1. Consider the function $f(x) = \sin(x) + \cos(x)$ at the nodes $\{-\pi,\ -\pi/2,\ 0\}$. Answer the following:

   (a) (6 marks) Write down the Vandermonde matrix V, and find the interpolation polynomial using V.
   Solution: Since there are three nodes, the interpolation polynomial must be a quadratic polynomial, *i.e.* $n = 2$. Therefore, the interpolation will be, $p_2(x) = a_0 + a_1x + a_2x^2$ in the Natural or Taylor basis. Here, we are given that: $x_0 = -\pi$, $-\pi/2$ and $x_2 = 0$. Also, we have: $f(x_0) = \sin(x_0) + \cos(x_0) = \sin(-\pi) + \cos(-\pi) = -1$, $f(x_1) = \sin(x_1) + \cos(x_1) = \sin(-\pi/2) + \cos(-\pi/2) = -1$ and $f(x_2) = \sin(x_2) + \cos(x_2) = \sin(0) + \cos(0) = 1$. Therefore, the Vandermonde matrix V and the constant matrix b are

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} = \begin{pmatrix} 1 & -\pi & \pi^2 \\ 1 & -\pi/2 & \pi^2/4 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} . \checkmark$$

   Here, since $\det V = \pi^3/4 \neq 0$, there exists inverse matrix. Using calculator (or any other software, like Mathematica, Maple, etc.), we find,

$$V^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1/\pi & -4/\pi & 3/\pi \\ 2/\pi^2 & -4/\pi^2 & 2/\pi^2 \end{pmatrix} \approx \begin{pmatrix} 0 & 0 & 1 \\ 0.3183 & -1.273 & 0.9549 \\ 0.2026 & -0.4053 & 0.2026 \end{pmatrix} .$$

   Therefore, the Taylor coefficients are

$$x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = V^{-1}b = \begin{pmatrix} 0 & 0 & 1 \\ 1/\pi & -4/\pi & 3/\pi \\ 2/\pi^2 & -4/\pi^2 & 2/\pi^2 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6/\pi \\ 4/\pi^2 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1.9099 \\ 0.4053 \end{pmatrix} .$$

   Comparing both sides, we get: $a_0 = 1$, $a_1 = 6/\pi \approx 1.9099$ and $a_2 = 4/\pi^2 \approx 0.4053$. Therefore, the interpolation polynomial is

$$p_2(x) = \frac{4}{\pi^2}x^2 + \frac{6}{\pi}x + 1 \approx 0.4053x^2 + 1.9099x + 1 . \quad \checkmark$$

   (b) (6 marks) Evaluate the Lagrange bases for the given nodes, and find the Lagrange interpolation polynomial.
   Solution: Since $n = 2$, there are three Lagrange basis elements $l_0(x)$, $l_1(x)$ and $l_2(x)$ and each of these are quadratic in x. Therefore, using the given nodes, we calculate,

$$\begin{aligned}
l_0(x) &= \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) = \left(\frac{x + \pi/2}{-\pi + \pi/2}\right)\left(\frac{x - 0}{-\pi - 0}\right) = \frac{2}{\pi^2}x\left(x + \frac{\pi}{2}\right) , \quad \checkmark \\
l_1(x) &= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) = \left(\frac{x + \pi}{\pi - \pi/2}\right)\left(\frac{x - 0}{-\pi/2 - 0}\right) = -\frac{4}{\pi^2}x\left(x + \pi\right) , \quad \checkmark \\
\text{and} \quad l_2(x) &= \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) = \left(\frac{x + \pi}{0 + \pi}\right)\left(\frac{x + \pi/2}{0 + \pi/2}\right) = \frac{2}{\pi^2}\left(x + \pi\right)\left(x + \frac{\pi}{2}\right) . \quad \checkmark
\end{aligned}$$

   The interpolation polynomial using Lagrange basis is

$$\begin{aligned}
p_2(x) &= \sum_{k=0}^{k=2} l_k(x)f(x_k) = l_0(x)\cancelto{-1}{f(x_0)} + l_1(x)\cancelto{-1}{f(x_1)} + l_2(x)\cancelto{1}{f(x_2)} , \\
&= -\frac{2}{\pi^2}x\left(x + \frac{\pi}{2}\right) \times 1 + \frac{4}{\pi^2}x\left(x + \pi\right)(-1) + \frac{2}{\pi^2}\left(x + \pi\right)\left(x + \frac{\pi}{2}\right) \times 1 . \quad \checkmark
\end{aligned}$$

   (This expression is the answer in Lagrange basis)

   (c) (8 marks) Compute the Newton basis elements needed to find the interpolation polynomial, and compute the Newton interpolation polynomial.
   Solution: The Newton basis elements are: $n_0(x)$ and $n_k(x) = (x - x_0)(x - x_1)\cdots(x - x_{k-1})$ for $k = 0, 1, \cdots, n$. In this case $n = 2$ because there are only three nodes which are: $x_0 = -\pi$, $x_1 = -\pi/2$ and $x_2 = 0$. $f[x_0] = f(x_0) = \sin(x_0) + \cos(x_0) = \sin(-\pi) + \cos(-\pi) = -1$, $f[x_1] = f(x_1) = \sin(x_1) + \cos(x_1) = \sin(-\pi/2) + \cos(-\pi/2) = -1$ and $f[x_2] = f(x_2) = \sin(x_2) + \cos(x_2) = \sin(0) + \cos(0) = 1$ Therefore, the Newton bases are:

$$\begin{aligned}
n_0(x) &= 1 , \\
n_1(x) &= (x - x_0) = \big(x - (-\pi)\big) = (x + \pi) , \\
n_2(x) &= (x - x_0)(x - x_1) = \big(x - (-\pi)\big)\big(x - (-\pi/2)\big) = (x + \pi)(x + \pi/2) . \quad \checkmark
\end{aligned}$$

