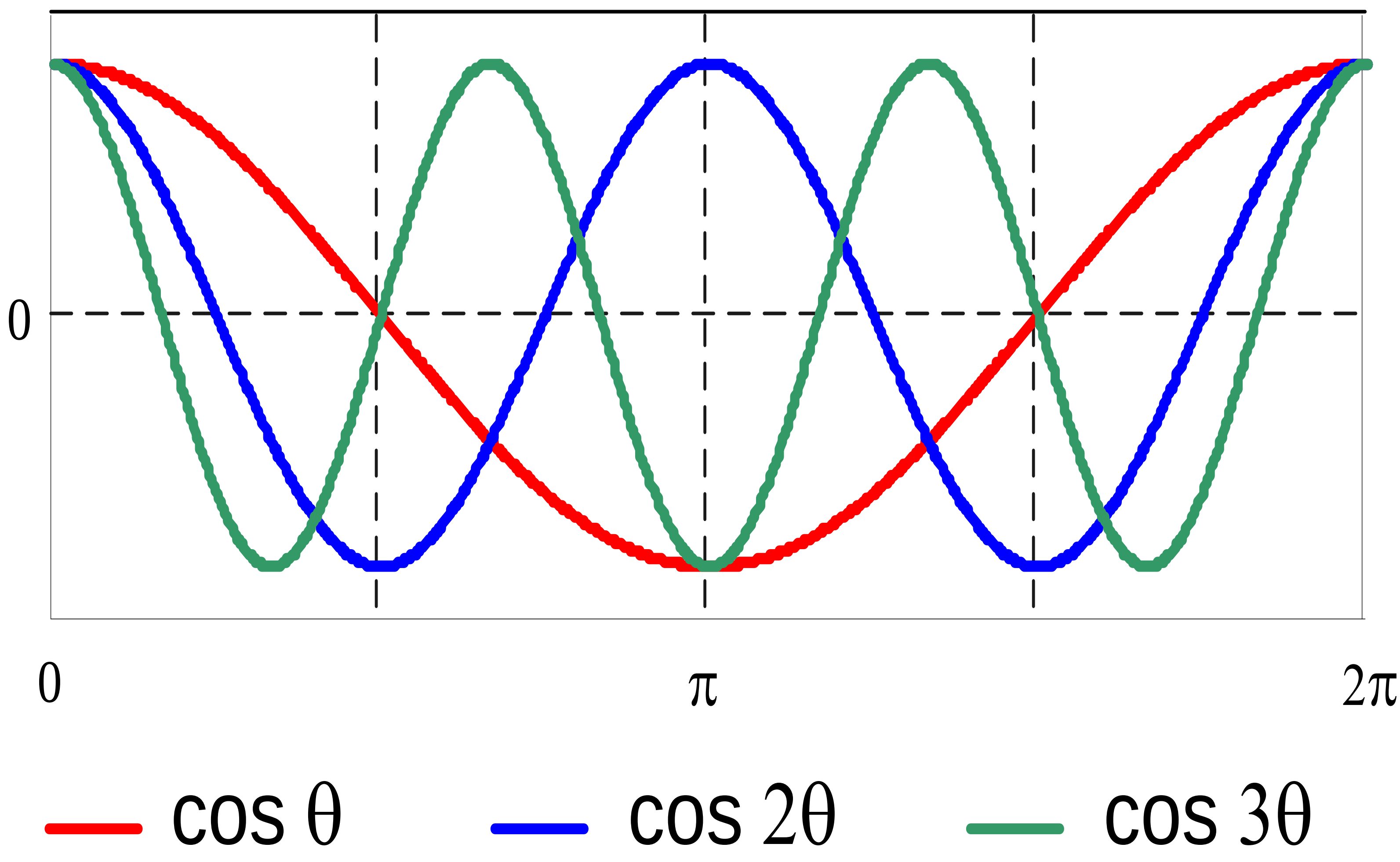
$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

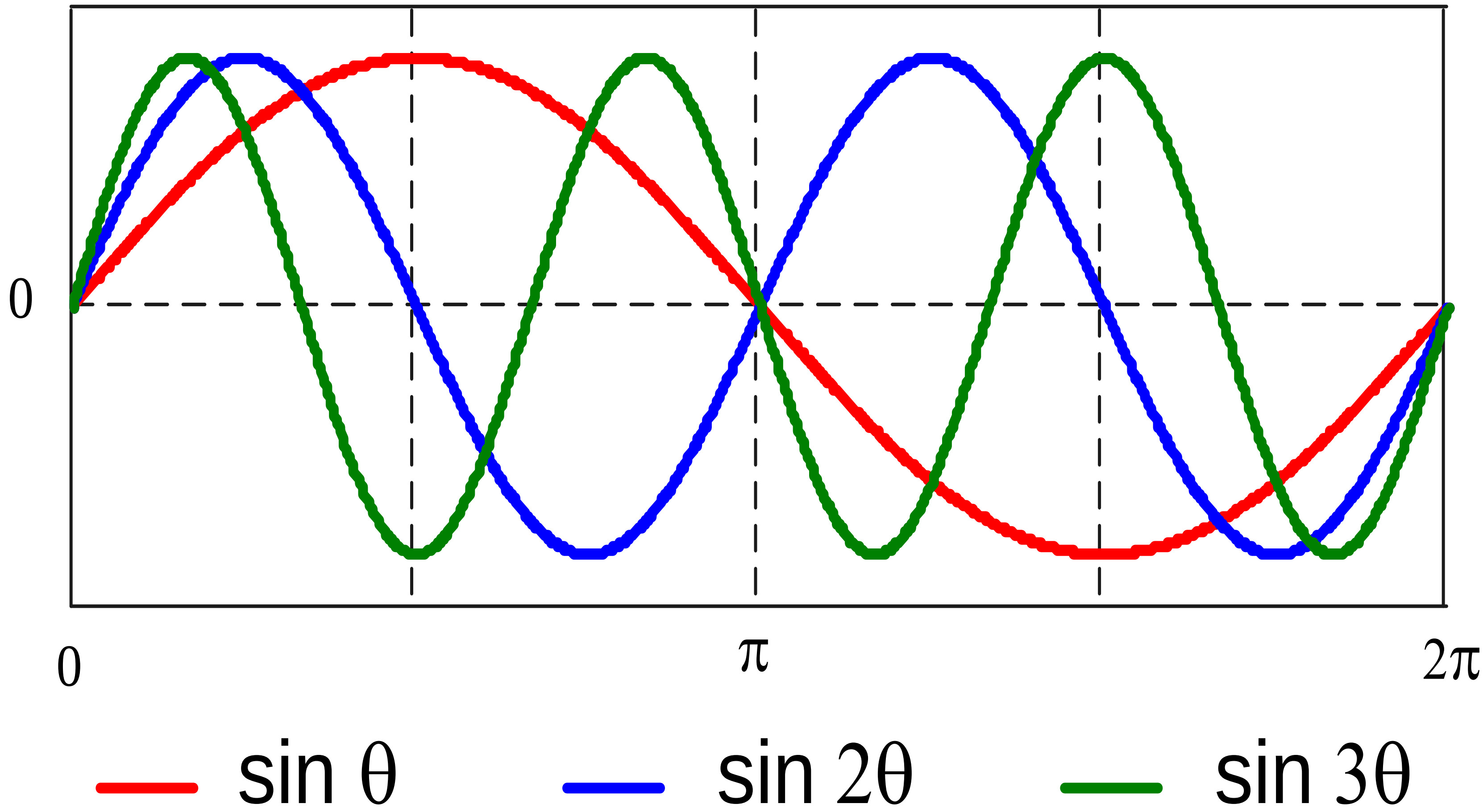
$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

What kind of trigonometric (series) functions are we talking about?

$$\cos\theta, \cos 2\theta, \cos 3\theta \cdots \quad \text{and}$$

$$\sin\theta, \sin 2\theta, \sin 3\theta \cdots$$


0
0
π
2π
cos θ
cos 2θ
cos 3θ


0
0
π
2π
sin θ
sin 2θ
sin 3θ

We want to determine the coefficients,

a_n **and** b_n **.**

Let us first remember some useful integrations.

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \pi \qquad n = m$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)\theta d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)\theta d\theta$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta d\theta = 0$$

for all values of *m*.

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)\theta \, d\theta - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta = 0 \quad n \neq m$$

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta = \pi \quad n = m$$

Determine a_0

Integrate both sides of (1) from

$-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$= \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta\right] d\theta$$

$$\int_{-\pi}^{\pi} f(\theta)d\theta$$

$$= \int_{-\pi}^{\pi} a_0 d\theta + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} a_n \cos n\theta \right) d\theta$$

$$+ \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} b_n \sin n\theta \right) d\theta$$

$$\int_{-\pi}^{\pi} f(\theta)d\theta = \int_{-\pi}^{\pi} a_0 d\theta + 0 + 0$$

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

a_0 is the average (dc) value of the function, $f(\theta)$.

