INTEGRAL CALCULUS
AND
ORDINARY DIFFERENTIAL EQUATIOSNS

MULTIPLE INTEGRATION

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Multiple Integration

- **Multiple Integration:** The integrals of functions of more than one variable are known as **multiple integrals** and are evaluated by a process involving iterated integrals.

- **Partial Integration:** The process in which the integration is performed with respect to one variable treating the other variable(s) as constant is called partial integration.

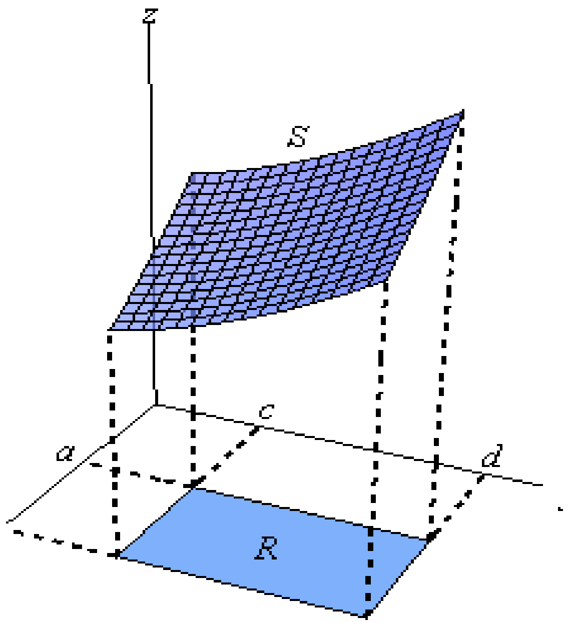
- **Iterated Integral:** A definite integral which is evaluated stage by stage using partial integration is called an iterated (successive or repeated) integral.

- **Double Integrals:** The double integral may be defined geometrically in much the same way as the definite Riemann integral.

- **Double Integrals over the rectangular region:**

If R is the region defined by $R = \{(x, y) | a \le x \le b, c \le y \le d\}$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy, \text{where} s = f(x,y).$$



Double Integrals over the rectangular region:

Example: Evaluate the iterated integral $\int_0^1 \int_1^2 (x^2 + xy)dxdy$

Solution: $\int_0^1 \int_1^2 (x^2 + xy)dxdy = \int_0^1 \left[\int_1^2 (x^2 + xy)dx\right] dy$

$$= \int_0^1 \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_{x=1}^{x=2} dy$$

$$= \int_0^1 \left[\frac{8-1}{3} + \frac{(4-1)y}{2}\right] dy$$

$$= \int_0^1 \left[\frac{7}{3} + \frac{3y}{2}\right] dy .$$

$$= \left[\frac{7}{3} y + \frac{3}{2}\frac{y^2}{2}\right]_{y=0}^{y=1}$$

$$= \frac{37}{12}$$

$$
\begin{aligned}
\int_{11}^{14}\int_{7}^{10} \left(x^2+4y\right)\,dy\,dx &= \int_{11}^{14}\left[x^2y+2y^2\right]_{y=7}^{y=10}\,dx \\
&= \int_{11}^{14}\left(3x^2+102\right)dx \\
&= \left[x^3+102x\right]_{x=11}^{x=14} \\
&= 1719.
\end{aligned}
$$

$$
\begin{aligned}
\int_1^3 \int_2^4 9x^3y^2\,dy\,dx &= \int_1^3 \left[3x^3y^3\right]_{y=2}^{y=4} dx \\
&= \int_1^3 \left[3x^3(4^3) - 3x^3(2^3)\right] dx \\
&= \int_1^3 168x^3\,dx \\
&= \left[42x^4\right]_{x=1}^{x=3} \\
&= (42(3^4)) - (42(1^4)) \\
&= 3360
\end{aligned}
$$

H.W Practice:

Evaluate the followings:

