'Assignment - 03'

Amirun Nahin

ID: 23201416

Sec: 06

Course: CSE330

Semester: Summer 2025

Date: 23-07-2025

Answer to the que. no - 01

given that,

$n=5$ and the interval is $[-2,3]$

We know, the chebyshev's points are defined as,

$$\phi_j = \frac{(2j+1)\pi}{2(n+1)} \quad \text{where } j=0,1,2\ldots,n$$

So,

$$\phi_0 = \frac{(2\cdot 0+1)\pi}{2(5+1)}$$

$$= \frac{\pi}{12}$$

$$\phi_1 = \frac{(2\cdot 1+1)\pi}{2(5+1)}$$

$$= \frac{3\pi}{12}$$

$$\phi_2 = \frac{(2\cdot 2+1)\pi}{2(5+1)}$$

$$= \frac{5\pi}{12}$$

$$\phi_3 = \frac{(2\cdot 3+1)\pi}{2(5+1)}$$

$$= \frac{7\pi}{12}$$

$$\phi_4 = \frac{(2\cdot 4+1)\pi}{2(5+1)}$$

$$= \frac{9\pi}{12}$$

$$\phi_5 = \frac{(2\cdot 5+1)\pi}{2(5+1)}$$

$$= \frac{11\pi}{12}$$

so, the chebyshev points are,

$$\phi_0 = \frac{\pi}{12}$$

$$\phi_1 = \frac{3\pi}{12}$$

$$\phi_2 = \frac{5\pi}{12}$$

$$\phi_3 = \frac{7\pi}{12}$$

$$\phi_4 = \frac{9\pi}{12}$$

$$\phi_5 = \frac{11\pi}{12}$$

Now, for chebyshev nodes in the interval $[-2,3]$

$$x_j = \frac{a+b}{2} + \frac{|b-a|}{2} \cos\phi_j$$

$$\Rightarrow x_j = \frac{-2+3}{2} + \frac{|3-(-2)|}{2} \cos\phi_j$$

$$\Rightarrow x_j = \frac{1}{2} + \frac{5}{2}\cos\phi_j$$

here $j = 0, 1, 2, 3, 4, 5$

$$x_0 = \frac{1}{2} + \frac{5}{2}\cos\phi_0$$
$$= \frac{1}{2} + \frac{5}{2}\cos\left(\frac{\pi}{12}\right)$$
$$= 2.915$$

$$x_1 = \frac{1}{2} + \frac{5}{2}\cos\phi_1$$
$$= \frac{1}{2} + \frac{5}{2}\cos\frac{3\pi}{12}$$
$$= 2.268$$

$$x_2 = \frac{1}{2} + \frac{5}{2}\cos\phi_2$$
$$= \frac{1}{2} + \frac{5}{2}\cos\phi_2$$
$$= \frac{1}{2} + \frac{5}{2}\cos\frac{5\pi}{12}$$
$$= 1.147$$

$$x_3 = \frac{1}{2} + \frac{5}{2}\cos\phi_3$$
$$= \frac{1}{2} + \frac{5}{2}\cos\frac{7\pi}{12}$$
$$= -0.147$$

$$x_4 = \frac{1}{2} + \frac{5}{2}\cos\phi_4$$
$$= \frac{1}{2} + \frac{5}{2}\cos\frac{9\pi}{12}$$
$$= -1.268$$

$$x_5 = \frac{1}{2} + \frac{5}{2}\cos\phi_5$$
$$= \frac{1}{2} + \frac{5}{2}\cos\frac{11\pi}{12}$$
$$= -1.915$$

Finally, the chebyshev nodes are,

$$x_0 = 2.915$$
$$x_1 = 2.268$$
$$x_2 = 1.147$$
$$x_3 = -0.147$$
$$x_4 = -1.268$$
$$x_5 = -1.915$$

(Ans.)

Answer to the que. no -02

given function,

$$f(x) = \sin x - \cos x$$

and the nodes, $[0, \pi/2]$

(i)

firstly, we have to compute lagrange basis. As there are two nodes $x_0 = 0$, $x_1 = \pi/2$, we have to compute, $l_0(x)$ and $l_1(x)$.

