Variational Tail Bounds for Norms of Random Vectors and Matrices

Sohail Bahmani

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Abstract

We propose a variational tail bound for norms of random vectors under moment assumptions on their one-dimensional marginals. We also propose a simplified version of the bound that parametrizes the "aggregating" distribution in the proposed variational bound by considering a certain pushforward of the Gaussian distribution. Furthermore, we show that the proposed method reproduces some of the well-known bounds on norms of Gaussian random vectors, as well as a recent concentration inequality for the spectral norm of sum of independent and identically distributed positive semidefinite matrices.

1 Introduction

We revisit the fundamental problem of bounding norms of a random vector $X \in \mathbb{R}^d$. In particular, for a prescribed norm $\|\cdot\|$ we propose a new variational upper bound to $\|X\|$ assuming suitable conditions on the moments of the one-dimensional marginals of X.

By viewing ||X|| as a supremum of linear functionals indexed by the unit ball of the dual norm $||\cdot||_*$, variety of techniques that are developed to bound supremum of stochastic processes can be used to bound ||X||. For example, we can resort to generic chaining [Tal14] or its variants to formulate an upper bound for ||X||. If X has a Gaussian distribution, Talagrand's celebrated majorizing measures theorem [Tal87] guarantees that these upper bounds are optimal up to constant factors. However, the bounds obtained through generic chaining are usually abstract in nature and not easy to evaluate since one needs to determine an optimal (or nearly-optimal) admissible sequence of the subsets of the unit ball of $||\cdot||_*$. In cases where X is a sum of independent and identically distributed (i.i.d.) random vectors, methods developed in empirical processes theory can be applied. Namely, using the symmetrization (see, e.g., [BLM13, Lemma 11.4]) approach, the supremum of the corresponding empirical process can be bounded in terms of the Rademacher complexity of the set of linear functionals indexed by the the unit ball of $||\cdot||_*$ (see [MRT19, Ch. 3], [SB14, Ch. 26], and [BLM13, Ch. 3] for the definition of Rademacher complexity and the related historical remarks).

A distinct alternative to the mentioned methods is a variational approach, commonly known as the PAC-Bayesian argument [SW97; McA98], that is primarily developed and

applied to analyze the generalization error in statistical learning theory (see, e.g. [LS02; Cat07; AB07; LLS10]). More recently, PAC-Bayesian arguments are used to derive non-vacuous generalization bounds for neural networks that perfectly fit the training samples [DR17; ZVAA+19]. In addition to their application in statistical learning theory, PAC-Bayesian arguments are also used in robust statistics [AC11; Cat12; CG17; Giu18] as well as non-asymptotic random matrix theory [Oli16; Zhi24]. In its most common formulation (see, e.g., [Alq24, Theorem 2.1]), to bound $\sup_{\theta \in \Theta} f_{\theta}(X)$ for a random variable X and given a parameter space Θ , the PAC-Bayesian argument uses the duality between the cumulant generating function and the Kullback-Leibler (KL) divergence. Specifically, the variational characterization of the KL divergence [Alq24, Lemma 2.2] guarantees that

$$\mathbb{E}_{\theta \sim \mathbb{P}} \left(f_{\theta}(x) \right) \le \log \mathbb{E}_{\theta \sim \mathbb{P}_{ref}} \left(e^{f_{\theta}(x)} \right) + D_{KL}(\mathbb{P}, \mathbb{P}_{ref}), \tag{1}$$

where $D_{\mathrm{KL}}(\mathbb{P}, \mathbb{P}_{\mathrm{ref}})$ denotes the KL divergence between the arbitrary "posterior" \mathbb{P} and the reference probability measure or "prior" $\mathbb{P}_{\mathrm{ref}}$ assuming that \mathbb{P} is absolutely continuous with respect to $\mathbb{P}_{\mathrm{ref}}$. The inequality (1) holds for any x, and in particular for x = X. Under suitable measurability conditions, $\mathbb{E}_{\theta \sim \mathbb{P}_{\mathrm{ref}}}\left(e^{f_{\theta}(X)}\right)$, as a random variable, can be bounded probabilistically using Markov's inequality. It then suffices to choose a suitable reference measure and a suitable class \mathcal{P} of aggregating posteriors such that $\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\theta\sim\mathbb{P}}\left(f_{\theta}(x)\right) \approx \sup_{\theta\in\Theta}f_{\theta}(X)$ and $\sup_{\mathbb{P}\in\mathcal{P}}D_{\mathrm{KL}}(\mathbb{P},\mathbb{P}_{\mathrm{ref}})$ is small in certain precise sense. While the PAC-Bayes arguments are usually based on the change of measure inequality (1), alternative change of measure inequalities that use other probability metrics and divergences are also considered in the literature [OH21].

Some components of the PAC-Bayesian arguments complicate and constrain the application of this method. For example, the fundamental inequality (1) implicitly requires $f_{\theta}(X)$ to have a finite exponential moment (in a neighborhood of the origin). More importantly, we need to have an approximation for the KL divergence term in (1) that is both accurate and "calculation-friendly". These requirements add to the subtlety of choosing a suitable reference measure \mathbb{P}_{ref} and a suitable class of aggregating posteriors \mathcal{P} . For example, Zhivotovskiy [Zhi24, Theorem 1] has established a matrix concentration inequality using the PAC-Bayesian argument in which the spectral norm is expressed as the supremum of a bilinear form, and the set \mathcal{P} consists of carefully chosen truncated non-isotropic Gaussian distributions for the left and right factors of the bilinear form.

Focusing on tail bounds for the norm of random vectors (or matrices), we propose an alternative variational bound that has simpler structure and applies under fewer constraints compared to the PAC-Bayesian bounds. The role of the main components of the proposed bound are also more transparent.

2 Problem Setup and the Main Lemma

Let $\|\cdot\|$ be a norm in \mathbb{R}^d whose dual is denoted by $\|\cdot\|_*$. Furthermore, denote the canonical centered unit balls of $\|\cdot\|$ and $\|\cdot\|_*$ respectively by \mathcal{B} and \mathcal{B}_* . Throughout, we denote the boundary of a set \mathcal{S} by $\partial \mathcal{S}$. We also use the notation $\|u\|_M = \sqrt{u^{\mathsf{T}} M u}$ for any positive semidefinite matrix M. We also interpret p! for $p \in \mathbb{R}$ as the standard extension of the

factorial to real numbers using the Gamma function. We use the expression $\alpha \lesssim \beta$ to suppress some absolute constant c > 0 in the inequality $\alpha \leq c\beta$.

Let us also introduce some notation that we use to formulate our results. For a probability measure \mathbb{P}_0 over \mathbb{R}^d we denote the expectation with respect to $U \sim \mathbb{P}_0$ by \mathbb{E}_0 , and define

$$\nu(\mathbb{P}_0) \stackrel{\text{def}}{=} \sup_{x \in \partial \mathcal{B}} \frac{1}{\mathbb{P}_0(|\langle U, x \rangle| \ge 1)}.$$

The central part of our results is the following elementary lemma.