1. $\int_0^1 \int_0^2 (x+2)dydx$
2. $\int_2^4 \int_0^3 (x+y)dxdy$
3. $\int_0^1 \int_x^y xy\,dydx$
4. $\int_0^1 \int_{y^2}^y (x^2y+xy^2)dxdy$
5. $\int_1^2 \int_1^y (\frac{1}{x}+\frac{1}{y})dxdy$
6. $\int_0^1 \int_0^{\sqrt{x}} ye^{x^2}\,dy\,dx$
7. $\int_0^{\sqrt{\frac{\pi}{2}}} \int_0^{x^2} x\cos y\,dy\,dx$
8. $\int_0^1 \int_0^{x^2} (x^2+y)dydx$
9. $\int_0^{\pi/2} \int_0^2 r\sqrt{4-r^2}\,dr\,d\theta$
10. $\int_0^1 \int_{-x}^x (x^2-y^2)dy\,dx$
11. $\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\cos\theta\,drd\theta$

Home Work

Iterated Integral (P-993) Example # 4, 5, 6

Page – 999 Ex # 15 – 21, 27, 28, 29, 34

Calculus– James Stewart - 8th edition

Double Integrals over the non-rectangular region:

(a) If R is the region defined by $R = \{(x, y) | a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}$, then

$$\iint_R f(x, y) dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y) dy dx .$$

(b) If R is the region defined by $R = \{(x, y) | c \leq y \leq d, g_1(y) \leq x \leq g_2(y)\}$, then

$$\iint_R f(x, y) dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) dx dy .$$

Double Integrals over the non-rectangular region:

Example: Evaluate $\iint_R y^2 x\, dA$ over the rectangle

$$R = \{(x, y)| -3 \le x \le 2, 0 \le y \le 1\}.$$

Solution: $\iint_R y^2 x\, dA$

$$= \int_{-3}^{2} \int_{0}^{1} y^2 x dy dx\ .$$

$$= \int_{-3}^{2} x \left[\frac{y^3}{3}\right]_0^1 dx$$

$$= \frac{1}{3} \int_{-3}^{2} x\, dx$$

$$= \frac{1}{3} \left[\frac{x^2}{2}\right]_{-3}^{2}$$

$$= -\frac{5}{6}$$

$$\int_0^1 \int_{x^2}^{x} (xy^2)\,d\underline{y}\,dx$$

$$= \int_0^1 \left(\frac{1}{3} x y^3 \Big|_{y=x^2}^{y=x} \right) dx$$

$$= \int_0^1 \left(\frac{1}{3} x^4 - \frac{1}{3} x^7 \right) dx$$

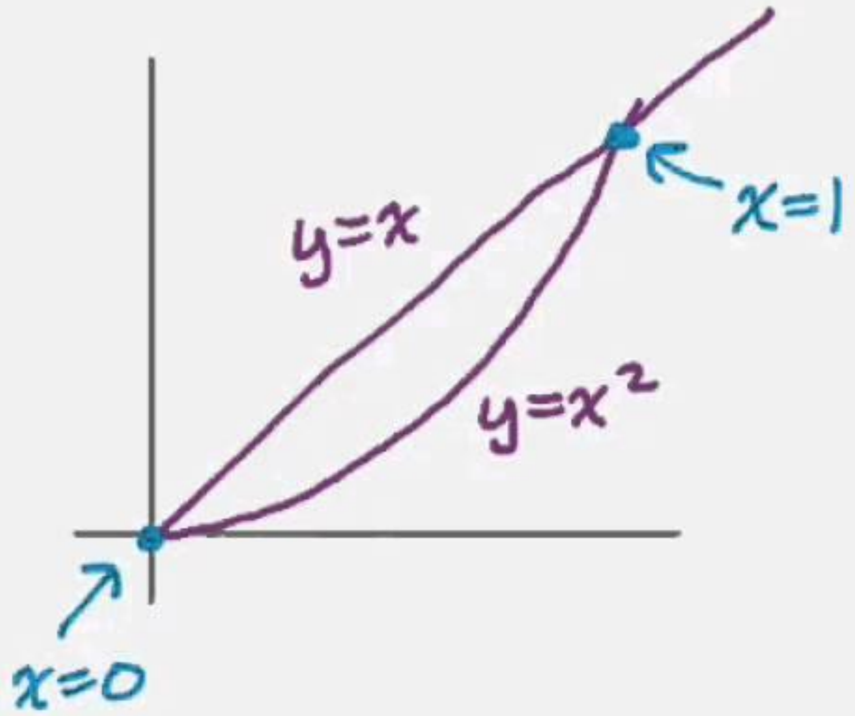
$$= \frac{1}{15} x^5 - \frac{1}{24} x^8 \Big|_0^1 = \frac{1}{15} - \frac{1}{24}$$

$$= \frac{9}{360} = \boxed{\frac{1}{40}}$$

Bounds:

