. Consider the following linear system:

$$6y + 2z = 10$$
$$3x + 2y + z = 6$$
$$4x + 5y + 2z = 9$$

1. (3 marks) From the given system, identify the matrices A, x and b such that this system can be expressed as a matrix equation. Also, explain how to find this system has a unique solution or not.
2. (2 marks) Write down $Aug(A)$ for this system and explain why the Gaussian elimination method fails to solve this system. Also explain how we can overcome the problem to actually solve it. (You don't have to solve this system for this question)
3. (2+3 marks) Find the upper triangular matrix U and compute the solution of this system by Gaussian elimination method.

1.

Here,

$$6y + 2z = 10$$

$$3x + 2y + z = 6$$

$$4x + 5y + 2z = 9$$

$$A = \begin{bmatrix} 0 & 6 & 2 \\ 3 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$b = \begin{bmatrix} 10 \\ 6 \\ 9 \end{bmatrix}$$

$Here,\ det(A) = 2\ which\ is\ not\ zero\ (non\ singular)\ and\ square\ matrix.$

$So,\ it\ is\ solvable\ or\ have\ unique\ solution.$

2.

$Augmented\ Matrix,\ Aug(A) \rightarrow$

$$\left[\begin{array}{ccc|c} 0 & 6 & 2 & 10 \\ 3 & 2 & 1 & 6 \\ 4 & 5 & 2 & 9 \end{array}\right]$$

Gaussian elimination fails if we ever hit a zero on the diagonal, even if the matrix A is non-singular. Here, the first element a_{11} is 0.

We know that if any diagonal element is zero, the multiplier for the subsequent row operation will be undefined. As a result, the Gaussian elimination method fails there. To overcome this, we need to apply pivoting, which is swapping rows or columns.

3.

$Augmented\ Matrix,\ Aug(A) \rightarrow$

$$\left[\begin{array}{ccc|c} 0 & 6 & 2 & 10 \\ 3 & 2 & 1 & 6 \\ 4 & 5 & 2 & 9 \end{array}\right]$$

$After\ swapping\ row 1\ and\ row\ 2,\ the\ augmented\ matrix,\ Aug(A)\ \rightarrow$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 6 \\ 0 & 6 & 2 & 10 \\ 4 & 5 & 2 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 6 \\ 0 & 6 & 2 & 10 \\ 4 & 5 & 2 & 9 \end{array}\right] \qquad \left[r'_3 = r_3 - \tfrac{4}{3} \times r_1\ ;\ m_{31} = \tfrac{4}{3}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 6 \\ 0 & 6 & 2 & 10 \\ 0 & \frac{7}{3} & \frac{2}{3} & 1 \end{array}\right] \qquad \left[r'_3 = r_3 - \frac{7}{18} r_2\ ;\ m_{32} = \frac{\frac{7}{3}}{6} = \frac{7}{18}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 6 \\ 0 & 6 & 2 & 10 \\ 0 & 0 & -\frac{1}{9} & -\frac{26}{9} \end{array}\right]$$

Upper triangular matrix $\rightarrow$

$$U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & -\frac{1}{9} \end{bmatrix}$$

So,

$$z = \frac{-\frac{26}{9}}{-\frac{1}{9}} = 26$$

$$y = \frac{10-(2)(26)}{6} = -7$$

$$x = \frac{6-2(-7)-1(26)}{3} = -2$$

B. A linear system is described by the following equations:

$$\begin{gathered} a+b+c=6 \\ 2a+3b+4c=20 \\ 3a+4b+2c=17 \end{gathered}$$

1. (3 marks) Construct the Frobenius matrices $F^{(1)}$ and $F^{(2)}$ from this system.
2. (2 marks) Evaluate the unit lower triangular matrix L using the Frobenius matrices.
3. (5 marks) Now find the solution of the linear system using LU decomposition method.

1.

Here,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 17 \end{bmatrix}$$

$$A^{(1)} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2; m_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{1} = 3$$

$$F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$A^{(2)} = F^{(1)} \times A^{(1)}$$

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$m_{32} = \frac{a_{32}}{a_{21}} = \frac{1}{1} = 1$$

$$F^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

2.

$Unit\ lower\ triangular\ matrix \rightarrow$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

3.

$Find\ A^{(3)} from\ F^{(2)} \times A^{(2)} \rightarrow$

$A^{(3)} = F^{(2)} \times A^{(2)}$

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A^{(3)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} = U$$

$So,$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}}_{U}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 17 \end{bmatrix}$$

$y_1 = 6,\ y_2 = 8,\ y_3 =- 9$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ -9 \end{bmatrix}$$

$c = 3,\ b = 2,\ a = 1$