Functions

Course Code: CSC 1204 Course Title: Discrete Mathematics

Department of Computer Science

Faculty of Science and Technology

Lecture No:	08	Semester:	Summer 2019-2020

Lecture Outline

2.3 Functions

- Definition of Function
- Domain, Codomain, Range, Image, Preimage,
- One-to-one function
- Onto function
- One-to-one correspondence
- Inverse Functions
- Compositions of Functions
- Floor function
- Ceiling Function

Objectives and Outcomes

- **Objectives**: To understand what is function, domain, codomain, range, image, preimage; to understand different types of functions. To understand Inverse Function, Compositions of Functions, Floor function, and Ceiling Function.
- **Outcomes**: Students are expected to be able to explain different types of functions with examples, be able to determine whether a function is one-to-one, onto, and/or one-to-one correspondence, be able to determine whether a function is invertible and find out the inverse of a function, be able to apply floor and ceiling functions. Students are expected to be able to determine whether a function is invertible, be able find out the inverse of a function if the function is invertible, be able to find the composite functions of two given functions, be able to apply floor and ceiling functions.

Functions

- Definition 1: Let A and B be nonempty sets.

  A function *f* from *A* to *B* is an assignment of exactly one element of *B* to each element of *A*.

  We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write $f: A \to B$
- Note: Functions are sometimes called *mappings* or *transformations*.

Functions

- Functions are specified in many different ways.
- Sometimes we explicitly state the assignments, as in Figure 1.
- Often we give a formula, such as $f(x) = x + 1$, to define a function.
- Other times we use a computer program to specify a function.

Functions

- A function $f: A \rightarrow B$ can also be defined in terms of a **relation** from A to B. [we will cover Relation in final term]
- A relation from A to B is just a subset of $A \times B$.
- A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B. This function is defined by the assignment $f(a)=b$, where (a, b) is the unique ordered pair in the relation that has a as its first element.

FIGURE 1: Assignment of Grades in a Discrete Mathematics Class

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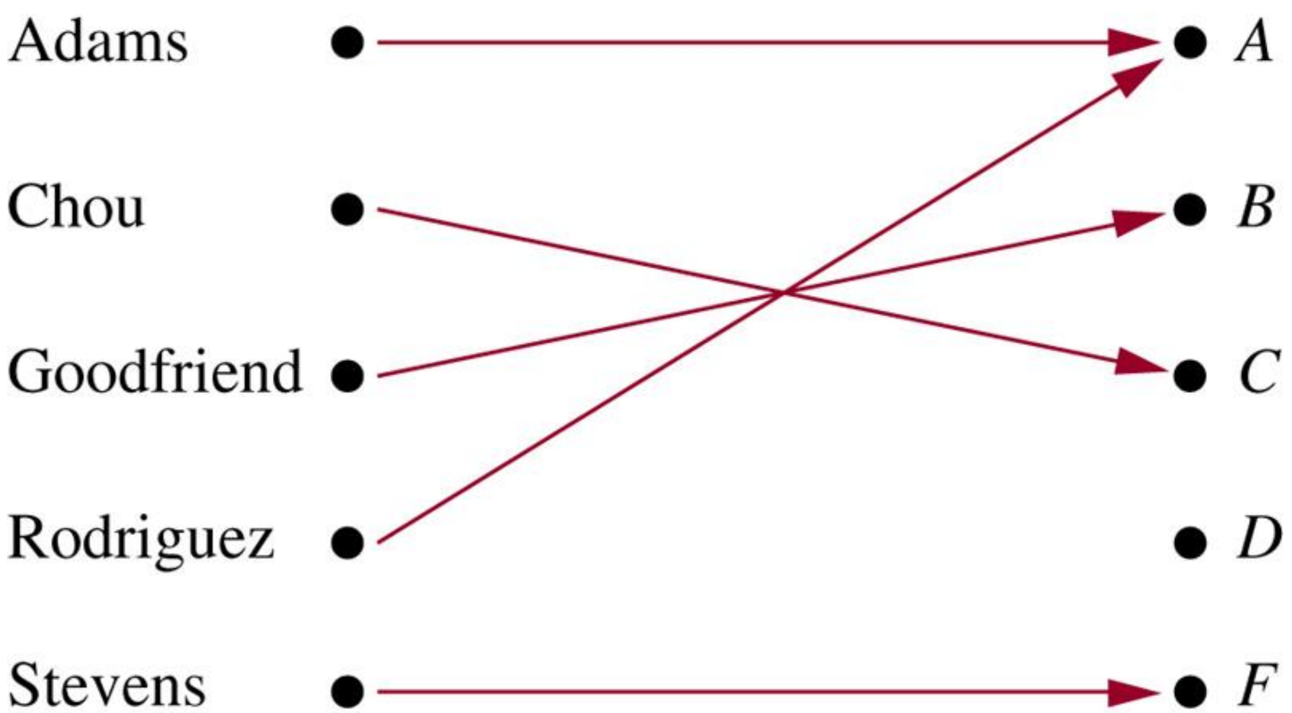

