UNIT-2.

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

* Formation of Partial Differential Equations by
  - elimination of arbitrary constants
  - elimination of arbitrary functions.
* Lagrange's method to solve first order linear equations
* Standard type methods to solve non-linear equations.

01) PARTIAL DIFFERENTIAL EQUATION: Any equation of the form $f(z, x, y, z_x, z_y, z_{xx}, z_{yy}, z_{xy}, z_{yx} \ldots) = 0$ where is z is dependent variable and others (x, y) are independent variable.

02) ORDER OF PDE: It is order of the highest order partial derivative

Ex: First order PDE:

03) NOTATION

$z_x = \frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q = z_y$, $z_{xx} = \frac{\partial^2 z}{\partial x^2} = r$,

$\frac{\partial^2 z}{\partial x \partial y} = z_{yx}$ $\cancel{\frac{\partial}{\partial x}} = z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = s$, $\frac{\partial^2 z}{\partial y^2} = t$.

04) LINEAR DIFFERENTIAL EQUATION: A PDE is said to linear if it satisfies the following conditions.

(i) the degree of the dependent variables and its derivatives is 1.

(ii) No term contains product of dependent variable or its derivatives

Eg: $z_{xx} + z_{yy} = 0$, $yz_x + xz_y = 2$

05) QUASI LINEAR DE: A PDE is said to be quasi linear if

(i) degree of the highest order derivative is 1

(ii) No products of partial derivatives of highest order are present.

Eg: (i) $z_{xx} + 3z_{yy} = 0$

$(z_{xx}) + (3y)^2 = 2$

$z_{xx} \cdot z_{yy} = 1$ Neither linear nor Quasi they

(6) FORMATION OF PDE:

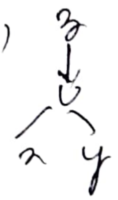
Recall Chain rules in Partial Difference

(i)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

(ii)



$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

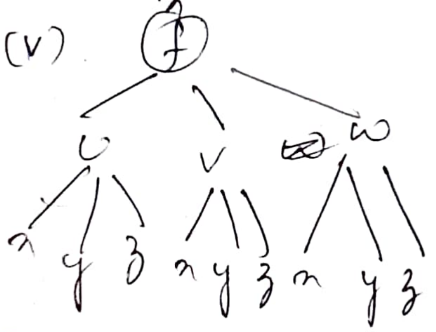
(iii)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

(iv)

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{dy}{dx}.$$

(v)

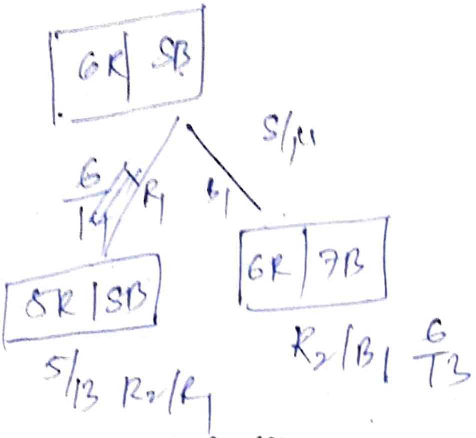


$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial y}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial z}$$

Prob I) A box contains 6 red balls and 8 blue balls. Two balls are drawn from the box successively, what is the probability that the second ball drawn is ~~white~~ red? without replacement



$P(R_2) = \frac{6}{14} \times \frac{5}{13} + \frac{8}{14} \cdot \frac{6}{13}$.

$P(R_2) = P(R_1) \cdot P(R_2/R_1) + P(B_1) + P(R_2/B_1)$

↓

total probability.

09th August 2019

FORMATION OF PARTIAL DIFFERENTIAL EQUATION:

a) By elimination of arbitrary constants:

$f(x, y, z, a, b) = 0 \longrightarrow (1)$

differentiate (1) partially w.r.t x.

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot p = 0 \longrightarrow (2)$

differentiate (1) w.r.t y.

$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0 \longrightarrow (3)$.

using equations (1), (2), (3), eliminate the arbitrary constants a & b. The resultant D.E is a first order order PDE.

Note: If the number of arbitrary constants is more than the number of independent variables. Then the PDE obtained is of second or higher order.

Problems

01) $z = ax^2 + by^2$

02) $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$ (α = constant)

03) $z = ax + by + a^2 + b^2$

04) $z = ax + by + \frac{a}{b} - b$.

