Sapienza
Università di Roma

Group Project

Risk Measures Analysis

Course in Models for Risk and Forecasting
Prof. Candila Vincenzo
Prof. Marino Mario

Sapienza Università di Roma
Dipartimento Memotef

Authors

Menga Alessandra - 1496982
Sciacca Daniele - 1716027
Sortino Ivan - 1759424

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1 Introduction

The purpose of this analysis is to investigate the stock's behaviour of three different American pharmaceutical companies across the last thirteen years. We test different models both in the univariate and multivariate cases, to reach a good forecasting of two risk measures: *Value at Risk* (VaR) and *Expected Shortfall* (ES). We work on the Profit & Loss distribution represented by the daily returns of the stocks. Due to the nature of the considered companies, we want also to check if the models are able to handle periods of particular instability linked to disease that have hit United States, namely *N1H1* (swine flu), *Zika* fever and most recent *Covid-19*. Then, we proceed to model the dependence between the considered stock by exploiting copulas.

2 Data description

The analysis starts considering a time span of thirteen years, starting from January 01, 2008 up to January 01, 2021, 3273 observations in all. We focus on three period of great interest, characterised by the diffusion of new diseases.
The first period pointed out starts[1] from the beginning of 2009 up to August 10, 2010. These years were characterised by the outspread worldwide of the the so called Swine Flu, caused by a sub-type of influenza virus named H1N1. The second relevant time frame was instead characterised by the Zika virus, starting from April 2015 up to November 2016. The virus is spread by daytime-active mosquitoes, and it is mostly diffused in central America, especially Mexico. After the virus reached States in 2016, many US companies started researching a vaccine. Lastly, the third period regards the Covid-19 pandemic, starting from March 2020 (when the World Health Organization declared it as pandemic) up to the end of January 2021,last day of the analysis. The instability linked to the Covid-19 results to be the greater among the three considered periods, showing large decreasing in stock prices of the companies specified below.
We select different pharmaceutical companies, choosing the three with the biggest market capitalization on the New York Stock Exchange (NYSE)[2].
The first one is Johnson & Johnson (*JNJ*) that, with $445.70 billions worth market capitalization, makes it the biggest pharmaceutical company worldwide. It is an American multinational corporation founded in 1886 that develops medical devices, pharmaceuticals and consumer packaged goods. Its stock is traded on NYSE and it is a component of the Dow Jones Industrial Average Index. One of the main branch of the company is the research and provision for vaccine worldwide, that played an important role in 2009 during the Swine Flu epidemics, providing promptly medicines to avoid spread of infections. Recently, Johnson & Johnson committed over $1 billion toward the development of a not-for-profit COVID-19 vaccine, partnered with Beth Israel Deaconess Medical Center. Its closely related history with research give us an explanation on why its stock price tends to fluctuate more during the spread of new disease, reflecting the positive belief of investors in the main role that the

[1] All epidemics periods are considered according to the World Health Organization (WHO) website https://www.who.int/emergencies/diseases/en/

[2] Data gathered at the end of may 2021 from https://companiesmarketcap.com/pharmaceuticals/largest-pharmaceutical-companies-by-market-cap/

company will play in the forthcoming instability.
The second company considered is Pfizer (*PFE*) with a market capitalization of $216.79 billions. It is an America multinational headquartered in Manhattan, developing and producing medicines and vaccines for immunology, oncology, cardiology, endocrinology, and neurology. The company has several blockbuster drugs or products generating more than $1 billion in annual revenues. Pfizer is traded on NYSE and it was a component of the Dow Jones Industrial Average stock market index from 2004 to August 2020. It plays a key role in the recent Covid-19 pandemic research for the vaccine, investing 2 billion dollars of its own funds without recurring to US federal *Operation Warp Speed* vaccine development program aids[3]. The choice to leave out the politics from the research reflects the high value and trust that Pfizer has from its investors, granting also them great revenues and hopes for future incomes.
The last company considered is Merck & Co. (*MRK*). It is an American multinational pharmaceutical company headquartered in New Jersey. It is named after the Merck family, which set up the Merck Group in Germany in 1668. The well-known Merck & Co. was established as an American affiliate in 1891. The company develops medicines, vaccines, biologic therapies and animal health products. Its stock is traded on NYSE and it has a market capitalization (cap) equal to $192.15 billions, making it the fifth largest pharmaceutical company by market cap.

3 Data analysis

The analysis considers the closing price of the three stocks mentioned above; the data are imported from Yahoo finance database. Starting from these, we calculate the daily log-returns for each series. The plots of closing price and log returns of the three companies are reported in figures 1, 2, 3

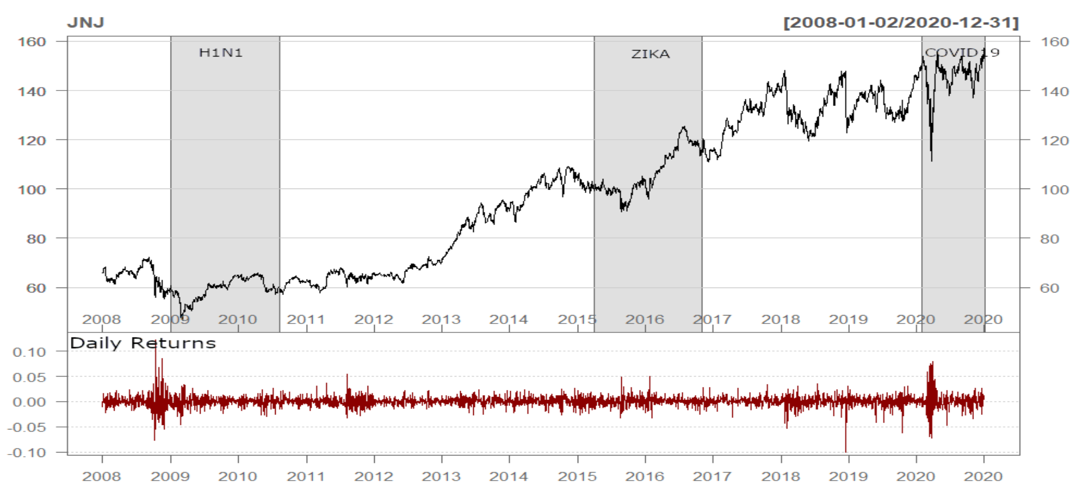


Figure 1: Johnson & Johnson stock price (top) and daily log returns (below).

[3]Czachor, Emily (November 9, 2020). "Pfizer Avoided R&D Funding From Trump's Operation Warp Speed Because of Bureaucracy, Politics". Newsweek. Retrieved May 18, 2021.

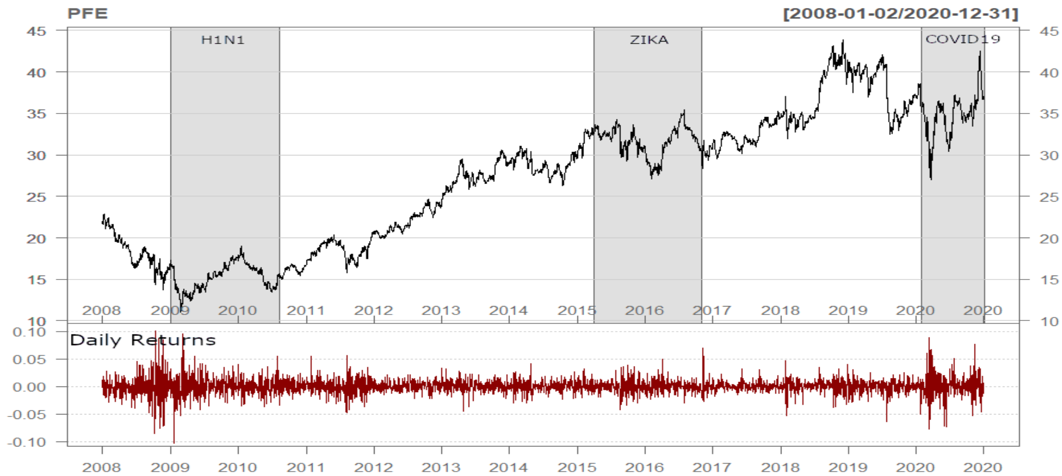


Figure 2: Pfizer stock price (top) and daily log returns (below).

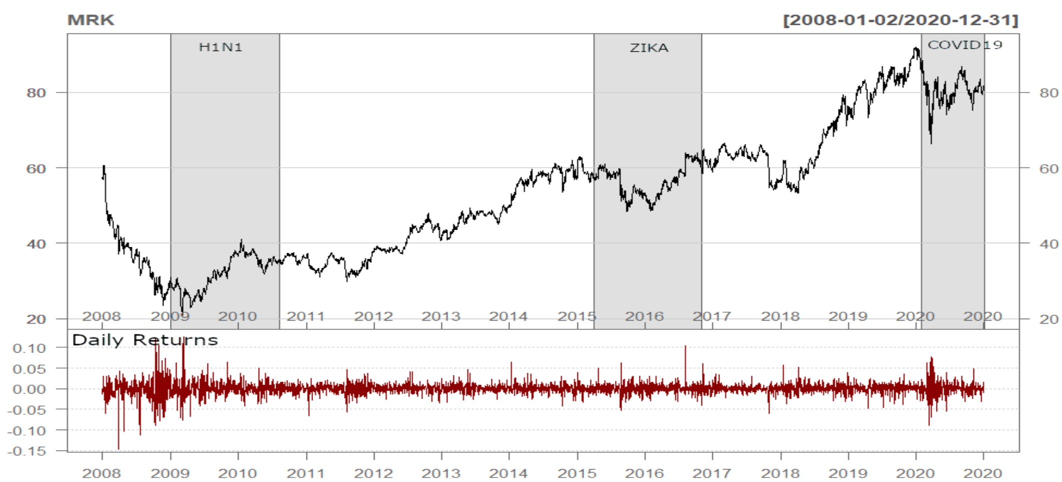


Figure 3: Merck stock price (top) and daily log returns (below).