Lemma 1. Given a random variable $X \in \mathbb{R}^d$, for all $u \in \mathbb{R}^d$ and $p \ge 1$ let

$$M_X(u,p) \stackrel{\text{def}}{=} \mathbb{E}\left(\left|\langle u, X \rangle\right|^p\right)$$
.

Then, for $t \geq 0$, with probability at least $1 - e^{-t}$ we have

$$||X|| \le \inf_{p \ge 1, \mathbb{P}_0} e^{t/p} \left(\mathbb{E}_0 \left(M_X(U, p) \right) \right)^{1/p} \nu^{1/p} (\mathbb{P}_0) . \tag{2}$$

Proof. For $x \in \mathbb{R}^d$ define

$$Q_0(x) \stackrel{\text{def}}{=} \mathbb{P}_0 \left(|\langle U, x \rangle| \ge ||x|| \right) ,$$

and observe that

$$\nu(\mathbb{P}_0) = \sup_{x \in \partial \mathcal{B}} \frac{1}{Q_0(x)}.$$

It follows from Markov's inequality that for any $p \geq 1$ and $x \in \mathbb{R}^d$ we have

$$||x||^{p} \leq \mathbb{E}_{0}\left(\left|\left\langle U, x \right\rangle\right|^{p}\right) / Q_{0}(x)$$

$$\leq \mathbb{E}_{0}\left(\left|\left\langle U, x \right\rangle\right|^{p}\right) \nu(\mathbb{P}_{0}). \tag{3}$$

Furthermore, using Markov's inequality this time with respect to X, with probability at least $1 - e^{-t}$ we have

$$\mathbb{E}_{0}(|\langle U, X \rangle|^{p}) \leq e^{t} \mathbb{E} \mathbb{E}_{0}(|\langle U, X \rangle|^{p})$$
$$= e^{t} \mathbb{E}_{0}(M_{X}(U, p)),$$

where we implicitly assumed that we can change the order of the expectations on the right-hand side of the first line.¹ Then, (2) follows by applying the bound above in (3) evaluated at x = X, taking the p-th root, and optimizing with respect to $p \ge 1$ and \mathbb{P}_0 .

Let us evaluate the general result of Lemma 1 in some special cases where $M_X(u, p)$ is replaced by some explicit upper bound. Here, we focus on two simple examples for norms of Gaussian random vectors. We emphasize that the lemma can be applied under other suitable moment conditions as well. For example, we may assume that X whose covariance matrix is Σ , has sub-exponential marginals, i.e., for some $\eta > 0$ and all $p \ge 1$, $u \in \mathbb{R}^d$ we have

$$M_X(u,p) \le p! \left(\eta \|u\|_{\Sigma} \right)^p . \tag{4}$$

In fact, by straightforward changes in the derivations of Example 2 below, we can recover, up to constant factors, the results in [Zhi24, Proposition 3] on the tail bounds for $||X||_2$ under the condition (4).

¹For example, it suffices that ||X|| has finite moments of every order.

Example 1 (Polyhedral norm of a standard Gaussian random vector). Suppose that $X \in \mathbb{R}^d$ is distributed as Normal(0, I), for which we have

$$M_X(u,2p) \le 2^p p! ||u||_2^{2p}$$
.

We examine the case where \mathcal{B} , the unit ball of the norm $\|\cdot\|$, is a symmetric polytope with nonempty interior. In this scenario, \mathcal{B}_* is also a polytope and we can write

$$||X|| = \max_{u \in \text{ext}(\mathcal{B}_*)} \langle u, X \rangle,$$

where

$$\operatorname{ext}(\mathcal{B}_*) = \{ \pm u_1, \dots, \pm u_N \},\,$$

is the set of extreme points of \mathcal{B}_* . Since any norm can be approximated by polyhedral norms to arbitrary precision, this example provides general insights about the nature of the bound (2).

We choose \mathbb{P}_0 to be the law of

$$U = \underset{u \in \rho_0 \operatorname{ext}(\mathcal{B}_*)}{\operatorname{argmax}} \langle u, G \rangle,$$

where $G \sim \mathsf{Normal}(0, I)$ and $\rho_0 > 0$ is a parameter to be specified. Let $[N] \stackrel{\text{def}}{=} \{1, \ldots, N\}$. For each $k \in [N]$ we define $c_k \geq 1$ as the smallest real number such that for every $x \in \partial \mathcal{B}$, for at least k different $i \in [N]$ we have $c_k \langle w, u_i \rangle \geq 1$. Formally,

$$c_k = \inf \left\{ c \ge 1 \colon \inf_{x \in \partial \mathcal{B}} \sum_{i \in [N]} \mathbb{1}(c|\langle u_i, x \rangle| \ge 1) \ge k \right\}.$$
 (5)

In particular, $c_1 = 1$ due to the fact that \mathcal{B} is the polar set of \mathcal{B}_* . Furthermore, we define

$$\pi_k = \min_{I \subseteq [N]:|I|=k} \sum_{i \in I} \mathbb{P}(U = \rho_0 u_i), \qquad (6)$$

which is the smallest probability assigned to any k-subset of $\{\rho_0 u_1, \ldots, \rho_0 u_N\}$ under \mathbb{P}_0 . Observe that π_k does not depend on the scaling parameter ρ_0 . Choosing $\rho_0 = c_k$ and recalling the definition of $\nu(\mathbb{P}_0)$, we have

$$\nu(\mathbb{P}_0) \le \pi_k^{-1} \, .$$

Therefore, using the fact that $p! \leq p^p$, it follows from Lemma 1 that

$$||X|| \le \inf_{p \ge 1, k \in [N]} e^{t/2p} \left(\sqrt{2p} \left(\sum_{i \in [N]} 2 \, \mathbb{P}_0(U = c_k u_i) ||u_i||_2^{2p} \right)^{1/(2p)} c_k \pi_k^{-1/(2p)} \right) , \qquad (7)$$

with probability at least $1 - e^{-t}$. This bound can be further simplified to

$$||X|| \le \inf_{p \ge 1, k \in [N]} e^{t/2p} \left(\sqrt{2p} c_k \pi_k^{-1/(2p)} \right) \max_{i \in [N]} ||u_i||_2 \le \sqrt{2e} \max_{i \in [N]} ||u_i||_2 \min_{k \in [N]} c_k \max \left\{ 1, \sqrt{t - \log \pi_k} \right\} ,$$
(8)

