3.1. MATRIX

A matrix is defined as a rectangular array (or arrangement in rows or columns) of scalars subject to certain rules of operations.

If mn numbers (real or complex) or functions are arranged in the form of a rectangular array A having m rows (horizontal lines) and n columns (vertical lines) then A is called an $m \times n$ matrix. Each of the mn numbers is called an element of the matrix.

There are different notations of enclosing the elements constituting a matrix *viz.* [ ], ( ). || || but square bracket is generally used. An $m \times n$ matrix is also called a matrix of order $m \times n$.

3.3. TYPES OF MATRICES

(1) **Real Matrix.** A matrix is said to be real if all its elements are real numbers.

e.g., $\begin{bmatrix} \sqrt{5} & -3 & 1 \\ 0 & -\sqrt{2} & 7 \end{bmatrix}$ is a real matrix.

(2) **Square Matrix.** A matrix in which the number of rows is equal to the number of columns is called a **square matrix,** otherwise, it is said to be a **rectangular matrix.**

Thus, a matrix $A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$ and a rectangular matrix if $m \neq n$.

A square matrix having n rows and n columns is called "a square matrix of order n" or an n-rowed square matrix",

e.g., $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3-rowed square matrix.

The elements a_{11}, a_{22}, a_{33} of a square matrix are called its **diagonal elements** and the diagonal along which these elements lie is called the **principal** or **leading diagonal.**

In a square matrix $A = [a_{ij}]$,

(*i*) for elements along the principal diagonal, $i = j$

(*ii*) for elements above the principal diagonal, $i < j$

(*iii*) for elements below the principal diagonal, $i > j$

(*iv*) for non-diagonal elements, $i \neq j$

The sum of the diagonal elements of a square matrix is called its **trace** or **spur.** Thus, trace of the n rowed square matrix $A = [a_{ij}]$ is $a_{11} + a_{22} + a_{33} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ij}$

(3) **Row Matrix.** A matrix having only one row and any number of columns *i.e.,* a matrix of order $1 \times n$ is called a **row matrix.** *e.g.* $[2 \;\; 5 \;\; -3 \;\; 0]$ is a row matrix.

(4) **Column Matrix.** A matrix having only one column and any number of rows *i.e.,* a matrix of order $m \times 1$ is called a **column matrix.** *e.g.* $\begin{bmatrix} \sqrt{2} \\ 0 \\ -1 \end{bmatrix}$ is a column matrix.

(5) **Null Matrix.** A matrix in which each element is zero is called a **null matrix** or **void matrix** or a **zero matrix.** A null matrix of order $m \times n$ is denoted by $O_{m \times n}$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{3 \times 2}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = O_{2 \times 4}$$

(6) **Sub-matrix.** A matrix obtained from a given matrix A by deleting some of its rows or columns or both is called a **sub-matrix of A.**

Thus, $B = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix}$ is a sub-matrix of $A = \begin{bmatrix} 0 & -1 & 2 & 5 \\ 3 & 5 & 0 & 7 \\ 1 & 6 & 4 & -2 \end{bmatrix}$

obtained by deleting the first row, second and fourth columns of A.

(7) **Diagonal Matrix. A square matrix** in which all non-diagonal elements are zero is called a **diagonal matrix.**

Thus, $A = [a_{ij}]_{n \times n}$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a diagonal matrix.

An n-rowed diagonal matrix is briefly written as diag. $[d_1, d_2, \ldots, d_n]$, where $d_1, d_2, \ldots, d_n$ are the diagonal elements. Thus, the above diagonal matrix A can be written as diag. $[2, -1, 0]$.

(8) **Scalar Matrix.** A diagonal matrix in which all the diagonal elements are equal to a scalar, say k, is called a **scalar matrix.**

Thus a scalar matrix is a square matrix in which all non-diagonal elements are zero and all diagonal elements are equal to a scalar, say k.

i.e., $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ k & \text{when } i = j \end{cases}$

For example, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$ are scalar matrices.

(9) **Unit Matrix or Identity Matrix.** A scalar matrix in which each diagonal element is unity (*i.e.,* 1) is called a **unit matrix or identity matrix.**

Thus, a unit matrix is a square matrix in which all non-diagonal elements are zero and all diagonal elements are equal to 1.

i.e., $A = [a_{ij}]_{n \times n}$ is a unit matrix if $a_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases}$

A unit matrix of order n is denoted by I_n. If the order is evident, it may be simply denoted by I.

