Differential Geometry

14.11.2024

L-1

7 Lectures 3 Problem sessions

Set $X \rightarrow$ topological set (X, ϕ are both open & closed)

A Topology τ, on a ~~set X~~ non-empty set X is a collection of subsets of X, ($\in \tau$) called elements in τ are open sets, such that

a) X, ϕ are open $X, \phi \in \tau$

b) arbitrary union of open sets are open $\Rightarrow$ (any union of sets in τ gives another set in τ)

$X_1, \ldots, X_n \subset \tau$, $\cup X_i \in \tau$,

c) finite intersection of open sets are open $\rightarrow$ (any intersection of sets in τ gives another set in τ)

$(X, \tau) \rightarrow$ topological space & $\tau \rightarrow$ a topology on X

Example:

1) $X = \{a, b, c\}$

topology on set $X \rightarrow$ $\tau = \{ X, \phi, \{a, b\}, \{b, c\}, \{b\} \}$

2) $\mathbb{R}^1$

τ = collection of usual open sets in $\mathbb{R}$ in Real analysis is a topology

ets are topo...

Map: $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ is continuous if

topological space

for every open set $V \subset Y$, $\underbrace{f^{-1}(V)}_{\text{pre-image}}$ is open in X.

$V \in \tau_Y$ $\quad$ $f^{-1}(V) \in \tau_X$

X, T এর জায়গা থাকে τ থাকে, τ_X, τ_Y

open set → যাদেরকে open বলব এটা defn

★ এভাবে defn দাঁড় করালে real line এর intuitive continuity define করা যাবে

Homeomorphic: $f : X \longrightarrow Y$

① f is bijective
② f continuous
③ f^{-1} continuous

(then) f is a homeomorphism between X and Y.

X ও Y একই রকম আচরণ ধারেকাছের প্রেক্ষিতে

Locally Euclidean space:

A topological space X is called locally Euclidean of dimension n if $\underbrace{\forall p \in X}_{\text{for all points in } X}$ we have an open neighbourhood $\underbrace{p \in U}_{\text{containing } p}$ s.t. $\underbrace{\varphi : U \longrightarrow \mathbb{R}^n \text{ is a homeomorphism.}}$

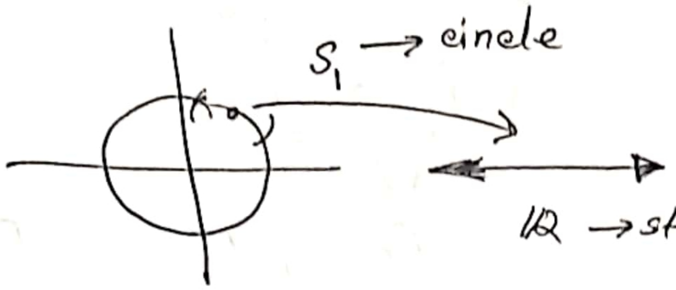
a homeomorphism exists from U to $\mathbb{R}^n$

a homeomorphism: a bijective map which itself & its inverse are continuous → as defn topologi

Joining Tu

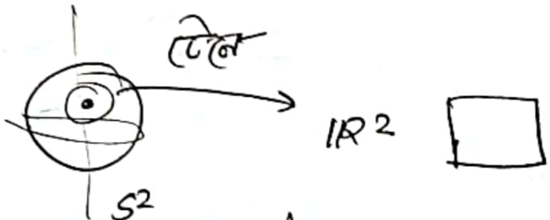
Example:

1)



Topological homeomorphism → deform করে এমন কিছু পেলে তারা homemorphic, কিছু ~~কাটা~~ without cutting/ disconnecting

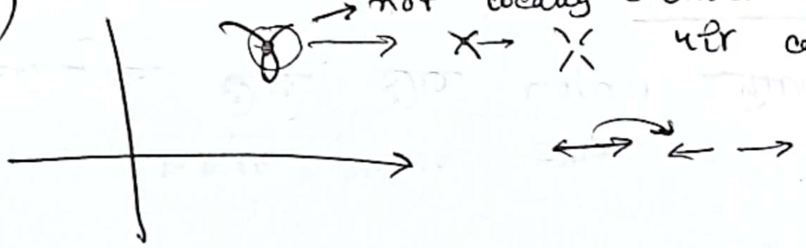
2)



S^2 → locally Euclidean with dimension 2

sphere surface

3)



→ not locally euclidean

4টা component হয়ে যাবে point বাদ দিলে

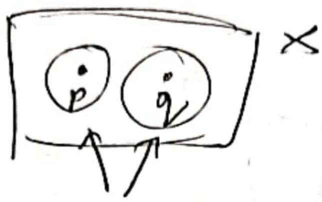
→ কিন্তু R^1 এ দুইটা হবে

Topological Manifold:

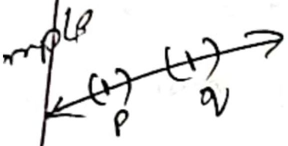
A topological ~~manifold~~ space X is called a topological manifold n if X is

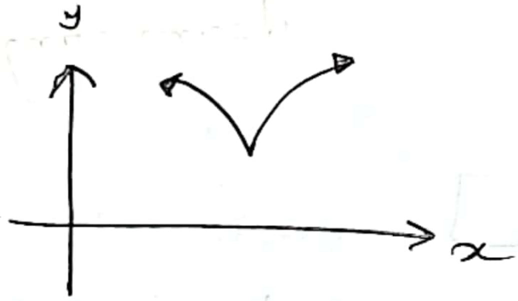
1. Hausdorff
2. Second countable → countable # basis
3. locally euclidean (n space)

Hausdorff:



for any pair of points in X, if open sets around them can be found s.t. the sets dont intersect then Hausdorff





~~2 D~~ $\mathbb{R}^2$ থেকে আসা ~~subspalle~~ subspace topology

topological basis

~~Smooth~~ Countable basis: ~~কি~~ কিছু open set থাকলে এদের union করে করে বাকিদের ~~বানা~~ বানাতে পারব

Example:

$\beta = \{(a,b) : a < b\}$ → সব open set কে এদের union করে বানানো হয়

তাই β is a basis for $\mathbb{R}$

Second countable: যদি countable সংখ্যক basis থাকে

$$\bar{\beta} = \{(a,b) : a<b \text{ and } a,b \in \mathbb{Q}\}$$

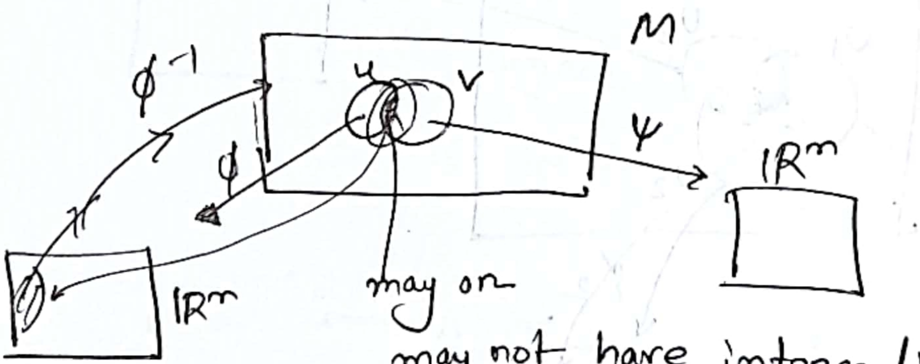
As $\mathbb{Q}$ is countable, so $\bar{\beta}$ is a countable that forms a basis of $\mathbb{R}$. Then $\mathbb{R}$ is 2^{nd} countable.