You may integrate both sides of (1) from

$\mathbf{0}$ to $\mathbf{2\pi}$ instead.

$$\int_0^{2\pi} f(\theta)\,d\theta$$

$$= \int_0^{2\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta\right] d\theta$$

It is alright as long as the integration is performed over one period.

$$\int_0^{2\pi} f(\theta)\,d\theta$$

$$= \int_0^{2\pi} a_0\,d\theta + \int_0^{2\pi}\left(\sum_{n=1}^{\infty} a_n \cos n\theta\right)d\theta$$

$$+ \int_0^{2\pi}\left(\sum_{n=1}^{\infty} b_n \sin n\theta\right)d\theta$$

$$\int_0^{2\pi} f(\theta)\,d\theta = \int_0^{2\pi} a_0\,d\theta + 0 + 0$$

$$\int_0^{2\pi} f(\theta)\,d\theta = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

Determine a_n

Multiply (1) by $\cos m\theta$

and then Integrate both sides from

$-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\cos m\theta \, d\theta$$

$$= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \right] \cos m\theta \, d\theta$$

Let us do the integration on the right-hand-side one term at a time.

First term,

$$\int_{-\pi}^{\pi} a_0 \cos m\theta \, d\theta = 0$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \cos m\theta \, d\theta$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \cos m\theta \, d\theta = a_m \pi$$

Third term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin n\theta \cos m\theta \, d\theta = 0$$

Therefore,

$$\int_{-\pi}^{\pi} f(\theta)\cos m\theta d\theta = a_m \pi$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos m\theta d\theta \qquad m = 1, 2, \cdots$$

Determine b_n

Multiply (1) by $\sin m\theta$

and then Integrate both sides from

$-\pi$ to π

$$\int_{-\pi}^{\pi} f(\theta)\sin m\theta \, d\theta$$

$$= \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta\right]\sin m\theta \, d\theta$$

Let us do the integration on the right-hand-side one term at a time.

First term,

$$\int_{-\pi}^{\pi} a_0 \sin m\theta \, d\theta = 0$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \sin m\theta \, d\theta$$

Second term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\theta \sin m\theta d\theta = 0$$

Third term,

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin n\theta \sin m\theta \, d\theta = b_m \pi$$

Therefore,

$$\int_{-\pi}^{\pi} f(\theta)\sin m\theta d\theta = b_m \pi$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin m\theta d\theta \quad m = 1, 2, \cdots$$

The coefficients are:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos m\theta\,d\theta \qquad m = 1, 2, \cdots$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin m\theta\,d\theta \qquad m = 1, 2, \cdots$$

We can write *n* in place of *m*:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta \qquad n = 1, 2, \cdots$$

The integrations can be performed from

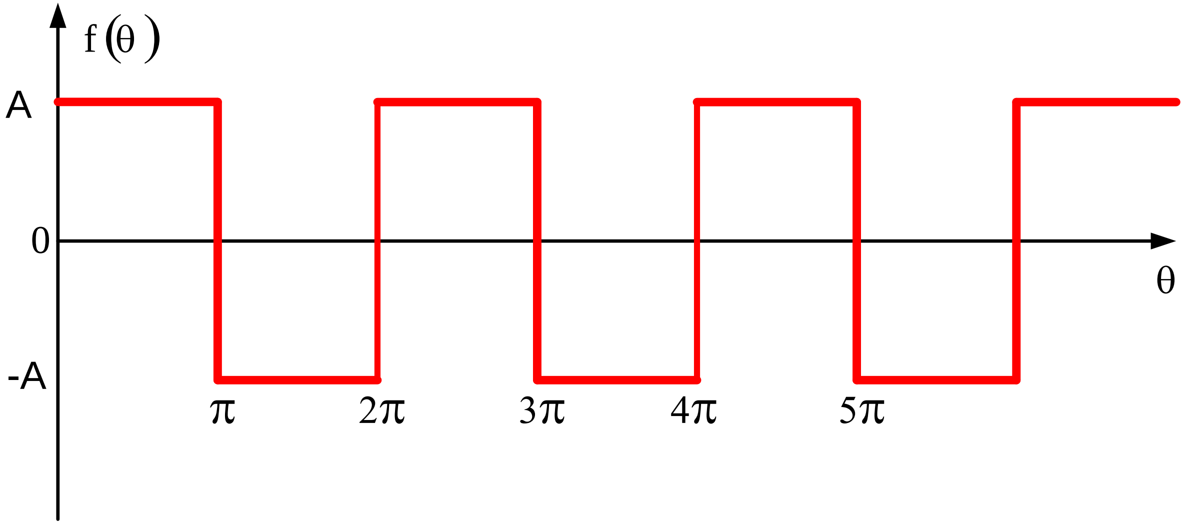
$\mathbf{0}$ **to** $\mathbf{2\pi}$ **instead.**

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta \quad n = 1, 2, \cdots$$

Example 1. Find the Fourier series of the following periodic function.



$$f(\theta) = A \quad when \quad 0 < \theta < \pi$$

$$= -A \quad when \quad \pi < \theta < 2\pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta \\
&= \frac{1}{2\pi}\left[\int_0^{\pi} f(\theta)d\theta + \int_{\pi}^{2\pi} f(\theta)d\theta\right] \\
&= \frac{1}{2\pi}\left[\int_0^{\pi} A\,d\theta + \int_{\pi}^{2\pi} -A\,d\theta\right] \\
&= 0
\end{aligned}
$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\cos n\theta\, d\theta + \int_{\pi}^{2\pi}(-A)\cos n\theta\, d\theta\right]$$

$$= \frac{1}{\pi}\left[A\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[-A\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\sin n\theta\, d\theta + \int_{\pi}^{2\pi}(-A)\sin n\theta\, d\theta\right]$$

$$= \frac{1}{\pi}\left[-A\frac{\cos n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[A\frac{\cos n\theta}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[1 + 1 + 1 + 1\right]$$

