CHAPTER 19

Differential Equations

Section-A JEE Advanced/ IIT-JEE

C MCQs with One Correct Answer

1. A solution of the differential equation *(1999 – 2 Marks)*

$\left(\dfrac{dy}{dx}\right)^2 - x\dfrac{dy}{dx} + y = 0$ is

(a) $y = 2$ (b) $y = 2x$
(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

2. If $x^2 + y^2 = 1$, then *(2000S)*
(a) $yy'' - 2(y')^2 + 1 = 0$ (b) $yy'' + (y')^2 + 1 = 0$
(c) $yy'' + (y')^2 - 1 = 0$ (d) $yy'' + 2(y')^2 + 1 = 0$

3. If $y(t)$ is a solution of $(1+t)\dfrac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is equal to *(2003S)*
(a) $-1/2$ (b) $e + 1/2$
(c) $e - 1/2$ (d) $1/2$

4. If $y = y(x)$ and $\dfrac{2+\sin x}{y+1}\left(\dfrac{dy}{dx}\right) = -\cos x$, $y(0) = 1$, then $y\left(\dfrac{\pi}{2}\right)$ equals *(2004S)*
(a) 1/3 (b) 2/3 (c) –1/3 (d) 1

5. If $y = y(x)$ and it follows the relation $x\cos y + y\cos x = \pi$ then $y''(0) =$ *(2005S)*
(a) 1 (b) –1 (c) $\pi - 1$ (d) $-\pi$

6. The solution of primitive integral equation $(x^2 + y^2)\, dy = xy\, dx$ is $y = y(x)$. If $y(1) = 1$ and $(x_0) = e$, then x_0 is equal to *(2005S)*

(a) $\sqrt{2(e^2-1)}$ (b) $\sqrt{2(e^2+1)}$

(c) $\sqrt{3}\, e$ (d) $\sqrt{\dfrac{e^2+1}{2}}$

7. For the primitive integral equation $ydx + y^2dy = x\, dy$; $x \in R, y > 0, y = y(x), y(1) = 1$, then $y(-3)$ is *(2005S)*
(a) 3 (b) 2 (c) 1 (d) 5

8. The differential equation $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y}$ determines a family of circles with *(2005S)*

(a) variable radii and a fixed centre at (0, 1)
(b) variable radii and a fixed centre at (0, –1)
(c) fixed radius 1 and variable centres along the x–axis.
(d) fixed radius 1 and variable centres along the y–axis.

9. The function $y = f(x)$ is the solution of the differential equation

$\dfrac{dy}{dx} + \dfrac{xy}{x^2-1} = \dfrac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then

$\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)\,d(x)$ is *(JEE Adv. 2014)*

(a) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}$ (b) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}$

(c) $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{4}$ (d) $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{2}$

10. If $y = y(x)$ satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx$, $x > 0$ and $y(0) = \sqrt{7}$, then $y(256) =$ *(JEE Adv. 2018)*
(a) 3 (b) 9
(c) 16 (d) 80

D MCQs with One or More than One Correct

1. The order of the differential equation whose general solution is given by
$y = (C_1 + C_2)\cos(x + C_3) - C_4 e^{x+C_5}$, where C_1, C_2, C_3, C_4, C_5, are arbitrary constants, is *(1998 – 2 Marks)*
(a) 5 (b) 4 (c) 3 (d) 2

2. The differential equation representing the family of curves $y^2 = 2c\left(x + \sqrt{c}\right)$, where c is a positive parameter, is of *(1999 – 3 Marks)*
(a) order 1 (b) order 2 (c) degree 3 (d) degree 4

3. A curve $y = f(x)$ passes through (1, 1) and at $P(x, y)$, tangent cuts the x–axis and y–axis at A and B respectively such that BP : AP = 3 : 1, then *(2006 – 5M, –1)*
(a) equation of curve is $xy' - 3y = 0$
(b) normal at (1, 1) is $x + 3y = 4$
(c) curve passes through (2, 1/8)
(d) equation of curve is $xy' + 3y = 0$

4. If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0) = 0$, then *(2012)*
(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$
(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

5. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0.$ Then the equation of the curve is *(JEE Adv. 2013)*
(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$
(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

6. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statement is (are) true? *(JEE Adv. 2015)*
(a) $y(-4) = 0$ (b) $y(-2) = 0$
(c) $y(x)$ has a critical point in the interval (–1, 0)
(d) $y(x)$ has no critical point in the interval (–1, 0)

