A. Consider the following function, $f(x) = 7e^{-5x}$. Based on these, answer the following questions:

1. (2 marks) Approximate the derivative of $f(x)$ at $x_0 = 1.0$ with step size $h = 0.1$ using the forward and central difference methods up to 5 significant figures.
2. (4 marks) Determine the upper bound of truncation error of $f(x)$ at $x_0 = 1.0$. Using step size $h = 0.1$ for the backward and central difference methods up to 5 significant figures.
3. (4 marks) Compute $f'(1.1)$ with step size $h = 0.1$ using backward difference method, and also calculate the relative error. Use 5 significant figures.

1.

Forward Difference

$$f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$$

$$= \frac{f(1+0.1)-f(1)}{0.1}$$

$$= \frac{f(1.1)-f(1)}{0.1}$$

$$= \frac{(7e^{-5\times1.1})-(7e^{-5\times1})}{0.1}$$

$$=-\ 0.18558 \qquad (5\ s.f.)$$

Central Difference

$$f'(x_0) = \frac{f(x_0+h)-f(x_0-h)}{2h}$$

$$= \frac{f(1+0.1)-f(1-0.1)}{0.1}$$

$$= \frac{f(1.1)-f(0.9)}{0.1}$$

$$= \frac{(7e^{-5\times1.1})-(7e^{-5\times0.9})}{0.1}$$

$$=-\ 0.49156 \qquad (5\ s.f.)$$

2.

Backward Difference

$$Truncation\ Error\ = \frac{f^{(2)}(\xi)}{2}(h)$$

Central Difference

$$Truncation\ Error\ = \frac{f^{(3)}(\xi)}{6}(h)^2$$

Given that-

$$f(x) = 7e^{-5x}$$

$$\Rightarrow f'(x) = -35e^{-5x}$$

$$\Rightarrow f^{2}(x) = 175e^{-5x}$$

$$\Rightarrow f^{3}(x) = -875e^{-5x}$$

Now,

For Backward Difference,

$$Interval: \left[x_0 - h,\ x_0\right] = [1 - 0.1,\ 1] = [0.9,\ 1]$$

$$\therefore f^{2}(x = 0.9) = 1.9441 \ (5\ s.f.)$$ ✅

$$\therefore f^{2}(x = 1) = 1.1791 \ (5\ s.f.)$$ ❌

$$\therefore Upper\ bound\ of\ Truncation\ Error = \frac{f^{(2)}(\xi)}{2}(h)$$

$$= \frac{1.9441}{2} \times 0.1$$

$$= 0.097205 \quad (5\ s.f.)$$

For Central Difference,

$$Interval: \left[x_0 - h,\ x_0 + h\right] = [1 - 0.1,\ 1 + 0.1] = [0.9,\ 1.1]$$

$$\therefore f^{3}(x = 0.9) = -9.7204 \ (5\ s.f.)$$ ❌

$$\therefore f^{3}(x = 1.1) = -3.5759 \ (5\ s.f.)$$ ✅

$$\therefore Upper\ bound\ of\ Truncation\ Error = \frac{f^{(3)}(\xi)}{6}(h)^2$$

$$= \frac{-3.5759}{6}(0.1)^2$$

$$= -5.95983 \times 10^{-3} \quad (5\ s.f.)$$

3.

Backward Difference

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h}$$

$$f'(1.1) = \frac{f(1.1) - f(1.1 - 0.1)}{0.1}$$

$$= \frac{f(1.1) - f(1)}{0.1}$$

$$= \frac{\left(7e^{-5\times1.1}\right) - \left(7e^{-5\times1}\right)}{0.1}$$

$$= -\ 0.18558 \qquad (5\ s.f.)$$

Given that-

$$f(x) = 7e^{-5x}$$

$$\Rightarrow f'(x) = -\ 35e^{-5x}$$

$$\Rightarrow f'(x = 1.1) = -\ 35e^{-5\times1.1} = -\ 0.14304 \qquad (5\ s.f.)$$

$$\therefore Relative\ Error = \left|\frac{f'(x) - Backward\ Difference}{f'(x)}\right|$$

