4

Linear Graphs

Among the graphical tools for developing and representing a model of a dynamic system, linear graphs take an important place. In particular, state-space models of lumped-parameter dynamic systems; regardless of whether they are mechanical, electrical, fluid, thermal, or multidomain (mixed) systems; can be conveniently developed by *linear graphs*. Interconnected line segments (called branches) are used in a linear graph to represent a dynamic model. The term "linear graph" stems from this use of line segments, and does not mean that the system itself has to be linear. Particular advantages of using linear graphs for model development and representation are: they allow visualization of the system structure (even before formulating an analytical model); they help identify similarities (structure, performance, etc.) in different types of systems; they are applicable for multidomain systems (the same approach is used in any domain); and they provide a unified approach to model multifunctional devices (e.g., a piezoelectric device which can function as both a sensors and an actuator). This chapter presents the use of linear graphs in the development of analytical models for mechanical, electrical, fluid, and thermal systems.

4.1 Variables and Sign Convention

Linear graphs systematically use through variables and across variables in providing a unified approach for the modeling of dynamic systems in multiple domains (mechanical, electrical, fluid, thermal). In accomplishing this objective it is important to adhere to standard and uniform conventions so that that there will not be ambiguities in a given linear graph representation. In particular, a standard sign convention must be established. These issues are discussed in this section.

4.1.1 Through Variables and Across Variables

Each branch in the linear graph model has one *through variable* and one *across variable* associated with it. Their product is the power variable. For instance, in a hydraulic or pneumatic system, a pressure "across" an element causes some change of fluid flow "through" the element. The across variable is the pressure, the through variable is the flow. Table 4.1 lists the through and across variable pairs for the four domains considered in the present treatment.

4.1.2 Sign Convention

Consider Figure 4.1 where a general basic element (strictly, a single-port element, as discussed later) of a dynamic system is shown. In the linear graph representation, as shown in Figure 4.1b, the element is shown as a branch (i.e., a line segment). One end of any branch is

TABLE 4.1

Through and Across Variable Pairs in Several Domains

System Type (domain)	Through Variable	Across Variable
Hydraulic/pneumatic	Flow rate	Pressure
Electrical	Current	Voltage
Mechanical	Force/torque	Velocity/angular velocity
Thermal	Heat transfer	Temperature

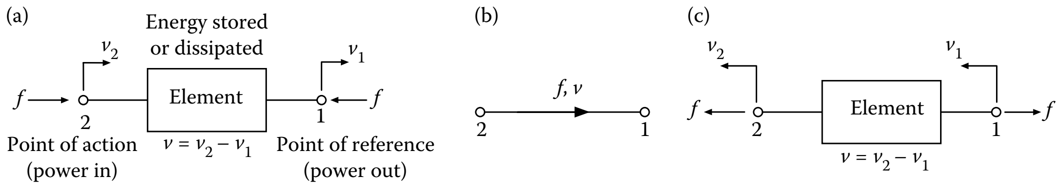

FIGURE 4.1
Sign convention for a linear graph. (a) A basic element and positive directions of its variables. (b) Linear graph branch of the element. (c) An alternative sign convention.

selected as the *point of reference* and the other end automatically becomes the *point of action* (see Figure 4.1a and c). The choice is somewhat arbitrary, and may reflect the physics of the actual system. An *oriented* branch is one to which a direction is assigned, using an arrowhead, as in Figure 4.1b. The arrow head denotes the positive direction of power flow at each end of the element. By convention, the positive direction of power is taken as "into" the element at the point of action, and "out of" the element at the point of reference. According to this convention, the arrowhead of a branch is always pointed toward the point of reference. In this manner the reference point and the action point are easily identified.

The across variable is always given relative to the point of reference. It is also convenient to give the through variable f and the across variable v as an ordered pair (f v) on one side of the branch, as in Figure 4.1b. Clearly, the relationship between f and v (the constitutive relation or physical relation, as discussed in Chapters 2 and 3) can be linear or nonlinear. The parameter of the element (e.g., mass, capacitance) is shown on the other side of the branch. It should be noted that the direction of a branch does not represent the positive direction of f or v. For example, when the positive directions of both f and v are changed, as in Figure 4.1a and c, the linear graph remains unchanged, as in Figure 4.1b, because the positive direction of power flow remains the same. In a given problem, the positive direction of any one of the two variables f and v should be preestablished for each branch. Then the corresponding positive direction of the other variable is automatically determined by the convention used to orient linear graphs. It is customary to assign the same positive direction for f (and v) and the power flow at the point of action (i.e., the convention shown in Figure 4.1a is customary, not Figure 4.1c). Then the positive directions of the variables at the point of reference are automatically established.

Note that in a branch (line segment), the through variable (f) is transmitted through the element with no change in value; it is the "through" variable. The absolute value of the across variable, however, changes across the element (from v_2 to v_1, in Figure 4.1a). In fact, it is this change ($v = v_2 - v_1$) across the element that is called the across variable. For example, v_2 and v_1 may represent electric potentials at the two ends of an electric element (e.g., a resistor) and then v represents the voltage across the element. According, the across variable, is measured relative to the point of reference of the particular element.

According to the sign convention shown in Figure 4.1, the work done on the element at the point of action (by an external device) is positive (i.e., power flows in), and work done by the element at the point of reference (on an external load or environment) is positive (i.e., power flows out). The difference in the work done on the element and the work done by the element (i.e., the difference in the work flow at the point of action and the point of reference) is either stored as energy (e.g., kinetic energy of a mass; potential energy of a spring; electrostatic energy of a capacitor; electromagnetic energy of an inductor), which has the capacity to do additional work; or dissipated (e.g., mechanical damper; electrical resistor) through various mechanisms manifested as heat transfer, noise, and other phenomena.

In summary:

1. An element (a single-port element) is represented by a line segment (branch). One end is the point of action and the other end is the point of reference.
2. The through variable f is the same at the point of action and the point of reference of an element; the across variable differs, and it is this difference (value relative to the point of reference) that is called the across variable v.
3. The variable pair (f, v) of the element is shown on one side of the branch. Their relationship (constitutive relation) can be linear or nonlinear. The parameter of the element is shown on the other side of the branch.
4. Power flow p is the product of the through variable and the across variable. By convention, at the point of action, f and p are taken to be positive in the same direction; at the point of reference, f is positive in the opposite direction.
5. The positive direction of power flow p (or energy or work) is into the element at the point of action; and out of the element at the point of reference. This direction is shown by an arrow on the linear graph branch (an oriented branch).
6. The difference in the energy flows at the two ends of the element is either stored (with capacity to do further work) or dissipated, depending on the element type.

Linear graph representation is particularly useful in understanding rates of energy transfer (power) associated with various phenomena, and dynamic interactions in a physical system (mechanical, electrical, fluid, etc.) can be interpreted in terms of power transfer. Power is the product of a through variable (a generalized force or current) and the corresponding across variable (a generalized velocity or voltage). For example, consider a mechanical system. The total work done on the system is, in part, used as stored energy (kinetic and potential) and the remainder is dissipated. Stored energy can be completely recovered when the system is brought back to its original state (i.e., when the cycle is completed). Such a process is *reversible.* On the other hand, dissipation corresponds to irreversible energy transfer that cannot be recovered by returning the system to its initial state. (A fraction of the mechanical energy lost in this manner could be recovered, in principle, by operating a heat engine, but we shall not go into these thermodynamic details which are beyond the present scope). Energy dissipation may appear in many forms including temperature rise (a molecular phenomenon), noise (an acoustic phenomenon), or work used up in wear mechanisms.

4.2 Linear Graph Elements

Many types of basic elements exist, which can be used in the development of a linear graph for a dynamic system. In this section we will discuss two types of basic elements in the categories of single-port elements and two-port elements. Analogous elements in these categories exist across the domains (mechanical, electrical, fluid, and thermal) for the most part.

4.2.1 Single-Port Elements

Single-port (or, *single energy port)* elements are those that can be represented by a single branch (single line segment) of linear graph. These elements possess only one power (or energy) variable; hence the name "single-port." They have two terminals. The general form of these elements is shown in Figure 4.1b.

In modeling mechanical systems we require three passive single-port elements, as shown in Figure 4.2. These lumped-parameter mechanical elements are mass (or inertia), spring, and dashpot/damper. Although translatory mechanical elements are presented in Figure 4.2, corresponding rotary elements are easy to visualize: f denotes an applied torque and v the relative angular velocity in the same direction. Note that the linear graph of an inertia element has a broken line segment. This is because, through an inertia, the force does not physically travel from one end of its linear graph branch to the other end, but rather the force "felt" at the two ends. This will be further discussed in Example 4.1.

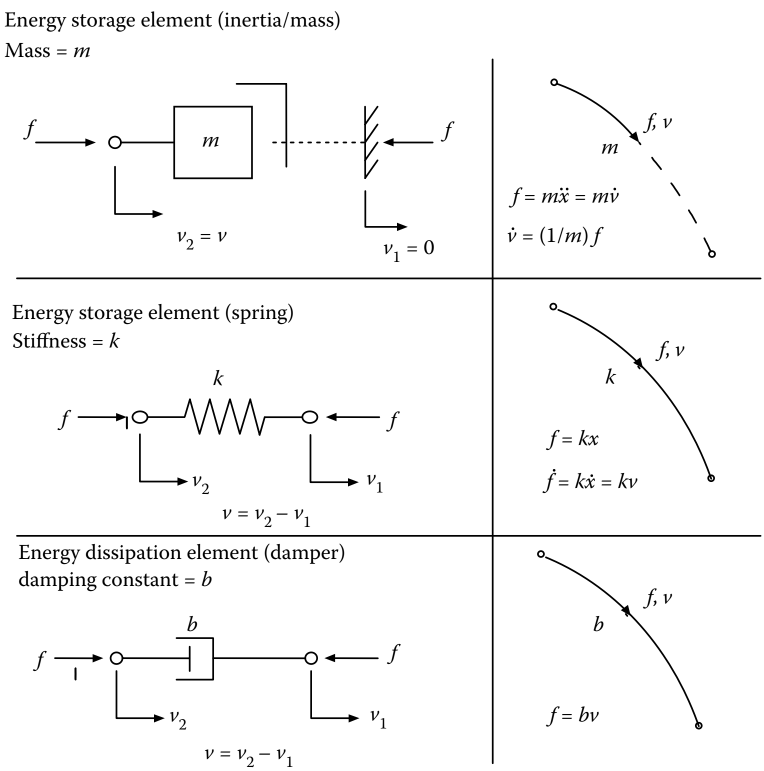

FIGURE 4.2
Single-port mechanical system elements and their linear-graph representations.

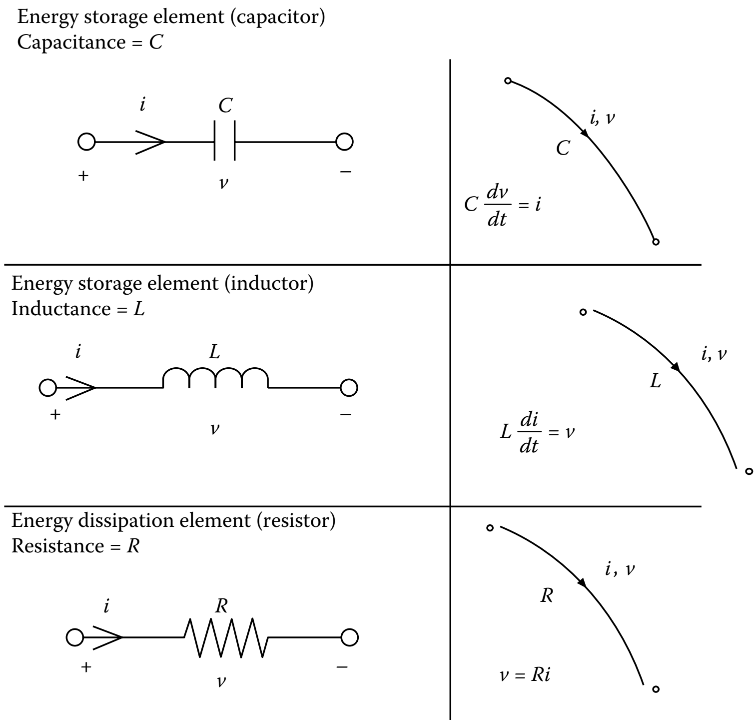

FIGURE 4.3
Single-port electrical system elements and their linear-graph representations.

Analogous single-port electrical elements may be represented in a similar manner. These are shown in Figure 4.3.

4.2.1.1 Source Elements

In linear graph models, system *inputs* are represented by *source elements*. There are two types of sources, as shown in Figure 4.4.

a. *T*-type source (e.g., force source, current source):
The independent variable (i.e., the source output, which is the system input) is the through variable f. The arrow head indicates the positive direction of f.

Note: For a *T*-type source, the sign convention that the arrow gives the positive direction of f still holds. However, the sign convention that the arrow is from the point of action to the point of reference (or the direction of the drop in the across variable) does not hold.

b. *A*-type source (e.g., velocity source, voltage source):
The independent variable is the across variable v. The arrow head indicates the positive direction of the "drop" in v. *Note*: + and − signs are also indicated, where the drop in v occurs from + to − terminals.

Note: For an *A*-type source, the sign convention that the arrow is from the point of action to the point of reference (or the direction of the drop in the across variable) holds.

However, the sign convention that the arrow gives the positive direction of f does not hold.

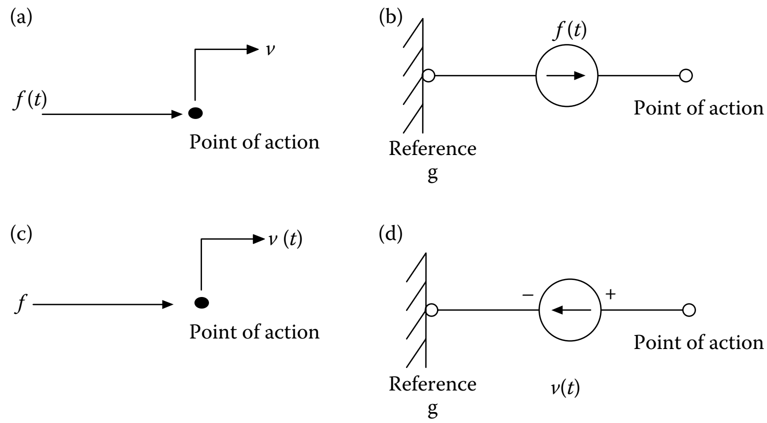

FIGURE 4.4
(a) *T*-type source (through variable input). (b) Linear graph representation of a *T*-type source. (c) *A*-type source. (d) Linear graph representation of an *A*-type source.

An ideal force source (a through-variable source) is able to supply a force input that is not affected by interactions with the rest of the system. The corresponding relative velocity across the force source, however, will vary as determined by the dynamics of the overall system. It should be clear that the direction of $f(t)$ as shown in Figure 4.4a is the applied force. The reaction on the source would be in the opposite direction. An ideal velocity source (across-variable source) supplies a velocity input independent of the system to which it is applied. The corresponding force is, of course, determined by the system dynamics.

4.2.1.2 Effects of Source Elements

We have noted that the source variable (independent variable or input variable) of a source is unaffected by the dynamics of the system to which the source is connected. But the covariable (dependent variable) will change. Another property associated with source elements is identified next.

Source elements can serve as a means of inhibiting interactions between systems. Specifically, it follows from the definition of an ideal source that the dynamic behavior of a system is not affected by connecting a new system in series with an existing *T*-type source (e.g., force source or current source) or in parallel with an existing *A*-type source (e.g., velocity source or voltage source). Conversely, then, the original system is not affected by removing the connected new system, in each case. These two situations are illustrated in Figure 4.5. In general, linking (networking) a subsystem will change the order of the overall system (because new dynamic interactions are introduced) although the two situations in Figure 4.5 are examples where this does not happen. Another way to interpret these situations is to consider the original system and the new system as two uncoupled subsystems driven by the same input source. In this sense, the order of the overall system is the sum of the order of the individual (uncoupled) subsystems.

