Set-A

Consider a function $f(x) = ln(x)$ which is continuous on the interval $[1, 3]$. Use this function to answer the following:

1. (1+1 marks) Find the $I_1(f)$ and $I_2(f)$ values for the function.
2. (2 marks) Evaluate the numerical integral $C_{1,3}$ by using the Composite Newton-Cotes formula.
3. (2.5+2.5+1 marks) Compute the upper bound error for $I_1(f)$ and $I_2(f)$ of the function. State which method yields the better result?

Set-B

Consider a function $f(x) = e^{-x}$ which is continuous on the interval $[1, 3]$. Use this function to answer the following:

1. (1+1 marks) Find the $I_1(f)$ and $I_2(f)$ values for the function.
2. (2 marks) Evaluate the numerical integral $C_{1,3}$ by using the Composite Newton-Cotes formula.
3. (2.5+2.5+1 marks) Compute the upper bound error for $I_1(f)$ and $I_2(f)$ of the function. State which method yields the better result?

Set-A

1.

$Given,\ f(x) = ln(x)$

$$f(a) = f(1) = ln(1) = 0$$

$$f(b) = f(3) = ln(3) = 1.099$$

$$f(m) = f\left(\frac{a+b}{2}\right) = f\left(\frac{1+3}{2}\right) = ln(2) = 0.693$$

$$I(f) = \int_1^3 (ln(x))dx = 1.296 \qquad [Actual\ Integration]$$

$$I_1(f) = \frac{b-a}{2}[f(a) + f(b)]$$

$$\Rightarrow I_1(f) = \frac{3-1}{2}[0 + 1.099] = 1.099$$

$$I_2(f) = \frac{b-a}{6}[f(a) + 4f(m) + f(b)]$$

$$\Rightarrow I_2(f) = \frac{3-1}{6}[0 + (4 \times 0.693) + 1.099]$$

$$\Rightarrow I_2(f) = \frac{1}{3}[3.871] = 1.290$$

2.

$$h = \frac{b-a}{m} = \frac{3-1}{3} = \frac{2}{3}$$

$If\ m = 3,\ then\ need\ to\ find\ x_0\ to\ x_3\ nodes$

$$x_0 = a = 1$$

$$x_1 = x_0 + h = 1 + \frac{2}{3} = \frac{5}{3}$$

$$x_2 = x_1 + h = \frac{5}{3} + \frac{2}{3} = \frac{7}{3}$$

$$x_3 = x_2 + h = \frac{7}{3} + \frac{2}{3} = \frac{9}{3} = 3$$

$Composite\ Newton\ Cotes\ formulae \rightarrow$

$$C_{1,m}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \ldots\ldots + 2f(x_{m-1}) + f(x_m)\right]$$

$$So,\ C_{1,3}(f) = \frac{\frac{2}{3}}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{\frac{2}{3}}{2}\left[f(1) + 2f\left(\frac{5}{3}\right) + 2f\left(\frac{7}{3}\right) + f(3)\right]$$

$$= \frac{1}{3}\left[(ln(1)) + 2\left(ln\left(\frac{5}{3}\right)\right) + 2\left(ln\left(\frac{7}{3}\right)\right) + (ln(3))\right]$$

$$= \frac{1}{3} \times 3.815 = 1.272$$

3.

For I_1, *upper bound error* $\rightarrow$

$$\left|I - I_n\right| \le \left|\frac{f^{(n+1)}\xi}{(n+1)!}\right| \int_a^b \left|\left(x - x_0\right)\left(x - x_1\right)\ldots\ldots\left(x - x_n\right)\right| dx$$

$$\Rightarrow \left|I - I_1\right| \le \left|\frac{f^{(2)}\xi}{2!}\right| \int_1^3 \left|\left(x - x_0\right)\left(x - x_1\right)\right| dx$$

$$\Rightarrow \left|I - I_1\right| \le \left|\frac{f^{(2)}\xi}{2!}\right| \int_1^3 |(x - a)(x - b)|\, dx$$

$$Now,\ finding\ the\ max\ of \left|\frac{f^{(2)}\xi}{2!}\right| within\ [1, 3]$$

$$= \left|\frac{f^2\xi}{2!}\right| = \left|\frac{1}{2} \times \left(-\frac{1}{\xi^2}\right)\right| \qquad \left[where\ f^2(\xi) = -\frac{1}{\xi^2}\right]$$

$$= \left|\frac{1}{2} \times \left(-\frac{1}{3^2}\right)\right| = \frac{1}{18}$$

$$Now,\ finding\ the\ max\ of \int_0^2 |(x - a)(x - b)|\, dx\ \ within\ [1, 3]$$

$$= \int_1^3 |(x - a)(x - b)|\, dx$$

$$= \int_1^3 |(x - 1)(x - 3)|\, dx$$

$$= \frac{4}{3}$$

$$So,\ upper\ bound\ of\ error\ \le \frac{1}{18} \times \frac{4}{3} \approx 0.0741$$