05) $z = a \ln \left(\frac{b(y-1)}{1-x} \right)$

06) $2z = (x+a)^{1/2} + (y-a)^{1/2} + b$

01) $z = ax^2 + by^2$

$z = ax^2 + by^2 \rightarrow ①$

diff ① wrt x partially | diff ① wrt y partially

$\frac{\partial z}{\partial x} = p = 2ax \Rightarrow a = \frac{p}{2x}$ | $\frac{\partial z}{\partial y} = q = 2by \Rightarrow b = \frac{q}{2y}$

$① \Rightarrow z = \frac{px^2}{2x} + \frac{qy^2}{2y} = \frac{px}{2} + \frac{qy}{2} \Rightarrow \boxed{px + qy = 2z}$

02) $(x-a)^2 + (y-b)^2 = z^2 \cot^2\alpha$ ($\alpha \rightarrow$ constant) $\rightarrow ①$

diff ① wrt x partially

$2(x-a) = 2zp\cot^2\alpha \Rightarrow (x-a) = pz\cot^2\alpha$.

diff ① wrt y partially

$2(y-b) = 2zq\cot^2\alpha \Rightarrow (y-b) = qz\cot^2\alpha$.

$① \Rightarrow p^2z^2\cot^4\alpha + q^2z^2\cot^4\alpha = z^2\cot^2\alpha$

$$\boxed{p^2 + q^2 = \tan^2\alpha}$$

03) $z = ax + by + a^2 + b^2$

$p = a + 0$

$q = b + 0$ $\quad z = px + qy + p^2 + q^2$

04) $z = ax + by + \frac{a}{b} - b$.

$p = a$. $\quad z = px + qy + \frac{p}{q} - q$

$q = b$

05) $z = a\ln\left(\frac{b(y-1)}{1-x}\right)$

$p = a \times \frac{(1-x)}{b(y-1)} \frac{\partial}{\partial x}\left(\frac{b(y-1)}{1-x}\right)$

$= a \times \frac{(1-x)}{b(y-1)} \quad \frac{0 - b(y-1)(-1)}{(1-x)^2} \quad p = \frac{a(1-x)}{(1-x)^2} = \frac{a}{1-x}$

$\boxed{p = \frac{a}{1-x}}$ $a = (1-x)p$ —②

w.r.t y $q = a \times \frac{\cancel{1-x}}{b(y-1)}$ $\frac{\cancel{(1-x)} b - b(y-1)(0)}{\cancel{(1-x)^2}} = \frac{a \times b \times (1-x)}{y-1}$

$\boxed{q = \frac{a}{y-1}}$ $a = (y-1)q$ —③

from 2&3, $p - xp = qy - q \Rightarrow px + qy = p + q$

06) $2z = (x+a)^{1/2} + (y-a)^{1/2} + b$

$2p = \frac{1}{2}(x+a)^{-1/2} \Rightarrow p = \frac{1}{4(x+a)^{1/2}} \Rightarrow (x+a)^{1/2} = \frac{1}{4p}$

$x + a = \frac{1}{16p^2}$ $\boxed{a = \frac{1}{16p^2} - x}$

$2q = \frac{1}{2}(y-a)^{-1/2} \Rightarrow q = \frac{1}{4(y-a)^{1/2}} \Rightarrow (y-a)^{1/2} = \frac{1}{4q}$

$(y-a)^2 = \frac{1}{16q^2}$

$\boxed{a = y - \frac{1}{16q^2}}$

$\frac{1}{16p^2} - x = y - \frac{1}{16q^2}$

$$\boxed{x + y = \frac{1}{16}\left[\frac{1}{p^2} + \frac{1}{q^2}\right]}$$

07) $ax + by + cz = 1$

$a + cp = 1$ $c = \frac{1-a}{p}$ $\frac{1-a}{p} = \frac{1-b}{q}$

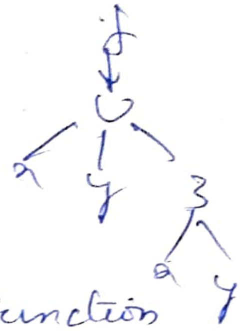
$b + cq = 1$ $c = \frac{1-b}{q}$ $q - aq = p - bp$

w.r.t x $a + cz_x = 0 \xrightarrow{\text{w.r.t } x} cz_{xx} = 0 \Rightarrow z_{xx} = 0 \Rightarrow \boxed{r = 0}$

w.r.t y $b + cz_y = 0 \xrightarrow{\text{w.r.t } y} cz_{yy} = 0 \Rightarrow z_{yy} = 0 \Rightarrow$

b) By elimination of arbitrary function

Case 1 $z = f(v)$ where $v = v(x, y, z)$



$$z_x = p = f'(v_x + v_z \cdot p) \longrightarrow (2)$$

$$z_y = q = f'(v_y + v_z q) \longrightarrow (3)$$

Using eq (1), (2), 3, we eliminate the arbitrary function to obtain 1st order PDE

$$\frac{z_x}{z_y} = \frac{p}{q} = \frac{v_x + v_z p}{v_y + v_z q}$$

~~Using~~ **Case 2**: Elimination of two arbitrary functions.