The descriptive statistics performed on the returns are shown in table 1. All the three series result to be fairly symmetric and leptokurtic, presenting the typical heaviness of the tails, common in financial time series. We can note that Pfizer has less kurtosis than the other two time series.

	JNJ	PFE	MRK
Mean	0.00027	0.00016	0.00011
Standard Deviation	0.01156	0.01480	0.01614
Skewness	-0.07569	-0.12203	-0.37649
Kurtosis	12.12462	6.16205	10.57617

Table 1: Descriptive statistics performed on returns.

We also perform some tests on the returns in order to check normality, correlation among them and stationarity. The values of statistics are reported in 2. The Jarque-Bera (J-B) test checks the normality of the data: the null hypothesis concerns the normality of the distribution; since all the test statistics result really higher than the critical value (5,991), we reject the gaussian hypothesis for the returns. Next, we inspect the independence, shown graphically in 4, and perform Ljung-Box (L-B) test on squared returns, since we are interested in the second moment of the returns, in order to model their variance. The null hypothesis of the L-B test checks the independence among the data. The test is performed with $lag = 20$; it rejects the null hypothesis for all the series since the test statistic is greater than the critical value (31.41), suggesting that there is Arch effect as confirmed by the result of the Arch test (table 2). We can also state that the series are stationary looking at the augmented Dickey-fuller (D-F) test performed on the returns; indeed all the test statistics fall above the critical value of -2.93, by taking the absolute values.

	JNJ	**PFE**	**MRK**
Jarque-Bera test	20081.7039*	5195.8258*	15355.6121*
Ljung-Box test	82.0669*	71.1654*	60.5712*
Dickey-Fuller test	-15.8711	-16.5053	-17.5431
Arch test	760.13*	654.68*	392.74*

Table 2: J-B and D-F test performed on returns, L-B on squared returns. The star (*) denotes that the null hypothesis is rejected.

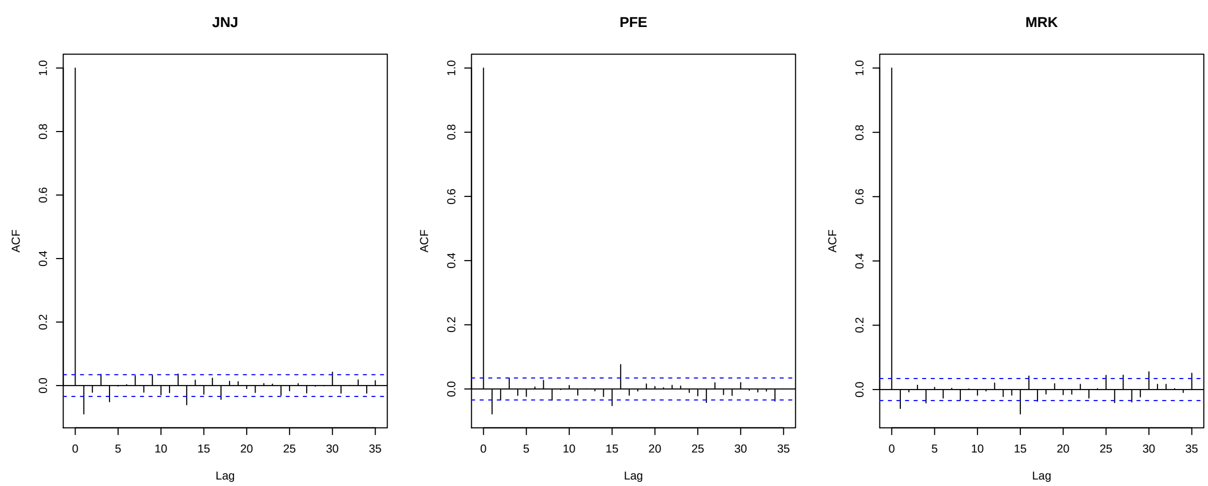


Figure 4: Autocorrelation plot on returns.

4 Models and risk measures

The aim of this work is to find the best estimate of two risk measures, the Value at Risk (VaR) and the Expected Shortfall (ES). The former is a quantile of the loss distribution; it measures the maximum expected loss in a certain time horizon (usually one day) given a confidence level (tipically 95% or 99%). The latter is the average of VaRs considering the tail of the loss distribution not inspected by the VaR. Moreover, we decide to work considering the Profit and Loss (P&L) distribution instead of Loss distribution. The choice is motivated by practical reasons. Several models are used for this purpose; we can split them out into three main approaches: parametric, semi-parametric and non parametric one.

For the parametric approach, we decide to implement various GARCH models, since they are better suited for modelling financial time series characterised by volatility clustering effect. Over standard GARCH models with different distributions, we considered also Glosten-Jagannathan-Runkle GARCH (*GJR-GARCH*) to take in account the potential asymmetric impact on volatility that negative shocks have with respect to positive ones. Lastly, we consider different Mixing Data Sampling GARCH models (*GARCH-MIDAS*) and Double Asymmetric GARCH-MIDAS (*DAGM*; these models take into account a Macroeconomic Variables (*MVs*) to estimate more faithfully the volatility. We compute each -MIDAS including (skew) and excluding (noSkew) the skewness parameter γ (see appendix A, formula (1) and (2)) to include in the short-run equation an eventual asymmetric response of conditional variance to returns' negative changes. We pair each MIDAS model with three different Macroeconomic Variables, discarding those in which the long-run component parameter θ (see appendix A) results to be non significant, meaning that it does not carry any relevant information for a better estimation of the volatility.

The first MV refers to the *Consumer Sentiment Index* (*CSI*) calculated by the University of Michigan[4] on a monthly basis. All the data are bestowed by the Federal Reserve of San Louis. This Index is a statistical measurement of the overall health of the economy, determined by consumer opinion. It takes into account people's feelings regarding their current financial health, the health of the economy in the short-term and the prospects for longer-term economic growth. It is widely considered to be a useful economic indicator and a great reference in term of investor inclination. The second MV considered is the categorical *Economic Policy Uncertainty Index* (*EPUI*) referring to Health care[5], calculated on monthly basis. It includes a range of sub-indices based on news data gathered from the *Access World News database*, composed by over 2,000 US newspapers. The last Macroeconomic Variable is *Equity Market Volatility Tracker* (*EMVT*) based on Healthcare Policy [6]. This Index have a monthly frequency and moves with the *VIX* index[7] and with the realized volatility of returns on

[4]University of Michigan, University of Michigan: Consumer Sentiment [UMCSENT], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/UMCSENT, June 1, 2021.

[5]Baker, Scott R., Bloom, Nick and Davis, Stephen J., Economic Policy Uncertainty Index: Categorical Index: Health care [EPUHEALTHCARE], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/EPUHEALTHCARE, June 1, 2021.

[6]Baker, Scott R., Bloom, Nick and Davis, Stephen J., Equity Market Volatility Tracker: Healthcare Policy [EMVHEALTHCAREPOL], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/EMVHEALTHCAREPOL, June 2, 2021.

[7]VIX is the ticker symbol and the popular name for the Chicago Board Options Exchange's CBOE Volatility Index, a popular measure of the stock market's expectation of volatility based on S&P 500 index options.

the Standard and Poor 500 (*S&P500*). Since all the indices considered so far are not stationary and seasonally adjusted as reported on the respective website, we perform the first difference in order to make them stationary and to exploit the data in GARCH-MIDAS models.
The second class of models is referred to semi-parametric approach that attempts to model directly quantile through quantile regression. They result powerful since they do not need a specification assumption for the returns distribution, avoiding to incur in some misspecification error. We use different type of Conditional Autoregressive Value at Risk (*CAViaR*) models and Linear ARCH models (*L-ARCH*), also considering the latter paired with a MIDAS part (*L-ARCH-MIDAS*), to take into account the information brought by the different Macroeconomic Variables considered before. The CAViaR models implemented are: the CAViaR-Symmetric Absolute Value (*CAViaR-SAV*), CAViaR-Asymmetric Slope (*CAViaR-AS*) and CAViaR-Indirect GARCH (*CAViaR-IG*).
In the end, we take into account a non-parametric approach exploiting the Historical Simulation model (HS). We consider different windows size (w) to reach a better evaluation of the VaR.