$$= \frac{4A}{n\pi} \quad \text{when n is odd}$$

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[-1+1+1-1\right]$$

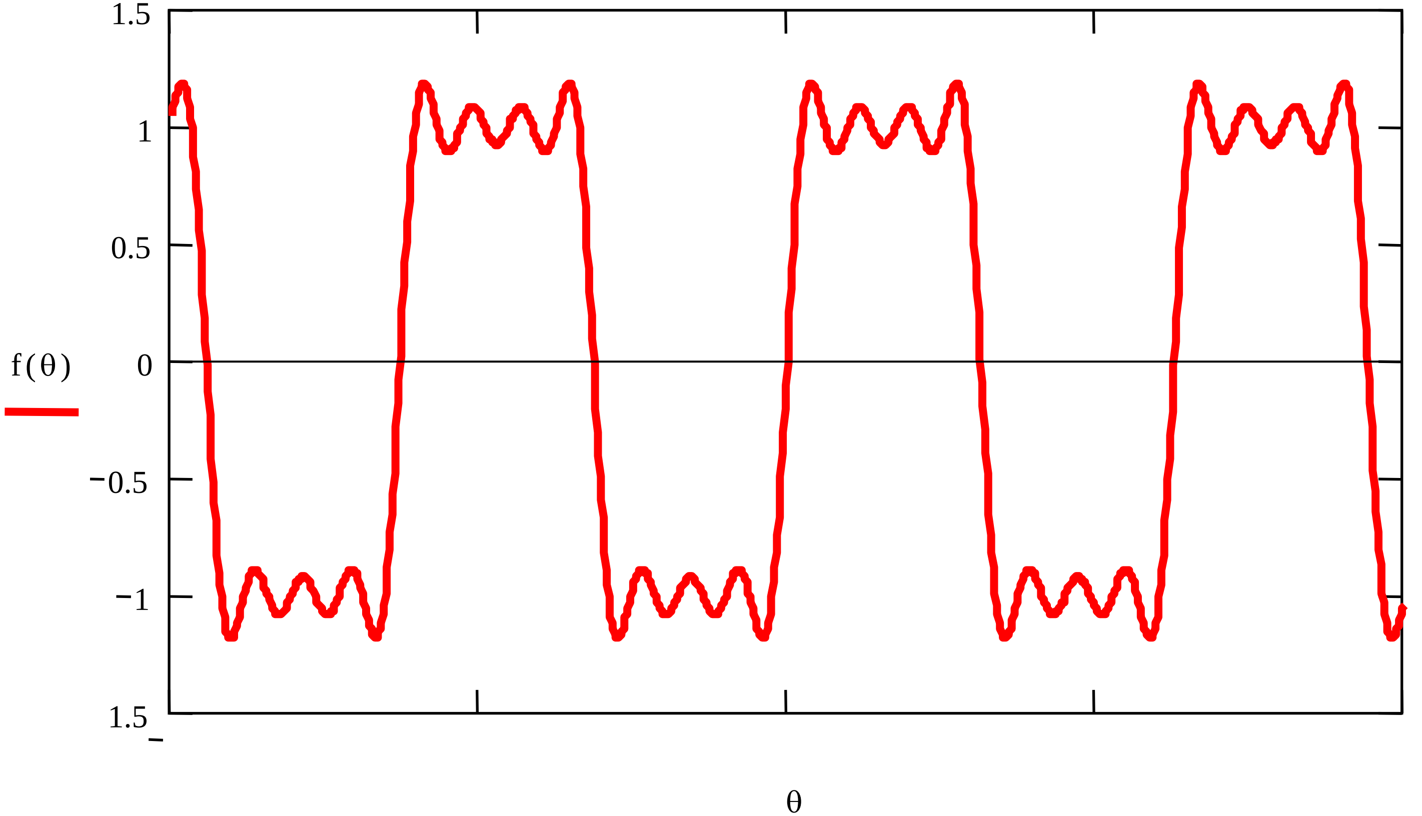
$$= 0 \quad \text{when n is even}$$

Therefore, the corresponding Fourier series is

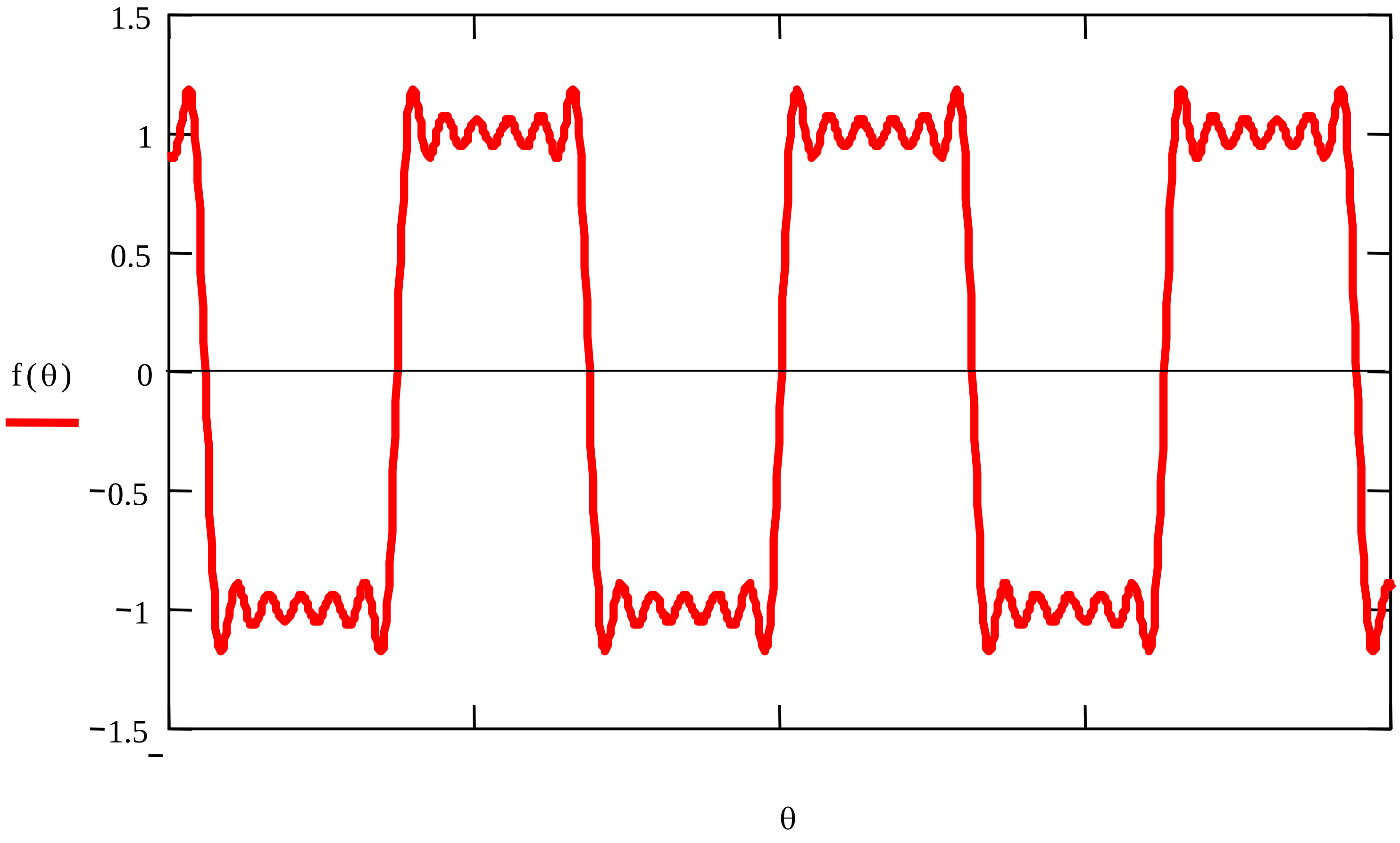
$$\frac{4A}{\pi}\left(\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta+\frac{1}{7}\sin 7\theta+\cdots\right)$$

In writing the Fourier series we may not be able to consider infinite number of terms for practical reasons. The question therefore, is – how many terms to consider?

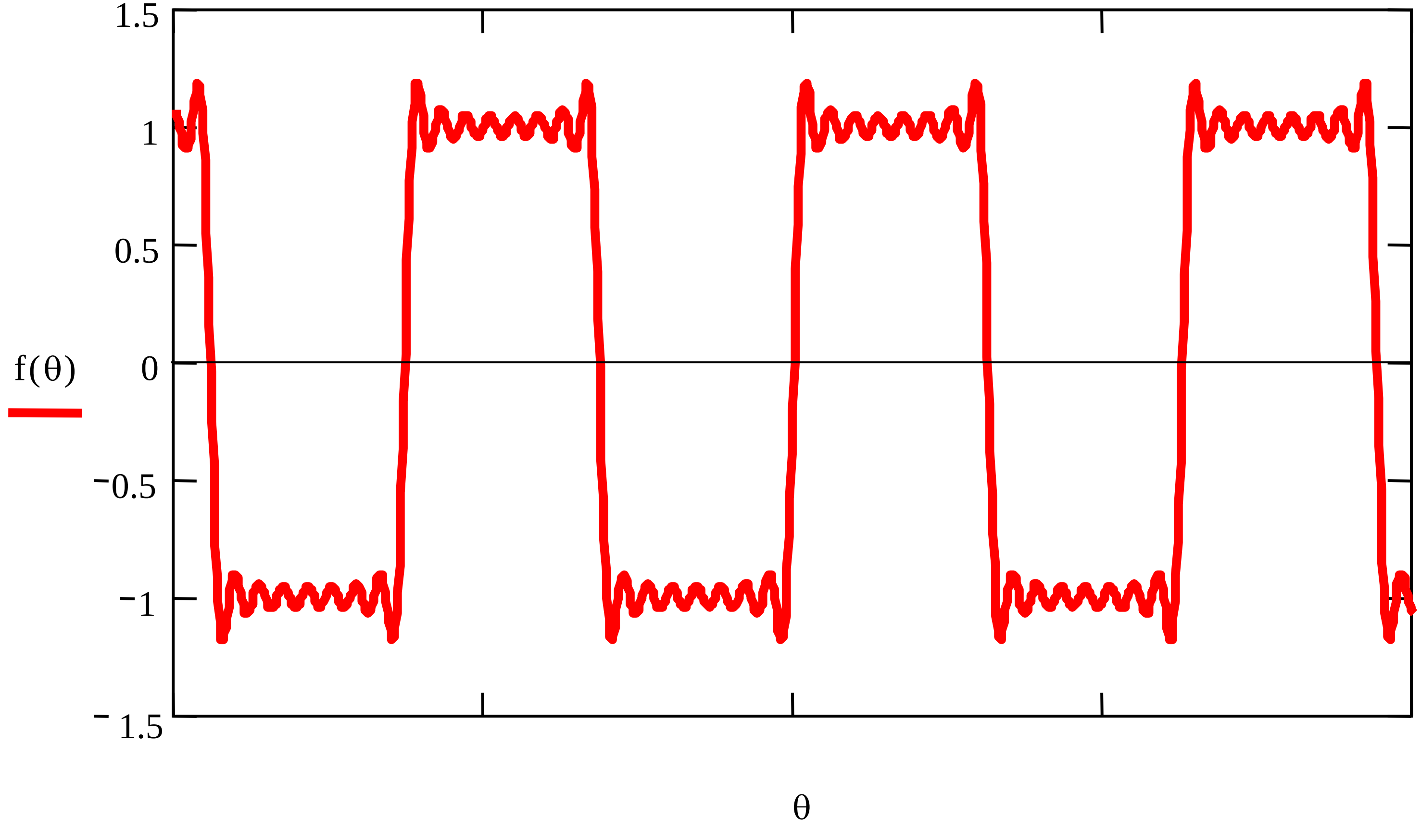
When we consider 4 terms as shown in the previous slide, the function looks like the following.



