'Assignment-06'

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Answer to the question no-1

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Given,

$$S = \left\{ \frac{1}{\sqrt{5}}(2,-1,0)^T, \frac{1}{\sqrt{30}}(1,2,-5)^T, \frac{1}{\sqrt{24}}(2,4,2)^T \right\}$$

We know that, the set S will be orthonormal if

$$\vec{s_i} \cdot \vec{s_j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

So, the matrix,

$$S = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{24}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{4}{\sqrt{24}} \\ 0 & \frac{-5}{\sqrt{30}} & \frac{2}{\sqrt{24}} \end{pmatrix}$$

we identify,

$$u_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{-5}{\sqrt{30}} \end{pmatrix}, \quad u_3 = \begin{pmatrix} \frac{2}{\sqrt{24}} \\ \frac{4}{\sqrt{24}} \\ \frac{2}{\sqrt{24}} \end{pmatrix}$$

Now,

$$u_1^T \cdot u_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 0 \end{pmatrix} = \frac{4}{5} + \frac{1}{5} = 1$$

$\therefore u_1^T u_1 = 1$

$$u_2^T \cdot u_2 = \begin{pmatrix} \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{-5}{\sqrt{30}} \end{pmatrix} = \frac{1}{30} + \frac{4}{30} + \frac{25}{30} = 1$$

$\therefore u_2^T \cdot u_2 = 1$

$$u_3^T \cdot u_3 = \begin{pmatrix} \frac{2}{\sqrt{24}} & \frac{4}{\sqrt{24}} & \frac{2}{\sqrt{24}} \end{pmatrix} \times \begin{pmatrix} \frac{2}{\sqrt{24}} \\ \frac{4}{\sqrt{24}} \\ \frac{2}{\sqrt{24}} \end{pmatrix} = \frac{4}{24} + \frac{16}{24} + \frac{4}{24} = 1$$

$\therefore u_3^T \cdot u_3 = 1$

and

$$u_1^T \cdot u_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{-5}{\sqrt{30}} \end{pmatrix} = \frac{2}{\sqrt{150}} - \frac{2}{\sqrt{150}} + 0 = 0$$

$\therefore u_1^T \cdot u_2 = 0$

$$u_1^T \cdot u_3 = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{pmatrix} \times \begin{pmatrix} \frac{2}{\sqrt{24}} \\ \frac{4}{\sqrt{24}} \\ \frac{2}{\sqrt{24}} \end{pmatrix} = \frac{4}{\sqrt{120}} - \frac{4}{\sqrt{120}} + 0 = 0$$

$\therefore u_1^T \cdot u_3 = 0$

$$u_2^T \cdot u_3 = \begin{pmatrix} \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{pmatrix} \times \begin{pmatrix} \frac{2}{\sqrt{24}} \\ \frac{4}{\sqrt{24}} \\ \frac{2}{\sqrt{24}} \end{pmatrix} = \frac{2}{\sqrt{720}} + \frac{8}{\sqrt{720}} - \frac{10}{\sqrt{720}} = 0$$

$\therefore u_2^T \cdot u_3 = 0$

As we see that the condition of orthonormality is true for all the elements of S.

Therefore, Yes, the set S is orthonormal.

(b)

given,

$$f(x) = \sin x \quad \text{and} \quad x_0 = 4, x_1 = 9, x_2 = -6$$

So,

$$f(x_0) = f(4) = \sin(4) = -0.757$$
$$f(x_1) = f(9) = \sin(9) = 0.412$$
$$f(x_2) = f(-6) = \sin(-6) = 0.279$$

Now we have to evaulate the best fit straight line using discrete square method. So, we identify the matrix

$$A = \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \\ 1 & x_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 9 \\ 1 & -6 \end{pmatrix}$$

$$x = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix} = \begin{pmatrix} -0.757 \\ 0.412 \\ 0.279 \end{pmatrix}$$

So, the equation,

$$Ax = b$$

$$\Rightarrow A^T \cdot A x = A^T \cdot b$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 4 & 9 & -6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 9 \\ 1 & -6 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 9 & -6 \end{pmatrix} \begin{pmatrix} -0.757 \\ 0.412 \\ 0.279 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 7 \\ 7 & 133 \end{pmatrix} \times \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} -0.066 \\ -0.994 \end{pmatrix}$$

the equations,

$$7a_0 + 133a_1 = -0.994 \quad \text{—(i)}$$

$$3a_0 + 7a_1 = -0.066 \quad \text{—(ii)}$$

solving (i), (ii),

$$a_0 = -0.005$$

$$a_1 = -0.007$$

$\therefore$ the best fit straight line,

$$P_1(x) = -0.005 - 0.007x$$

(Ans.)

