A. Consider the following function, $v(t) = -\frac{3}{4}t^2 + \frac{19}{2}t - 6$ and $t_0 = 2, t_1 = 4, t_2 = 6$. Based on these, answer the following questions:

1. (4+1 marks) Find an interpolating polynomial for velocity that passes through the given data points using the Vandermonde Matrix method. Additionally, determine an approximate acceleration value at t=7 seconds.
2. (4 marks) Determine an interpolating polynomial for velocity that passes through the given data points using the Lagrange method.
3. (1 mark) If an additional data point is added in the given scenario, which method should be utilized to determine the new interpolating polynomial? Additionally, what will be the degree of the new polynomial?

1.

$$Given\ that\ v(t) = -\frac{3}{4}(t)^2 + \frac{19}{2}(t) - 6$$

$So,$

$$t_0 = 2,\ v(t_0) = -\frac{3}{4}(t_0)^2 + \frac{19}{2}(t_0) - 6$$

$$\Rightarrow v(2) = -\frac{3}{4}(2)^2 + \frac{19}{2}(2) - 6$$

$$= 10$$

$$t_1 = 4,\ v(t_1) = -\frac{3}{4}(t_1)^2 + \frac{19}{2}(t_1) - 6$$

$$\Rightarrow v(4) = -\frac{3}{4}(4)^2 + \frac{19}{2}(4) - 6$$

$$= 20$$

$$t_2 = 6,\ v(t_2) = -\frac{3}{4}(t_2)^2 + \frac{19}{2}(t_2) - 6$$

$$\Rightarrow v(6) = -\frac{3}{4}(6)^2 + \frac{19}{2}(6) - 6$$

$$= 24$$

Polynomial

$$P_n(t) = a_0t^0 + a_1t^1 + a_2t^2 + \dots\dots\dots\dots + a_nt^n$$

$$= a_0 + a_1t + a_2t^2 + \dots\dots\dots\dots + a_nt^n$$

$$P_2(t) = a_0 + a_1t + a_2t^2;$$

$$P_2(t_0) = P_2(2) = a_0 + a_12 + a_22^2$$

$$\Rightarrow a_0 + 2a_1 + 4a_2 = 10$$

$$P_2(t_1) = P_2(4) = a_0 + a_14 + a_24^2$$

$$\Rightarrow a_0 + 4a_1 + 16a_2 = 20$$

$$P_2(t_2) = P_2(6) = a_0 + a_16 + a_26^2$$

$$\Rightarrow a_0 + 6a_1 + 36a_2 = 24$$

Matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 24 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 20 \\ 24 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{4} & 2 & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 24 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ \frac{19}{2} \\ -\frac{3}{4} \end{bmatrix}$$

$So,\ Polynomial \rightarrow$

$$P_2(t) = \ -6 + \frac{19}{2}t - \frac{3}{4}t^2$$

$Acceleration\ value\ at\ t\ =\ 7 \rightarrow$

$$P_2{}'(t) = \frac{19}{2} - \frac{3}{2}t\ = \frac{19}{2} - \frac{3}{2} \times 7\ = \ -1$$

2.

$Degree\ =\ 2;$

$$P_2(t) = f\left(t_0\right)l_0(t)\ +\ f\left(t_1\right)l_1(t)\ +\ f\left(t_2\right)l_2(t)$$

$$l_0(t)\ = \frac{\left(t-t_1\right)\times\left(t-t_2\right)}{\left(t_0-t_1\right)\times\left(t_0-t_2\right)} = \frac{(t-4)\times(t-6)}{(2-4)\times(2-6)} = \frac{1}{8}(t\ -\ 4) \times (t\ -\ 6)$$

$$l_1(t)\ = \frac{\left(t-t_0\right)\times\left(t-t_2\right)}{\left(t_1-t_0\right)\times\left(t_1-t_2\right)} = \frac{(t-2)\times(t-6)}{(4-2)\times(4-6)} =- \frac{1}{4}(t\ -\ 2) \times (t\ -\ 6)$$

$$l_2(t) = \frac{(t-t_0)\times(t-t_1)}{(t_2-t_0)\times(t_2-t_1)} = \frac{(t-2)\times(t-4)}{(6-2)\times(6-4)} = \frac{1}{8}(t-2)\times(t-4)$$

So, Polynomial $\rightarrow$

$$P_2(t) = f(t_0)l_0(t) + f(t_1)l_1(t) + f(t_2)l_2(t)$$

$$= \left(10 \times \frac{1}{8}(t-4)\times(t-6)\right) + \left(20 \times \left(-\frac{1}{4}\right)(t-2)\times(t-6)\right)$$

$$+ \left(24 \times \frac{1}{8}(t-2)\times(t-4)\right)$$

$$=- 6 + \frac{19}{2}t - \frac{3}{4}t^2$$

3. *If new data point is added, Newton's method should be used. New degree* $= 3$.

B. Read the following questions and answer accordingly:

1. (4 marks) Given the nodes $[-\frac{\pi}{2}, 0, \frac{\pi}{2}]$, determine an interpolating polynomial of the appropriate degree using Newton's divided difference method for the function $f(x) = xsin(x)$.
2. (2 marks) Utilize the interpolating polynomial to estimate the value at $\frac{\pi}{4}$, and determine the percentage relative error at $\frac{\pi}{4}$.
3. (4 marks) Insert a new node π into the given set of nodes and determine the interpolating polynomial of the appropriate degree.