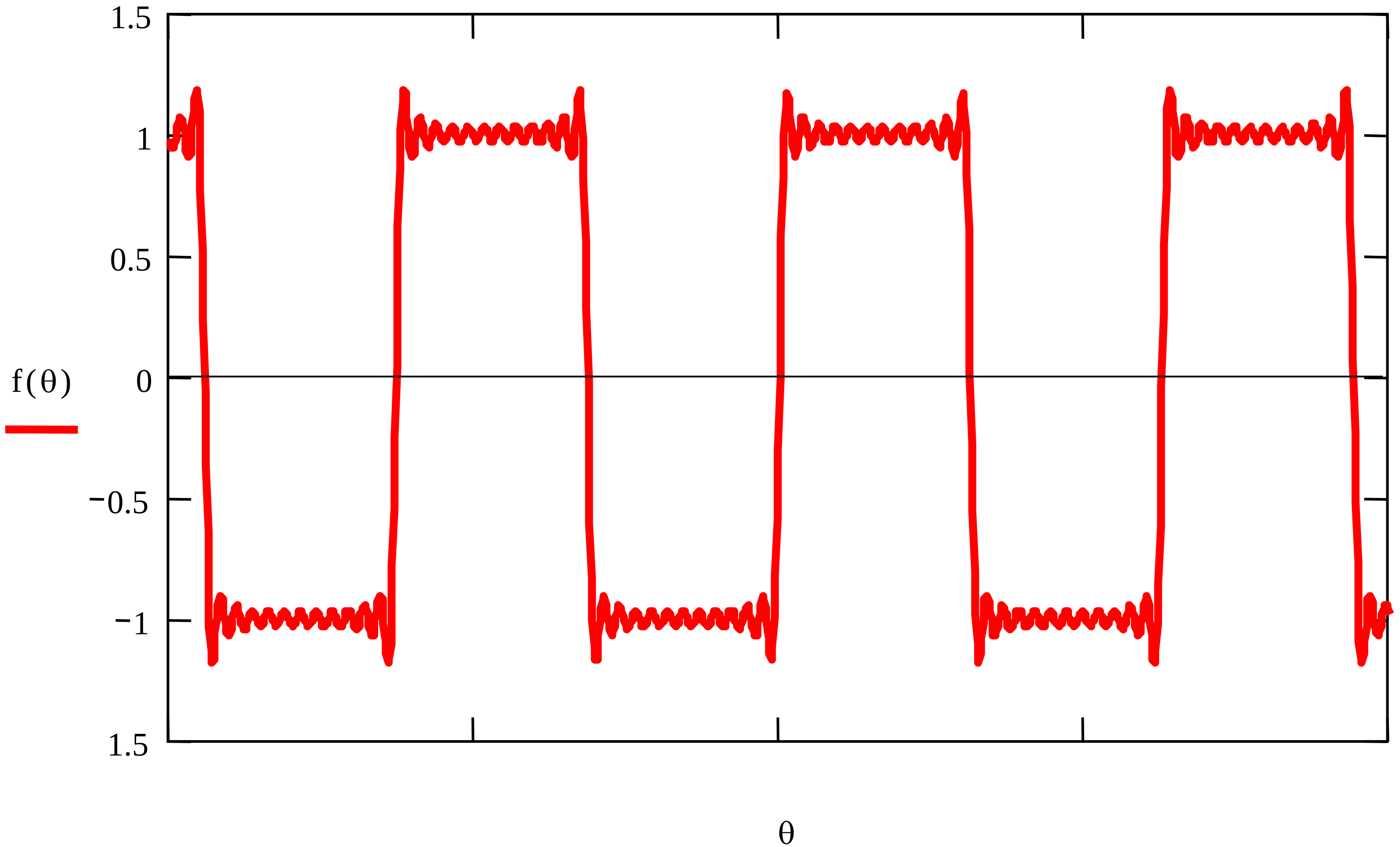
When we consider 6 terms, the function looks like the following.



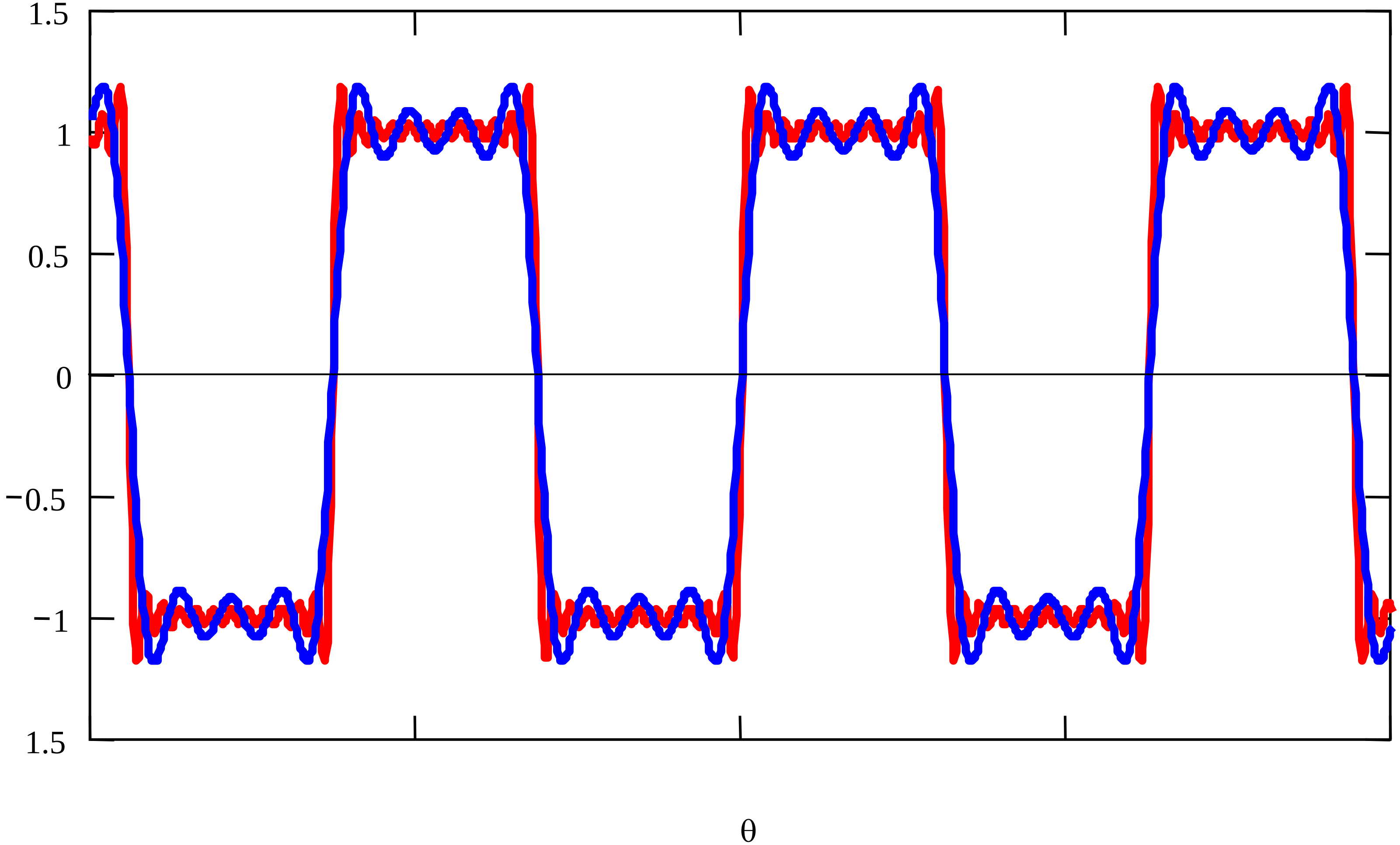
When we consider 8 terms, the function looks like the following.



