Solution: Assignment-6

#1 (a) Here: $\vec{u}_1 = \frac{1}{\sqrt{5}}(2,-1,0)$; $\vec{u}_2 = \frac{1}{\sqrt{30}}(1,2,-5)$ & $u_3 = \frac{1}{\sqrt{24}}(2,4,2)$.

Now. $|\vec{u}_1| = \left[\left(\frac{1}{\sqrt{5}}\right)^2(2^2+(-1)^2+0)\right]^{1/2} = 1$

$|\vec{u}_2| = \left[\left(\frac{1}{\sqrt{30}}\right)^2(1^2+2^2+(-5)^2)\right]^{1/2} = 1$.

& $|\vec{u}_3| = \sqrt{\left(\frac{1}{\sqrt{24}}\right)^2(2^2+4^2+2^2)} = 1$.

Also, $\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{\sqrt{5}\cdot\sqrt{30}}(\cancel{2\cdot 1} - \cancel{1\cdot 2} - 0.5) = 0$.

$\vec{u}_2 \cdot \vec{u}_3 = \frac{1}{\sqrt{30}\cdot\sqrt{24}}(\cancel{1\cdot 2} + \cancel{2\cdot 4} - \cancel{5\cdot 2}) = 0$

& $\vec{u}_3 \cdot \vec{u}_1 = \frac{1}{\sqrt{30}\sqrt{5}}(\cancel{2\cdot 2} - \cancel{4\cdot 1} + \cancel{2\cdot 0}) = 0$

Since, each vector has norm or magnitude 1 and all pairs have dot product zero, the set S is an orthonormal set.

(b) The data values are:

$x_0 = 4 \Rightarrow f(x_0) = \sin 4 = -0.757$.

$x_1 = 9 \Rightarrow f(x_1) = \sin 9 = 0.412$

& $x_2 = -6 \Rightarrow f(x_2) = \sin(-6) = 0.279$.

Since, the best fit curve is a straight line, we have, $n=1$.

Hence, the matrices are:

$$A = \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \\ 1 & x_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 9 \\ 1 & -6 \end{pmatrix} ; \; X = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = ?? \;\; \& \; b = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} -0.757 \\ 0.412 \\ 0.279 \end{pmatrix}.$$

Now, from $Ax = b$, we calculate.

$$A^T A x = A^T b$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 4 & 9 & -6 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 1 & 9 \\ 1 & -6 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 9 & -6 \end{pmatrix}\begin{pmatrix} -0.757 \\ 0.412 \\ 0.279 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 7 \\ 7 & 133 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} -0.066 \\ -0.994 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 7 & 133 \end{pmatrix}^{-1} \begin{pmatrix} -0.066 \\ -0.994 \end{pmatrix}$$

$$= \begin{pmatrix} 0.38 & -0.02 \\ -0.02 & 8.5\times 10^{-3} \end{pmatrix} \begin{pmatrix} -0.066 \\ -0.994 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} -5.2\times 10^{-3} \\ -7.2\times 10^{-3} \end{pmatrix} = \begin{pmatrix} -0.0052 \\ -0.0072 \end{pmatrix}$$

So the best fit straight line is

$$\boxed{P_1(x) = a_0 + a_1 x = -0.0052 - 0.0072x}$$

#2 (a) From the given data, we find, $n=2$, and hence.

$$A = \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} ; \quad b = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

and $x = (a_0 \ a_1 \ a_2)^T = ??$.

From A, we identify: $u_1 = (1\ 1\ 1\ 1)^T$; $u_2 = (0\ 4\ -1\ 1)^T$ & $u_3 = (0\ 16\ 1\ 1)^T$.

Since there are only 3 columns (because $n=2$), we have only 3 linearly independent vectors.

(b) $p_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \|p_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2 \Rightarrow \boxed{q_1 = \frac{1}{2}(1\ 1\ 1\ 1)^T}$

$$p_2 = u_2 - (u_2^T q_1) q_1 = \begin{pmatrix} 0 \\ 4 \\ -1 \\ 1 \end{pmatrix} - \left[(0\ 4\ -1\ 1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right] \left(\frac{1}{2}\right)^2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow p_2 = \begin{pmatrix} 0 \\ 4 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{4}(4) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \end{pmatrix}$$

And $|p_2| = \sqrt{(-1)^2 + 3^2 + (-2)^2 + 0} = \sqrt{14} \Rightarrow \boxed{q_2 = \frac{p_2}{|p_2|} = \frac{1}{\sqrt{14}} \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \end{pmatrix}}$