For verification, we can recover, up to the constant factors, the well-known bound for the ℓ_{∞} -norm of $X \sim \mathsf{Normal}(0, I)$ which states that

$$||X||_{\infty} \le \sqrt{2\log(2d)} + \sqrt{t},$$

holds with probability at least $1 - e^{-t}$ (see, e.g., [BLM13, Example 5.7]). This special case is captured in our formulation, by choosing N = d and u_1, \ldots, u_d to be the canonical unit basis vectors in \mathbb{R}^d . Then, due to symmetry, we have $\mathbb{P}_0(U = c_k u_1) = \cdots = \mathbb{P}_0(U = c_k u_d) = 1/(2d)$. Thus

$$\sum_{i \in [d]} 2 \, \mathbb{P}_0(U = c_k u_i) \|u_i\|_2^{2p} = 1 \,,$$

and choosing k = 1 we have $c_1 = 1$ and $\pi_1 = 1/(2d)$. Therefore, the derived bound (7) simplifies to

$$||X||_{\infty} \le \inf_{p \ge 1} e^{t/2p} \left(\sqrt{2p} (2d)^{1/(2p)} \right)$$

 $\le \sqrt{2e(\log(2d) + t)}.$

Any norm can be approximated by polyhedral norms. However, using such an approximation is not always desirable because it ignores any special structure that the norm $\|\cdot\|$ might have, and may produce bounds that are not explicit in terms of the parameters that describe the law of the random vector. Lemma 1 can still provide more explicit bounds in some special cases. As a classic example, we consider the case of bounding the ℓ_2 norm of a Gaussian random vector.

Example 2 (Euclidean norm of a Gaussian random vector). Let $X \in \mathbb{R}^d$ is a $\mathsf{Normal}(0,\Sigma)$ random vector for which we have

$$M_X(u, 2p) \le 2^p p! ||u||_{\Sigma}^{2p}.$$

We are interested in bounding $||X||_2$, the ℓ_2 -norm of X. Therefore, in this example, both \mathcal{B} and \mathcal{B}_* specialize to the unit ℓ_2 ball.

We apply Lemma 1 with \mathbb{P}_0 being Normal $(0, \kappa_0^2 I)$ for some $\kappa_0 > 0$. Writing $U = \kappa_0 G$ for a standard Gaussian random vector $G \in \mathbb{R}^d$, it follows from Lemma 2 below in the appendix that

$$\mathbb{E}_0\left(M_X(u,2p)\right) \le 2^p p! \, \kappa_0^{2p} \, \mathbb{E}\left(\left(G^\mathsf{T} \Sigma G\right)^p\right)$$

$$\leq 2^{p} p! \,\kappa_0^{2p} \left(\operatorname{tr}(\Sigma) + p \|\Sigma\|_{\operatorname{op}} \right)^{p} \,. \tag{9}$$

Furthermore, recalling that \mathcal{B} is the unit ℓ_2 ball, for any $x \in \partial \mathcal{B}$ we can write

$$\mathbb{P}_0(|\langle U, x \rangle| \ge 1) = \mathbb{P}(\kappa_0 | g | \ge 1)$$
$$= 2\Phi(-\frac{1}{\kappa_0}),$$

where $g \in \mathbb{R}$ is a standard Gaussian random variable, and $\Phi(\cdot)$ denotes the standard Gaussian cumulative distribution function (CDF). It follows from straightforward calculus and the inequality $2\pi < 3e$ that $\Phi(-z) - e^{-2z^2/3}$ is decreasing in $z \geq 0$, which implies that for all $z \geq 0$ we have

$$\Phi(-z) \ge e^{-2z^2/3}/4. \tag{10}$$

Recalling the definition of $\nu(\mathbb{P}_0)$, we obtain

$$\nu(\mathbb{P}_0) \le 2e^{2/(3\kappa_0^2)}.$$

Invoking Lemma 1 and applying (9), we conclude that with probability at least $1-e^{-t}$, we have

$$||X|| \leq \inf_{p \geq 1, \kappa_0 > 0} e^{t/(2p)} \kappa_0 \sqrt{2p} \left(\operatorname{tr}(\Sigma) + p ||\Sigma||_{\operatorname{op}} \right)^{1/2} 2^{1/(2p)} e^{1/(3p\kappa_0^2)}$$

$$= \sqrt{\frac{4e}{3}} \inf_{p \geq 1} e^{(t + \log(2))/(2p)} \left(\operatorname{tr}(\Sigma) + p ||\Sigma||_{\operatorname{op}} \right)^{1/2}$$

$$\leq \sqrt{\frac{8e}{3}} \left(2 \operatorname{tr}(\Sigma) + \frac{1}{\log(2)} ||\Sigma||_{\operatorname{op}} t \right)^{1/2}$$

$$< 4\sqrt{\operatorname{tr}(\Sigma)} + 4\sqrt{||\Sigma||_{\operatorname{op}} t},$$

where third line is obtained by evaluating the argument of the infimum at $p = 1 + t/\log(2)$. Therefore, up to constant factors^b, Lemma 1 recovers the tail bound for the ℓ_2 -norm of X obtained via the Gaussian concentration inequality (see e.g., [BLM13, Theorem 5.5]), i.e., with probability at least $1 - e^{-t}$, we have

$$||X||_2 \le \sqrt{\operatorname{tr}(\Sigma)} + \sqrt{2||\Sigma||_{\operatorname{op}}t}$$
.

^aHere we favored simplicity of the lower bound over its accuracy. More accurate lower bounds for the quantile function, or the *Mills ratio* of the standard Gaussians [see e.g., [Wai19, Exercise 2.2] [BLM13, Exercise 7.8], [GU14], and references therein] can be used instead.

^bThe provided bounds are not optimized in terms of the constant factors.

3 Simplified Bounds via Pushforward of Gaussians

The tail bound of Lemma 1 holds in rather general situations, but it is not entirely calculation-friendly, mainly due to its implicit form and the minimization with respect to \mathbb{P}_0 . Assuming that X has sub-Gamma marginals, by characterizing \mathbb{P}_0 as a certain pushforward of the standard Gaussian measure, the following theorem provides a bound that can be optimized using a few scalar parameters. In particular, one can recover the bounds in Example 1 and Example 2 using this theorem.

Theorem 1. Let $X \in \mathbb{R}^d$ be a zero-mean random variable with covariance matrix Σ and sub-Gamma marginals in the sense that for some constants $\eta_1, \eta_2 \geq 0$, for every $p \geq 1$ and $u \in \mathbb{R}^d$ we have

$$\mathbb{E}(|\langle u, X \rangle|^{2p}) \le p! (\eta_1 ||u||_{\Sigma})^{2p} + (2p)! (\eta_2 ||u||_*)^{2p}.$$

Furthermore, for parameters $\rho_0 \geq 0$ and $\kappa_0 > 0$, let

$$f_0(x) = \inf_{y} \frac{\kappa_0}{2} ||x - y||_{\Sigma^{-1/2}}^2 + \rho_0 ||y||,$$