$$= \left|\frac{-0.14304 - (-0.18558)}{-0.14304}\right|$$

$$= \left|\frac{-0.14304 + 0.18558}{-0.14304}\right|$$

$$= 0.29739 \qquad (5\ s.f.)$$

B. Read the following questions and answer accordingly:

1. (5 marks) Deduce an expression for $D_h^{(1)}$ from D_h by replacing h with $\frac{h}{3}$ using Richardson extrapolation method.
2. (3 marks) If $f(x) = -9e^{7x} + 4x^3$, compute $D_{0.2}^{(1)}$ at $x = 2.7$ using Richardson extrapolation method up to 4 significant figures and calculate the truncation error.
3. (2 marks) Following B(2), if $f(x) = -9e^{7x} + 4x^3$, compute $D_{0.2}^{(2)}$ at $x = 2.7$ using Richardson extrapolation method up to 4 significant figures .

1.

$$D_h = \frac{f(x+h)-f(x-h)}{2h}$$

<u>From Taylor series-</u>

$$f(x) = f(x_0) + f^1(x_0)(x - x_0) + \frac{f^2(x_0)}{2!}(x - x_0)^2 + \frac{f^3(x_0)}{3!}(x - x_0)^3 + \dots\dots\dots$$

$Now,$

$$f(x + h) = f(x) + f^1(x)(h) + \frac{f^2(x)}{2!}(h)^2 + \frac{f^3(x)}{3!}(h)^3 + \frac{f^4(x)}{4!}(h)^4 + \frac{f^5(x)}{5!}(h)^5 + O(h^6)$$

$$f(x - h) = f(x) - f^1(x)(h) + \frac{f^2(x)}{2!}(h)^2 - \frac{f^3(x)}{3!}(h)^3 + \frac{f^4(x)}{4!}(h)^4 - \frac{f^5(x)}{5!}(h)^5 + O(h^6)$$

$So,$ $\left[5^{th}\ order\ approximation\right]$

$$D_h = \frac{f(x+h)-f(x-h)}{2h}$$

$$\Rightarrow D_h = \frac{1}{2h}\left(2f^1(x)(h) + 2\frac{f^3(x)}{3!}(h)^3 + 2\frac{f^5(x)}{5!}(h)^5 + O(h^7)\right)$$

$$D_h = f^1(x) + \frac{f^3(x)}{3!}(h)^2 + \frac{f^5(x)}{5!}(h)^4 + O(h^6) \qquad -\ (1)$$

$$D_{\frac{h}{3}} = f^1(x) + \frac{f^3(x)}{3!}\left(\frac{h}{3}\right)^2 + \frac{f^5(x)}{5!}\left(\frac{h}{3}\right)^4 + O(h^6) \qquad -\ (2)$$

$Now,\ 4 \times (2) - (1) \rightarrow$

$$9 \times D_{\frac{h}{3}} - D_h = 9 \times \left(f^1(x) + \frac{f^3(x)}{3!}\left(\frac{h}{3}\right)^2 + \frac{f^5(x)}{5!}\left(\frac{h}{3}\right)^4 + O(h^6)\right)$$
$$- \left(f^1(x) + \frac{f^3(x)}{3!}(h)^2 + \frac{f^5(x)}{5!}(h)^4 + O(h^6)\right)$$

$$= 8f^{1}(x) + 0 - \frac{8}{9}\frac{f^{5}(x)}{5!}(h)^{4} + O\left(h^{6}\right)$$

$$\frac{9\times D_{\frac{h}{3}} - D_{h}}{8} = f^{1}(x) - \frac{1}{9}\frac{f^{5}(x)}{5!}(h)^{4} + O\left(h^{6}\right)$$

$$\therefore D_{h}^{(1)} = f^{1}(x) - \frac{1}{9}\frac{f^{5}(x)}{5!}(h)^{4} + O\left(h^{6}\right)$$

2.