$For\ I_2\ ,\ upper\ bound\ error \rightarrow$

$$\left|I - I_n\right| \le \left|\frac{f^{(n+1)}\xi}{(n+1)!}\right| \int_a^b \left|\left(x - x_0\right)\left(x - x_1\right)\ldots\ldots\left(x - x_n\right)\right| dx$$

$$\Rightarrow \left|I - I_2\right| \le \left|\frac{f^{(3)}\xi}{3!}\right| \int_1^3 \left|\left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right)\right| dx$$

$$\Rightarrow \left|I - I_2\right| \le \left|\frac{f^{(3)}\xi}{3!}\right| \int_1^3 \left|(x - a)\left(x - \frac{a+b}{2}\right)(x - b)\right| dx$$

$Now,\ finding\ the\ max\ of \left|\frac{f^{(3)}\xi}{3!}\right| within\ [1, 3]$

$$= \left|\frac{f^3\xi}{3!}\right| = \left|\frac{1}{6} \times \left(\frac{2}{\xi^3}\right)\right| \qquad \left[where\ f^3(\xi) = \frac{2}{\xi^3}\right]$$

$$= \left|\frac{1}{6} \times \left(\frac{2}{3^3}\right)\right| = \frac{1}{81}$$

$Now,\ finding\ the\ max\ of \int_1^3 \left|(x - a)\left(x - \frac{a+b}{2}\right)(x - b)\right| dx\ \ within\ [1, 3]$

$$= \int_1^3 \left|(x - a)\left(x - \frac{a+b}{2}\right)(x - b)\right| dx$$

$$= \int_1^3 \left|(x - 1)\left(x - \frac{1+3}{2}\right)(x - 3)\right| dx$$

$$= \int_1^3 |(x - 1)(x - 2)(x - 3)|\ dx$$

$$= 0.0015 \times 10^{-12}$$

$So,\ upper\ bound\ of\ error\ \le \frac{1}{81} \times 0.0015 \times 10^{-12} \approx 1.852 \times 10^{-17}$

$\therefore Since\ upper\ bound\ of\ error\ I_2(f) < I_1(f),\ I_2(f)\ have\ better\ result.$

Set-B

1.

$Given,\ f(x) = e^{-x}$

$f(a) = f(1) = e^{-1} = 0.368$

$f(b) = f(3) = e^{-3} = 0.0498$

$f(m) = f\left(\frac{a+b}{2}\right) = f\left(\frac{1+3}{2}\right) = e^{-2} = 0.1353$

$I(f) = \int_1^3 \left(e^{-x}\right)dx = 0.3181$ $\quad$ $[Actual\ Integration\ ]$

$I_1(f) = \frac{b-a}{2}\ [f(a) + f(b)]$

$\Rightarrow I_1(f) = \frac{3-1}{2}\ [0.368 + 0.0498] = 0.4178$

$I_2(f) = \frac{b-a}{6}\ [f(a) + 4\,f(m) + f(b)]$

$\Rightarrow I_2(f) = \frac{3-1}{6}\ [0.368 + (4 \times 0.1353) + 0.0498]$

$\Rightarrow I_2(f) = \frac{1}{3}\ [0.959] = 0.3197$

2.

$h = \frac{b-a}{m} = \frac{3-1}{3} = \frac{2}{3}$

$If\ m = 3,\ then\ need\ to\ find\ x_0\ to\ x_3\ nodes$

$x_0 = a = 1$

$x_1 = x_0 + h = 1 + \frac{2}{3} = \frac{5}{3}$

$x_2 = x_1 + h = \frac{5}{3} + \frac{2}{3} = \frac{7}{3}$

$x_3 = x_2 + h = \frac{7}{3} + \frac{2}{3} = \frac{9}{3} = 3$

$Composite\ Newton\ Cotes\ formulae \rightarrow$

$C_{1,m}(f) = \frac{h}{2}\left[f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + 2f\left(x_3\right) + \ldots\ldots + 2f\left(x_{m-1}\right) + f\left(x_m\right)\right]$