5 Backtesting results

In order to understand which of the aforementioned models better estimate the risk measures, we continue the analysis by fitting them on our financial time series. As said in the previous section, we work on three different approaches: parametric, semi-parametric and non parametric. Referring to the parametric one, we consider the following GARCH(1,1)[8] class of models with two different distributions (N for normal and T for t-Student) for the innovation term: GARCH-N, GARCH-T, GJR-GARCH-N, GJR-GARCH-T. To model the variance involving also the MVs as additional volatility determinant, we use GM-N-noSkew-*MVs*, GM-N-Skew-*MVs*, GM-T-noSkew-*MVs*, GM-T-Skew-*MVs*, DAGM-N-noSkew-*MVs*, DAGM-N-Skew-*MVs*, DAGM-T-noSkew-*MVs*, DAGM-T-Skew-*MVs*. Referring to the semi-parametric approach, we use CAViaR-IG, CAViaR-SAV, CAViaR-AS, L-ARCH, L-ARCH-MIDAS-*MVs* models. The non-parametric approach includes Historical Simulation (HS) with different window size: w=125, w=250, w=375 and w=500[9]). These models are used to perform the *backtest*, namely a set of statistical procedures designed to check if real losses are in line with estimated risk measures. In particular we focus on two hypothesis tests: the *Proportion Of Failure (POF)* test (LR_{uc}), which inspects if the theoretical VaR violations are equal to the estimated ones, and the *conditional coverage* test (LR_{cc}), composed by the sum of POF test and the independence test; the latter examines if the VaR violation at time t depends on the outcome at time $(t-1)$. We have a good model in terms of VaR predictability if it accepts the null hypothesis of the two aforementioned tests. The analysis is conducted at two confidence levels of the risk measures: 95% and 99%.
Looking at tables 3, 4 and 5, we can notice other two columns, AE and ES: the former represents the ratio between the actual VaR violations and the expected ones; its ideal value is around one. The latter represents if the martingale difference property for the Expected Shortfall is satisfied; this happens when its value is equal to zero. In the same column in brackets, the number of ES

[8] We decide to not model the conditional mean of the returns, since, looking at the figure 4, we can observe the absence of autocorrelation among them.

[9] The windows size, considering the trading calendar, refer to 6 months, 12 months, 18 months and 24 months.

violations are reported. As already said, the only exclusion criterion for models before the backtesting procedure regards -MIDAS models: when they present a non-significant parameter *theta*, they are discarded. Indeed, observing tables 3, 4 and 5, not all these models are presented. Moreover, we can notice that, while for the parametric approach a -MIDAS model rejected at 95% of confidence level is automatically rejected also at 99%, this could not be true for the L-ARCH-MIDAS belonging to the semi-parametric approach, being this a quantile regression: it can happens that a model accepted at a given confidence level, could be discarded at another one. For example, taking into account PFE series, L-ARCH-MIDAS-MVs are accepted at 95% but not at 99%; this means that the parameter associated to that macroeconomic variable is not significant at that confidence level.

Confidence level	95%				99%			
PFE	LR_{uc}	LR_{cc}	AE	ES(viol)	LR_{uc}	LR_{cc}	AE	ES(viol)
parametric								
GARCH-N	1.0550	3.2690	0.9230	-0.0002 (82)	4.827*	13.209*	1.405	-0.0001 (29)
GARCH-T	0.0860	1.3670	0.9780	0.0004 (22)	0.017	13.924*	0.978	0.0002 (3)
GJR-GARCH-N	0.8930	2.0980	0.9290	-0.0001 (81)	8.804*	15.741*	1.558	-0.0001 (28)
GJR-GARCH-T	0.0010	0.9520	1.0020	0.0003 (23)	0.237	4.927	0.917	0.0002 (4)
GM-N-noSkew-*CSI*	0.2880	1.1040	0.9590	-0.0001 (84)	7.088*	18.528*	1.497	-0.0001 (30)
GM-N-Skew-*CSI*	1.0550	1.6560	0.9230	-0.0001 (87)	11.675*	17.843*	1.65	-0.0001 (26)
DAGM-N-noSkew-*CSI*	0.0860	1.3670	0.9780	-0.0001 (84)	9.722*	20.081*	1.589	-0.0001 (30)
DAGM-N-Skew-*CSI*	1.4210	2.1470	0.9100	-0.0001 (85)	16.027*	18.648*	1.772	-0.0001 (28)
DAGM-N-noSkew-*EPUI*	0.6100	2.4820	0.9410	-0.0002 (80)	7.088*	14.577*	1.497	-0.0001 (27)
DAGM-N-Skew-*EPUI*	0.3820	2.0430	0.9530	-0.0001 (81)	12.708*	18.635*	1.68	-0.0001 (31)
DAGM-N-noSkew-*EMVT*	1.4210	3.8790	0.9100	-0.0001 (81)	8.804*	19.513*	1.558	-0.0001 (29)
DAGM-N-Skew-*EMVT*	1.4210	1.6480	0.9100	-0.0001 (83)	17.205*	25.36*	1.803	-0.0001 (27)
DAGM-T-noSkew-*CSI*	42.6590*	44.9120*	0.5440	0.0004 (20)	17.511*	22.003*	0.367	0.0003 (2)
DAGM-T-Skew-*CSI*	40.1740*	42.1860*	0.5560	0.0003 (23)	13.789*	17.669*	0.428	0.0003 (2)
DAGM-T-noSkew-*EMVT*	49.2820*	52.2070*	0.5130	0.0004 (21)	13.789*	17.669*	0.428	0.0002 (3)
semi-parametric								
CAViaR-IG	0.0120	1.6200	1.0080	-0.0000 (51)	0.002	13.42*	1.008	0.0001 (10)
CAViaR-SAV	0.0180	1.9410	0.9900	-0.0000 (56)	0.017	4.246	0.978	0.0001 (9)
CAViaR-AS	0.0010	1.7110	1.0020	-0.0000 (58)	0.002	4.015	1.008	0.0001 (11)
L-ARCH	0.0860	2.2370	0.9780	0.0000 (59)	0.32	0.985	1.1	0.0001 (14)
L-ARCH-MIDAS-*CSI*	0.0120	0.8890	1.0080	0.0000 (61)	-	-	-	-
L-ARCH-MIDAS-*EPUI*	0.0350	2.4420	1.0140	0.0000 (61)	-	-	-	-
L-ARCH-MIDAS-*EMVT*	0.0720	3.4060	1.0200	0.0000 (61)	-	-	-	-
non-parametric								
HS (w = 125)	5.4690*	13.1670*	1.1840	-0.0002 (100)	27.3880*	43.9780*	2.0470	-0.0001 (51)
HS (w = 250)	6.9930*	22.0200*	1.2090	-0.0001 (87)	9.3580*	24.1710*	1.5810	-0.0001 (32)
HS (w = 375)	5.4690*	26.1340*	1.1840	-0.0001 (82)	6.7550*	11.0180*	1.4880	-0.0001 (30)
HS (w = 500)	6.2080*	26.0570*	1.1970	-0.0002 (88)	8.4490*	12.3240*	1.5500	-0.0001 (29)

Table 3: The critical values of LR_{uc} and LR_{cc}, at $\alpha = 0.05$, are 3.841 and 5.991 respectively. The star (*) denotes that the null hypothesis is rejected at the 5% significance level.

When this happens, we indicate the lack of a model with a ”-” in the tables. Considering all the remaining models, we calculate the one-day-ahead VaR on the whole sample.
At 95%, looking at the *parametric approach*, it is clear that for PFE series, all parametric models are good except for the DAGM models with a t-student distribution. This suggests that, when we assume the normal distribution for MIDAS models, the considered MVs used to model the volatility, better predict the VaR. Indeed, the AE values related to this models are near to one and the null

hypothesis of POF and Conditional Coverage tests are accepted. This do not held anymore for MRK series, since the only two models passing[10] the test are the GARCH-T and the GJR-GARCH-T models. Furthermore, a different situation appears in the table 5 of JNJ, where the only MIDAS models accepted are the DAGM with normal distribution, suggesting that we are able to better predict VaR if we consider the asymmetric response of the volatility to the MVs. It is worth noting that DAGM-N-*EMVT* model does not pass the POF test but, we know that the VaR violations are independent because the value of the LR_{cc} is less than its critical value.
With regards to the *semi-parametric approach*, we can observe that all the models perform well for the considered series.