7. Consider the family of all circles whose centers lie on the straight line $y = x$. If this family of circle is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' $\left(\text{here } y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}\right)$, then which of the following statements is (are) true? *(JEE Adv. 2015)*
(a) $P = y + x$ (b) $P = y - x$
(c) $P + Q = 1 - x + y + y' + (y')^2$ (d) $P - Q = x + y - y' - (y')^2$

8. Let f : (0, ∞) → ℝ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then *(JEE Adv. 2016)*
(a) $\lim_{x\to 0+} f'\left(\frac{1}{x}\right) = 1$ (b) $\lim_{x\to 0+} xf\left(\frac{1}{x}\right) = 2$
(c) $\lim_{x\to 0+} x^2 f'(x) = 0$ (d) $|f(x)| \le 2$ for all $x \in (0, 2)$

9. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point (1, 3). Then the solution curve *(JEE Adv. 2016)*
(a) intersects $y = x + 2$ exactly at one point
(b) intersects $y = x + 2$ exactly at two points
(c) intersects $y = (x + 2)^2$
(d) does NOT intersect $y = (x + 3)^2$

10. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that
$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$$
for all $x \in [0, \infty)$. Then, which of the following statement (s) is (are) TRUE? *(JEE Adv. 2018)*
(a) The curve $y = f(x)$ passes through the point (1, 2)
(b) The curve $y = f(x)$ passes through the point (2, –1)
(c) The area of the region
$\left\{(x, y) \in [0,1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\right\}$ is $\frac{\pi - 2}{4}$
(d) The area of the region
$\left\{(x, y)\} \in [0,1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\right\}$ is $\frac{\pi - 1}{4}$

11. Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the *y-axis* at Y_p. If PY_p has length 1 for each point P on Γ, then which of the following options is/are correct? *(JEE Adv. 2019)*
(a) $y = -\log_e\left(\frac{1 + \sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$
(b) $xy' - \sqrt{1-x^2} = 0$
(c) $y = \log_e\left(\frac{1 + \sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$
(d) $xy' + \sqrt{1-x^2} = 0$

E Subjective Problems

1. If $(a+bx)\,e^{y/x}=x$, then prove that $x^3\dfrac{d^2y}{dx^2}=\left(x\dfrac{dy}{dx}-y\right)^2$ *(1983 – 3 Marks)*

2. A normal is drawn at a point $P(x, y)$ of a curve. It meets the x–axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is

$$y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$$ *(1994 – 5 Marks)*

Find the equation of such a curve passing through $(0, k)$.

3. Let $y=f(x)$ be a curve passing through $(1, 1)$ such that the triangle formed by the coordinate axes and the tangent at any point of the curve lies in the first quadrant and has area 2. Form the differential equation and determine all such possible curves. *(1995 – 5 Marks)*

4. Determine the equation of the curve passing through the origin, in the form $y=f(x)$, which satisfies the differential equation $\dfrac{dy}{dx}=\sin(10x+6y)$. *(1996 – 5 Marks)*

5. Let $u(x)$ and $v(x)$ satisfy the differential equation $\dfrac{du}{dx}+p(x)\,u=f(x)$ and $\dfrac{dv}{dx}+p(x)\,v=g(x)$, where $p(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1)>v(x_1)$ for some x_1 and $f(x)>g(x)$ for all $x>x_1$, prove that any point (x, y) where $x>x_1$, does not satisfy the equatons $y=u(x)$ and $y=v(x)$. *(1997 – 5 Marks)*

6. A curve passing through the point $(1, 1)$ has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x–axis. Determine the equation of the curve. *(1999 – 10 Marks)*

7. A country has a food deficit of 10%. Its population grows continously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self–sufficient in food after n years, where n is the smallest integer bigger than or equal to $\dfrac{\ln 10-\ln 9}{\ln(1.04)-0.03}$. *(2000 – 10 Marks)*

8. A hemispherical tank of radius 2 metres is initially full of water and has an outlet of 12 cm^2 cross–sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t)=0.6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank. (Hint : Form a differential equation by relating the decrease of water level to the outflow). *(2001 – 10 Marks)*

9. A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = $k>0$). Find the time after which the cone is empty. *(2003 – 4 Marks)*