Some Function Terminology

Definition 2: If f is a function from A to B, we say that **A** is the **domain** of f and **B** is the **codomain** of f.

- If $f(a)=b$, a is the ***preimage*** of b and b is the ***image*** of a.

- ***Range*** of f *is* the set of all images of elements of *A.*
- Also, if f is a function from A to B, we say that f maps from A to B.

Some Function Terminology

- If *f* is a function from A to B, we write $f: A \rightarrow B$
  - *A* is the *domain* of *f*
  - *B* is the *codomain* of *f*
  - If $f(a)=b$,
    - *a is called the preimage* of *b*
    - *b is called the image* of *a*
- ***Range*** of *f :* the set of all images of elements of *A*

Range versus Codomain

- The range of a function might ***not*** be its whole codomain.
- The codomain is the set that the function is ***declared*** to map all domain values into.
- The range is the particular set of values in the codomain the function ***actually*** maps elements of the domain to.

Range versus Codomain: *Example* (*See the* FIGURE 1 in the previous slide)

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {*A, B, C,D,F*}".
- At this point, you know *f*'s codomain is: {*A, B, C,D,F*}, and it's range is unknown!
- Suppose the grades turn out all *A*s and *B*s.
- Then the range of *f* is {*A, B*}", but it's codomain is still {*A, B, C,D,F*}.

Example 1

- **What** are the **domain**, **codomain**, and **range** of the function that assigns grades to students of Discrete Math class as follows?

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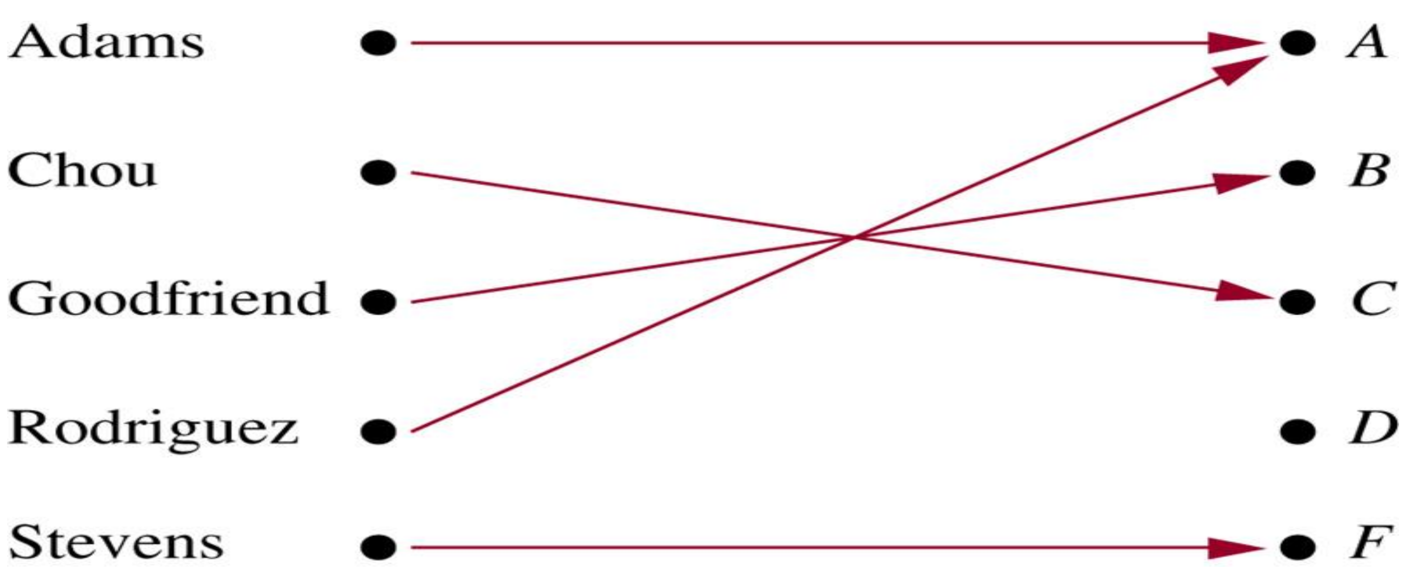