With $(z)_+ \stackrel{\text{def}}{=} \max\{z,0\}$ and $G \sim \text{Normal}(0,I)$ being a standard Gaussian random variable in \mathbb{R}^d , let

$$\underline{\lambda}_{\Sigma}(\kappa_0, \rho_0) = \inf_{x \in \partial \mathcal{B}} \mathbb{E}\left(\left(\left|\langle \nabla f_0(\Sigma^{1/2}G), x \rangle \right| - 1\right)_+\right),$$

and define

$$\Omega_{\Sigma}(p, \kappa_0, \rho_0) = \eta_1 \sqrt{p} \min \left\{ \rho_0 \|\Sigma\|_{\square}, \ \kappa_0 \left(\operatorname{tr}(\Sigma) + p \|\Sigma\|_{\operatorname{op}} \right)^{1/2} \right\} + 2\eta_2 p \rho_0,$$

and

$$\overline{\nu}_{\Sigma}(\kappa_0, \rho_0) = \min \left\{ \frac{\rho_0}{\underline{\lambda}_{\Sigma}(\kappa_0, \rho_0)}, \frac{\kappa_0^2 \operatorname{rad}^2(\mathcal{B})}{\underline{\lambda}_{\Sigma}^2(\kappa_0, \rho_0)} + 1 \right\},\,$$

where $||M||_{\square} = \sup_{u \in \mathcal{B}_*} u^{\mathsf{T}} M u$ for any $d \times d$ symmetric matrix M, and $\operatorname{rad}(\mathcal{B}) = \sup_{x \in \mathcal{B}} ||x||_2$ denotes the radius of \mathcal{B} as measured by the ℓ_2 norm. Then, for every $t \geq 0$, with probability at least $1 - e^{-t}$, we have

$$||X|| \le \inf_{\substack{p \ge 1 \\ \rho_0 \ge 0 \\ \kappa_0 > 0}} e^{t/(2p)} \Omega_{\Sigma}(p, \kappa_0, \rho_0) \overline{\nu}_{\Sigma}^{1/(2p)}(\kappa_0, \rho_0).$$
(11)

It is worth mentioning that using duality we can show that the mapping $x \mapsto \nabla f_0(x)$ can be expressed equivalently as

$$\nabla f_0(x) = \underset{u \in g_0 B_*}{\operatorname{argmin}} \frac{1}{2} \| \kappa_0 \Sigma^{-1/2} x - u \|_{\Sigma^{1/2}}^2.$$

Specifically, $\nabla f_0(\Sigma^{1/2}G)$ is a projection of $\kappa_0 G$ onto $\rho_0 \mathcal{B}_*$ in the sense that

$$\nabla f_0(\Sigma^{1/2}G) = \underset{u \in \rho_0 \mathcal{B}_*}{\operatorname{argmin}} \frac{1}{2} \|\kappa_0 G - u\|_{\Sigma^{1/2}}^2.$$

This characterization also shows that the law of $U = \nabla f_0(\Sigma^{1/2}G)$ is a mixture of a truncated Gaussian distribution on the interior of $\rho_0 \mathcal{B}_*$ and another distribution on the boundary of $\rho_0 \mathcal{B}_*$ that becomes more concentrated around the extreme points as κ_0 increases (relative to ρ_0).

Proof. Taking $U = \nabla f_0(\Sigma^{1/2}G)$, it suffices to find the appropriate approximations for the terms that appear in (2). The dual representation of $f_0(x)$, i.e.,

$$f_0(x) = \sup_{u \in \rho_0 \mathcal{B}_*} \langle u, x \rangle - \frac{1}{2\kappa_0} ||u||_{\Sigma^{1/2}}^2,$$

reveals that for all $x \in \mathbb{R}^d$ we have

$$\left\|\nabla f_0(x)\right\|_* \le \rho_0.$$

Furthermore, we have $\nabla^2 f_0(x) \leq \kappa_0 \Sigma^{-1/2}$ and $\nabla f_0(0) = 0$, thus

$$\|\nabla f_0(x)\|_{\Sigma}^2 = \left\| \int_0^1 \nabla^2 f_0(\tau x) x d\tau \right\|_{\Sigma}^2$$

$$\leq \int_0^1 \|\nabla^2 f_0(\tau x) x\|_{\Sigma}^2 d\tau$$

$$= \int_0^1 x^{\mathsf{T}} \Sigma^{-1/2} \left(\Sigma^{1/2} \nabla^2 f_0(\tau x) \Sigma^{1/2} \right)^2 \Sigma^{-1/2} x d\tau$$

$$\leq \kappa_0^2 \|x\|_2^2,$$

where the second line follows from the Jensen's inequality.

Therefore, we have

$$\|\nabla f_0(\Sigma^{1/2}G)\|_{\Sigma}^2 \le \min \{\rho_0^2 \|\Sigma\|_{\square}, \kappa_0^2 \|G\|_{\Sigma}^2 \}$$
.

Putting the derived bounds together, using Lemma 2, and recalling the definition of $\Omega_{\Sigma}(p, \kappa_0, \rho_0)$, we obtain

$$\left(\mathbb{E}_0\left(M_X(\nabla f_0(\Sigma^{1/2}G), 2p)\right)\right)^{1/(2p)} \le \Omega_\Sigma(p, \kappa_0, \rho_0). \tag{12}$$

Furthermore, using the second moment method (see Lemma 3 below in the appendix) we have

$$\mathbb{P}\left(\left|\left\langle \nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| \ge 1\right) \ge \frac{\left(\mathbb{E}\left(\left|\left\langle \nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| - 1\right)_+\right)^2}{\mathbb{E}\left(\left(\left|\left\langle \nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| - 1\right)_+^2\right)},$$

and thus

$$\nu(\mathbb{P}_0) \le \sup_{x \in \partial \mathcal{B}} \frac{\mathbb{E}\left(\left(\left|\left\langle \nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| - 1\right)_+^2\right)}{\left(\mathbb{E}\left(\left|\left\langle \nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| - 1\right)_+\right)^2}.$$

To further simplify the right-hand side of the inequality above we consider two situations. In the first situation, for $x \in \partial \mathcal{B}$ we use the inequality

$$\mathbb{E}\left(\left(\left|\left\langle\nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| - 1\right)_+^2\right) \le \rho_0 \,\mathbb{E}\left(\left(\left|\left\langle\nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| - 1\right)_+\right)\,,$$

which implies

$$\nu(\mathbb{P}_0) \le \sup_{x \in \partial \mathcal{B}} \frac{\rho_0}{\left(\mathbb{E}\left(\left|\left\langle \nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| - 1\right)_+\right)} \\ \le \frac{\rho_0}{\underline{\lambda}_{\Sigma}(\kappa_0, \rho_0)}.$$

In the second situation, the Gaussian Poincaré inequality² [BLM13, Theorem 3.20] and the fact that $\nabla^2 f_0(z) \leq \kappa_0 \Sigma^{-1/2}$ provide the bound

$$\operatorname{var}\left(\left(\left|\left\langle \nabla f_0(\Sigma^{1/2}G), x\right\rangle\right| - 1\right)_+\right) \le \mathbb{E}\left(\left\|\Sigma^{1/2}\nabla^2 f_0(\Sigma^{1/2}G)x\right\|_2^2\right)$$
$$\le \kappa_0^2 \|x\|_2^2,$$

which in turn implies

$$\nu(\mathbb{P}_0) \le \frac{\kappa_0^2 \operatorname{rad}^2(\mathcal{B})}{\underline{\lambda}_{\Sigma}^2(\kappa_0, \rho_0)} + 1.$$

Therefore, recalling the definition of $\overline{\nu}_{\Sigma}(\kappa_0, \rho_0)$, we have shown that

$$\nu(\mathbb{P}_0) \le \overline{\nu}_{\Sigma}(\kappa_0, \rho_0) \,. \tag{13}$$

Then (11) follows by applying the inequalities (12) and (13) on the bound provided by Lemma 1.

