THE Z-TRANSFORM

DEFINITION:

$$X(z) = \mathcal{Z}\{x(k)\} \triangleq \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \dots$$

Z-TRANSFORM OF COMMON SIGNALS:

K: time index
T: sampling time

- IMPULSE:

$$\mathcal{Z}\{\delta(k)\} = 1$$

- UNIT STEP:

$$\mathcal{Z}\{\mu(k)\} = \frac{z}{z-1}$$

- UNIT RAMP:

$$\mathcal{Z}\{kT\} = \frac{zT}{(z-1)^2}$$

- EXPONENTIAL:

$$\mathcal{Z}\{e^{-akT}\} = \frac{z}{z - e^{-aT}}$$

- SINUSOID:

$$\mathcal{Z}\{\cos(\omega kT)\} = \frac{z^2 - z\cos(\omega T)}{z^2 - 2z\cos(\omega T) + 1}$$

Z-TRANSFORM OPERATIONS:

0~2 properties are used most often

⓪ SUMMATION AND SCALING:

$$\mathcal{Z}\{ax(k) + by(k)\} = aX(z) + bY(z)$$

① DELAY (BACKWARD SHIFT):

$$\mathcal{Z}\{y(k-n)\} = z^{-n}Y(z)$$

② ADVANCE (FORWARD SHIFT):

$$\mathcal{Z}\{y(k+n)\} = z^{n}Y(z) - z^{n}y(0) - \dots - zy(n-1)$$

③ CUMULATIVE SUMMATION ("INTEGRATION"):

$$\mathcal{Z}\left\{\sum_{i=0}^{k} y(i)\right\} = Y(z) \cdot \frac{z}{z-1}$$

Discrete time integration of y

④ COMPLEX FREQUENCY SHIFT:

$$\mathcal{Z}\{a^k y(k)\} = Y\left(\frac{z}{a}\right)$$

⑤ MULTIPLICATION BY k:

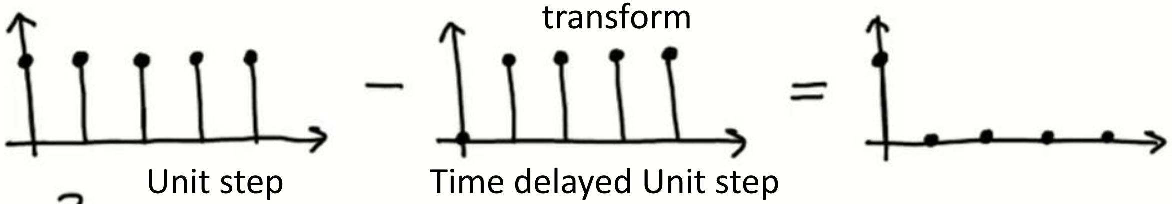
$$\mathcal{Z}\{ky(k)\} = -z\frac{dY(z)}{dz}$$

EXAMPLES OF USING THE Z-TRANSFORM:

Although we can use the table to get the transform of impulse, we will try to get it using properties and step transform

- EXAMPLE 1:

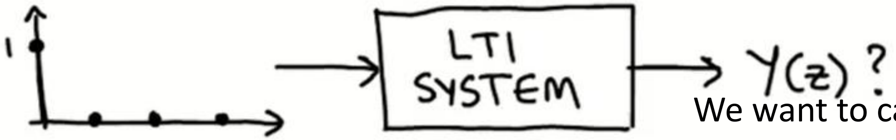
$\delta(k) = \mu(k) - \mu(k-1)$



$$\begin{aligned}
\mathcal{Z}\{\delta(k)\} &= \mathcal{Z}\{\mu(k) - \mu(k-1)\} \\
&= \mathcal{Z}\{\mu(k)\} - \mathcal{Z}\{\mu(k-1)\} && [\text{PROPERTY } 0] \\
&= \mathcal{Z}\{\mu(k)\} - z^{-1}\mathcal{Z}\{\mu(k)\} && [\text{PROPERTY } 1] \\
&= \frac{z}{z-1} - z^{-1} \cdot \frac{z}{z-1} && \left[\mathcal{Z}\{\mu(k)\} = \frac{z}{z-1}\right] \\
&= \frac{z-1}{z-1} = 1
\end{aligned}$$

- EXAMPLE 2: Difference equation

$y(k) = 0{,}95\,y(k-1) + r(k); \quad r(k) = \delta(k)$



We want to calculate the impulse response of the system

$$\mathcal{Z}\{y(k)\} = \mathcal{Z}\{0{,}95\,y(k-1) + \delta(k)\} = 0{,}95\,\mathcal{Z}\{y(k-1)\} + \mathcal{Z}\{\delta(k)\} \quad [\text{PROPERTY } 0]$$

$$\Rightarrow Y(z) = 0{,}95\,z^{-1}Y(z) + 1 \quad [\text{PROPERTY } 1;\ \mathcal{Z}\{\delta(k)\} = 1]$$

$$\Rightarrow (1 - 0{,}95\,z^{-1})Y(z) = 1 \Rightarrow Y(z) = \frac{1}{1 - 0{,}95\,z^{-1}} = \frac{z}{z - 0{,}95}$$