Smooth manifold:

C^∞) Chart compatibility: ~~Topolo~~ $(U,\phi) \rightarrow$ chart

locally Euclidean space হয়

Let (U,ϕ) & (V,ψ) be two charts of কোন a topological Manifold M, they are called C^∞-com[patible] if



may or may not have intersection

i) intersection না থাকলে তাহলেই C^∞ compatible

ii) intersection থাকলে, $\psi \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \psi(U \cap V)$

$C^\infty \rightarrow$ can be differentiat[ed] ∞ times

$$\psi \circ \phi^{-1} : \phi(U \cap V) \subset \mathbb{R}^n \longrightarrow \psi(U \cap V) \subset \mathbb{R}^n$$

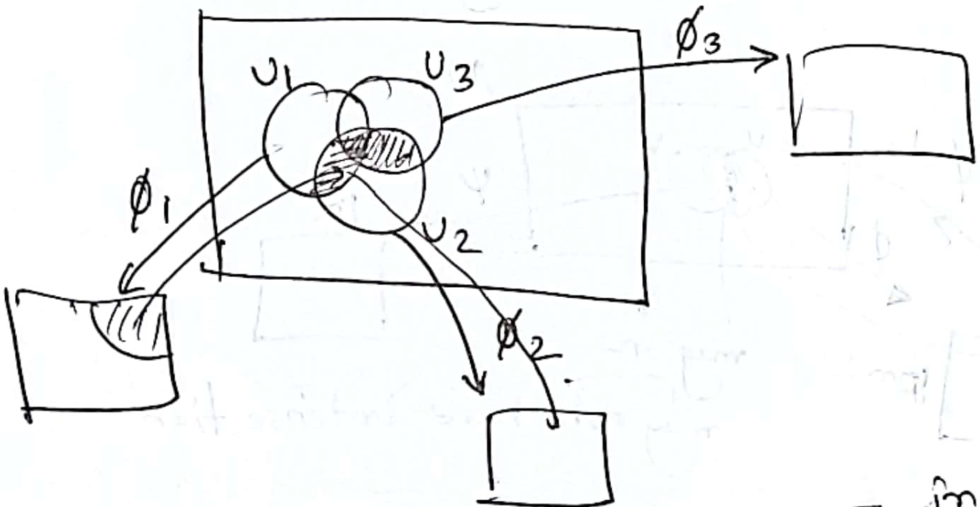
$$\phi \circ \psi^{-1} : \psi(U \cap V) \subset \mathbb{R}^n \longrightarrow \phi(U \cap V)$$

দুটি যদি C^∞ হয়,

তাহলে তারা C^∞ compatible.

– intersected region এর map যেন C^∞ হয়

If (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible and, (U_2, ϕ_2) and (U_3, ϕ_3) are C^∞-compatible does NOT imply (U_1, ϕ_1), (U_3, ϕ_3) $\nRightarrow$ C^∞-comp



সবার intersection region এর জন্য,

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$$ কিন্তু

এদের intersection আলাদা, কিন্তু বাদ আছে

R^n diff,

$\underbrace{f(x,y)} = (x+y,\ x^2+xy)$

$\psi_0\phi^{-1}$:

$\mathbb{R}^n \longrightarrow \mathbb{R}^n$

Maximum atlas:

$M \rightarrow$ a topological manifold

$\{(U_\alpha, \phi_\alpha)\}$ s.t. $\cup U_\alpha = M$

সকল pairwise compatible chart এর largest collection, নতুন chart add করা যাবেনা।

আছে এই property থাকে, smooth

Smooth manifold:

① Topological manifold

② has a C^∞ - compatible maximal atl

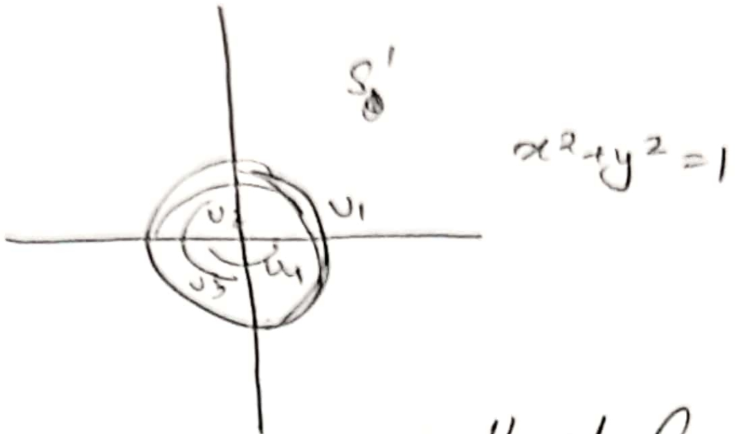
㊉ M is a smooth manifold if

① M is a topological manifold

② It has an C^∞ atlas $\{(U_\alpha, \phi_\alpha)\} \rightarrow C^\infty$ at

$\cup U_\alpha = M$

We will prove S_1 is a smooth manifold



The mapping to $\mathbb{R}^1$ give Hausdorff & 2nd countability. So we need atlas.