4.2.2 Two-Port Elements

A two-port element has two energy ports and two separate, yet coupled, branches corresponding to them. These elements can be interpreted as a pair of single-port elements whose net power is zero. A transformer (mechanical, electrical, fluid, etc.) is a two-port

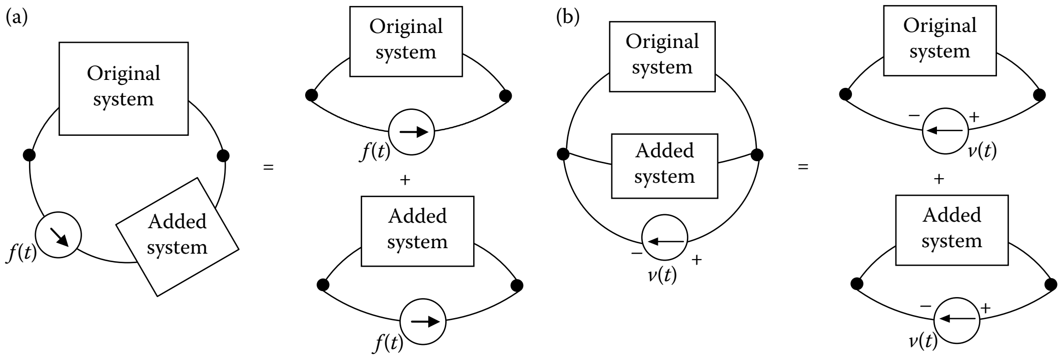

FIGURE 4.5
(a) Two systems connected in series to a *T*-type source. (b) Two systems connected in parallel to an *A*-type source.

element. Also, a mechanical gyrator is a two-port element. Examples of mechanical transformers are a lever and pulley for translatory motions and a meshed pair of gear wheels for rotation. A gyrator is typically an element that displays gyroscopic properties. We shall consider only the linear case; i.e., ideal transformers and gyrators only. The extension to the nonlinear case should be clear.

4.2.2.1 Transformer

In an ideal transformer, the across variables in the two ports (branches) are changed without dissipating or storing energy in the process. Hence the through variables in the two ports will also change. Examples of mechanical, electrical, and fluid transformers are shown in Figure 4.6a through d. The linear graph representation of a transformer is given in Figure 4.6e.

In Figure 4.6e, as for a single-port passive element, the arrows on the two branches (line segments) give the positive direction of power flow (i.e., when the product of the through variable and the across variable for that segment is positive). One of the two ports may be considered the input port and the other the output port. Let

v_i and f_i = across and through variables at the input port

v_o and f_o = across and through variables at the output port

The (linear) transformation ratio r of the transformer is given by

$$v_o = r v_i \tag{4.1}$$

Due to the conservation of power we have:

$$f_i v_i + f_o v_o = 0 \tag{4.2}$$

By substituting Equation 4.1 into Equation 4.2 gives

$$f_o = -\frac{1}{r} f_i \tag{4.3}$$

Here r is a dimensional parameter. The two constitutive relations for a transformer are given by Equations 4.1 and 4.3.

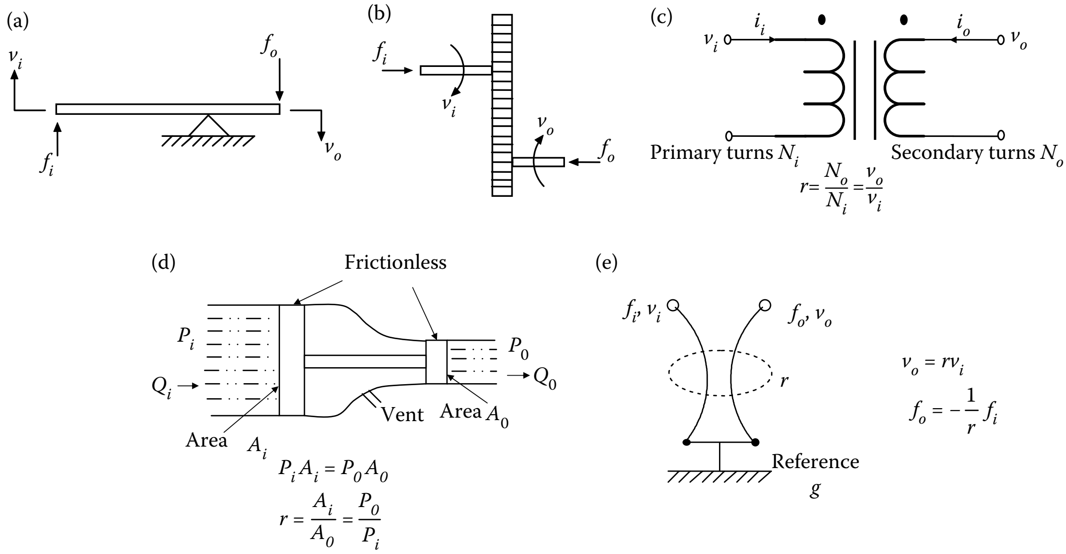

FIGURE 4.6
Transformer. (a) Lever. (b) Meshed gear wheels. (c) Electrical transformer. (d) Fluid transformer. (e) Linear graph representation.

4.2.2.2 Electrical Transformer

As shown in Figure 4.6c, an electrical transformer has a primary coil, which is energized by an ac voltage (v_i), a secondary coil in which an ac voltage (v_o) is induced, and a common core, which helps the linkage of magnetic flux between the two coils. Note that a transformer converts v_i to v_o without making use of an external power source. Hence it is a passive device, just like a capacitor, inductor, or resistor. The turn ratio of the transformer:

$$r = \frac{\text{number of turns in the secondary coil } (N_o)}{\text{number of turns in the primary coil } (N_i)}$$

In Figure 4.6c, the two dots on the top side of the two coils indicate that the two coils are wound in the same direction.

In a pure and ideal transformer, there will be full flux linkage without any dissipation of energy. Then, the flux linkage will be proportional to the number of turns. Hence

$$\lambda_o = r\lambda_i \tag{4.4}$$

where λ denotes the flux linkage in each coil. Differentiation of Equation 4.4, noting that the induced voltage in coil is given by the rate of charge of flux, gives

$$v_o = r\,v_i \tag{4.1}$$

For an *ideal transformer*, there is no energy dissipation and also the signals will be in phase. Hence, the output power will be equal to the input power; thus,

$$v_o i_o = v_i i_i \tag{4.2b}$$

Hence, the current relation becomes

$$i_o = \frac{1}{r} i_i \tag{4.3b}$$

4.2.2.3 Gyrator

An ideal gyroscope is an example of a mechanical *gyrator* (Figure 4.7a). It is simply a spinning top that rotates about its own axis at a high angular speed ω (positive in the x direction) and assumed to remain unaffected by other small motions that may be present. If the moment of inertia about this axis of rotation (x in the shown configuration) is J, the corresponding angular momentum is $h=J\omega$, and this vector is also directed in the positive x direction, as shown in Figure 4.7b.

Suppose that the angular momentum vector h is given an incremental rotation $\delta\theta$ about the positive z axis, as shown. The free end of the gyroscope will move in the positive y direction as a result. The resulting change in the angular momentum vector is $\delta h=J\omega\delta\theta$ in the positive y direction, as shown in Figure 4.7b. Hence the rate of change of angular momentum is

$$\frac{\delta h}{\delta t} = \frac{J\omega\delta\theta}{\delta t} \tag{i}$$

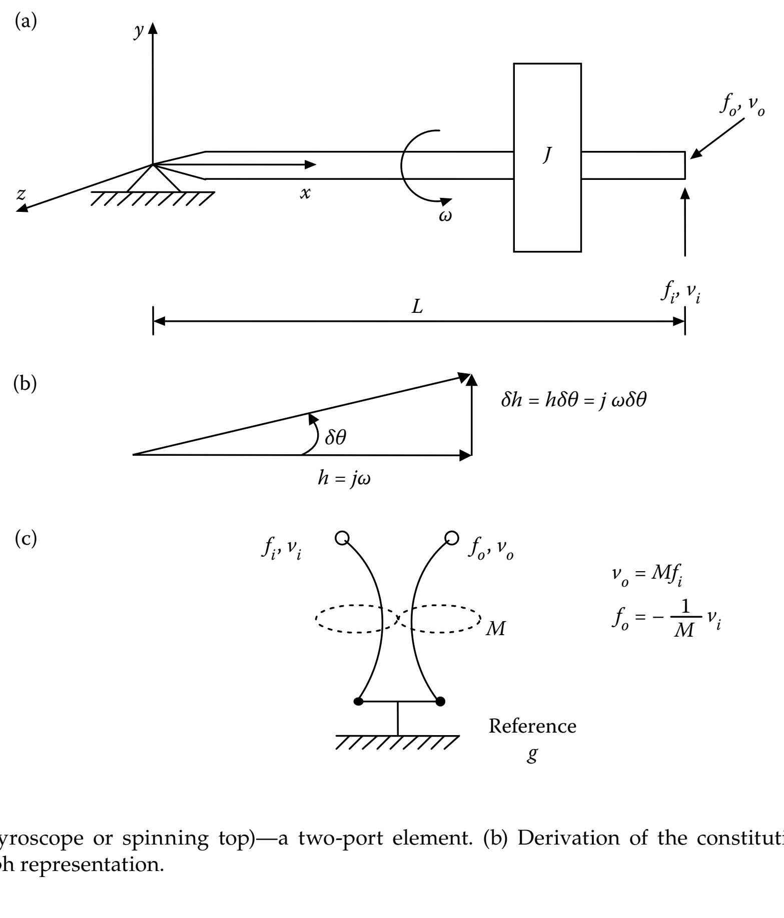

FIGURE 4.7
(a) Gyrator (gyroscope or spinning top)—a two-port element. (b) Derivation of the constitutive equations. (c) Linear-graph representation.

where δt is the time increment of the motion. Hence, in the limit, the rate of change of angular momentum is

$$\frac{dh}{dt} = J\omega\frac{d\theta}{dt} \tag{ii}$$

If the velocity given to the free end of the gyroscope, in the positive y direction, to generate this motion is v_i (which will result in a force f_i at that point, in the positive y direction) the corresponding angular velocity about the positive z axis is

$$\frac{d\theta}{dt} = \frac{v_i}{L} \tag{iii}$$

in which L is the length of the gyroscope. Substitute Equation (iii) in Equation (ii). The rate of change of angular momentum is

$$\frac{dh}{dt} = \frac{J\omega v_i}{L} \tag{4.5}$$

about the positive y direction. By Newton's second law, to sustain this rate of change of angular momentum, it will require a torque equal to $J\omega v_i/L$ in the same direction. If the corresponding force at the free end of the gyroscope is denoted by f_o in the positive z-direction, the corresponding torque is f_oL acting about the negative y-direction. It follows that

$$-f_oL = \frac{J\omega v_i}{L} \tag{4.6}$$

This may be expressed as

$$f_o = -\frac{1}{M}v_i \tag{4.7}$$

By the conservation of power (Equation 4.2) for an ideal gyroscope, it follows from Equation 4.7 that

$$v_o = Mf_i \tag{4.8}$$

in which, the gyroscope parameter

$$M = \frac{L^2}{J\omega} \tag{4.9}$$

Note: M is a "mobility" parameter (velocity/force), as discussed in Chapter 5.

Equations 4.7 and 4.8 are the constitutive equations of a gyrator. The linear graph representation of a gyrator is shown in Figure 4.7c.

4.3 Linear Graph Equations

Three types of equations have to be written to obtain an analytical model from a linear graph:

1. Constitutive equations for all the elements that are not sources (inputs).
2. Compatibility equations (loop equations) for all the independent closed paths.
3. Continuity equations (node equations) for all the independent junctions of two or more branches.

Constitutive equations of elements have been discussed in detail in Chapter 2 and earlier in the present chapter. In the examples in Chapter 2, compatibility equations and continuity equations were not used explicitly because the system variables were chosen to satisfy these two types of equations. In modeling of complex dynamic systems, systematic approaches, which can be computer-automated, will be useful. In that context, approaches are necessary to explicitly write the compatibility equations and continuity equations. The related approaches and issues are discussed next.

4.3.1 Compatibility (Loop) Equations

A loop in a linear graph is a closed path formed by two or more branches. A loop equation (compatibility equation) is obtained by summing all the across variables along the branches of the loop is zero. This is a necessary condition because, at a given point in the linear graph there must be a unique value for the across variable, at a given time. For example, a mass and a spring connected to the same point must have the same velocity at a particular time, and this point must be intact (i.e., does not break or snap); hence, the system is "compatible."

4.3.1.1 Sign Convention

1. Go in the counter-clockwise direction of the loop.
2. In the direction of a branch arrow the across variable drops. This direction is taken to be positive (except in a *T*-source, where the arrow direction indicates an increase in its across variable, which is the negative direction).

The arrow in each branch is important, but we need not (and indeed cannot) always go in the direction of the arrows in the branches that form a loop. If we do go in the direction of the arrow in a branch, the associated across variable is considered positive; when we go opposite to the arrow, the associated across variable is considered negative.

4.3.1.2 Number of "Primary" Loops

Primary loops are a "minimal" set of loops from which any other loop in the linear graph can be determined. A primary loop set is an "independent" set. It will generate all the independent loop equations.

Note: Loops closed by broken-line (inertia) branches should be included as well in counting the primary loops.

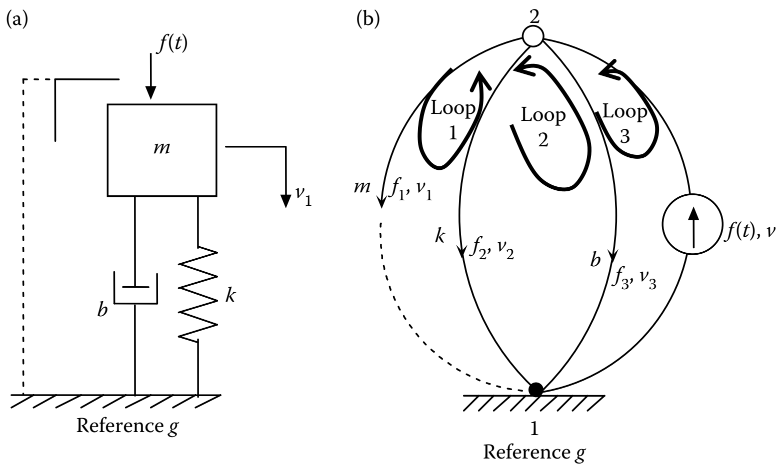

FIGURE 4.8
(a) A mass-spring-damper system. (b) Linear graph having two nodes and three primary loops.

Example 4.1

Figure 4.8 shows a mass-spring-damper system and its linear graph. Each element in the linear graph forms a branch. As noted before, an inertia element is connected to the reference point (ground *g*) by a dotted line because the mass is not physically connected to ground, but all measurements must be referenced to the ground reference point. This reference point "feels" the inertia force of the mass. To understand this further, suppose that we push a free mass upwards by our hands, imparting it an acceleration. The required force is equal to the inertia force, which is the product of mass and acceleration. An equal force is transmitted to the ground though our feet. Clearly, the mass itself is not directly connected to the ground, yet the force applied to the mass and the force "felt" at the ground are equal. Hence the force "appears" to travel directly through the mass element to the ground. Similarly, in Figure 4.8, the input force from the "force source" also travels to ("felt at") the reference point.

In this example, there are three primary loops. Note that loops closed by broken-line (inertia) branches are included in counting primary loops. The primary loop set can be chosen as ($b-k$, $m-b$, and $m-f$), or as ($b-k$, $m-b$, and $f-k$), or any three closed paths.

One obvious choice of primary loops in this example is what is marked in Figure 4.8: Loop 1 ($m-k$), Loop 2 ($k-b$), Loop 3 ($b-f$). The corresponding loop equations are

$$\text{Loop 1 equation: } v_1 - v_2 = 0$$

$$\text{Loop 2 equation: } v_2 - v_3 = 0$$

$$\text{Loop 3 equation: } v_3 - v = 0$$

Once one has selected a primary set of loops (three loops in this example), any other loop will depend on this primary set. For example, an $m-k$ loop can be obtained by algebraically adding the $m-b$ loop and $b-k$ loop (i.e., subtracting the $b-k$ loop from the $m-b$ loop). Similarly, the $f-m$ loop is obtained by adding the $f-b$ and $b-m$ loops. That is:

$$m-k \text{ loop} = (m-b \text{ loop}) - (b-k \text{ loop}); \text{ or Loop } 1 = (m-b \text{ loop}) - \text{Loop } 2$$

$$f-m \text{ loop} = (f-b \text{ loop}) + (b-m \text{ loop}); \text{ or } f-m \text{ loop} = \text{Loop } 3 + (b-m \text{ loop})$$

We can verify these relations using the three loop equations written before together with the following loop equations:

$$m-b \text{ Loop equation (or } b-m \text{ Loop equation): } v_1-v_3=0$$

$$f-m \text{ Loop equation: } v_1-v=0$$

This example illustrates that the primary loop set becomes an "independent" set, which is the minimum number of loops required to obtain all the independent loop equations.