Thus, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

(10) **Upper Triangular Matrix.** A square matrix in which all the elements **below** the principal diagonal are zero is called an **upper triangular matrix.**

Thus, $A = [a_{ij}]_{n \times n}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

For example, $\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular matrix.

(11) **Lower Triangular Matrix.** A square matrix in which all the elements **above** the principal diagonal are zero is called a **lower triangular matrix.**

Thus $A = [a_{ij}]_{n \times n}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

For example, $\begin{bmatrix} -1 & 0 & 0 \\ 5 & 6 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ is a lower triangular matrix.

(12) **Triangular Matrix.** A square matrix in which all the elements either below or above the principal diagonal are zero is called a triangular matrix. Thus, a triangular matrix is either upper triangular or lower triangular.

(13) **Equal Matrices.** Two matrices A and B are said to be equal (written as A = B) if and only if they have the same order and their corresponding elements are equal.

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$, then A = B if and only if

(*i*) $m = p$ and $n = q$ (*ii*) $a_{ij} = b_{ij}$ for all i and j.

(14) **Single Element Matrix.** A matrix having only one element is called a single element matrix. Thus any matrix [3] is a single element matrix.

(15) **Singular and Non-singular Matrices.** A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

For example, $A = \begin{bmatrix} 2 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ is a singular matrix since $|A| = 0$.

(16) **Tridiagonal Matrix.** Tridiagonal matrix is the matrix having non-zero entries only in the leading diagonal, sub-diagonal and super diagonal. In other words, we can define it as:

A real matrix $A = [a_{ij}]$ is said to be tridiagonal if $a_{ij} = 0$ for $|i - j| > 1$

For example, $A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$ is a tridiagonal matrix.

3.12. SYMMETRIC MATRIX

A square matrix $A = \{a_{ij}\}$ is said to be symmetric if $\mathbf{A}' = \mathbf{A}$ *i.e.*, if the transpose of the matrix is equal to the matrix itself.

Thus, for a symmetric matrix $A = [a_{ij}]$, $a_{ij} = a_{ji}$

For example, $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 4 \end{bmatrix}$, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ are symmetric matrices.

3.13. SKEW-SYMMETRIC MATRIX (OR ANTI-SYMMETRIC MATRIX)

A square matrix $A = \{a_{ij}\}$ is said to be skew-symmetric if $\mathbf{A}' = -\mathbf{A}$ *i.e.*, if the transpose of the matrix is equal to the negative of the matrix.

Thus, for a skew-symmetric matrix $A = [a_{ij}]$, $a_{ij} = -a_{ji}$.

Putting $j = i$, $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$ or $a_{ii} = 0$ for all i.

Thus, all diagonal elements of a skew-symmetric matrix are zero.

For example, $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ are skew-symmetric matrices.

3.15. ORTHOGONAL MATRIX

A square matrix A is called an orthogonal matrix if $AA' = A'A = I$ or $A' = A^{-1}$.

Note. If A and B are any two orthogonal matrices, AB will also be an orthogonal matrix.

3.16. NILPOTENT MATRIX

A square matrix A is said to be nilpotent if $A^2 = O$ *e.g.* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

It will be of index p if p is the least positive integer such that $A^p = O$.

3.17. IDEMPOTENT MATRIX

A square matrix A is idempotent if $A^2 = A$ *e.g.* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A square matrix A is said to be idempotent of period n if n is the least positive integer such that $A^{n+1} = A$.

3.18. INVOLUTARY MATRIX

A square matrix A is said to be involutary if $A^2 = I$ where I is a unit matrix.

e.g. $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$.

3.35. COMPLEX MATRICES

If *at least one* element of a matrix is a complex number $a + ib$, where a, b are real and $i = \sqrt{-1}$, then the matrix is called a *complex matrix*.

The matrix obtained by replacing the elements of a complex matrix A by the corresponding conjugate complex numbers is called the *conjugate of the matrix* A and is denoted by $\overline{A}$.