$U_1 = \{(x,y) \in S^1 : x > 0\}$

$U_2 = \{(x,y) \in S^1 : y > 0\}$

$U_3 = \{(x,y) \in S^1 : x < 0\}$

$U_4 = \{(x,y) \in S^1 : y < 0\}$

$$\bigcup_{i=1}^{4} U_i = S^1$$

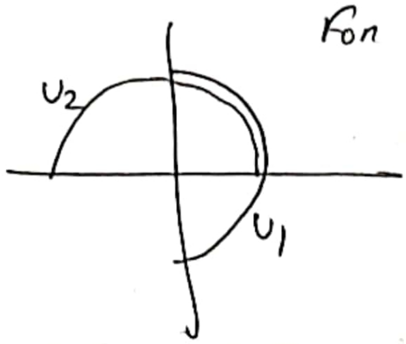
$\phi_1 : U_1 \longrightarrow \mathbb{R}$, $\phi_1(x,y) = y$

$\phi_2 : U_2 \longrightarrow \mathbb{R}$, $\phi_2(x,y) = x$

$\phi_3 : U_3 \longrightarrow \mathbb{R}$, $\phi_3(x,y) = y$

$\phi_4 : U_4 \longrightarrow \mathbb{R}$, $\phi_4(x,y) = x$

${}^4C_2(6)$ pairs need to be compatibility checked



For U_1, U_2,

$$\phi_2 \circ \phi_1^{-1} : \underbrace{\phi_1(U_1 \cap U_2)}_{(0,1)} \longrightarrow \phi_2(U_1 \cap U_2)$$

$$\phi_1(x,y) = y$$

$$\phi_1^{-1}(y) = (\sqrt{1-y^2}, y)$$

$$\phi_2 \circ \phi_1^{-1}(x) = \phi_2(\sqrt{1-x^2}, x) = \sqrt{1-x^2}, \quad x \in (0,1)$$

$$\in C^\infty(0,1)$$

Similarly ~~all oth~~ $\phi_1 \circ \phi_2^{-1}$ is also C^∞.

$\therefore (U_1, \phi_1)$ & (U_2, ϕ_2) are C^∞ compatible

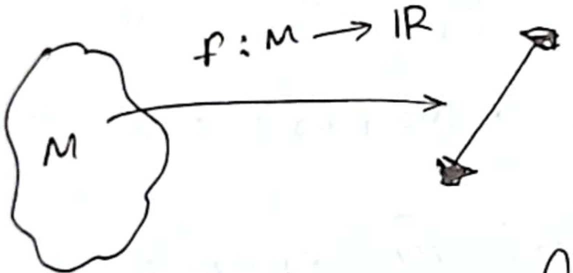
Such can be proved for every other pair.

S_0, $S_1 = \{(U_1, \phi_1), (U_2, \phi_2), (U_3, \phi_3), (U_4, \phi_4)\}$

is a smooth manifold

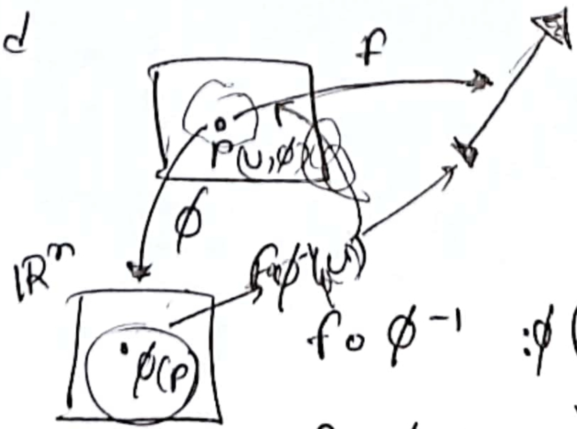
* দুইটা manifold Cartesian ~~cross~~ product নিলে নতুন manifold তৈরি হয়

Calculus on curved coordinates:



Smooth fn ~~fun~~ between manifolds & real numbers:-

A function $f : M \longrightarrow \mathbb{R}$ is called smooth at $p \in M$, if there is a chart (U, ϕ) s.t. $p \in U$ and



$f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbb{R}$

$\downarrow$

$f \circ \phi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$

f is smooth if $f\circ\phi^{-1}: \mathbb{R}^n \to \mathbb{R}$ is smooth $\phi(p)$.

সাপেক্ষে f এর

যেহেতু M এর smoothness জানিনা, তাই $p\in M$ কে $\mathbb{R}^n$ এ নেব ϕ দিয়ে, তারপর $f\circ\phi^{-1}$ এর smoothness check করব।

(V,ψ) এর সাপেক্ষেও smooth হবে- যদি $p\in U\cap V$ হবে যদি ϕ,ψ C^∞ compatible

independent of chart → $f\circ\psi^{-1} = f\circ\phi^{-1}\circ\underbrace{\boxed{\phi\circ\psi^{-1}}}_{\mathbb{R}^n\to\mathbb{R}^n}$

a fn with smooth comp

So, smoothness depend করে f ও M এর উপর

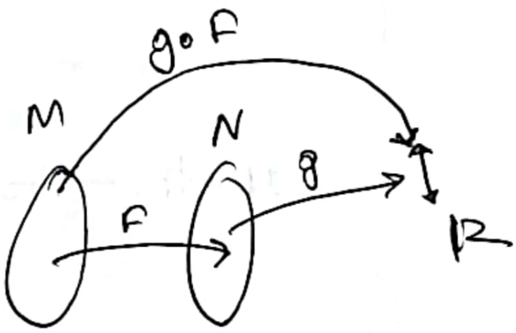
smooth → C^∞

Let $F: M\to N$ be a map from manifold M to N.

Pull back of function:

Let $g: N\to\mathbb{R}$

$(g\circ F): M\to\mathbb{R}$



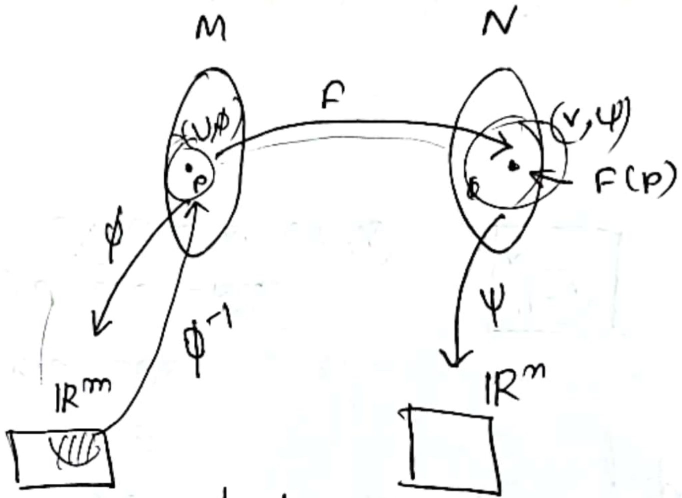
M এর উপর fn

Pullback: $F^*g = g\circ F$

$$(F^* g)(p) = g \circ \underbrace{F(p)}_{}$$

Just a map

Smooth map between Manifolds:



→ topological continuity

A continuous map $F: M \to N$ is smooth ~~at~~ $\forall\, p \in M$ if $\exists\,(U, \phi)$ containing p and (V, ψ) containing $F(p)$ such that

$$\psi \circ F \circ \phi^{-1} : \phi(U) \longrightarrow \psi(V)$$

$$\mathbb{R}^m \longrightarrow \mathbb{R}^n \text{ is smooth at } \phi(p)$$

~~If this is true for~~

are

t true

C Homeomorphism

Diffeomorphism :

$M, N \to$ smooth manifolds

orphism

, N

$F : M \longrightarrow N$ is a diffeomorphism if | same for homeo

necessary condition

bijech ① F is bijective

cont ② F is smooth

is cont ③ F^{-1} is smooth

From topology it can be proved that M & N have same local Euclidean dimension

হতেই পারবে smooth

\# Let M be a smooth manifold and

Topological manifold

C^∞ maximal atlas

$U \subset M$ be an open set of M, then we can show

① U is Hausdorff, second countabe

② U is locally Euclidean

③ $\{(U \cap U_\alpha, \phi_\alpha|_{U \cap U_\alpha})\}$

a reduced chart

So ~~manifolds~~ open sets within manifolds are ~~sm~~ also manifolds with same dimensions.