4.3.2 Continuity (Node) Equations

A node is the point where two or more branches meet. A node equation (or, continuity equation) is created by equating to zero the sum of all the through variables at a node. This holds in view of the fact that a node can neither store nor dissipate energy; in effect saying, "what goes in must come out." Hence, a node equation dictates the continuity of the through variables at a node. For this reason one must use proper signs for the variables when writing either node equations or loop equations. The sign convention that is used is: The through variable "into" the node is positive.

The meaning of a node equation in the different domains is:

Mechanical systems: Force balance; equilibrium equation; Newton's third law; etc.

Electrical systems: Current balance; Kirchoff's current law; conservation of charge; etc.

Hydraulic systems: Conservation of matter.

Thermal systems: Conservation of energy.

Example 4.2

Revisit the problem given in Figure 4.8. The system has two nodes. Corresponding node equations are identical, as given below.

$$\text{Node 2 equation: } -f_1-f_2-f_3+f=0$$

$$\text{Node 1 equation: } f_1+f_2+f_3-f=0$$

This example illustrates the following important result:

Required number of node equations = Total number of nodes – 1.

Example 4.3

Consider the *L-C-R* electrical circuit shown in Figure 4.9a. Its linear graph is drawn as in Figure 4.9b. It should be clear that this electrical system is analogous to the mechanical system of Figure 4.8. The system has three primary loops; one primary node; and a voltage source. We may select any three loops as primary loops; for example, (*v-L*, *L-C*, *C-R*) or (*v-C*, *L-C*, *C-R*) or (*v-L*, *v-C*, *v-R*), etc. No matter what set we choose, we will get the same "equivalent" loop equations. In particular, note that the across variables for all four branches of this linear graph are the same.

For example select Loop 1: *L-v*; Loop 2: *C-L*; Loop 3: *R-C* as the primary loops, as shown in Figure 4.9b. The necessary loop equation (three) and the node equations (one) are given below, with our standard sign convention.

$$\text{Loop 1 equation: } -v_1+v=0$$

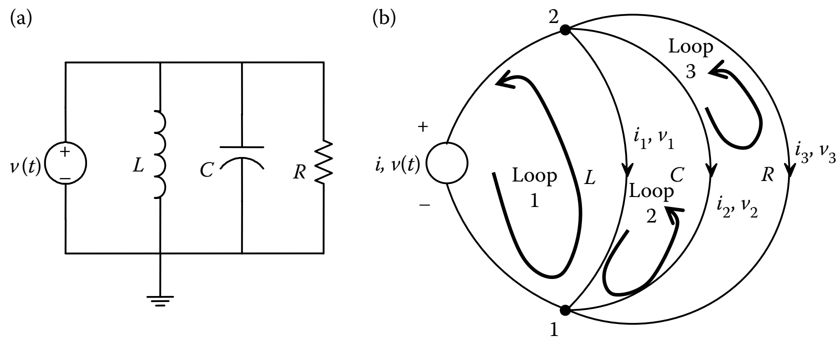

FIGURE 4.9
(a) An *L-C-R* circuit. (b) Its linear graph.

$$\text{Loop 2 equation: } -v_2+v_1=0$$

$$\text{Loop 3 Equation: } -v_3+v_2=0$$

$$\text{Node 2 equation: } i-i_1-i_2-i_3+=0$$

4.3.3 Series and Parallel Connections

If two elements are connected in series, their through variables are the same but the across variables are not the same (they add algebraically). If two elements are connected in parallel, their across variables are the same but the through variables are not the same (they add algebraically). These facts are given in Table 4.2.

Let us consider two systems with a spring (k) and a damper (b), and an applied force ($f(t)$). In Figure 4.10a they are connected in parallel, and in Figure 4.10b they are connected in series. Their linear graphs are as shown in the figures. Note that the linear graph in (a) has two primary loops (two elements in parallel with the force source), whereas in (b) it has only one loop, corresponding to all elements in series with the force source. In Table 4.1 we note the differences in their node and loop equations. These observations should be intuitively clear, without even writing the loop and node equations.

4.4 State Models from Linear Graphs

We can obtain a state model of a dynamic system from its linear graph. Each branch in the linear graph is a "model" of an actual system element of the system, with an associated " constitutive relation," As discussed in Chapter 2, for a mechanical system it is justifiable to use the velocities of independent inertia elements and the forces through independent stiffness (spring) elements as state variables. Similarly, for an electrical system, voltages across independent capacitors and currents through independent inductors are appropriate state variables. In general then, in the linear graph approach we use:

State variables: Across variables of independent A-type elements and through variables of independent T-type elements.

TABLE 4.2

Series-Connected Systems and Parallel-Connected Systems

Series System	Parallel System
Through variables are the same.	Across variables are the same.
Across variables are not the same (they add algebraically).	Through variables are not the same (they add algebraically).

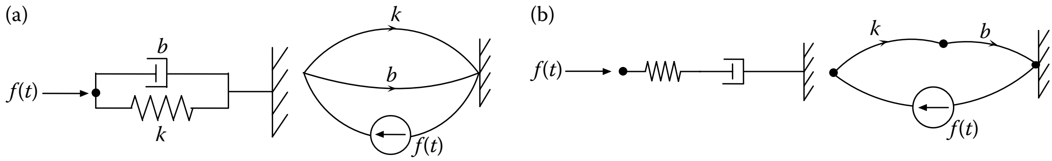

FIGURE 4.10
Spring-damper systems with a force source and their linear graphs. (a) Elements in parallel. (b) Elements in series.

In obtaining an analytical model from the linear graph of a system we write three types of equations:

1. Constitutive equations for all the branches that are not source (input) elements
2. Compatibility equations for the independent loops
3. Continuity equations for the independent nodes

This approach will be further elaborated in this section.

4.4.1 System Order

It is known that A-type elements and T-type elements are energy storage elements. The *system order* is given by the number of independent energy-storage elements in the system. This is also equal to the number state variables; the order of the state-space model; the number of initial conditions required to solve the response of the analytical model; and the order of the input–output differential equation model.

The total number of energy storage elements in a system can be greater than the system order because some of these elements might not be independent.

4.4.2 Sign Convention

The important first step of developing a state-space model using linear graphs is indeed to draw a linear graph for the considered system. A sign convention should be established, as discussed before. The sign convention which we use is as follows:

1. Power flows into the action point and out of the reference point of an element (branch). This direction is shown by the branch arrow (which is an oriented branch). *Exception*: In a source element power flows out of the action point.
2. Through variable (f), across variable (v), and power flow (fv) are positive in the same direction at an action point. At reference point, v is positive in the same

direction given by the linear-graph arrow, but f is taken positive in the opposite direction.

3. In writing node equations: Flow into a node is positive.
4. In writing loop equations: Loop direction is counter-clockwise. A potential (A-variable) "drop" is positive (same direction as the branch arrow. *Exception*: In a T-source the arrow is in the direction in which the A-variable increases).

Note: Once the sign convention is established, the actual values of the variables can be positive or negative depending on their actual direction.

4.4.3 Steps of Obtaining a State Model

The following are the systematic steps for obtaining a set of state equations (a state-space model) from a linear graph:

1. Choose as state variables: Across variables for independent A-type elements and through variables for independent T-type elements.
2. Write constitutive equations for independent energy storage elements. This will give the *state-space shell.*
3. Do similarly for the remaining elements (dependent energy storage elements and dissipation—D-type—elements, transformers, etc.).
4. Write compatibility equations for the primary loops.
5. Write continuity equations for the primary nodes (total number of nodes–1).
6. In the state-space shell, retain state and input variables only. Eliminate all other variables using the loop and node equations and extra constitutive equations.

4.4.4 General Observation

Now some general observations are made with regard to a linear graph in terms of its geometric (topological) characteristics (nodes, loops, branches), elements, unknown and known variables, and relevant equations (constitutive, compatibility, and continuity).

First let

$$\text{Number of sources} = s$$

$$\text{Number of branches} = b$$

Since each source branch has one unknown variable (because one variable is the known input to the system—the source output) and all other passive branches have two unknown variables each, we have:

$$\text{Total number of unknown variables} = 2b - s \tag{4.10}$$

Since each branch other than a source branch provides one constitutive equation, we have:

$$\text{Number of constitutive equations} = b - s \tag{4.11}$$

Let

$$\text{Number of primary loops} = \ell$$

Since each primary loop give a compatibility equation, we have:

$$\text{Number of loop (compatibility) equations} = \ell$$

Let

$$\text{Number of nodes} = n$$

Since one of these nodes does not provide an extra node equation, we have:

$$\text{Number of node (continuity) equations} = n - 1 \tag{4.12}$$

Hence,

$$\text{Total number of equations} = (b - s) + \ell + (n - 1) = b + \ell + n - s - 1$$

To uniquely solve the analytical model we must have:

$$\text{Number of unknowns} = \text{Number of equations or } 2b - s = b + \ell + n - s - 1$$

Hence we have the result

$$\ell = b - n + 1 \tag{4.13}$$

This topological result must be satisfied by any linear graph.

4.4.5 Topological Result

As shown before, Equation 4.13 must be satisfied by a linear graph. Now we will prove by induction that this topological result indeed holds for any linear graph.

Consider Figure 4.11. Using the notation: Number of sources $= s$; Number of branches $= b$; Number of nodes $= n$; we proceed with the following steps.

Step 1: Start with Figure 4.11a: For this graph: $\ell = 1$, $b = 2$, $n = 2$.

Hence, Equation 4.13 is satisfied.

Step 2: Add new loop to Figure 4.11a by using m nodes and $m+1$ branches, as in Figure 4.11b. For this new graph we have: $\ell = 2$; $n = 2 + m$; $b = 2 + m + 1 = m + 2$.

Hence, Equation 4.13 is still satisfied.

Note: $m = 0$ is a special case.

Step 3: Start with a general linear graph having ℓ loops, b ranches, and n nodes that satisfies Equation 4.13. This is the general case of Step 1—Figure 4.11a.

Add a new loop by using m nodes and $m+1$ branches (as in Step 2).

We have

$$\ell \rightarrow \ell + 1; \quad n \rightarrow n + m; \quad b \rightarrow b + m + 1$$

Equation 4.12 is still satisfied by these new values.

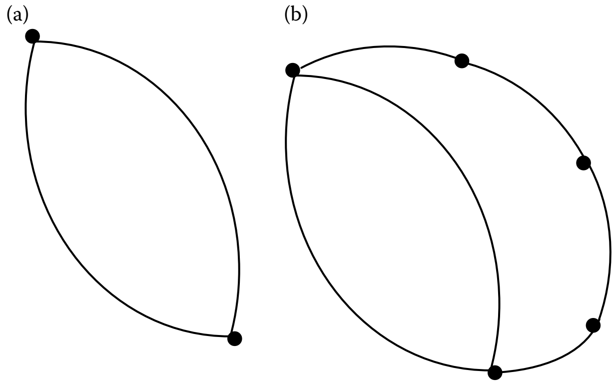

FIGURE 4.11
Proof of topological result. (a) Single loop with two branches. (b) Adding new branches to create a new loop.

Hence, by induction, Equation 4.13 is true in general.

Now we will present five mechanicalsystem examples to illustrate the development of state-space models using linear graphs. Since the approach is unified and uniform across various domains, the same approach is applicable in electrical, fluid, thermal, and multidomain (i.e., mixed) systems, as will be demonstrated later.

Example 4.4

Let us develop a state-space model for the system shown in Figure 4.8, using its linear graph. There are four branches and one source. Thus, $2b-s=7$; we will need seven equations to solve for unknowns. Note from Figure 4.8 that there are three primary loops. In particular, in this example we have:

Number of line branches $b=4$
Number of nodes $n=2$
Number of sources $s=1$
Number of primary loops $l=3$
Number of unknowns $= v_1, f_1, v_2, f_2, v_3, f_3, v=7$
(*Note*: $f(t)$, the input variable, is known)
Number of constitutive equations (one each for m, k, b) $=b-s=3$
Number of node equations $=n-1=1$
Number of loop equations $=3$ (because there are three primary loops)

We have:

Total number of equations = constitutive equations + node equations + loop equations $=3+1+3=7$.

Hence the system is solvable (seven unknowns and seven equations).
The steps of obtaining the state model are given next.

Step 1. Select state variables: Velocity v_1 of mass m and force f_2 of spring k ➔ $x_1=v_1$; $x_2=f_2$
Input variable = applied forcing function (force source) $f(t)$.
Step 2. Constitutive equations for *m* and *k*: These generate the state-space shell (model skeleton):

$$\text{From Newton's second law: } \dot{v}_1=(1/m)f_1 \tag{i}$$

$$\text{Hooke's law for spring: } \dot{f}_2=kv_2 \tag{ii}$$

Step 3. Remaining constitutive equation (for damper):

$$f_3 = bv_3 \tag{iii}$$

Step 4. Node and loop equations:

$$\text{Node equation (for Node 2): } f - f_1 - f_2 - f_3 = 0 \tag{iv}$$

$$\text{Loop equation for Loop 1: } v_1 - v_2 = 0 \tag{v}$$

$$\text{Loop equation for Loop 2: } v_2 - v_3 = 0 \tag{vi}$$

$$\text{Loop equation for loop 3: } v_3 - v = 0 \tag{vii}$$

Step 5. Eliminate auxiliary variables:
To obtain state model, retain v_1 and f_2 and eliminate the auxiliary variables f_1 and v_2 in Equations (i) and (ii).

From Equation (v): $v_2 = v_1$

From Equations (iv) and (iii): $f_1 = -f_2 - bv_3 + f \rightarrow f_1 = -f_2 - bv_1 + f$ (from Equations (vi) and (v))

Substituting these into the state-space shell (Equations (i) and (ii)) we get the state model:

$$\dot{v}_1 = -\frac{b}{m}v_1 - \frac{1}{m}f_2 + \frac{1}{m}f \rightarrow$$

$$\dot{f}_2 = k\dot{v}$$

with the state vector $x = [x_1 \quad x_2]^T = [v_1 \quad f_2]^T$ and the input vector $u = f(t)$.

The model matrices, in the usual notation, are:

$$\boldsymbol{A} = \begin{bmatrix} -b/m & -1/m \\ k & 0 \end{bmatrix}; \quad \boldsymbol{B} = \begin{bmatrix} 1/m \\ 0 \end{bmatrix}$$

Note that this is a second-order system, as clear from the fact that the state vector $\boldsymbol{x}$ is a second-order vector and, further, from the fact that the system matrix $\boldsymbol{A}$ is a 2×2 matrix. Also, note that in this system, the input vector $\boldsymbol{u}$ has only one element, $f(t)$. Hence it is actually a scalar variable, not a vector.

The velocity (v) of the force source is not a state variable, and we need not use Equation (vii). When v and $f(t)$ are positive, for example, power from the source flows out into Node 2.

Example 4.5

A dynamic absorber is a passive vibration-suppression device, which is mounted on the vibrating area of the dynamic system. By properly tuning (selecting the parameters of) the absorber, it is possible to "absorb" most of the power supplied by an unwanted excitation (e.g., support motion, imbalance in rotating parts) in sustaining the absorber motion such that, in steady operation, the vibratory motions of the main system are inhibited. In practice, there should be some damping present in the absorber to dissipate the energy flowing into the absorber, without generating excessive motions in the absorber mass. In the example shown in Figure 4.12a, the main system and the absorber are modeled as simple oscillators with parameters (m_2, k_2, b_2) and (m_l, k_l, b_l), respectively. The linear graph of this system can be drawn in the usual manner, as shown in Figure 4.12b. The external excitation (system input) is the velocity $u(t)$ of the support. We note the following:

Number of branches $= b = 7$
Number of nodes $= n = 4$
Number of sources $= s = 1$

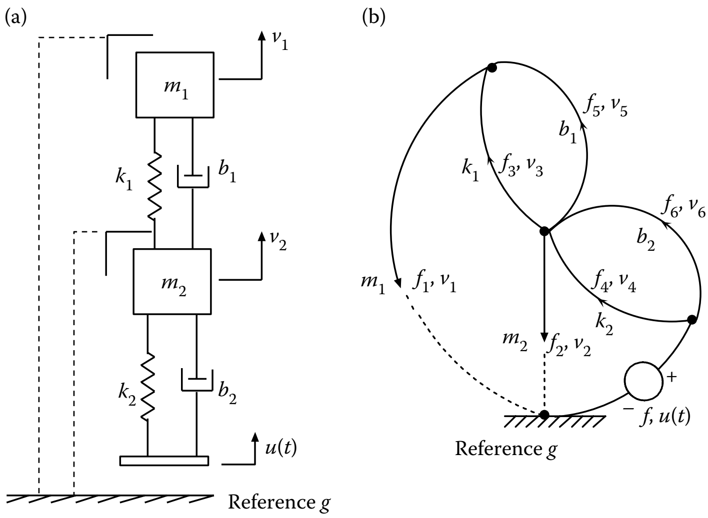

FIGURE 4.12
(a) A mechanical system with a shock absorber. (b) Linear graph of the system.