We can recover the bound (8) in Example 1 from Theorem 1 up to a factor of order $\sqrt{\log c_k}$ that can be ignored if c_k is not large relative to π_k^{-1} . The parameters of the problem in this example are $\eta_1 = \sqrt{2}$, $\eta_2 = 0$, and $\Sigma = I$. By taking $\kappa_0 \to \infty$, we have $f_0(x) = \rho_0 ||x||$ which corresponds to the choice

$$U = \nabla f_0(G) = \underset{u \in \rho_0 \{ \pm u_1, \dots, \pm u_N \}}{\operatorname{argmax}} \langle u, G \rangle,$$

as considered in Example 1. Recalling the definitions (5) and (6), we focus on the case $\rho_0 = 2c_k$ for some $k \in [N]$. Because

$$\|\Sigma\|_{\square} = \sup_{u \in \mathcal{B}_*} u^{\mathsf{T}} \Sigma u$$

We apply the inequality to function $g \mapsto (|\langle \nabla f_0(\Sigma^{1/2}g), x \rangle| - 1)_+$ which is weakly differentiable for $\kappa_0 < \infty$.

$$= \sup_{u \in \mathcal{B}_*} \|u\|_2^2$$
$$= \max_{i \in [N]} \|u_i\|_2^2$$

we have

$$\Omega_{\Sigma}(p, \infty, 2c_k) = 2\sqrt{2p}c_k \max_{i \in [N]} \|u_i\|_2^2.$$

Next, we bound $\overline{\nu}_{\Sigma}(\infty, 2c_k)$. Let $I_k(x) = \{i \in [N] : c_k | \langle u_i, x \rangle | \geq 1 \}$. We have

$$\lambda_{\Sigma}(\infty, 2c_k) = \inf_{x \in \partial B} \sum_{i \in [N]} 2 \mathbb{P}_0 (U = 2c_k u_i) (2c_k |\langle u_i, x \rangle| - 1)_+$$

$$\geq \inf_{x \in \partial B} \sum_{i \in I_k(x)} 2 \mathbb{P}_0 (U = 2c_k u_i) c_k |\langle u_i, x \rangle|$$

$$\geq 2\pi_k,$$

using which we obtain the bound

$$\overline{\nu}_{\Sigma}(\infty, 2c_k) \leq \frac{c_k}{\pi_k}$$
.

Therefore, the bound (11) provided by Theorem 1 implies

$$||X|| \le 2 \max_{i \in [N]} ||u_i||_2^2 \inf_{\substack{p \ge 1 \\ k \in [N]}} e^{t/(2p)} \sqrt{2p} c_k \left(\frac{c_k}{\pi_k}\right)^{1/(2p)}$$
$$\le \sqrt{8e} \max_{i \in [N]} ||u_i||_2^2 \min_{k \in [N]} c_k \max\left\{1, \sqrt{t - \log(\pi_k) + \log(c_k)}\right\}.$$

Let $k_{\star} \in [N]$ be the minimizer of $\min_{k \in [N]} c_k \max \{1, \sqrt{t - \log \pi_k}\}$ in (8). If we have $\log c_{k_{\star}} \lesssim t - \log \pi_{k_{\star}}$, our derivations confirm that, up to constant factors, Theorem 1 recovers the bound (8) of Example 1.

Next we will show how Example 2 can be recovered from Theorem 1. Again we have $\eta_1 = \sqrt{2}$ and $\eta_2 = 0$. Also, recall that in this example \mathcal{B} and \mathcal{B}_* coincide with the unit ℓ_2 -ball. By taking $\rho_0 \to \infty$ we have $f_0(x) = \kappa_0 ||x||_{\Sigma^{-1/2}}^2/2$, and thus

$$U = \nabla f_0(\Sigma^{1/2}G) = \kappa_0 G.$$

Clearly,

$$\Omega_{\Sigma}(p, \kappa_0, \rho_0) = \sqrt{2p\kappa_0} \left(\operatorname{tr}(\Sigma) + p \|\Sigma\|_{\operatorname{op}} \right)^{1/2}.$$

Furthermore, since $\langle G, x \rangle$ has the standard Gaussian distribution we can write

$$\underline{\lambda}_{\Sigma}(\kappa_0, \infty) = \mathbb{E}\left(\left(\kappa_0 \left| \langle G, x \rangle \right| - 1\right)_{+}\right)$$

$$= \frac{2}{\sqrt{2\pi}} \int_{1/\kappa_0}^{\infty} (\kappa_0 z - 1) e^{-z^2/2} dz$$

$$\geq \frac{2}{\sqrt{2\pi}} \int_{1/\kappa_0}^{\infty} (\kappa_0 z - 1) e^{-((z - \kappa_0^{-1})^2 + \kappa_0^{-2})} dz$$

$$= \frac{2\kappa_0}{\sqrt{2\pi}} e^{-\kappa_0^{-2}} \int_0^{\infty} s e^{-s^2} ds$$

$$= \frac{\kappa_0}{\sqrt{2\pi}} e^{-\kappa_0^{-2}}$$

where we used the inequality $z^2/2 \leq (z - \kappa_0^{-1})^2 + \kappa_0^{-2}$ on the third line, and the change of variable $s = z - \kappa_0^{-1}$ on the fourth line. Applying the inequality above in the definition of $\overline{\nu}_{\Sigma}(\kappa_0, \infty)$ yields

$$\overline{\nu}_{\Sigma}(\kappa_0, \infty) \le 2\pi e^{1/\kappa_0^2} + 1 < 8e^{1/\kappa_0^2}.$$

Therefore, the bound (11) of Theorem 1 implies

$$||X|| = ||X||_{2} \le \inf_{\substack{p \ge 1 \\ \kappa_{0} > 0}} e^{t/(2p)} \sqrt{2p} \kappa_{0} \left(\operatorname{tr}(\Sigma) + p ||\Sigma||_{\operatorname{op}} \right)^{1/2} 8^{1/(2p)} e^{1/(2p\kappa_{0}^{2})}$$

$$= \sqrt{2e} \inf_{p \ge 1} e^{(t+\log 8)/(2p)} \left(\operatorname{tr}(\Sigma) + p ||\Sigma||_{\operatorname{op}} \right)^{1/2}$$

$$\le 4\sqrt{e} \left(2\operatorname{tr}(\Sigma) + \frac{1}{\log 8} ||\Sigma||_{\operatorname{op}} t \right)^{1/2},$$

where the third line follows by evaluating the argument of the infimum at $p = 1 + t/\log 8$ and the fact that $\|\Sigma\|_{\text{op}} \leq \text{tr}(\Sigma)$. Therefore, up to constant factors, we have recovered the same bound on $\|X\|_2$ in Example 2 through Theorem 1.

