A. Answer the following questions:

1. (2 marks) Prove that the set, $S = \{\frac{1}{\sqrt{5}}(2,-1,0)^T, \frac{1}{\sqrt{30}}(1,2,-5)^T, \frac{1}{\sqrt{24}}(2,4,2)^T\}$ is orthonormal.
2. (4 marks) Consider the values of $f(x) = \frac{7}{20}x^3 - \frac{3}{2}x^2 - \frac{17}{20}x + 3$ at the points $x_0 = 0, x_1 = 4, x_2 = -1$. Now, evaluate the best fit straight line using Discrete Square Approximation for the given function.
3. (3+6+1+2+2 marks) Consider the coordinates: $(x, f(x)) = (0,3), (4,-2), (-1,2), (1,1)$.
   In the following, you are asked to find the best fit ploynomial of degree 2 by QR decomposition method showing the following steps:
   (i) find out the linearly independent column matrices, (ii) using Gram-Schmidt process construct the orthonomal column matrices (or vectors) from the the linearly independent column vectors, (iii) write down the matrices Q and R. Finally, find the coeffcients, let $x = (a_0, a_1, a_2)^T$, and write down the polynomial $p_2(x)$.

1. $S = \left\{\frac{1}{\sqrt{5}}(2, -1, 0)^T, \frac{1}{\sqrt{30}}(1, 2, -5)^T, \frac{1}{\sqrt{24}}(2, 4, 2)^T,\right\}$

$$S = \begin{array}{c} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{24}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{4}{\sqrt{24}} \\ \frac{0}{\sqrt{5}} & \frac{-5}{\sqrt{30}} & \frac{2}{\sqrt{24}} \end{bmatrix} \\ \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \end{array}$$

$$Normality \rightarrow u_1^T . u_1 = \frac{1}{5}(4 + 1) = 1$$

$$Normality \rightarrow u_2^T . u_2 = \frac{1}{30}(1 + 4 + 25) = 1$$

$$Normality \rightarrow u_3^T . u_3 = \frac{1}{24}(4 + 16 + 4) = 1$$

$$Orthogonality \rightarrow u_1^T . u_2 = \frac{\sqrt{6}}{30}(2 - 2 - 0) = 0$$

$$Orthogonality \rightarrow u_1^T . u_3 = \frac{\sqrt{30}}{60}(4 - 4 + 0) = 0$$

$$Orthogonality \rightarrow u_2^T . u_3 = \frac{\sqrt{5}}{60}(2 + 8 - 10) = 0$$

$Yes,\ set\ S\ is\ orthonormal.$

2.

$Given,\ f(x) = \frac{7}{20}x^3 - \frac{3}{2}x^2 - \frac{17}{20}x + 3$

x_i	0	4	-1
$f(x_i)$	3	-2	2

$straight\ line \rightarrow a_0 + a_1 x$

$P_1(x_0) = a_0 + a_1(x_0) \Rightarrow f(x_0) = a_0 + a_1(0) = 3$

$P_1(x_1) = a_0 + a_1(x_1) \Rightarrow f(x_1) = a_0 + a_1(4) = -2$

$P_1(x_2) = a_0 + a_1(x_2) \Rightarrow f(x_2) = a_0 + a_1(-1) = 2$

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

$multiplying\ A^T\ on\ the\ both\ sides \rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 17 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \end{bmatrix}$$

$Now\ apply\ Gaussian\ elimination\ /\ LU/\ inverse\ method\ to\ find\ the\ values\ of$ $a_0\ and\ a_1$

$Applying\ inverse\ method \rightarrow$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 17 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{27}{14} \\ \frac{-13}{14} \end{bmatrix}$$

$\therefore a_0 = \frac{27}{14},\ a_1 = \frac{-13}{14}$

$\therefore P_1(x) = \frac{27}{14} - \frac{13}{14}x$

3.

