Some Practice Problems

1.) Lecture Note Form: $F = \pm (0.d_1 d_2 \ldots d_m)_\beta \times \beta^e$ —— (1)

Normalized Form: $F = \pm (1.d_1 d_2 \ldots d_m)_\beta \times \beta^e$ —— (2)

Denormalized Form: $F = \pm (0.1 d_1 d_2 \ldots d_m)_\beta \times \beta^e$ —— (3)

$\beta = 2$, $m = 4$, $-3 \le e \le 6$

(a) What are the maximum numbers that can be stored in the system by the 3 forms defined above?

(b) What are the non-negative minimum numbers that can be stored in the system by 3 forms?

(c) Using Eq (1), find all the decimal numbers for $e = -1$, Plot them on a real line and show if the number line is equally spaced or not.

d) how many different sets of numbers can we store in this system?

1st) $\Rightarrow 2^{m-1} \times$ number of $e \Rightarrow 2^3 \times 10$

2nd & 3rd) $\Rightarrow 2^m \times$ number of $e \Rightarrow 2^4 \times 10$

int Sol^n: (a)

1st) $(0.1111)_2 \times 2^{\overset{e_{max}}{6}} \Rightarrow (2^{-1}+2^{-2}+2^{-3}+2^{-4}) \times 2^6$

2nd) $(1.1111)_2 \times 2^6 \Rightarrow (1+2^{-1}+2^{-2}+2^{-3}+2^{-4}) \times 2^6$

3rd) $(0.1\ 1111)_2 \times 2^6 \Rightarrow (2^{-1}+2^{-2}+2^{-3}+2^{-4}+2^{-5}) \times 2^6$

অবশ্যই decimal এ convert করবে Ans লেখার সময়। এখানে $\beta = 2$ (যেহেতু) তার জায়গায় $2^{-1}, 2^{-2}, \ldots$ এভাবে লিখছি। ($\beta^{-1}, \beta^{-2}, \beta^{-3} \ldots$ general rule to convert into decimal)

(non-negative)
(b)

1) $(0.1\ 000)_2 \times 2^{\overset{e_{min}}{-3}} \Rightarrow 2^{-1} \times 2^{-3} = \frac{1}{16}$

2) $(1.0000)_2 \times 2^{-3} \Rightarrow 1 \times 2^{-3} = \frac{1}{8}$

3) $(0.1\ 0000)_2 \times 2^{-3} \Rightarrow 2^{-1} \times 2^{-3} = \frac{1}{16}$

* (negative) consider করলে smallest = (− largest) হবে।

(c) একটা definite e এর জন্য সব possible floating point লিখে decimal এ convert করবে। Then, তাদের মধ্যে difference বের করবে। দেখা যাবে, same e এর জন্য difference same. তাই equally spaced.

given that, $\beta = 2$, $m = 4$, $e = -1$ (using $\underline{\text{1st eq}^n}$)

smallest number for this definite e is

$$(0.1\underbrace{000}_{}\,)_2 \times 2^{-1} \Rightarrow 2^{-1} \times 2^{-1} = \frac{1}{4}$$