4 Spectral Norm of Sum of Random Positive Semidefinite Matrices

We mentioned previously that we may easily adapt Lemma 1 to apply under different moment conditions. Furthermore, the special structure of the norm of interest $\|\cdot\|$ can be leveraged to adjust the choice of \mathbb{P}_0 in Lemma 1. As an example, we use Lemma 1 to recover the result of Zhivotovskiy [Zhi24, Theorem 1] up to the constant factors.

Theorem 2. Let Y_1, \ldots, Y_n be i.i.d. copies of a random positive semidefinite matrix $Y \in \mathbb{R}^{\ell \times \ell}$ whose mean is $\Sigma = \mathbb{E} Y$ and for some parameter $\eta \geq 1$ and every $p \geq 1$ and $u \in \mathbb{R}^{\ell}$ obeys

$$\left(\mathbb{E}\left(\left|u^{\mathsf{T}}Yu\right|^{p}\right)\right)^{1/p} \leq \eta p \|u\|_{\Sigma}^{2}.$$

Denoting the effective rank of Σ by

$$r_{\rm eff}(\Sigma) \stackrel{\text{def}}{=} \frac{\operatorname{tr} \Sigma}{\|\Sigma\|_{\rm op}},$$

with probability at least $1 - e^{-t}$, we have

$$\left\| \frac{1}{n} \sum_{i \in [n]} Y_i - \Sigma \right\|_{\text{op}} \lesssim \eta \|\Sigma\|_{\text{op}} \sqrt{\frac{2r_{\text{eff}}(\Sigma) + t}{n}} \max \left\{ 1, \sqrt{\frac{r_{\text{eff}}(\Sigma) + t/2}{n}} \right\}.$$

Proof. Let $X = \sum_{i=1}^{n} (Y_i - \Sigma)$, and \mathbb{P}_0 be the law of $U = \kappa_0^2 G G^{\mathsf{T}}$ where G is a standard Gaussian random vector in \mathbb{R}^{ℓ} independent of $(Y_i)_{i \in [n]}$. With $\mathbb{M}^{\ell}_{\text{sym}}$ denoting the set of $\ell \times \ell$ real symmetric matrices, it follows from Lemma 1 that

$$||X||_{\text{op}} \le \sup_{M \in \mathbb{M}_{\text{sym}}^{\ell}: ||M||_{\text{op}} = 1} e^{t/p} \kappa_0^2 \left(\mathbb{E} \left(|G^{\mathsf{T}} X G|^p \right) \right)^{1/p} \left(\mathbb{P} \left(\kappa_0^2 |G^{\mathsf{T}} M G| \ge 1 \right) \right)^{-1/p}. \tag{14}$$

First, we bound $(\mathbb{E}(|G^{\mathsf{T}}XG|^p))^{1/p}$. With $\varepsilon_1, \ldots, \varepsilon_n$ being a sequence of i.i.d. Rademacher random variables independent of everything else, the standard symmetrization argument guarantees

$$\left(\mathbb{E}\left(\left|u^{\mathsf{T}}Xu\right|^{p}\right)\right)^{1/p} \leq 2\left(\mathbb{E}\left(\left|\sum_{i\in[n]}\varepsilon_{i}u^{\mathsf{T}}Y_{i}u\right|^{p}\right)\right)^{1/p}.$$

Since $\sum_{i \in [n]} \varepsilon_i u^{\mathsf{T}} Y_i u$ is a sum of independent symmetric random variables, it follows from a moment bound due to Latała [Lat97, Corollary 2] that

$$(\mathbb{E}(|u^{\mathsf{T}}Xu|^{p}))^{1/p} \lesssim \sup_{q \in [\max\{2, p/n\}, p]} \frac{p}{q} \left(\frac{n}{p}\right)^{1/q} (\mathbb{E}(|u^{\mathsf{T}}Yu|^{q}))^{1/q}$$

$$\lesssim \sup_{q \in [\max\{2, p/n\}, p]} \eta p \left(\frac{n}{p}\right)^{1/q} \|u\|_{\Sigma}^{2}$$

$$\lesssim \eta \max\{\sqrt{np}, p\} \|u\|_{\Sigma}^{2}.$$

Using the inequality above together with Lemma 2 we obtain

$$(\mathbb{E}(|G^{\mathsf{T}}XG|^{p}))^{1/p} \lesssim \eta \max\{\sqrt{np}, p\} \left(\mathbb{E}(\|G\|_{\Sigma}^{2p})\right)^{1/p}$$

$$\lesssim \eta \max\{\sqrt{np}, p\} \left(\operatorname{tr}(\Sigma) + p\|\Sigma\|_{\operatorname{op}}\right).$$
(15)

Next, we bound $\mathbb{P}\left(\kappa_0^2 | G^{\mathsf{T}} M G| \geq 1\right)$ for $M \in \mathbb{M}^{\ell}_{\mathrm{sym}}$ with $\|M\|_{\mathrm{op}} = 1$. We use a simple form of a general technique due to Chatterjee [Cha19, Lemma 1.2]. Let $\widetilde{G} = (I - vv^{\mathsf{T}})G + \widetilde{g}v$ where v is the singular vector of M corresponding to the singular value 1, and $\widetilde{g} \sim \mathrm{Normal}(0,1)$ is independent of G. Since G and \widetilde{G} are both standard Gaussian random vectors, we have

$$2 \mathbb{P}\left(\kappa_0^2 | G^{\mathsf{T}} M G| \ge 1\right) = \mathbb{P}\left(\kappa_0^2 | G^{\mathsf{T}} M G| \ge 1\right) + \mathbb{P}_0\left(\kappa_0^2 | \widetilde{G}^{\mathsf{T}} M \widetilde{G}| \ge 1\right)$$
$$\ge \mathbb{P}\left(\kappa_0^2 | \widetilde{G}^{\mathsf{T}} M \widetilde{G} - G^{\mathsf{T}} M G| \ge 2\right)$$

$$\begin{split} &= \mathbb{P}\left(\kappa_0^2 \left| \left\langle \widetilde{G}, v \right\rangle^2 - \left\langle G, v \right\rangle^2 \right| \ge 2\right) \\ &= \mathbb{P}\left(\kappa_0^2 \left| \left\langle \widetilde{G} + G, v \right\rangle \right| \left| \left\langle \widetilde{G} - G, v \right\rangle \right| \ge 2\right) \,. \end{split}$$