Confidence level	95%				99%			
MRK	LR_{uc}	LR_{cc}	AE	ES (viol)	LR_{uc}	LR_{cc}	AE	ES (viol)
parametric								
GARCH-N	10.4240*	10.7400*	0.7640	-0.0003 (72)	11.6750*	14.9110*	1.6500	-0.0002 (33)
GARCH-T	0.4900	0.5070	0.9470	0.0003 (23)	2.4340	4.8610	1.2830	0.0003 (5)
GJR-GARCH-N	8.8080*	11.6440*	0.7820	-0.0003 (81)	19.6620*	24.2790*	1.8640	-0.0001 (39)
GJR-GARCH-T	2.0760	2.4930	0.8920	0.0003 (28)	1.9540	2.3250	1.2530	0.0004 (5)
GM-N-noSkew-*CSI*	9.8690*	10.1440*	0.7700	-0.0003 (78)	18.4160*	20.7600*	1.8330	-0.0001 (34)
GM-N-Skew-*CSI*	9.3300*	9.3310*	0.7760	-0.0002 (80)	17.2050*	17.2090*	1.8030	-0.0001 (38)
GM-N-Skew-*EMVT*	23.3970*	24.9900*	0.6540	-0.0001 (60)	2.9630	3.2430	1.3140	-0.0000 (28)
GM-T-noSkew-*CSI*	58.0390*	60.0090*	0.4770	0.0003 (22)	12.1500*	15.7600*	0.4580	0.0004 (5)
DAGM-N-noSkew-*CSI*	9.8690*	10.7820*	0.7700	-0.0002 (79)	16.0270*	18.6480*	1.7720	-0.0001 (33)
DAGM-N-Skew-*CSI*	10.9960*	11.0170*	0.7580	-0.0002 (79)	16.0270*	16.0280*	1.7720	-0.0001 (39)
DAGM-N-Skew-*EPUI*	8.8080*	9.5790*	0.7820	-0.0002 (76)	16.0270*	21.2700*	1.7720	-0.0001 (37)
DAGM-T-noSkew-*CSI*	55.0180*	56.7620*	0.4890	0.0003 (22)	12.1500*	15.7600*	0.4580	0.0004 (5)
DAGM-T-Skew-*CSI*	50.6780*	52.1140*	0.5070	0.0003 (27)	13.7890*	17.6690*	0.4280	0.0004 (5)
DAGM-T-Skew-*EPUI*	55.0180*	56.7620*	0.4890	0.0003 (26)	9.2640*	12.3940*	0.5190	0.0004 (5)
semi-parametric								
CAViaR-IG	0.0450	2.6270	0.9840	-0.0001 (59)	0.0170	4.2460	0.9780	0.0001 (10)
CAViaR-SAV	0.0120	0.7990	1.0080	-0.0001 (55)	0.0020	4.0150	1.0080	0.0002 (8)
CAViaR-AS	0.0010	0.7280	1.0020	-0.0001 (58)	0.0020	4.0150	1.0080	0.0002 (10)
L-ARCH	0.0180	0.1450	0.9900	-0.0001 (53)	0.0170	0.9960	0.9780	0.0001 (14)
L-ARCH-MIDAS-*CSI*	0.0010	0.9520	1.0020	-0.0000 (56)	0.5400	3.7740	1.1300	0.0001 (10)
L-ARCH-MIDAS-*EPUI*	-	-	-	-	0.3200	3.7360	1.1000	0.0001 (13)
non-parametric								
HS (w = 125)	6.5930*	30.5040*	1.2030	-0.0001 (96)	12.3190*	21.8800*	1.6740	-0.0001 (36)
HS (w = 250)	1.2020	21.6670*	1.0850	-0.0001 (78)	4.5340*	9.4320*	1.3950	-0.0001 (26)
HS (w = 375)	1.9890	18.6610*	1.1100	-0.0001 (78)	2.2080	12.1710*	1.2710	-0.0001 (19)
HS (w = 500)	1.9890	18.6610*	1.1100	-0.0002 (88)	4.5340*	17.4320*	1.3950	-0.0000 (20)

Table 4: The critical values of LR_{uc} and LR_{cc}, at $\alpha = 0.05$, are 3.841 and 5.991 respectively. The star (*) denotes that the null hypothesis is rejected at the 5% significance level.

These results are actually expected, since we do not assume any kind of distribution for both CAViaR and L-ARCH/L-ARCH-MIDAS models.
A diametrically opposite situation occurs in the *non-parametric* technique, where none of Historical Simulation passes the tests. The only exceptions regard HS (w = 250), HS (w = 375) and HS (w = 500) for MRK and JNJ that pass only the POF test. It worth noting that, despite these latter models pass the test and have an AE value close to one, they do not pass the conditional coverage

[10] In this context, passing means that it accepts the null hypothesis.

test; being this composed by LR_{uc} and LR_{ind}, high results suggest that VaR violations are dependent. At 99% of confidence level, none of the MIDAS models belonging to the parametric approach pass the test for PFE, MRK and JNJ, except for the GM-N-Skew-*EMVT* of MRK. The GJR-GARCH-T model represents the best one in this approach for all the series. It would seem that at this confidence level, the three series prefer the Student-t distribution, suggesting that the empirical distribution of returns has high kurtosis. In addition, one can see that quantile models, CAViaR-IG/SAV/AS and L-ARCH/L-ARCH-MIDAS-MVs, generally perform better in terms of AE ratio, which is always closer to one compared with that of GARCH, GJR-GARCH, GM and DAGM models. The VaR estimates of the latter are consistently overly optimistic or too conservative reflecting the high variability of the AE ratio, which varies from a minimum of 0.367 to a maximum of 1.8640. Conversely, the AE ratio of the former varies in a range much closer to one: it swings between 0.9780 and 1.1300, which allows a more accurate risk assessment. All the HS with different windows size seem to be not able to predict correctly the VaR.

Confidence level	95%				99%			
JNJ	LR_{uc}	LR_{cc}	AE	ES (viol)	LR_{uc}	LR_{cc}	AE	ES (viol)
parametric								
GARCH-N	6.0110*	8.0890*	0.8190	-0.0002 (78)	14.8850*	16.9060*	1.7420	-0.0001 (29)
GARCH-T	0.7450	1.2330	0.9350	0.0002 (23)	0.1560	0.9120	1.0690	0.0002 (8)
GJR-GARCH-N	4.8250*	5.1090	0.8370	-0.0001 (74)	9.7220*	11.4010*	1.5890	-0.0001 (28)
GJR-GARCH-T	0.4900	0.5530	0.9470	0.0002 (25)	0.1560	0.9120	1.0690	0.0002 (7)
GM-N-noSkew-*CSI*	5.2050*	7.0590*	0.8310	-0.0002 (77)	17.2050*	19.3710*	1.8030	-0.0001 (30)
GM-N-Skew-*CSI*	6.8800*	7.4040*	0.8070	-0.0001 (73)	13.7780*	15.7290*	1.7110	-0.0001 (29)
GM-N-noSkew-*EPUI*	6.0110*	7.1180*	0.8190	-0.0002 (81)	16.0270*	18.1210*	1.7720	-0.0001 (36)
GM-N-Skew-*EPUI*	4.1100*	5.6540	0.8490	-0.0001 (79)	10.6790*	12.4240*	1.6190	-0.0001 (34)
GM-T-noSkew-*CSI*	47.9090*	48.1800*	0.5190	0.0002 (22)	13.7890*	13.9100*	0.4280	0.0002 (8)
GM-T-Skew-*CSI*	47.9090*	48.7770*	0.5190	0.0002 (27)	13.7890*	13.9100*	0.4280	0.0002 (8)
GM-T-Skew-*EMVT*	46.5610*	47.4890*	0.5260	0.0002 (25)	15.5730*	15.6770*	0.3970	0.0002 (8)
DAGM-N-noSkew-*CSI*	1.8440	2.1480	0.8980	-0.0002 (80)	12.7080*	14.5890*	1.6800	-0.0001 (31)
DAGM-N-Skew-*CSI*	3.7750	3.7750	0.8550	-0.0001 (78)	11.6750*	11.6880*	1.6500	-0.0001 (29)
DAGM-N-Skew-*EPUI*	3.7750	3.9520	0.8550	-0.0001 (79)	16.0270*	18.1210*	1.7720	-0.0001 (33)
DAGM-N-Skew-*EMVT*	5.6000*	5.6360	0.8250	-0.0001 (81)	14.8850*	16.9060*	1.7420	-0.0001 (34)
DAGM-T-Skew-*EMVT*	13.4527*	17.5097*	0.7333	-0.0001 (24)	4.8641*	12.2676*	0.6416	-0.0001 (8)
semi-parametric								
CAViaR-IG	0.0010	0.0070	1.0020	-0.0001 (62)	0.0490	0.7630	1.0390	0.0001 (10)
CAViaR-SAV	0.0030	0.4550	0.9960	-0.0001 (56)	0.0020	0.6750	1.0080	0.0001 (9)
CAViaR-AS	0.0030	0.4550	0.9960	-0.0001 (57)	0.0020	0.6750	1.0080	0.0001 (10)
L-ARCH	0.0350	0.8420	1.0140	-0.0001 (67)	0.0490	0.7630	1.0390	0.0001 (13)
L-ARCH-MIDAS-*CSI*	0.0010	1.7110	1.0020	-0.0001 (64)	-	-	-	-
L-ARCH-MIDAS-*EPUI*	0.0180	1.9410	0.9900	-0.0001 (65)	-	-	-	-
L-ARCH-MIDAS-*EMVT*	0.0120	0.3670	1.0080	0.0000 (61)	-	-	-	-
non-parametric								
HS(w = 125)	10.5530*	28.5750*	1.2590	-0.0002 (114)	28.8570*	32.2960*	2.0780	-0.0001 (47)
HS(w = 250)	1.0360	19.6040*	1.0790	-0.0002 (97)	14.4820*	36.0340*	1.7360	-0.0001 (33)
HS(w = 375)	1.5720	25.9940*	1.0980	-0.0001 (90)	10.3060*	34.1910*	1.6120	-0.0000 (25)
HS(w = 500)	2.9650	20.2750*	1.1350	-0.0001 (92)	15.6190*	41.4650*	1.7670	-0.0000 (26)

Table 5: The critical values of LR_{uc} and LR_{cc}, at $\alpha = 0.05$, are 3.841 and 5.991 respectively. The star (*) denotes that the null hypothesis is rejected at the 5% significance level.

From these results it is evident how quantile models produce better forecasts. This may be due to several reasons: firstly, the quantile regression approach is free of any restrictive hypothesis on the error term distribution. Secondly, this procedure is a more robust tool with respect to the tails and outliers of the data. In the end, CAViaR and L-ARCH/L-ARCH-MIDAS directly model the quantiles of the return distributions that, in a quantile estimation problem, may be more reasonable. We also consider the Expected Shortfall. After a sanity check, where we control if the ES is always less than VaR at each time t, we verify if the martingale difference property is satisfied or not. As observed in ES columns, this happens for all series and for each model in both confidence levels, since the values are approximately equal to zero. This suggest that the distribution of ES violations has zero mean. In brackets we report the number of total ES violations. It is worth noting that, in parametric approach, when we assume a Student-t distribution for the innovation term, the ES violations result to be less than those observed for the normal one. It can be due to the presence of fat tails typical of the Student-t, suggesting that extreme events are more likely than those expected assuming a Gaussian distribution.