\# $M_{n\times n} \to$ set of all $n \times n$ matrices

$M_{n\times n} \equiv \mathbb{R}^{n^2} \to$ যেকোনো open set হলে $M_{n\times n}$ ও open set

as R^{n^2} is a manifold, so

$M_{n\times n}$ is also a manifold

$$\det : M_{n\times n} \equiv \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$$

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

$\det(\underbrace{a, b, c, d}_{\mathbb{R}^4}) = ad - bc \rightarrow$ a polynomial hence smooth

So, det is a smooth map

$\det^{-1}\{\mathbb{R}\setminus\{0\}\} \longrightarrow$ all non-singular $n\times n$ matrices

General Linear Group $= GL(n, \mathbb{R})$

$\downarrow$

set of all invertible matrices

$\Downarrow$

open in $\mathbb{R}^{n^2}$

So, $GL(n, \mathbb{R})$ is an open set in $\mathbb{R}^{n^2}$.

$\Downarrow$

is a manifold of dim n^2

Elements in $GL(n, \mathbb{R})$ can be multiplied & Inverted

$A, B \in GL(n, \mathbb{R})$, $A \cdot B \in GL(n, \mathbb{R})$ A^{-1}

Smooth mapping from $\ldots \times M_{n\times n}$

prod : $\boxed{GL(n, \mathbb{R}) \times GL(n, \mathbb{R})} \longrightarrow GL(n, \mathbb{R})$

$inv: GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R})$ } continuous map

$inv(A) = A^{-1}$

$GL(n, \mathbb{R}) \longrightarrow$ একটি manifold

product, inverse define করা যায় ও

group structure follow করে

~~যেহেতু~~ এই সব prod, inv smooth

এরকম manifold কেই বলা হয় Lie groups.

Lie Group:

① Topological Manifold and group

② Product & inverse map from the group structure will also be smooth.

Field আমরা $\mathbb{R}$ use করেছি।

$GL(n, \mathbb{C})$ is also a Lie group

* Topological space এর তার group multiplication op এর উপর group property hold করলে তা Topological group

* Topological group $\neq$ Topological manifold

points to endpoints.
while a circle does
them.

Topology

Hausdorff
separable

Mapping between T. spaces

Bases

Second countable

Continuous mapping between T spaces

Homeomorphisms

Locally Euclidean space

+

Topological Manifold

Charts (U, ϕ)

(C^∞) Chart compatibility

Maximal Atlas (C^∞)

Smooth Manifold

Chart compatibility is not transitive

Any open subset of a smooth manifold is another smooth manifold having maximal atlas $\bigcup \{(U \cap U_\alpha, \phi_\alpha|_{U \cap U_\alpha})\}$

Smooth ~~mapping~~ function from any T. manifold to $\mathbb{R}$

Pullback
$F^* g : M \to \mathbb{R}$
$g : N \to \mathbb{R}$
$F : M \to N$

Smooth maps between T. manifolds

Diffeomorphism $\subset$ Homeomorphisms

$GL(n, \mathbb{R}) \equiv M_{n \times n}$ invertible $=$ open subset of $\mathbb{R}^{n^2}$ $\longrightarrow$ Lie group ($GL(n, \mathbb{C})$ too)

A differentiable mapping from $\mathbb{R}^1$ to a manifold M is called a curve

$u \to \{x^i(u)\}$ then $\frac{\partial x^i}{\partial u}$ exists

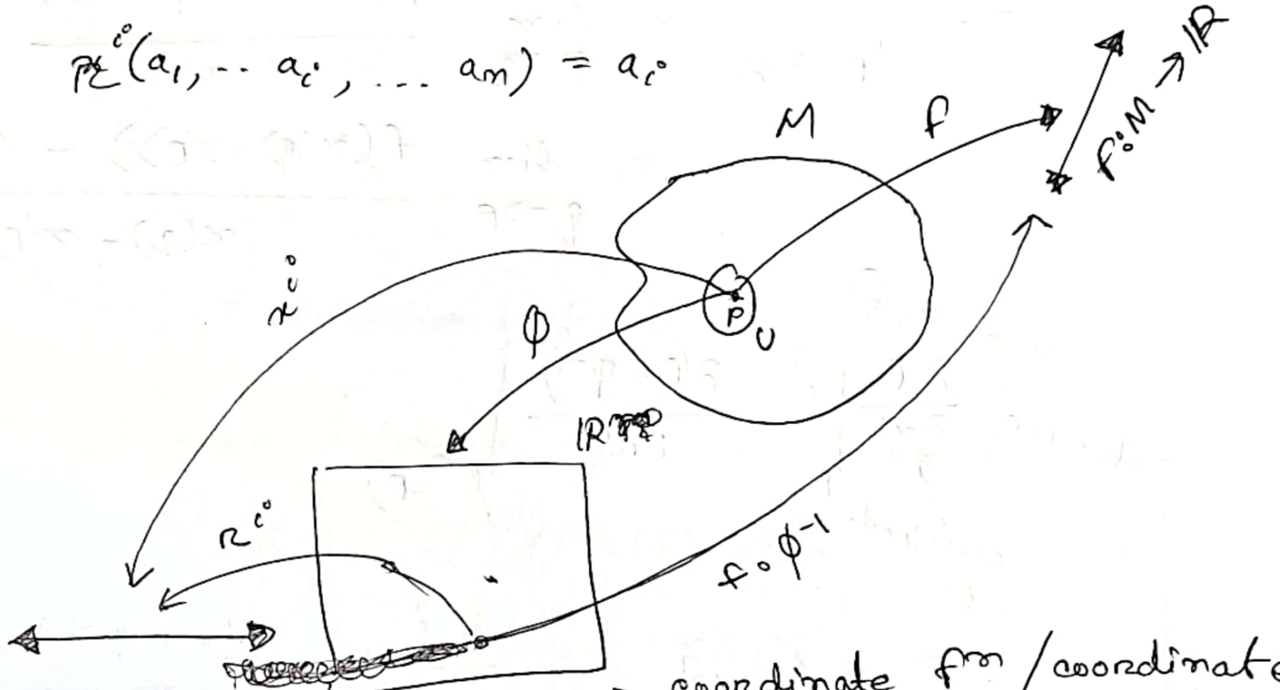
L – 2 16.11.24

no added properties, only ~~ordere~~ cartesian product

$R^n = \{(a_1, \dots, a_n) : a_i \in \mathbb{R},\ i \in 1, \dots, n\}$

$r^i : \mathbb{R}^n \longrightarrow \mathbb{R}$

$r^i(a_1, \dots a_i, \dots a_m) = a_i$



coordinate fn / coordinate chart

$x^i = r^i \circ \phi : U \to \mathbb{R}$

$\phi(p) = \underbrace{(x^1(p), x^2(p), \dots, x^n(p))}_{\text{co-ordinates of } p}$