Differentiating twice or more, the elimination process results in a PDE of 2nd or higher order

01) $z = f(x^2+y^2+z^2) \longrightarrow (1)$

diff (1) wrt $x \rightarrow$ $z_x = p = f'(2x + 0 + 2z p) \longrightarrow (2)$

diff (1) wrt $y \rightarrow$ $z_y = q = f'(0 + 2y + 2zq) \longrightarrow (3)$

$$\frac{(2)}{(3)} = \frac{z_x}{z_y} = \frac{p}{q} = \frac{2x + 2zp}{2y + 2zq}$$

$$\Rightarrow \frac{z_x}{z_y} = \frac{p}{q} = \frac{x + zp}{y + zq}$$

$$py + zpq = qx + zpq$$

$$\boxed{py = qx} \longrightarrow py - qx = 0$$

02) $xyz = f(x^2+y^2+z^2) \longrightarrow (1)$

diff (1) wrt $x \Rightarrow yz + xyp = f' \cdot [2x + 2zp]$

diff (1) wrt $y \Rightarrow xz + xyq = f'[2y + 2zq]$

$$\frac{y(z + xp)}{x(z + yq)} = \frac{x + zp}{y + zq}.$$

$z\check{y} + \cancel{z^2 q} + x\check{y}p + xzpq = x\check{z} + \cancel{z^2 p} + \cancel{xy q} + zypq.$

$z(y-x) + z^2(q-p) + xy\cancel{(p-q)} + \cancel{(p-q)}\ (x-y)pqz = 0.$

$y^2 z + xy^2 p + yz^2 q + xyzpq = x^2 z + x^2 yq + xz^2 p + xyzpq$

$(x^2-y^2)z + (x^2 y - yz^2)q + (xz^2 - xy^2)p = 0$ //

03) $xyz = f(x+y+z)$

$yz + xyp = f'[1+p]$

$xz + xyq = f'(1+q)$

$\dfrac{yz+xyp}{xz+xyq} = \dfrac{1+p}{1+q} \Rightarrow$

$yz + xyp + qyz + xypq = xz + pxz + xyq + xypq$

$(x-y)z + (xy - yz)q + (xz - yz)p = 0$ //

04) $z = (x+y)\phi(x^2-y^2)$ → 1

$z_x = p = (1)\phi + (x+y)\phi'(2x)$ → 2

$z_y = q = (1)\phi + (x+y)\phi'(-2y)$ → 3

2 - 3

$p-q = \phi'(x+y)(2x+2y)$

$\boxed{\phi' = \dfrac{p-q}{2(x+y)^2}}$ → 4

Substitute 4 in 2

$p = \phi + \cancel{(x+y)} \times \dfrac{(p-q) \times \cancel{2}x}{\cancel{2} \times (x+y)^{\cancel{2}}}$

$p = \phi + x\dfrac{p-q}{(x+y)} \Rightarrow \phi = p - \dfrac{x(p-q)}{x+y}$ → 5

Substitute (5) in (1)

$$z = (x+y)\left[p - \frac{x(p-q)}{(x+y)}\right]$$

$$= \cancel{(x+y)}\left[\frac{\cancel{px}+py-\cancel{px}+qx}{\cancel{(x+y)}}\right]$$

$$\boxed{z = py+qx}$$

05) $xy+yz+zx = f\left(\frac{z}{x+y}\right) \longrightarrow (1)$

~~$y+yp+z+xp = f'\left(\frac{-z}{(x+y)^2}\right) \longrightarrow (2)$~~

$$y+yp+z+xp = f'\cdot\left[\frac{(x+y)p - z(1)}{(x+y)^2}\right] \rightarrow (2)$$

$$x+z+yq+xq = f'\cdot\left[\frac{(x+y)q - z(1)}{(x+y)^2}\right] \rightarrow (3)$$