Thus, if $A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$, then $\overline{A} = \begin{bmatrix} 2-3i & 7i \\ 5 & 1+i \end{bmatrix}$

It is easy to see that the *conjugate of the transpose of* A *i.e.*, $(\overline{A'})$ and the *transposed conjugate of* A *i.e.*, $(\overline{A})'$ are equal. Each of them is denoted by A^*.

Thus, $(\overline{A'}) = (\overline{A})' = A^*$.

A square matrix $A = [a_{ij}]$ is said to be *Hermitian* if $A^* = A$ or $a_{ij} = \overline{a}_{ji}$.

A square matrix $A = [a_{ij}]$ is said to be *skew-Hermitian* if $A^* = -A$ or $a_{ij} = -\overline{a}_{ji}$.

In a Hermitian matrix, the diagonal elements are all real, while every other element is the conjugate complex of the element in the transposed position. For example,

$A = \begin{bmatrix} 5 & 2+i & -3i \\ 2-i & -3 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ is a Hermitian matrix.

In a skew-Hermitian matrix, the diagonal elements are zero or purely imaginary numbers of the form $i\beta$, where β is real. Every other element is the negative of the conjugate complex of the element in the transposed position.

Elementary Transformations:-

(i) Interchanging any two rows and columns. Indicated by R_{ij} or C_{ij}.

(ii) Multiplication of the elements of any row R_i (or Col. C_i) by K (non zero).

(iii) Addition of a constant multiplication of elements of any row R_j to the corresponding elements of any other row R_i i.e. $R_i + KR_j$.

To find A^{-1} using E- row transformation:

Q 1: Find A^{-1} of A = $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ using E- row transformation.

Sol: We have A = I A

Or A ~ I (Rank of A = order of A = Rank of I_3)

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\3 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & -5 & -8\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & -3 & 1\end{bmatrix} \quad R_3 \to R_3 + 5R_2$$

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\5 & -3 & 1\end{bmatrix} \quad R_3 \to (1/2)\, R_3$$

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\5/2 & -3/2 & 1/2\end{bmatrix} \quad R_1 \to R_1 - 3R_3 \;\&\; R_2 \to R_2 - 2R_3$$

$$\begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}-15/2 & 11/2 & -3/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1/2 & -1/2 & 1/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix} = A^{-1}$$

Q 2: Find A^{-1} of $A = \begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$ using E- row transformation.

Rank of Matrix

Def – The **rank of a matrix** is said to be **r** if

- **(i)** It has at least one non zero minor of order r.
- **(ii)** Every minor of A of order higher than r is zero.

Echelon Form – A matrix is said to be in **Echelon form** if,

(i) all the zero rows occur below nonzero rows;

(ii) the number of zeros before the first nonzero element in a row is less than the number of such zeros in the next row.

(iii) The first non zero element in every nonzero row is 1.

Note – Rank = No. of nonzero rows in Echelon form.

Echelon Form: (Row operation)

Q1: Find the rank of: A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ By Echelon form

Sol: we have A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$. Using $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ Using $R_3 \to R_3 - R_2$ $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **Echelon form**

Hence rank of A = number of non zero rows = 2

<u>**Normal Form**</u> – The matrix obtained in three of the forms viz. I, $\begin{bmatrix} I \\ 0 \end{bmatrix}$, $\begin{bmatrix} I & 0 \end{bmatrix}$, $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is called **Normal form**.

Note – Rank in Normal form = Order of Identity Matrix.

1) Reduce the matrix A to the Echelon and Normal form and find its rank where, $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 3 & 1 & 1 \\ 1 & 3 & 5 & 5 \end{bmatrix}$.

Sol – Since $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 3 & 1 & 1 \\ 1 & 3 & 5 & 5 \end{bmatrix}$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 1 & 3 & 3 \end{bmatrix}$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$R_3 \to R_3 - R_1$
(Echelon Form)

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 \to C_2 - 2C_1$
$C_3 \to C_3 - 2C_1$
$C_4 \to C_4 - 2C_1$

$R_2 \to (-1) R_2$

$C_3 \to C_3 - 3C_2$
$C_4 \to C_4 - 3C_2$

$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ **Rank A = 2 = order of I_2**

(Normal Form)

<u>Linear Equations</u>:-

(i) Consistent – A system of equations is said to be **consistent** if they have one or more solutions.