So,

$$l_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$= \frac{x - \pi/2}{0 - \pi/2}$$

$$= \frac{x - \pi/2}{-\pi/2}$$

$$= \frac{2x-\pi}{x} \times \frac{-x}{\pi}$$

$$= \frac{-(2x-\pi)}{\pi}$$

$$= \frac{\pi-2x}{\pi}$$

$$= 1-\frac{2x}{\pi}$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0}$$

$$= \frac{x-0}{\pi/2-0}$$

$$= \frac{2x}{\pi}$$

So, $l_0(x) = 1-\frac{2x}{\pi}$

$l_1(x) = \frac{2x}{\pi}$

and, their derivatives are,

$$l_0'(x) = \frac{d}{dx}\left(1-\frac{2x}{\pi}\right)$$

$$= -\frac{2}{\pi}$$

$$l_1'(x) = \frac{d}{dx}\left(\frac{2x}{\pi}\right)$$

$$= \frac{2}{\pi}$$

$\therefore\ l_0'(x) = -\frac{2}{\pi}$, $l_1'(x) = \frac{2}{\pi}$ (Ans.)

(ii)

Now, we have to fine Hermite bases. As there are two nodes, we have to find. $h_0(x)$, $\hat{h}_0(x)$, $h_1(x)$ and $\hat{h}_1(x)$. So, we know,

$$h_0(x) = \left[1 - 2(x - x_0)\, l_0'(x_0)\right] l_0^2(x)$$

$$= \left[1 - 2(x-0) \cdot -\frac{2}{\pi}\right] \left(1 - \frac{2x}{\pi}\right)^2$$

$$= \left(1 - 2x \cdot \left(-\frac{2}{\pi}\right)\right) \cdot \left(1 - \frac{2x}{\pi}\right)^2$$

$$= \left(1 + \frac{4x}{\pi}\right)\left(1 - \frac{2x}{\pi}\right)^2$$

and

$$\hat{h}_0(x) = (x - x_0)\, l_0^2(x)$$

$$= (x-0)\left(1 - \frac{2x}{\pi}\right)^2$$

$$= x\left(1 - \frac{2x}{\pi}\right)^2$$

Again,

$$h_1(x) = \left[1 - 2(x - x_1)\, l_1'(x_1)\right] l_1^2(x)$$

$$= \left[1 - 2\left(x - \frac{\pi}{2}\right) \cdot \frac{2}{\pi}\right] \cdot \left(\frac{2x}{\pi}\right)^2$$

$$= \left[1 - \frac{4}{\pi}\left(x - \frac{\pi}{2}\right)\right] \cdot \left(\frac{2x}{\pi}\right)^2$$

$$= \left[1 - \frac{4x}{\pi} + 2\right]\left(\frac{2x}{\pi}\right)^2$$

$$h_1(x) = \left(3 - \frac{4x}{\pi}\right)\left(\frac{2x}{\pi}\right)^2 = \left(3 - \frac{4x}{\pi}\right) \cdot \frac{4x^2}{\pi^2}$$

and,

$$\hat{h}_1(x) = (x - x_1)\, l_1^2(x)$$

$$= \left(x - \frac{\pi}{2}\right)\left(\frac{2x}{\pi}\right)^2$$

$$= \left(x - \frac{\pi}{2}\right) \cdot \frac{4x^2}{\pi^2}$$

So, the hermite bases are,

$$h_0(x) = \left(1 + \frac{4x}{\pi}\right)\left(1 - \frac{2x}{\pi}\right)^2$$

$$\hat{h}_0(x) = x\left(1 - \frac{2x}{\pi}\right)^2$$

$$h_1(x) = \left(3 - \frac{4x}{\pi}\right)\frac{4x^2}{\pi^2}$$

$$\hat{h}_1(x) = \left(x - \frac{\pi}{2}\right)\frac{4x^2}{\pi^2}$$

(Ans.)

(iii)

As there are two nodes $x_0 = 0$, $x_1 = \frac{\pi}{2}$, here $n=1$. We know, the hermite polynomial

$$P_{2n+1}(x) = \sum_{k=0}^{n} \left[h_k(x) f(x_k) + \hat{h}_k(x) f'(x_k) \right]$$

the degree of our polynomial is, $(2\cdot 1+1) = 3$.

$$P_3(x) = \sum_{k=0}^{1} \left[h_k(x) f(x_k) + \hat{h}_k(x) f'(x_k) \right]$$

$$= h_0(x) f(x_0) + \hat{h}_0(x) f'(x_0) + h_1(x) f(x_1) + \hat{h}_1(x) f'(x_1)$$

Now, our function,

$$f(x) = \sin x - \cos x$$

and, $$f'(x) = \frac{d}{dx}(\sin x - \cos x)$$

$$= \cos x + \sin x$$

Now,

$$f(x_0) = \sin 0 - \cos 0$$

$$= -1$$

$$f(x_1) = \sin\frac{\pi}{2} - \cos\frac{\pi}{2}$$

$$= 1$$

$$f'(x_0) = \cos 0 + \sin 0 = 1$$

$$f'(x_1) = \cos x_1 + \sin x_1$$
$$= \cos\frac{\pi}{2} + \sin\frac{\pi}{2}$$
$$= 1$$