And finally:

$$p_3 = u_3 - (u_3^T q_1) q_1 - (u_3^T q_2) q_2$$

$$= \begin{pmatrix} 0 \\ 16 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{1}{4}\right) \underbrace{\left[(0 \; 16 \; 1 \; 1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right]}_{=18} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{1}{14}\right) \underbrace{\left[(0 \; 16 \; 1 \; 1) \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \end{pmatrix} \right]}_{=+46} \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 16 \\ 1 \\ 1 \end{pmatrix} - \frac{9}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{23}{7} \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} -17 \\ 23 \\ 43 \\ -49 \end{pmatrix}$$

$$\therefore |p_3| = \sqrt{\frac{1}{(14)^2}\left((-17)^2 + 23^2 + 43^2 + (-49)^2\right)} = \sqrt{\frac{181}{7}} \Rightarrow |p_3| \approx 5.085.$$

$$\therefore q_3 = \frac{p_3}{|p_3|} = \frac{1}{5.085} \begin{pmatrix} -17/14 \\ 23/14 \\ 43/14 \\ -49/14 \end{pmatrix} \Rightarrow \boxed{q_3 = \begin{pmatrix} -0.239 \\ 0.323 \\ 0.604 \\ -0.688 \end{pmatrix}}$$

(c) Here: $R = \begin{pmatrix} u_1^T q_1 & u_2^T q_1 & u_3^T q_1 \\ 0 & u_2^T q_2 & u_3^T q_2 \\ 0 & 0 & u_3^T q_3 \end{pmatrix}$

From Part-(b), we already have: $u_2^T q_1 = 2$ ✓; $u_3^T q_1 = 9$ ✓ and $u_3^T q_2 = 12.294$ ✓

Now, $u_1^T q_1 = (1\;1\;1\;1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{2} = 2$ ✓

$u_2^T q_2 = (0\;4\;-1\;1) \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{14}} = 3.742$ ✓

and $u_3^T q_3 = (0\;16\;1\;1) \begin{pmatrix} -0.239 \\ 0.323 \\ 0.604 \\ -0.688 \end{pmatrix} = 5.084$ ✓

$$\Rightarrow \boxed{R = \begin{pmatrix} 2 & 2 & 9 \\ 0 & 3.742 & 12.294 \\ 0 & 0 & 5.084 \end{pmatrix}}$$ ✓

(d) $Rx = \begin{pmatrix} 2 & 2 & 9 \\ 0 & 3.742 & 12.294 \\ 0 & 0 & 5.084 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2a_0 + 2a_1 + 9a_2 \\ 3.742a_1 + 12.294a_2 \\ 5.084a_2 \end{pmatrix}$

and ~~Q^Tb~~ Now, the Q matrix is

$$Q = \{q_1\ q_2\ q_3\} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{14}} & -0.239 \\ \frac{1}{2} & \frac{3}{\sqrt{14}} & 0.323 \\ \frac{1}{2} & -\frac{2}{\sqrt{14}} & 0.604 \\ \frac{1}{2} & 0 & -0.688 \end{pmatrix} = \begin{pmatrix} 0.5 & -0.267 & -0.239 \\ 0.5 & 0.802 & 0.323 \\ 0.5 & -0.535 & 0.604 \\ 0.5 & 0 & -0.688 \end{pmatrix}$$

$$\therefore Q^Tb = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ -0.267 & 0.802 & -0.535 & 0 \\ -0.239 & 0.323 & 0.604 & -0.688 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3.475 \\ -0.843 \end{pmatrix}$$

(e) Since : $Rx = Q^Tb$, we have

$$\begin{pmatrix} 2a_0 + 2a_1 + 9a_2 \\ 3.742a_1 + 12.294a_2 \\ 5.084a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3.475 \\ -0.843 \end{pmatrix}$$

Comparing $\Rightarrow a_2 = -\frac{.843}{5.084} \Rightarrow \boxed{a_2 = -0.166}$

And $3.742a_1 + 12.294a_2 = -3.475$

$\Rightarrow a_1 = \frac{-3.475 - 12.294(-0.166)}{3.742} \Rightarrow \boxed{a_1 = -0.383}$

Finally: $2a_0 + 2a_1 + 9a_2 = 2$

$\Rightarrow a_0 = \frac{2 - 2(-0.383) - 9(-0.166)}{2} \Rightarrow \boxed{a_0 = 2.130}$

Therefore, the polynomial $p_2(x)$ is

$$\boxed{p_2(x) = a_0 + a_1x + a_2x^2 = 2.13 - 0.383x - 0.166x^2}$$