(ii) Inconsistent – A system of equations is said to be **inconsistent** if a system of equation has no solution.

<u>System of Linear Equations</u>

There are two types of systems:

(1) Homogeneous Linear equations (AX = B = O)

(2) **Non Homogeneous Linear equations (AX = B ≠ O)**

***Working rule to solve homogeneous:**

(1) Express the given system in matrix form as AX = B = O

(2) System is homogeneous. Then find rank of A

(3) If rank of A = n = number of variables. Then unique solution or trivial solution or zero solution.

(4) If rank of A < n = number of variables Then infinite solutions
Independent solutions will be n – r.

Q1: Solve: x + 3 y – 2z = 0, 2 x – y + 4 z = 0, x – 11 y + 14 z = 0.

Sol: We have x + 3 y – 2z = 0,

2 x – y + 4 z = 0,

x – 11 y + 14 z = 0

Write the given system in matrix form as:

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$ or AX = O

Which is homogeneous

A = $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$ $R_2 \rightarrow R_2 - 2R_1$ & $R_3 \rightarrow R_3 - R_1$

~ $\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$ $R_3 \rightarrow R_3 - 2R_2$

~ $\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ (Echelon form)

Here rank of A = 2 < n =3 $\rightarrow$ infinite solutions

Independent variable will be: n – r = 3 – 2 = 1

Solution will be: AX = O

$$\begin{bmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$x + 3y - 2z = 0$ ……..(1)

$-7y + 8z = 0$ …………(2)

$\Rightarrow$ $7y = 8z$ Let $z = c$

$\Rightarrow$ $7y = 8c$ $\Rightarrow$ $y = 8c/7$

By (1) $x + 3y - 2z = 0$

$x + 24c/7 - 2c = 0$

$\Rightarrow$ $x = -10c/7$, $y = 8c/7$, $z = c$

***Working rule to solve non homogeneous Linear equations:**

(1) Express the given system in matrix form as $AX = B \neq O$

(2) System is non homogeneous. Then find rank of A and rank of A augmented matrix B (r [A: B])

(3) If $r(A) \neq r[A: B]$ $\Rightarrow$ No solution (Inconsistent)

(4) If $r(A) = r[A: B]$ $\Rightarrow$ Solution exists

 (a) If $r(A) = r[A: B] = n$ = number of variables. Then unique solution

 (b) If $r(A) = r[A: B] < n$ .Then infinite solutions Independent variables will be $n - r$.

Q 1: Solve: $2x - y + 3z = 8$, $-x + 2y + z = 4$, $3x + y - 4z = 0$.

Sol: We have $2x - y + 3z = 8$, $-x + 2y + z = 4$, $3x + y - 4z = 0$

Matrix form is : $\begin{bmatrix}2 & -1 & 3\\ -1 & 2 & 1\\ 3 & 1 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}8\\ 4\\ 0\end{bmatrix} \neq O$ (Non homogeneous)

$$[A: B] = \begin{bmatrix}2 & -1 & 3: & 8\\ -1 & 2 & 1: & 4\\ 3 & 1 & -4: & 0\end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix}-1 & 2 & 1: & 4\\ 2 & -1 & 3: & 8\\ 3 & 1 & -4: & 0\end{bmatrix} \quad R_2 \to R_2 + 2R_1,\ R_3 \to R_3 + 3R_1$$

$$\sim \begin{bmatrix} -1 & 2 & 1: & 4 \\ 0 & 3 & 5: & 16 \\ 0 & 7 & -1: & 12 \end{bmatrix} \quad R_2 \rightarrow (1/3)\, R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1: & 4 \\ 0 & 1 & 5/3: & 16/3 \\ 0 & 7 & -1: & 12 \end{bmatrix} \quad R_3 \rightarrow R_3 - 7R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1: & 4 \\ 0 & 1 & 5/3: & 16/3 \\ 0 & 0 & -38/3: & -76/3 \end{bmatrix}$$

Here r (A) = 3 = r [A: B] = n

Solution exists and unique.