Ans. to the que. no-02

(a)

given,

$$f(0)=3, \quad f(4)=-2, \quad f(-1)=2, \quad f(1)=1$$

We identify the matrices A, x and b in the form

$Ax=b$ where,

$$A=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad x=\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \quad b=\begin{pmatrix} 3 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

here, A is the overdetermined system.

so, the independent column vectors u_1, u_2 and u_3 are,

$$u_1=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2=\begin{pmatrix} 0 \\ 4 \\ -1 \\ 1 \end{pmatrix}, \quad u_3=\begin{pmatrix} 0 \\ 16 \\ 1 \\ 1 \end{pmatrix}$$

(Ans.)

ⓑ

So,

we have, u_1, u_2 and u_3 from 'a'.

Now, let,
$$P_1 = u_1$$
$$P_2 = u_2 - (u_2^T q_1) q_1$$
$$P_3 = u_3 - (u_3^T q_1) q_1 - (u_3^T q_2) q_2$$
$$\vdots$$
$$P_n = u_n - \sum_{i=1}^{n-1} (u_n^T q_i) q_i \,.$$

So, from Gram-Schmidt process let's find q_1; q_2, q_3.

$$\therefore P_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$\therefore q_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

and,
$$P_2 = u_2 - (u_2^T q_1) q_1$$
$$= \begin{pmatrix} 0 \\ 4 \\ -1 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 0 & 4 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right] \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 4 \\ -1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 4 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \end{pmatrix}$$

$$\therefore q_2 = \frac{p_2}{|p_2|} = \frac{1}{\sqrt{14}} \times \begin{pmatrix} -1 \\ 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \\ 0 \end{pmatrix}$$

and finally

$$p_3 = u_3 - (u_3^T q_1) q_1 - (u_3^T q_2) q_2$$

Now,

$$(u_3^T q_1) q_1 = \left[\begin{pmatrix} 0 & 16 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right] \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 9/2 \\ 9/2 \\ 9/2 \\ 9/2 \end{pmatrix}$$

So, $$(u_3^T q_2) q_2 = \left[\begin{pmatrix} 0 & 16 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \\ 0 \end{pmatrix} \right] \begin{pmatrix} -1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \\ 0 \end{pmatrix}$$

$$= \frac{46}{\sqrt{14}} \begin{pmatrix} -1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \\ 0 \end{pmatrix} = \begin{pmatrix} -23/7 \\ 69/7 \\ -46/7 \\ 0 \end{pmatrix}$$

So, $$P_3 = \begin{pmatrix} 0 \\ 16 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 9/2 \\ 9/2 \\ 9/2 \\ 9/2 \end{pmatrix} - \begin{pmatrix} -23/7 \\ 69/7 \\ -46/7 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -17/14 \\ 23/14 \\ 43/14 \\ -7/2 \end{pmatrix}$$

$$\therefore q_3 = \frac{P_3}{|P_3|} = \frac{1}{\sqrt{(-17/14)^2 + (23/14)^2 + (43/14)^2 + (-7/2)^2}} \times \begin{pmatrix} -17/14 \\ 23/14 \\ 43/14 \\ -7/2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{\frac{289}{14} + \frac{529}{196} + \frac{1849}{196} + \frac{49}{4}}} \times \begin{pmatrix} -17/14 \\ 23/14 \\ 43/14 \\ -7/2 \end{pmatrix}$$

$$= \frac{14}{\sqrt{5068}} \times \begin{pmatrix} -17/14 \\ 23/14 \\ 43/14 \\ -7/2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5068}} \times \begin{pmatrix} -17 \\ 23 \\ 43 \\ -49 \end{pmatrix}$$

$$= \begin{pmatrix} -17/\sqrt{5068} \\ 23/\sqrt{5068} \\ 43/\sqrt{5068} \\ -49/\sqrt{5068} \end{pmatrix}$$

Now, the Q matrix,

$$Q = \begin{pmatrix} 1/2 & -1/\sqrt{14} & -17/\sqrt{5068} \\ 1/2 & 3/\sqrt{14} & 23/\sqrt{5068} \\ 1/2 & -2/\sqrt{14} & 43/\sqrt{5068} \\ 1/2 & 0 & -49/\sqrt{5068} \end{pmatrix}$$

(Ans.)