6 Model Confidence Set procedure

After the backtesting technique, which scrutinise the model performances in terms of accuracy and efficiency, the successive step of our analysis is the *Model Confidence Set* (*MCS*) procedure. It starts from an initial set M_0 of m competing models (in our case it is composed by all the models considered in the backtesting procedure) and results in a (hopefully) smaller set of superior models (i.e. SSM) denoted by $M^*_{1-\alpha}$. The best scenario occurs when the final set consists of a single model. Formally, the MCS procedure relies on a sequence of statistical tests, which are based on the so called asymmetric VaR loss function. Two different test statistics are taken into account in order to test the null hypothesis of *Equal Predictive Ability* (*EPA*)[11]: the T_R statistics, based on a variable $\bar{d}_{i,j}$, $i, j \in M_0$, that allows us to measure the sample loss difference between model i and model j, and T_{MAX}, based on $\bar{d}_{i,\cdot}$, that is a measure of model i's sample loss with respect to that across all models loss average. Since we are interested in comparing model by model, we choose to consider only the T_R statistics in our analysis.

We implement the MCS procedure for both confidence level of 95% and 99%. In order to have a wider H_0 acceptance region, we consider a significance level of 0.01% for T_R statistics. This lets EPA hypothesis to be accepted more often when the algorithm compare the models. In this way, having a larger SSM, we are able to rank the models from the best to the worst.

Despite the low significance level, some models are rejected due to their very worst ability to predict VaR; these models are reported in tables 6, 7 and 8, containing the MCS result, with ”-”. Models rejected at both confidence levels are reported in the bottom of the same tables. In the first column of each table we can find the models belonging to the set $M^*_{1-\alpha}$ of SSM; the second and the fifth column (denoted by Rank_R) contain the ranking; the third and the sixth column (denoted by v_R) indicate the value of T_R statistic; finally, the fourth and the seventh column report the p-value for the EPA hypothesis.

By observing data, we can notice that, the parametric approach for PFE series works well as high-

[11] When the null hypothesis is accepted, it means that two models have the same ability to predict the VaR

lighted in the backtesting table: the DAGM-N-Skew-*CSI* model results to be the best one both for 95% and 99%; however, it does not pass the backtesting at 99%. This has not to surprise us, since the two procedures of backtesting and MCS work in different ways[12]. The DAGM models with a Student-t distribution for PFE occupy the last positions of the ranking in both confidence levels. However, we can conclude that in general, for PFE, the parametric approach seems to work better than the semi-parametric and non-parametric ones, in which some models are even discarded.

Model Confidence Set Procedure

Confidence level (VaR)	95%			99%		
PFE	Rank_R	v_R	MCS_R	Rank_R	v_R	MCS_R
parametric						
GARCH-N	12	1.4318	0.9700	13	1.8406	0.8060
GARCH-T	10	1.2128	0.9950	15	1.8850	0.7830
GJR-GARCH-N	4	0.7549	1.0000	12.	1.8183	0.8340
GJR-GARCH-T	3	0.4641	1.0000	2	0.0322	1.0000
GM-N-noSkew-*CSI*	9	1.2071	0.9950	9	1.3073	0.9880
GM-N-Skew-*CSI*	6	0.9210	1.0000	6	0.9247	0.9990
DAGM-N-noSkew-*CSI*	15	1.6687	0.9160	3	0.5261	1.0000
DAGM-N-Skew-*CSI*	1	-0.4377	1.0000	1	-0.0322	1.0000
DAGM-N-noSkew-*EPUI*	14	1.5650	0.9420	10	1.4337	0.9740
DAGM-N-Skew-*EPUI*	5	0.8775	1.0000	16	1.8914	0.7780
DAGM-N-noSkew-*EMVT*	13	1.4433	0.9690	5	0.9177	1.0000
DAGM-N-Skew-*EMVT*	2	0.4377	1.0000	4	0.6399	1.0000
DAGM-T-noSkew-*CSI*	23	3.9790	0.0090	22	4.0140	0.0140
DAGM-T-Skew-*CSI*	24	4.2624	0.0050	17	2.0936	0.6330
DAGM-T-noSkew-*EMVT*	25	4.3933	0.0040	23	5.6326	0.0000
semi-parametric						
CAViaR-IG	7	0.9660	1.0000	14	1.8575	0.8060
CAViaR-SAV	11	1.2418	0.9920	7	0.9317	0.9990
CAViaR-AS	8	1.1997	0.9990	8	0.9321	0.9990
L-ARCH	19	2.5198	0.3520	11	1.7987	0.8340
L-ARCH-MIDAS-*CSI*	16	2.1061	0.6470	-	-	-
L-ARCH-MIDAS-*EPUI*	17	2.3629	0.4640	-	-	-
L-ARCH-MIDAS-*EMVT*	18	2.4886	0.3680	-	-	-
non-parametric						
HS (w = 125)	20	3.0121	0.1260	19	2.6121	0.2740
HS (w = 250)	21	3.0732	0.1090	18	2.5035	0.3430
HS (w = 375)	22	3.2467	0.0740	20	2.7903	0.1870
HS (w = 500)	-	-	-	21	2.9626	0.1330
Rejected models:	NONE					

Table 6: Superior set of models provided by the MCS procedure. Significance level for the EPA hypothesis 0.01.

A different situation regards MRK, where the semi-parametric approach contains the same best model both for 95% and 99%: being this the L-ARCH-MIDAS-*CSI* model, we can state that the *Consume Sentiment Index* helps us to predict well the VaR. Despite the L-ARCH-MIDAS-*EPUI* covers the second position at 99% of confidence level, it is discarded at 95%. The same anomaly can

[12] According to us, we prefer to rely on MCS procedure, being this based on the difference between the actual loss and the expected one among models; despite this, we also take into account the backtesting procedure to check violation.

be observed for the CAViaR-IG that represents the seventh best model at 95%, but it ranks in last positions at 99%. This behaviour can be associated to the fact that, being these quantile models, fat tails can affect strongly the results. As seen before in the table 4 of backtesting, the parametric approach does not work well as the semi-parametric approach; the models are spread on all the ranking, covering position between the fifth and the twenty-third. Historical Simulations occupy the bottom of the ranking.
We can find a similar situation for JNJ financial series; the semi-parametric models cover the first nine positions considering both the confidence level: even if the best model at 99% is the semi-parametric CAViaR-SAV, at 95% the best model results to be the parametric GJR-GARCH-T. It is worth noting that, while the L-ARCH-MIDAS-*EPUI* at 95% pass the backtesting, it is discarded as model in the MCS procedure at the same level of confidence. Considering the parametric approach, despite some models cover the first positions, some others fall into the second half of the ranking, even covering the last position.

Model Confidence Set Procedure

Confidence level (VaR)	95%			99%		
MRK	Rank_R	v_R	MCS_R	Rank_R	v_R	MCS_R
parametric						
GARCH-N	16	3.3488	0.0450	11	1.9364	0.7540
GARCH-T	9	2.2402	0.5270	7	1.6543	0.9140
GJR-GARCH-N	13	2.4848	0.3780	10	1.8564	0.8570
GJR-GARCH-T	5	1.1986	0.9980	6	1.4894	0.9640
GM-N-noSkew-*CSI*	12	2.4406	0.3780	15	2.0953	0.6430
GM-N-Skew-*CSI*	8	1.9068	0.7670	8	1.7422	0.8840
GM-N-Skew-*EMVT*	14	2.8827	0.1720	5	1.4708	0.9640
GM-T-noSkew-*CSI*	22	6.6529	0.0000	23	5.3924	0.0000
DAGM-N-noSkew-*CSI*	10	2.3543	0.5270	14	2.0630	0.7280
DAGM-N-Skew-*CSI*	6	1.8362	0.9980	9	1.7993	0.8570
DAGM-N-Skew-*EPUI*	11	2.4238	0.3870	12	1.9420	0.7500
DAGM-T-noSkew-*CSI*	23	6.6542	0.0000	24	5.4964	0.0000
DAGM-T-Skew-*CSI*	20	4.8851	0.0010	21	4.7473	0.0010
DAGM-T-Skew-*EPUI*	21	5.0186	0.0000	22	4.9201	0.0010
semi-parametric						
CAViaR-IG	7	1.8720	0.7900	20	3.8327	0.0170
CAViaR-SAV	2	0.6828	1.0000	4	1.3401	0.9830
CAViaR-AS	3	0.6830	1.0000	3	1.3400	0.9830
L-ARCH	4	0.9764	0.9980	13	1.9711	0.7280
L-ARCH-MIDAS-*CSI*	1	-0.6828	1.0000	1	-1.1650	1.0000
L-ARCH-MIDAS-*EPUI*	-	-	-	2	1.1650	1.0000
non-parametric						
HS (w = 125)	15	3.0911	0.1060	16	2.7208	0.2210
HS (w = 250)	17	3.3743	0.0410	17	2.8747	0.1680
HS (w = 375)	19	3.5250	0.0270	18	2.9265	0.1510
HS (w = 500)	18	3.4036	0.0360	19	3.0371	0.1120
Rejected models:	NONE					

Table 7: Superior set of models provided by the MCS procedure. Significance level for the EPA hypothesis 0.01.

