'Assignment-04'

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Answer to the ques. no- 01

ⓐ

given function,

$$f(x) = x^3 - x^2 - 4x + 4$$

the function has three roots and one root is $x_* = 1$.

So, $(x-1)$ is a factor of $f(x)$. Hence, $f(x)$ is divisible by $(x-1)$.

$$x-1\,)\; x^3 - x^2 - 4x + 4 \;(\, x^2 - 4$$
$$(-)\; \cancel{x^3} - \cancel{x^2}$$
$$-\cancel{4x} + \cancel{4}$$
$$(-)\; -\cancel{4x} + \cancel{4}$$
$$0$$

So,

$$x^2 - 4 = 0$$

$$\Rightarrow x = \pm 2$$

So, the remaining two roots of the function $f(x)$ are $+2$ and -2. (Ans.)

(b)

We have,

$$f(x) = x^3 - x^2 - 4x + 4$$

Now, we have to construct two fixed point functions $g(x)$ such that,

$$f(x) = 0$$

$$\Rightarrow x^3 - x^2 - 4x + 4 = 0$$

firstly,

$$x^3 - x^2 - 4x + 4 = 0$$

$$\Rightarrow x^3 - x^2 - 4x = -4$$

$$\Rightarrow x(x^2 - x - 4) = -4$$

$$\Rightarrow x = \frac{-4}{x^2 - x - 4} = g_1(x)$$

$$\therefore g_1(x) = \frac{-4}{x^2 - x - 4}$$

again,

$$x^3 - x^2 - 4x + 4 = 0$$

$$\Rightarrow x^2 = x^3 - 4x + 4$$

$$\Rightarrow x = \pm\sqrt{x^3 - 4x + 4}$$

$$\Rightarrow x = \sqrt{x^3 - 4x + 4}$$ [using only non-negative]

$$\therefore\ g_2(x) = \sqrt{x^3 - 4x + 4}$$

So, our two fixed point functions are,

$$g_1(x) = \frac{-4}{x^2 - x - 4}$$

$$g_2(x) = \sqrt{x^3 - 4x + 4} \qquad \text{(Ans.)}$$

(c)

from 'b'

$$g_1(x) = \frac{-4}{x^2 - x - 4}$$

$$g_2(x) = \sqrt{x^3 - 4x + 4}$$

we have, $x_* = 1, -2, 2$ for $f(x) = x^3 - x^2 - 4x + 4$.

Now, to know the convergence rate,

$$\lambda = \left| \frac{d\, g(x)}{dx} \right|_{x = x_*}$$

firstly, for $g_1(x)$,

$$\lambda = \left| \frac{d}{dx}\, g_1(x) \right|_{x = 1, -2, 2}$$

$$\lambda = \left| \frac{d}{dx}\left(\frac{-4}{x^2 - x - 4} \right) \right|_{x = 1, -2, 2}$$

$$\Rightarrow \lambda = \left| (-4) * \frac{-1(2x-1)}{(x^2 - x - 4)^2} \right|_{x = 1, -2, 2}$$

$$\Rightarrow \lambda = \left| \frac{4(2x-1)}{(x^2 - x - 4)^2} \right|_{x = 1, -2, 2}$$

Now, for $x_* = 1$,

$$\lambda = |g_1'(x_*)| = |g_1'(1)| = \left| \frac{4(2 \cdot 1 - 1)}{(1^2 - 1 - 4)^2} \right| = \frac{1}{4} = 0.25$$

$$\Rightarrow \lambda = 0.25$$

So, this is linearly converging.

for $x_* = -2$,

$$\lambda = |g_1'(x_*)| = |g_1'(-2)| = \left| \frac{4\{2 \cdot (-2) - 1\}}{\{(-2)^2 - (-2) - 1\}} \right| = |-5| = 5$$

$$\Rightarrow \lambda = 5 > 1$$

this ~~root~~ is diverging.

for $x_* = 2$,

$$\lambda = |g_1'(x_*)| = |g_1'(2)| = \left| \frac{4(2\cdot 2-1)}{(2^2-2-1)^2} \right| = 3$$

$\Rightarrow \lambda = 3 > 1$

$\therefore$ this ~~root~~ is diverging.

Therefore, $g_1(x)$ converges to $x_* = 1$ and the convergence rate, $\lambda = 0.25$.

