MCQ: Choose Only One Answer

A function is to be interpolated using four nodes: $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$ by Lagrange method. Answer the following:

1. The Vandermonde Matrix corresponding to these data values is of order

   A. 1×1. **B.** 2×2. **C.** 3×3. **D.** 4×4.

   1. **D**

2. The interpolation polynomial for the given system is of order 3 when solved by Vandermonde method. If the system is solved by the Lagrange method, how many Lagrange basis elements will be needed?

   A. 2. **B.** 3. **C.** 4. **D.** 5.

   2. **C**

3. The Lagrange basis element, $l_1(x)$, for the given system is a

   A. cubic function of x. **B.** quadratic function of x. **C.** linear function of x. **D.** None of the above.

   3. **A**

4. The Newton basis element $n_2(x)$ is a

   A. cubic function of x. **B.** quadratic function of x. **C.** linear function of x. **D.** None of the above.

   4. **B**

Problem solving

5. (3 marks) The Taylor series expansion of $f(x) = \sin x + \cos x$ around $x_0 = 0$ is given by

$$f(x) = \sin(x) + \cos(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \cdots .$$

If $x = 0.1$, how many terms from the Taylor expansion you need to consider to agree with exact value of $f(0.1)$ up to six significant figure. What is the degree of the corresponding Taylor polynomial? Hint: first find the exact value and then consider the terms to match with that value.

Solution: The exact value within 6 significant figures is: $f(0.1) = \sin(0.1) + \cos(0.1) = 1.094837\cdots = 1.09484$. Lets now compute $f(0.1)$ by using the Taylor expansion. We get,

$$f(0.1) = \begin{cases} 1 & & ; \text{ This is NOT equal to the exact value.} \\ 1 + 0.1 & = 1.1 & ; \text{ This is NOT equal to the exact value.} \\ 1 + 0.1 - \frac{(0.1)^2}{2!} & = 1.095 & ; \text{ This is NOT equal to the exact value.} \\ 1 + 0.1 - \frac{(0.1)^2}{2!} - \frac{(0.1)^3}{3!} & = 1.09483 & ; \text{ This is NOT equal to the exact value.} \\ 1 + 0.1 - \frac{(0.1)^2}{2!} - \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!} & = 1.09484 & ; \text{ This is EQUAL to the exact value.} \end{cases}$$

Clearly, we need to keep up to the fifth term from the Taylor expansion, and it is a degree four Polynomial, $p_4(x)$.

6. (3 marks) Using the properties of the Lagrange bases, $l_i(x_j) = 1$ if $i = j$ and 0 if $i \neq j$ for $i, j = 0, 1, 2, 3, 4$, show that the Lagrange basis element, $l_2(x)$, can be expressed as

$$l_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} .$$

Solution: From the given information, clearly the degree $n = 4$. Therefore, the properties of the Lagrange bases satisfies: $l_2(x) = 0$ at $x = x_0 = x_1 = x_3 = x_4$ and $l_2(x) = 1$ at $x = x_2$. Hence, we can write, $l_2(x_0) = l_2(x_1) = l_2(x_3) = l_2(x_4) = 0$ and $l_2(x_2) = 1$. Hence according to the properties of a polynomial x_0, x_1, x_3 and x_4 are roots of $l_2(x)$ and so it can be written as

$$l_2(x) \propto (x - x_0)(x - x_1)(x - x_3)(x - x_4) \quad \Longrightarrow \quad l_2(x) = A(x - x_0)(x - x_1)(x - x_3)(x - x_4) ,$$

where A is the proportionality constant. Now using $l_2(x_2) = 1$, we obtain,

$$l_2(x_2) = A(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4) = 1 \quad \Longrightarrow \quad A = \frac{1}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} .$$

Substituting this, we can express $l_2(x)$ as

$$l_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} . \quad \checkmark$$