$0 \le x \le 1$

$x^2 \le y \le x$



Double Integrals over the non-rectangular region:

Example: Evaluate $\iint_R xy\, dA$ over the region

$$R = \{(x, y): \frac{1}{2}x \le y \le \sqrt{x}, 2 \le x \le 4\}.$$

Solution: $\iint_R x\, y dA = \int_2^4 \int_{\frac{x}{2}}^{\sqrt{x}} xydydx$

$$= \int_2^4 x \left[\frac{y^2}{2}\right]_{\frac{x}{2}}^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_2^4 x[x - \frac{x^2}{4}]\, dx$$

$$= \frac{1}{2}\int_2^4 [x^2 - \frac{x^3}{4}]dx]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{16}\right]_2^4 = \frac{11}{6}$$

H.W Practice:

Evaluate the following:

1. $\iint_R (x^2 + xy^3)\, dA$ over the rectangle $R = \{(x, y): 0 \leq x \leq 1, 1 \leq y \leq 2\}$.
2. $\iint_R (xy - y^2) dA$ where R is rectangle whose vertices are $(-1,0), (0,0), (0,1)$, and $(-1,1)$.
3. $\iint_R (2x + y)\, dA$ over the rectangle $R = \{(x, y)|3 \leq x \leq 5, 1 \leq y \leq 2\}$.
4. $\iint_R (x^2 + y^2)\, dA$ where R is rectangle whose vertices are $(0,1), (1,1), (1,2)$and$(0,2)$.
5. $\iint_R (x + y)\, dA$, where R is the region bounded by $y = 1, y = x^2$ $and\ x \geq 0$.
6. $\iint_R x\, dA$ over the triangular region R enclosed by the lines
   $x + 2y = 2, x = 0$ and $y = 0$.

Home Work

Double Integral over general regions (P- 1001) Example # 1, 3

Page- 1008 Ex # 1-4, 7, 8, 9, 17, 18

Calculus– James Stewart - 8th edition

APPLICATION OF DOUBLE INTEGRALS

- The **AREA** A of a region **R** in the xy-plane is given by the double integral

$$A = \iint_R dx\, dy$$

- The **VOLUME** V beneath the surface z= f(x,y)>0 and above a region **R** in the xy-plane is

$$V = \iint_R f(x, y)\, dx\, dy$$

Application of Double Integrals:

Area: Plane area of a closed bounded region *R* is $A = \iint_R dA$.

Example: Using double integrals, find the finite area bounded by the following curves $y = -x^2$and$y = x - 2$.

Solution: $A = \iint_R 1\, dA$

$$= \int_{x=-2}^{x=1} \int_{y=x-2}^{-x^2} 1 dy dx$$

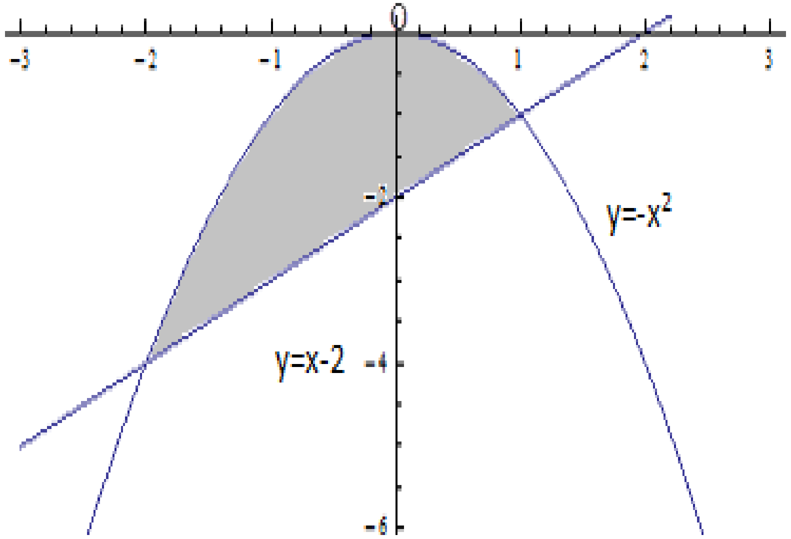
$$= \int_{x=-2}^{x=1} [y]_{-2}^{1}\, dx$$

$$= \int_{x=-2}^{x=1} [-x^2 - x + 2] dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$

$$= \frac{11}{6}.$$

$$-x^2 = x - 2$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2, 1$$



Application of Double Integrals:

Area: Plane area of a closed bounded region R is $A = \iint_R dA$.

