CHAPTER 12

Differentiation

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx} =$ .... *(1982 - 2 Marks)*

2. If $f_r(x)$, $g_r(x)$, $h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$

and $F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$ then $F'(x)$ at $x = a$ is

.... *(1985 - 2 Marks)*

3. If $f(x) = \log_x(\ln x)$, then $f'(x)$ at $x = e$ is .... *(1985 - 2 Marks)*

4. The derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x = \frac{1}{2}$ is .... *(1986 - 2 Marks)*

5. If $f(x) = |x-2|$ and $g(x) = f[f(x)]$, then $g'(x) =$ .... for $x > 20$ *(1990 - 2 Marks)*

6. If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\frac{dy}{dx} =$ .... *(1996 - 1 Mark)*

B True/ False

1. The derivative of an even function is always an odd function. *(1983 - 1 Mark)*

C MCQs with One Correct Answer

1. If $y^2 = P(x)$, a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ equals *(1988 - 2 Marks)*

   (a) $P'''(x) + P'(x)$ (b) $P''(x)P'''(x)$
   (c) $P(x)P'''(x)$ (d) a constant

2. Let $f(x)$ be a quadratic expression which is positive for all the real values of x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x, *(199 larks)*

   (a) $g(x) < 0$ (b) $g(x) > 0$
   (c) $g(x) = 0$ (d) $g(x) \geq 0$

3. If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to *(1994)*

   (a) $(\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$
   (b) $\tan x (\sin x)^{\tan x - 1}.\cos x$
   (c) $(\sin x)^{\tan x}\sec^2 x \log \sin x$
   (d) $\tan x (\sin x)^{\tan x - 1}$

4. If $x^2 + y^2 = 1$ then *(2000)*

   (a) $yy'' - 2(y')^2 + 1 = 0$ (b) $yy'' + (y')^2 + 1 = 0$
   (c) $yy'' + (y')^2 - 1 = 0$ (d) $yy'' + 2(y')^2 + 1 = 0$

5. Let $f: (0, \infty) \to R$ and $F(x) = \int_0^x f(t)dt$. If $F(x^2) = x^2(1+x)$, then $f(4)$ equals *(2001S)*

   (a) 5/4 (b) 7 (c) 4 (d) 2

6. If y is a function of x and $\log(x+y) - 2xy = 0$, then the value of $y'(0)$ is equal to *(2004S)*

   (a) 1 (b) -1 (c) 2 (d) 0

7. If $f(x)$ is a twice differentiable function and given that $f(1) = 1; f(2) = 4, f(3) = 9$, then *(2005S)*

   (a) $f''(x) = 2$ for $\forall\ x \in (1, 3)$
   (b) $f''(x) = f'(x) = 5$ for some $x \in (2, 3)$
   (c) $f''(x) = 3$ for $\forall\ x \in (2, 3)$
   (d) $f''(x) = 2$ for some $x \in (1, 3)$

8. $\frac{d^2x}{dy^2}$ equals *(2007 -3 marks)*

   (a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

   (c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

9. Let $g(x) = \log f(x)$ where $f(x)$ is twice differentible positive function on $(0, \infty)$ such that $f(x+1) = x f(x)$. Then, for $N = 1, 2, 3, \ldots\ldots\ldots$ *(2008)*

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(a) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots..+\frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots..+\frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots..+\frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots..+\frac{1}{(2N+1)^2}\right\}$

10. Let $f : [0, 2] \to R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let

$F(x) = \int_0^{x^2} f\left(\sqrt{t}\right)dt$ for $x \in [0,2]$. If $F'(x) = f'(x)$ for all

$x \in (0,2)$, then $F(2)$ equals *(JEE Adv. 2014)*

(a) $e^2 - 1$ (b) $e^4 - 1$
(c) $e - 1$ (d) e^4

D MCQs with One or More than One Correct

1. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in \mathbb{R}$. Then *(JEE Adv. 2016)*

