PROBABILITY

Probability is simply the measure of the likelihood that an event will occur. And, in the form of a number, the probability is from 0 (impossible) to 1 (certain). The sum of all probabilities of all the events in a sample space is equal to 1.

Sample Space:

A sample space is a set of all possible outcomes of an experiment. The sample space is denoted by the symbol, S

Example:

- When you flip a coin then your sample space is $S = \{H, T\}$ where H = "Heads" & T = "Tails"
- When you roll a dice then your sample space is $S = \{1, 2, 3, 4, 5, 6\}$
- Now lets say we are tossing 2 coins together then our $S = \{HH, HT, TH, TT\}$

In general, if you have "n" coins, then the possible number of outcomes/ Sample space = 2^n.

Example: If you toss 3 coins, "n" is taken as 3. Therefore, the possible number of outcomes = 2^3 = 8

Hence Sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Event:

An event is a subset of the sample space. The event is denoted by the symbol, E. So we could say $E \subseteq S$ is called an event.

Example

- In our previous example for rolling a die, our sample space, $S = \{1, 2, 3, 4, 5, 6\}$ so subset E could be $\{1, 3, 5\}$ which represents the set of odd numbers and
  or $\{2, 4, 6\}$ which represents the set of even numbers.
- $E = \{J, K, G\}$ the E is the event of getting a face card.

Example:

A fair die is rolled, What will be the probability that the outcome will be an odd number?

A = {1, 3, 5}

Since there are three values of A meaning there are 3 possible outcomes out of 6 possibilities Hence P(A) = 3 / 6 = 1 / 2

Properties of Probability

Lets denoted multiple events by capital letters $A, B, C....Z$ etc for sample space S.

1) $$P(A) = \frac{\text{No. of outcomes favourable to the occurrence of } A}{\text{Total number of equally likely outcomes}} = \frac{n(A)}{n(S)}$$
2) $1 \leq\ P(A) \leq\ 0$ for any events
3) $P(S) = 1$

Example

1) Each and every letter of the word "EVERYDAY" is inscribed on a separate card. One card is drawn randomly. What is the probability of getting an E?
   We know that E occurs two times in the word, EVERYDAY. Therefore,
   Sample space, S = {E, V, E, R, Y, D, A, Y} = 9
   Let A be the event of getting the letter "E" = (E1 and E2)
   P(A) = 2/9

2) There is a standard deck of 52 cards and 1 card is drawn at random.
   (i) Find the probability that it is a face card
   We know that there are three face cards = K, Q, and J.
   3 cards from each suit meaning = 3 * 4
   P(face card) = 12/52

Complement Probability

$$P(A^c) = 1 - P(A)$$

1) Find the probability that it is not a face card
   P(of not getting face card) = 1 - P(face card)
   P(of not getting face card) = 1 - 12/52
   Therefore, probability of not getting a face card = 40/52 or 10/13

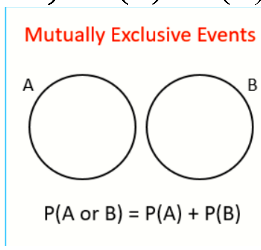
2) What is the probability of at least one "H" in five tosses of a coin?
   Recall when you have 5 coins, then the possible number of outcomes will be 2^5 = 32 So the sample space S will have 32 outcomes
   $\mathrm{P}(at\ least\ one\ H) = 1 - \mathrm{P}(no\ H) = 1 - \frac{1}{32} = \frac{31}{32}$.

Theorem:

I. General Addition Rule:

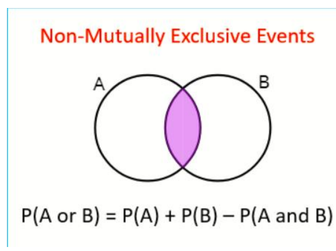
1) set of mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$
$$P(A \cup B \cup ... Z) = P(A) + P(B) + ...P(Z)$$



2) (non-disjoint) events is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



EXAMPLES

1) When one card is drawn from a well-shuffled deck of 52 playing cards, what are the probabilities of getting:

i) a jack, queen, king, or ace:

$$P(jack \cup queen \cup king \cup ace) = P(jack) + P(queen) + P(king) + P(ace)$$
$$= \frac{4}{52} + \frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{16}{52}$$

ii) a ace or a heart

The sample space S consists of the 52 cards.

Let A be the event of drawing a ace so $P(A) = \frac{4}{52}$

Let B be the event of drawing a heart so $P(B) = \frac{13}{52}$

We are interested in the probability of drawing a ace or a heart.