$So,\ C_{1,3}(f) = \frac{\frac{2}{3}}{2}\left[f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + f\left(x_3\right)\right]$

$$= \frac{\frac{2}{3}}{2}\left[f(1) + 2f\left(\frac{5}{3}\right) + 2f\left(\frac{7}{3}\right) + f(3)\right]$$

$$= \frac{1}{3}\left[\left(e^{-1}\right) + 2\left(e^{-\frac{5}{3}}\right) + 2\left(e^{-\frac{7}{3}}\right) + \left(e^{-3}\right)\right]$$

$$= \frac{1}{3} \times 0.9894 = 0.3298$$

3.

For I_1, upper bound error →

$$\left|I - I_n\right| \le \left|\frac{f^{(n+1)}\xi}{(n+1)!}\right| \int_a^b \left|\left(x - x_0\right)\left(x - x_1\right)\ldots\ldots\left(x - x_n\right)\right| dx$$

$$\Rightarrow \left|I - I_1\right| \le \left|\frac{f^{(2)}\xi}{2!}\right| \int_1^3 \left|\left(x - x_0\right)\left(x - x_1\right)\right| dx$$

$$\Rightarrow \left|I - I_1\right| \le \left|\frac{f^{(2)}\xi}{2!}\right| \int_1^3 |(x - a)(x - b)|\, dx$$

Now, finding the max of $\left|\frac{f^{(2)}\xi}{2!}\right|$ *within* $[1, 3]$

$$= \left|\frac{f^2\xi}{2!}\right| = \left|\frac{1}{2} \times \left(e^{-\xi}\right)\right| \qquad \left[where\ f^2(\xi) = e^{-\xi}\right]$$

$$= \left|\frac{1}{2} \times \left(e^{-3}\right)\right| = 0.0166$$

Now, finding the max of $\int_0^2 |(x - a)(x - b)|\, dx$ *within* $[1, 3]$

$$= \int_1^3 |(x - a)(x - b)|\, dx$$

$$= \int_1^3 |(x - 1)(x - 3)|\, dx$$

$$= \frac{4}{3}$$

So, upper bound of error $\le 0.0166 \times \frac{4}{3} \approx 0.02213$

$For\ I_2\ ,\ upper\ bound\ error \rightarrow$

$$\left|I - I_n\right| \leq \left|\frac{f^{(n+1)}\ \xi}{(n+1)!}\right| \int_a^b \left|\left(x - x_0\right)\left(x - x_1\right) \ldots\ldots \left(x - x_n\right)\right| dx$$

$$\Rightarrow \left|I - I_2\right| \leq \left|\frac{f^{(3)}\ \xi}{3!}\right| \int_1^3 \left|\left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right)\right| dx$$

$$\Rightarrow \left|I - I_2\right| \leq \left|\frac{f^{(3)}\ \xi}{3!}\right| \int_1^3 \left|(x - a)\left(x - \frac{a+b}{2}\right)(x - b)\right| dx$$

$Now,\ finding\ the\ max\ of\ \left|\frac{f^{(3)}\ \xi}{3!}\right|\ within\ [1, 3]$

$$= \left|\frac{f^{3}\ \xi}{3!}\right| = \left|\frac{1}{6} \times \left(- e^{-\xi}\right)\right| \qquad \left[where\ f^{3}(\xi) = - e^{-\xi}\right]$$

$$= \left|\frac{1}{6} \times \left(- e^{-\xi}\right)\right| = 0.0083$$

$Now,\ finding\ the\ max\ of \int_1^3 \left|(x - a)\left(x - \frac{a+b}{2}\right)(x - b)\right| dx\ \ within\ [1, 3]$

$$= \int_1^3 \left|(x - a)\left(x - \frac{a+b}{2}\right)(x - b)\right| dx$$

$$= \int_1^3 \left|(x - 1)\left(x - \frac{1+3}{2}\right)(x - 3)\right| dx$$

$$= \int_1^3 |(x - 1)(x - 2)(x - 3)|\ dx$$

$$= 0.0015 \times 10^{-12}$$

$So,\ upper\ bound\ of\ error\ \ \leq 0.0083 \times 0.0015 \times 10^{-12} \approx 1.245 \times 10^{-17}$

$\therefore Since\ upper\ bound\ of\ error\ I_2(f)\ <\ I_1(f),\ I_2(f)\ \ have\ better\ result.$