So, এখানের থেকে $\phi = (x^1, \dots, x^n)$

$$\left.\frac{\partial f}{\partial x^i}\right|_p = \left.\frac{\partial (f \circ \phi^{-1})}{\partial r^i}\right|_{\phi(p)}$$

fn এর respect এ fn

একটি $f(x,y) \to \dfrac{\partial f}{\partial(x)} \leftarrow$ ধরি এর fn, $x(a,b) = a$ only thi

$$\frac{\partial f}{\partial x}\Big|_p = \lim_{q\to p} \frac{f(x(q),y(p)) - f(x(p),y(p))}{x(q)-x(p)}$$

$$\frac{\partial f}{\partial x}\Big|_{p=(a,b)} = \lim_{x\to a} \frac{f(x,b)-f(a,b)}{x-a}$$

$$= \lim_{q\to p} \frac{f(x(q),y(p)) - f(x(p),y(p))}{x(q)-x(p)}$$

Define নতুন definition $\left\{ \dfrac{\partial f}{\partial x^1}\Big|_p : \dfrac{\partial(f\circ\varphi^{-1})}{\partial r^i}\Big|_{\phi(p)} \right.$

co-ordinate fn

$$x^i, x^j \qquad \frac{\partial x^i}{\partial x^j} \;\; \frac{\partial(x^i\circ\varphi^{-1})}{\partial r^j}$$

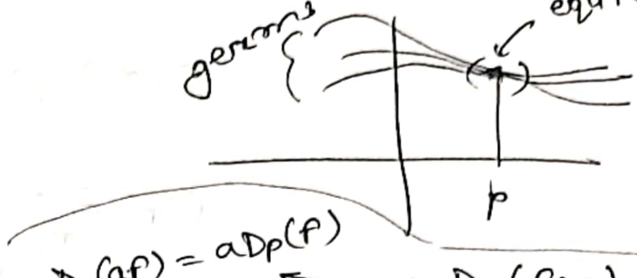
$$= \frac{\partial(r^i\circ\phi\circ\phi^{-1})}{\partial r^j}$$

$$= \frac{\partial r^i}{\partial r^j}\Big|_{\phi(p)}$$

$$= \underbrace{\delta^i_j}_{\text{Kronecker-delta}} = \begin{cases} 1 : i=j \\ 0 : i\neq j \end{cases}$$

vector space $\longrightarrow$ module (?)

$C_p^\infty(M) \to$ set of all fns smooth at $p \in M$
$\to f: M \to \mathbb{R}$



equivalent fns if values within neighbourhood are same

$D_p(af) = aD_p(f)$ $\qquad$ $D_p(f+g) = D_p(f) + D_p(g)$

A derivation at $p \in M$ is a linear map $D_p : C_p^\infty(M) \to \mathbb{R}$
such that,

$$D_p(fg) = f(p)D_p(g) + D_p(f)g(p)$$ } Leibnitz property

a vector space over $\mathbb{R}$ $\leftarrow$ $D_p(M) :=$ collection of all derivation at p

If we take $D_p, D_p' \in D_p(M)$

$$(D_p + D_p')(f) := D_p(f) + D_p'(f)$$

$$(D_p + D_p') : C_p^\infty(M) \longrightarrow \mathbb{R}$$

Example of elements in $D_p(M)$:
$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \left.\frac{\partial (f \circ \phi^{-1})}{\partial r^i}\right|_{\phi(p)}$$

$$\frac{\partial}{\partial x^i}(f+g) = \left.\frac{\partial}{\partial r^i}((f+g)\circ \phi^{-1})\right|_{\phi(p)}$$

$$= \left.\frac{\partial}{\partial r^i}(f\circ\phi^{-1} + g\circ\phi^{-1})\right|_{\phi(p)}$$

$$= \left.\frac{\partial f}{\partial x^i}\right|_p + \left.\frac{\partial g}{\partial x^i}\right|_p$$

HW

$$\frac{\partial}{\partial x^i}(fg) = \frac{\partial}{\partial x^i}\left((f\circ\phi^{-1})(g\circ\phi^{-1})\right)$$

$$= (f\circ\phi^{-1})(\phi(p))\,\frac{\partial(g\circ\phi^{-1})}{\partial x^i}\Big|_{\phi(p)} + \frac{\partial(f\circ\phi^{-1})}{\partial x^i}\Big|_{\phi(p)}(g\circ\phi^{-1})(\phi(p))$$

$$= f(p)\,\frac{\partial g}{\partial x^i}\Big|_p + \frac{\partial f}{\partial x^i}\Big|_p\, g(p)$$

So,

$$\frac{\partial}{\partial x^i}\Big|_p \in D_p(M).$$

Now, we claim $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}_{i=1}^{m}$ is a basis for $D_p(M)$

T_pM (Tangent space at p $= D_p(M)$)

কারণ partial derivative গুলোর tangent space form করে

যেকোন direction এর $\frac{\partial}{\partial x^i}$ → একেক দিকের tangent

manifold এ আমরা $\mathbb{R}^n$ এর মতন tangent space দেখেছি আঁকিনা

আমরা এখন থেকে $T_pM = D_p(M)$ লিখব

For any chart, D_p's are isomorphic | Smoothness depends on ~~chart~~ the atlas

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(#) Let $F: M \to N$ be a smooth map

Push forward, $F_{*}: T_p(M) \longrightarrow T_{F(p)}(N)$

Let $X \in T_pM$,

$X: C_p^\infty(M) \to \mathbb{R}$

$$F_*(X)h := X(\underbrace{h \circ F}_{\text{smooth fn from } M \text{ to } \mathbb{R}}) \quad \forall\ h \in C^\infty_{F(p)}(N)$$

