1. (3+3 marks) Compute the Chebyshev points and then the Chebyshev nodes for $n = 5$ for the interval $[-2, 3]$.
Solution: Since $n = 5$, there will be 6 Chebyshev points to be computed by using

$$\phi_j = \frac{(2j+1)\pi}{2(n+1)} \; ; \quad j = 0, 1, \cdots, n \quad \text{and} \quad n = 5 \; .$$

Therefore, we easily find the Chebyshev points as

$$\phi_0 = \frac{\pi}{12},\ \phi_1 = \frac{3\pi}{12},\ \phi_2 = \frac{5\pi}{12},\ \phi_3 = \frac{5\pi}{12},\ \phi_4 = \frac{7\pi}{12},\ \phi_5 = \frac{9\pi}{12} \text{ and } \phi_6 = \frac{11\pi}{12} \; .\checkmark$$

To compute the Chebyshev nodes, we notice that the interval $I \equiv= [a, b] = [-2, 3]$ is not symmetric with respect to the origin. Here, we have $a = -2$ and $b = 3$. Hence the circle is not centered at the origin as well. The center is shifted by $(a + b)/2$ and the radius of the circle is $(b - a)/2$. Therefore, the Chebyshev nodes will be given by

$$\begin{aligned}
x_j &= \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) \cos\phi_j \; , \\
&= \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right) \cos\left[\frac{(2j+1)\pi}{2(n+1)}\right]; \quad \text{with } n = 5 \text{ and } j = 0, 1, \cdots, n. \\
&= \left(\frac{-2+3}{2}\right) + \left(\frac{3-(-2)}{2}\right) \cos\left[\frac{(2j+1)\pi}{2(5+1)}\right]; \quad \text{with } n = 5 \text{ and } j = 0, 1, \cdots, n. \\
&= 0.5 + 2.5\cos\left[\frac{(2j+1)\pi}{12}\right] \; , \quad \text{for } j = 0, 1, \cdots, 5.
\end{aligned}$$

Keeping up to 5 significant figures, the Chebyshev nodes are (using the above equation),

$$x_0 = 2.9148,\ x_1 = 2.2678,\ x_2 = 1.1471,\ x_3 = -0.14705,\ x_4 = -1.2678 \text{ and } x_5 = -1.9148.\checkmark$$

2. (6+6+2 marks) Consider the function $f(x) = \sin x - \cos x$ and the nodes $[0,\ \pi/2$. Compute: (i) the Lagrange bases and their derivatives, (ii) the Hermite bases and then (iii) find the Hermite interpolation polynomial.
Solution: (i) Here we have only two nodes: $x_0 = 0$ and $x_1 = \pi/2$. Therefore, $n = 1$. The function and its derivative at the nodal points are:

$$f(x_0) = \sin 0 - \cos 0 = -1,\ f'(x_0) = \cos 0 + \sin 0 = 1,\ f(x_1) = \sin\pi/2 - \cos\pi/2 = 1 \text{ and } f'(x_1) = \cos\pi/2 + \sin\pi/2 = 1.$$

Now, the Lagrange bases and their derivatives are:

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - \pi/2}{0 - \pi/2} = 1 - \frac{2x}{\pi} \quad \Longrightarrow \quad l_0'(x) = -\frac{2}{\pi} \; .$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{\pi/2 - 0} = \frac{2x}{\pi} \quad \Longrightarrow \quad l_1'(x) = \frac{2}{\pi} \; .$$