Example: Using double integrals, find the finite area bounded by the following curves $y = x^2$ and $y = x + 6$.

$$x^2 = x + 6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$

Solution: $A = \iint_R 1\, dA$

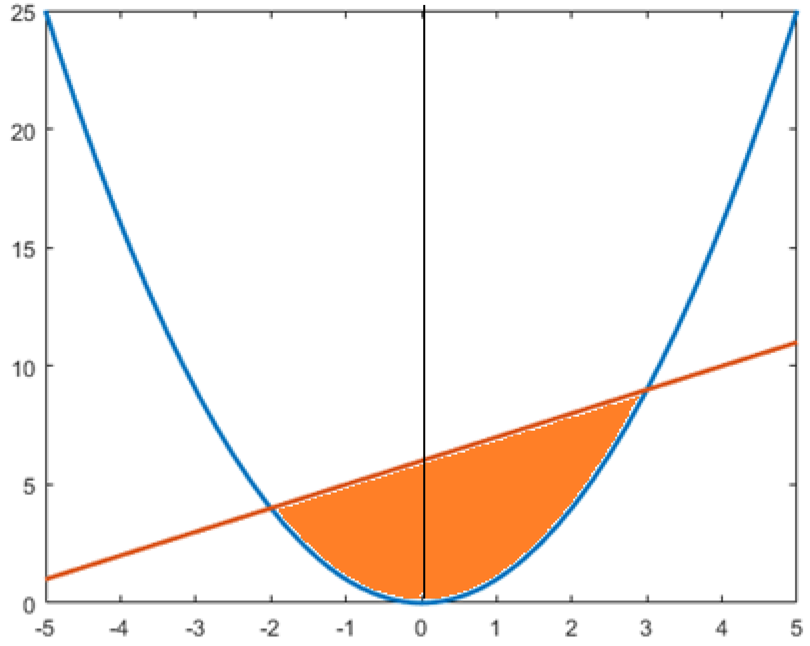
$$= \int_{x=-2}^{x=3} \int_{y=x^2}^{y=x+6} 1 dy dx$$

$$= \int_{x=-2}^{x=3} [y]_{x^2}^{x+6}\, dx$$

$$= \int_{x=-2}^{x=3} [x + 6 - x^2] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{125}{6}.$$



$$
\begin{aligned}
\int_0^1 \int_{x^2}^{x} (x+3)\,dy\,dx &= \int_0^1 (xy+3y)\Big|_{y=x^2}^{y=x}\,dx \\
&= \int_0^1 \left[\left(x(x)+3(x)\right)-\left(x(x^2)+3(x^2)\right)\right]\,dx \\
&= \int_0^1 \left[3x-2x^2-x^3\right]\,dx \\
&= \left[\frac{3x^2}{2}-\frac{2x^3}{3}-\frac{x^4}{4}\right]_{x=0}^{x=1} \\
&= \frac{7}{12}.
\end{aligned}
$$

H.W Practice:

Sketch the region and **using** double integrals, find the finite area bounded by the following curve (s).

1. $y = 2x - x^2$ and x-axis
2. $x^2 = 4y, 8y = x^2 + 16$
3. $y = -x, x = 0,\ y = 2$

Iterated Integrals in Three Variables:

Example: Evaluate $\int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} xy^2z^3\,dzdydx$

Solution: $\int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} xy^2z^3\,dzdydx$

$$= \int_{-1}^{2}\int_{0}^{3} xy^2\left[\frac{z^4}{4}\right]_0^2 dydx$$

$$= 4\int_{-1}^{2}\int_{0}^{3} xy^2\,dydx$$

$$= 4\int_{-1}^{2} x\left[\frac{y^3}{3}\right]_0^3 dx$$

$$= 36\int_{-1}^{2} x\,dx$$

$$= 36\left[\frac{x^2}{2}\right]_{-1}^{2}$$

$$= 54$$

$$
\begin{aligned}
\text{Volume} &= \int_0^{\ell} dx \int_0^{\ell - x} dy \int_0^{\ell - x - y} dz \\
&= \int_0^{\ell} dx \int_0^{\ell - x} (\ell - x - y)\, dy \\
&= \int_0^{\ell} \left(l^2 - 2\ell x + x^2 - \frac{(\ell - x)^2}{2} \right) dx \\
&= \ell^3 - \ell\ell^2 + \frac{\ell^3}{3} - \left[\frac{\ell^2 x}{2} - \frac{\ell x^2}{2} + \frac{x^3}{6} \right]_0^{\ell} \\
&= \frac{\ell^3}{3} - \frac{\ell^3}{6} = \frac{\ell^3}{6}
\end{aligned}
$$