this $d_1 = 1$ fixed in 1st convention

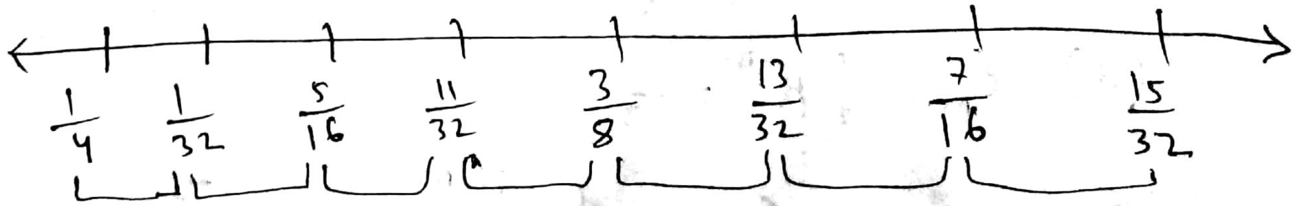
so 3 bit এ সব combination নিতে হবে

$\therefore 2^3 = 8$ টা combination for one e

1) $(0.1\,000)_2 \times 2^{-1} \Rightarrow \frac{1}{4}$
2) $(0.1\,001)_2 \times 2^{-1} \Rightarrow (2^{-1} + 2^{-4}) \times 2^{-1} \Rightarrow \frac{9}{32}$ $\Big\} \rightarrow \text{gap} = \frac{1}{32}$
3) $(0.1\,010)_2 \times 2^{-1} \Rightarrow \frac{5}{16}$
4) $(0.1\,011)_2 \times 2^{-1} \Rightarrow \frac{11}{32}$
5) $(0.1\,100)_2 \times 2^{-1} \Rightarrow \frac{3}{8}$
6) $(0.1\,101)_2 \times 2^{-1} \Rightarrow \frac{13}{32}$ $\Big\} \rightarrow \frac{1}{32}$
7) $(0.1\,110)_2 \times 2^{-1} \Rightarrow \frac{7}{16}$
8) $(0.1\,111)_2 \times 2^{-1} \Rightarrow \frac{15}{32}$ $\Big\} \rightarrow \text{gap} = \frac{1}{32}$

$\boxed{\text{সব gap} = \frac{1}{32} \text{ আসবে}}$

Real Line:



সব difference $\frac{1}{32}$ (equally spaced for definite e)

—x—

2) $\beta = 2,\ m = 4,\ e_{min} = -1,\ e_{max} = 2$

a) compute minimum of $|x|$ for normalized form.

(b) compute machine epsilon for normalized form

(c) State what you can see about the relation between Machine Epsilon and exponent

(d) compute ϵ_M for ~~done~~ denormalized form.

(e) compute maximum delta value for general form/convention which is discussed in lecture note.

hint! (a) $\underbrace{(1.0000)_2}_{\text{normalized}} \times 2^{-1} \Rightarrow \frac{1}{2}$ [denormalized হলে $(0.10000)_2 \times 2^{-1}$ হত] $\frac{1}{4}$

(b) $\epsilon_M = \frac{1}{2}\beta^{-m} = \frac{1}{2} \times 2^{-4} = \frac{1}{32}$

(c) ϵ_M doesn't depend on exponent. It depends on β and m.

(d) $\epsilon_M = \frac{1}{2}\beta^{-m} = \frac{1}{32}$

(e) $\max \delta = \epsilon_M = \frac{1}{2}\beta^{1-m} = \frac{1}{2} \times 2^{-3} = \frac{1}{16}$

—x—

3. $f(x) = xe^{x}$. An interpolating polynomial, $P_3(x)$ is computed by using Taylor expansion. Answer the following:

(a) Using Taylor expansion write $f(x)$ as an infinite series, centered at $x_0 = 1$

or, expand /write $f(x)$ upto the 5th term, centered at $x_0 = 1$

Soln:

$$f(x) = xe^{x} \rightarrow f(x_0) = f(1) = 1 \cdot e^{1} = e$$

$$f^{(1)}(x) = xe^{x} + e^{x} \rightarrow f^{(1)}(1) = e + e = 2e$$

$$f^{(2)}(x) = xe^{x} + e^{x} + e^{x} \rightarrow f^{(2)}(1) = 3e$$

$$f^{(3)}(x) = xe^{x} + e^{x} + e^{x} + e^{x} \rightarrow f^{(3)}(1) = 4e$$

$$f^{(4)}(x) = xe^{x} + 4e^{x} \rightarrow f^{(4)}(1) = 5e$$

$$\vdots$$

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{f^{4}(x_0)}{4!}(x - x_0)^4 + \cdots$$

at $x_0 = 1$,

$$f(x) = \underbrace{e}_{a_0} + \underbrace{2e}_{a_1}(x-1) + \underbrace{\frac{3e}{2}}_{a_2}(x-1)^2 + \underbrace{\frac{4e}{6}}_{a_3}(x-1)^3 + \underbrace{\frac{5e}{24}}_{a_4}(x-1)^4 + \cdots$$