Now, the probability of drawing a ace or a heart is

$$P(E) = P(Ace \cup Heart) = P(Ace) + P(Heart) - P(Ace \cap Heart)$$

= 4/52 + 13/52 - 1/52 = 16/52 = **4/13**

II. General Multiplication Rule:

1) set of mutually exclusive events

$$P(A \cap B) = 0$$

When two events (call them "A" and "B") are Mutually Exclusive it is **impossible** for them to happen together:

2) set of independent events

$$P(A \cap B) = P(A)\ P(B)$$
$$P(A \cap B \cap ... Z) = P(A)\ P(B) \ ......P(Z)$$

3) If the events are not independent, one event affects the probability for the other event. In this case conditional probability must be used. The conditional probability of B given that A occurs, or on condition that A occurs, is written P[B | A].

$$P[A \cap B] = P[A] \times P[B \mid A] = P[B] \times P[A \mid B]$$

EXAMPLES

1) A bag contains 6 black marbles and also 4 blue marbles. Two of the marbles are drawn from the bag, without any replacement. What will be the probability that both of the marbles will be of colour blue?
Let A = the marble 1 is blue and
B = be marble 2 is blue.
There are ten marbles inside the bag, so the probability for drawing marble 1 so $P(A) = 4/10$.
After the first blue ball is drawn now there are nine marbles inside the bag, so $P(B \mid A) = 3/9$.
Hence $P(2\ blue\ balls) = (4/10) * (3/9) = 2/15$

2) Suppose we draw two cards from the deck, without replacement. What is the probability that two hearts are drawn?
Since there are 13 hearts among the 52 cards, the $P(first\ heart) = 13/52$
On second draw, the number of total cards and the desired suit is one less than it was for the first pick. Hence $P(second\ heart \mid first\ heart) = 12/51$
Hence $P(2\ hearts) = \left(\frac{13}{52}\right) * \left(\frac{12}{51}\right)$

3) What is the probability of drawing a queen GIVEN that the card drawn is of suit hearts

$$P(Q \mid H) = \frac{P(Q \cap H)}{P(H)}$$

Note there are total 4 Queens in 52 deck of cards so $\frac{4}{52}$ where each QUEEN is from Hearts, diamonds, clubs and spades so 1 QUEEN from HEARTS then $\frac{1}{52}$

Hence $\frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}$

4) What is the probability of drawing a QUEEN given that the card drawn is a Face Card

$$P(Q \mid F) = \frac{P(Q \cap F)}{P(F)}$$

Note there are total 4 Queens in 52 deck of cards so $\frac{4}{52}$

There are **12 face cards (4 Kings, 4 queens, and 4 jacks)**

36 numbered cards (2's through 10's……. 9 x 4)

4 ace cards

Hence $\frac{\frac{4}{52}}{\frac{12}{52}} = \frac{1}{3}$

5) A single card is drawn from a deck. A card is selected at random, find the probability of selecting a

a) King given that it is a red card

$$P(King\ given\ that\ it\ is\ a\ red\ card) = P(King|red) = \frac{P(King\ and\ red)}{P(red)}$$

$$= \frac{2}{52} \Big/ \frac{26}{52} = \frac{2}{26} = \frac{1}{13}$$

b) red card given that it is a King

$$P(red\ card\ given\ that\ it\ is\ a\ King) = P(red|King) = \frac{P(red\ and\ King)}{P(King)}$$

$$= \frac{2}{52} \Big/ \frac{4}{52} = \frac{2}{4} = \frac{1}{2}$$

c) Queen given that it is a Heart

$$P(queen\ card\ given\ that\ it\ is\ a\ heart) = P(queen|heart) = \frac{P(queen\ and\ heart)}{P(heart)}$$

$$= \frac{1}{52} \Big/ \frac{13}{52} = \frac{1}{13}$$

6) The results of a survey of a group of 100 people having insurances with a certain company are as follows: 40% have both home and car insurances with the company. The probability that person selected at random from this group, has a car insurance is 0.8. What is the probability that a person selected at random has a home insurance given that he has a car insurance?