Observe that, by construction, $\langle \widetilde{G} + G, v \rangle / \sqrt{2}$ and $\langle \widetilde{G} - G, v \rangle / \sqrt{2}$ are independent standard Gaussian random variables. Therefore, the above inequality implies that

$$\mathbb{P}\left(\kappa_0^2 | G^{\mathsf{T}} M G| \ge 1\right) \ge \frac{1}{2} \,\mathbb{P}\left(\frac{1}{\sqrt{2}} \left| \langle \widetilde{G} + G, v \rangle \right| \ge \frac{1}{\kappa_0}\right) \,\mathbb{P}\left(\frac{1}{\sqrt{2}} \left| \langle \widetilde{G} - G, v \rangle \right| \ge \frac{1}{\kappa_0}\right) \\
= \frac{1}{2} \left(2\Phi\left(-\kappa_0^{-1}\right)\right)^2 \\
\ge \frac{1}{8} e^{-4/(3\kappa_0^2)}, \tag{16}$$

where we used (10) on the third line. Applying the bounds (15) and (16) in (14) we obtain

$$||X||_{\text{op}} \lesssim e^{t/p} \eta \max\{\sqrt{np}, p\} \kappa_0^2 \left(\text{tr}(\Sigma) + p ||\Sigma||_{\text{op}} \right) 8^{1/p} e^{4/(3\kappa_0^2 p)}$$

$$= C \eta ||\Sigma||_{\text{op}} e^{(t + \log(8))/p} \sqrt{\max\{n, p\}} \left(p^{-1/2} r_{\text{eff}}(\Sigma) + p^{1/2} \right) \kappa_0^2 p e^{4/(3\kappa_0^2 p)},$$

where C represents a positive absolute constant. Minimizing the right-hand side of the inequality with respect to $\kappa_0 > 0$, choosing $p = r_{\text{eff}}(\Sigma) + t/\log(8)$, and using the facts that $r_{\text{eff}}(\Sigma) \geq 1$ and $\log(8) > 2$ we conclude that with probability at least $1 - e^{-t}$ we have

$$\left\| \frac{1}{n} \sum_{i \in [n]} Y_i - \Sigma \right\|_{\text{op}} = \frac{1}{n} \|X\|_{\text{op}} \lesssim \eta \|\Sigma\|_{\text{op}} \sqrt{\frac{2r_{\text{eff}}(\Sigma) + t}{n}} \max \left\{ 1, \sqrt{\frac{r_{\text{eff}}(\Sigma) + t/2}{n}} \right\}.$$

Supporting Lemmas

Lemma 2. Let $B \in \mathbb{R}^{\ell \times \ell}$ be a symmetric matrix and $G \in \mathbb{R}^{\ell}$ be a standard Gaussian random variable. Then, for $p \geq 1$ we have

$$\mathbb{E}\left(|G^{\mathsf{T}}BG|^{p}\right) \leq \|B\|_{*}^{p} \prod_{i=1}^{\lfloor p \rfloor} \left(1 + \frac{2(p-i)\|B\|_{\mathrm{op}}}{\|B\|_{*}}\right), \tag{17}$$

where $||B||_*$ denotes the nuclear norm of B. The inequality above can be simplified to

$$\mathbb{E}(|G^{\mathsf{T}}BG|^p) \le (\|B\|_* + p\|B\|_{\mathrm{op}})^p$$
.

Proof. With B_+ and B_- respectively denote the positive semidefinite parts of B and -B, let $B_{\natural} = B_+ + B_-$ which is a positive semidefinite matrix itself. It follows from the triangle inequality that

$$|G^{\mathsf{T}}BG| \leq G^{\mathsf{T}}B_{\mathsf{b}}G$$
.

Using the Gaussian integration by parts (or Stein's lemma) we have

$$\mathbb{E}\left(\left|G^{\mathsf{T}}BG\right|^{p}\right) \leq \mathbb{E}\left(\left(G^{\mathsf{T}}B_{\natural}G\right)^{p}\right) \\
= \operatorname{tr}(B_{\natural}) \,\mathbb{E}\left(\left(G^{\mathsf{T}}B_{\natural}G\right)^{p-1}\right) + 2(p-1) \,\mathbb{E}\left(\left(G^{\mathsf{T}}B_{\natural}^{2}G\right)\left(G^{\mathsf{T}}B_{\natural}G\right)^{p-2}\right) \\
\leq \left(\operatorname{tr}(B_{\natural}) + 2(p-1)\|B_{\natural}\|_{\operatorname{op}}\right) \,\mathbb{E}\left(\left(G^{\mathsf{T}}B_{\natural}G\right)^{p-1}\right) \\
\leq \prod_{i=1}^{\lfloor p \rfloor} \left(\operatorname{tr}(B_{\natural}) + 2(p-i)\|B_{\natural}\|_{\operatorname{op}}\right) \,\mathbb{E}\left(\left(G^{\mathsf{T}}B_{\natural}G\right)^{p-\lfloor p \rfloor}\right),$$

where the third line follows from the fact that $B_{\natural}^2 \leq \|B_{\natural}\|_{\text{op}} B_{\natural}$, and the third lines holds by recursion. Since $p-|p|\in[0,1)$ and $z\mapsto z^{p-|p|}$ is concave over $z\geq 0$, Jensen's inequality guarantees that

$$\mathbb{E}\left((G^{\mathsf{T}}B_{\natural}G)^{p-\lfloor p\rfloor}\right) \leq (\operatorname{tr}(B_{\natural}))^{p-\lfloor p\rfloor}.$$

Combining the derived inequalities, we obtain

$$\mathbb{E}\left(|G^{\mathsf{T}}BG|^{p}\right) \leq \left(\operatorname{tr}(B_{\natural})\right)^{p} \prod_{i=1}^{\lfloor p \rfloor} \left(1 + \frac{2(p-i)\|B_{\natural}\|_{\operatorname{op}}}{\operatorname{tr}(B_{\natural})}\right)$$

The result follows by observing that $||B_{\natural}||_{\text{op}} = ||B||_{\text{op}}$ and $\text{tr}(B_{\natural}) = ||B||_{*}$. To obtain the simplified version of (17), we can apply the AM–GM inequality to the right-hand side of (17) which yields

$$\mathbb{E}\left(|G^{\mathsf{T}}BG|^{p}\right) = \|B\|_{*}^{p} \left(1 + (2p - 1 - \lfloor p \rfloor) \frac{\|B\|_{\mathrm{op}}}{\|B\|_{*}}\right)^{\lfloor p \rfloor}$$

$$\leq \left(\|B\|_{*} + p\|B\|_{\mathrm{op}}\right)^{p},$$

as desired.