INVERSE Z-TRANSFORM USING TABLES:

- EXAMPLE 1:

$$Y(z) = \frac{z}{z-0{,}95} \; ; \; y(k)?$$

IN Z-TRANSFORM TABLE: $\mathcal{Z}\{e^{-akT}\} = \frac{z}{z-e^{-aT}}$

IF $e^{-aT} = 0{,}95 \Rightarrow e^{-akT} = (e^{-aT})^k = 0{,}95^k$

$$\Rightarrow y(k) = \begin{cases} 0 & , k < 0 \\ 0{,}95^k & , k \geq 0 \end{cases} = 0{,}95^k \mu(k)$$

Because it should be only in the positive time

- EXAMPLE 2:

$$X(z) = \frac{z}{(z-0{,}5)(z+0{,}3)} \; ; \; x(k)?$$

We divide first by z

$$\frac{X(z)}{z} = \frac{1}{(z-0{,}5)(z+0{,}3)} = \frac{A_1}{(z-0{,}5)} + \frac{A_2}{(z+0{,}3)}$$

Then use partial fraction expansion

$$A_1 = \left.\frac{1}{z+0{,}3}\right|_{z=0{,}5} = 1{,}25 \; ; \; A_2 = \left.\frac{1}{z-0{,}5}\right|_{z=-0{,}3} = -1{,}25$$

We calculate the residuals

$$\Rightarrow \frac{X(z)}{z} = \frac{1{,}25}{z-0{,}5} - \frac{1{,}25}{z+0{,}3} \Rightarrow X(z) = 1{,}25 \cdot \frac{z}{z-0{,}5} - 1{,}25 \cdot \frac{z}{z+0{,}3}$$

Multiply by z

$$\Rightarrow x(k) = \begin{cases} 0 & , k < 0 \\ 1{,}25(0{,}5)^k - 1{,}25(-0{,}3)^k & , k \geq 0 \end{cases} = \left[1{,}25(0{,}5)^k - 1{,}25(-0{,}3)^k\right]\mu(k)$$

INVERSE Z-TRANSFORM USING LONG DIVISION:

EXAMPLE:

$$X(z) = \frac{z}{(z-0{,}5)(z+0{,}3)} = \frac{z}{z^2 - 0{,}2z - 0{,}15} \quad ; \quad x(k)?$$

The idea is to find an infant series that when multiplied by the denominator will give us the numerator

$$\begin{array}{r|l} & z^{-1} + 0{,}2z^{-2} + 0{,}19z^{-3} + \dots \\ \hline z^2 - 0{,}2z - 0{,}15 & z \quad \text{numerator} \\ \text{denominator} & \underline{z - 0{,}2 - 0{,}15z^{-1}} \\ & \quad 0{,}2 + 0{,}15z^{-1} \\ & \quad \underline{0{,}2 - 0{,}04z^{-1} - 0{,}03z^{-2}} \\ & \qquad 0{,}19z^{-1} + 0{,}03z^{-2} \end{array}$$

$$\Rightarrow X(z) = z^{-1} + 0{,}2z^{-2} + 0{,}19z^{-3} + \dots$$

COMPARE WITH Z-TRANSFORM DEFINITION:

$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots$$

$$\Rightarrow x(k) = 0,\ 1,\ 0{,}2,\ 0{,}19,\ \dots$$

This method is good to find the few first values of a discrete signal **<u>but not to find the an analytical description of the signal</u>**

INVERSE Z-TRANSFORM USING DELAY OPERATION:

And we want to calculate the output

For that we will use **z transform and inverse z transform**

EXAMPLE:

Difference equation — Apply impulse to the input

$$y(k) = 0{,}95\,y(k-1) + r(k-1)\ ;\quad r(k) = \delta(k)\ ;\quad y(k)\,?$$

$$Y(z) = 0{,}95\,z^{-1}Y(z) + z^{-1}\cdot 1 \qquad \text{[APPLIED Z-TRFM; PROPERTY 1 ; } \mathcal{Z}\{\delta(k)\} = 1\text{]}$$

$$\Rightarrow (1 - 0{,}95z^{-1})Y(z) = z^{-1}$$

Not in the table so we multiply by z to power -1

$$\Rightarrow Y(z) = \frac{z^{-1}}{1-0{,}95z^{-1}} = \frac{1}{z-0{,}95} = z^{-1}\cdot\underbrace{\left(\frac{z}{z-0{,}95}\right)}_{X(z)} = z^{-1}X(z)$$

$$x(k) = 0{,}95^k\mu(k) \qquad \text{[Z-TRFM TABLES]}$$

$$y(k) = x(k-1) \qquad \text{[PROPERTY 1]}$$

$$= 0{,}95^{k-1}\mu(k-1)$$

$$= \begin{cases} 0 & , k < 1 \\ 0{,}95^{k-1} & , k \geq 1 \end{cases}$$