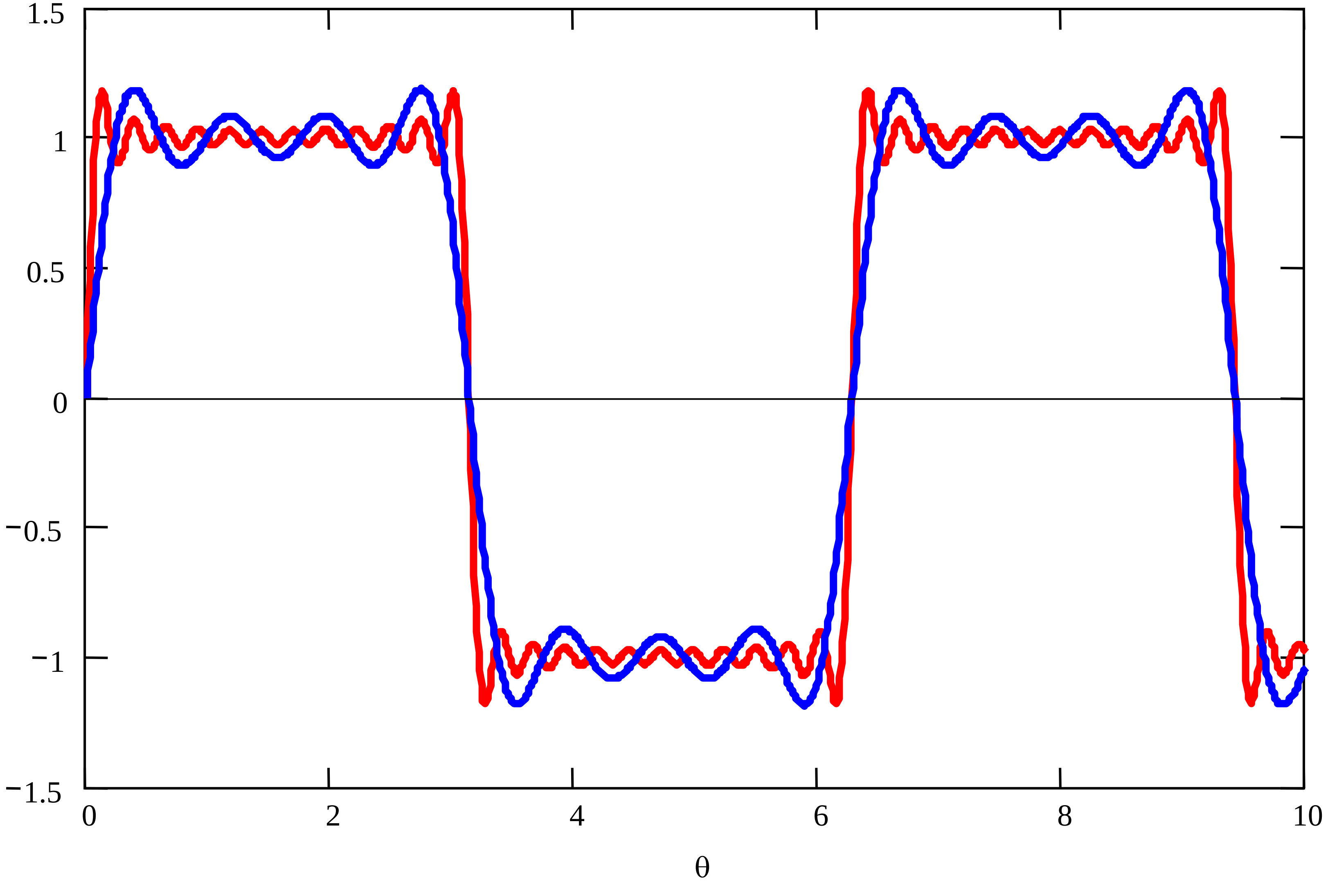
When we consider 12 terms, the function looks like the following.



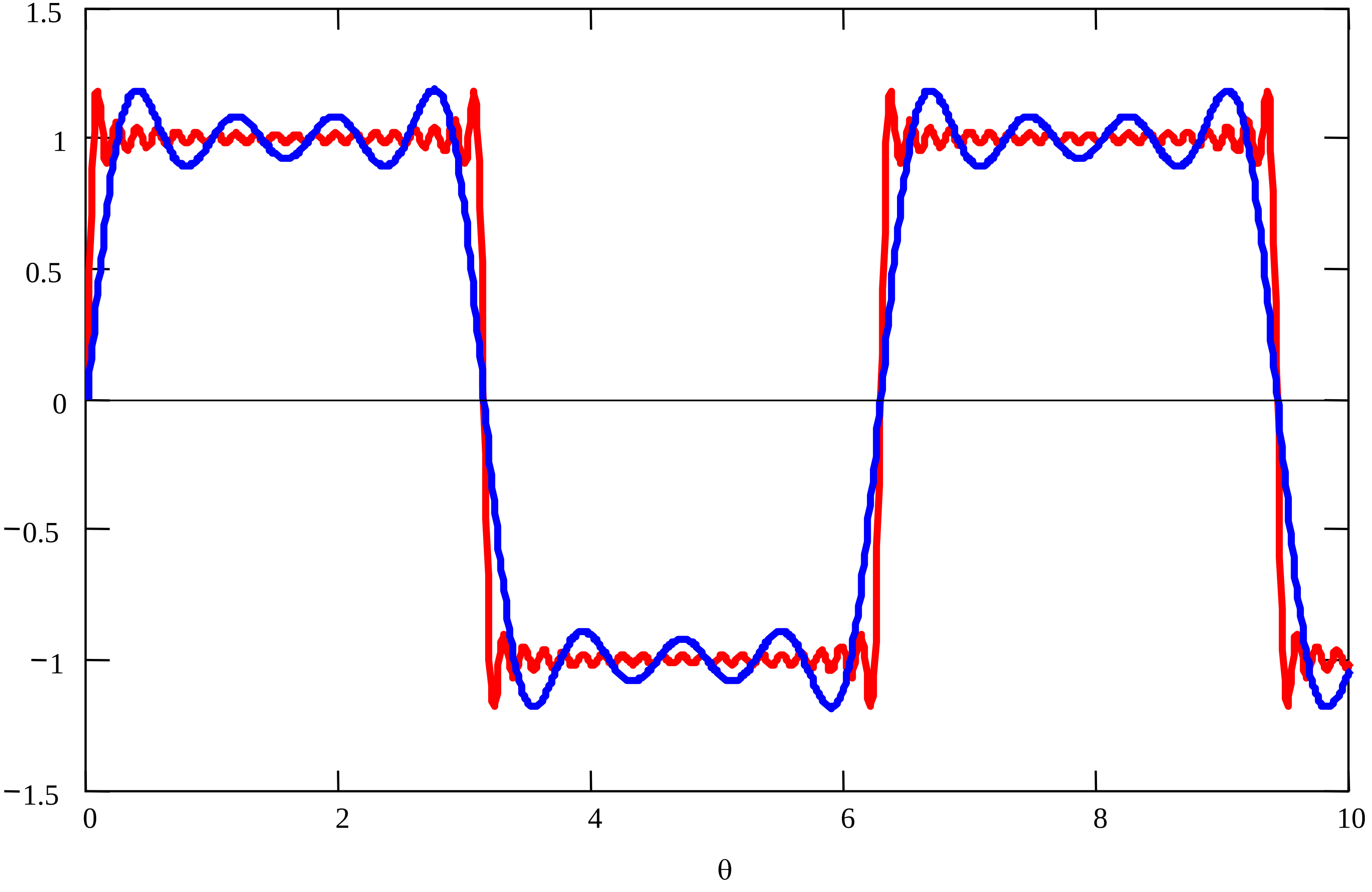
The red curve was drawn with 12 terms and the blue curve was drawn with 4 terms.



The red curve was drawn with 12 terms and the blue curve was drawn with 4 terms.



