Practice Problems Chapter 4: Non-Linear Equations

1. Consider a fixed point function $g(x) = (9x - 1)^{\frac{1}{3}}$. The corresponding nonlinear function $f(x)$ has a solution $x_* \in$ R.
   a. Show that $g(x)$ would lead to linear convergence if x > $\frac{1}{9}(1 + \sqrt{27})$. ** *hint: use the condition for linear convergence* **
   b. Starting from $x_0 = 2.5$, find the value of x_* after 5 iterations while keeping up to 5 significant figures.
2. Use Secant method to estimate the root of the following function $f(x) = x^3 - 3x^2 + x$ with initial the initial values of $x_0 = 0.35$ & $x_1 = 0.3$. Show your result with the error in a tabular format for the first 5 iterations.
3. Consider the following function $f(x) = x^2\sqrt{(2x + 3)}$. Now, find the solution of $f(x) = 0$ by doing 5 iterations using the Newton's Raphson method starting with $x_0 = 1.5$. Keep your answers up to 3 significant figures.
4. Consider the function, $f(x) = x^3 - x^2 - 9x + 9$. Answer the following:
   a. State the exact roots of f (x).
   b. Construct three different fixed point functions g(x) such that f (x) = 0. (Make sure that one of the g(x)'s that you constructed converges to at least a root).
   c. Find the convergence rate/ratio for each g(x) constructed in the previous part and also find which root it is converging to.
   d. Find the approximate root, x*, of the above function using fixed point iterations up to 4 significant figures within the error bound of 1×10^{-3} using $x_0 = 0$ and any fixed point function g(x) from part(b) that converges to the root(s).
5. Use Newton's method to find the root, x_*, of the equation, $f(x) = x^2 e^{-x} - 0.6$, up to machine epsilon of 1×10^{-4} starting with $x_0 = 0.2$.
6. Let $f(x) = x^3 + 4x^2 - 10 = 0$, which can be written as $g(x) = x$ for some function $g(x)$.
   a. By manipulating $f(x) = 0$, find at least three expressions for $g(x)$ such that $g(x) = x$.
   b. The given function, $f(x)$, has one real root which is $x_\star = 1.36523$. The other two roots are complex and ignore those two roots. Now evaluate the rate λ for the three function, $g(x)$, you evaluated in the previous part using the real root. Are these three function, $g(x)$, converging or diverging?
7. Use Newton's method to find the solutions for (i) $f(x) = \sqrt{x} - \cos(x)$ and (ii) $f(x) = x^2 - 2xe^{-x} + e^{2x}$ starting with $x_0 = 2.0000$ within 10^{-5}.
8. Consider the nonlinear equation, $f(x) = x^3 - 7x^2 + 4x + 12$. Answer the following:
   a. Find the roots of the given function. Note all three distinct roots are real in this case.
   b. Construct two different fixed point functions for the given function.
   c. Find out if the fixed point functions you evaluated in the previous part are converging or diverging. If converging, which root it is converging to.
   d. Construct a superlinear converging function $g(x)$ for the given function and computer six iterations starting from (i) $x_0 = 4$ and (ii) $x_0 = 0$. Which root the $g(x)$ seems to be converging to?
9. A function is given by $f(x) = x^3 + 2x^2 - x - 2$. Answer the following:
   a. By manipulating $f(x) = 0$, find at least three expressions for $g(x))$ such that $g(x) = x$.
   b. The three roots of $f(x)$ are ±1 and -2. For all your $g(x)$'s, compute the rate λ to find if it is converging to any of the roots.

10. Use Newton's method to find the solution of $f(x) = 1 - 4x\cos(x) + 2x^2 + \cos(2x) = 0$ within 10^{-5} for $0 \leqslant x \leqslant 1$ starting with $x_0 = 0.25$.
11. Use Newton's method to find the root, $x_\star$, of the function $f(x) = x^2 e^{-x} - 0.5$ up to machine epsilon 1.0×10^{-4} starting with $x_0 = 0.2$.
12. A function is given by: $f(x) = x^6 - x^3 - 2$ which has two real roots and the other roots are complex. Answer the following:
    a. Construct two fixed point function $g(x)$ such that $f(x) = 0$.
    b. Compute the rate $\lambda$for the fixed point functions constructed above, and which root it is converting to or diverging
    c. Starting from $x_0 = 60$, and the converging fixed point function $g(x)$ that you constructed in the previous part to find the root of the above function accurate up to 3 decimal places.
13. Find the root of the equation, $f(x) = xe^x - 1$ using fixed point iteration accurate up to machine epsilon of 1.0×10^{-5}. Use the fixed point function $g(x) = e^{-x}$ and start with $x_0 = 0$.
14. Use a secant method to find the root of the equation, $f(x) = 2x^3 + 7x^2 - 14x + 5$. Find the root accurate up to 4 decimal places starting with $x_0 = -5.5$ and $x_1 = -4.5$.