The Newton coefficients, a_k, can be computed using the Divided-Difference method, which is

$$a_k \equiv f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0} \ , \quad \text{where } k = 0, 1, \cdots, n \text{ and } a_0 = f[x_0] = f(x_0).$$

Now using the fact that for single argument, $f[x_k] = f(x_k)$, we compute the Newton coefficients as,

$$\begin{aligned}
a_0 &= f[x_0] = f(x_0) = -1 \ , \\
a_1 &= f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{-1+1}{-\pi/2 + \pi} = 0 \ , \\
f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1 - (-1)}{0 + \pi/2} = \frac{4}{\pi} \ , \\
a_2 &= f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{4/\pi - (-4/\pi)}{0 - (-\pi)} = \frac{4}{\pi^2} \ . \quad \checkmark
\end{aligned}$$

Using above, write down the interpolation polynomial, as

$$p_2(x) = a_0 n_0(x) + a_1 n_1(x) + a_2 n_2(x) = -1 + \frac{4}{\pi^2}(x + \pi)(x + \pi/2) \ . \quad \checkmark$$

(This is the answer in Newton basis)

(d) (5 marks) Using Cauchy's theorem, compute the upper bound of the interpolation error.
Solution: Here, by definition, $I = [a, b] = [x_0, x_2] = [-\pi, 0]$. Therefore the upper bound of the interpolation is,

$$\begin{aligned}
\text{Upper Bound} &= \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot w_n(x) \right|_{n=2, \max, \xi \in (x_0, x_n), \, x \in [x_0, x_n]} \ , \\
&\leq \frac{1}{3!} \left| f^{(3)}(\xi) \right|_{\max\limits_{\xi \in (x_0, x_2)}} \cdot |w_2(x)|_{\max\limits_{x \in [x_0, x_2]}} \ , \\
&= \frac{1}{6} \left| \frac{d^3 f(\xi)}{d\xi^3} \right|_{\max\limits_{\xi \in (x_0, x_2)}} \cdot \left| \Big((x + \pi)(x + \pi/2)x \Big) \right|_{\max\limits_{x \in [x_0, x_2]}} \ , \\
&= \frac{1}{6} \left| \frac{d^3}{d\xi^3} \Big(\sin \xi + \cos \xi \Big) \right|_{\max\limits_{\xi \in (x_0, x_2)}} \cdot \left| \Big(x^3 + \frac{3\pi}{2} x^2 + \frac{\pi^2}{2} x \Big) \right|_{\max\limits_{x \in [x_0, x_2]}} \ , \\
\text{Now, } \left| \frac{d^3}{d\xi^3} \Big(\sin \xi + \cos \xi \Big) \right|_{\max\limits_{\xi \in (x_0, x_2)}} &\leq \left| \cos \xi \right|_{\max\limits_{\xi \in [-\pi, 0]}} + \left| \sin \xi \right|_{\max\limits_{\xi \in [-\pi, 0]}} = |\cos(0)| + \left| \sin\left(\frac{\pi}{2}\right) \right| = 2 \ , \\
\text{And } \frac{d^2}{dx^2} \Big(x^3 + \frac{3\pi}{2} x^2 + \frac{\pi^2}{2} x \Big) &= 3x^2 + 3\pi x + \frac{\pi^2}{2} = 0 \ , \\
\text{Critical points,} \quad x_c &= \left(-1 \pm \frac{1}{\sqrt{3}} \right) \frac{\pi}{2} = -0.663897, \ -2.477696 \ . \\
\therefore \ |w_2(x_c)| &= |\mp 1.49179| = 1.49179 \ . \\
\text{Hence, Upper Bound} &\leq \frac{1}{6} \times 2 \times 1.49479 \approx 0.49726 \ . \quad \checkmark
\end{aligned}$$

(e) (5 marks) Now add a new node $\pi/2$ to the above nodes, and find the interpolating polynomial using the appropriate method.
Solution: Adding a new node $x_3 = \pi/2$, we have, $f[x_3] = f(x_3) = \sin(\pi/2) + \cos(\pi/2) = 1$. Therefore, we just compute the additional Divided-Difference terms to calculate the new Newton coefficient a_3. Hence, we find,

$$\begin{aligned}
f[x_2, x_3] &= \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{1 - 1}{\pi/2 - 0} = 0 \ , \\
f[x_1, x_2, x_3] &= \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{0 - 4/\pi}{\pi/2 - (-\pi/2)} = -\frac{4}{\pi^2} \ , \\
a_3 &= f[x_0, x_1.x_2.x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-4/\pi^2 - 4/\pi^2}{\pi/2 - (-\pi)} = -\frac{16}{3\pi^3} \ . \quad \checkmark
\end{aligned}$$

Therefore, after adding a new node, we have now $n = 3$, and the Newton interpolation polynomial is,

$$p_3(x) = p_2(x) + a_3 n_3(x) = p_2(x) - \frac{16}{3\pi^3} x(x + \pi/2)(x + \pi) \ . \quad \checkmark$$