Number of independent loops $= l = 4$
Number of unknowns $= 2b - s = 13$
Number of constitutive equations $= b - s = 6$
Number of node equations $= n - 1 = 3$
Number of loop equations $= 4$

The four loop equations are provided by the four independent loops of the linear graph.
Check: Number of unknowns $= 2b - s = 13$

$$\text{Number of equations} = (b-s) + (n-1) + l = 6 + 3 + 4 = 13.$$

Hence the analytical model is solvable.

Step 1. Since the system has four independent energy storage elements (m_1, m_2, k_1, k_2) it is a fourth-order system. The state variables are chosen as the across variables of the two masses (velocities v_1 and v_2) and the through variables of the two springs (forces f_1 and f_2). Hence

$$\mathbf{x} = [x_1, \quad x_2 \quad x_3 \quad x_4]^T = [v_1, \quad v_2 \quad f_3 \quad f_4]^T$$

The input variable is $u(t)$.
Step 2. The skeleton state equations (model shell) are:

$$\text{Newton's second law for mass } m_1\text{: } \dot{v}_1 = \frac{1}{m_1} f_1$$

$$\text{Newton's second law for mass } m_2\text{: } \dot{v}_2 = \frac{1}{m_2} f_2$$

$$\text{Hooke's law for spring } k_1\text{: } \dot{f}_3 = k_1 v_3$$

$$\text{Hooke's law for spring } k_2\text{: } \dot{f}_4 = k_2 v_4$$

Step 3. The remaining constitutive equations:

$$\text{For damper } b_1\text{: } f_5 = b_1 v_5$$

$$\text{For damper } b_2\text{: } f_6 = b_2 v_6$$

Step 4. The node equations:

$$-f_1 + f_3 + f_5 = 0$$

$$-f_3 - f_5 - f_2 + f_4 + f_6 = 0$$

$$-f_4 - f_6 + f = 0$$

The loop equations:

$$v_1 - v_2 + v_3 = 0$$

$$v_2 - u + v_4 = 0$$

$$-v_4 + v_6 = 0$$

$$-v_3 + v_5 = 0$$

Step 5. Eliminating the auxiliary variables in the state-space shell.
The following state equations are obtained:

$$\dot{v}_1 = -(b_1/m_1)v_1 + (b_1/m_1)v_2 + (1/m_1)f_3$$
$$\dot{v}_2 = (b_1/m_2)v_1 - [(b_1 + b_2)/m_2]v_2 - (1/m_2)f_3 + (1/m_2)f_4 + (b_2/m_2)u(t)$$
$$\dot{f}_3 = -k_1 v_1 + k_1 v_2$$
$$\dot{f}_4 = -k_2 v_2 + k_2 u(t)$$

This corresponds to:

$$\text{System matrix: } \boldsymbol{A} = \begin{bmatrix} -b_1/m_1 & b_1/m_1 & 1/m_1 & 0 \\ b_1/m_2 & -(b_1 + b_2)/m_1 & -1/m_2 & 1/m_2 \\ -k_1 & k_1 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \end{bmatrix}$$

$$\text{Input distribution matrix: } \boldsymbol{B} = \begin{bmatrix} 0 \\ b_2/m_2 \\ 0 \\ k_2 \end{bmatrix}$$

Example 4.6

Commercial motion controllers are digitally controlled (microprocessor-controlled) high-torque devices capable of applying a prescribed motion to a system. Such controlled actuators can be considered as velocity sources. Consider an application where a rotatory motion controller is used to position an object, which is coupled through a gear box. The system is modeled as in Figure 4.13. We will develop a state-space model for this system using the linear graph approach.

Step 1. Note that the two inertia elements m_1 and m_2 are not independent, and together comprise one storage element. Thus, along with the stiffness element, there are only two independent energy storage elements. Hence the system is second order. Let us choose as state variables, v_1

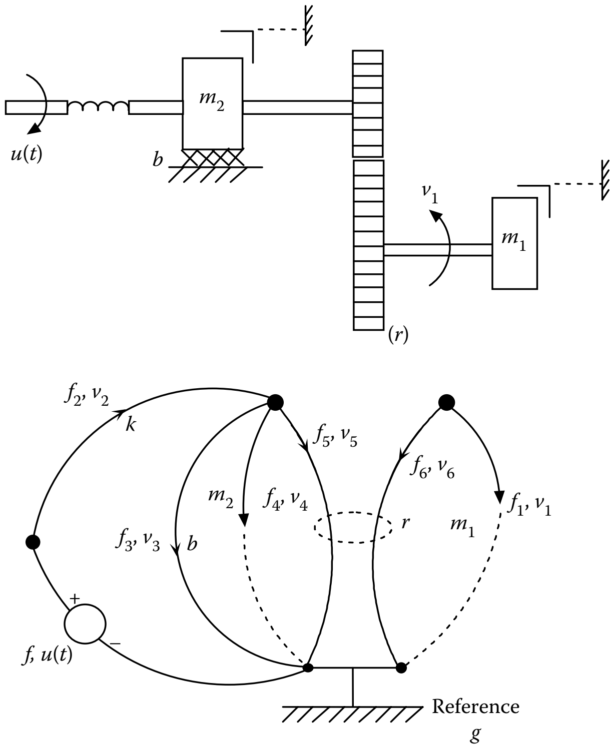

FIGURE 4.13
(a) Rotary-motion system with a gear transmission. (b) Linear graph of the system.

and f_2—the across variable of one of the inertias (because the other inertia will be "dependent") and the through variable of the spring.

$$\text{Let } x_1 = v_1 \quad \text{and} \quad x_2 = f_2$$

$$\text{Hence } [x_1 \;\; x_2]^T = [v_1 \;\; f_2]^T$$

Step 2. The constitutive equations for m_1 and k: $\dot{v}_1 = \frac{1}{m} f_1$; $\dot{f}_2 = kv_2$
Step 3. The remaining constitutive equations:

$$\text{For damper: } f_3 = bv_3$$

$$\text{For the "dependent" inertia } m_2\text{: } \dot{v}_4 = \frac{1}{m_2} f_4$$

$$\text{For the transformer (pair of meshed gear wheels): } v_6 = rv_5;\; f_6 = -\frac{1}{r} f_5$$

Step 4. The node equations:

$$-f_6 - f_1 = 0$$

$$f - f_2 = 0$$

$$f_2 - f_3 - f_4 - f_5 = 0$$

The loop equations:

$$v_6 - v_1 = 0$$

$$v_3 - v_4 = 0$$

$$v_4 - v_5 = 0$$

$$-v_2 + u(t) - v_3 = 0$$

Step 5. Eliminate the auxiliary variables.

Using equations from Steps 3 and 4, the auxiliary variable f_l can be expressed as:

$$f_1 = \frac{1}{r}\left[f_2 - \frac{b}{r}v_1 - \frac{m_2}{r}\dot{v}_1\right]$$

The auxiliary variable v_2 can be expressed as:

$$v_2 = -\frac{1}{r}v_1 + u(t)$$

By substituting these equations into the state-space shell we obtain the following two state equations:

$$\dot{v}_1 = -\left[\frac{b}{(m_1r^2 + m_2)}\right]v_1 + \left[\frac{r}{(m_1r^2 + m_2)}\right]f_2$$

$$\dot{f}_2 = -\frac{k}{r}v_1 + ku(t)$$

Note that the system is second-order; only two state equations are present. The corresponding system matrix and the input-gain matrix (input distribution matrix) are:

$$\boldsymbol{A} = \begin{bmatrix} -b/m & r/m \\ -k/m & 0 \end{bmatrix}; \boldsymbol{B} = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

where $m = m_1r^2 + m_2 =$ *equivalent inertia* of m_l and m_2 when determined at the location of inertia m_2.

Example 4.7

a. List several advantages of using linear graphs in developing a state-space model of a dynamic system.
b. Electrodynamic shakers are commonly used in the dynamic testing of products. One possible configuration of a shaker/test-object system is shown in Figure 4.14a. A simple, linear, lumped-parameter model of the mechanical system is shown in Figure 4.14b.

Note that the driving motor is represented by a torque source T_m. Also, the following parameters are indicated:

J_m = equivalent moment of inertia of motor rotor, shaft, coupling, gears, and the shaker platform
r_1 = pitch circle radius of the gear wheel attached to the motor shaft
r_2 = pitch circle radius of the gear wheel rocking the shaker platform
l = lever arm from the center of the rocking gear to the support location of the test object

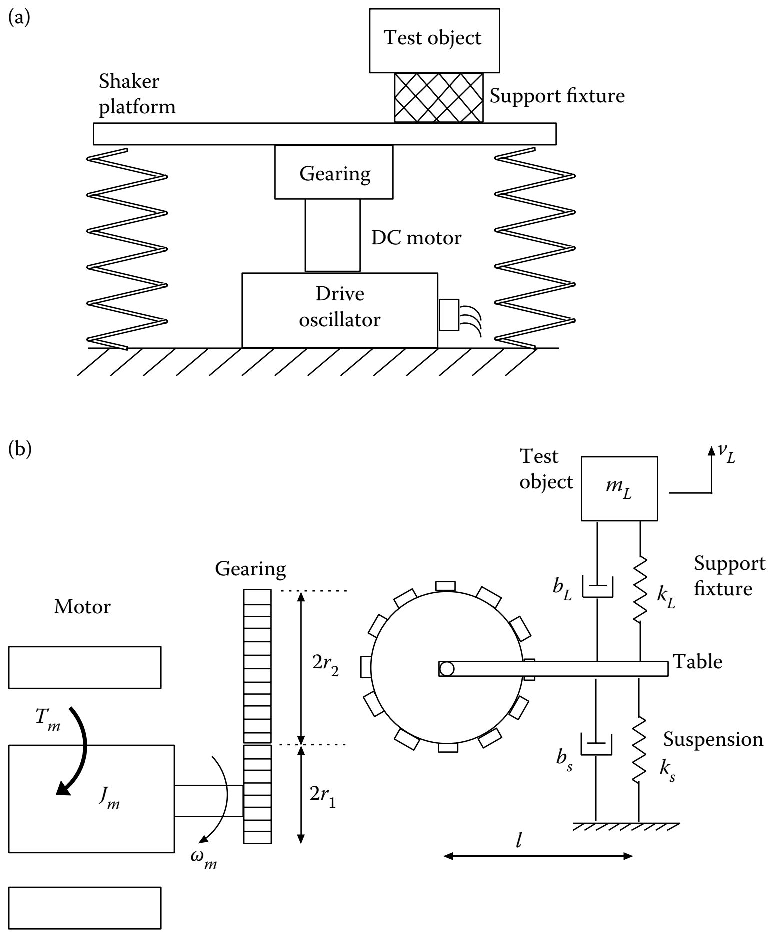

FIGURE 4.14
(a) A dynamic-testing system. (b) A model of the dynamic testing system.

m_L = equivalent mass of the test object and its support fixture
k_L = stiffness of the support fixture
b_L = equivalent viscous damping constant of the support fixture
k_s = stiffness of the suspension system of the shaker table
b_s = equivalent viscous damping constant of the suspension system.

Since the inertia effects are lumped into equivalent elements it may be assumed that the shafts, gearing, platform and the support fixtures are light. The following variables are of interest:

ω_m = angular speed of the drive motor
v_L = vertical speed of motion of the test object
f_L = equivalent dynamic force of the support fixture (force in spring k_L)
f_s = equivalent dynamic force of the suspension system (force in spring k_s).

i. Obtain an expression for the motion ratio:

$$r = \frac{\text{vertical movement of the shaker table at the test object support location}}{\text{angular movement of the drive motor shaft}}$$

ii. Draw a linear graph to represent the dynamic model.

iii. Using $\boldsymbol{x} = [\omega_m, f_s, f_L, v_L]^T$ as the state vector, $\boldsymbol{u} = [T_m]$ as the input, and $\boldsymbol{y} = [v_L\ f_L]^T$ as the output vector, obtain a complete state-space model for the system. For this purpose you must use the linear graph drawn in (ii).

Solution

a. Linear graphs:

- Use physical variables as states.
- Provide a generalized and unified approach for mechanical, electrical, fluid, and thermal systems. Hence they can be conveniently used in multidomain (i.e., mixed) systems.
- Provide a unified approach to model multifunctional devices (e.g., a piezoelectric device which can function as both a sensors and an actuator).
- Show the directions of power flow in various parts of the system.
- Provide a graphical representation of the system model.
- Allow visualization of the system structure (even before formulating an analytical model).
- Help identify similarities (structure, performance, etc.) in different types of systems.
- Provide a systematic approach to automatically (using computer) generate state equations.

b. (i) Let θ_m = rotation of the motor (drive gear).
Hence, rotation of the output gear $= (r_1/r_2)\theta_m$
Hence, displacement of the table at the test object support point $= l\,(r_1/r_2)\theta_m$
Hence, $r = l\,(r_1/r_2)$
(ii) The linear graph of the system is drawn as in Figure 4.15.
(iii) Constitutive equations:

State-space shell:

$$J_m \frac{d\omega_m}{dt} = T_2$$

$$\frac{df_s}{dt} = k_s v_5$$

$$\frac{df_L}{dt} = k_L v_L$$

$$m_L \frac{dv_L}{dt} = f_9$$

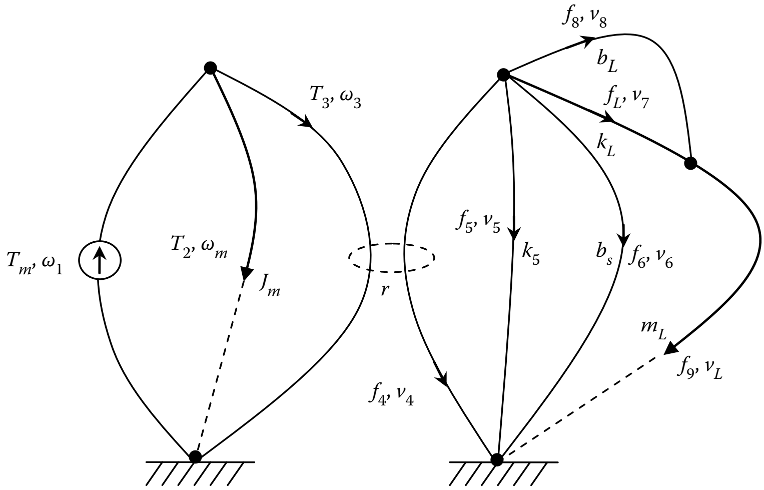

FIGURE 4.15
Linear graph of the shaker system.

Others:

$$v_4 = r\omega \quad \text{where} \quad r = \frac{r_1}{r_2} l$$

$$f_4 = -\frac{1}{r} T_3$$

$$f_6 = b_s v_6$$

$$f_8 = b_L v_8$$

Continuity (Node) equations

$$T_m - T_2 - T_3 = 0$$

$$-f_4 - f_s - f_6 - f_L - f_8 = 0$$

$$f_L + f_8 - f_9 = 0$$

Compatibility (Loop) equations:

$$-\omega_m + \omega_3 = 0$$

$$-v_4 + v_5 = 0$$

$$-v_5 + v_6 = 0$$

$$-v_6 + v_7 + v_L = 0$$

$$-v_7 + v_8 = 0$$

Elimination/substitution results in the following:

$$\begin{aligned} J_m \frac{d\omega_m}{dt} &= T_2 = T_m - T_3 = T_m + rf_4 = T_m + r(-f_s - f_6 - f_L - f_8) \\ &= T_m - r(f_s + b_s v_6 + f_L + b_L v_8) \\ &= T_m - r(f_s + f_L + b_s v_6 + b_L v_8) \end{aligned}$$

$$v_6 = v_5 = v_4 = r\omega_3 = r\omega_m$$

$$v_8 = v_7 = v_6 - v_L = v_5 - v_L = v_4 - v_L = r\omega_3 - v_L = r\omega_m - v_L$$

Hence,

$$J_m \frac{d\omega_m}{dt} = T_m - r(f_s + f_L) - r^2 b_s \omega_m - r b_L (r\omega_m - v_L) \tag{i}$$

$$\frac{df_s}{dt} = k_s v_5 = k_s v_4 = k_s r\omega_3 = k_s r\omega_m \tag{ii}$$

$$\frac{df_L}{dt} = k_L v_7 = k_L(v_6 - v_L) = k_L(v_4 - v_L) = k_L(r\omega_3 - v_L) = k_L(r\omega_m - v_L) \tag{iii}$$

$$m_L \frac{dv_L}{dt} = f_9 = f_L + f_8 = f_L + b_L v_8 = f_L + b_L(r\omega_m - v_L) \tag{iv}$$

In summary, we have the following state equations:

$$J_m \frac{d\omega_m}{dt} = T_m - rf_s - rf_L - r^2(b_s + b_L)\omega_m + rb_L v_L$$

$$\frac{df_s}{dt} = rk_s\omega_m$$

$$\frac{df_L}{dt} = rk_L\omega_m - k_L v_L$$

$$m_L \frac{dv_L}{dt} = f_L + rb_L\omega_m - b_L v_L$$

with v_L and f_L as the outputs.