Iterated Integrals in Three Variables:

Example: Evaluate $\int_0^3 \int_0^1 \int_{-1}^1 (x^2 + yz) dz dy dx$

Solution: $\int_0^3 \int_0^1 \int_{-1}^1 (x^2 + yz) dz dy dx$

$$= \int_0^3 \int_0^1 \left(x^2 z + y\frac{z^2}{2}\right)_{z=-1}^{z=1} dy\, dx$$

$$= \int_0^3 \int_0^1 2x^2\, dy\, dx$$

$$= \int_0^2 2x^2 [y]_0^1\, dx$$

$$= \int_0^3 2x^2 \cdot 1 dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^3$$

$$= 18$$

H.W Practice:

Evaluate the following iterated integral

1. $\int_0^2 \int_{-3}^0 \int_{-1}^1 (x^2 + yz)dzdydx$
2. $\int_1^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2)dxdydz$
3. $\int_0^1 \int_0^{y^2} \int_0^{x+y} x\ dzdxdy$
4. $\int_0^1 \int_0^x \int_0^{x-y} xdzdydx$
5. $\int_0^2 \int_{-1}^{y^2} \int_{-1}^z yzdxdzdy$
6. $\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} zr\ dzdrd\theta$
7. $\int_0^{2\pi} \int_0^{\pi} \int_0^a r^3 \sin\theta\ drd\theta d\varphi,$

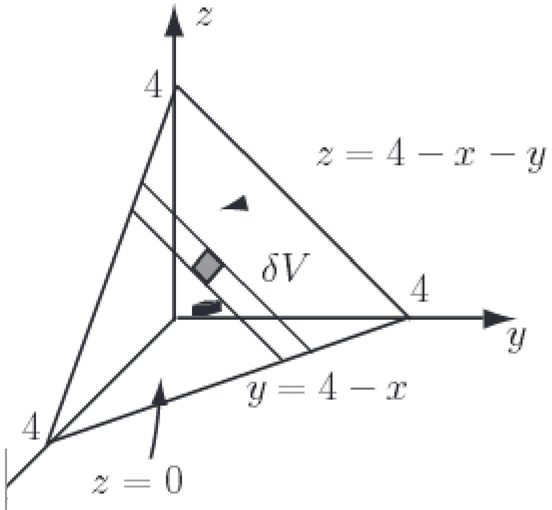
Home Work

Triple Integral (P-1030) Example # 1

Page- 1037 Ex # 3-7

Application of Triple Integrals:

The volume of a closed bounded region R in space is $V = \iiint_R dV$.



Example V is the (irregular) tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 4$. (see diagram)

(a) Express $\int_V f(x, y, z)\,\mathrm{d}V$ (where f is a function of x, y and z as a triple integral.

(b) Hence find $\int_V x\,\mathrm{d}V$.

Solution

The tetrahedron is divided into a series of slices parallel to the yz-plane and each slice is divided into a series of vertical strips. For each strip, the bottom is at $z = 0$ and the top is on the plane $x + y + z = 4$ i.e. $z = 4 - x - y$. So the integral up each strip is given by $\int_{z=0}^{4-x-y} f(x, y, z)\,\mathrm{d}z$ and this (inner) integral will be a function of x and y.

This, in turn, is integrated over all strips which form the slice. For each value of x, one end of the slice will be at $y = 0$ and the other end at $y = 4 - x$. So the integral over the slice is $\int_{y=0}^{4-x}\int_{z=0}^{4-x-y} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y$ and this (intermediate) integral will be a function of x.

Finally, integration is carried out over x. The limits on x are $x = 0$ and $x = 4$. Thus the triple integral is $\int_{x=0}^{4}\int_{y=0}^{4-x}\int_{z=0}^{4-x-y} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ and this (outer) integral will be a constant.