$h \circ F: M \to \mathbb{R}$, $h \circ F \in C_p^\infty(M)$

$$\therefore F_*(X) \in C^\infty_{F(p)}(N) \ (!)(?)$$

linearity:

$$F_*(X)(g+h) = X((g+h)\circ F) = X(g \circ F + h \circ F)$$

$$= X((g\circ F) + (h \circ F))$$

$$= F_*(X)g + F_*(X)h$$

$$F_*(X)(gh) = X(gh \circ F)$$

$$= X((g\circ F)(h \circ F))\big|_p$$

$$= X(g\circ F)(h\circ F)\big|_p + (g\circ F)\big|_p X(h \circ F)$$

Leibnitz property

$$= (F_*(X)g)h\big|_{F(p)} + g\big|_{F(p)} (F_*(X)h)$$

$F_*(X) : C^\infty(N) \to \mathbb{R}$

$\underbrace{F_*}(X) \in T_{F(p)}(N)$

push-forward

$\hookrightarrow$ pushes T_pM in M to $T_{F(p)}N$ in N

$F_*(\ ) : T_pM \longrightarrow T_{F(p)}N$

vector space $\longrightarrow$ vector space

mapping

To prove linearity, we need

$$F_*(X+Y) = F_*(X) + F_*(Y)$$

$$F_*(X+Y)h = (X+Y)(h \circ F)$$

$$= X(h \circ F) + Y(h \circ F)$$

$$= F_*(X)h + F_*(Y)h$$

F_*

$G_{*,F(p)} \circ F_{*,p} = (G \circ F)_{*,p}$

(#) If $F : M \to N$ is a diffeomorphism

$$T_p(M) \cong T_{F(p)} N$$

$F : M \to N$ and $F^{-1} : N \to M$ are smooth

$F \circ F^{-1} = \mathbb{I}_N \quad (F^{-1} \circ F) = \mathbb{I}_M$

$(F \circ F^{-1})_{*,F(p)} = \mathbb{I}_{*,F(p)}$

$(F^{-1} \circ F)_{*,p} = \mathbb{I}_{*,p}$

$\Rightarrow F^{-1}{}_{*,F(p)} \circ F_{*,p} = \mathbb{I}_{*,p}$

$F_{*,p} : T_p M \to T_{F(p)} N$

$\therefore T_p M \cong T_{F(p)} N$

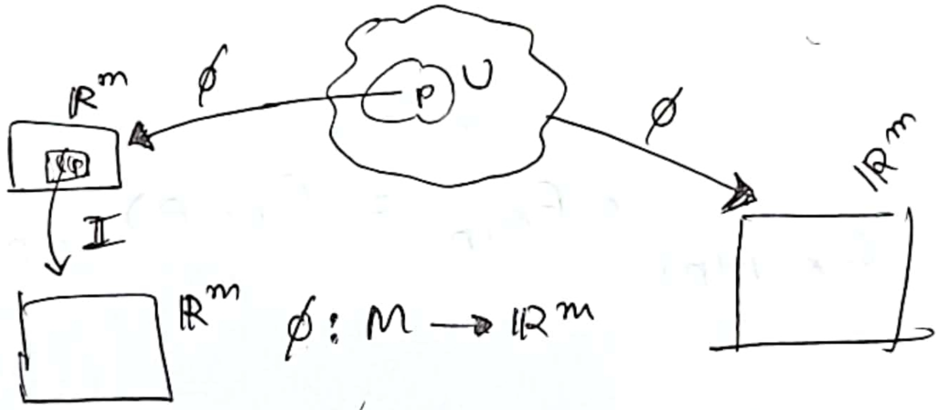
$$\mathbb{I}_M : M \longrightarrow M$$

$$\mathbb{I}_{*,p} : T_pM \longrightarrow T_{(\mathbb{I})(p)=p}M$$

$$\mathbb{I}_{*,p}(X)(h) = X(h \circ \mathbb{I}) = X(h)$$

$p \in M$

Let (U, ϕ) be a chart containing p. $p \in U$



$\phi : M \to \mathbb{R}^m$

$$T_p(U) \cong T_{\phi(p)}(\mathbb{R}^m) \cong \mathbb{R}^m$$

$$T_pM = \{D : C_p^\infty(M) \to \mathbb{R}\}$$
$$= \{D : C_p^\infty(U) \to \mathbb{R}\}$$

$C_p^\infty(M)$: All fn defnd on M s.t. f is smooth at p

So, $D_p(M) = D_p(U)$ exactly same as only one point

$$C_p^\infty(M) \cong C_p^\infty(U)$$

→ fn হয়, $\mathbb{R}$ এর কাছে

$$\left\{ \frac{\partial}{\partial x^i}\bigg|_p \right\}_{i=1}^{m}, \quad \frac{\partial}{\partial x^i}\bigg|_p \in T_pM$$

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^j}\right) \in T_pM$$

$$\sum_i c^i \frac{\partial}{\partial x^i}\bigg|_p = 0$$

$$\Rightarrow \sum_i c^i \frac{\partial}{\partial x^i}\bigg|_p (x^j) = 0(x^j)$$

$$c^j = 0, \quad \forall j = 1, \dots, m$$

So, these are linearly independent

& There are M of $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}_{i=1}^{M}$. So they form a basis T_pM.

→ if p is varied over U then T_pM becomes a vector field

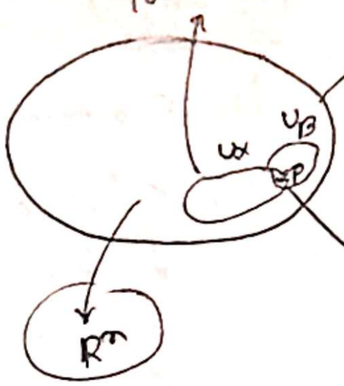
point → vector } vector field

Topological space + Chart → থেকে [illegible]

L - 3 — 05.12.24

$\phi_\alpha = (x^1, \ldots, x^n)$

$\phi_\beta = (y^1, \ldots, y^n)$



Then,

$T_p M \cong \mathbb{R}^n$ — isomorphic হলেও information loss নেই

(U_α, ϕ_α)

T_pM এর basis $U_{\alpha,\beta}$ এই দুই চার্ট এর co-ordinate f^n এর derivation থেকে হবে

$$\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\} \in T_pM \longrightarrow \text{forms a basis set of } T_pM$$

for $p \in U_\alpha$

$$\underbrace{\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}}_{\phi_\alpha} \text{ \& } \underbrace{\left\{ \frac{\partial}{\partial y^i}\Big|_p \right\}}_{\phi_\beta}$$

