Solution : Assignment-7

#1 (a) Here: $f(x) = xe^{-x} + x^3 - \sin x$.

(i) $g_1(x) = e^{x}(\sin x - x^3) = x \Rightarrow \sin x - x^3 = xe^{-x} \Rightarrow \boxed{xe^{-x} + x^3 - \sin x = 0}$

(ii) $g_2(x) = \ln\left(\frac{x}{\sin x - x^3}\right) = x \Rightarrow \frac{x}{\sin x - x^3} = e^{x} \Rightarrow xe^{-x} = \sin x - x^3$
$\Rightarrow \boxed{xe^{-x} + x^3 - \sin(x) = 0}$

(iii) $g_3(x) = \sin^{-1}(xe^{-x} + x^3) = x \Rightarrow xe^{-x} + x^3 = \sin x$
$\Rightarrow \boxed{xe^{-x} + x^3 - \sin x = 0}$

(b) For superlinear $g(x)$, we have to use Newton's method.

$\therefore g(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g(x) = x - \frac{xe^{-x} + x^3 - \sin x}{(1-x)e^{-x} + 3x^2 - \cos x}$

Now: $\delta = 1 \times 10^{-5} = 0.00001$. Let's choose $x_0 = 0.5$

k	x_k	$x_{k+1} = g(x_k)$	$f(x_k)$	less than δ
0	0.5	0.79121	0.05110	NO
1	0.79121	0.67878	0.14776	NO
2	0.67878	0.64070	0.02920	"
3	0.64070	0.63610	0.00285	"
4	0.63610	0.63603	0.00004	"
5	0.63603	—	3×10^{-6}	Yes

Therefore $\boxed{x_5 \approx x_p = 0.63603}$

#2 (a) Here: $x_0 = 0$; $x_1 = 0.05$, $x_2 = 0.10$, $x_3 = 0.15$, $x_4 = 0.20$.

$f(x_0) = 0$; $f(x_1) = 1.3$, $f(x_2) = 2.5$; $f(x_3) = 3.8$; $f(x_4) = 5.1$

Need to find: $P_1(x) = a_0 + a_1 x$.

Hence: $A = \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{pmatrix}$; $x = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ and $b = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{pmatrix}$

$\therefore A = \begin{pmatrix} 1 & 0 \\ 1 & 0.05 \\ 1 & 0.10 \\ 1 & 0.15 \\ 1 & 0.20 \end{pmatrix}$ & $b = \begin{pmatrix} 0 \\ 1.3 \\ 2.5 \\ 3.8 \\ 5.1 \end{pmatrix}$

$\therefore A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0.05 & 0.10 & 0.15 & 0.20 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0.05 \\ 1 & 0.10 \\ 1 & 0.15 \\ 1 & 0.20 \end{pmatrix} = \begin{pmatrix} 5 & 0.5 \\ 0.5 & 0.075 \end{pmatrix}$

and $A^T b = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0.05 & 0.10 & 0.15 & 0.20 \end{pmatrix} \begin{pmatrix} 0 \\ 1.3 \\ 2.5 \\ 3.8 \\ 5.1 \end{pmatrix} = \begin{pmatrix} 12.7 \\ 1.905 \end{pmatrix}$

Therefore: $(A^T A)x = A^T b \Rightarrow \begin{pmatrix} 5 & 0.5 \\ 0.5 & 0.075 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 12.7 \\ 1.905 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 5 & 0.5 \\ 0.5 & 0.075 \end{pmatrix}^{-1} \begin{pmatrix} 12.7 \\ 1.905 \end{pmatrix} = \begin{pmatrix} 0.6 & -4 \\ -4 & 40 \end{pmatrix} \begin{pmatrix} 12.7 \\ 1.905 \end{pmatrix} = \begin{pmatrix} 0 \\ 25.4 \end{pmatrix}$

So, $a_0 = 0$ & $a_1 = 25.4$.

Hence: $\boxed{P_1(x) \equiv F(x) = 25.4x}$ & $\boxed{k = 25.4}$ is the spring constant.

(b) $A \equiv (u_1 \; u_2) \Rightarrow u_1 = (1\;1\;1\;1\;1)^T$ & $u_2 = (0\;1.3\;2.5\;3.8\;5.1)^T$

$\therefore p_1 = u_1 = (1\;1\;1\;1\;1)^T \Rightarrow |p_1| = \sqrt{5} \Rightarrow q_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

$p_2 = u_2 - (u_2^T q_1) q_1 = \begin{pmatrix} 0 \\ 1.3 \\ 2.5 \\ 3.8 \\ 5.1 \end{pmatrix} - \frac{1}{5} \left[(0\;1.3\;2.5\;3.8\;5.1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 1.3 \\ 2.5 \\ 3.8 \\ 5.1 \end{pmatrix} - \frac{12.7}{5} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2.54 \\ -1.24 \\ -0.04 \\ 1.26 \\ 2.56 \end{pmatrix} \Rightarrow |p_2| = 4.02 \Rightarrow q_2 = 4.0164$

$$\therefore q_2 = \begin{pmatrix} -0.6324 \\ -0.3087 \\ -0.00996 \\ 0.3137 \\ 0.6374 \end{pmatrix} \Rightarrow Q = \begin{pmatrix} \frac{1}{\sqrt{5}} & -0.6324 \\ \frac{1}{\sqrt{5}} & -0.3087 \\ \frac{1}{\sqrt{5}} & -0.00996 \\ \frac{1}{\sqrt{5}} & 0.3137 \\ \frac{1}{\sqrt{5}} & 0.6374 \end{pmatrix}$$

$$\therefore R = Q^T A = \begin{pmatrix} \sqrt{5} & \frac{\sqrt{5}}{10} \\ 0 & 0.158 \end{pmatrix} \quad \& \quad Q^T Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $\delta = 10^{-6}$.

#3 (a) $I(f) = \int_{-0.15}^{0.25} (xe^{-x} + x^3 \sin x)\,dx = \underline{-0.00495}$.

(b) $I_2(f) = \left(\frac{0.25+0.15}{6}\right)\left(f(-0.15) + 4f(0.05) + f(0.25)\right) = \underline{-0.00496}$.

(c) $C_{1,4}(f) = \left(\frac{0.1}{2}\right)\left[f(-0.15) + 2f(-0.05) + 2f(0.05) + 2f(0.15) + f(0.25)\right]$

$= \underline{-0.00546}$.