Solution of Example 1

- **Solution:**
- Let G be the function that assigns grade to a student of Discrete Mathematics class.
- The **domain of G** is the set { Adams, Chou, Goodfriend, Rodriguez, Stevens}
- The **codomain of G** is the set { A, B, C, D, F}
- The **range of G** is the set { A, B, C, F}
  - Because each grade except D is assigned to some student

Example 2

- Let R be the relation consisting of ordered pairs (Abdul, 22), (Brenda,24), (Carla,21), (Desire,22), (Eddie,24), and (Felicia,22), where each pair consists of a graduate student and the age of this student. *What is the function that this relation determines?*

- **Solution**: This relation defines the function f, where with f(Abdul)= 22, f(Brenda)=24, f(Carla)=21, f(Desire)=22, f(Eddie)= 24, and f(Felicia)=22.
- **Here, domain** is the set { Abdul, Brenda, Carla, Desire, Eddie, Felicia }
- To define the function f, we need to specify a codomain. Here, we can take the **codomain** to be the set of positive integers
- **Range** is the set {21,22,24}

Functions

Definition 3: Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x)\, f_2(x)$$

Example 6

- Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x)=x^2$ and $f_2(x)=x-x^2$. What are the functions f_1+f_2 and $f_1 f_2$?
- ***Solution***:

$$(f_1+f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4$$

Functions

- **Definition 4**: Let f be a function from the set A to the set B, and let S is a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S.
- We denote the image of S by $f(S)$.

$$f(S) = \{t \mid \exists s \in S(t=f(s))\}$$

Example 7

- Let A = {a, b, c, d, e} and B = {1, 2, 3, 4} with f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1, f(e) = 1.

  What is the image of the subset S = {b, c, d}?

- **Solution**:
- The image of the subset S = {b, c, d} is the set

  $f(S) = \{1, 4\}$

One-to-One Functions

- **Definition 5**: A function f is *one-to-one* or *injective*, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
- A function $f: A \to B$ is said to be one-to-one if all the elements in the domain A have **distinct images**.
- We can express that f is one-to-one Using quantifiers as $\forall a \forall b\, (f(a) = f(b) \to a = b)$, *or equivalently,*

  $\forall a \forall b\, (a \neq b \to f(a) \neq f(b))$, *where the universe of discourse is the domain of the function f*

Example 8

- Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with $f(a)=4$, $f(b)=5$, $f(c)= 1$, and $f(d)=3$ is one-to-one.
- **Solution**: The function is one-to-one because every element of domain has a distinct image.
  - The function f is one-to-one because f takes on different values at the four elements of its domain.