$\frac{(2)}{(3)}$

$$\Rightarrow \frac{y+yp+z+xp}{x+z+yq+xq} = \frac{(x+y)p-z}{(x+y)q-z}$$

$$\frac{(y+z)+p(x+y)}{x+z+q(x+y)} = \frac{(x+y)p-z}{(x+y)q-z}$$

$$(x+y)(y+z)q + (x+y)^2pq - (y+z)z - pz(x+y)$$

$$= (x+y)p(x+z) - z(x+z) \cancel{+(x+y)^2pq-qz}$$

$$+(x+y)^2pq - qz(x+$$

$(x+y)(y+z)q + \cancel{(x+y)^2pq} - yz$

$q[xy+xz+y^2+yz]+pq(x+y)^2-yz-z^2$

~~Case 2~~ 1) $z=f(x)\,g(y)$

$z_x=p=f'\cdot g$

$z_y=q=f\cdot g'$ $\Big\}\Rightarrow pq=(f'g')(fg)$ $\boxed{pq=sz}$

$z_{xx}=r=f''g$

$z_{xy}=s=f'g'$

$z_{yy}=t=f\cdot g''$

2) $z=f(x)+e^y g(x)$

$z_x=f'+e^y g'=p$

$z_y=q=e^y g$

$z_{xx}=f''+e^y g''$

$z_{xy}=s=e^y g'$

$z_{yy}=t=e^y g$

$q=t$

$\boxed{q-t}=0$

3) $z=f(x-iy)+g(x-iy)$

$z_x=p=f'+g'$

$z_y=q=f'(-i)+g'(-i)$

$\cancel{z_{xx}=f''}$

$\cancel{z_x+z_y=(f'+g')(1-i)}$

$q=p(-i)$

$q+pi=0$

Homework

01) $z = f(x+y)\,g(x-y)$

02) $z = xf(ax+by) + g(ax+by)$

03) $z = f_1(y+2x) + f_2(y-3x)$

01) $z = f(x+y)\,g(x-y) \Rightarrow z_x = p = f'g + fg'$, $z_y = f'g + g'(-1)f$

~~$z_x = p = f'(x+y)\,g'(x-y)$~~ $\Rightarrow z_x = \cancel{p = f'g'}$ $\quad z_y = f'g - fg'$

$z_x = p = f'g + fg'$

~~$z_y = q = f'g'$~~ $\quad \cancel{z_y} =$

$z_y = q = f'g - fg'$

$z_{xx} = r = f'g' + gf'' + fg'' + g'f'$

$z_{yy} = t = f'g'(-1) + gf'' + fg''(-1) - g'f'$

$z_{xy} = s = f''g + f'g'(-1) + fg''(-1) + g'f' = f''g - fg''$

$p + q = 2f'g$

$p - q = 2fg'$

$z_{xx} = fg'' + gf'' + 2f'g'$

$z_{yy} = fg'' + gf'' - 2f'g'$

08th July 2019

Case 3 : Formation of PDE by elimination of arbitrary function of Specific Functions

$$F(U,V) = 0 \longrightarrow (1)$$

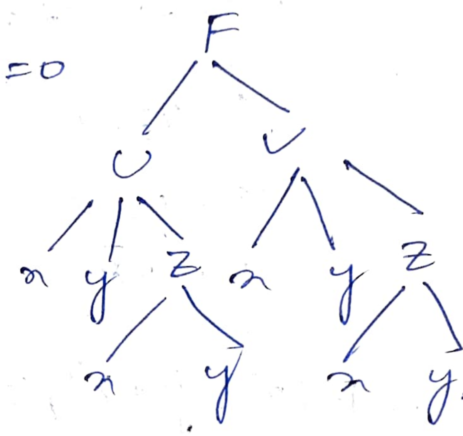
where $U = U(x,y,z)$, $V = V(x,y,z)$

differentiate (1) partially wrt x

$$\frac{\partial F}{\partial U}\left[\frac{\partial U}{\partial x} + \frac{\partial U}{\partial z} p\right] + \frac{\partial F}{\partial V}\left[\frac{\partial V}{\partial x} + \frac{\partial V}{\partial z} p\right] = 0$$

differentiate (1) partially wrt y

$$\frac{\partial F}{\partial U}\left[\frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} q\right] + \frac{\partial F}{\partial V}\left[\frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} q\right] = 0$$



has a solution if

$$\begin{vmatrix} U_x + U_z p & V_x + V_z p \\ U_y + U_z q & V_y + V_z q \end{vmatrix} = 0$$

$\Rightarrow \boxed{Pp + Qq = R}$

where P, Q, R are functions of x, y.