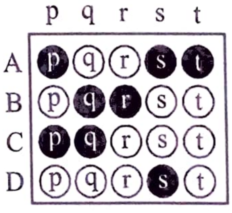
10. A curve 'C' passes through $(2, 0)$ and the slope at (x, y) as $\dfrac{(x+1)^2+(y-3)}{x+1}$. Find the equation of the curve. Find the area bounded by curve and x–axis in fourth quadrant. *(2004 – 4 Marks)*

11. If length of tangent at any point on the curve $y=f(x)$ intecepted between the point and the x–axis is of length 1. Find the equation of the curve. *(2005 – 4 Marks)*

F Match the Following

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



1. Match the statements/expressions in **Column I** with the open intervals in **Column II**. (2009)

Column I	**Column II**
(A) Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2 + y' + y = 0$	(p) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(B) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(q) $\left(0, \frac{\pi}{2}\right)$
(C) Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(r) $\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
(D) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s) $\left(0, \frac{\pi}{8}\right)$
	(t) $(-\pi, \pi)$

H Assertion & Reason Type Questions

1. Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2) = \frac{2}{\sqrt{3}}$.

STATEMENT-1: $y(x) = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$ and

STATEMENT-2: $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$

(2008)

(a) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 **is** a correct explanation for STATEMENT - 1
(b) STATEMENT - 1 is True, STATEMENT - 2 is True; STATEMENT - 2 **is NOT** a correct explanation for STATEMENT - 1
(c) STATEMENT - 1 is True, STATEMENT - 2 is False
(d) STATEMENT - 1 is False, STATEMENT - 2 is True

I Integer Value Correct Type

1. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in \boldsymbol{R}$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is *(2011)*

2. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation
$$\frac{dy}{dx} = (2+5y)(5y-2),$$
then the value of $\lim_{x \to -\infty} f(x)$ is ______. *(JEE Adv. 2018)*

3. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation
$$f(x+y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}$$
Then, the value of $\log_e(f(4))$ is ______. *(JEE Adv. 2018)*

Section-B JEE Main / AIEEE

1. The order and degree of the differential equation $\left(1+3\frac{dy}{dx}\right)^{2/3}=4\frac{d^3y}{dx^3}$ are [2002]

(a) $(1, \frac{2}{3})$ (b) $(3, 1)$
(c) $(3, 3)$ (d) $(1, 2)$

2. The solution of the equation $\frac{d^2y}{dx^2}=e^{-2x}$ [2002]

(a) $\frac{e^{-2x}}{4}$ (b) $\frac{e^{-2x}}{4}+cx+d$
(c) $\frac{1}{4}e^{-2x}+cx^2+d$ (d) $\frac{1}{4}e^{-4x}+cx+d$

3. The degree and order of the differential equation of the family of all parabolas whose axis is x - axis, are respectively. [2003]

(a) 2, 3 (b) 2, 1 (c) 1, 2 (d) 3, 2.

4. The solution of the differential equation $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$, is [2003]

(a) $xe^{2\tan^{-1}y}=e^{\tan^{-1}y}+k$ (b) $(x-2)=ke^{2\tan^{-1}y}$
(c) $2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+k$ (d) $xe^{\tan^{-1}y}=\tan^{-1}y+k$

5. The differential equation for the family of circle $x^2+y^2-2ay=0$, where a is an arbitrary constant is [2004]

(a) $(x^2+y^2)y'=2xy$ (b) $2(x^2+y^2)y'=xy$
(c) $(x^2-y^2)y'=2xy$ (d) $2(x^2-y^2)y'=xy$

6. Solution of the differential equation $ydx+(x+x^2y)dy=0$ is [2004]

(a) $\log y=Cx$ (b) $-\frac{1}{xy}+\log y=C$
(c) $\frac{1}{xy}+\log y=C$ (d) $-\frac{1}{xy}=C$

7. The differential equation representing the family of curves $y^2=2c(x+\sqrt{c})$, where $c>0$, is a parameter, is of order and degree as follows : [2005]

(a) order 1, degree 2 (b) order 1, degree 1
(c) order 1, degree 3 (d) order 2, degree 2

8. If $x\frac{dy}{dx}=y(\log y-\log x+1)$, then the solution of the equation is [2005]

(a) $y\log\left(\frac{x}{y}\right)=cx$ (b) $x\log\left(\frac{y}{x}\right)=cy$
(c) $\log\left(\frac{y}{x}\right)=cx$ (d) $\log\left(\frac{x}{y}\right)=cy$

9. The differential equation whose solution is $Ax^2+By^2=1$ where A and B are arbitrary constants is of [2006]

(a) second order and second degree
(b) first order and second degree
(c) first order and first degree
(d) second order and first degree

10. The differential equation of all circles passing through the origin and having their centres on the x-axis is [2007]

(a) $y^2=x^2+2xy\frac{dy}{dx}$ (b) $y^2=x^2-2xy\frac{dy}{dx}$
(c) $x^2=y^2+xy\frac{dy}{dx}$ (d) $x^2=y^2+3xy\frac{dy}{dx}$.