As always, Historical Simulations do not perform well in estimating VaR. Model Confidence Set procedure confirms what seen in the backtesting, with some exceptions: quantile models seem to have best performances in both confidence levels. Non-parametric models remains the worse ones.

Model Confidence Set Procedure

Confidence level (VaR)	95%			99%		
JNJ	Rank_R	v_R	MCS_R	Rank_R	v_R	MCS_R
parametric						
GARCH-N	20	2.7982	0.2160	14	2.1642	0.5840
GARCH-T	13	1.9213	0.7750	12	1.7078	0.8720
GJR-GARCH-N	15	2.0414	0.7750	9	1.3390	0.9950
GJR-GARCH-T	1	-0.4089	1.0000	4	0.3699	1.0000
GM-N-noSkew-*CSI*	18	2.4501	0.4160	15	2.1871	0.5570
GM-N-Skew-*CSI*	11	1.7902	0.9680	8	1.2519	0.9950
GM-N-noSkew-*EPUI*	16	2.2576	0.5460	11	1.6782	0.8860
GM-N-Skew-*EPUI*	3	0.5882	1.0000	7	1.1560	0.9990
GM-T-noSkew-*CSI*	25	5.8036	0.0000	23	3.4350	0.0330
GM-T-Skew-*CSI*	23	5.2205	0.0010	18	2.8064	0.1780
GM-T-Skew-*EMVT*	24	5.4997	0.0000	17	2.5695	0.2840
DAGM-N-noSkew-*CSI*	17	2.4109	0.5460	16	2.2491	0.5090
DAGM-N-Skew-*CSI*	10	1.7736	0.9680	10	1.4705	0.9560
DAGM-N-Skew-*EPUI*	2	0.4089	1.0000	5	0.8425	1.0000
DAGM-N-Skew-*EMVT*	12	1.8090	0.9570	13	1.7631	0.8390
DAGM-T-Skew-*EMVT*	14	1.9855	0.7750	-	-	-
semi-parametric						
CAViaR-IG	4	0.8026	1.0000	6	0.9819	1.0000
CAViaR-SAV	5	0.8059	1.0000	1	-0.0211	1.0000
CAViaR-AS	9	1.5753	0.9680	2	0.0211	1.0000
L-ARCH	7	1.3032	0.9980	3	0.2497	1.0000
L-ARCH-MIDAS-*CSI*	8	1.5502	0.9980	-	-	-
L-ARCH-MIDAS-*EMVT*	6	1.1980	1.0000	-	-	-
non-parametric						
HS (w = 125)	-	-	-	22	3.1490	0.0730
HS (w = 250)	21	2.8352	0.1990	21	2.9448	0.1190
HS (w = 375)	22	2.8943	0.1840	20	2.9347	0.1260
HS (w = 500)	19	2.7778	0.2210	19	2.9208	0.1310
Rejected models:	L-ARCH-MIDAS-*EPUI*					

Table 8: Superior set of models provided by the MCS procedure. Significance level for the EPA hypothesis 0.01.

7 Copulas

In this section we focus on the joint behaviour of the considered stocks. To achieve this aim, we use an important probabilistic tool, the *copula*, which allows us to inspect the dependence among the losses. From a statistical point of view, a copula is a multivariate cumulative distribution function for a random vector, for which the marginal probability distribution of each variable is a standard uniform one. This type of analysis is performed on residuals[13] obtained by modelling the conditional variance of the returns with a specific model. The choice falls on GJR-GARCH-T since the backtesting and MCS procedures show a good performance of the model in terms of VaR predictability. Once extrapolated residuals, we evaluate pseudo-observation on them in order to have standard uniform realisations to calibrate the copula. In figure 5 we report the plot, containing the standard uniform realisations, useful to visualise the dependence in multivariate data: it shows a positive dependence among our three financial series; in particular it would seem that PFE and MRK

[13]We use residuals instead of returns since the former represents the stochastic part of the GARCH class models, whereas the latter represents the deterministic part.

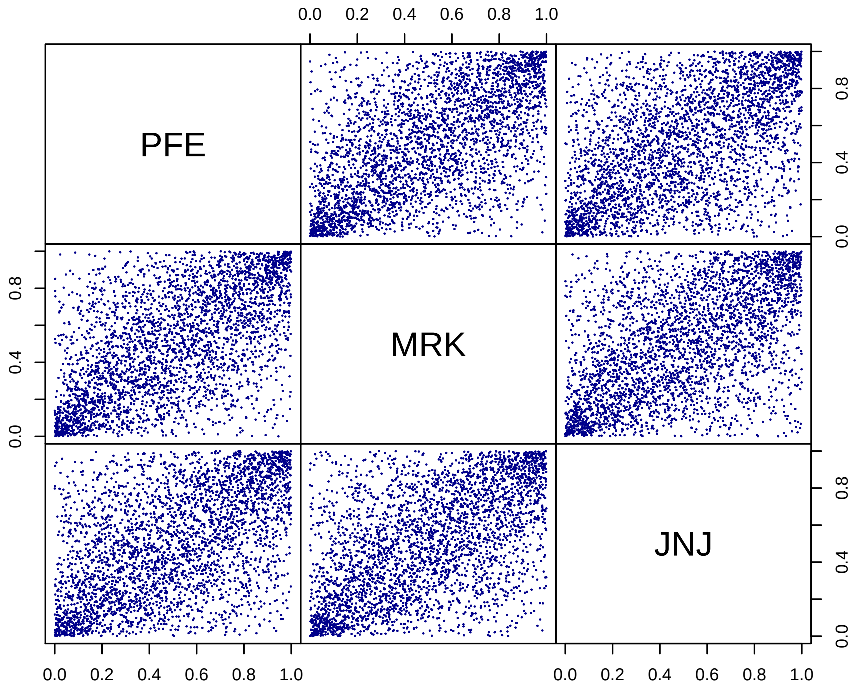


Figure 5: Pseudo-observations' paired plot.

have more positive dependence than the other combinations, due to their less dispersion.

Different types of copulas are taken into account: *normal-copula*, *t-copula*, *Gumbel-copula*,*Clayton-copula* and*Frank-copula*. The first two belong to *elliptical copulas*, which are implicit distribution function, since they are extrapolated by elliptical multivariate distribution. Instead, the last copulas are known as *Archimedean copulas*, namely copulas for which no assumption on distributions is taken into account; for this reason they are also called explicit. Their strength lies in the possibility to model the dependence among random variable with only one parameter.

copulas:	Normal	t	Gumbel	Clayton	Frank
likelihoods	1410.37	1513.82	1260.62	1153.03	1359.80
AIC	-2814.75	-3019.63	-2519.23	-2304.06	-2717.59

Table 9: Selection methods.

In table 9 we report the values of likelihoods and Akaike Information Criterion (AIC) to understand what is the best copula. If we look at data, the best copula results to be the t-copula, taking the minimum value for the AIC and the maximum one for the likelihoods.

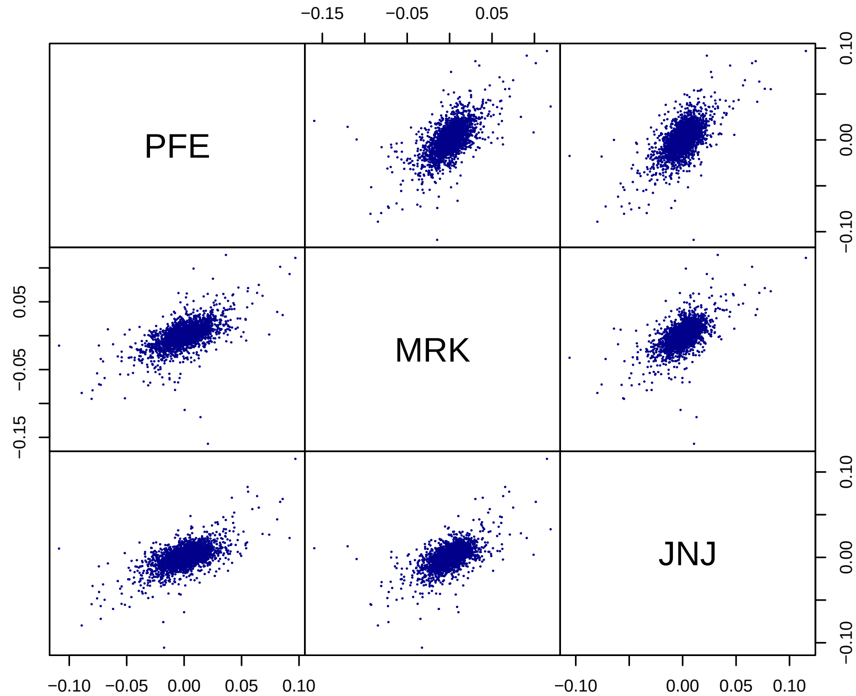


Figure 6: Returns' paired plot.

The result suggested by the table 9 is not striking since, as shown in figure 6, that reports the joint pair of returns for each considered stock, data arrange themselves on an ellipse. This observation suggests us that they could have an elliptical copula.