In the standard notation: $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$ and $\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$ where

$$\boldsymbol{x} = [\omega_m,\ f_s,\ f_L,\ v_L]^T, \quad \boldsymbol{u} = [T_m], \quad \boldsymbol{y} = [v_L \quad f_L]^T$$

$$\boldsymbol{A} = \begin{bmatrix} -\dfrac{r^2}{J_m}(b_s + b_L) & -\dfrac{r}{J_m} & -\dfrac{r}{J_m} & -\dfrac{rb_L}{J_m} \\ rk_s & 0 & 0 & 0 \\ rk_L & 0 & 0 & -k_L \\ \dfrac{rb_L}{m_L} & 0 & \dfrac{1}{m_L} & -\dfrac{b_L}{m_L} \end{bmatrix}; \quad \boldsymbol{B} = \begin{bmatrix} 1/J_m \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \boldsymbol{D} = 0$$

Example 4.8

A robotic sewing system consists of a conventional sewing head. During operation, a panel of garment is fed by a robotic hand into the sewing head. The sensing and control system of the robotic hand ensures that the seam is accurate and the cloth tension is correct in order to guarantee the quality of the stitch. The sewing head has a frictional feeding mechanism, which pulls the fabric in a cyclic manner away from the robotic hand, using a toothed feeding element. When there is slip between the feeding element and the garment, the feeder functions as a *force source* and the applied force is assumed cyclic with a constant amplitude. When there is no slip, however, the feeder functions as a *velocity source*, which is the case during normal operation. The robot hand has inertia. There is some flexibility at the mounting location of the hand on the robot. The links of the robot are assumed rigid and some of its joints can be locked to reduce the number of degrees of freedom, when desired.

Consider the simplified case of a single-degree-of-freedom robot. The corresponding robotic sewing system is modeled as in Figure 4.16. Here the robot is modeled as a single moment of inertia J_r which is linked to the hand with a light rack-and-pinion device with its speed transmission parameter given by:

$$\frac{\text{Rack Tanslatory Movement}}{\text{Pinion Rotatory Movement}} = r$$

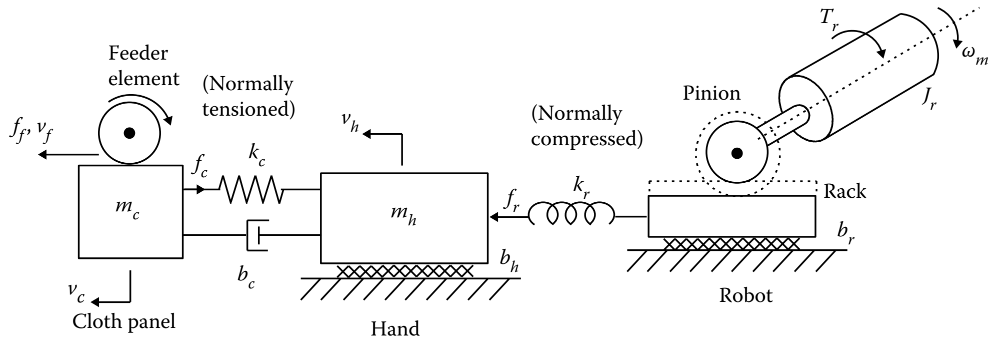

FIGURE 4.16
A robotic sewing system.

The drive torque of the robot is T_r and the associated rotatory speed is ω_r. Under conditions of slip the feeder input to the cloth panel is force f_f , and with no slip the input is the velocity v_f. Various energy dissipation mechanisms are modeled as linear viscous damping of damping constant b (with corresponding subscripts). The flexibility of various system elements is modeled by linear springs with stiffness k. The inertia effects of the cloth panel and the robotic hand are denoted by the lumped masses m_c and m_h, respectively, having velocities v_c and v_h, as shown in Figure 4.16.

Note: The cloth panel is normally in tension with tensile force f_c. In order to push the panel, the robotic wrist is normally in compression with compressive force f_r.

First consider the case of the feeding element with slip:

a. Draw a linear graph for the model shown in Figure 4.16, orient the graph, and mark all the element parameters, through variables and across variables on the graph.
b. Write all the constitutive equations (element physical equations), independent node equations (continuity), and independent loop equations (compatibility). What is the order of the model?
c. Develop a complete state-space model for the system. The outputs are taken as the cloth tension f_c, and the robot speed ω_r, which represent the two variables that have to be measured to control the system. Obtain the system matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$.

Now consider the case where there is no slip at the feeder element:

d. What is the order of the system now? How is the linear graph of the model modified for this situation? Accordingly, modify the state-space model obtained earlier to represent the present situation and from that obtain the new model matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$.
e. Generally comment on the validity of the assumptions made in obtaining the model shown in Figure 4.16 for a robotic sewing system.

Solution

a. Linear graph of the system is drawn as in Figure 4.17. Since in this case the feeder input to the cloth panel is force f_f, a T-source, the arrow of the source element should be retained but the + and − signs (used for an A-source) should be removed.
b. In the present operation f_f is an input. This case corresponds to a fifth-order model, as will be clear from the development given below.

   Constitutive equations:

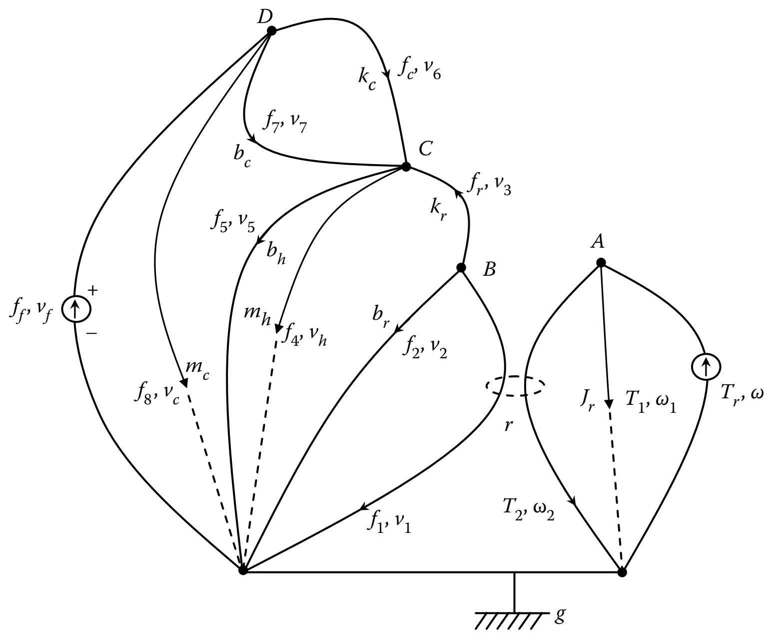

FIGURE 4.17
Linear graph of the robotic sewing system.

$$\left.\begin{aligned} J_r \frac{d\omega_r}{dt} &= T_1 \\ \frac{df_r}{dt} &= k_r v_3 \\ m_h \frac{dv_h}{dt} &= f_4 \\ \frac{df_c}{dt} &= k_c v_6 \\ m_c \frac{dv_c}{dt} &= f_8 \end{aligned}\right\} \text{State-space shell}$$

$$v_1 = r\omega_2$$

$$f_1 = -\frac{1}{r} T_2$$

$$f_2 = -b_r v_2$$

$$f_5 = -b_h v_5$$

$$f_7 = -b_c v_7$$

Continuity equations (Node equations):

$$\text{Node } A\text{: } T_r - T_1 - T_2 = 0$$

$$\text{Node } B\text{: } -f_1 - f_2 - f_r = 0$$

$$\text{Node } C\text{: } f_r + f_c + f_7 - f_5 - f_4 = 0$$

$$\text{Node } D\text{: } -f_c + f_f - f_8 - f_7 = 0$$

Compatibility equations (Loop equations):

$$-\omega + \omega_1 = 0$$
$$-\omega_1 + \omega_2 = 0$$
$$-v_1 + v_2 = 0$$
$$-v_1 + v_3 + v_h = 0$$
$$-v_h + v_5 = 0$$
$$-v_6 + v_7 = 0$$
$$-v_h - v_7 + v_c = 0$$
$$-v_c + v_f = 0$$

c. Eliminate unwanted variables as follows:

$$\begin{aligned} T_1 &= T_r - T_2 = T_r + rf_1 = T_r + r(-f_2 - f_r) \\ &= T_r - rb_r v_2 - rf_r = T_r - rb_r v_1 - rf_r \\ &= T_r - rb_r r\omega_2 - rf_r \\ &= T_r - r^2 b_r \omega_2 - rf_r \end{aligned}$$

$$v_3 = v_1 - v_h = r\omega_2 - v_h = r\omega_r - v_h$$

$$\begin{aligned} f_4 &= f_r + f_c + f_7 - f_5 = f_r + f_c + b_c v_7 - b_h v_5 \\ &= f_r + f_c + b_c(v_c - v_h) - b_h v_h \end{aligned}$$

$$v_6 = v_7 = v_c - v_h$$

$$f_8 = f_f - f_c - f_7 = f_f - f_c - b_c v_7 = f_f - f_c - b_c(v_c - v_h)$$

State-space model:

$$J_r \frac{d\omega_r}{dt} = -r^2 b_r \omega_r - r f_r + T_r$$

$$\frac{df_r}{dt} = k_r (r\omega_r - v_h)$$

$$m_h \frac{dv_h}{dt} = f_r - (b_c + b_h) v_h + f_c + b_c v_c$$

$$\frac{df_c}{dt} = k_c (-v_h + v_c)$$

$$m_c \frac{dv_c}{dt} = b_c v_h - f_c - b_c v_c + f_f$$

with $\boldsymbol{x} = \begin{bmatrix} \omega_r & f_r & v_h & f_c & v_c \end{bmatrix}^T$; $\boldsymbol{u} = \begin{bmatrix} T_r & f_f \end{bmatrix}^T$; $\boldsymbol{y} = \begin{bmatrix} f_c & \omega_r \end{bmatrix}^T$

$$\boldsymbol{A} = \begin{bmatrix} -r^2 b_r/J_r & -r/J_r & 0 & 0 & 0 \\ rk_r & 0 & -k_r & 0 & 0 \\ 0 & 1/m_h & -(b_c + b_h)/m_h & 1/m_h & b_c/m_h \\ 0 & 0 & -k_c & 0 & k_c \\ 0 & 0 & b_c/m_c & -1/m_c & -b_c/m_c \end{bmatrix};$$

$$\boldsymbol{B} = \begin{bmatrix} 1/J_r & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1/m_c \end{bmatrix}; \quad \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad \boldsymbol{D} = 0$$

d. In this case, v_f is an input, which is an *A*-source. The corresponding element in the linear graph given in Figure 4.17 should be modified to account for this. Specifically, the direction of the arrow of this source element should be reversed (because it is an *A*-source) and the + and – signs (used for an *A*-source) should be retained. Furthermore, the inertia element m_c ceases to influence the dynamics of the overall system because, $v_c = v_f$ in this case and is completely specified. This results from the fact that any elements connected in parallel with an *A*-source have no effect on the rest of the system. Accordingly, the branch representing the m_c element should be removed from the linear graph.

Hence, we now have a fourth-order model, with

State vector $\boldsymbol{x} = \begin{bmatrix} \omega_r & f_r & v_h & f_c \end{bmatrix}^T$; Input vector $\boldsymbol{u} = \begin{bmatrix} T_r & v_f \end{bmatrix}^T$

State model:

$$J_r \frac{d\omega_r}{dt} = -r^2 b_r \omega_r - rf_r + T_r$$

$$\frac{df_r}{dt} = k_r(r\omega_r - v_h)$$

$$m_h \frac{dv_h}{dt} = f_r - (b_c + b_h)v_h + f_c + b_c v_f$$

$$\frac{df_c}{dt} = k_c(-v_h + v_f)$$

The corresponding model matrices are:

$$\boldsymbol{A} = \begin{bmatrix} -r^2 b_r/J_r & -r/J_r & 0 & 0 \\ rk_r & 0 & -k_r & 0 \\ 0 & 1/m_h & -(b_c + b_h)/m_h & 1/m_h \\ 0 & 0 & -k_c & 0 \end{bmatrix}; \quad \boldsymbol{B} = \begin{bmatrix} 1/J_r & 0 \\ 0 & 0 \\ 0 & b_c/m_h \\ 0 & k_c \end{bmatrix};$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad \boldsymbol{D} = 0$$

e. In practice, the cloth panel is not a rigid and lumped mass; damping and flexibility effects are nonlinear; and conditions of pure force source and pure velocity source may not be maintained.

4.5 Miscellaneous Examples

Thus far in this chapter we have primarily considered the modeling of lumped-parameter mechanical systems—systems with inertia, flexibility, and mechanical energy dissipation. In view of the analogies that exist between mechanical, electrical, fluid, and thermal components and associated variables, there is an "analytical" similarity between these four types of physical systems. Accordingly, once we have developed procedures for modeling and analysis of one type of systems (say, mechanical systems) the same procedures may be extended (in an "analogous" manner) to the other three types of systems. This fact is exploited in the use of linear graphs in modeling mechanical, electrical, fluid, and thermal systems, in a unified manner, using essentially the same procedures. Furthermore, for this reason, a unified and integrated procedure is provided through linear graphs to model multidomain (mixed systems); for example electro-mechanical or mechatronic systems—systems that use a combination of two or more types of physical components (mechanical, electrical, fluid, and thermal) in a convenient manner. In this section first we will introduce two useful components: amplifier and dc motor, which are useful in electrical, electro-mechanical, and other types of multidomain systems. We will end the section with examples.

4.5.1 Amplifiers

An amplifier is a common component, primarily in an electrical system or electrical subsystem. Purely mechanical, fluid, and thermal amplifiers have been developed and envisaged as well. Two common characteristics of an amplifier are:

1. They accomplish tasks of signal amplification.
2. They are active devices (i.e., they need an external power to operate).

3. They are not affected (ideally) by the load which they drive (i.e., loading effects are small).
4. They have a decoupling effect on systems (i.e., the desirable effect of reducing dynamic interactions between components).

Electrical signals voltage, current, and power are amplified using voltage amplifiers, current amplifiers, and power amplifiers, respectively. Operational amplifiers (opamps) are the basic building block in constructing these amplifiers. Particularly, an opamp, with feedback provides the desirable characteristics of: very high input impedance, low output impedance, and stable operation. For example, due to its impedance characteristics, the output characteristics of a good amplifier are not affected by the device (load) that is connected to its output. In other words electrical loading errors are negligible.

Analogous to electrical amplifiers, a mechanical amplifier can be designed to provide force amplification (a *T*-type amplifier) or a fluid amplifier can be designed to provide pressure amplification (an *A*-type amplifier). In these situations, typically, the device is active and an external power source is needed to operate the amplifier (e.g., to drive a motor-mechanical load combination).

4.5.1.1 Linear Graph Representation

In its linear graph representation, an amplifier is considered as a "dependent source" element or a "modulated source" element. Specifically, the amplifier output depends on (modulated by) the amplifier input, and is not affected by the dynamics of any devices that are connected to the output of the amplifier (i.e., the load of the amplifier). This is the ideal case. In practice some loading error will be present (i.e., the amplifier output will be affected by the load which it drives).

The linear graph representations of an across-variable amplifier (e.g., voltage amplifier, pressure amplifier) and a through-variable amplifier (e.g., current amplifier, force amplifier) are shown in Figure 4.18a and b, respectively. The pertinent constitutive equations in the general and linear cases are given as well in the figures.