Problems

1

(1) $F(xy + z^2, x + y + z) = 0$

$U = xy + z^2$

$V = x + y + z$

The given relation is of the form $F(U,V) = 0$

$$\frac{\partial U}{\partial x} = y \quad \frac{\partial U}{\partial y} = x \quad \frac{\partial U}{\partial z} = 2z \qquad \frac{\partial z}{\partial x}.$$

$$\frac{\partial V}{\partial x} = 1 \quad \frac{\partial V}{\partial y} = 1 \quad \frac{\partial V}{\partial z} = 1$$

$$\frac{\partial F}{\partial U}\left[y + 2zp\right] + \frac{\partial F}{\partial V}\left[1 + p\right] = 0$$

$$\frac{\partial F}{\partial U}\left[x + 2zq\right) + \frac{\partial F}{\partial V}\left(1 + q\right] = 0$$

$$\begin{vmatrix} y+2pz & 1+p \\ x+2qz & 1+q \end{vmatrix} = 0$$ The system has a non trivial solution when $|\,| = 0$.

$$(1+q)(y+2pz) - (1+p)(x+2qz) = 0.$$

$$y+2pz+qy+\cancel{2pqz} - [x+2qz+px+\cancel{2pqz}] = 0$$

$$y-x+2pz-2qz+qy-px=0$$

$$p(2z-x)+q(y-2z) = x-y$$

$$P = 2z-x$$
$$Q = y-2z$$
$$R = x-y$$

Q2) $F(x^2+y^2+z^2,\ z^2-2xy)=0$

$u = x^2+y^2+z^2 \qquad v = z^2-2xy$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = -2y$

$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = -2x$

$\frac{\partial u}{\partial z} = 2z \qquad \frac{\partial v}{\partial z} = 2z$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right] = 0$$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right] = 0$$

$$\frac{\partial F}{\partial u}\left[2x+2zp\right] + \frac{\partial F}{\partial v}\left[-2y+2zp\right] = 0$$

$$\frac{\partial F}{\partial u}\left[2y+2zq\right] + \frac{\partial F}{\partial v}\left[-2x+2zq\right] = 0$$

$$2\begin{vmatrix} x+zp & -y+zp \\ y+zq & -x+zq \end{vmatrix} = 0.$$

$$(x+zp)(-x+zq) - [(-y+zp)(y+zq)] = 0.$$

$$-x^2 + xzq - pzx + \cancel{pqz^2} = -y^2 - yzq + pyz + \cancel{pqz^2}$$

$$x^2 - y^2 = xzq + yzq - pzx - pyz = 0.$$

$$\underset{R}{x^2 - y^2} = \underset{P}{(-xz - yz)}\,p + \underset{Q}{(xz + yz)}\,q$$

$$x - y = -zp + zq$$

$$\frac{y-x}{z} = (p-q)$$

3) $F(y/x,\ x^2+y^2+z^2) = 0$

$$v = x^2+y^2+z^2$$

$$\frac{\partial F}{\partial x}\Big[\quad u = y/x$$

$$\frac{\partial u}{\partial x} = \frac{-y(1)}{x^2} \qquad \frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\left[\frac{-y}{x^2} + 0\right] + \frac{\partial F}{\partial v}\left[2x + 2zp\right]$$

$$\frac{\partial u}{\partial y} = \frac{x(1)}{x^2} = \qquad \frac{\partial F}{\partial u}\left[\frac{1}{x} + 0\right] + \frac{\partial F}{\partial v}\left[2y + 2zq\right]$$

$$\frac{\partial u}{\partial z} = 0.$$

$$\begin{vmatrix} -\frac{y}{x^2} & 2x+2zp \\ \frac{1}{x} & 2y+2zq \end{vmatrix}$$

$$\frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial v}{\partial y} = 2y \qquad -\frac{y}{x^2}(2y+2zq) = \frac{1}{x}(2x+2zp)$$

$$\frac{\partial v}{\partial z} = 2z \qquad -\frac{y}{x}(y+zq) = x+zp$$

$$-y^2 - yzq = x^2 + xzp.$$

$$xzp + yzq = -(x^2+y^2)$$

Q4) $F(ax+by+cz,\ x^2+y^2+z^2)=0$

$$\frac{\partial F}{\partial u}[a+cp]+\frac{\partial F}{\partial v}(2x+2zp)=0$$

$$\frac{\partial F}{\partial u}[b+cq)+\frac{\partial F}{\partial v}(2y+2zq)=0$$

$$\begin{vmatrix} a+cp & x+zp \\ b+cq & y+zq \end{vmatrix}$$

$$(a+cp)(y+zq)-(x+zp)(b+cq)=0$$

$$ay+azq+cpy+\cancel{czpq}-[xb+xcq+zpb+\cancel{zcpq}]=0.$$

$$p(cy-zb)+q(az-xc)=xb-ay$$

Q5) $F(x^2+y^2,\ z-xy)=0$

$$\frac{\partial F}{\partial u}[2x+0]+\frac{\partial F}{\partial v}[-y+p)=0$$

$$\frac{\partial F}{\partial u}[2y+0]+\frac{\partial F}{\partial v}[-x+q)=0$$

$$\begin{vmatrix} x & -y+p \\ y & -x+q \end{vmatrix}=0 \qquad -x^2+xq-y(-y+p)=0$$

$$-x^2+xq+y^2-yp=0.$$

$$\therefore -yp+xq=x^2-y^2$$

21st August 2019

SOLUTION OF FIRST ORDER PDE:

Lagrange's PDE:

$$P(x,y,z)p+Q(x,y,z)q=R(x,y,z)$$

$$Pp+Qq=R$$

1^{st} Order PDE: $f(x,y,z,p,q)=0 \rightarrow (*)$

Complete solution: $\phi(x,y,z,a,b)=0$ which satisfies $(*)$ where a,b are arbitrary constants.