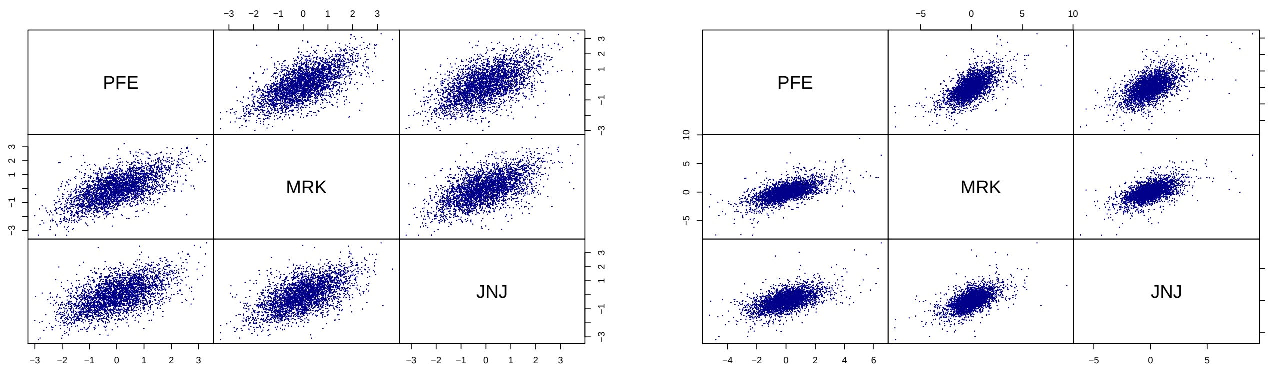
	t-Copula coeff.	**Kendall's tau**	**Lower cond. tail dep.**
PFE-MRK	0.6103	0.4179	0.1802
PFE-JNJ	0.5538	0.3737	0.1480
MRK-JNJ	0.5719	0.3876	0.1576
	df = 7.7649		

Table 10: Dependence measures.

After the choice of the t-copula, whose parameters are reported in the first column of table 10, we decide to evaluate further measures of dependence: Kendall's tau and the lower conditional tail dependence[14], reported in the second and in the third column of same table. Kendall's tau suggests what initially said, namely the existence of a positive dependence. Lower conditional tail dependence tells us that there is a slight dependence in the left tail of distribution for all the considered pairs. The next step of the analysis is to model the marginals. To do this, we proceed in simulating a copula by considering the parameter of the t-copula estimated ex-ante and applying the quantile

[14]Working on Profit & Loss, we consider the left tail of distribution to catch the dependence among losses.

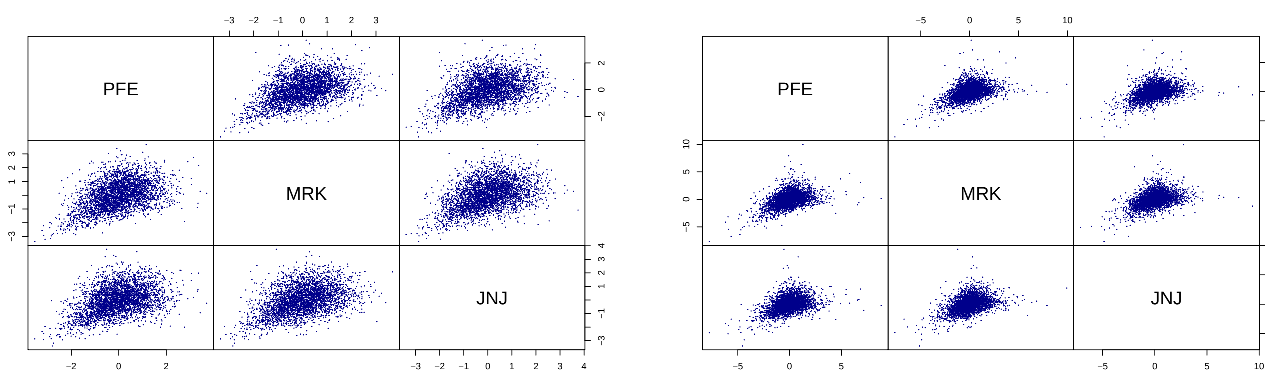
transformation on simulated data. Such transformation concerns two distribution: normal and Student-t.



(a) (b)

Figure 7: Plot (a) Paired plot considering a t-copula with normal distribution for the marginals; Plot (b) Paired plot considering a t-copula with Student-t distribution for the marginals.

Looking at the figure 7(b), we can note that Student-t distribution for the marginals better describes the dispersion of data shown in figure 6; indeed, comparing with figure 7(a), that considers a normal distribution for marginals, data seem more concentrated in the middle, with few values far from the main cloud of points.



(a) (b)

Figure 8: Plot (a) Paired plot considering a Clayton-copula with normal distribution for the marginals; Plot (b) Paired plot considering a Clayton-copula with Student-t distribution for the marginals.

For comparison purposes, plotting the Clayton-copula, the worst one from table 10, immediately catches the eyes that the cloud of points does not draw an ellipse both for the normal and the Student-t distribution for marginals, but rather it has few realizations in the bottom left part of the graph. Much more realizations are concentrated in the middle.

Considering the aggregated loss associated to our three financial series, the last step concerns the use of the t-copula with a Student-t distribution for marginals to simulate the future aggregate loss. In order to do this we split the series in two parts: the in-sample (from January 03, 2008 to December 31, 2019)[15] and the out-of sample one (from January, 2020 to January, 2021). Exploiting the t-copula with the Student-t marginals, we obtain the residuals, then used together with the GJR-GARCH coefficient (estimated previously) to simulate 350 paths of the loss distribution. After aggregating them, we compute the VaR and the ES at 95% and 99% confidence level. All these results are reported in figures 9 and 10: the yellow and the red lines represent the VaR and the ES respectively; the grey area, delimited by blue line, reports the confidence interval at 95% for the simulated loss; finally, the in-sample part is divided from the out-of-sample one by a vertical line.

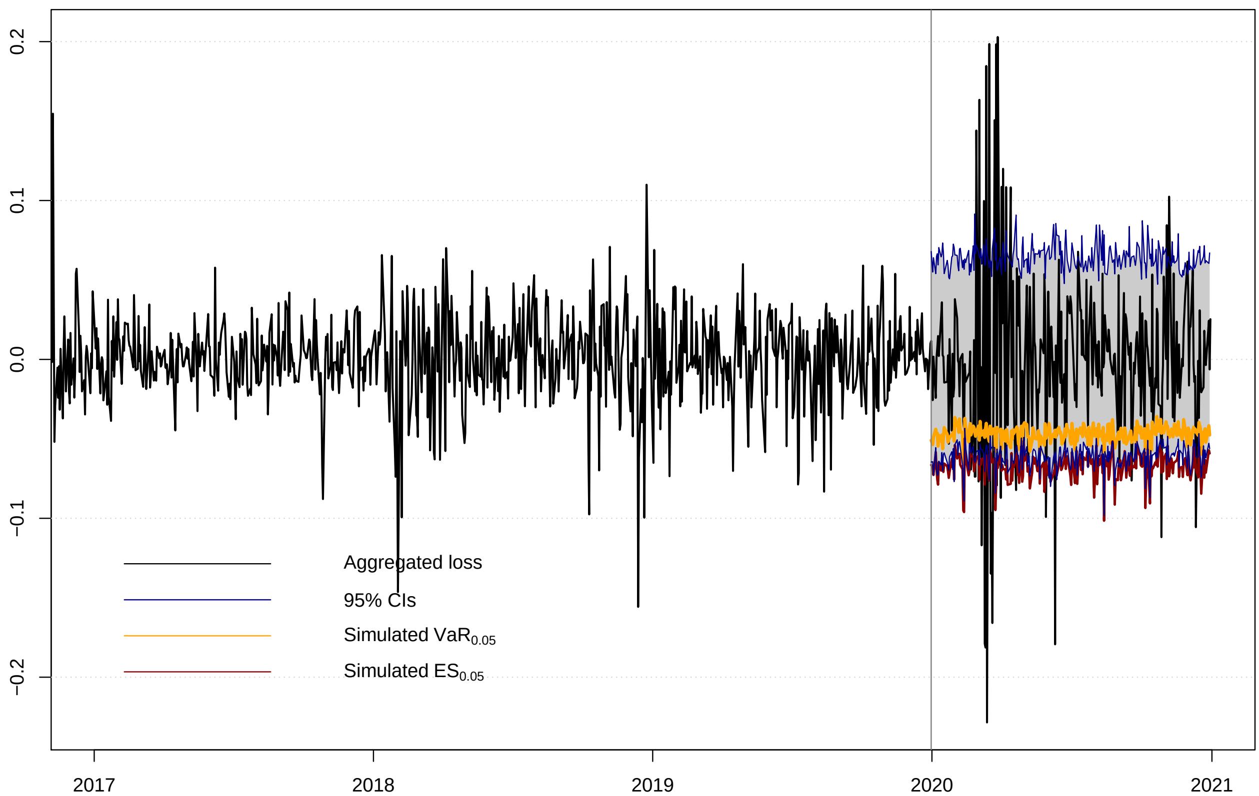


Figure 9: Simulated aggregated VaR and ES at confidence level of 0.95

[15]In figures 9 and 10 only the last three years of in-sample part are reported.