(b) Find the values of a_0, a_1, a_2 and a_3 if the function is interpolated by a degree 3 Polynomial $P_3(x)$

$\Rightarrow$ $a_0 = f(x_0) = e$

$a_1 = \frac{f^{(1)}(x_0)}{1!} = 2e$

$a_2 = \frac{f^{(2)}(x_0)}{2!} = \frac{3e}{2}$

$a_3 = \frac{f^{(3)}(x_0)}{3!} = \frac{4e}{6}$

(c) compute $f(0.1)$ and $P_3(0.1)$ up to seven significant figures.

$f(x) = xe^x$

$\Rightarrow$ $f(0.1) = (0.1) \times e^{0.1} = 0.\overbrace{1105171}^{7\ sf}$

$P_3(x) = e + 2e(x-1) + \frac{3e}{2}(x-1)^2 + \frac{4e}{6}(x-1)^3$

$P_3(0.1) = e + 2e(0.1-1) + \frac{3e}{2}(0.1-1)^2 + \frac{2e}{3}(0.1-1)^3$

$= -0.\underbrace{1929980}_{7\ sf}$

$\Rightarrow$ if $x_0 = 0$ were given, then $P_3(0.1)$ would be positive.

d) find the percent error for interpolating $f(0.1)$ by $P_3(0.1)$

$$\text{percent error} = \frac{|f(0.1) - P_3(0.1)|}{f(0.1)} \times 100\%$$

percent error বললে এই formula use করলে হবে।

(e) Find the maximum error using 4th degree $P_4(x)$ Taylor's Polynomial while approximating $f(0.2)$ on interval $[0, 0.2]$ [given $x_0 = 0$]

$$f(x) = xe^x$$

$$f^{(5)}(x) = xe^x + 5e^x$$

$$f^{(5)}(0.2) = 0.2 \times e^{0.2} + 5 \times e^{0.2} = 6.3513$$ max value on interval $[0, 0.2]$

$$f(x) = P_4(x) + \frac{f^{(5)}(\xi)}{5!}(x - x_0)^5$$

$$|f(x) - P_4(x)| = \frac{(\xi e^{\xi} + 5e^{\xi})}{120}(x - 0)^5$$

at $x = 0.2$,

$$|f(0.2) - P_4(0.2)| \leq \frac{6.3513}{120} \times (0.2)^5$$

$$\leq 1.6937 \times 10^{-5}$$

— x —

Same type math

Practice for some other $f(x)$ such as

$$f(x) = \ln(1+x)$$

$$f(x) = \sin x$$

$$f(x) = \cos x$$

$$f(x) = \sin\left(3x + \frac{\pi}{4}\right)$$

$$f(x) = e^{x} + e^{-x} \quad \text{or, } f(x) = e^{\pm x}$$

$$\text{or, } f(x) = e^{|x|}$$

→ Ques এ x_0 দেওয়া থাকবে।

→ Max error / Lagrange form of Remainder

তবে খেয়াল রাখবে— $f^{(n+1)}(\xi)$ এর max value নিয়ে upper bound করবে always.