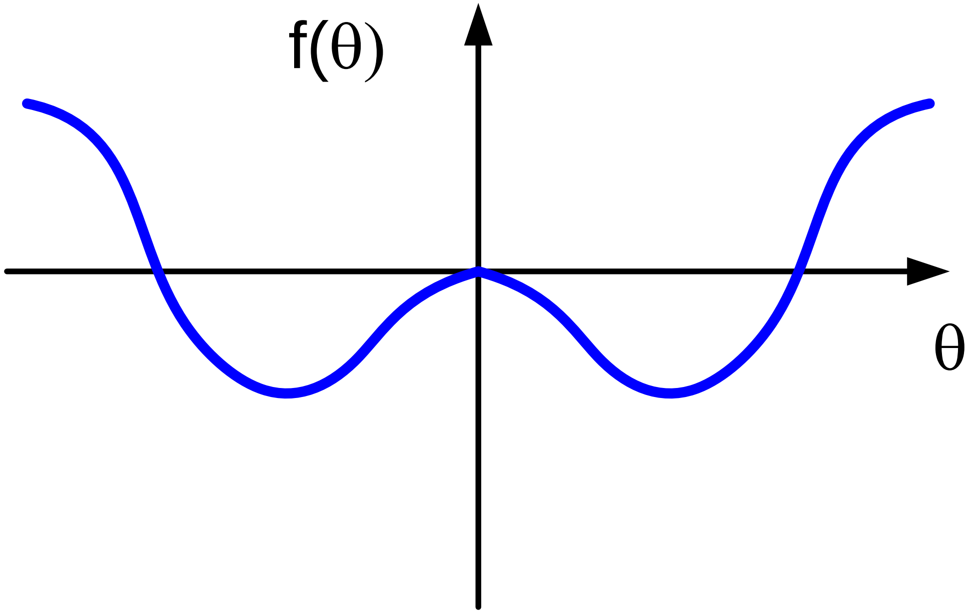
The red curve was drawn with 20 terms and the blue curve was drawn with 4 terms.



Even and Odd Functions

(We are not talking about even or odd numbers.)

Even Functions

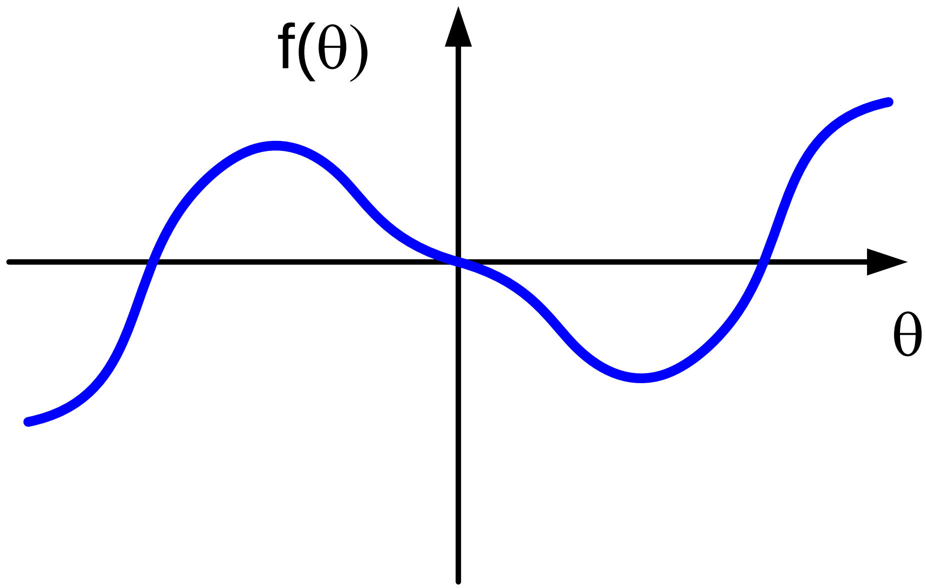


The value of the function would be the same when we walk equal distances along the X-axis in opposite directions.

Mathematically speaking -

$$f(-\theta) = f(\theta)$$

Odd Functions

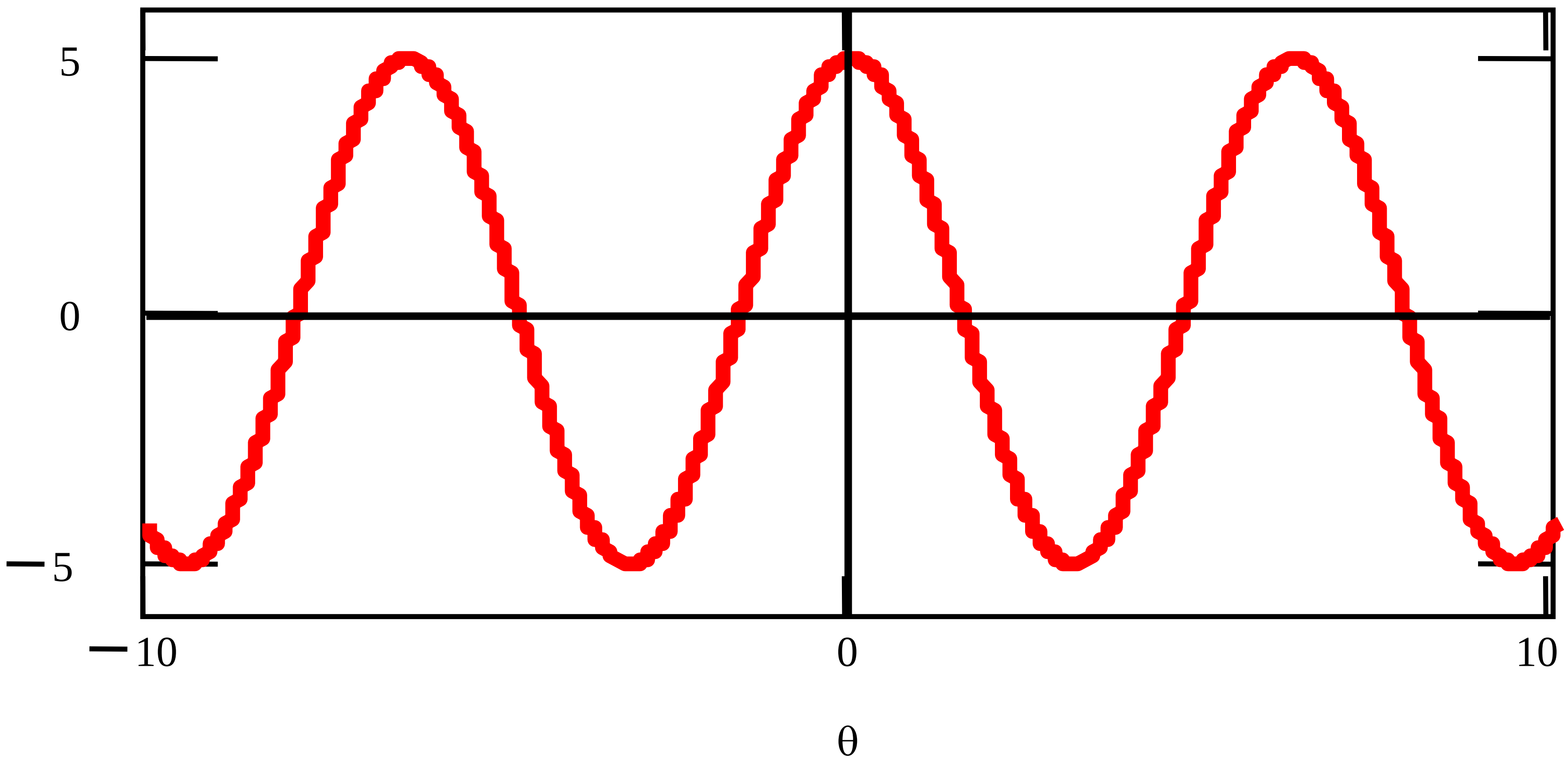


The value of the function would change its sign but with the same magnitude when we walk equal distances along the X-axis in opposite directions.