Solution will be : AX = B

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 5/3 \\ 0 & 0 & -38/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16/3 \\ -76/3 \end{bmatrix}$$

-x + 2 y + z = 4 …………(1)

y + 5z/3 = 16 / 3 …………(2)

-38z / 3 = -76 / 3 ………..(3) $\Rightarrow$ z ? On solving we get x = y = z = 2

Q 2: Solve: x + y + z = 6, x + 2y +3 z = 14, x + 4y +7 z = 30.

<u>Eigen Values and Eigen Vectors</u>:-

Characteristic Matrix – Suppose A is a square matrix then (A – λI) is called **characteristic matrix**.
Characteristic Polynomial – The det $|A - \lambda I|$ is said to be **characteristic polynomial**.
Characteristic Equation – The equation $|A - \lambda I|$ = 0 is said to be **characteristic equation**.
Characteristic Roots – The roots of equation $|A - \lambda I|$ = 0 are called **characteristic roots or eigen values**.

Note – (i) A and A' have same eigen values.

(ii) The sum of eigen values of a matrix is equal to trace of the matrix.

(iii) The product of eigen values of a matrix is equal to determinant of the matrix.

(iv) If $\lambda_1, \lambda_2 - - - - \lambda_n$ are eigen values of A then

(a) $K\lambda_1, K\lambda_2 - - - - K\lambda_n$ are eigen values of KA.

(b) $\lambda_1^m, \lambda_2^m - - - - \lambda_n^m$ are eigen values of A^m.

(c) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} - - - - \frac{1}{\lambda_n}$ are eigen values of A^{-1}.

1. Find Eigen values and Eigen vectors of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ **.**

Sol – Since $|A-\lambda I| = \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)(5-\lambda)$

$\therefore$ Characteristic equation is $|A-\lambda I| = 0$

$\Rightarrow (3-\lambda)(2-\lambda)(5-\lambda) = 0 \qquad \Rightarrow \lambda = 2,3,5$

Eigenvectors of A corresponding to the eigen value λ is given by

$(A-\lambda I)X = 0$ (Non-zero solution)

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--------} \quad \{1\}$$

(i) For $\lambda = 2$

From eq {1}, $\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \Rightarrow x_1 + x_2 + 4x_3 = 0$

$6x_3 = 0 \Rightarrow x_3 = 0$

$\therefore x_1 + x_2 + 0 = 0 \Rightarrow x_1 = -x_2 = -K (x_2 = K, say)$

$\therefore$ Eigenvector = $\begin{bmatrix} -K \\ K \\ 0 \end{bmatrix} = X_1 = K\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(ii) For $\lambda = 3$

From eq {1}, $\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \Rightarrow x_1 + 4x_3 = 0, -x_2 + 6x_3 = 0, 2x_3 = 0$

$Let x_1 = 1, x_2 = x_3 = 0$

Then, Eigenvector = $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(iii) For $\lambda = 5$

From eq {1}, $\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \Rightarrow -2x_1 + x_2 + 4x_3 = 0, -3x_2 + 6x_3 = 0$

$$\frac{x_1}{6+12} = \frac{x_2}{0+12} = \frac{x_3}{6-0} \Rightarrow \frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6} \Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$\therefore$ Eigenvector = $X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

<u>Linear Dependence and Linear Independence</u> :-

Def – Vectors X_1, X_2,-------- X_n are said to be **linear dependent** if

(i) All the vectors are of same order.

(ii) $\exists \lambda_1, \lambda_2 ---- \lambda_n (not all zero) s.t. \lambda_1 X_1 + \lambda_2 X_2 + ----- + \lambda_n X_n = 0$ otherwise **linear independent.**

1. **Find whether or not the following set of vectors is linear dependent or independent (1, 2, 4), (2, -1, 3), (0, 1, 2), (-3, 7, 2). Find the relation between them.**

Sol – Let $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$ ------------ $\{1\}$

Then $\lambda_1(1,2,4) + \lambda_2(2,-1,3) + \lambda_3(0,1,2) + \lambda_4(-3,7,2) = 0$

$\Rightarrow \lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$

$\Rightarrow 2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$

$\Rightarrow 4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$

This is a Homogeneous system and can be written in matrix form as

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{or} \qquad A\lambda = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (R_3 \to R_3 - R_2)$$

Row by column multiplication, we get

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0, \; -5\lambda_2 + \lambda_3 + 13\lambda_4 = 0, \; \lambda_3 + \lambda_4 = 0$$

Let λ_4 = t, then λ_3 = -t, $\lambda_2 = \frac{12}{5}$ t and $\lambda_1 = \frac{-9}{5}$ t.