suppose, $f^{n+1}(x) = e^{-x}$ at interval $[0, 0.2]$

$f^{n+1}(0) = 1 \rightarrow$ max value

but if $f^{n+1}(x) = e^{x}$

$f^{n+1}(0.2) = 1.2214 \rightarrow$ max value

4.) $f(x) = xe^x$, we are going to find interpolating polynomial by using the Vandermonde matrix method:

a) Construct the Vandermonde matrix V if $f(x)$ passes through the nodes $\{-1, 0, 1\}$

$$x_0 = -1 \rightarrow f(x_0) = (-1)e^{-1} = -0.3679$$

$$x_1 = 0 \rightarrow f(x_1) = 0$$

$$x_2 = 1 \rightarrow f(x_2) = e = 2.7183$$

as 3 nodes $\rightarrow$ $\underline{3 \times 3 \text{ matrix}}$ $\rightarrow$ $\boxed{P_2(x) \text{ বের করতে হবে}}$

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

b) Compute $\det(V)$

$$\Rightarrow 1(0-0) + 1(1-0) + 1(1-0)$$

$$\Rightarrow 2$$

c) inverse matrix, $V^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{pmatrix}$

write expression of $P_2(x) = ?$

d) $A = V^{-1} \cdot F$, $F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix}$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{pmatrix} \begin{pmatrix} -0.3679 \\ 0 \\ 2.7183 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1.5431 \\ 1.1752 \end{pmatrix}$$

$\therefore P_2(x) = a_0 + a_1 x + a_2 x^2$

$= 1.5431x + 1.1752x^2$

e) $P_2(0.25) = 1.5431 \times 0.25 + 1.1752 \times (0.25)^2 = \_\_\_\_$

$f(0.25) = 0.25 \times e^{0.25} = \_\_\_\_$

f) percent error $= \dfrac{|f(0.25) - P_2(0.25)|}{f(0.25)} \times 100\%$

Practice same type 2 math for different f(x) or different nodes

প্রশ্ন: $f(x) = e^{x} + e^{-x}$ nodes $\{-1, 0, 1\}$

$f(x) = \cos x$ nodes $\{0, 2\pi, 4\pi\}$

⋮

5.) Construct an appropriate polynomial for the following data using Hermite basis by following the question:

	x	$f(x)$	$f'(x)$
$x_0 \rightarrow$	-1	0	1
$x_1 \rightarrow$	0	1	0
$x_2 \rightarrow$	1	0	1

a) find the Lagrange basis from given data

b) compute Hermite basis

c) find expression of interpolating Hermite polynomial

d) use expression to determine the approximate value at $x = 0.5$

hint soln: a) $l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2}$

$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{1(-1)} = \frac{x^2-1}{-1}$

$= 1-x^2$

$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)x}{2}$

b) as 3 nodes $\longrightarrow n+1=3$

$\therefore n=2$

We know $(2n+1)$ degree এর Hermite polynomial বানাতে হয় $\therefore P_5(x)$ বের করতে হবে।

$$P_5(x) = f(\overset{0}{x_0})\,h_0(x) + f(\overset{1}{x_1})\,h_1(x) + f(\overset{0}{x_2})\,h_2(x) +$$

$$\underbrace{f'(x_0)}_{1}\,\hat{h}_0(x) + \underbrace{f'(x_1)}_{0}\,\hat{h}_1(x) + \underbrace{f'(x_2)}_{1}\,\hat{h}_2(x)$$

$$= h_1(x) + \hat{h}_0(x) + \hat{h}_2(x)$$

hermite basis $h_k(x)$ & $\hat{h}_k(x)$

we know,

মুখস্থ করতে হবে

$$\begin{cases} h_k(x) = l_k^2(x)\left[1 - 2(x - x_k)\, l_k'(x_k)\right] \\ \hat{h}_k(x) = l_k^2(x)\,(x - x_k) \end{cases}$$