General solution: $F(u,v)=0$ where $u=c_1$, $v=c_2$ forms complete solution of $(*)$ such that u, v are independent.

Method of obtaining General solution:

① Rewrite the given first order PDE in standard form $Pp+Qq=R$.

② Form the Lagrange's Auxilliary Equations as $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

③ Nature of solution to the simultaneous DE's is of the form $u(x,y,z)=c_1$, $v(x,y,z)=c_2$ where u & v are linearly independent. These solutions can be obtained through the following cases.

Case 1: One of the variables is absent or cancels out from the set of auxiliary equations.

Case 2: If $u=c_1$ is known, but $v=c_2$ is not possible by case 1, then use $u=c_1$ to get $v=c_2$.

Case 3: Introducing Lagrange's multipliers P_1, Q_1, R_1 which are functions of x,y,z, are constants. Each of the fractions

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}.$$

If P_1, Q_1, R_1 are selected such that $P_1P+Q_1Q+R_1R=0$ then $P_1dx+Q_1dy+R_1dz=0$ can be integrated.

Case 4: Multipliers may be chosen such that, the numerator $P_1dx+Q_1dy+R_1dz$ is an exact differential of the denominator $P_1P+Q_1Q+R_1R$. Then combining the fraction $\frac{P_1dx+Q_1dy+R_1dz}{P_1P+Q_1Q+R_1R}$ with fractions of simultaneous D.E, you can get an ~~solution~~ integral.

The general solution of Lagranges PDE is $F(U,V)=0$.

01) $xp+yq=3z \equiv Pp+Qq=R$

$P=x, Q=y, R=3z$

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \frac{dx}{x}=\frac{dy}{y}=\frac{dz}{3z}$.

$\frac{dx}{x}=\frac{dy}{y} \Rightarrow$ ~~$\log x = \log y$~~ $+$ $\ln x = \ln y + \ln c_1$

$\ln\frac{x}{y}=\ln c_1$

$x/y=c_1$

$\Rightarrow U=c_1$

$\frac{dy}{y}=\frac{dz}{3z} \Rightarrow \ln y=\frac{1}{3}\ln z \pm \frac{1}{3}\ln c_1 \Rightarrow 3\ln y=\ln\frac{z}{c_2}$

$y^3=\frac{z}{c_2}$

$\frac{y^3}{z}=\frac{z}{c_2}$

$\Rightarrow V=c_2$

$\therefore$ General solution is $F(U,V)=0$ $F(\frac{x}{y}, y^3/z)=0$

02) $yzp-xzq=xy \equiv Pp+Qq=R$

$P=yz, Q=-xz, R=xy$

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \frac{dx}{yz}=\frac{dy}{-xz}=\frac{dz}{xy}$

$\frac{dx}{yz}=\frac{dy}{-xz} \Rightarrow -xdx=ydy \Rightarrow xdx+ydy=0 \Rightarrow \frac{x^2}{2}+\frac{y^2}{2}=\frac{c_1}{2}$

$x^2+y^2=c_1$, $U=c_1$

Consider $\frac{dx}{yz}=\frac{dz}{xy} \Rightarrow xdx=zdz \Rightarrow \frac{x^2}{2}-\frac{z^2}{2}=\frac{c_2}{2} \Rightarrow x^2-z^2=c_2 \Rightarrow V=c_2$

$\therefore$ general solution is $F(U,V)=0$ $F(x^2+y^2, x^2-z^2)=0$

03) $p-q=\ln(x+y)$

$P=1, Q=-1, R=\ln(x+y)$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\ln(x+y)}$

$\frac{dx}{1} = \frac{dy}{-1}$

$x = -y + c_1 \Rightarrow x + y = c_1 \Rightarrow u = c_1$

$\frac{dx}{1} = \frac{dz}{\ln c_1}$

$x \ln c_1 = z + c_2$

$x \ln(x+y) - z = c_2$

$\Rightarrow v = c_2$

$\therefore$ G.S is $F(u,v) = 0$

$F(x+y, x\ln(x+y) - z) = 0.$

Q4) $z(z^2 + xy)(px - qy) = x^4$

$[xz(z^2 + xy)]p + [-yz(z^2 + xy)]q = x^4$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$