FIGURE for Example 8 :
A *One-to-One* Function

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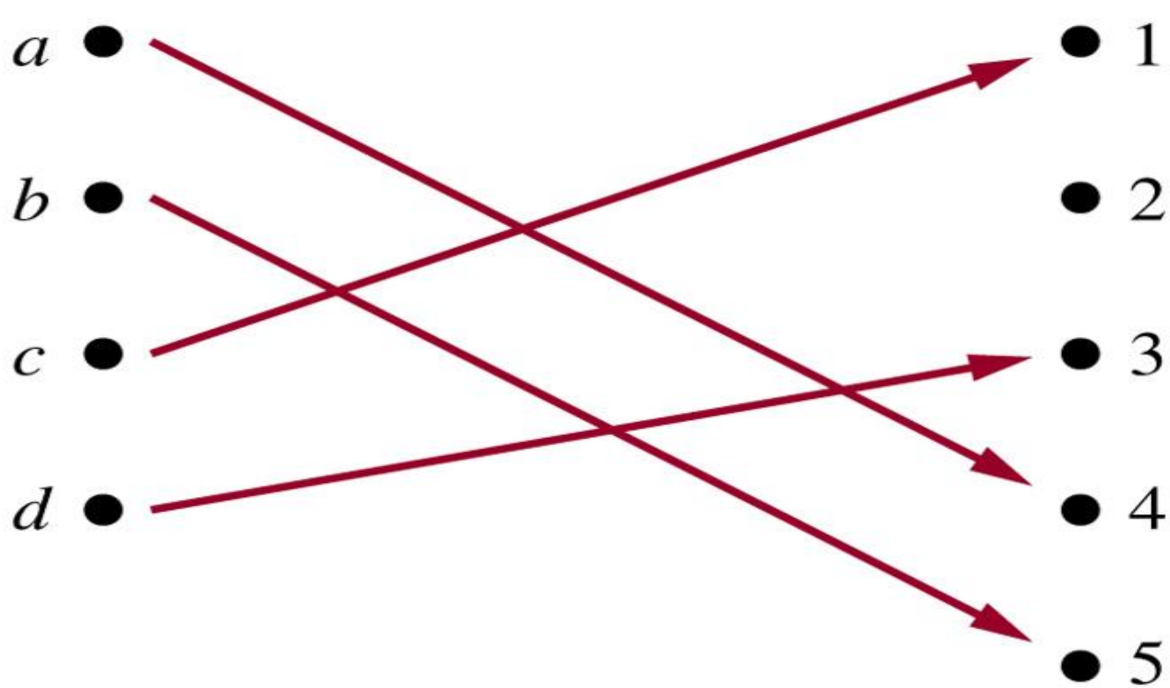

Example 9

Example 9: Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is ***not*** one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 = -1$
(*i.e.* 1 and –1 have same image 1)

Class Work

Determine whether the function $f(x) = x^2$ from the set of **positive integers** to the set of **positive integers** is one-to-one.

Example 10

- Determine whether the function $f(x) = x + 1$ from the set of real numbers to the set of real numbers is one-to-one.

- **Solution**: The function $f(x) = x + 1$ is a one-to-one function. Since $x + 1 = y + 1$, when $x = y$
- For any real number x, there is a distinct image, just 1 bigger than x; so, the function is one-to-one.

Example : One-to-one function

- Let A = {1, 2, 3} and B = {a, b, c, d}, and let f(1) = a, f(2) = b, f(3) = d. Then *f* is injective, since the different elements 1, 2, 3 in A are assigned to the different elements a, c, d respectively in B
- Note: Every element of domain has a distinct image. So, the function is one-to-one.

Onto Function

- **Definition 7**: A function f from A to B is ***onto*** or ***surjective***, iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
- A function $f: A \rightarrow B$ is said to be an ***onto*** function if **each element of B** is the image of ***some*** element of A
  - i.e., if B = range of f
- **Note:** A function is onto if every element of codomain has preimage(s).

Example 11

- Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f(a)=3, f(b)=2, f(c)=1, and f(d)=3. Is f an onto?

  [see Figure on next slide]
- **Solution**: Because all three elements of the codomain are images of elements in the domain, *f* is onto.

