'Assignment 02'

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Ans. to the que no -01 (a)

given, the function,

$$f(x) = \sin(x) + \cos(x)$$

and the nodes,

$$x_0 = -\pi$$
$$x_1 = -\pi/2$$
$$x_2 = 0$$

So, we get,

$$f(x_0) = -1$$
$$f(x_1) = -1$$
$$f(x_2) = 1$$

as there are three nodes, the degree of the polynomial is 2.

$$\therefore \; P_2(x) = a_0 + a_1 x + a_2 x^2$$

Now, we know, vandermonde matrix form,

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -\pi & (-\pi)^2 \\ 1 & -\pi/2 & (-\pi/2)^2 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

here, vandermonde matrix is,

$$V = \begin{pmatrix} 1 & -\pi & \pi^2 \\ 1 & -\pi/2 & \pi^2/4 \\ 1 & 0 & 0 \end{pmatrix}$$

Now, to find the co-efficients,

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & -\pi & \pi^2 \\ 1 & -\pi/2 & \pi^2/4 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0.3183 & -1.273 & 0.9549 \\ 0.2026 & -0.405 & 0.2026 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1.9096 \\ 0.405 \end{pmatrix}$$

So, the interpolation polynomial is,

$$P_2(x) = 1 + 1.9096x + 0.405x^2$$

(Ans.)

(b)

As there are three nodes, the degree of the polynomial is 2. So, the lagrange interpolation polynomial is,

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

Now, the lagrange basis are,

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{(x+\pi/2)\cdot x}{\left(-\pi+\frac{\pi}{2}\right)(-\pi)}$$

$$= \frac{x^2+\frac{\pi x}{2}}{\left(\frac{-\pi}{2}\right)\cdot(-\pi)}$$

$$= \frac{x^2+\frac{\pi x}{2}}{\frac{\pi^2}{2}} = \frac{x^2+\pi x}{\pi^2}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$= \frac{(x+\pi)(x)}{\left(-\frac{\pi}{2}+\pi\right)(-\pi/2)}$$

$$= \frac{x^2+\pi x}{\frac{\pi}{2}\cdot(-\pi/2)}$$

$$= \frac{x^2+\pi x}{\frac{-\pi^2}{4}}$$

$$= \frac{4(x^2+\pi x)}{-\pi^2}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{(x+\pi)(x+\pi/2)}{\pi \cdot \pi/2}$$

$$= \frac{x^2+\frac{\pi x}{2}+\pi x+\frac{\pi^2}{2}}{\frac{\pi^2}{2}}$$

$$= \frac{2x^2+\pi x+2\pi x+\pi^2}{\cancel{2}} \times \frac{\cancel{2}}{\pi^2}$$

$$= \frac{2x^2+3\pi x+\pi^2}{\pi^2}$$

So, the lagrange basis are,

$$l_0(x) = \frac{x^2+\pi x}{\pi^2}, \quad l_1(x) = \frac{4(x^2+\pi x)}{-\pi^2}, \quad l_2(x) = \frac{2x^2+3\pi x+\pi^2}{\pi^2}$$

Now, from (a),

$$f(x_0) = -1$$
$$f(x_1) = -1$$
$$f(x_2) = 1$$

So, the lagrange interpolation polynomial is,

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$= -\frac{(x^2+\pi x)}{\pi^2} + \frac{4(x^2+\pi x)}{\pi^2} + \frac{2x^2+3\pi x+\pi^2}{\pi^2}$$

(Ans.)

(c)

We know, for $n=2$, the newton's interpolation polynomial is,

$$P_2(x) = a_0 n_0(x) + a_1 n_1(x) + a_2 n_2(x)$$

$$= a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \quad [\because n_0(x)=1]$$

Now, to find the newton's basis elements,

$x_0 = -\pi$; $\overbrace{f[x_0]}^{a_0} = -1$

$\overbrace{f[x_0,x_1]}^{a_1} = \frac{-1+1}{-\frac{\pi}{2}+\pi} = 0$

$\overbrace{f[x_0,x_1,x_2]}^{a_2} = \frac{\frac{4}{\pi}-0}{\pi} = \frac{4}{\pi^2}$

$x_1 = -\pi/2$; $f[x_1] = -1$

$f[x_1,x_2] = \frac{2}{\pi/2} = \frac{4}{\pi}$

$x_2 = 0$, $f[x_2] = 1$

So, we get,

$a_0 = -1$

$a_1 = 0$

$a_2 = \frac{4}{\pi^2}$

So, the Newton interpolation polynomial is.

$$P_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

$$= -1 + 0\cdot(x+\pi) + \frac{4}{\pi^2}(x+\pi)(x+\pi/2)$$

$$= -1 + \frac{4}{\pi^2}(x+\pi)(x+\pi/2).$$

(Ans.)

(d)

We have, the polynomial, and the function,

$$P_2(x) = -1 + \frac{4}{\pi^2}(x+\pi)(x+\frac{\pi}{2}), \quad f(x) = \sin x + \cos x$$

from Cauchy's theorem,

$$\text{upperbound} \le \left| \frac{1}{(n+1)!} f^{(n+1)}(\xi) \cdot W_n(x) \right|_{\substack{max \\ \xi, x \in [x_0, x_n]}}$$

$$\le \left| \frac{1}{3!} f'''(\xi) W_2(x) \right|_{\substack{max \\ \xi, x \in [x_0, x_n]}}$$

$$\le \frac{1}{3!} \left| f'''(\xi) \right|_{\substack{max \\ \xi \in (-\pi, 0)}} \cdot \left| W_2(x) \right|_{\substack{max \\ x \in [-\pi, 0]}}$$

$$\le \frac{1}{6} \left| f'''(\xi) \right|_{\substack{max \\ \xi \in (-\pi, 0)}} \cdot \left| W_2(x) \right|_{\substack{max \\ x \in [-\pi, 0]}}$$