(c)

We know, the matrix,

$$R = \begin{pmatrix} u_1^T q_1 & u_2^T q_1 & u_3^T q_1 \\ 0 & u_2^T q_2 & u_3^T q_2 \\ 0 & 0 & u_3^T q_3 \end{pmatrix}$$

So, the matrix elements of R are,

$$u_1^T q_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = 2$$

$$u_2^T q_1 = \begin{pmatrix} 0 & 4 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = 2$$

$$u_3^T q_1 = \begin{pmatrix} 0 & 16 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = 9$$

$$u_2^T q_2 = \begin{pmatrix} 0 & 4 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \\ 0 \end{pmatrix} = \sqrt{14}$$

$$u_3^T q_2 = \begin{pmatrix} 0 & 16 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} -1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \\ 0 \end{pmatrix} = \frac{46}{\sqrt{14}}$$

$$u_3^T q_3 = \begin{pmatrix} 0 & 16 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} -17/\sqrt{5068} \\ 23/\sqrt{5068} \\ 43/\sqrt{5068} \\ -49/\sqrt{5068} \end{pmatrix} = \frac{362}{\sqrt{5068}}$$

$$\therefore \text{matrix, } R = \begin{pmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & 46/\sqrt{14} \\ 0 & 0 & \frac{362}{\sqrt{5068}} \end{pmatrix}$$

(Ans.)

(d)

We have to compute Rx and $Q^T b$.

So,

$$Rx = \begin{pmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & \frac{46}{\sqrt{14}} \\ 0 & 0 & \frac{362}{\sqrt{5068}} \end{pmatrix} x$$

and,

$$Q^T b = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ \frac{-1}{\sqrt{14}} & \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{14}} & 0 \\ -17/\sqrt{5068} & \frac{23}{\sqrt{5068}} & \frac{43}{\sqrt{5068}} & \frac{-49}{\sqrt{5068}} \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{6}\,2 \\ \frac{-13}{\sqrt{14}} \\ \frac{-60}{\sqrt{5068}} \end{pmatrix}$$

$$\therefore\ Q^T b = \begin{pmatrix} 2 \\ \frac{-13}{\sqrt{14}} \\ \frac{-60}{\sqrt{5068}} \end{pmatrix}$$ (Ans.)

(e)

given,

$$x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Now,

$$Ax = b$$

$$\Rightarrow QR.x = b$$

$$\Rightarrow Q^T.Q.R.x = Q^T.b \quad [\text{as } Q^T.Q = I]$$

$$\Rightarrow I.Rx = Q^T.b$$

$$\Rightarrow Rx = Q^T.b$$

So,

$$\begin{pmatrix} 2 & 2 & 9 \\ 0 & \sqrt{14} & \frac{46}{\sqrt{14}} \\ 0 & 0 & \frac{362}{\sqrt{5068}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{-13}{\sqrt{14}} \\ \frac{-60}{\sqrt{5068}} \end{pmatrix} \quad [\text{from 'C'}]$$

Using backward substitution method,

$$\frac{362}{\sqrt{5068}} a_2 = \frac{-60}{\sqrt{5068}}$$

$$\Rightarrow 362\, a_2 = \frac{-60}{\cancel{\sqrt{5068}}} \times \cancel{\sqrt{5068}}$$

$$\Rightarrow a_2 = \frac{-60}{362} \Rightarrow a_2 = -0.166$$

and,

$$\sqrt{14}\,a_1 + \frac{46}{\sqrt{14}}a_2 = \frac{-13}{\sqrt{14}}$$

$$\Rightarrow \sqrt{14}\,a_1 = \frac{-13}{\sqrt{14}} - \frac{46}{\sqrt{14}} \times (-0.166)$$

$$\Rightarrow a_1 = -0.383$$

finally,

$$2a_0 + 2a_1 + 9a_2 = 2$$

$$\Rightarrow a_0 = \frac{2 - 2a_1 - 9a_2}{2}$$

$$\Rightarrow a_0 = \frac{2 - 2(-0.383) - 9(-0.166)}{2}$$

$$\Rightarrow a_0 = 2.13$$

So, the polynomial,

$$P_2(x) = 2.13 - 0.383x - 0.166x^2$$

(Ans.)