Let event H: people with home insurance,

event C: people with can insurance

We are given $P(C) = 0.8$ and $P(H \text{ } and \text{ } C) = 0.4$
$P(H|C)$ = person have a home insurance (H) given that this person has a car insurance (C).
Hence $P(H|C) = P(H \text{ } and \text{ } C) / P(C) = 0.4 / 0.8 = ½$

7) A group of 200 Students were asked whether they played football or basketball. Among the group, 120 said they played football, 50 said they played basketball and 20 said they played both football and basketball.
a) What is the probability that a students selected at random from the group plays football given that he plays basketball?

$$P(football \mid basketball) = \frac{P(football \text{ } and \text{ } basketball)}{P(basketball)} = \frac{20/200}{50/200} = \frac{20}{50} = \frac{2}{5}$$

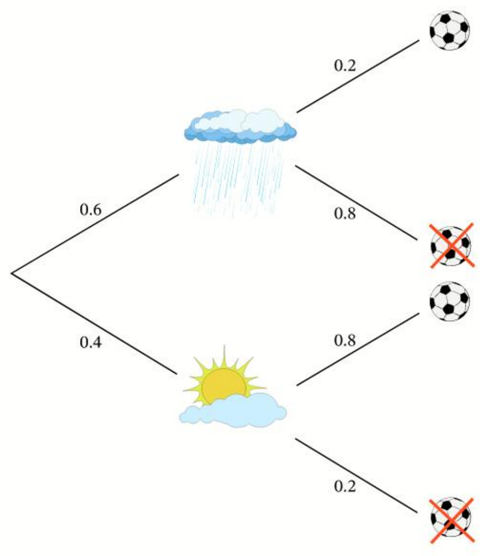
b) What is the probability that a students selected at random from the group plays basketball given that he plays football?

$$P(basketball \mid football) = \frac{P(basketball \text{ } and \text{ } football)}{P(football)} = \frac{20/200}{120/200} = \frac{20}{120} = \frac{1}{6}$$

c) What is the probability that a students selected at random from the group plays football given that he plays one game only.
The number of students who play one game only is (120−20)+(50−20) = 130
$P(One \text{ } Game) = 130 / 200 = 0.65$
$P(F \text{ } and \text{ } O) = (120 – 20) / 200 = 0.5$

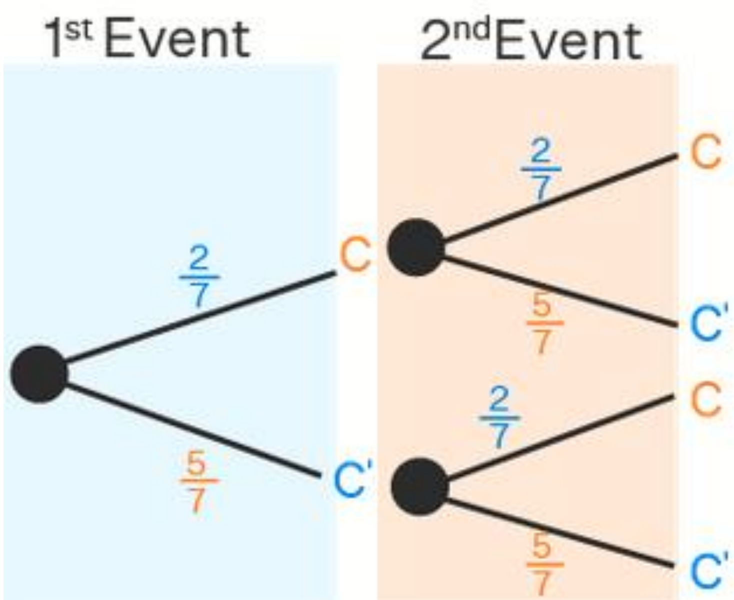
$$P(football \mid one \text{ } game) = \frac{P(football \text{ } and \text{ } one \text{ } game)}{P(one \text{ } game)} = \frac{0.50}{0.65}$$

8) The probability that it rains on a given day is 0.6. If it rains, the probability that a group of friends play football is 0.2. If it does not rain, the probability that they play football rises to 0.8. Find the probability that
a) it rains on a given day and the friends play football.
b) it does not rain on a given day and the friends play football.
c) the friends will play football on a given day?



a) $P(R \cap F) = P(R \mid F)\, P(F) = P(F \mid R)\, P(R)$ = (0.20)(0.60)
b) $P(NR \cap F) = P(NR \mid F)\, P(F) = P(F \mid NR)\, P(NR)$ = (0.80)(0.40)
c) $P(F) = P(F \mid R)\, P(R) + P(F \mid NR)\, P(NR)$ = (0.20)(0.60) + (0.80)(0.40)