$$h_1(x) = l_1^2(x)\left[1 - 2(x - x_1)\, l_1'(x_1)\right]$$

$$= (1 - x^2)^2\left[1 - 2x \times (-2x)\right]$$

$$= (1 - x^2)^2 (1 + 4x^2)$$

$$\hat{h}_0(x) = l_0^2(x)\,(x - x_0)$$

$$= \frac{x^2(x-1)^2}{4}(x+1)$$

$$\hat{h}_2(x) = l_2^2(x)\,(x - x_2)$$

$$= \frac{x^2(x+1)^2}{4}(x-1)$$

c) $P_5(x) = \underbrace{h_1(x) + \hat{h}_0(x) + \hat{h}_2(x)}_{\text{value গুলো বসায় দিব।}}$

d) $P_5(0.5) = h_1(0.5) + \hat{h}_0(0.5) + \hat{h}_2(0.5)$

—x—

6.) Construct an appropriate polynomial for the data using Newton's divided difference method :—

a) $f(x) = \sin x \rightarrow$ nodes $\left\{0, \frac{\pi}{2}, \pi\right\}$
find values of a_0, a_1, a_2

b) write down interpolating polynomial.
→ as 3 nodes → $P_2(x) = a_0 + a_1 x + a_2 x^2$

c) Add a new node $\frac{3\pi}{2}$ to the above nodes, and find new interpolant.

d) Write down the interpolation error term for the above polynomial, and identify the polynomial $w(x)$

e) Estimate the upper Bound of the interpolation error between given $f(x) = \sin x$ and interpolant $P_3(x)$ with four nodes.

a) $f(x) = \sin x$

$x_0 = 0 \longrightarrow f(x_0) = 0$

$x_1 = \frac{\pi}{2} \longrightarrow f(x_1) = 1$

$x_2 = \pi \longrightarrow f(x_2) = 0$

Ques. এ $f(x) = \sin x$ না দিয়ে 3 nodal points $(0,0)$, $(\frac{\pi}{2},1)$, $(\pi,0)$ এভাবে বলা থাকতে পারে।

$(x,y) \Rightarrow (x, f(x))$

$(x_0, f(x_0))$, $(x_1, f(x_1))$, এরকম বুঝায় $(x_2, f(x_2))$

$$P_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

$$= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$

$x_0 = 0 \rightarrow f[x_0] = 0$

$$f[x_0,x_1] = \frac{1-0}{\frac{\pi}{2}-0} = \frac{2}{\pi}$$

$$f[x_0,x_1,x_2] = \frac{-\frac{2}{\pi}-\frac{2}{\pi}}{\pi-0} = \frac{-4}{\pi^2}$$

$x_1 = \frac{\pi}{2} \rightarrow f[x_1] = 1$

$$f[x_1,x_2] = \frac{-1}{\frac{\pi}{2}} = -\frac{2}{\pi}$$

$x_2 = \pi \rightarrow f[x_2] = 0$

$\therefore a_0 = f[x_0] = 0$

$a_1 = f[x_0,x_1] = \frac{2}{\pi}$

$a_2 = f[x_0,x_1,x_2] = -\frac{4}{\pi^2}$

b) $P(x) = \frac{2}{\pi}x - \frac{4}{\pi^2}x\left(x-\frac{\pi}{2}\right)$

c) nodes $\left\{ 0, \frac{\pi}{2}, \pi, \underbrace{\overbrace{\frac{3\pi}{2}}^{x_3}}_{\text{extra}} \right\}$

$$f(x_3) = \sin\left(\frac{3\pi}{2}\right) = -1$$

x_i	$f[x_i]$			
0	0			
		$\frac{2}{\pi}$		
$\frac{\pi}{2}$	1		$-\frac{4}{\pi^2}$	
		$-\frac{2}{\pi}$		$\frac{0-\left(-\frac{4}{\pi^2}\right)}{\frac{3\pi}{2}-0} = \frac{8}{3\pi^3}$
π	0		0	
		$\frac{-1}{\frac{3\pi}{2}-\pi} = -\frac{2}{\pi}$		
$\frac{3\pi}{2}$	-1			

$$\therefore P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= P_2(x) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$$

$$= \frac{2}{\pi}x - \frac{4}{\pi^2}x\left(x-\frac{\pi}{2}\right) + \frac{8}{3\pi^3}(x-0)\left(x-\frac{\pi}{2}\right)(x-\pi)$$