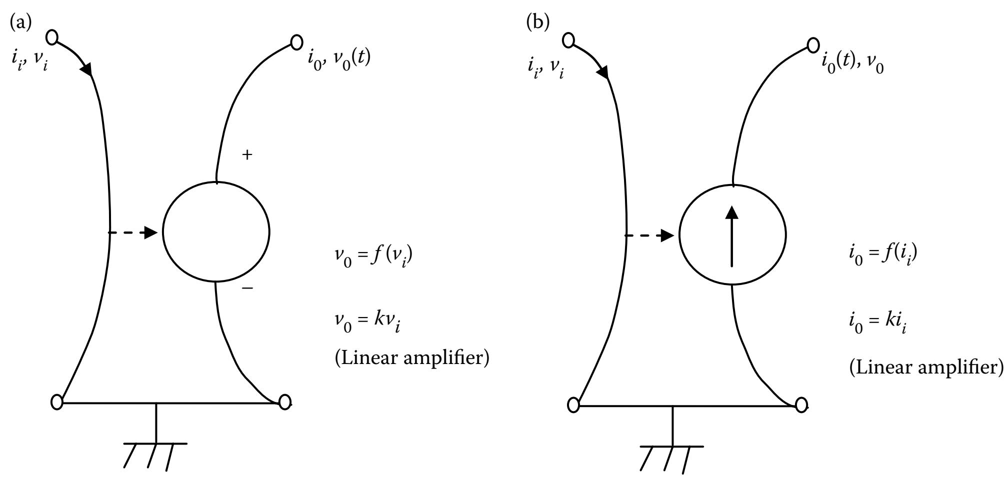

FIGURE 4.18
Linear graph representation of (a) an across-variable amplifier (*A*-type amplifier) and (b) a through-variable amplifier (*T*-type amplifier).

4.5.2 DC Motor

The dc motor is a commonly used electrical actuator. It converts dc electrical energy into mechanical energy. The principle of operation is based on the fact that when a conductor carrying current is placed in a magnetic field, a force is generated (Lorentz's law). It is this force, which results from the interaction of two magnetic fields, that is presented as the magnetic torque in the rotor of the motor.

A dc motor has a stator and a rotor (armature) with windings which are excited by a field voltage v_f and an armature voltage v_a, respectively. The equivalent circuit of a dc motor is shown in Figure 4.19a, where the field circuit and the armature circuit are shown separately, with the corresponding supply voltages. This is the separately excited case. If the stator filed is provided by a permanent magnet, then the stator circuit that is shown in Figure 4.19a is simply an equivalent circuit, where the stator current i_f can be assumed constant. Similarly, if the rotor is a permanent magnet, what is shown in Figure 4.19a is an equivalent circuit where the armature current i_a can be assumed constant. The magnetic torque of the motor is generated by the interaction of the stator field (proportional to i_f) and the rotor field (proportional to i_a) and is given by

$$T_m = k i_f i_a \tag{4.14}$$

A back-electromotive force (back e.m.f) is generated in the rotor (armature) windings to oppose its rotation when these windings rotate in the magnetic field of the stator (Lenz's law). This voltage is given by

$$v_b = k' i_f \omega_m \tag{4.15}$$

where i_f = field current; i_a = armature current; ω_m = angular speed of the motor.

Note: For perfect transfer of electrical energy to mechanical energy in the rotor we have

$$T_m \omega_m = i_a v_b \tag{4.16}$$

This is an electro-mechanical transformer.

$$\text{Field circuit equation: } v_f = R_f i_f + L_f \frac{di_f}{dt} \tag{4.17}$$

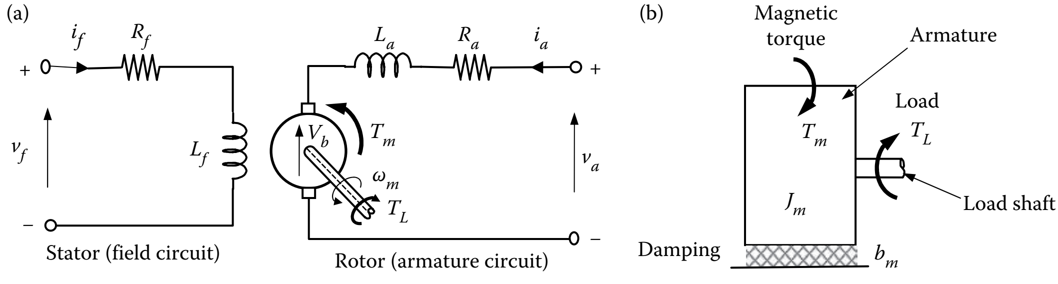

FIGURE 4.19
(a) Equivalent circuit of a dc motor (separately excited). (b) Armature mechanical loading.

where v_f = supply voltage to stator; R_f = resistance of the field windings; L_f = inductance of the field windings.

$$\text{Armature (rotor) circuit equation: } v_a = R_a i_a + L_a \frac{di_a}{dt} + v_b \tag{4.18}$$

where v_a = armature supply voltage; R_a = armature winding resistance; L_a = armature leakage inductance.

Suppose that the motor drives a load whose equivalent torque is T_L. Then from Figure 4.19b.

$$\text{Mechanical (load) equation: } J_m \frac{d\omega_m}{dt} = T_m - T_L - b_m\omega_m \tag{4.19}$$

where J_m=moment of inertia of the rotor; b_m=equivalent (mechanical) damping constant for the rotor; T_L=load torque.

In field control of the motor, the armature supply voltage v_a is kept constant and the field voltage v_f is controlled. In armature control of the motor, the field supply voltage v_f is kept constant and the armature voltage v_a is controlled.

Example 4.9

A classic problem in robotics is the case of robotic hand gripping and turning a doorknob to open a door. The mechanism is schematically shown in Figure 4.20a. Suppose that the actuator of the robotic hand is an armature-controlled dc motor. The associated circuit is shown in Figure 4.20b.

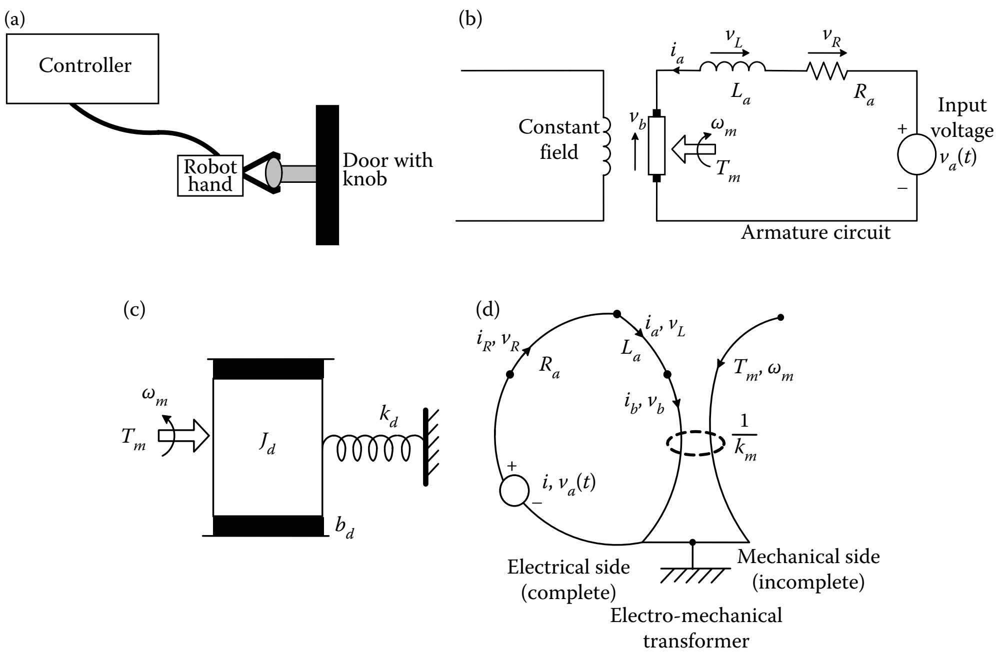

FIGURE 4.20
(a) Robotic hand turning a doorknob. (b) Armature-controlled dc motor of the robotic hand. (c) Mechanical model of the hand-doorknob system. (d) Incomplete linear graph.

The field circuit provides a constant magnetic field to the motor, and is not important in the present problem. The armature (with motor rotor windings) circuit has a back e.m.f. v_b, a leakage inductance L_a, and a resistance R_a. The input signal to the robotic hand is the armature voltage $v_a(t)$ as shown. The rotation of the motor (at an angular speed ω_m) in the two systems of magnetic field generates a torque T_m (which is negative as marked in Figure 4.20b during normal operation). This torque (magnetic torque) is available to turn the doorknob, and is resisted by the inertia force (moment of inertia J_d), the friction (modeled as linear viscous damping of damping constant b_d) and the spring (of stiffness k_d) of the hand-knob-lock combination. A mechanical model is shown in Figure 4.20c. The dc motor may be considered as an ideal electromechanical transducer which is represented by a linear graph transformer. The associated equations are

$$\omega_m = \frac{1}{k_m} v_b \tag{4.20}$$

$$T_m = -k_m i_b \tag{4.21}$$

Note: The negative sign in Equation 4.21 arises due to the specific sign convention. The linear graph may be easily drawn, as shown in Figure 4.20d, for the electrical side of the system.

Answer the following questions:

a. Complete the linear graph by including the mechanical side of the system.
b. Give the number of branches (b), nodes (n), and the independent loops (l) in the completed linear graph. Verify your answer.
c. Take current through the inductor (i_a), speed of rotation of the door knob (ω_d), and the resisting torque of the spring within the door lock (T_k) as the state variables, the armature voltage $v_a(t)$ as the input variable, and ω_d and T_k as the output variables. Write the independent node equations, independent loop equations, and the constitutive equations for the completed linear graph. Clearly show the state-space shell. Also verify that the number of unknown variables is equal to the number of equations obtained in this manner.
d. Eliminate the auxiliary variables and obtain a complete state-space model for the system, using the equations written in (c) above.

Solution

a. The complete linear graph is shown in Figure 4.21.
b. $b=8$, $n=5$, $l=4$ for this linear graph. It satisfies the topological relationship $l=b-n+1$

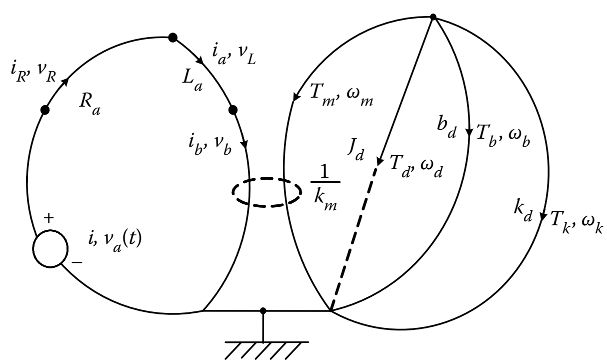

FIGURE 4.21
The complete linear graph of the system.

c. Independent node equations:

$$i - i_R = 0$$
$$i_R - i_a = 0$$
$$i_a - i_b = 0$$
$$-T_m - T_d - T_b - T_k = 0$$

Independent loop equations:

$$v_a(t) - v_R - v_L - v_b = 0$$
$$\omega_m - \omega_d = 0$$
$$\omega_d - \omega_b = 0$$
$$\omega_b - \omega_k = 0$$

Constitutive equations:

$$\left.\begin{aligned} L_a \frac{di_a}{dt} &= v_L \\ J_d \frac{d\omega_d}{dt} &= T_d \\ \frac{dT_k}{dt} &= k_d \omega_k \end{aligned}\right\} \text{State-space shell}$$

$$\left.\begin{aligned} v_R &= R_a i_R \\ T_b &= b_d \omega_b \end{aligned}\right\} \text{Auxiliary constitutive equations}$$

$$\left.\begin{aligned} \omega_m &= \frac{1}{k_m} v_b \\ T_m &= -k_m i_b \end{aligned}\right\} \text{Electro-mechanical transformer}$$

Note: There are 15 unknown variables (i, i_R, i_a, i_b, T_m, T_d, T_b, T_k, v_R, v_L, v_b, ω_m, ω_d, ω_b, ω_k) and 15 equations.

$$\text{Number of unknown variables} = 2b - s = 2 \times 8 - 1 = 15$$

$$\text{Number of independent node equations} = n - 1 = 5 - 1 = 4$$

$$\text{Number of independent loop equations} = l = 4$$

$$\text{Number of constitutive equations} = b - s = 8 - 1 = 7$$

$$\text{Check: } 15 = 4 + 4 + 7$$

d. Eliminate the auxiliary variables from the state-space shell, by substitution:

$$\begin{aligned} v_L &= v_a(t) - v_R - v_b = v_a(t) - R_a i_a - k_m \omega_m \\ &= v_a(t) - R_a i_a - k_m \omega_d \end{aligned}$$

$$T_d = -T_k - T_m - T_b = -T_k + k_m i_b - b_d \omega_b$$
$$= k_m i_a - b_d \omega_d - T_k$$
$$\omega_k = \omega_b = \omega_d$$

Hence, we have the state-space equations:

$$L_a \frac{di_a}{dt} = -R_a i_a - k_m \omega_d + v_a(t)$$
$$J_d \frac{d\omega_d}{dt} = k_m i_a - b_d \omega_d - T_k$$
$$\frac{dT_k}{dt} = k_d \omega_d$$

with $\boldsymbol{x} = [i_a \quad \omega_d \quad T_k]^T$, $\boldsymbol{u} = [v_a(t)]$, and $\boldsymbol{y} = [\omega_d \quad T_k]^T$ we have the state-space model

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

The model matrices are:

$$\boldsymbol{A} = \begin{bmatrix} -R_a/L_a & -k_m/L_a & 0 \\ k_m/J_d & -b_d/J_d & -1/J_d \\ 0 & k_d & 0 \end{bmatrix}; \quad \boldsymbol{B} = \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix}; \quad \boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \boldsymbol{D} = 0$$

Note 1: This is a multidomain (electro-mechanical model).
Note 2: Multifunctional devices (e.g., a piezoelectric device that serves as both actuator and sensor) may be modeled similarly, using an electro-mechanical transformer (or, through the use of the "reciprocity principle").

4.5.3 Linear Graphs of Thermal Systems

Thermal systems have temperature (T) as the across variable, as it is always measured with respect to some reference (or as a temperature difference across an element), and heat transfer (flow) rate (Q) as the through variable. Heat source and temperature source are the two types of source elements. The former is more common. The latter may correspond to a large reservoir whose temperature is virtually not affected by heat transfer into or out of it. There is only one type of energy (thermal energy) in a thermal system. Hence there is only one type (A-type) energy storage element with the associated state variable, temperature. As discussed in Chapter 2, there is no T-type element in a thermal system.

4.5.3.1 Model Equations

In developing the model equations for a thermal system, the usual procedure is followed as for any other system. Specifically we write:

1. Constitutive equations (for thermal resistance and capacitance elements)
2. Node equations (the sum of heat transfer rate at a node is zero)
3. Loop equations (the sum of the temperature drop around a closed thermal path is zero)

Finally, we obtain the state-space model by eliminating the auxiliary variables that are not needed.

Example 4.10

A traditional Asian pudding is made by blending roughly equal portions by volume of treacle (a palm honey similar to maple syrup), coconut milk, and eggs, spiced with cloves and cardamoms, and baking in a special oven for about 1 hour. The traditional oven uses charcoal fire in an earthen pit that is well insulated, as the heat source. An aluminum container half filled with water is placed on fire. A smaller aluminum pot containing the dessert mixture is placed inside the water bath and covered fully with an aluminum lid. Both the water and the dessert mixture are well stirred and assumed to have uniform temperatures. A simplified model of the oven is shown in Figure 4.22a.

Assume that the thermal capacitances of the aluminum water container, dessert pot, and the lid are negligible. Also, the following equivalent (linear) parameters and variables are defined:

C_r = thermal capacitance of the water bath
C_d = thermal capacitance of the dessert mixture
R_r = thermal resistance between the water bath and the ambient air
R_d = thermal resistance between the water bath and the dessert mixture
R_c = thermal resistance between the dessert mixture and the ambient air, through the covering lid
T_r = temperature of the water bath
T_d = temperature of the dessert mixture
T_S = ambient temperature
Q = input heat flow rate from the charcoal fire into the water bath.

a. Assuming that T_d is the output of the system, develop a complete state-space model for the system. What are the system inputs?
b. In (a) suppose that the thermal capacitance of the dessert pot is not negligible, and is given by C_p. Also, as shown in Figure 4.22b, thermal resistances R_{p1} and R_{p2} are defined for the two interfaces of the pot. Assuming that the pot temperature is maintained uniform at T_p show how the state-space model of part (a) should be modified to include this improvement. What parameters do R_{p1} and R_{p2} depend on?
c. Draw the linear graphs for the systems in (a) and (b). Indicate in the graph only the system parameters, input variables, and the state variables.