Now,

$$\left| f'''(\xi) \right|_{\substack{max \\ \xi\in(-\pi,0)}} = \left| \frac{d^3}{d\xi^3}(\sin\xi + \cos\xi) \right|_{\substack{max \\ \xi\in(-\pi,0)}}$$

$$= \left| \frac{d^2}{d\xi^2}\left(\frac{d}{d\xi}(\sin\xi + \cos\xi) \right) \right|_{\substack{max \\ \xi\in(-\pi,0)}}$$

$$= \left| \frac{d}{d\xi}\left(\frac{d}{d\xi}(\cos\xi - \sin\xi) \right) \right|_{\substack{max \\ \xi\in(-\pi,0)}}$$

$$= \left| \frac{d}{d\xi}(-\sin\xi - \cos\xi) \right|_{\substack{max \\ \xi\in(-\pi,0)}}$$

$$= \left| \sin\xi - \cos\xi \right|_{\substack{max \\ \xi\in(-\pi,0)}}$$

Now, from triangular inequality,

$$\left| \sin\xi - \cos\xi \right|_{\substack{max \\ \xi\in(-\pi,0)}} \le \left| \sin\xi \right|_{\substack{max \\ \xi\in(-\pi,0)}} + \left| -\cos\xi \right|_{\substack{max \\ \xi\in(-\pi,0)}}$$

$$= \left| \sin(-\pi/2) \right|_{\cancel{\substack{max \\ \xi\in(-\pi,0)}}} + \left| -\cos 0 \right|_{\substack{max \\ \xi\in(-\pi,0)}}$$

$$= 1 + 1$$

$$= 2$$

Now,

$$|w_2(x)|_{\substack{max\\ x\in[-\pi,0]}} = \left|(x-x_0)(x-x_1)(x-x_2)\right|_{\substack{max\\ x\in[-\pi,0]}}$$

$$= \left| x(x+\pi)(x+\frac{\pi}{2})\right|_{\substack{max\\ x\in[-\pi,0]}}$$

Now, to find the critical value,

$$w_2'(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left[x(x+\pi)(x+\frac{\pi}{2})\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[(x^2+\pi x)(x+\frac{\pi}{2})\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[x^3+\frac{\pi x^2}{2}+\pi x^2+\frac{\pi^2 x}{2}\right] = 0$$

$$\Rightarrow 3x^2+\frac{\pi}{\cancel{2}}\cdot\cancel{2}x+2\pi x+\frac{\pi^2}{2} = 0$$

$$\Rightarrow 3x^2+\pi x+2\pi x+\frac{\pi^2}{2} = 0$$

$$\Rightarrow 6x^2+6\pi x+\pi^2 = 0$$

$$\therefore x_1 = -0.6638, \quad w_n(x_1) = -1.491$$

$$x_2 = -2.477, \quad w_n(x_2) = 1.491$$

So, $|W_n(x)|_{max} = 1.491$

So, the upperbound will be

$$\text{upperbound} \leq \frac{1}{6} \cdot 2 \cdot 1.491$$

$$= 0.497$$

(Ans.)

(e)

Adding new node $\frac{\pi}{2}$ to the existing nodes give us the nodes $\{-\pi, \frac{-\pi}{2}, 0, \frac{\pi}{2}\}$. As there are four ~~three~~ nodes the degree of the polynomial is three and the Newton interpolation polynomial is,

$$P_3(x) = P_2(x) + a_3 n_3(x)$$

$$= P_2(x) + a_3 (x - x_0)(x - x_1)(x - x_2)$$

$$= P_2(x) + a_3 (x + \pi)(x + \frac{\pi}{2}) \cdot x$$

from (c) we got,

$$P_2(x) = -1 + \frac{4}{\pi^2}(x + \pi)(x + \frac{\pi}{2})$$

Now, new basis element a_3 will be.

$$a_3 \equiv f[x_0, x_1, x_2, x_3] = \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0}$$

from (c), we have,

$$f[x_0] = -1, \quad f[x_1] = -1, \quad f[x_2] = 1$$

Now, $x_3 = \pi/2$, so, $f[x_3] = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$

Again,

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

and,
$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$$

$$= \frac{1-1}{\frac{\pi}{2} - 0}$$

$$= 0$$

$$f[x_1, x_2] = \frac{4}{\pi} \quad (\text{from 'c'})$$

$$\therefore f[x_1, x_2, x_3] = \frac{0 - 4/\pi}{\frac{\pi}{2} + \pi/2} = \frac{-4/\pi}{\pi} = \frac{-4}{\pi^2}$$

we also have,

$$a_2 \equiv f[x_0, x_1, x_2] = \frac{4}{\pi^2}$$

finally, we have

$$a_3 \equiv f[x_0, x_1, x_2, x_3] = \frac{-\frac{4}{\pi^2} - \frac{4}{\pi^2}}{\frac{\pi}{2} + \pi}$$

$$= \frac{\frac{-8}{\pi^2}}{\frac{3\pi}{2}}$$

$$= \frac{-8}{\pi^2} \times \frac{2}{3\pi}$$

$$= \frac{-16}{3\pi^3}$$

So, the new interpolation polynomial is,

$$P_3(x) = -1 + \frac{4}{\pi^2}(x+\pi)\left(x+\frac{\pi}{2}\right) - \frac{16}{3\pi^3}x(x+\pi)\left(x+\frac{\pi}{2}\right)$$

(Ans.)