Finally our hermite interpolation polynomial,

$$P_3(x) = h_0(x)\, f(x_0) + \hat{h}_0(x)\, f'(x_0) + h_1(x)\, f_1(x) + \hat{h}_1(x)\, f'(x_1)$$
$$= (-1)\, h_0(x) + 1\cdot \hat{h}_0(x) + 1\cdot h_1(x) + 1\cdot \hat{h}_1(x)$$
$$= -\left(1+\frac{4x}{\pi}\right)\left(1-\frac{2x}{\pi}\right)^2 + x\left(1-\frac{2x}{\pi}\right)^2 + \left(3-\frac{4x}{\pi}\right)\frac{4x^2}{\pi^2} + \left(x-\frac{\pi}{2}\right)\cdot\frac{4x^2}{\pi^2}$$

(Ans.)

Answer to the que. no - 03

(a)

given function,

$$f(x) = x\ln(3x) + x^2$$

and, $x_0 = 2$ and step size $h = 0.1$.

Now, we have to find the approximation of the derivative of $f(x)$ at x_0 with step size h using central difference method.

∴ we know, for central difference method,

$$D_h = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$= \frac{f(2+0.1) - f(2-0.1)}{2 \times 0.1}$$

$$= \frac{f(2.1) - f(1.9)}{0.2}$$

Now,

$$f(2.1) = 2.1 \times \ln(3 \times 2.1) + (2.1)^2$$

$$= (2.1 \times 1.8405) + 4.41$$

$$= \cancel{3.860}\ 3.8651 + 4.41$$

$$= 8.2751$$

and, $f(1.9) = (1.9 \times \ln(3 \times 1.9)) + (1.9)^2$

$$= (1.9 \times 1.7405) + 3.61$$

$$= 3.3070 + 3.61$$

$$= 6.917$$

$$\therefore D_h = \frac{(8.2751 - 6.917)}{0.2}$$

$$= \frac{1.3581}{0.2}$$

$$= 6.7905$$

(Ans.)

(b)

We have to calculate the upper bound of the truncation error at $x_0 = 2$ and $h = 0.1$ using central difference method.

We know, for central difference method

$$\text{upperbound of truncation error} = \left| -\frac{h^2}{6} f'''(\xi) \right|_{\max \; \xi \in [x_0 - h, x_0 + h]}$$

for $x_0 = 2$ and $h = 0.1$,

$$\text{upperbound of tuncation error} \leq \left| -\frac{h^2}{6} f'''(\xi) \right|_{\max \; \xi \in [1.9, 2.1]}$$

$$= \frac{h^2}{6} \left| f'''(\xi) \right|_{\max \; \xi \in [1.9, 2.1]}$$

$$= \frac{(0.1)^2}{6} \cdot \left| \frac{d^3}{d\xi^3} (f(\xi)) \right|_{\max \; \xi \in [1.9, 2.1]}$$

$$= \frac{0.01}{6} \cdot \left| \frac{d^3}{d\xi^3} \left(\xi \ln(3\xi) + \xi^2 \right) \right|_{\max \; \xi \in [1.9, 2.1]}$$

$$= 1.6667 \times 10^{-3} \times \left| \frac{d^2}{d\xi^2} \left(\cancel{\xi} \cdot \frac{1}{\cancel{3\xi}} \cdot \cancel{3} + \ln 3\xi + 2\xi \right) \right|_{\max \ldots}$$

$$= 1.6667 \times 10^{-3} \times \left| \frac{d^2}{d\xi^2} \left(1 + \ln 3\xi + 2\xi \right) \right|_{\max \ldots}$$

$$= 1.6667\times10^{-3} \times \left| \frac{d}{d\xi}\left(0 + \frac{1}{3\xi}\cdot 3 + 2\right) \right|_{\substack{max \\ \xi\in[1.9,\,2.1]}}$$

$$= 1.6667\times10^{-3} \times \left| \frac{d}{d\xi}\left(\frac{1}{\xi}\right) \right|_{\substack{max \\ \xi\in[1.9,\,2.1]}}$$

$$= 1.6667\times10^{-3} \times \left| -\frac{1}{\xi^2} \right|_{\substack{max \\ \xi\in[1.9,\,2.1]}}$$

$$= 1.6667\times10^{-3} \times \left| -\frac{1}{(1.9)^2} \right|$$

$$= 1.6667\times10^{-3} \times \frac{1}{(1.9)^2}$$

$$= 1.6667\times10^{-3} \times \frac{1}{3.61}$$

$$= 1.6667\times10^{-3} \times 0.27701$$

$$= 4.6169\times10^{-4}$$

Finally, the upperbound of the truncation error,

$$\leq 4.6169\times10^{-4}$$

(Ans.)