$f(x) = \sin x \longrightarrow f^{(4)}(x) = \sin x$

d) By Cauchy's thm, interpolation error term,

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{(x-x_0)(x-x_1) \cdots (x-x_n)}_{w(x)}$$

we see $P_3(x)$ above

so $n+1 = 4$

$$\therefore |f(x) - P_3(x)| = \frac{f^{(4)}(\xi)}{4!} (x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$\boxed{\text{max error} \leq \frac{1}{4!} \times w(x) \text{ এর } \max}$$

$$= \underbrace{\frac{\sin(\xi)}{4!}}_{} \underbrace{(x-0)\left(x-\frac{\pi}{2}\right)(x-\pi)\left(x-\frac{3\pi}{2}\right)}_{w(x)}$$

২ টাও আলাদা আলাদা max বের করতে হবে।

$\sin(\xi)$ max value $\underline{1}$ at interval $\left[0, \frac{3\pi}{2}\right]$

e) $w(x) = x\left(x-\frac{\pi}{2}\right)(x-\pi)\left(x-\frac{3\pi}{2}\right)$

$w'(x) = 0$ হবে x এর critical value বের করবো।

এরপর — we have to find $w(x)$ at critical x and corner x (then find max value)

$\beta = 2,\ m = 4,\ e = [-1, 2]$

→ min non-negative value for convention 1

$$(0.1000)_2 \times 2^{-1} = 2^{-1} \times 2^{-1}$$
$$= \frac{1}{4}$$

→ min value for convention 1.

(− max value)

$$-(0.1111)_2 \times 2^2$$
$$-(2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}) \times 2^2$$
$$-3.75$$

→ max value for conv 1 ⇒ $(0.1111)_2 \times 2^2$

3.75

→ how many combination of values possible (1st Conv)

↓

$$2^{m-1} \times \text{number of } e$$
$$\Rightarrow 2^3 \times 4 \Rightarrow 32$$

$x = (6.25)_{10}$ $\quad \beta = 2, m = 3, \; e \in [-1, 2]$

$fl(x) = ?$ (normalized form)

$$(6.25)_{10} = (110.01)_2$$

$$= (1.\underline{1001})_2 \times 2^2$$

as $m = 3$, rounded to

$$(1.100)_2 \times 2^2 = 1.5 \times 4$$

$$= 6$$

$\therefore fl(x) = (6)_{10}$

rounding error, $\delta = \dfrac{|fl(x) - x|}{|x|} \times 100\%$

$$= \frac{|6 - 6.25|}{6.25} \times 100\%$$

$$= 4\%$$

→ upper bound of rounding error ⟶ Machine epsilon, ϵ_M

6.25 কে de normalized form এ লিখা যাবে

না কারণ $(0.11001)_2 \times 2^3$

but $e_{max} = 2$ দেয়া

তাই $e = 3$ use করা যাবে

না।

→ Suppose,

Ques এ যদি 5 টা nodes দেয়া থাকে but বের করতে বলা হয় 2nd order polynomial, তাহলে 5 টা nodes consider না করে Just 3 টা nodes consider করতে হবে। (confused হওয়া যাবে না এক্ষেত্রে)

→ Practice sheet এ এরকম type এর math দেয়া আছে। Please check those maths and solve the problems.

x_i $\quad$ $f(x_i)$

→ time and velocity curve থাকলে যদি acceleration বের করতে বলা হয় তাহলে velocity এর function/polynomial টা বের করবো। then velocity এর derivative নিলে acceleration পাবো। কারণ

$$a = \frac{dv}{dt}$$

Practice All maths which are provided in my class Lecture notes.

Practice all maths which are provided in the practice sheets

Practice all maths which are given as assignment

Best of Luck.