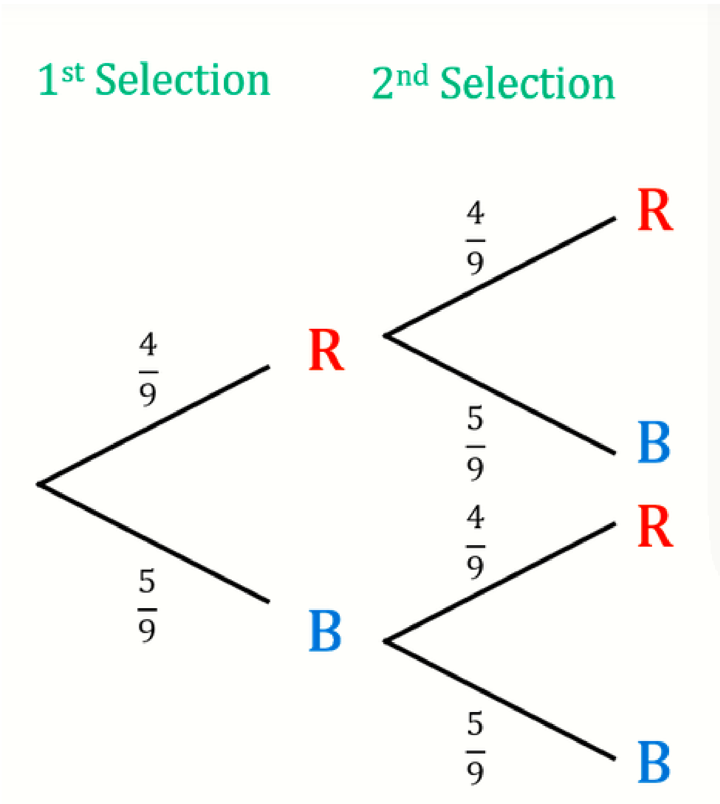
9) The letters of the word "CHICKEN" are written on 7 cards. Kelly chooses a card and replaces it and chooses another one. Find the probability that only one of the chosen cards will have the letter C on it.



$$P(C, not\ C) + P(not\ C, C) = (2 / 7)(5 / 7) + (5 / 7)(2 / 7) = 20 / 49$$

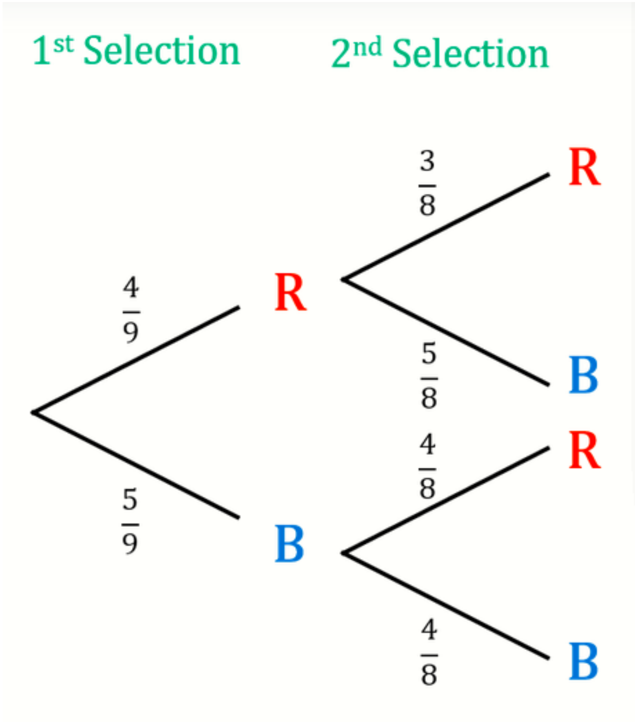
10) A bag contains 4 **red** balls and 5 **blue** balls. Raheem picks 2 balls at random.

a) Calculate the probability that he selects the same coloured ball each time, given that after each time a ball is selected, **it is replaced**.