Solution

a. For the water bath:

$$C_w \frac{dT_w}{dt} = Q - \frac{1}{R_w}(T_w - T_a) - \frac{1}{R_d}(T_w - T_d) \tag{i}$$

For the dessert mixture:

$$C_d \frac{dT_d}{dt} = \frac{1}{R_d}(T_w - T_d) - \frac{1}{R_c}(T_d - T_a) \tag{ii}$$

Equations (i) and (ii) are the state equations with:

$$\text{State vector } \boldsymbol{x} = [T_w, \quad T_d]^T$$

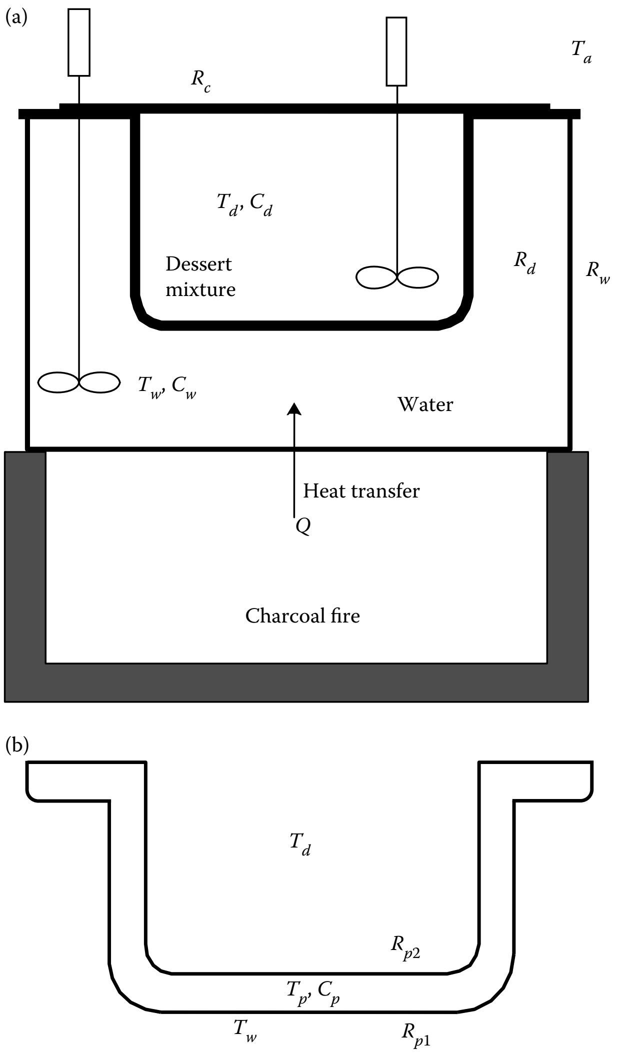

FIGURE 4.22
(a) A simplified model of an Asian dessert oven. (b) An improved model of the dessert pot.

$$\text{Input vector } \boldsymbol{u} = [Q, \quad T_a]^T$$

$$\text{Output vector } \boldsymbol{y} = [T_d]^T$$

The corresponding matrices of the state-space model are:

$$\boldsymbol{A} = \begin{bmatrix} -\dfrac{1}{C_w}\left(\dfrac{1}{R_w}+\dfrac{1}{R_d}\right) & \dfrac{1}{C_w R_d} \\ \dfrac{1}{C_d R_d} & -\dfrac{1}{C_d}\left(\dfrac{1}{R_d}+\dfrac{1}{R_c}\right) \end{bmatrix}; \quad \boldsymbol{B} = \begin{bmatrix} \dfrac{1}{C_w} & \dfrac{1}{C_w R_w} \\ 0 & \dfrac{1}{C_d R_c} \end{bmatrix}; \quad \boldsymbol{C} = [0 \quad 1]; \quad \boldsymbol{D} = [0 \quad 0]$$

b. For the dessert pot:

$$C_p \frac{dT_p}{dt} = \frac{1}{R_{p1}}(T_w - T_p) - \frac{1}{R_{p2}}(T_p - T_d) \tag{iii}$$

Equations (i) and (ii) have to be modified as

$$C_w \frac{dT_w}{dt} = Q - \frac{1}{R_w}(T_w - T_a) - \frac{1}{R_{p1}}(T_w - T_p) \tag{i*}$$

$$C_d \frac{dT_d}{dt} = \frac{1}{R_{p2}}(T_w - T_d) - \frac{1}{R_c}(T_d - T_a) \tag{ii*}$$

The system has become third order now, with the state Equations (i*), (ii*), and (iii) and the corresponding state vector:

$$\boldsymbol{x} = \begin{bmatrix} T_w & T_d & T_p \end{bmatrix}^T$$

But $\boldsymbol{u}$ and $\boldsymbol{y}$ remain the same as before. Matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ have to be modified accordingly. The resistance R_{pi} depends on the heat transfer area A_i and the heat transfer coefficient h_i. Specifically,

$$R_{pi} = \frac{1}{h_i A_i}$$

c. The linear graph for (a) is shown in Figure 4.23a. The linear graph for (b) is shown in Figure 4.23b.

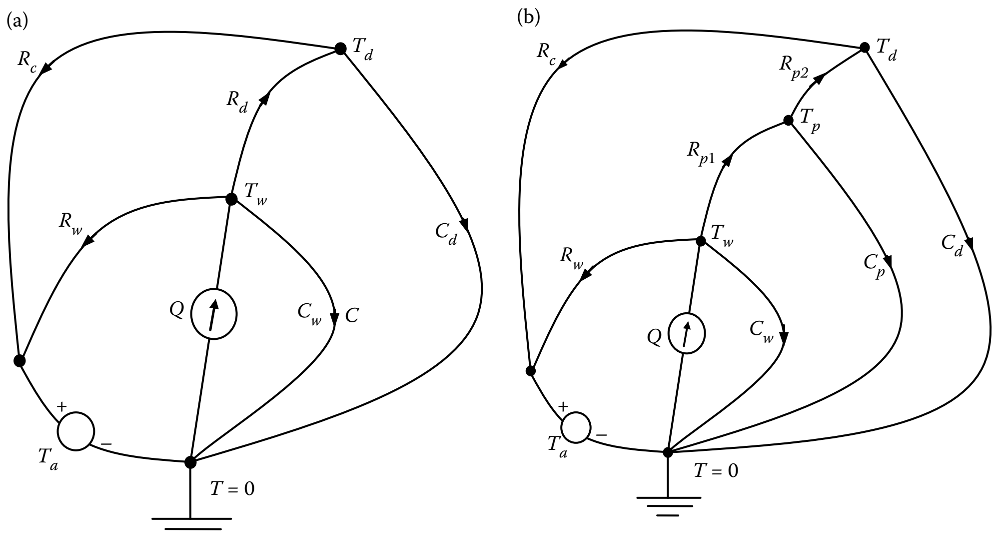

FIGURE 4.23
Linear graph of the (a) simplified model and (b) improved model.

Problems

PROBLEM 4.1

Select the correct answer for each of the following multiple-choice questions:

i. A through variable is characterized by
   (a) being the same at both ends of the element
   (b) being listed first in the pair representation of a linear graph
   (c) requiring no reference value
   (d) all the above
ii. An across variable is characterized by
   (a) having different values across the element
   (b) being listed second in the pair representation
   (c) requiring a reference point
   (d) all the above
iii. Which of the following could be a through variable?
   (a) pressure
   (b) voltage
   (c) force
   (d) all the above
iv. Which of the following could be an across variable?
   (a) motion (velocity)
   (b) fluid flow
   (c) current
   (d) all the above
v. If angular velocity is selected as an element's across variable, the accompanying through variable is
   (a) force
   (b) flow
   (c) torque
   (d) distance
vi. The equation written for through variables at a node is called
   (a) a continuity equation
   (b) a constitutive equation
   (c) a compatibility equation
   (d) all the above
vii. The functional relation between a through variable and its across variable is called
   (a) a continuity equation
   (b) a constitutive equation
   (c) a compatibility equation
   (d) a node equation
viii. The equation that equates the sum of across variables in a loop to zero is known as
   (a) a continuity equation
   (b) a constitutive equation
   (c) a compatibility equation
   (d) a node equation
ix. A node equation is also known as
   (a) an equilibrium equation
   (b) a continuity equation
   (c) the balance of through variables at the node
   (d) all the above

x. A loop equation is
   (a) a balance of across variables
   (b) a balance of through variables
   (c) a constitutive relationship
   (d) all the above

PROBLEM 4.2

A linear graph has ten branches, two sources, and six nodes.

i. How many unknown variables are there?
ii. What is the number of independent loops?
iii. How many inputs are present in the system?
iv. How many constitutive equations could be written?
v. How many independent continuity equations could be written?
vi. How many independent compatibility equations could be written?
vii. Do a quick check on your answers.

PROBLEM 4.3

The circuit shown in Figure P4.3 has an inductor L, a capacitor C, a resistor R, and a voltage source $v(t)$. Considering that L is analogous to a spring, and C is analogous to an inertia, follow the standard steps to obtain the state equations. First sketch the linear graph denoting the currents through and the voltages across the elements L, C, and R by (f_1, v_1), (f_2, v_2) and (f_3, v_3), respectively, and then proceed in the usual manner.

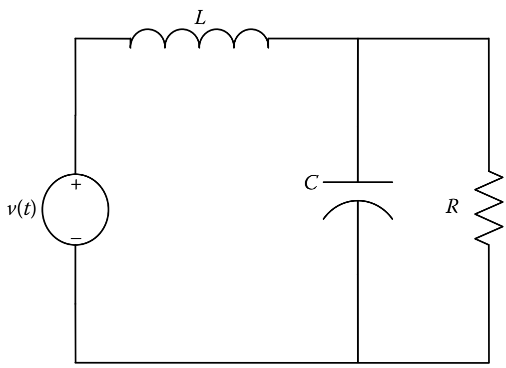

FIGURE P4.3
An electrical circuit.

i. What is the system matrix and what is the input distribution matrix for your choice of state variables?
ii. What is the order of the system?
iii. Briefly explain what happens if the voltage source $v(t)$ is replaced by a current source $i(t)$.

PROBLEM 4.4

Consider an automobile traveling at a constant speed on a rough road, as sketched in Figure P4.4a. The disturbance input due to road irregularities can be considered as a velocity source $u(t)$ at the tires in the vertical direction. An approximate one-dimensional model shown in Figure P4.4b may be used to study the "heave" (up and down) motion

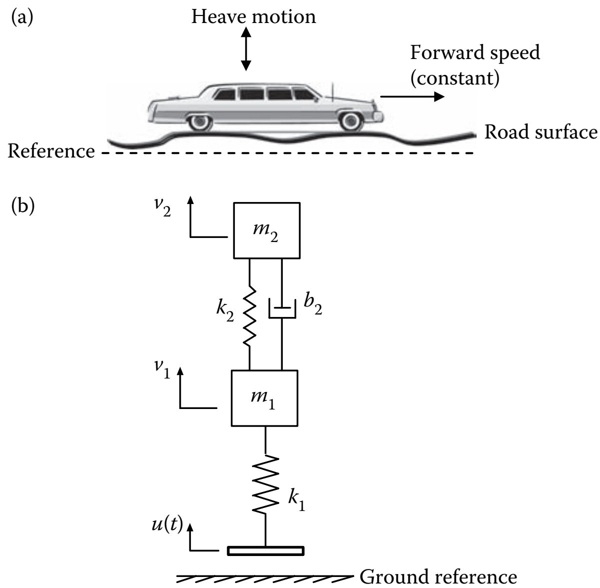

FIGURE P4.4
(a) An automobile traveling at constant speed. (b) A crude model of the automobile for the heave motion analysis.

of the automobile. Note that v_1 and v_2 are the velocities of the lumped masses m_1 and m_2, respectively.

a. Briefly state what physical components of the automobile are represented by the model parameters k_1, m_1, k_2, m_2, and b_2. Also, discuss the validity of the assumptions that are made in arriving at this model.
b. Draw a linear graph for this model, orient it (i.e., mark the directions of the branches), and completely indicate the system variables and parameters.
c. By following the step-by-step procedure of writing constitutive equations, node equations and loop equations, develop a complete state-space model for this system. The outputs are v_1 and v_2. What is the order of the system?
d. If instead of the velocity source $u(t)$, a force source $f(t)$ which is applied at the same location, is considered as the system input, draw a linear graph for this modified model. Obtain the state equations for this model. What is the order of the system now?

Note: In this problem you may assume that the gravitational effects are completely balanced by the initial compression of the springs with reference to which all motions are defined.

PROBLEM 4.5

Suppose that a linear graph has the following characteristics:
n=number of nodes
b=number of branches (segments)
s=number of sources
l=number of independent loops.

Carefully explaining the underlying reasons, answer the following questions regarding this linear graph:

a. From the topology of the linear graph show that $l = b - n + 1$.
b. What is the number of continuity equations required (in terms of n)?
c. What is the number of lumped elements including source elements in the model (expressed in terms of b and s)?
d. What is the number of unknown variables, both state and auxiliary, (expressed in terms of b and s)? Verify that this is equal to the number available equations, and hence the problem is solvable.

PROBLEM 4.6

An approximate model of a motor-compressor combination used in a process control application is shown n Figure P4.6.

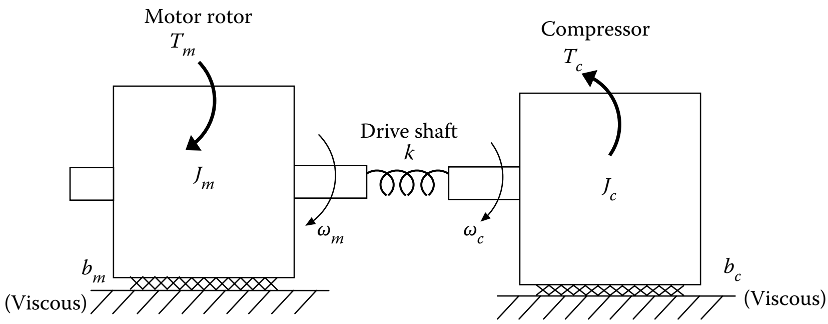

FIGURE P4.6
A model of a motor-compressor unit.

Note that T, J, k, b, and ω denote torque, moment of inertia, torsional stiffness, angular viscous damping constant, and angular speed, respectively, and the subscripts m and c denote the motor rotor and compressor impeller, respectively.

a. Sketch a translatory mechanical model that is analogous to this rotatory mechanical model.
b. Draw a linear graph for the given model, orient it, and indicate all necessary variables and parameters on the graph.
c. By following a systematic procedure and using the linear graph, obtain a complete state-space representation of the given model. The outputs of the system are compressor speed ω_c and the torque T transmitted through the drive shaft.

PROBLEM 4.7

A model for a single joint of a robotic manipulator is shown in Figure P4.7. The usual notation is used. The gear inertia is neglected and the gear reduction ratio is taken as 1:r (*Note*: $r < 1$).

a. Draw a linear graph for the model, assuming that no external (load) torque is present at the robot arm.
b. Using the linear graph derive a state model for this system. The input is the motor magnetic torque T_m and the output is the angular speed ω_r of the robot arm. What is the order of the system?

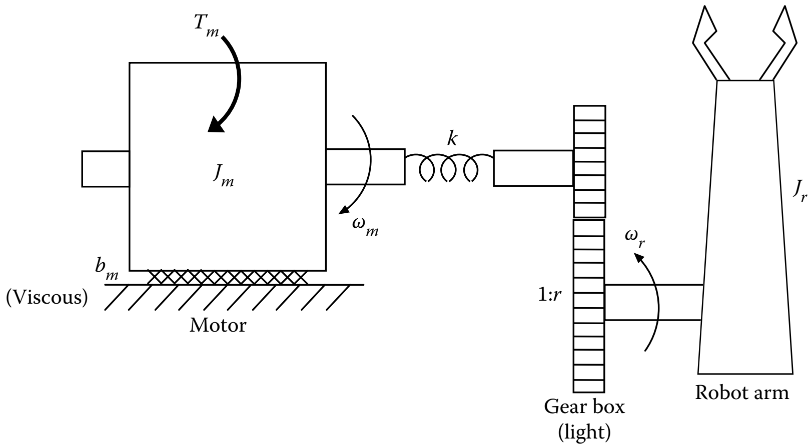

FIGURE P4.7
A model of a single-degree-of-freedom robot.

c. Discuss the validity of various assumptions made in arriving at this simplified model for a commercial robotic manipulator.