11. The soluton of the differential equation $\frac{dy}{dy}=\frac{x+y}{x}$ satisfying the condition $y(1)=1$ is [2008]

(a) $y=\ln x+x$ (b) $y=x\ln x+x^2$
(c) $y=xe^{(x-1)}$ (d) $y=x\ln x+x$

12. The differential equation which represents the family of curves $y=c_1e^{c_2x}$, where c_1, and c_2 are arbitrary constants, is [2009]

(a) $y''=y'y$ (b) $yy''=y'$
(c) $yy''=(y')^2$ (d) $y'=y^2$

13. Solution of the differential equation $\cos x\,dy=y(\sin x-y)\,dx,\ 0<x<\frac{\pi}{2}$ is [2010]

(a) $y\sec x=\tan x+c$ (b) $y\tan x=\sec x+c$
(c) $\tan x=(\sec x+c)\,y$ (d) $\sec x=(\tan x+c)\,y$

14. If $\frac{dy}{dx}=y+3>0$ and $y(0)=2$, then $y(\ln 2)$ is equal to : [2011]

(a) 5 (b) 13
(c) -2 (d) 7

15. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T-t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then the scrap value $V(T)$ of the equipment is **[2011]**

(a) $I - \frac{kT^2}{2}$ (b) $I - \frac{k(T-t)^2}{2}$

(c) e^{-kT} (d) $T^2 - \frac{1}{k}$

16. The population $p(t)$ at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5\,p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is : **[2012]**

(a) $2 \ln 18$ (b) $\ln 9$ (c) $\frac{1}{2} \ln 18$ (d) $\ln 18$

17. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is **[JEE M 2013]**

(a) 2500 (b) 3000 (c) 3500 (d) 4500

18. Let the population of rabbits surviving at time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ equals: **[JEE M 2014]**

(a) $600 - 500\,e^{t/2}$ (b) $400 - 300\,e^{-t/2}$

(c) $400 - 300\,e^{t/2}$ (d) $300 - 200\,e^{-t/2}$

19. Let y(x) be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x, (x \geq 1)$. Then y (e) is equal to: **[JEE M 2015]**

(a) 2 (b) 2e (c) e (d) 0

20. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)\,dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to : **[JEE M 2016]**

(a) $\frac{2}{5}$ (b) $\frac{4}{5}$ (c) $-\frac{2}{5}$ (b) $-\frac{4}{5}$

21. If $(2 + \sin x)\frac{dy}{dx} + (y + 1)\cos x = 0$ and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to : **[JEE M 2017]**

(a) $\frac{4}{3}$ (b) $\frac{1}{3}$

(c) $-\frac{2}{3}$ (d) $-\frac{1}{3}$

22. Let $y = y(x)$ be the solution of the differential equation $\sin x \frac{dy}{dx} + y\cos x = 4x, x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to : **[JEE M 2018]**

(a) $\frac{-8}{9\sqrt{3}}\pi^2$ (b) $-\frac{8}{9}\pi^2$ (c) $-\frac{4}{9}\pi^2$ (d) $\frac{4}{9\sqrt{3}}\pi^2$

23. If $y = y(x)$ is the solution of the differential equation, $x\frac{dy}{dx} + 2y = x^2$ satisfying $y(a) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to: **[JEE M 2019 – 9 Jan (M)]**

(a) $\frac{7}{64}$ (b) $\frac{1}{4}$

(c) $\frac{49}{16}$ (d) $\frac{13}{16}$

24. The solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$ $(x \neq 0)$ with $y(1) = 1$, is: **[JEE M 2019 – 9 April (M)]**

(a) $y = \frac{4}{5}x^3 + \frac{1}{5x^2}$ (b) $y = \frac{x^3}{5} + \frac{1}{5x^2}$

(c) $y = \frac{x^2}{4} + \frac{3}{4x^2}$ (d) $y = \frac{3}{4}x^2 + \frac{1}{4x^2}$