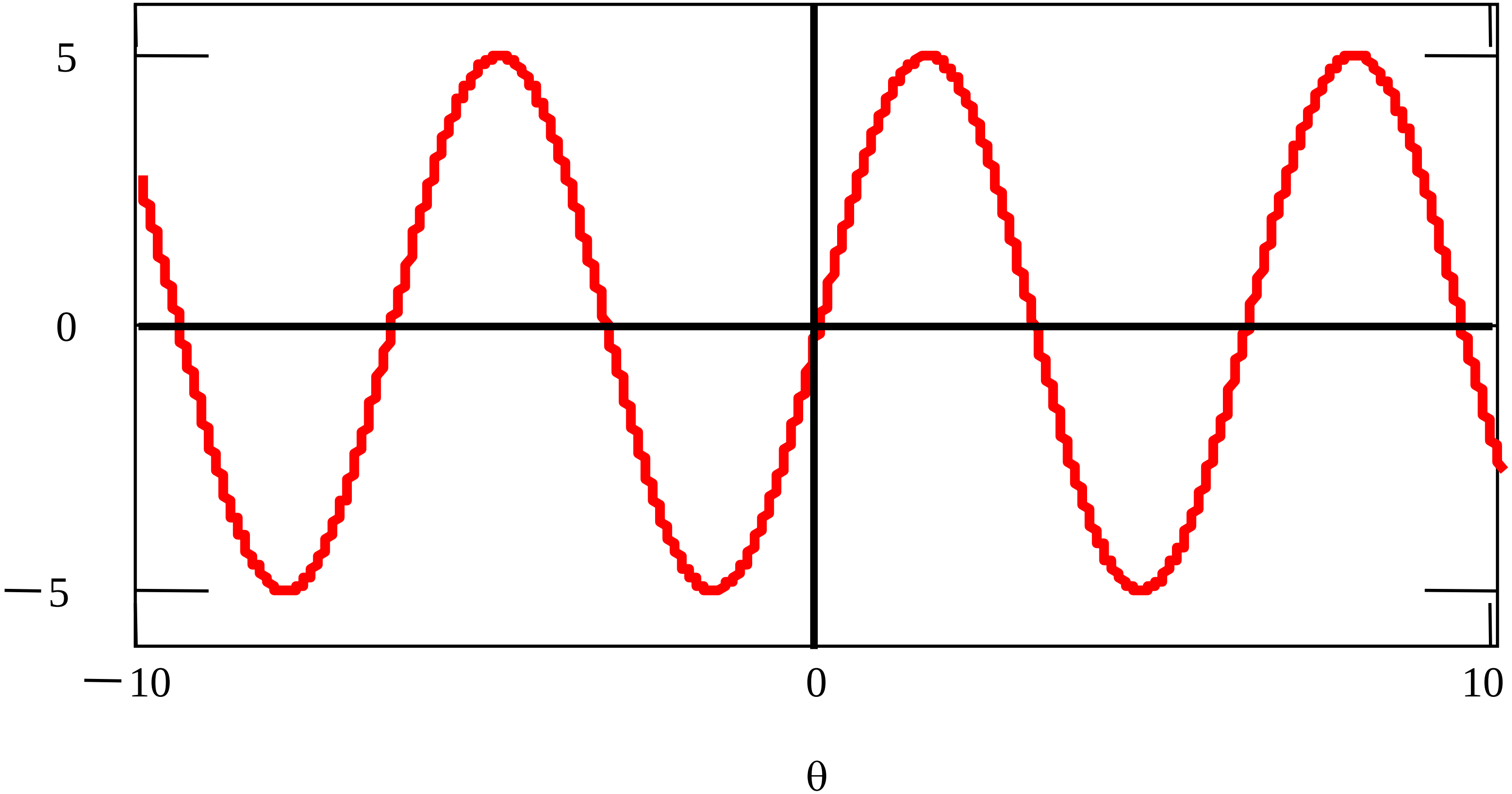
Mathematically speaking -

$$f(-\theta) = -f(\theta)$$

Even functions can solely be represented by cosine waves because, cosine waves are even functions. A sum of even functions is another even function.



Odd functions can solely be represented by sine waves because, sine waves are odd functions. A sum of odd functions is another odd function.



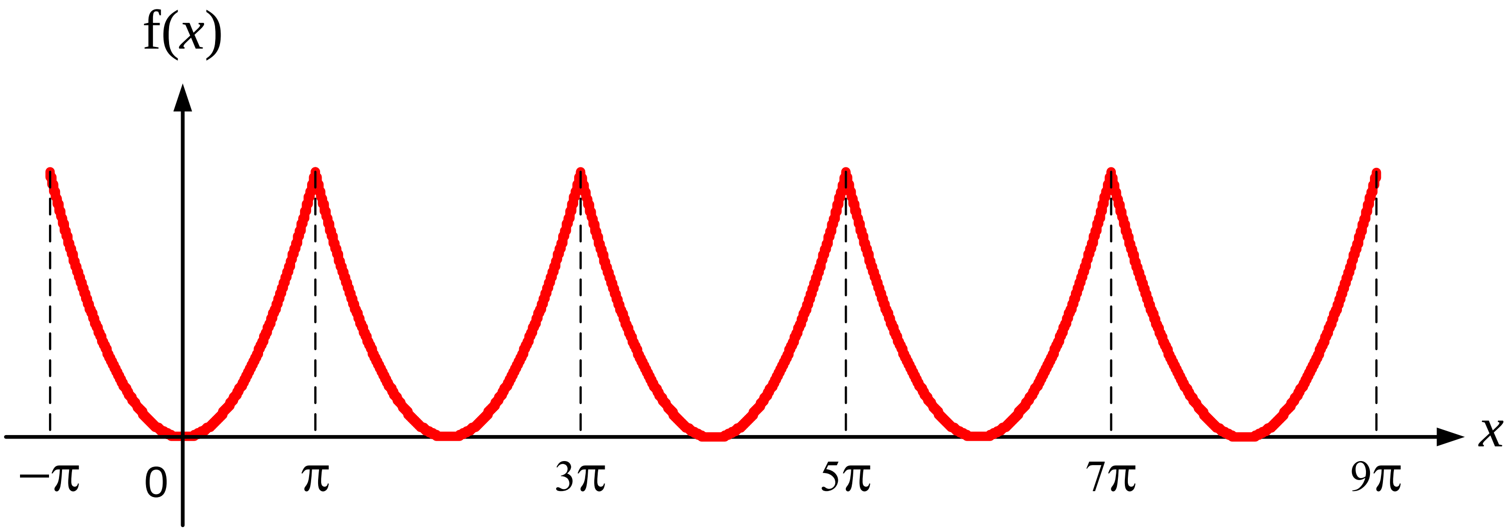
The Fourier series of an even function $f(\theta)$ is expressed in terms of a cosine series.

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

The Fourier series of an odd function $f(\theta)$ is expressed in terms of a sine series.

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

Example 2. Find the Fourier series of the following periodic function.



$$f(x) = x^2 \quad \textit{when} \quad -\pi \leq x \leq \pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{x=-\pi}^{x=\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x^2 \cos nx dx\right]$$

Use integration by parts. Details are shown in your class note.

$$a_n = \frac{4}{n^2}\cos n\pi$$

$$a_n = -\frac{4}{n^2} \quad \text{when n is odd}$$

$$a_n = \frac{4}{n^2} \quad \text{when n is even}$$

This is an even function.

Therefore, $b_n = 0$

The corresponding Fourier series is

$$\frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right)$$

Functions Having Arbitrary Period

Assume that a function $f(t)$ has period, T. We can relate angle (θ) with time (t) in the following manner.

$$\theta = \omega t$$

ω is the angular velocity in radians per second.

$$\omega = 2\pi f$$

f **is the frequency of the periodic function,**

$f(t)$

$$\theta = 2\pi f t$$ **where** $$f = \frac{1}{T}$$

Therefore, $$\theta = \frac{2\pi}{T} t$$

$$\theta = \frac{2\pi}{T} t \qquad d\theta = \frac{2\pi}{T} dt$$

Now change the limits of integration.

$$\theta = -\pi \qquad -\pi = \frac{2\pi}{T} t \qquad t = -\frac{T}{2}$$

$$\theta = \pi \qquad \pi = \frac{2\pi}{T} t \qquad t = \frac{T}{2}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt$$

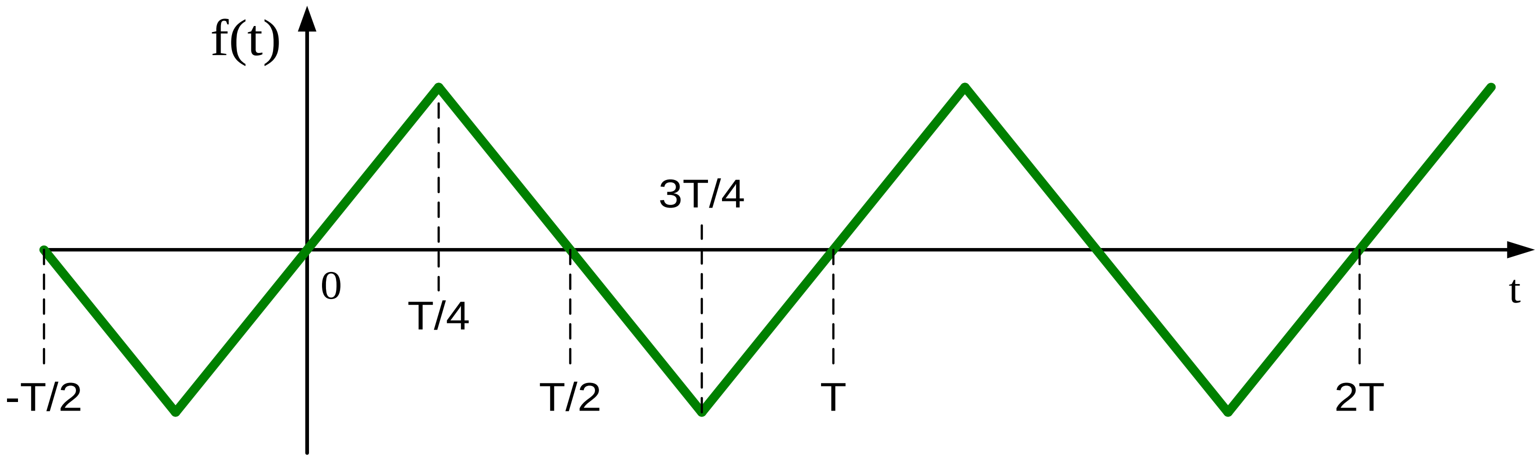
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta \qquad n = 1, 2, \cdots$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2\pi n}{T}t\right)dt \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta \qquad n = 1, 2, \cdots$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi n}{T}t\right)dt \qquad n = 1, 2, \cdots$$