Therefore, the Hermite bases are:

$$\begin{aligned}
h_0(x) &= [1 - 2(x - x_0)l_0'(x_0)]\, l_0^2(x) = \Big[1 - 2(x-0)(-2/\pi)\Big]\left(1 - \frac{2x}{\pi}\right)^2 = \left(1 + \frac{4x}{\pi}\right)\left(1 - \frac{2x}{\pi}\right)^2 \; . \\
\widehat{h}_0(x) &= (x - x_0) l_0^2(x) = (x - 0)\left(1 - \frac{2x}{\pi}\right)^2 = x\left(1 - \frac{2x}{\pi}\right)^2 \; . \\
h_1(x) &= [1 - 2(x - x_1)l_1'(x_1)]\, l_1^2(x) = \left[1 - 2(x - \pi/2)\frac{2}{\pi}\right]\left(\frac{2x}{\pi}\right)^2 = \left(3 - \frac{4x}{\pi}\right)\left(\frac{2x}{\pi}\right)^2 \; . \\
\widehat{h}_1(x) &= (x - x_1) l_1^2(x) = \Big[x - \pi/2\Big]\left(\frac{2x}{\pi}\right)^2 = \left(x - \frac{\pi}{2}\right)\left(\frac{2x}{\pi}\right)^2 \; .
\end{aligned}$$

Using ass these, the Hermite polynomial becomes,

$$\begin{aligned}
p_3(x) &= h_0(x)\cancelto{-1}{f(x_0)} + h_1(x)\cancelto{1}{f(x_1)} + \widehat{h}_0(x)\cancelto{1}{f'(x_0)} + \widehat{h}_0(x)\cancelto{1}{f'(x_1)}, \\
&= -h_0(x) + h_1(x) + \widehat{h}_0(x) + \widehat{h}_1(x) \; . \\
&= -\left(1 + \frac{4x}{\pi}\right)\left(1 - \frac{2x}{\pi}\right)^2 + \left(3 - \frac{4x}{\pi}\right)\left(\frac{2x}{\pi}\right)^2 + x\left(1 - \frac{2x}{\pi}\right)^2 + \left(x - \frac{\pi}{2}\right)\left(\frac{2x}{\pi}\right)^2 , \\
&= \left(1 - \frac{2x}{\pi}\right)^2\left[-1 + \left(1 - \frac{4}{\pi}\right)x\right] + \frac{4x^2}{\pi^2}\left[\left(3 - \frac{\pi}{2}\right) + \left(1 - \frac{4}{\pi}\right)x\right] \; .\checkmark
\end{aligned}$$

3. A function is given by $f(x) = x\ln(3x) + x^2$. Now answer the following up to five significant figures.

   (a) (4 marks) Approximate the derivative of $f(x)$ at $x_0 = 2$ with step size $h = 0.1$ using the central difference method.

   Solution: Here we have: $x_0 = 2$ and $h = 0.1$. Therefore, $x_0+h = 2.1$ and $x_0-h = 1.9$. Hence the approximate derivative using central difference method is

$$
\begin{aligned}
D_h &= \frac{f(x_0+h) - f(x_0-h)}{2h}\,, \\
&= \frac{(x_0+h)\ln(-3(x_0+h)) + (x_0+h)^2 - (x_0-h)\ln(-3(x_0-h)) - (x_0-h)^2}{2h}\,, \\
&= \frac{2.1\,\ln(3\times 2.1) + (2.1)^2 - 1.9\,\ln(3\times 1.9) - (1.9)^2}{2\times 0.1}\,, \\
&= 6.7913\,. \quad \checkmark
\end{aligned}
$$

   (b) (6 marks) Calculate the upper bound of the truncation error of $f(x)$ at $x_0 = 2$ using $h = 0.1$ using the central difference method.

   Solution: For the central difference method, the upper bound of the truncation is given by

$$
\begin{aligned}
\text{Upper Bound} &= \left|\frac{h^2}{3!} f'''(\xi)\right|_{\max,\ \xi\in(x_0-h,x_0+h)}\,, \\
&= \frac{(0.1)^2}{6}\left|\frac{d^3}{d\xi^3}\left(\xi\ln(3\xi) + \xi^2\right)\right|_{\max,\ \xi\in(1.9,2.1)}\,, \\
&= \frac{(0.1)^2}{6}\left|-\frac{1}{\xi^2}\right|_{\max,\xi\in(1.9,2.1)}\,, \\
&= \frac{(0.1)^2}{6}\frac{1}{(1.9)^2}\,, \\
&= 4.6168\times 10^{-4}\,. \quad \checkmark
\end{aligned}
$$