FIGURE for Example 11: An *Onto* Function

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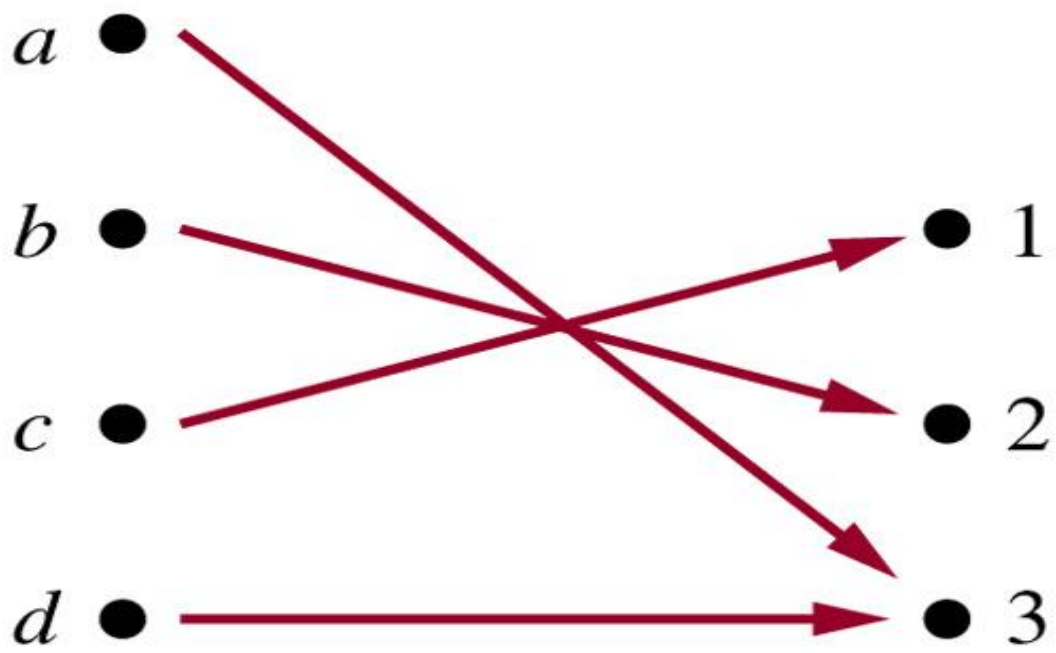

Example 12

- Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?
- **Solution**: The function f is not onto, because there is no integer x with $x^2 = -1$, for instance.

- Note: The elements of the codomain that are negative integers (– 1, – 2, – 3 etc.) do not have any preimage.

Class Work

Is the function $f(x) = x^2$ from the set of positive integers to the set of positive integers onto?

One-to-one correspondence (bijection)

- **Definition 8**: A function *f* is a ***one-to-one correspondence*** or a ***bijection*** if it is both one-to-one and onto.

- **Example**: Let ***f*** be the function from A to B where A={1, 2, 3, 4} and B = {a, b, c, d} with *f*(1)=d, *f*(2)=b, *f*(3)=c, and *f*(4)=a, then *f* is bijective function.
  - *f* is one-to-one since the every element of domain has a distinct image
  - *f* is onto since every element of B is the image of some element in A.
  - Hence *f* is a *bijective function (or, one-to-one correspondence)*
- Practice yourself: Example 14