Lemma 3 (Second Moment Method). Let $\xi \in \mathbb{R}$ be a random variable. For any $s \in \mathbb{R}$ we have

$$\mathbb{P}(\xi \ge s) \ge \frac{\left(\mathbb{E}(\xi - s)_{+}\right)^{2}}{\mathbb{E}((\xi - s)_{+}^{2})},$$

where $(x)_{+} \stackrel{\text{def}}{=} \max\{x,0\}$. In particular, if $\mathbb{E} \xi \geq 0$, then for any $\rho \in (0,1)$ we have

$$\mathbb{P}(\xi \ge \rho \,\mathbb{E}\,\xi) \ge \frac{(1-\rho)^2 (\mathbb{E}\,\xi)^2}{\operatorname{var}(\xi) + (1-\rho)^2 (\mathbb{E}\,\xi)^2}.$$

Proof. By the Cauchy–Schwarz inequality we have

$$\mathbb{P}(\xi \ge s) \mathbb{E}\left((\xi - s)_+^2\right) = \mathbb{E}\left(\mathbb{1}(\xi \ge s)\right) \mathbb{E}\left((\xi - s)_+^2\right)$$

$$\geq \left(\mathbb{E}\left((\xi-s)_+\right)\right)^2$$
,

which yields the main inequality. If we further know that $\mathbb{E} \xi \geq 0$, then choosing $s = \rho \mathbb{E} \xi$ yields

$$\mathbb{P}(\xi \ge \rho \,\mathbb{E}\,\xi) \ge \frac{\left(\mathbb{E}\,(\xi - \rho \,\mathbb{E}\,\xi)_{+}\right)^{2}}{\mathbb{E}\,((\xi - \rho \,\mathbb{E}\,\xi)^{2})}$$
$$\ge \frac{(1 - \rho)^{2}(\mathbb{E}\,\xi)^{2}}{\operatorname{var}(\xi) + (1 - \rho)^{2}(\mathbb{E}\,\xi)^{2}},$$

where the second line follows by applying the Jensen's inequality in the numerator and using the assumption $\mathbb{E} \xi \geq 0$.

References

- [Alq24] P. Alquier. "User-friendly introduction to PAC-Bayes bounds". Foundations and Trends in Machine Learning 17: 2 (2024), pp. 174–303 (cit. on p. 2).
- [AB07] J.-Y. Audibert and O. Bousquet. "Combining PAC-Bayesian and generic chaining bounds". *Journal of Machine Learning Research* 8 (2007), pp. 863–889 (cit. on p. 2).
- [AC11] J.-Y. Audibert and O. Catoni. "Robust linear least squares regression". *The Annals of Statistics* 39: 5 (2011), pp. 2766–2794 (cit. on p. 2).
- [BLM13] S. Boucheron, G. Lugosi, and P. Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford: Oxford University Press, Feb. 2013 (cit. on pp. 1, 5, 6, 9).
- [Cat07] O. Catoni. "PAC-Bayesian supervised classification: the thermodynamics of statistical learning". *IMS Lecture Notes Monograph Series*. Vol. 56. 2007 (cit. on p. 2).
- [Cat12] O. Catoni. "Challenging the empirical mean and empirical variance: a deviation study". Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 48: 4 (2012), pp. 1148–1185 (cit. on p. 2).
- [CG17] O. Catoni and I. Giulini. Dimension-free PAC-Bayesian bounds for matrices, vectors, and linear least squares regression. 2017. arXiv: 1712.02747 [math.ST] (cit. on p. 2).
- [Cha19] S. Chatterjee. "A general method for lower bounds on fluctuations of random variables". *The Annals of Probability* 47: 4 (2019), pp. 2140–2171 (cit. on p. 12).
- [DR17] G. K. Dziugaite and D. M. Roy. "Computing nonvacuous generalization bounds for deep (stochastic) neural networks with many more parameters than training data". Proceedings of the Thirty-Third Conference on Uncertainty in Artificial Intelligence, UAI 2017, Sydney, Australia, August 11-15, 2017. 2017 (cit. on p. 2).

- [GU14] A. Gasull and F. Utzet. "Approximating mills ratio". *Journal of Mathematical Analysis and Applications* 420: 2 (2014), pp. 1832–1853 (cit. on p. 6).
- [Giu18] I. Giulini. "Robust dimension-free Gram operator estimates". *Bernoulli* 24: 4B (Nov. 2018) (cit. on p. 2).
- [LS02] J. Langford and J. Shawe-Taylor. "PAC-Bayes & margins". Advances in Neural Information Processing Systems. Vol. 15. 2002 (cit. on p. 2).
- [Lat97] R. Latała. "Estimation of moments of sums of independent real random variables". The Annals of Probability 25: 3 (1997), pp. 1502–1513 (cit. on p. 12).
- [LLS10] G. Lever, F. Laviolette, and J. Shawe-Taylor. "Distribution-dependent PAC-Bayes priors". *Algorithmic Learning Theory*. Berlin, Heidelberg, 2010, pp. 119–133 (cit. on p. 2).
- [McA98] D. A. McAllester. "Some PAC-Bayesian theorems". Proceedings of the Eleventh Annual Conference on Computational Learning Theory. COLT'98. Madison, Wisconsin, USA, 1998, pp. 230–234 (cit. on p. 1).
- [MRT19] M. Mohri, A. Rostamizadeh, and A. Talwalkar. Foundations of machine learning. The MIT Press, Apr. 1, 2019. 504 pp. (cit. on p. 1).
- [OH21] Y. Ohnishi and J. Honorio. "Novel change of measure inequalities with applications to PAC-Bayesian bounds and Monte Carlo estimation". Proceedings of The 24th International Conference on Artificial Intelligence and Statistics. Vol. 130. Proceedings of Machine Learning Research. Apr. 2021, pp. 1711–1719 (cit. on p. 2).
- [Oli16] R. I. Oliveira. "The lower tail of random quadratic forms with applications to ordinary least squares". *Probability Theory and Related Fields* 166: 3-4 (Sept. 2016), pp. 1175–1194 (cit. on p. 2).
- [SB14] S. Shalev-Shwartz and S. Ben-David. *Understanding machine learning: from theory to algorithms*. Cambridge University Press, May 2014 (cit. on p. 1).
- [SW97] J. Shawe-Taylor and R. C. Williamson. "A PAC analysis of a Bayesian estimator". Proceedings of the Tenth Annual Conference on Computational Learning Theory. COLT '97. Nashville, Tennessee, USA, 1997, pp. 2–9 (cit. on p. 1).
- [Tal87] M. Talagrand. "Regularity of Gaussian processes". *Acta Mathematica* 159: 0 (1987), pp. 99–149 (cit. on p. 1).
- [Tal14] M. Talagrand. Upper and lower bounds for stochastic processes. Springer Berlin Heidelberg, 2014 (cit. on p. 1).
- [Wai19] M. J. Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*. Cambridge University Press, Feb. 21, 2019 (cit. on p. 6).
- [Zhi24] N. Zhivotovskiy. "Dimension-free bounds for sums of independent matrices and simple tensors via the variational principle". *Electronic Journal of Probability* 29 (Jan. 2024) (cit. on pp. 2, 3, 11).

[ZVAA+19] W. Zhou, V. Veitch, M. Austern, R. P. Adams, and P. Orbanz. "Non-vacuous generalization bounds at the ImageNet scale: a PAC-Bayesian compression approach". *International Conference on Learning Representations*. 2019 (cit. on p. 2).