(a) $g'(2) = \frac{1}{15}$ (b) $h'(1) = 666$
(c) $h(0) = 16$ (d) $h(g(3)) = 36$

2. For every twice differentiable function $f : \mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE? *(JEE Adv. 2018)*

(a) There exist $r, s \in \mathbb{R}$, where $r < s$, such that f is one-one on the open interval (r, s)

(b) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$

(c) $\lim_{x\to\infty} f(x) = 1$

(d) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \ne 0$

3. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE? *(JEE Adv. 2018)*

(a) $\sum_{j=1}^{5} \tan^2(f_j(0)) = 55$

(b) $\sum_{j=1}^{10} (1 + f_j'(0)) \sec^2(f_j(0)) = 10$

(c) For any fixed positive integer n, $\lim_{x\to\infty} \tan(f_n(x)) = \frac{1}{n}$

(d) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2(f_n(x)) = 1$

4. Let $f : (0, \pi) \to \mathbb{R}$ be a twice differentiable function such that

$$\lim_{t\to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x \text{ for all } x \in (0, \pi).$$

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE? *(JEE Adv. 2018)*

(a) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(b) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(c) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(d) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

E Subjective Problems

1. Find the derivative of $\sin(x^2 + 1)$ with respect to x from first principle. *(1978)*

2. Find the derivative of

$$f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5} & \text{when } x \ne 1 \\ -\dfrac{1}{3} & \text{when } x = 1 \end{cases}$$

at $x = 1$ *(1979)*

3. Given $y = \frac{5x}{3\sqrt{(1-x)^2}} + \cos^2(2x+1)$; Find $\frac{dy}{dx}$. *(1980)*

4. Let $y = e^{x \sin x^3} + (\tan x)^x$. Find $\frac{dy}{dx}$ *(1981 - 2 Marks)*

5. Let f be a twice differentiable function such that

$f''(x) = -f(x)$, and $f'(x) = g(x)$, $h(x) = [f(x)]^2 + [g(x)]^2$

Find $h(10)$ if $h(5) = 11$ *(1982 - 3 Marks)*

6. If α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3, 4 and 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where prime denotes the derivatives. *(1984 - 4 Marks)*

7. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$ *(1989 - 2 Marks)*

8. Find $\frac{dy}{dx}$ at $x = -1$, when

$$(\sin y)^{\sin\left(\frac{\pi}{2}x\right)} + \frac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x \tan(\ln(x+2)) = 0$$

(1991 - 4 Marks)

9. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$,

prove that $\frac{y'}{y} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$.

(1998 - 8 Marks)

H Assertion & Reason Type Questions

1. Let $f(x) = 2 + \cos x$ for all real x.

STATEMENT - 1 : For each real t, there exists a point c in $[t, t+\pi]$ such that $f'(c) = 0$ because

STATEMENT - 2 : $f(t) = f(t + 2\pi)$ for each real t.

(2007 -3 marks)

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(c) Statement-1 is True, Statement-2 is False

(d) Statement-1 is False, Statement-2 is True.

2. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x)\sin x$

STATEMENT - 1 : $\lim\limits_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$

and

STATEMENT - 2 : $f'(0) = g(0)$ *(2008)*

(a) Statement - 1 is True, Statement - 2 is True; Statement - 2 is a correct explanation for Statement - 1

(b) Statement - 1 is True, Statement - 2 is True; Statement - 2 **is NOT** a correct explaination for Statement - 1

(c) Statement - 1 is True, Statement - 2 is False

(d) Statement - 1 is False, Statement - 2 is True

I Integer Value Correct Type

1. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is *(2009)*

2. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is ***(2011)***

Section-B JEE Main / AIEEE

1. If $y = (x + \sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is [2002]

(a) n^2y
(b) $-n^2y$
(c) $-y$
(d) $2x^2y$

2. If $f(y) = e^y$, $g(y) = y$; $y > 0$ and

$F(t) = \int_0^t f(t-y)g(y)\,dy$, then [2003]

(a) $F(t) = te^{-t}$ (b) $F(t) = 1 - te^{-t}(1+t)$

(c) $F(t) = e^t - (1+t)$ (d) $F(t) = te^t$.