FIGURE: Examples of Different Types of Correspondences

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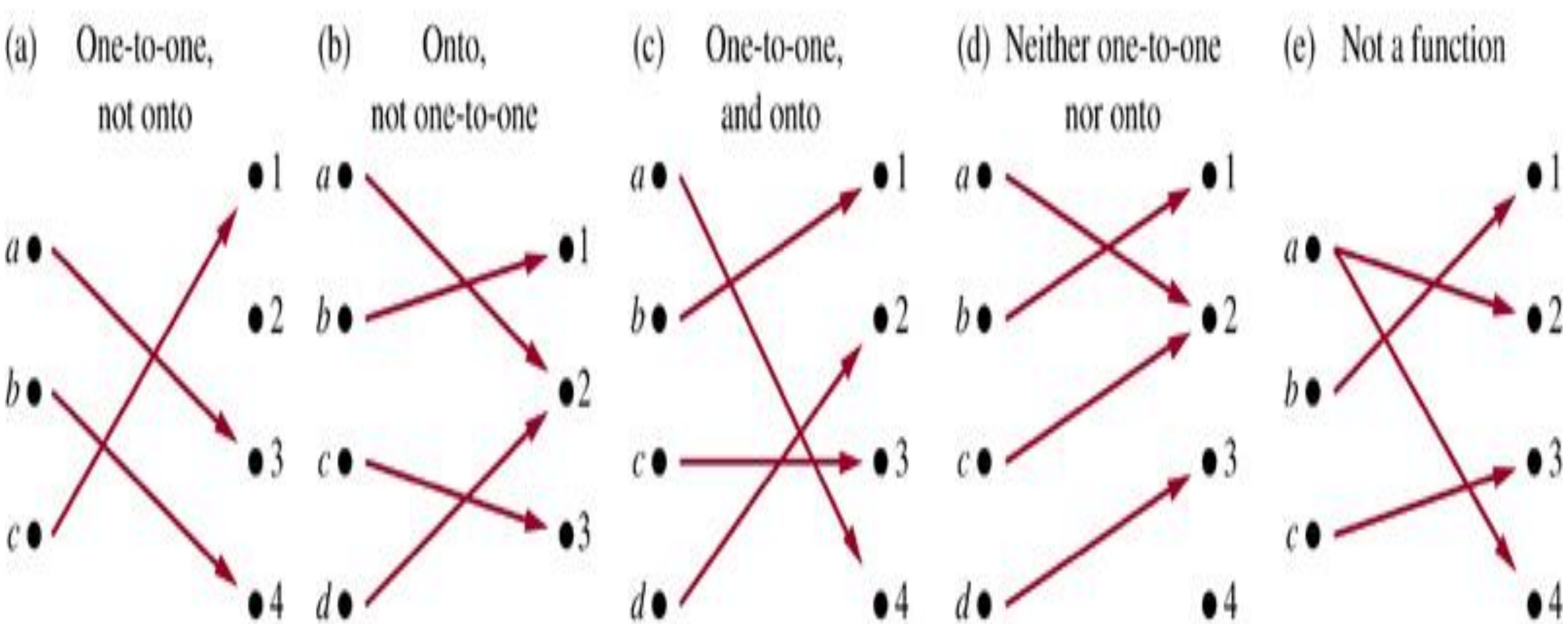

Inverse Functions

- **Definition 9**: Let f be a one-to-one correspondence from A to B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. Hence,

  $f^{-1}(b) = a$, when $f(a) = b$

FIGURE: The Function f^{-1} is the Inverse of Function f

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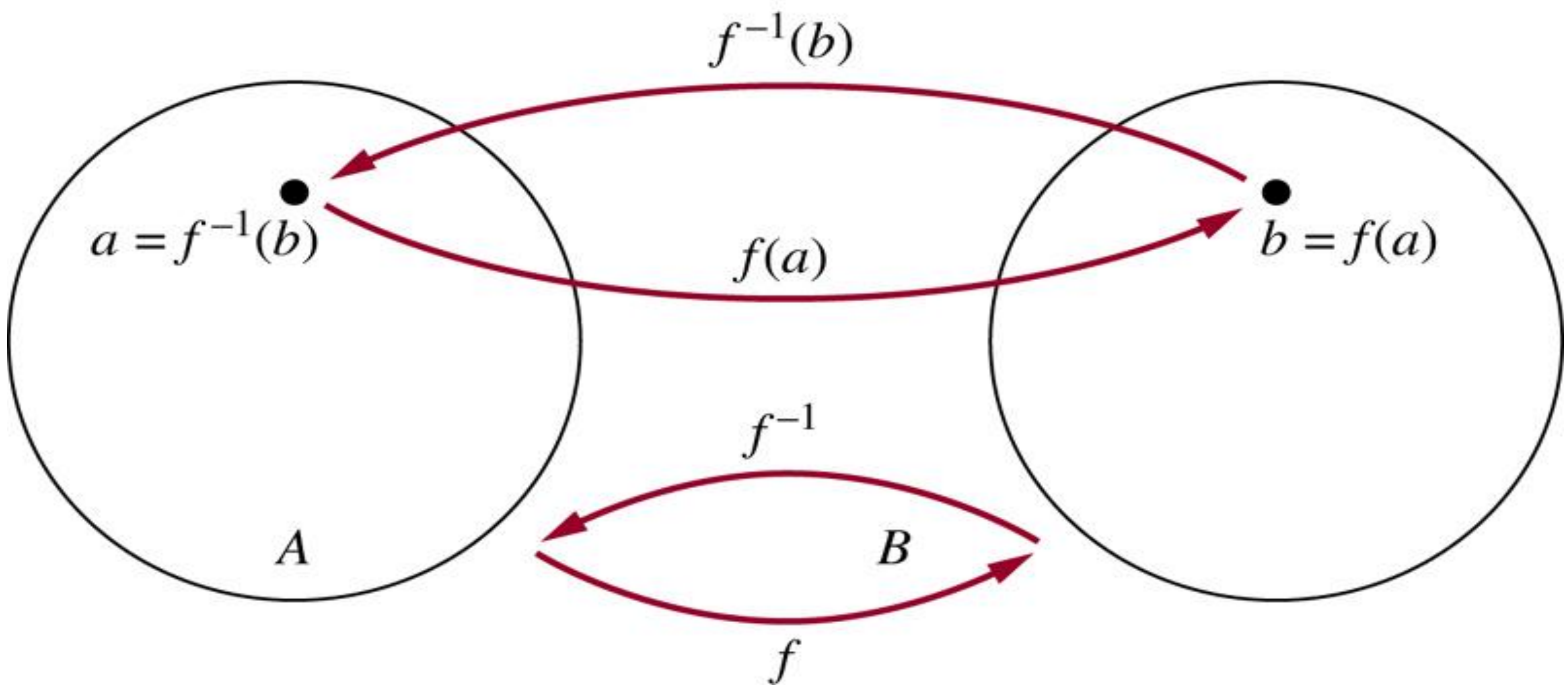