Again, for $g_2(x)$

$$\lambda = \left| \frac{d}{dx}\left(g_2(x)\right) \right|_{x_* = 1, -2, 2}$$

$$\Rightarrow \lambda = \left| \frac{d}{dx}\left(\sqrt{x^3 - 4x + 4}\right) \right|_{x_* = 1, -2, 2}$$

$$\Rightarrow \lambda = \left| \frac{1 \times (3x^2 - 4)}{2\sqrt{x^3 - 4x + 4}} \right|_{x_* = 1, -2, 2}$$

$$\Rightarrow \lambda = \left| \frac{3x^2 - 4}{2\sqrt{x^3 - 4x + 4}} \right|_{x_* = 1, -2, 2}$$

Now, for $x_* = 1$,

$$\lambda = |g_2'(x_*)| = |g_2'(1)| = \left| \frac{3 \cdot 1^2 - 4}{2\sqrt{1^3 - 4 \cdot 1 + 4}} \right| = \left| -\frac{1}{2} \right| = 0.5$$

$$\Rightarrow \lambda = 0.5$$

So, this ~~root~~ is linearly converging.

for $x_* = -2$,

$$\lambda = |g_2'(x_*)| = |g_2'(-2)| = \left| \frac{3 \cdot (-2)^2 - 4}{2\sqrt{(-2)^3 - 4(-2) + 4}} \right| = 2$$

$$\Rightarrow \lambda = 2 > 1$$

this is diverging.

for $x_* = 2$,

$$\lambda = |g_2'(x_*)| = |g_2'(2)| = \left| \frac{3 \cdot 2^2 - 4}{2\sqrt{2^3 - 4 \cdot 2 + 4}} \right| = 2$$

$$\Rightarrow \lambda = 2 > 1$$

this is diverging.

finally, $g_2(x)$ converges to $x_* = 1$ with convergence rate $\lambda = 0.5$. (Ans.)

Ans. to the que. no-02

given function,

$$f(x) = x^3 - x + \sin(x)$$

and, $x_0 = 1.5$, $\delta = 10^{-5}$

Now, to find the superlinear fixed point function $g(x)$, we can use Newton's method.

In this method,

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$\Rightarrow g(x) = x - \frac{x^3 - x + \sin(x)}{\frac{d}{dx}(x^3 - x + \sin(x))}$$

$$\therefore g(x) = x - \frac{x^3 - x + \sin x}{3x^2 - 1 + \cos x}$$

Now, let's check the critical value of the function $f(x)$, so,

$$f'(x) = 0$$

$$\Rightarrow \frac{d}{d(x)}(x^3 - x + \sin x) = 0$$

$$\Rightarrow 3x^2 - 1 + \cos x = 0$$

$$\therefore x = 0$$

Now, let's check two successive iteration starting with $x_0 = 1.5$ to x_1 and see if the critical value lies between the iterations.

$$x_1 = g(x_0) = g(1.5) = 1.5 - \frac{(1.5)^3 - 1.5 + \sin 1.5}{3(1.5)^2 - 1 + \cos(1.5)}$$

$$\Rightarrow \quad x_1 = 1.00651$$

$$x_2 = g(x_1) = g(1.00651) = 1.00651 - \frac{(1.00651)^3 - 1.00651 + \sin(1.00651)}{3(1.00651)^2 - 1 + \cos(1.00651)}$$

$$\Rightarrow x_2 = 0.673131$$

So, critical value is not in between x_1 and x_2.

Now, let's proceed with the Newton's method to find the root, given that $x_0 = 1.5$.

K	x_K	$f(x_K)$	Is $\lvert f(x_K) \rvert \leq \delta = 10^{-5}$?
0	$x_0 = 1.5$	2.87249	No
1	$x_1 = g(x_0)$ $\Rightarrow x_1 = 1.00651$	0.858118	No
2	$x_2 = g(x_1)$ $\Rightarrow x_2 = 0.673129$	0.255303	No

3	$x_3 = g(x_2)$ $\Rightarrow x_3 = 0.449411$	0.0757917	No
4	$x_4 = g(x_3)$ $\Rightarrow x_4 = 0.299806$	0.0224765	No
5	$x_5 = g(x_4)$ $\Rightarrow x_5 = 0.199930$	6.66233×10^{-3}	No
6	$x_6 = g(x_5)$ $\Rightarrow x_6 = 0.133305$	1.9744×10^{-3}	No
7	$x_7 = g(x_7)$ $\Rightarrow x_7 = 0.088875$	5.85049×10^{-4}	No
8	$x_8 = g(x_7)$ $\Rightarrow x_8 = 0.0592516$	1.73354×10^{-4}	No
9	$x_9 = g(x_8)$ $\Rightarrow x_9 = 0.0395015$	5.13649×10^{-5}	No
10	$x_{10} = g(x_9)$ $\Rightarrow x_{10} = 0.0263345$	1.52194×10^{-5}	No
11	$x_{11} = g(x_{10})$ $\Rightarrow x_{11} = 0.0175564$	4.50948×10^{-6}	Yes

So, $x_{11} = 0.0175564$ is our root within the bound 10^{-5} and 11 iterations are needed to find the root. (Ans.)