Hence the given vectors are linearly dependent.

Eqn {1} $\Rightarrow \frac{-9t}{5}X_1 + \frac{12t}{5}X_2 - tX_3 + tX_4 = 0 \quad \Rightarrow 9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$ is the required result**.**

2. Find the characteristic equation of $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and find A^{-1}. Also verify Cayley Hamilton Theorem and find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol – Since **Characteristic equation is** $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \qquad \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3I = 0$$

By Cayley Hamilton Theorem –

$A^3 - 5A^2 + 7A - 3I = 0$ ---------- $\{1\}$

We have to verify eqn {1},

LHS = $A^3 - 5A^2 + 7A - 3I$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}^2 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}^2 + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence Cayley Theorem is verified.

To find A^{-1} :-

Pre-multiplying eqn 1 by A^{-1}, we get

$$A^{-1}A^3 - 5A^{-1}A^2 + 7A^{-1}A - 3A^{-1}I = A^{-1}O$$

$$\Rightarrow A^2 - 5A + 7I - 3A^{-1} = O$$

$$\Rightarrow 3A^{-1} = A^2 - 5A + 7I$$

$$\Rightarrow A^{-1} = \frac{1}{3}\left(A^2 - 5A + 7I\right) \qquad \text{----------} \quad \{2\}$$

Now, $A^2 = A \bullet A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

From eqn {2}, $3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5 [A^3 - 5A^2 + 7A - 3I] + A [A^3 - 5A^2 + 7A - 3I] + A^2 + A + I$

$= A^5 \times O + A \times O + A^2 + A + I$

$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$ **(Hence the Result)**

Similarity Transformation / Digonalisation of a Matrix:-

Diagonalisation of a matrix A is the process of reduction A to a **diagonal form.** A is related to D by a **similarity transformation** such that $D = M^{-1}AM$ and A is reduced to the diagonal matrix D through modal matrix M. D is also called **spectral matrix** of A.

1. Reduce the matrix A = $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ **to diagonal form by similarity transformation.**

Hence find A^3.

Sol – $\because$ Characteristic equation is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 1-\lambda & -1 & 2 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1,2,3$$

Hence eigen values of A are 1, 2, 3.

Corresponding to λ = 1, let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector then, $(A - I) X_1 = O$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -x_2 + 2x_3 = 0, x_2 - x_3 = 0, 2x_3 = 0$$

$\therefore$ $x_1 = k_1$ (say), $x_2 = 0 = x_3$ $\quad\Rightarrow \quad X_1 = k_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Corresponding to λ = 2, let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector then, $(A - 2I)\, X_2 = O$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -x_1 - x_2 + 2x_3 = 0, -x_3 = 0, x_3 = 0$$

$\therefore$ $x_1 = k_2$ (say), $x_2 = -k_2$, $x_3 = 0 \Rightarrow \quad X_2 = k_2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Corresponding to λ = 3, let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector then, $(A - 3I)\, X_3 = O$

$$\Rightarrow \begin{bmatrix} -2 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -2x_1 - x_2 + 2x_3 = 0, -x_2 - x_3 = 0$$

$\therefore$ $x_2 = k_3$ (say), $x_3 = -k_3$, $x_1 = \dfrac{-3}{2}k_3 \quad \Rightarrow \quad X_3 = k_3\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

Hence **Modal matrix** is M = $\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$.

$$\because \quad |M| = -2 \text{ and Adj } M = \begin{bmatrix} -2 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore M^{-1} = \frac{AdjM}{|M|} = \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & -1 & -1 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$M^{-1}AM = \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & -1 & -1 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now, since $D = M^{-1}AM \Rightarrow A = MDM^{-1}$

$\Rightarrow A^2 = (MDM^{-1})(MDM^{-1}) = MD^2M^{-1}$ $\qquad (\because \; M^{-1}M = I)$

Similarly, $A^3 = M D^3M^{-1}$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1/2 \\ 0 & -1 & -1 \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -7 & 32 \\ 0 & 8 & -19 \\ 0 & 0 & 27 \end{bmatrix}.$$