Inverse Functions

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Example 16

- Let f be the function from {a, b, c} to {1, 2, 3} such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.

  Is f invertible, and if it is, what is its inverse?

- **Solution**: The function f is invertible because it is a one-to-one correspondence.
- The inverse function f^{-1} reverses the correspondence given by f. So $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$

Example 17

- Let ***f*: *Z*→*Z*** be such that $f(x) = x + 1$.

  Is *f* invertible? and if it is, what is its inverse?

- ***Solution***: The function *f* is one-to-one because every element of domain has a distinct image, just 1 bigger than that element. Again, the function *f* is onto because every element of codomain has a preimage, just 1 smaller than that element. Therefore, the function *f* is a one-to-one correspondence. So, the function *f* is invertible.

- To reverse the correspondence, suppose that y is the image of x, so that $y = x+1$. Then $x = y – 1$. This means that $y – 1$ is the unique element of Z that is sent to y by *f*. Consequently, $f^{-1}(y) = y – 1$

Example 18

- Let f be the function from **R** to **R** with $f(x) = x^2$.
  Is f invertible?

- **Solution**: Because $f(-2) = f(2) = 4$, f is not one-to-one.
  Since f is not one-to-one, it is not one-to-one correspondence. Hence, f is **not** invertible.

Compositions of Functions

- **Definition**: Let g be a function from the set $\boldsymbol{A}$ to the set $\boldsymbol{B}$ and let f be a function from the set $\boldsymbol{B}$ to the set $\boldsymbol{C}$.

  The composition of functions f and g, denoted by $\boldsymbol{f \circ g}$, is defined by $(f \circ g)(a) = f(g(a))$

- **Note:** $\boldsymbol{f \circ g}$ and $\boldsymbol{g \circ f}$ are not equal
- **Note:** The composition $\boldsymbol{f \circ g}$ can NOT be defined ***unless*** the range of g is a subset of the domain of f