Hence $\int_V f(x, y, z)\,\mathrm{d}V = \int_{x=0}^{4}\int_{y=0}^{4-x}\int_{z=0}^{4-x-y} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

Solution (contd.)

In the case where $f(x,y,z) = x$, the integral becomes

$$\begin{aligned}
\int_V f(x,y,z)\,\mathrm{d}V &= \int_{x=0}^{4}\int_{y=0}^{4-x}\int_{z=0}^{4-x-y} x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{4}\int_{y=0}^{4-x} [xz]_{z=0}^{4-x-y}\ \mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{4}\int_{y=0}^{4-x} [(4-x-y)x-0]\ \mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{4}\int_{y=0}^{4-x} \left[4x - x^2 - xy\right]\ \mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{4} \left[4xy - x^2y - \frac{1}{2}xy^2\right]_{y=0}^{4-x}\ \mathrm{d}x \\
&= \int_{x=0}^{4} \left[4x(4-x) - x^2(4-x) - \frac{1}{2}x(4-x)^2 - 0\right]_{y=0}^{4-x}\ \mathrm{d}x \\
&= \int_{x=0}^{4} \left[16x - 4x^2 - 4x^2 + x^3 - 8x + 4x^2 - \frac{1}{2}x^3\right]\ \mathrm{d}x \\
&= \int_{x=0}^{4} \left[8x - 4x^2 + \frac{1}{2}x^3\right]\ \mathrm{d}x \\
&= \left[4x^2 - \frac{4}{3}x^3 + \frac{1}{8}x^4\right]_0^4 = 4\times 4^2 - \frac{4}{3}\times 4^3 + \frac{1}{8}\times 4^4 - 0 \\
&= 64 - \frac{256}{3} + 32 = \frac{192 - 256 + 96}{3} = \frac{32}{3}
\end{aligned}$$

Mass and center of mass

The co-ordinates $(\overline{x}, \overline{y})$of the center of mass of a lamina occupying the region *D* and having density function $\rho(x, y)$ are

$$\overline{x} = \frac{1}{m} \iint\limits_{D} x\rho(x, y)\, dA$$

$$\overline{y} = \frac{1}{m} \iint\limits_{D} y\rho(x, y)\, dA$$

Where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

Mass and center of mass

Example: Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ. Where $\boldsymbol{D = \{(x, y) | 0 \le x \le 1, 0 \le y \le 2\}}$ **and** $\boldsymbol{\rho(x, y) = x + y}$.

Solution: $\boldsymbol{m = \int_0^2 \int_0^1 (x + y) dx\, dy}$

$$\boldsymbol{= \int_0^2 \left(\frac{x^2}{2} + xy\right)\Big|_0^1 dy}$$

$$\boldsymbol{= \int_0^2 \left(\frac{1}{2} + y\right) dy}$$

$$\boldsymbol{= \left(\frac{1}{2} y + \frac{y^2}{2}\right)\Big|_0^2}$$

$$\boldsymbol{= 3}$$

Mass and center of mass

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA$$

$$\bar{x} = \frac{1}{3}\int_0^2\int_0^1 x(x+y)dx\,dy$$

$$= \frac{1}{3}\int_0^2\int_0^1 \left(x^2+xy\right)dx\,dy$$

$$= \frac{1}{3}\int_0^2 \left(\frac{x^3}{3}+\frac{x^2}{2}y\right)\Big|_0^1 dy$$

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$$= \frac{1}{3}\int_0^2 \left(\frac{1}{3}+\frac{1}{2}y\right)dy$$

$$= \frac{1}{3}\left(\frac{1}{3}y+\frac{1}{2}\frac{y^2}{2}\right)\Big|_0^2$$

$$= \frac{5}{9}$$

Mass and center of mass

$$\bar{y} = \frac{1}{m}\iint_D \; y\rho(x, y)\, dA$$

$$\bar{y} = \frac{1}{3}\int_0^2 \int_0^1 y(x + y)dx\, dy$$

$$= \frac{1}{3}\int_0^2 \int_0^1 (xy + y^2)dx\, dy$$

$$=\frac{1}{3}\int_0^2 \left(\frac{x^2}{2}y + xy^2\right)\Big|_0^1 dy$$

`

$$= \frac{1}{3}\int_0^2 \left(\frac{1}{2}y + y^2\right) dy$$

$$= \frac{1}{3}\left(\frac{1}{2}\frac{y^2}{2} + \frac{y^3}{3}\right)\Big|_0^2$$

$$= \frac{11}{9}$$

Mass and center of mass

Class practice:

1. Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ. Where $\boldsymbol{D}$ is the rectangular region with vertices $(0,0),(1,0),(0,1),(1,1)$**and** $\boldsymbol{\rho(x,y)=x+y}$

2. Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ. Where $\boldsymbol{D=\{(x,y)|0\le x\le 1, 0\le y\le 2\}}$ **and** $\boldsymbol{\rho(x,y)=y^2}$.

3. Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ. Where $\boldsymbol{D=\{(x,y)|0\le x\le 1, 0\le y\le 1\}}$ **and** $\boldsymbol{\rho(x,y)=2x}$.

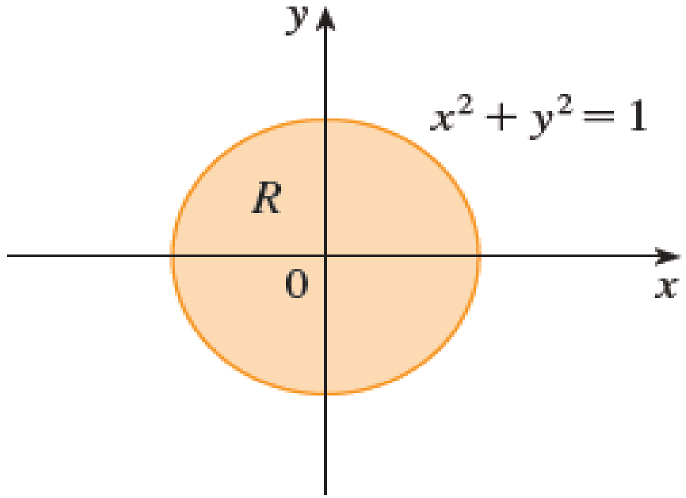
Home Work

Page-1017, Example # 2

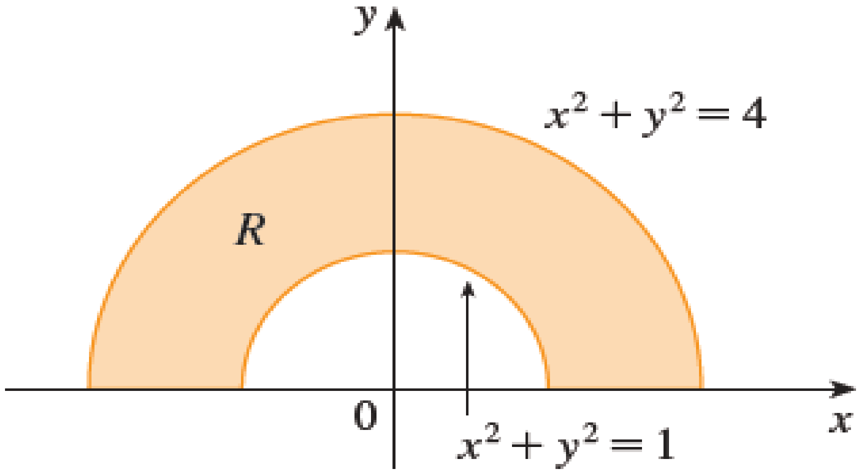
Page- 1024 Ex # 3-10

Double Integration in polar co-ordinate

Suppose that we want to evaluate a double integral $\iint_R f(x, y)dA$, where *R* is one of the regions shown in Figure. In either case the description of *R* in terms of rectangular coordinates is rather complicated, but *R* is easily described using polar coordinates.