$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}$

$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} \Rightarrow \frac{dx}{x} = \frac{dy}{-y} \Rightarrow \ln x + \ln y = \ln c_1$

$\ln xy = \ln c_1 \Rightarrow xy = c_1 \quad \Rightarrow u = c_1$

Consider $\frac{dx}{xz(z^2+xy)} = \frac{dz}{x^4}$

$\frac{dx}{z(z^2+c_1)} = \frac{dz}{x^3} \Rightarrow x^3 dx = (z^3 + c_1 z)dz.$

$\Rightarrow \frac{x^4}{4} - \frac{z^4}{4} - \frac{c_1 z^2}{2} = c_2$

$\frac{x^4}{4} - \frac{z^4}{4} - \frac{xyz^2}{2} = c_2$

$x^4 - z^4 - 2xyz^2 = 4c_2$

$x^4 - z^4 - 2xyz^2 = c$

$\Rightarrow v = c_2$

General solution is

$F(u,v) = 0$

$F(xy, x^4 - z^4 - 2xyz^2) = 0$

05) $(z-y)p+(x-z)q=y-x$

$Pp+Qq=R$

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \frac{dx}{P_1(z-y)}=\frac{dy}{Q_1(x-z)}=\frac{dz}{R_1(y-x)}$

choose Lagranges multipliers as 1, 1, 1

then $P_1P+Q_1Q+R_1R=1(z-y)+1(x-z)+1(y-x)=0$

Integrate $P_1dx+Q_1dy+R_1dz=0 \Rightarrow 1\,dx+1dy+1dz=0$

$\Rightarrow d(x+y+z)=0 \Rightarrow x+y+z=c_1 \Rightarrow U=c_1$

Another set of multipliers as x, y, z.

Then $P_1P+Q_1Q+R_1R=x(z-y)+y(x-z)+z(y-x)=0$

$\therefore$ Integrate $P_1dx+Q_1dy+R_1dz=0$

$xdx+ydy+zdz=0$

$d\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}\right)=0$

$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_2$

$V=c_2$

$\therefore$ The General solution

$f(U,V)=0$

$f\left(x+y+z, \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}\right)=0$

22nd August 2019

01) $(y+zx)p-(x+yz)q=x^2-y^2$

02) $x(y-z)p+y(z-x)q=z(x-y)$

03) $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

04) $(mz-ny)p+(nx-lz)q=ly-mx$.

05) $(x^2-y^2-yz)p+(x^2-y^2-zx)q=z(x-y)$

06) $(x^2-yz)p+(y^2-zx)q=z^2-xy$

$Pp+Qq=R$

01) $\frac{dx}{P_1(y+zx)}=\frac{dy}{Q_1(x+yz)}=\frac{dz}{R_1(x^2-y^2)}$

P_1, Q_1, R_1

$y, x, 1$

$y^2+xyz-x^2-xyz+x^2-y^2$

Integrate $ydx+xdy+dz=0$.

$d(xy)+dz=0$

$xy+z=c_1$

$U=c_1$

Again $x, y-z$

$x(y+zx)+y(-x-yz)+z(x^2-y^2)$

$\cancel{xy}+\cancel{x^2z}-\cancel{xy}-\cancel{y^2z}\ \cancel{\ldots}$

$=0$.

$xdx+ydy-zdz$.

$d\left(\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}\right)=0$

$\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}=c_2$

$V=c_2$

$\therefore$ The General solution is $f\left(xy+z,\ \frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}\right)=0$

(02) $x(y-z)p+y(z-x)q=z(x-y)$

$Pp+Qq=R$

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\frac{dx}{xy-xz}=\frac{dy}{yz-xy}=\frac{dz}{xz-yz}$

choose Lagranges multipliers as $1,1,1$

then

$P_1P+Q_1Q+R_1R$

$=1(xy-xz)+1(yz-xy)+(xz-yz)$

$=0$

Integrate $P_1dx+Q_1dy+R_1dz=0$

$1dx+1dy+1dz=0$

$d(x+y+z)=0$

$x+y+z=c_1$

$u=c_1$

choose

$P_1=\frac{1}{x}$ $Q_1=\frac{1}{y}$, $R=\frac{1}{z}$ as

Lagranges multipliers.

Then $P_1P+Q_1Q+R_1R$

$=\frac{1}{x}(x(y-z))+\frac{1}{y}(yz-xy)+\frac{1}{z}(x-y)$

$=0$.

Integrate

$P_1dx+Q_1dy+R_1dz=0$.

$\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$.

$d(\ln x+\ln y+\ln z)=0$.