x_i	0	4	-1	1
$f(x_i)$	3	-2	2	1

$P_2(x) \rightarrow a_0 + a_1 x + a_2 x^2$

$$P_2(x_0) = a_0 + a_1(x_0) + a_2(x_0)^2 \Rightarrow f(x_0) = a_0 + a_1(0) + a_2(0)^2 = 3$$

$$P_2(x_1) = a_0 + a_1(x_1) + a_2(x_1)^2 \Rightarrow f(x_1) = a_0 + 4a_1 + 16a_2 =- 2$$

$$P_2(x_2) = a_0 + a_1(x_2) + a_2(x_2)^2 \Rightarrow f(x_2) = a_0 + (-1)a_1 + (1)a_2 = 2$$

$$P_2(x_3) = a_0 + a_1(x_3) + a_2(x_3)^2 \Rightarrow f(x_2) = a_0 + (1)a_1 + (1)a_2 = 1$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$b = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \end{array}$$

$$p_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_1 = \frac{p_1}{|p_1|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$p_2 = u_2 - \left(u_2{}^T q_1\right) q_1 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} q_1$$

$$= \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$$

$$q_2 = \frac{p_2}{|p_2|} = \frac{1}{\sqrt{14}} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{-2}{\sqrt{14}} \\ \frac{0}{\sqrt{14}} \end{bmatrix}$$

$$p_3 = u_3 - \left(\left(u_3{}^T q_1\right) q_1 + \left(u_3{}^T q_2\right) q_2\right)$$

$$\left(u_3{}^T q_1\right) = \begin{bmatrix} 0 & 16 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 9$$

$$\left(u_3{}^T q_1\right) q_1 = 9\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ \frac{9}{2} \\ \frac{9}{2} \\ \frac{9}{2} \end{bmatrix}$$

$$\left(u_3{}^T q_2\right) = \begin{bmatrix} 0 & 16 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{-2}{\sqrt{14}} \\ \frac{0}{\sqrt{14}} \end{bmatrix} = \frac{23\sqrt{2}}{\sqrt{7}}$$

$$\left(u_3{}^T q_2\right) q_2 = \frac{23\sqrt{2}}{\sqrt{7}}\begin{bmatrix} \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{-2}{\sqrt{14}} \\ \frac{0}{\sqrt{14}} \end{bmatrix} = \begin{bmatrix} \frac{-23}{7} \\ \frac{69}{7} \\ \frac{-46}{7} \\ \frac{0}{7} \end{bmatrix}$$

$$p_3 = u_3 - \left(\left(u_3{}^T q_1\right) q_1 + \left(u_3{}^T q_2\right) q_2\right) = \begin{bmatrix} 0 \\ 16 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{9}{2} \\ \frac{9}{2} \\ \frac{9}{2} \\ \frac{9}{2} \end{bmatrix} + \begin{bmatrix} \frac{-23}{7} \\ \frac{69}{7} \\ \frac{-46}{7} \\ \frac{0}{7} \end{bmatrix}$$

$$p_3 = \begin{bmatrix} \frac{-17}{14} \\ \frac{23}{14} \\ \frac{43}{14} \\ \frac{-7}{2} \end{bmatrix}$$

$$q_3 = \frac{p_3}{|p_3|} = \frac{1}{5.08} \begin{bmatrix} \frac{-17}{14} \\ \frac{23}{14} \\ \frac{43}{14} \\ \frac{-7}{2} \end{bmatrix} = \begin{bmatrix} \frac{-17}{71.12} \\ \frac{23}{71.12} \\ \frac{43}{71.12} \\ \frac{-7}{10.16} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{-1}{\sqrt{14}} & \frac{-17}{71.12} \\ \frac{1}{2} & \frac{3}{\sqrt{14}} & \frac{23}{71.12} \\ \frac{1}{2} & \frac{-2}{\sqrt{14}} & \frac{43}{71.12} \\ \frac{1}{2} & \frac{0}{\sqrt{14}} & \frac{-7}{10.16} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{14}} & \frac{0}{\sqrt{14}} \\ \frac{-17}{71.12} & \frac{23}{71.12} & \frac{43}{71.12} & \frac{-7}{10.16} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & 12.3 \\ 0 & 0 & 5.08 \end{bmatrix}$$

$$R\,x = Q^T\,b$$

$$\begin{bmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & 12.3 \\ 0 & 0 & 5.08 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{14}} & \frac{0}{\sqrt{14}} \\ \frac{-17}{71.12} & \frac{23}{71.12} & \frac{43}{71.12} & \frac{-7}{10.16} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & 12.3 \\ 0 & 0 & 5.08 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3.475 \\ -0.844 \end{bmatrix}$$

$$a_2 = \frac{-0.844}{5.08} =- \ 0.166$$

$$a_1 = \frac{-3.475-(12.3)(-0.166)}{\sqrt{14}} =- \ 0.383$$

$$a_0 = \frac{2-(2)(-0.383)-(9)(-0.166)}{2} = \ 2.13$$

$P_2(x) = 2.13 - 0.383x - 0.166x^2$