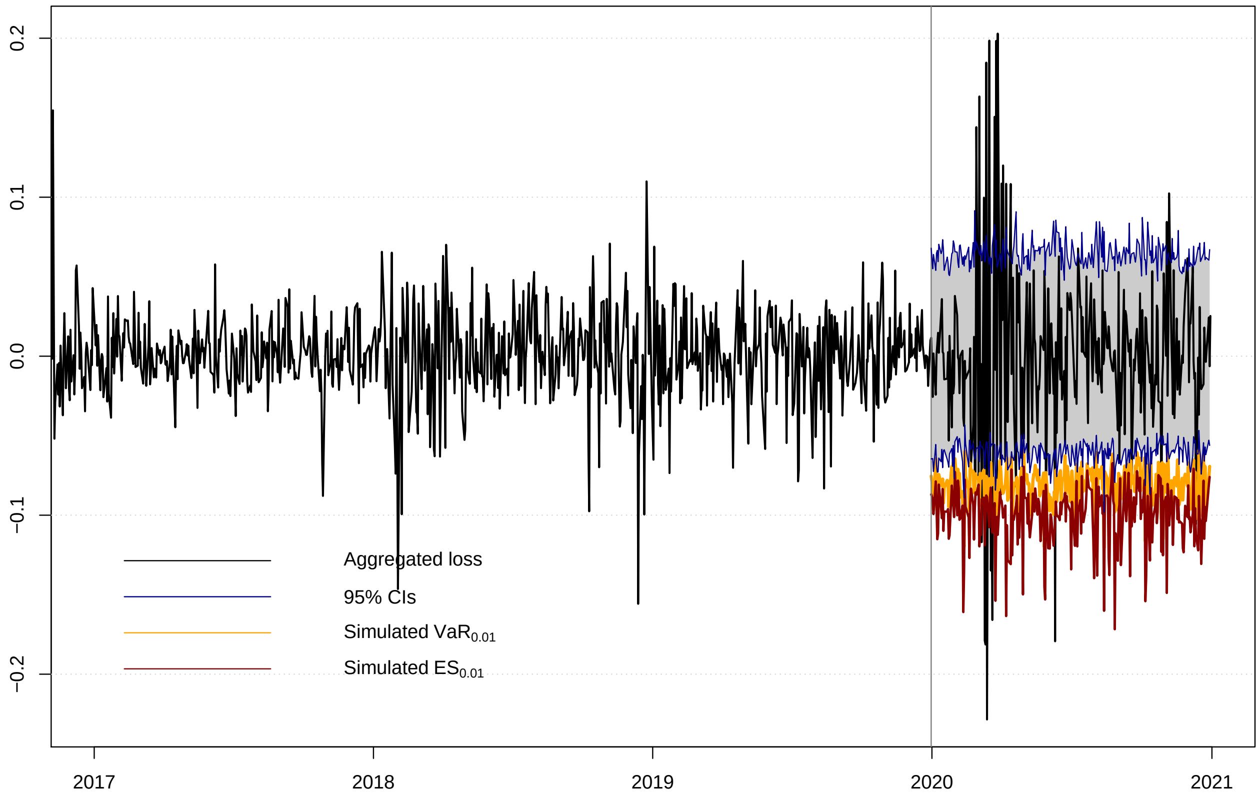


Figure 10: Simulated aggregated VaR and ES at confidence level of 0.99

Looking at the figures, we can notice that the model used has not been able to predict the period of high volatility linked to COVID-19 disease. Table 11 shows the backtesting procedure implemented on the aggregated VaR and ES for both confidence levels. As highlighted by the graphs and confirmed by the table, the method used to predict VaR and ES seems to be not correct: the null hypothesis of POF and Conditional Coverage test are not accepted; the AE ratio reports more actual violations than the expected ones, especially at 99%. It is difficult to state if the martingale difference is satisfied, due to the small size of the out-of-sample data.

Confidence level	95%				99%			
	LR_{uc}	LR_{cc}	AE	ES (viol)	LR_{uc}	LR_{cc}	AE	ES (viol)
Copula	14.9290*	15.2211*	2.2222	0.0706 (19)	25.5909*	30.0409*	5.5556	0.1022 (11)

Table 11: The critical values of LR_{uc} and LR_{cc}, at $\alpha = 0.05$, are 3.841 and 5.991 respectively. The star (*) denotes that the null hypothesis is rejected at the 5% significance level.

8 Conclusions

Taking into account PFE, MRK and JNJ, the parametric, semi-parametric and non-parametric approaches help to correctly foresee the VaR and the ES in the univariate field. The results are supported by backtesting and MCS procedures. In general the best approach results to be the parametric one due to its features. Passing to the multivariate field, copula is a good tool to study the dependence within a stock portfolio. By the way, the method used to predict the aggregated VaR and ES does not work well, in particular for high and unpredictable volatility periods, as well as the COVID-19 one.

Appendix

A Formulas

The appendix reports the formulas of the main models used. The General Conditional Heteroskedastic framework is define as:

$$r_t = \mu_t + \sigma_t z_t$$

where μ_t is the conditional mean, z_t is a sequence of i.i.d random variables with zero mean and unit variance and σ_t is the square root of the conditional volatility. Since we are interested in modelling this latter, the following models are used:

GARCH(1,1): $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2$

where ω, α and β are the parameters that need to be estimated. To ensure the positiveness of σ_t^2, it must hold $\omega > 0$, α_1, $\beta_1 \geq 0$.

GJR-GARCH(1,1): $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma r_{t-1}^2 I_{\{r_{t-1}<0\}}$

with $\omega > 0$, α, $\beta \geq 0$ and $(\alpha + \gamma) \geq 0$ in order to assure that $\sigma_t^2 > 0$.

GARCH MIDAS: $r_{it} = \sqrt{\tau_t g_{it}} z_{it} \quad i = 1, ..., N_t$

where r_{it} represents the log-return for day i of the long period t; N_t is the number of days in period t; $z_{it}|\mathcal{F}_{i-1,t} \sim \mathcal{N}(0, 1)$, where $\mathcal{F}_{i-1,t}$ denotes the information set up to day $i-1$ of period t. g_{it} is the short-run component and follows a unit-mean reverting GARCH(1,1) process. Its formula is:

$$g_{it} = (1 - \alpha - \beta) + \alpha \frac{r_{i-1,t}^2}{\tau_t} + \beta g_{i-1,t} \tag{1}$$

In case we are interested in modelling the possible asymmetric response of conditional variance to negative past returns, the short-run component becomes:

$$g_{it} = (1 - \alpha - \beta - \gamma\backslash 2) + (\alpha + \gamma I_{r_{i-1,t}<0}) \frac{r_{\{i-1,t\}}^2}{\tau_t} + \beta g_{i-1,t} \tag{2}$$

where $I_{\{.\}}$ is the indicator function and γ is the parameter for the skew effect. In the backtesting and MCS tables, when *noSkew* appears, it means that we are using formula 1 as short-run component; when we have *Skew*, the considered short-run component is given by formula 2.
τ_t represents the long-run component and provides the slow moving average level of volatility. It is given by

$$\tau_t = exp\Big\{ m + \theta \sum_{k=1}^{K} \delta_k(\omega) X_{t-k} \Big\}$$

where m plays the role of an intercept; θ represents the coefficient of interest associated to the Macroeconomic Variable; $\delta_k(\omega)$ is a proper function weighting the past K realizations of X_t.

DOUBLE ASYMMETRIC GARCH MIDAS: $r_{it} = \sqrt{\tau_t g_{it}} z_{it} \quad i = 1, ..., N_t$

where the short run component g_{it} has the same structure seen before.
Whereas, the long-run component is defined by:

$$\tau_t = exp\Big\{m + \theta^+ \sum_{k=1}^{K} \delta_k(\omega)^+ X_{t-k} I_{\{X_{t-k} \geq 0\}} + \theta^- \sum_{k=1}^{K} \delta_k(\omega)^- X_{t-k} I_{\{X_{t-k} < 0\}}\Big\}$$

where θ^+ and θ^- are associated to positive and negative Macroeconomic Variable (MV) values respectively; they evaluate separately the volatility associated with positive and negative shocks to MVs. $\delta_k^+(\omega)$ and $\delta_k^-(\omega)$ represent the different system of weights for positive and negative MVs realizations.

CAViaR-INDIRECT GARCH: $Q_{r_t}(\tau) = -\sqrt{\beta_0 + \beta_1 Q^2_{r_{t-1}}(\tau) + \beta_2 r^2_{t-1}}$

$Q_{r_t}(\tau)$ is the τ quantile referred to r_t distribution, where r_t are the returns at time t; β_i with $i = 0, 1, 2$, are the parameters need to be estimated. The same elements appear in the next two formulas.

CAViaR-SAV: $Q_{r_t}(\tau) = \beta_0 + \beta_1 Q_{r_{t-1}}(\tau) + \beta_2 |r_{t-1}|$

CAViaR-AS: $Q_{r_t}(\tau) = \beta_0 + \beta_1 Q_{r_{t-1}}(\tau) + (\beta_2 I_{\{r_{t-1} \geq 0\}} + \beta_3 I_{\{r_{t-1} < 0\}})|r_{t-1}|$

L-ARCH: $r_t = (\beta_0 + \beta_1 |r_{t-1}| + ... + \beta_q |r_{t-q}|) z_t$

where r_t is a log-return of an asset observed at time t and $z_t \sim_{iid} (0, 1)$; note that z_t is left with an unspecified distribution.

L-ARCH-MIDAS: $r_t = (\beta_0 + \beta_1 |r_{t-1}| + ... + \beta_q |r_{t-q}| + \theta \sum_{j=1}^{K} \delta_k(\omega) |MV_{t-j}|) z_t$

where the coefficient θ signals the impact of the weighted summation of the K realizations of the additional stationary variable MV_t. this latter can be the Macroeconomic Variable driving the log-returns or a proxy of volatility at lower frequency(i.e. weekly or monthly).

References

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[3] Roccioletti S.: *Backtesting Value at Risk and Expected Shortfall*, 2015.