So, for $v \in T_pM$

$$v = \sum_i a^i \frac{\partial}{\partial x^i}\Big|_p = \sum_j b^j \frac{\partial}{\partial y^j}$$

basis transformation

এরা f^n এর উপর act করবে, তাই প্রতিটি f^n দিতে হবে

$$\frac{\partial}{\partial x^i}\Big|_p = \sum_j c_i^j \frac{\partial}{\partial y^j}\Big|_p$$

$$\frac{\partial}{\partial x^i}\Big|_p (y^k) = \sum_j c_i^j \frac{\partial}{\partial y^j}\Big|_p (y^k)$$

$$\begin{bmatrix} \partial/\partial x^1 \\ \vdots \\ \partial/\partial x^n \end{bmatrix} = \begin{bmatrix} c_1^1 & c_1^2 & \cdots & c_1^n \\ \vdots & & & \vdots \\ c_n^1 & \cdots & \cdots & c_n^n \end{bmatrix} \begin{bmatrix} \partial/\partial y^1 \\ \vdots \\ \partial/\partial y^n \end{bmatrix}$$

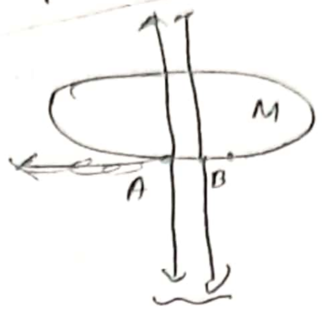
$$\therefore \; c_i^k = \frac{\partial y^k}{\partial x^i}\Big|_p$$

$\hookrightarrow$ forms the basis transform matrix

T_pM কে $\mathbb{R}^n$ এর isomorphic ধরলেও আবার ~~[illegible]~~ disjoint union করা কিন্তু $T_pM, T_qM, \ldots$ এগুলো কে disjoint সবগুলো এর set of ... 3 এগুলো domain...

$$\underline{TM} = \bigsqcup_{p \in M} T_pM$$

বড় set



TM is a collection of all these lines (T_pM),

these two are different

$v \in T_pM$

$v : C_p^\infty(M) \to \mathbb{R}$

$w \in T_qM$

$w : C_q^\infty(M) \to \mathbb{R}$

Natural Projection map, We define, a natural projection map, $\pi : TM \longrightarrow M$

Let, $v \in T_pM$ for some $p \in M$,

then, $\pi(v) = p$

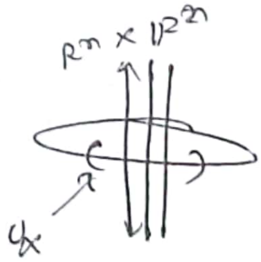
π is a surjective but not injective map.

$TM \to$ a set

Let (U_α, ϕ_α) be a co-ordinate chart of M. $\phi_\alpha = (x^1, x^2, \ldots x^n)$

$$T_pM = T_pU_\alpha$$

$$TU_\alpha = \bigsqcup_{p \in U_\alpha} T_pM$$



Let,

$v \in TU_\alpha \Rightarrow v \in T_pU_\alpha$ for some $p \in U_\alpha$

Then, $v = \sum c^i(v) \frac{\partial}{\partial x^i}\Big|_p$

We define a map, $\mathbb{R}^{2n} \to$ hence has a topology

To inherit the topology

bijection
$$\begin{cases} \tilde{\phi}_\alpha : TU_\alpha \to \underbrace{\phi_\alpha(U_\alpha) \times \mathbb{R}^n}_{\mathbb{R}^{2n}} & (\text{actually } \mathbb{R}^n \times \mathbb{R}^n \text{ as } \phi(p) \in \mathbb{R}^n) \\ \tilde{\phi}_\alpha(v) = (\phi_\alpha(p), c^1(v), \ldots, c^n(v)) \end{cases}$$

$$\tilde{\Psi}_\alpha = \tilde{\phi}_\alpha^{-1}\left(\phi(p), c^1, \ldots, c^n\right) = \sum_i c^i \frac{\partial}{\partial x^i}\Big|_p = v \in T_p U_\alpha = T_p M$$

ϕ^{-1}

$$\tilde{\phi}_\alpha \circ \tilde{\Psi}_\alpha = \tilde{\Psi}_\alpha \circ \phi_a = I$$

As $\tilde{\phi}_\alpha$ is a bijection to $\mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{R}^{2n}$, so TU_α inherits the topology $\mathbb{R}^{2n}$.

Let,

$V \subset U \subset M$, ~~$V, U \subset M$~~

$\phi(V) \times \mathbb{R}^n$, $\phi(U) \times \mathbb{R}^n$

$TV \subset TU$ subspace topology দিতে পারি V কে

অথবা $\tilde{\phi}$ দিয়ে V কে topology দেওয়া যায়।

Condition for a collection to be a basis of a topology

Let β be a collection of subsets of X,

β is a basis of a topology on X if

(i) $\bigcup_{B \in \beta} B = X$

(ii) If $A, B \in \beta$, $A \cap B \in \beta$

Let us define,

$\beta = \{A \mid A$ is open in TU_α, where U_α is a open co-ordinate chart$\}$

we will prove they form a basis of TM

$$TM = \bigcup_{\alpha \in M} T_\alpha M$$

$$= \Big(\bigcup_{p \in U_{\alpha_1}} T_p M\Big) \cup \Big(\bigcup_{p \in U_{\alpha_2}} T_p M\Big) \cup \dots$$

$$= \bigcup_{A \in \beta} A$$

Now, let,

$A, B \in \beta$, then A is open in TU_α

B is open in TU_β

$$\left.\begin{array}{l} A \cap B \subset TU_\alpha \\ A \cap B \subset TU_B \end{array}\right\} \quad A \cap B \subset TU_\alpha \cap TU_\beta = T(U_\alpha \cap U_\beta)$$

$$A \cap B = (A \cap B) \cap \Big(T(\overbrace{U_\alpha \cap U_\beta}^{\text{atlas এর ভেতর}})\Big)$$ এটা থাকবে

So, $A \cap B$ is open in $T(U_\alpha \cap U_\beta)$

So, β is a basis for TM, so TM is now a topology.

It can be proved from $\overbrace{TU_\alpha \to \phi(U_\alpha) \times \mathbb{R}^m}^{\text{has countable}}$

that TM β is a second countable basis.