3. If $f(x)=x^n$, then the value of [2003]

$$f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+\ldots\ldots\frac{(-1)^n f^n(1)}{n!} \text{ is}$$

(a) 1 (b) 2^n (c) 2^n-1 (d) 0

4. Let $f(x)$ be a polynomial function of second degree. If $f(1)=f(-1)$ and a,b,c are in A. P, then $f'(a), f'(b), f'(c)$ are in [2003]

(a) Arithmetic-Geometric Progression
(b) A.P
(c) G.P
(d) H.P.

5. If $x=e^{y+e^{y+e^{y+\ldots\infty}}}$, $x>0$, then $\frac{dy}{dx}$ is [2004]

(a) $\frac{1+x}{x}$ (b) $\frac{1}{x}$ (c) $\frac{1-x}{x}$ (d) $\frac{x}{1+x}$

6. The value of a for which the sum of the squares of the roots of the equation $x^2-(a-2)x-a-1=0$ assume the least value is [2005]

(a) 1 (b) 0 (c) 3 (d) 2

7. If the roots of the equation $x^2-bx+c=0$ be two consecutive integers, then b^2-4c equals [2005]

(a) -2 (b) 3 (c) 2 (d) 1

8. Let $f:R\to R$ be a differentiable function having $f(2)=6$, $f'(2)=\left(\frac{1}{48}\right)$. Then $\lim_{x\to 2}\int_6^{f(x)}\frac{4t^3}{x-2}dt$ equals [2005]

(a) 24 (b) 36 (c) 12 (d) 18

9. The set of points where $f(x)=\frac{x}{1+|x|}$ is differentiable is

(a) $(-\infty,0)\cup(0,\infty)$ (b) $(-\infty,-1)\cup(-1,\infty)$
(c) $(-\infty,\infty)$ (d) $(0,\infty)$ [2006]

10. If $x^m.y^n=(x+y)^{m+n}$, then $\frac{dy}{dx}$ is [2006]

(a) $\frac{y}{x}$ (b) $\frac{x+y}{xy}$ (c) xy (d) $\frac{x}{y}$

11. Let y be an implicit function of x defined by $x^{2x}-2x^x\cot y-1=0$. Then $y'(1)$ equals [2009]

(a) 1 (b) $\log 2$ (c) $-\log 2$ (d) -1

12. Let $f:(-1,1)\to\mathbf{R}$ be a differentiable function with $f(0)=-1$ and $f'(0)=1$. Let $g(x)=[f(2f(x)+2)]^2$. Then $g'(0)=$ [2010]

(a) -4 (b) 0 (c) -2 (d) 4

13. $\frac{d^2x}{dy^2}$ equals : [2011]

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

(c) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

14. If $y=\sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x=1$ is equal to : [JEE M 2013]

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$

(c) 1 (d) $\sqrt{2}$

15. If g is the inverse of a function f and $f'(x)=\frac{1}{1+x^5}$, then $g'(x)$ is equal to: [JEE M 2014]

(a) $\frac{1}{1+\{g(x)\}^5}$ (b) $1+\{g(x)\}^5$

(c) $1+x^5$ (d) $5x^4$

16. If $x=-1$ and $x=2$ are extreme points of $f(x)=\alpha\log|x|+\beta x^2+x$ then [JEE M 2014]

(a) $\alpha=2,\beta=-\frac{1}{2}$ (b) $\alpha=2,\beta=\frac{1}{2}$

(c) $\alpha=-6,\beta=\frac{1}{2}$ (d) $\alpha=-6,\beta=-\frac{1}{2}$

17. If for $x\in\left(0,\frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}.g(x)$, then g(x) equals : [JEE M 2017]

(a) $\frac{3}{1+9x^3}$ (b) $\frac{9}{1+9x^3}$

(c) $\frac{3x\sqrt{x}}{1-9x^3}$ (d) $\frac{3x}{1-9x^3}$