Example 4. Find the Fourier series of the following periodic function.



$$f(t) = t \quad when -\frac{T}{4} \le t \le \frac{T}{4}$$

$$= -t + \frac{T}{2} \quad when \quad \frac{T}{4} \le t \le \frac{3T}{4}$$

$$f(t+T) = f(t)$$

This is an odd function. Therefore, $a_n = 0$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{4}} t\sin\left(\frac{2\pi n}{T}t\right)dt$$

$$+\frac{4}{T}\int_{\frac{T}{4}}^{\frac{T}{2}}\left(-t+\frac{T}{2}\right)\sin\left(\frac{2\pi n}{T}t\right)dt$$

Use integration by parts.

$$b_n = \frac{4}{T}\left[2.\left(\frac{T}{2\pi n}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{2T}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$b_n = 0$ when n is even.

Therefore, the Fourier series is

$$\frac{2T}{\pi^2}\left[\sin\left(\frac{2\pi}{T}t\right)-\frac{1}{3^2}\sin\left(\frac{6\pi}{T}t\right)+\frac{1}{5^2}\sin\left(\frac{10\pi}{T}t\right)-\cdots\right]$$

The Complex Form of Fourier Series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

Let us utilize the Euler formulae.

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2i}$$

The nth harmonic component of (1) can be expressed as:

$$a_n \cos n\theta + b_n \sin n\theta$$

$$= a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} + b_n \frac{e^{jn\theta} - e^{-jn\theta}}{2i}$$

$$= a_n \frac{e^{jn\theta} + e^{-jn\theta}}{2} - ib_n \frac{e^{jn\theta} - e^{-jn\theta}}{2}$$

$$a_n \cos n\theta + b_n \sin n\theta$$

$$= \left(\frac{a_n - jb_n}{2}\right) e^{jn\theta} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\theta}$$

Denoting

$$c_n = \left(\frac{a_n - jb_n}{2}\right), \quad c_{-n} = \left(\frac{a_n + jb_n}{2}\right)$$

and $c_0 = a_0$

$$a_n \cos n\theta + b_n \sin n\theta$$

$$= c_n e^{jn\theta} + c_{-n} e^{-jn\theta}$$

The Fourier series for $f(\theta)$ can be expressed as:

$$f(\theta) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{jn\theta} + c_{-n} e^{-jn\theta}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$$

The coefficients can be evaluated in the following manner.

$$c_n = \left(\frac{a_n - jb_n}{2} \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta - \frac{j}{2\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) (\cos n\theta - j \sin n\theta) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-jn\theta} d\theta$$

$$c_{-n} = \left(\frac{a_n + jb_n}{2}\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta + \frac{j}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(\cos n\theta + j\sin n\theta)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{jn\theta}d\theta$$

$$c_n = \left(\frac{a_n - jb_n}{2} \right) \quad c_{-n} = \left(\frac{a_n + jb_n}{2} \right)$$

Note that c_{-n} is the complex conjugate of c_n. Hence we may write that

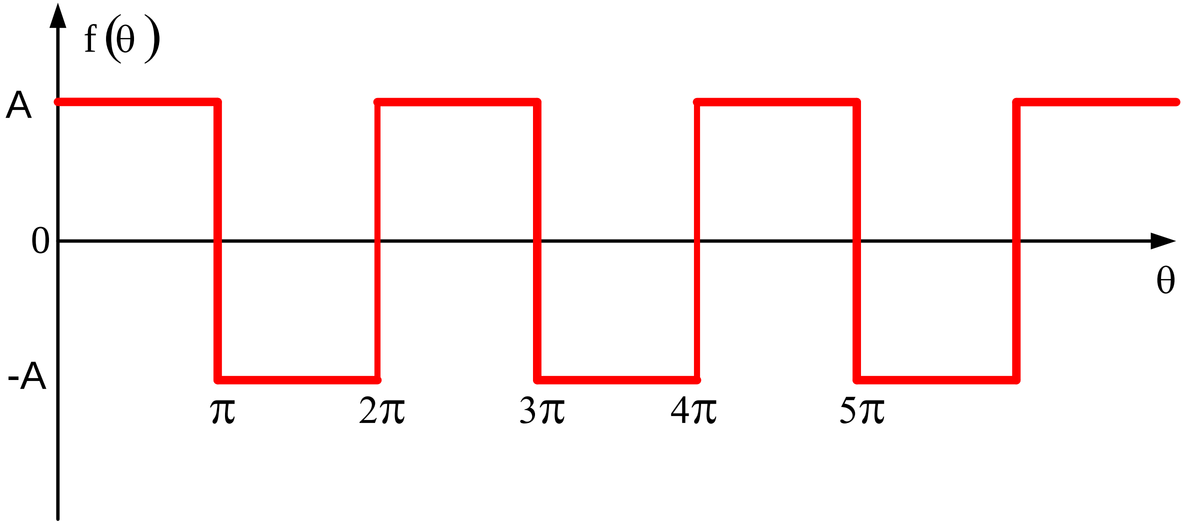
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-jn\theta} d\theta$$

$$n = 0, \pm 1, \pm 2, \cdots$$

The complex form of the Fourier series of $f(\theta)$ with period 2π is:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$$

Example 1. Find the Fourier series of the following periodic function.



$$f(\theta) = A \quad \textbf{\textit{when}} \quad 0 < \theta < \pi$$

$$= -A \quad \textbf{\textit{when}} \quad \pi < \theta < 2\pi$$

$$f(\theta + 2\pi) = f(\theta)$$

$$A := 5$$

$$f(x) := \begin{cases} A & \text{if } 0 \le x < \pi \\ -A & \text{if } \pi \le x \le 2\cdot\pi \\ 0 & \text{otherwise} \end{cases}$$

$$A_0 := \frac{1}{2\pi} \cdot \int_0^{2\pi} f(x)\, dx$$

$$A_0 = 0$$

$$n := 1 .. 8$$

$$A_n := \frac{1}{\pi} \cdot \int_0^{2\pi} f(x) \cdot \cos(n \cdot x) \, dx$$

$A_1 = 0$ $\quad A_2 = 0$ $\quad A_3 = 0$ $\quad A_4 = 0$

$A_5 = 0$ $\quad A_6 = 0$ $\quad A_7 = 0$ $\quad A_8 = 0$

$$B_n := \frac{1}{\pi} \cdot \int_0^{2\pi} f(x) \cdot \sin(n \cdot x)\, dx$$

$B_1 = 6.366$ $B_2 = 0$ $B_3 = 2.122$ $B_4 = 0$

$B_5 = 1.273$ $B_6 = 0$ $B_7 = 0.909$ $B_8 = 0$

Complex Form

$$f(\theta)=\sum_{n=-\infty}^{\infty} c_n e^{jn\theta}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-jn\theta}d\theta$$

$$n = 0, \pm 1, \pm 2, \cdots$$

$$C(n) := \frac{1}{2\pi}\cdot\int_{0}^{2\pi} f(x)\cdot e^{-1i\cdot n\cdot x}dx$$

$$C(n) := \frac{1}{2\pi} \cdot \int_0^{2\pi} f(x) \cdot e^{-1i \cdot n \cdot x}\, dx$$

$C(0) = 0$ $\quad$ $C(1) = -3.183i$ $\quad$ $C(2) = 0$ $\quad$ $C(3) = -1.061i$

$C(4) = 0$ $\quad$ $C(5) = -0.637i$ $\quad$ $C(6) = 0$ $\quad$ $C(7) = -0.455i$

$C(-1) = 3.183i$ $\quad$ $C(-2) = 0$ $\quad$ $C(-3) = 1.061i$

$C(-4) = 0$ $\quad$ $C(-5) = 0.637i$ $\quad$ $C(-6) = 0$ $\quad$ $C(-7) = 0.455i$