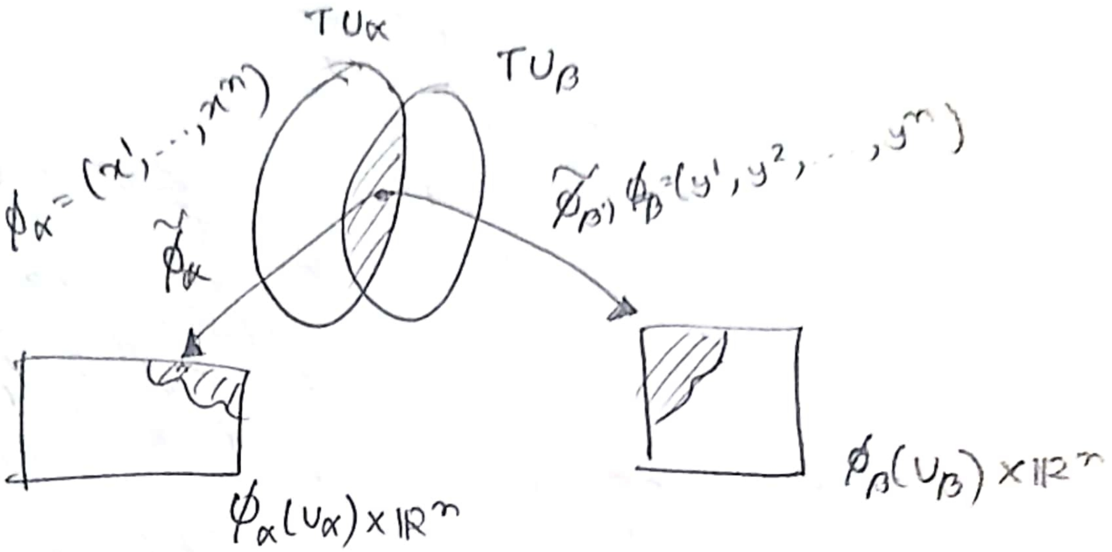
It can also be shown that it is Hausdorff

We need to show now that TM is locally Euclidean

Let, $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M

we claim, $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ will be an atlas of TM

$\tilde{\phi}_\alpha : TU_\alpha \to \phi(U_\alpha) \times \mathbb{R}^n$



Let, $v \in TU_\alpha \cap TU_\beta$. Then, $v \in T_p(M)$

$$v = \sum a^i \frac{\partial}{x^i}\Big|_p = \sum_j b^j \frac{\partial}{\partial y^j}\Big|_p$$

$$\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} \left(\left(\phi_\alpha(p), \overset{a^1,}{c^1(v)}, \ldots, \overset{a^n}{c^n(v)} \right) \right)$$

$$= \tilde{\phi}_\beta \left(\sum_i a^i \frac{\partial}{\partial x^i}\Big|_p \right) = \tilde{\phi}_\beta \left(\sum_j b^j \frac{\partial}{\partial y^j}\Big|_p \right)$$

$$= \cancel{(\phi_\beta(p),} \left(\phi_\beta(p), b^1, \ldots, b^n \right)$$

$$= \left(\underbrace{\phi_\beta \circ \phi_\alpha^{-1}}^{C^\infty\text{-comp}} \circ \phi_\alpha(p), b^1, \ldots, b^n \right)$$

$$\sum_i a^i \frac{\partial}{\partial x^i}\Big|_p (y^j) \quad \sum_j b^j \frac{\partial}{\partial y^j}\Big|_p (y^j)$$

$$\Rightarrow \quad b^j = \sum_i a^i \frac{\partial y^j}{\partial x^i}\Big|_p = \sum_i a^i \frac{\partial\, y^j \circ \phi_\alpha^{-1}}{\partial r^i}\Bigg|_{\phi_\alpha(p)}$$

$$y^j = r^j \circ \phi_\beta$$

$$\therefore\ b^j = \sum_i a^i \frac{\partial\, r^j \circ (\phi_\beta \circ \phi_\alpha^{-1})}{\partial r^i}\Bigg|_{\phi_\alpha(p)}$$

$$= \underbrace{\sum_i a^i \frac{\partial\, (\overbrace{\phi_\beta \circ \phi_\alpha^{-1}}^{C^\infty})^j}{\partial r^i}\Bigg|_{\phi_\alpha(p)}}_{\text{also } C^\infty}$$

So, b^j is C^∞

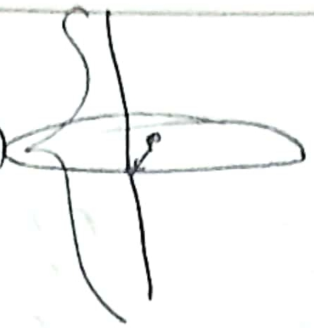
Similarly, a^i is also C^∞

So, TM is a smooth manifold of dimension $2n$.
if M is a smooth manifold.

$(TM, M, \pi) \longrightarrow$ is a tangent bundle of M

$$\begin{array}{l} TM \rightarrow \text{total space} \\ \quad \big\downarrow \pi \rightarrow \text{natural projection map} \\ M \rightarrow \text{base manifold} \end{array}$$

$T_pM = \pi^{-1}(p)$



we will fit a vector ~~to~~ swapping each T_pM to get vector bundle

$\downarrow$

is a fibre

$\cong$

$\mathbb{R}^m$

লাগবে $\begin{cases} \text{Section } 8.1 \text{ to } 8.7 \\ + \\ \text{Section} - 12 \rightarrow \text{Tangent bundle} \end{cases}$

pg - 129 $\downarrow$ 12.1 to 12.2

Section – 8.4

Section – 8.6, 8.7

$T_pM \cong \mathbb{R}^n$

Disjoint Union, $\sqcup$

Basis transform between two ~~two~~ charts on same T_pM

$$\frac{\partial}{\partial x^i} = \sum c_i^j \frac{\partial}{\partial y^j} ; \quad c_i^j = \frac{\partial y^j}{\partial x^i}$$

Push-forward using a curve, $c : \mathbb{R} \longrightarrow M$

$$c_{*,p}(\;) : C_p^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$c_{*}\left(\frac{\partial}{\partial t}\Big|_p\right) f = \frac{\partial}{\partial t}(f \circ c(t)) \mapsto p$$

Directional derivative along the curve

Collection ~~Set~~ of tangent spaces

$$TM = \bigsqcup_{p \in M} T_pM$$

$+$

Bases, $\mathcal{B} = \{A \mid A$ is open in TU_α, U_α is an open set of chart of $M\}$

Topology, TM

$+$

forms bases,

$+$

Manifold TM

Hausdorff

C^∞

Smooth Manifold TM

Natural projection map

$$\pi : T_pM \longrightarrow M$$

$$\pi(X) = p, \text{ if } X \in T_pM$$

give topology of $\mathbb{R}^{2n}$

locally Euclidean

Topology inheritance map

$$\tilde{\phi} : TU \longrightarrow \phi(U) \times \mathbb{R}^n$$

$$\tilde{\phi}(v) = (\phi(\pi(v)), c^1(v), \dots, c^n(v))$$

$$v = \sum c^i \frac{\partial}{\partial x^i} ; \; c^i = c^i(v), \; v \in T_pU$$

M

$$(TM, M, \pi)$$

Tangent bundle