A. Consider the following function, $f(x) = \frac{2x(x^2+x)-8(x+1)}{2}$. Based on these, answer the following questions:

1. (1 mark) Find the possible roots of $f(x)$ within the error bound (machine epsilon) of 1×10^{-2} using $x_0 = -10$.
2. (4 marks) State the exact roots of $f(x)$ and construct two different fixed point functions $g(x)$ such that $f(x) = 0$.
3. (5 marks) Compute the convergence rate of each fixed point function $g(x)$ obtained in the previous part, and state which is convergent or divergent.

1. $f(x) = \frac{2x(x^2+x)-8(x+1)}{2}$

$$\Rightarrow f(x) = \frac{2(x(x^2+x)-4(x+1))}{2}$$

$$\Rightarrow f(x) = x(x^2 + x) - 4(x + 1)$$

$$\Rightarrow f(x) = x^3 + x^2 - 4x - 4$$

$$\therefore f'(x) = 3x^2 + 2x - 4$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x^3+x^2-4x-4}{3x^2+2x-4}$$

k	x_k	$f(x_k) = x^3 + x^2 - 4x - 4$	$is\ \lvert f(x_k)\rvert < 10^{-2}?$
0	-10	-864	No
1	-6.8696	-253.512	No
2	-4.8224	-73.601	No
3	-3.5109	-20.907	No
4	-2.7055	-5.662	No
5	-2.2543	-1.357	No
6	-2.0529	-0.226	No
7	-2.0031	-0.012	No
8	-2.0000	-4.915×10^{-5}	Yes

$So,\ x_* = -2$

2. $f(x) = x^3 + x^2 - 4x - 4$

$actual\ root,\ x_* =- 2, - 1, 2\ [Using\ Calculator]$

$Construct\ two\ g(x) = x$

$(1)\ x^3 + x^2 - 4x - 4 = 0$

$\Rightarrow x^2 =- x^3 + 4x + 4$

$\Rightarrow x = \sqrt{- x^3 + 4x + 4}\ \left[g_1(x) = \sqrt{- x^3 + 4x + 4}\right]$

$(2)\ x^3 + x^2 - 4x - 4 = 0$

$\Rightarrow 4x = x^3 + x^2 - 4$

$\Rightarrow x = \frac{x^3+x^2-4}{4}\ \left[g_2(x) = \frac{x^3+x^2-4}{4}\ \right]$

3.

$(1)\ g_1(x) = \sqrt{- x^3 + 4x + 4}$

$$g_1'(x) = \frac{4-3x^2}{2\sqrt{-x^3+4x+4}}$$

$$\Rightarrow \lambda = \left|g'\left(x_*\right)\right| = \left|g_1'(- 1)\right| = 0.5\ (convergent)$$

$$\Rightarrow \lambda = \left|g'\left(x_*\right)\right| = \left|g_1'(- 2)\right| = 2\ (divergent)$$

$$\Rightarrow \lambda = \left|g'\left(x_*\right)\right| = \left|g_1'(2)\right| = 2\ (divergent)$$

$(2)\ g_2(x) = \frac{x^3+x^2-4}{4}$

$$g_2'(x) = \frac{3x^2+2x}{4}$$

$$\Rightarrow \lambda = \left|g'\left(x_*\right)\right| = \left|g_2'(- 1)\right| = 0.25\ (convergent)$$

$$\Rightarrow \lambda = \left|g'\left(x_*\right)\right| = \left|g_2'(- 2)\right| = 2\ (divergent)$$

$$\Rightarrow \lambda = \left|g'\left(x_*\right)\right| = \left|g_2'(2)\right| = 4\ (divergent)$$

B. Read the following questions and answer accordingly:

1. (2 marks) Consider the fixed point function, $g(x) = \frac{4x+2}{2\sqrt{x+1}}$. Show that to be super-linearly convergent, the root is $-\frac{3}{2}$.
2. (4 marks) For $f(x) = x^3 - 2x + 2$, where $x_0 = 0$. Showing the first 2 iterations state that is it possible to find the actual root of $f(x)$. If yes, then write the actual root and if no, then state the reason.
3. (4 marks) For $f(x) = -1 + xe^x$, find the solution of $f(x) = 0$ up to 5 iterations using Newton's method starting with $x_0 = 1.5$. (use 4 significant figures)

1.

$$Given,\ g(x) = \frac{4x+2}{2\sqrt{x+1}} = \frac{2x+1}{\sqrt{x+1}}$$

$$g'(x) = \frac{2x+3}{2(x+1)^{\frac{3}{2}}}$$

$$To\ be\ super\ linear\ convergent,\ \lambda = 0$$

$$\lambda = |g'(x)| = 0$$

$$\Rightarrow \frac{2x+3}{2(x+1)^{\frac{3}{2}}} = 0$$

$$\Rightarrow 2x + 3 = 0$$

$$\Rightarrow x = -\frac{3}{2}$$

$$(Showed)$$

2.

$$Given,\ f(x) = x^3 - 2x + 2,\ x_0 = 0$$

$$f'(x) = 3x^2 - 2$$

<u>1st iteration</u>:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = 2$$

2nd iteration:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 0$$

$Considering\ the\ roots,\ x_0 = 0,\ x_1 = 2, x_2 = 0\ this\ iterations\ get\ stuck\ in\ an$

$infinity\ loop.\ So,\ it\ is\ not\ possible\ to\ find\ the\ root.$

3.

$Given,\ f(x) \ =- 1 + xe^x,\ x_0 = 1.5$

$f'(x) = e^x + xe^x$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{-1+x_k e^{x_k}}{e^{x_k}+x_k e^{x_k}}$$

k	x_k
0	1.5
1	0.9893
2	0.6789
3	0.5766
4	0.5672
5	0.5671