1.

x_i	$f(x_i) = x_i sin(x_i)$
$x_0 =- \frac{\pi}{2}$	$\frac{\pi}{2}$
$x_1 = 0$	0
$x_2 = \frac{\pi}{2}$	$\frac{\pi}{2}$

$So,\ Polynomial \rightarrow$

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$Now\ need\ to\ find,$

$$f[x_0]$$

$$f[x_0, x_1]$$

$$f[x_0, x_1, x_2]$$

$x_0 = -\frac{\pi}{2}$	$f[x_0] = \frac{\pi}{2}$	$f[x_0, x_1] = \frac{0-\frac{\pi}{2}}{0-(-\frac{\pi}{2})}$ $= -1$	$f[x_0, x_1, x_2]$ $= \frac{1-(-1)}{\frac{\pi}{2}-(-\frac{\pi}{2})}$ $= \frac{2}{\pi}$
$x_1 = 0$	$f[x_1] = 0$	$f[x_1, x_2] = \frac{\frac{\pi}{2}-0}{\frac{\pi}{2}-0}$ $= 1$	
$x_2 = \frac{\pi}{2}$	$f[x_2] = \frac{\pi}{2}$		

$So,\ Polynomial \rightarrow$

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= \frac{\pi}{2} + \left((-1)\left(x - \left(-\frac{\pi}{2}\right)\right)\right) + \left(\frac{2}{\pi}\left(x - \left(-\frac{\pi}{2}\right)\right)(x)\right)$$

$$= \frac{\pi}{2} - \left(x + \frac{\pi}{2}\right) + \left(\frac{2x}{\pi}\left(x + \frac{\pi}{2}\right)\right)$$

$$= \frac{2}{\pi}x^2$$

2.

$From\ (1) \rightarrow$

$$P_2(x) = \frac{2}{\pi}x^2$$

$$\Rightarrow P_2\left(\frac{\pi}{4}\right) = \frac{2}{\pi}\left(\frac{\pi}{4}\right)^2 = \frac{\pi}{8}$$

$$f(x) = x sin(x) = \frac{\pi}{4} sin\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{8}$$

$relative\ error\ (in\ percentage) \rightarrow$

$$= \left|\frac{actual\ value - approx.\ value}{actual\ value}\right| \times 100$$

$$= \left|\frac{f(x) - P_2(x)}{f(x)}\right| \times 100$$

$$= \left|\frac{f\left(\frac{\pi}{4}\right) - P_2\left(\frac{\pi}{4}\right)}{f\left(\frac{\pi}{4}\right)}\right| \times 100$$

$$= \left|\frac{\frac{\pi\sqrt{2}}{8} - \frac{\pi}{8}}{\frac{\pi\sqrt{2}}{8}}\right| \times 100$$

$$\approx 29.26\%$$

3.

$after\ adding\ new\ data\ point \rightarrow$

x_i	$f(x_i) = x_i sin(x_i)$
$x_0 = -\frac{\pi}{2}$	$\frac{\pi}{2}$
$x_1 = 0$	0
$x_2 = \frac{\pi}{2}$	$\frac{\pi}{2}$
$x_3 = \pi$	0

$So,\ Polynomial \rightarrow$

$$P_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

Now need to find,

$$f[x_0]$$

$$f[x_0\,,\ x_1]$$

$$f[x_0\,,\ x_1\,,\ x_2]$$

$$f[x_0\,,\ x_1\,,\ x_2\,,\ x_3]$$

$x_0 =- \frac{\pi}{2}$	$f[x_0] = \frac{\pi}{2}$	$f[x_0\,,\ x_1] = \frac{0-\frac{\pi}{2}}{0-(-\frac{\pi}{2})}$ $=- 1$	$f[x_0\,,\ x_1\,,\ x_2]$ $= \frac{1-(-1)}{\frac{\pi}{2}-(-\frac{\pi}{2})}$ $= \frac{2}{\pi}$	$f[x_0\,,\ x_1\,,\ x_2\,,\ x_3]$ $= \frac{-\frac{2}{\pi}-\frac{2}{\pi}}{\pi-(-\frac{\pi}{2})}$ $=- \frac{8}{3\pi^2}$
$x_1 = 0$	$f[x_1] = 0$	$f[x_1\,,\ x_2] = \frac{\frac{\pi}{2}-0}{\frac{\pi}{2}-0}$ $= 1$	$f[x_1\,,\ x_2\,,\ x_3]$ $= \frac{-1-1}{\pi-0}$ $=- \frac{2}{\pi}$	
$x_2 = \frac{\pi}{2}$	$f[x_2] = \frac{\pi}{2}$	$f[x_2\,,\ x_3] = \frac{0-\frac{\pi}{2}}{\pi-\frac{\pi}{2}}$ $=- 1$		
$x_3 = \pi$	$f[x_3] = 0$			

So, Polynomial $\rightarrow$

$$P_2(x) = f[x_0] + f[x_0\,,\ x_1](x - x_0) + f[x_0\,,\ x_1\,,\ x_2](x - x_0)(x - x_1)$$

$$+ f[x_0\,,\ x_1\,,\ x_2\,,\ x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= \frac{\pi}{2} + \left((-1)\left(x - \left(-\frac{\pi}{2}\right)\right)\right) + \left(\frac{2}{\pi}\left(x - \left(-\frac{\pi}{2}\right)\right)(x)\right)$$

$$+ \left(-\frac{8}{3\pi^2}\right)\left(x - \left(-\frac{\pi}{2}\right)\right)(x - 0)\left(x - \frac{\pi}{2}\right)$$

$$= \frac{\pi}{2} - \left(x + \frac{\pi}{2}\right) + \left(\frac{2x}{\pi}\left(x + \frac{\pi}{2}\right)\right) - \left(\frac{8x}{3\pi^2}\left(x + \frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)\right)$$

$$= \frac{2x}{3\pi^2}\left(-4x^2 + 3\pi x + \pi^2\right)$$