$\ln x+\ln y+\ln z=\ln c_2$

$\ln xyz=\ln c_2$

$xyz=c_2$

$v=c_2$

The General solution is $f(u,v)=0$

$f(x+y+z,\ xyz)=0$.

(03) $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

$Pp+Qq=R$

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Lagranges multipliers $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$

then $P_1P+Q_1Q+R_1R$

$=\frac{x^2}{x^2}(y-z)+z-x+x-y=0$.

Integrate $P_1dx+Q_1dy+R_1dz=0$.

$\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz=0$

$d\left(-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)=0$.

$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=c_1$

$u=c_1$

Another set

$\frac{1}{x},\frac{1}{y},\frac{1}{z}$

$\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

$d(\ln x+\ln y+\ln z)=0$

$d(\ln xyz)=0$

$\ln xyz=\ln c_2$

$xyz=c_2$

$v=c_2$

The General solution is $F(u,v)=0$ $F\left(-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}, xyz\right)=0$

04) $(mz-ny)p+(nx-lz)q=ly-mx.$

$Pp+Qq=R$

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}.$$

Choose LMs as x, y, z. l, m, n

$x(mz-ny)+y(nx-lz)+z(ly-mx)=0$ $l(mz-ny)+m(nx-lz)+n(ly-mx)$

Integrate $xdx+ydy+zdz=0$

$d\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}\right)=0$

$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_1$

$u=c_1$

Integrate

$l\,dx+m\,dy+n\,dz=0$

$lx+my+nz=c_2$

$v=c_2$

The General solution is $F(u,v)$

$F\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}, lx+my+nz\right)$

05) $(x^2-y^2-yz)p+(x^2-y^2-zx)q=z(x-y)$

$Pp+Qq=R$

$$\frac{dx}{x^2-y^2-yz}=\frac{dy}{x^2-y^2-zx}=\frac{dz}{z(x-y)}.$$

Choose $1\ -1\ -1$

$x^2-y^2-yz-x^2+y^2+zx-zx+zy=0$

Integrate

$1dx-1dy-zdz=0$

$d\left(x-y-\frac{z^2}{2}\right)=0$

$x-y-\frac{z^2}{2}=c_1$

$2x-2y-z^2=c_1$

$u=c_1$

Not possible using case 3

So use case 4

We know $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}$

$$\frac{dz}{z(x-y)} = \frac{x\,dx - y\,dy}{x(x^2-y^2-yz) - y(x^2-y^2-zx)}$$

$$\frac{dz}{z(x-y)} = \frac{x\,dx - y\,dy}{(x^2-y^2)(x-y)}$$

$$\frac{dz}{z} = \frac{x\,dx - y\,dy}{x^2-y^2}$$

$$d(\ln z) = \frac{1}{2} d\ln(x^2-y^2)$$

$$\ln z^2 = \ln(x^2-y^2) + \ln c_2$$

$$z^2 = c_2(x^2-y^2)$$

$$\frac{z^2}{x^2-y^2} = c_2$$

The General solution is

$$F(u,v)=0$$

$$F\left(x-y-z,\ \frac{z^2}{x^2-y^2}\right)=0$$

06) ★ $(x^2-yz)p + (y^2-zx)q = z^2-xy$.

$$Pp + Qq = R$$

$$\cancel{\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y}}$$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{(x^2-yz)-(y^2-zx)} = \frac{dy-dz}{(y^2-zx)-(z^2-xy)}$$

$$\frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)}$$

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

$$d[\ln(x-y)] = d[\ln(y-z)]$$

$$\ln(x-y) - \ln(y-z) = \ln c_1$$

$$\ln\frac{x-y}{y-z} = \ln c_1$$

$$\frac{x-y}{y-z} = c_1$$

$$u = c_1$$

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2-yz) + y(y^2-zx) + z(z^2-xy)} = \frac{1\,dx + 1\,dy + 1\,dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx + y\,dy + z\,dy}{x^3+y^3+z^3-3xyz} = \frac{1\,dx + 1\,dy + 1\,dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx + y\,dy + z\,dy}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx + dy + dz}{x^2+y^2+z^2-xy-yz-zx)}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x+y+z} = dx + dy + dz$$

$$x\,dx + y\,dy + z\,dz = (x+y+z)(dx+dy+dz)$$

$$\frac{1}{2}d(x^2+y^2+z^2) = \frac{1}{2}d(x+y+z)^2$$

$$x^2+y^2+z^2 = (x+y+z)^2 + c_2$$

$$V = c_2$$

The General Solution is $F(U,V)=0$

$$F\left(\frac{x-y}{y-z},\ (x^2+y^2+z^2)-(x+y+z)^2\right)=0$$