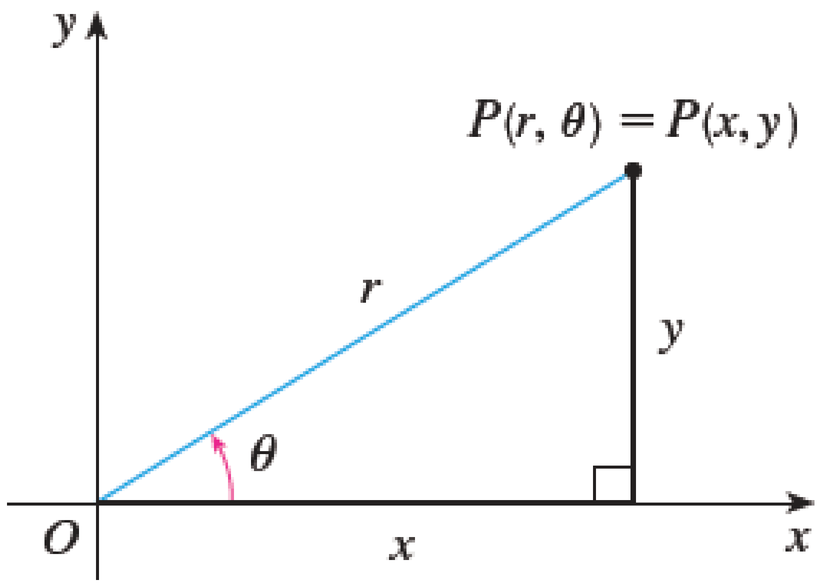
(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$



(b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$



Change to Polar Coordinates in a Double Integral

If f is continuous in a polar rectangle R given by $a \leq r \leq b, \alpha \leq \theta \leq \beta$

Then

$$\iint_R f(x,y)dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r sin\,\theta) r\,dr\,d\theta$$

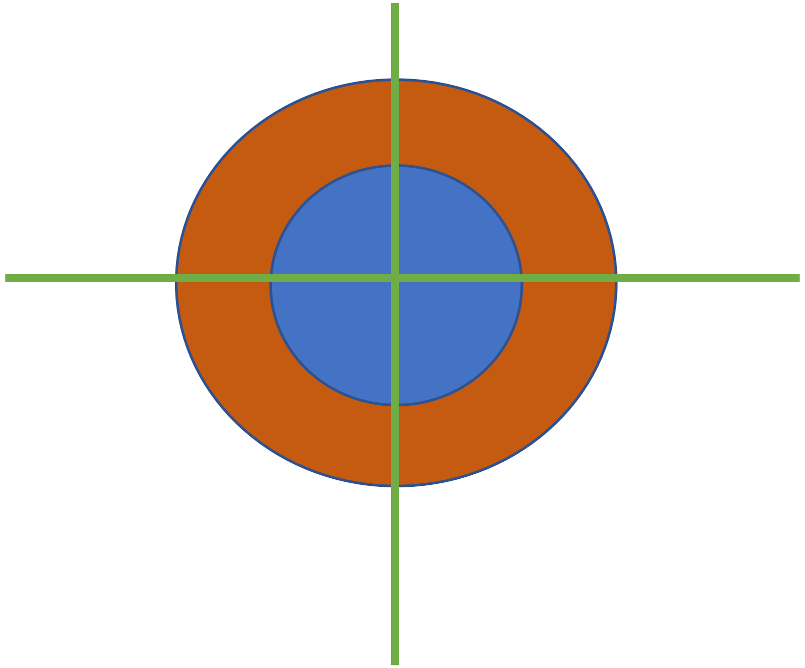
Example: Evaluate $\iint_R e^{-(x^2+y^2)} dA$ where R is the annulus $x^2 + y^2 \geq 1, x^2 + y^2 \leq 4.$

Solution: $\iint_R e^{-(x^2+y^2)} dA$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} e^{-r^2} r \, dr \, d\theta$$

$$= \int_0^{2\pi} (e^{-4} - e^{-1}) \, d\theta$$

$$= 2\pi(e^{-4} - e^{-1})$$



H.W Practice

1. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dxdy$ by changing it to polar coordinate.
2. Evaluate $\iint_R (x^2+y^2)dxdy$ where R is the region in the first quadrant bounded by the circle $x^2+y^2=9$.

Home Work

Double Integral in Polar Coordinates (P- 1010) Example # 1, 2, 3

P-1014 Ex # 7-11

Sample MCQ

1. Evaluate $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(a) $\frac{13}{2}$ (b) $\frac{27}{2}$ (c) …… (d)…….

2. Evaluate $\iint_R (x - 3y^2) dA$ where $R = \{(x, y) | 0 \leq x \leq 2, 1 \leq y \leq 2\}$

(a) 14 (b) 12 (c) -12 (d) 20

3. Evaluate $\int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dx \, dy \, dz$

$(a) \frac{27}{4}$ (b) $\frac{13}{2}$ (c) …….. (d) …….

4.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