$$D_{h} = \frac{f(x+h)-f(x-h)}{2h}$$

$$Considering\ \ h = 0.2\ and\ \frac{h}{2} = 0.1,$$

$$D_{h} = D_{0.2} = \frac{f(2.7+0.2)-f(2.7-0.2)}{2\times 0.2}$$

$$= \frac{f(2.9)-f(2.5)}{0.4}$$

$$= \frac{-5894140511.82314-(-358422997.07818)}{0.4}$$

$$=- 1.384 \times 10^{10} \qquad (4\ s.f.)$$

$$D_{\frac{h}{2}} = D_{0.1} = \frac{f(2.7+0.1)-f(2.7-0.1)}{2\times 0.1}$$

$$= \frac{f(2.8)-f(2.6)}{0.2}$$

$$= \frac{-2926943517.28982-(-721775336.34142)}{0.2}$$

$$=- 1.103 \times 10^{10} \qquad (4\ s.f.)$$

$$\therefore D_{h=0.2}^{(1)} = \frac{2^{2}\times D_{\frac{h}{2}} - D_{h}}{2^{2}-1} = \frac{4\times D_{\frac{h}{2}} - D_{h}}{4-1} = \frac{4\times D_{\frac{h}{2}} - D_{h}}{3}$$

$$= \frac{\left(4\times\left(-1.103\times10^{10}\right)\right)-\left(-1.384\times10^{10}\right)}{3} =- 1.009 \times 10^{10} (4\ s.f.)$$

Given that-

$$f(x) =- 9e^{7x} + 4x^{3}$$

$$\Rightarrow f'(x) =- 63e^{7x} + 12x^{2}$$

$$\Rightarrow f'(x = 2.7) =- 63e^{7\times2.7} + 12\times(2.7)^{2} =- 1.017 \times 10^{10} (4\ s.f.)$$

$$\therefore Truncation\ error = |Actual\ derivative - approx.\ derivative|$$

$$= \left|\left(- 1.017 \times 10^{10}\right) - \left(- 1.009 \times 10^{10}\right)\right|$$

$$= 0.08434 \times 10^{9} \quad (4\ s.f.)$$

3.

$$Considering\ h = 0.1\ and\ \frac{h}{2} = 0.05,$$

$$D_{\frac{0.2}{2}} = D_{0.1} = D_{h} = \frac{f(2.7+0.1)-f(2.7-0.1)}{2\times0.1}$$

$$= \frac{f(2.8)-f(2.6)}{0.2}$$

$$= \frac{-2926943517.28982-(-721775336.34142)}{0.2}$$

$$=- 1.103 \times 10^{10} \qquad (4\ s.f.)$$

$$D_{\frac{0.2}{4}} = D_{0.05} = D_{\frac{h}{2}} = \frac{f(2.7+0.05)-f(2.7-0.05)}{2\times0.05}$$

$$= \frac{f(2.75)-f(2.65)}{0.1}$$

$$= \frac{-2062582214.60328-(-1024247982.504723)}{0.1}$$

$$=- 1.038 \times 10^{10} \qquad (4\ s.f.)$$

$$\therefore D_{h=0.1}^{(1)} = \frac{2^2 \times D_{\frac{h}{2}} - D_h}{2^2 - 1} = \frac{4 \times D_{\frac{h}{2}} - D_h}{4-1} = \frac{4 \times D_{\frac{h}{2}} - D_h}{3}$$

$$= \frac{\left(4 \times \left(-1.038 \times 10^{10}\right)\right) - \left(-1.103 \times 10^{10}\right)}{3} = -\ 1.016 \times 10^{10} (4\ s.f.)$$

$So,$

$$D_{h=0.2}^{(2)} = \frac{2^4 \times D_{\frac{h}{2}}^{(1)} - D_h^{(1)}}{2^4 - 1} = \frac{16 \times D_{\frac{h}{2}}^{(1)} - D_h^{(1)}}{16-1} = \frac{16 \times D_{\frac{h}{2}}^{(1)} - D_h^{(1)}}{15}$$

$$= \frac{\left(16 \times \left(-1.016 \times 10^{10}\right)\right) - \left(-1.009 \times 10^{10}\right)}{15} = -\ 1.016 \times 10^{10} (4\ s.f.)$$