PROBLEM 4.8

Consider the rotatory feedback control system shown schematically by Figure P4.8a. The load has inertia J, stiffness K and equivalent viscous damping B as shown. The armature circuit for the dc fixed field motor is shown in Figure P4.8b.

The following relations are known:

The back e.m.f. $v_B = K_V\omega$

The motor torque $T_m = K_T i$

a. Identify the system inputs.
b. Write the linear system equations.

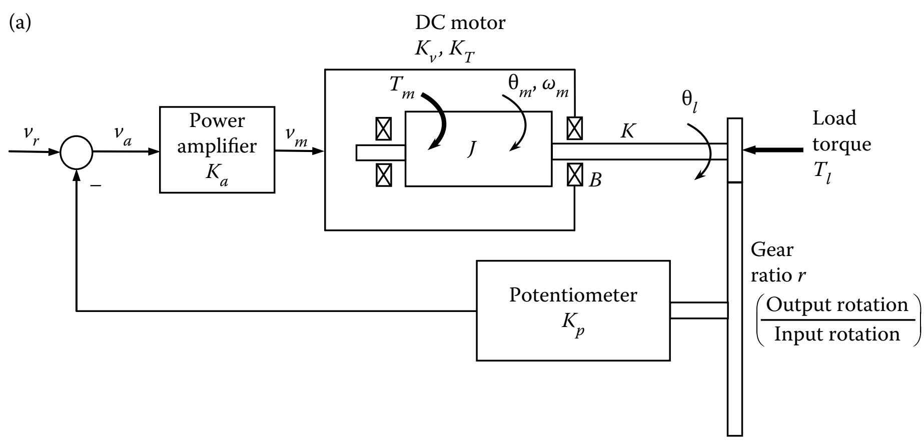

FIGURE P4.8
(a) A rotatory electromechanical system. (b) The armature circuit.

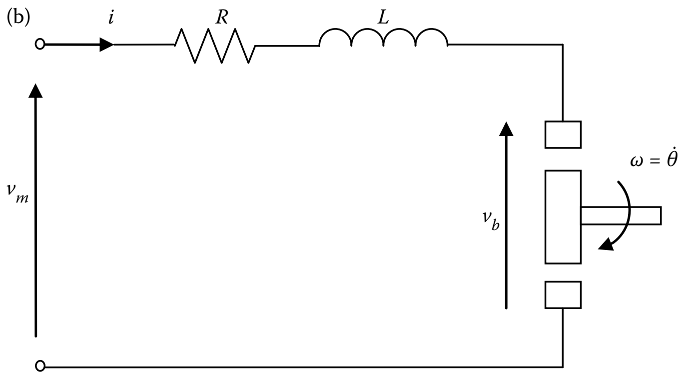

FIGURE P4.8 (continued)

PROBLEM 4.9

a. What is the main physical reason for oscillatory behavior in a purely fluid system? Why do purely fluid systems with large tanks connected by small-diameter pipes rarely exhibit an oscillatory response?
b. Two large tanks connected by a thin horizontal pipe at the bottom level are shown in Figure P4.9a. Tank 1 receives an inflow of liquid at the volume rate Q_i when the inlet valve is open. Tank 2 has an outlet valve, which has a fluid flow resistance of R_o and a flow rate of Q_o when opened. The connecting pipe also has a valve, and when opened, the combined fluid flow resistance of the valve and the thin pipe is R_p. The following parameters and variables are defined:

   C_1, C_2 = fluid (gravity head) capacitances of Tanks 1 and 2
   ρ = mass density of the fluid
   g = acceleration due to gravity
   P_1, P_2 = pressure at the bottom of Tanks 1 and 2
   P_0 = ambient pressure.

   Using $P_{10} = P_1 - P_0$ and $P_{20} = P_2 - P_0$ as the state variables and the liquid levels H_1 and H_2 in the two tanks as the output variables, derive a complete, linear, state-space model for the system.

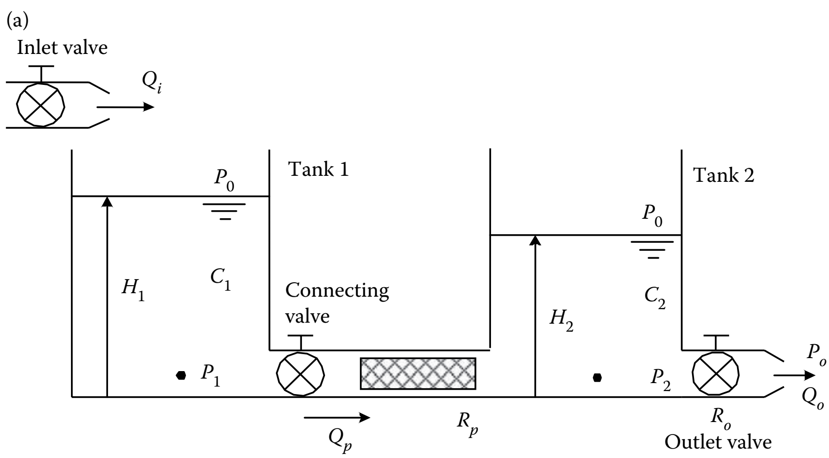

FIGURE P4.9
(a) An interacting two-tank fluid system. (b) A noninteracting two-tank fluid system.

c. Suppose that the two tanks are as in Figure P4.9b. Here Tank 1 has an outlet valve at its bottom whose resistance is R_t and the volume flow rate is Q_t when open. This flow directly enters Tank 2, without a connecting pipe. The remaining characteristics of the tanks are the same as in (b).
Derive a state-space model for the modified system in terms of the same variables as in (b).

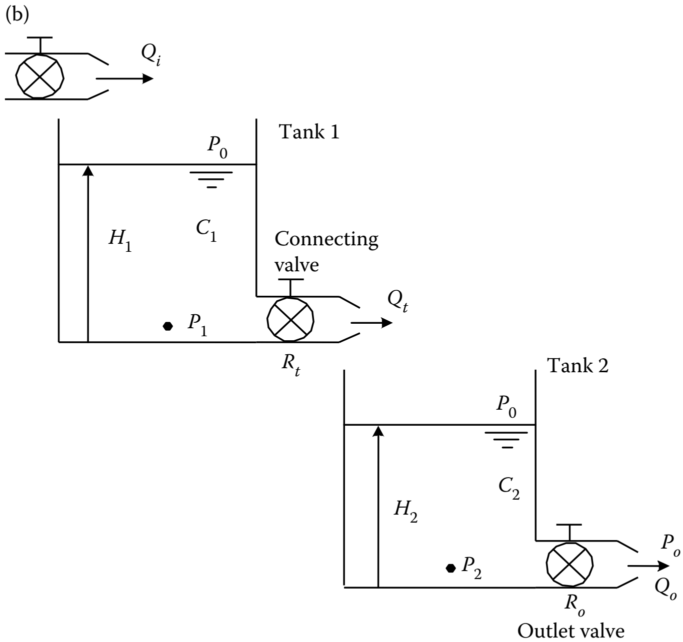

FIGURE P4.9 (continued)

PROBLEM 4.10

Give reasons for the common experience that in the flushing tank of a household toilet, some effort is needed to move the handle for the flushing action but virtually no effort is needed to release the handle at the end of the flush.

A simple model for the valve movement mechanism of a household flushing tank is shown in Figure P4.10. The overflow tube on which the handle lever is hinged, is assumed rigid. Also, the handle rocker is assumed light, and the rocker hinge is assumed frictionless.

The following parameters are indicated in the figure:

$r = l_v / l_h$ = the lever arm ratio of the handle rocker
m = equivalent lumped mass of the valve flapper and the lift rod
k = stiffness of spring action on the valve flapper.
The damping force f_{NLD} on the valve is assumed quadratic and is given by

$$f_{NLD} = a|v_{VLD}|v_{VLD}$$

where the positive parameter:
$a = a_u$ for upward motion of the flapper ($v_{NLD} \geq 0$)
$= a_d$ for downward motion of the flapper ($v_{NLD} < 0$)
with $a_u \gg a_d$

The force applied at the handle is $f(t)$, as shown.

We are interested in studying the dynamic response of the flapper valve. Specially, the valve displacement x and the valve speed v are considered outputs, as shown in Figure P4.10. Note that x is measured from the static equilibrium point of the spring where the weight mg is balanced by the spring force.

a. By defining appropriate through variables and across variables, draw a linear graph for the system shown in Figure P4.10. Clearly indicate the power flow arrows.
b. Using valve speed and the spring force as the state variables, develop a (nonlinear) state-space model for the system, with the aid of the linear graph. Specifically, start with all the constitutive, continuity, and compatibility equations, and eliminate the auxiliary variables systematically, to obtain the state-space model.
c. Linearize the state-space model about an operating point where the valve speed is $\bar{v}$. For the linearized model, obtain the model matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, in the usual notation. The incremental variables $\hat{x}$ and $\hat{v}$ are the outputs in the linear model, and the incremental variable $\hat{f}(t)$ is the input.
d. From the linearized state-space model, derive the input–output model (differential equation) relating $\hat{f}(t)$ and $\hat{x}$.

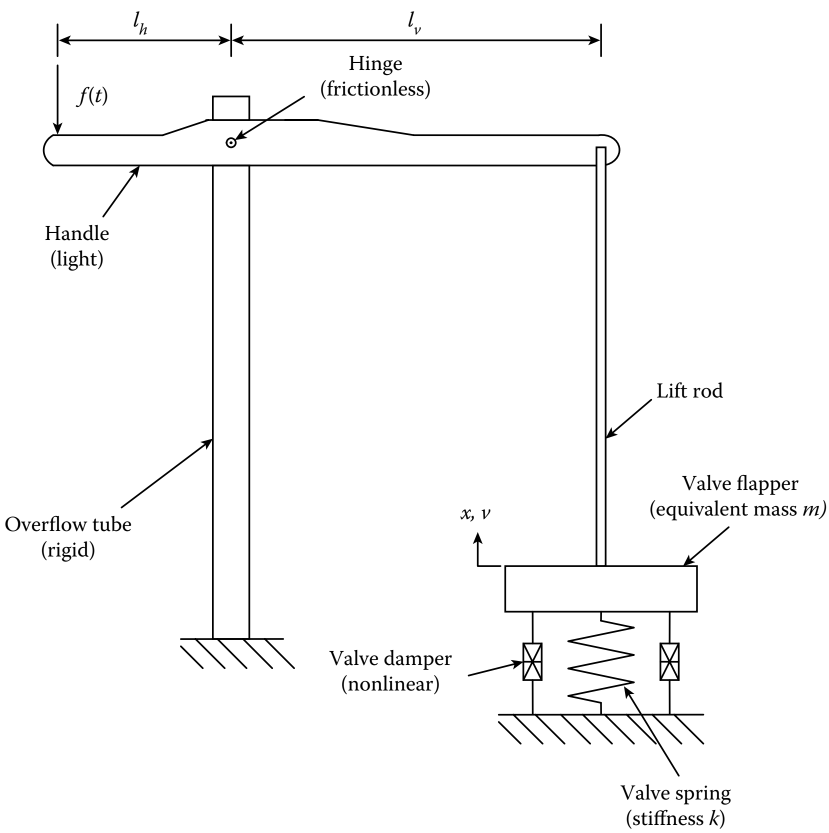

FIGURE P4.10
Simplified model of a toilet-flushing mechanism.

PROBLEM 4.11

A common application of dc motors is in accurate positioning of a mechanical load. A schematic diagram of a possible arrangement is shown in Figure P4.11. The actuator of the system is an armature-controlled dc motor. The moment of inertia of its rotor is J_r and the angular speed is ω_r. The mechanical damping of the motor (including that of its bearings) is neglected in comparison to that of the load.

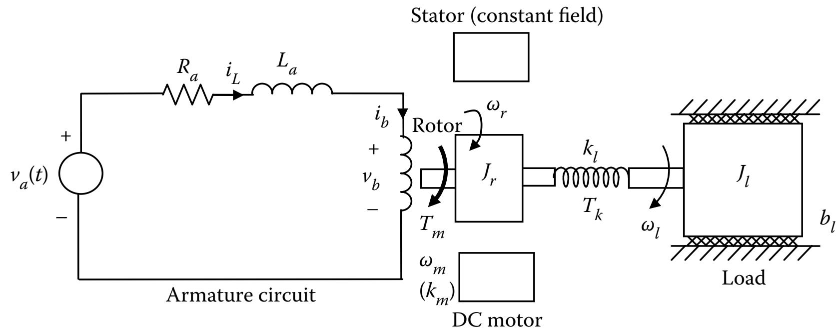

FIGURE P4.11
An electro-mechanical model of a rotatory positioning system.

The armature circuit is also shown in Figure P4.11, which indicates a back e.m.f. v_b (due to the motor rotation in the stator field), a leakage inductance L_a, and a resistance R_a. The current through the leakage inductor is i_L. The input signal is the armature voltage $v_a(t)$ as shown. The interaction of the rotor magnetic field and the stator magnetic field (*Note*: the rotor field rotates at an angular speed ω_m) generates a "magnetic" torque T_m which is exerted on the motor rotor.

The stator provides a constant magnetic field to the motor, and is not important in the present problem. The dc motor may be considered as an ideal electromechanical transducer which is represented by a linear-graph transformer. The associated equations are:

$$\omega_m = \frac{1}{k_m} v_b$$

$$T_m = -k_m i_b$$

where k_m is the torque constant of the motor. *Note*: The negative sign in the second equation arises due to the specific sign convention used for a transformer, in the conventional linear graph representation.

The motor is connected to a rotatory load of moment of inertia J_l using a long flexible shaft of torsional stiffness k_l. The torque transmitted through this shaft is denoted by T_k. The load rotates at an angular speed ω_l and experiences mechanical dissipation, which is modeled by a linear viscous damper of damping constant b_l.

Answer the following questions:

a. Draw a suitable linear graph for the entire system shown in Figure P4.11, mark the variables and parameters (you may introduce new, auxiliary variables but not new parameters), and orient the graph.
b. Give the number of branches (b), nodes (n), and the independent loops (l) in the complete linear graph. What relationship do these three parameters satisfy? How many independent node equations, loop equations, and constitutive equations

can be written for the system? Verify the sufficiency of these equations to solve the problem.

c. Take current through the inductor (i_L), speed of rotation of the motor rotor (ω_r), torque transmitted through the load shaft (T_k), and speed of rotation of the load (ω_l) as the four state variables; the armature supply voltage $v_a(t)$ as the input variable; and the shaft torque T_k and the load speed ω_l as the output variables. Write the independent node equations, independent loop equations, and the constitutive equations for the complete linear graph. Clearly show the state-space shell.
d. Eliminate the auxiliary variables and obtain a complete state-space model for the system, using the equations written in (c) above. Express the matrices ***A***, ***B***, ***C***, and ***D*** of the state-space model in terms of the system parameters R_a, L_a, k_m, J_r, k_l, b_l, and J_l only.

PROBLEM 4.12

Consider a multidomain engineering system that you are familiar with (in your projects, research, engineering practice, informed imagination, through literature which you have read, etc.). It should include the mechanical structural domain (i.e., with inertia, flexibility, and damping) and at least one other domain (e.g., electrical, fluid, thermal).

a. Using sketches, describe the system, by giving at least the following information:
   (i) The practical purpose and functions of the system.
   (ii) Typical operation/behavior of the system.
   (iii) System boundary.
   (iv) Inputs and outputs.
   (v) Characteristics of the main components of the system.
b. Sketch a lumped-parameter model of the system, by approximating any significant distributed effects using appropriate lumped elements, and showing how the lumped-parameter elements (including sources) are interconnected. You must justify your choice of elements and approximation decisions. Also, you must retain significant nonlinearities in the original system.
c. Develop an analytical model of the system by writing the necessary constitutive equations, continuity equations, and compatibility equations. The model should be at least fifth-order but not greater than tenth-order.
   Note: Draw a linear graph of the system (particularly if you plan to use the linear graph approach to obtain the analytical model).
d. Approximate the nonlinear elements by suitable linear elements.
e. Identify suitable state variables for the linear system and develop a complete state-space model (i.e., matrices ***A***, ***B***, ***C***, and ***D***) for the system.