FIGURE : The Composition of the Functions *f* and *g*

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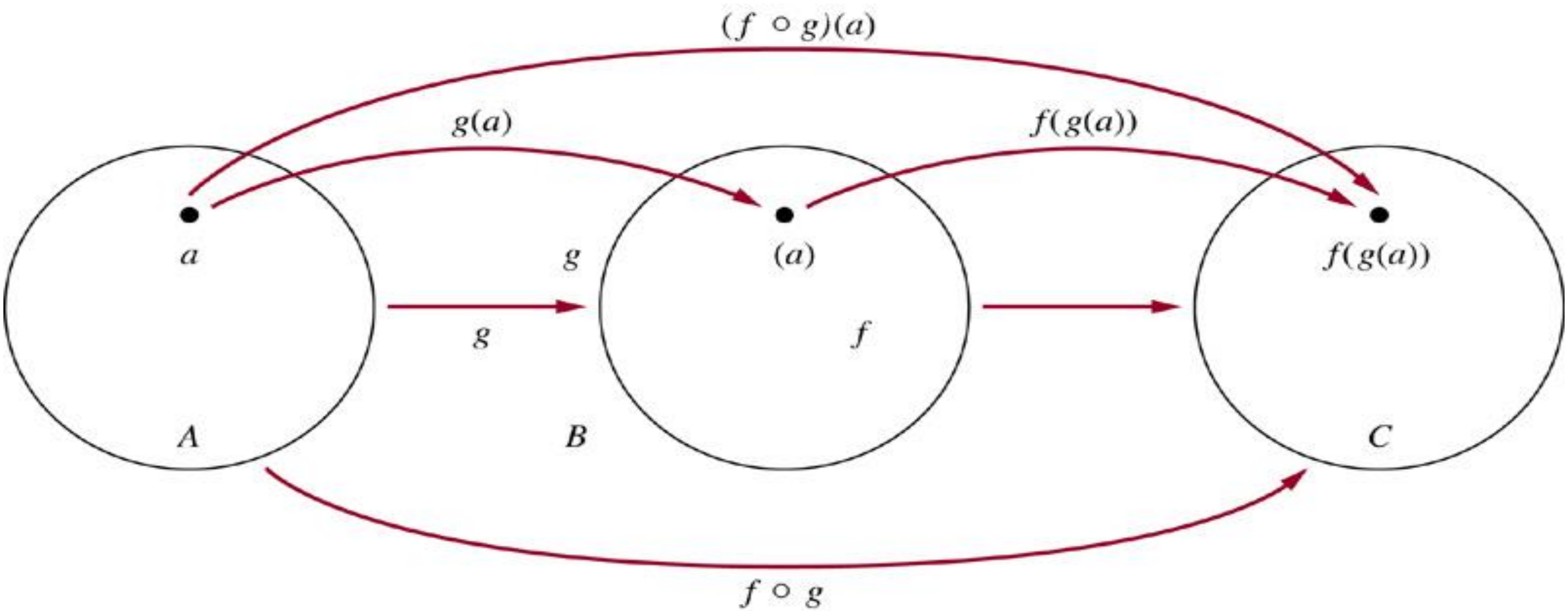

Modified Example 21

@ p. 141(6th ed.) @ p.149 (7th ed.)

- Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

  Find the composite functions of f and g.

Solution:

- The composition of the functions f and g is

  $(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2) + 3 = 6x + 7$
- The composition of the functions g and f *is*

  $(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3) + 2 = 6x + 11$
- Practice @ Home: Example 20 (p.149)

Some Important Functions

- Two important functions in discrete mathematics. These functions are often used when objects are counted. They play an important role in the analysis of the number of steps used by procedures to solve problems of a particular size.
  - **Floor function**
  - **Ceiling Function**
- The **floor** and **ceiling** functions map the **real numbers** onto the **integers** (**R**$\rightarrow$**Z**).

Floor Function

- The floor function assigns to the real number *x* the largest integer that is less than or equal to *x*.
  - Let *x* be a real number. The floor function rounds *x* down to the closest integer less than or equal to *x*.

- The value of the floor function at *x* is denoted by $\lfloor x \rfloor$

- **Examples:** $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$

Ceiling Function

- The ceiling function assigns to the real number *x* the smallest integer that is greater than or equal to *x*.
- The value of the ceiling function at x is denoted by $\lceil x \rceil$
- **Examples:** $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$

Examples of Ceiling Function

- How many **bytes** are required to encode 600 kilobits of data? [***Note***: Each byte is made up of 8 bits]
- Answer: $\lceil (600X1000)/8 \rceil$ = 75000 bytes

- How many bytes are required to encode 1001 bits of data?
- Answer: $\lceil 1001/8 \rceil$ = 126 bytes

Example of Floor Function

- In asynchronous transfer mode (ATM), data are organized into cells of 53 bytes. **How many ATM cells** can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?
- **Solution**: In 1 minute this connection can transmit 500.1000.60 bits = 30,000,000 bits

  Each ATM cell is 53 bytes long, which means that it is

  53.8 = 424 bits long

  Number of ATM cells that can be transmitted is

  $= \lfloor 30{,}000{,}000/424 \rfloor = 70{,}754$

Practice @ Home

- **Relevant odd-numbered exercises from text book**

THANK YOU