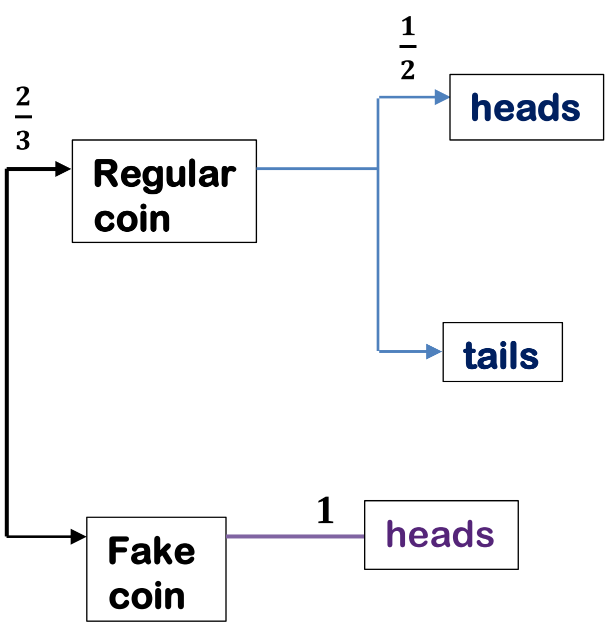
$$P(\text{same colored ball}) = P(2\ RED) + P(2\ BLUE) = \left(\frac{4}{9}\right)\left(\frac{4}{9}\right) + \left(\frac{5}{9}\right)\left(\frac{5}{9}\right) = \frac{16}{81} + \frac{25}{81} = \frac{41}{81}$$

b) Calculate the probability that he selects the same coloured ball each time, given that after each time a ball is selected, **it is not replaced**.



$$P(same\ colored\ ball) = P(2\ RED) + P(2\ BLUE) = \left(\frac{4}{9}\right)\left(\frac{3}{8}\right) + \left(\frac{5}{9}\right)\left(\frac{4}{8}\right) = \frac{12}{72} + \frac{20}{72} = \frac{32}{72}$$

11) A box contains three coins: two regular coins and one fake two-headed coin



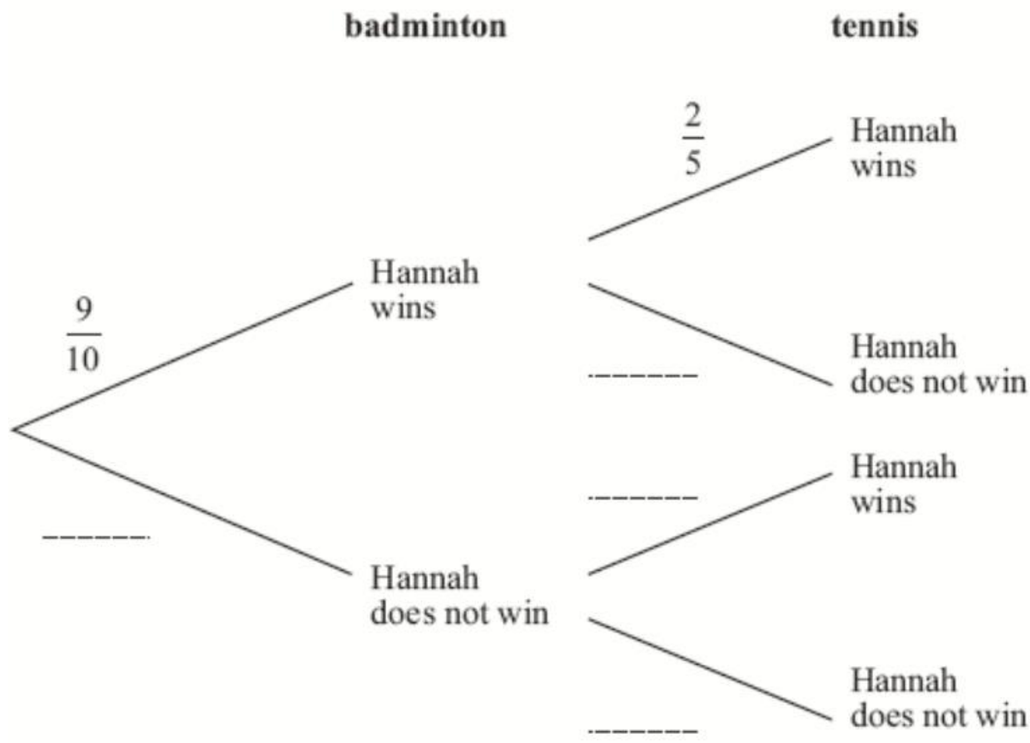
a) You pick a coin at random and toss it. What is the probability that it lands heads?

$$P(heads) = P(RC,H) + P(FC,H) = (\frac{2}{3}.\frac{1}{2}) + (\frac{1}{3}.1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

b) You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin GIVEN that it is heads?

$$P(FC \mid H) = \frac{P(FC,H)}{P(heads)} = \frac{\frac{1}{3}.1}{\frac{2}{3}} = \frac{1}{2}$$

12) Hannah is going to play one badminton match and one tennis match. The probability that she will win the badminton match is 9/10. The probability that she will win the tennis match is 2/5. Work out the probability that Hannah will win both matches.



P(wins 2 matches) $= (\frac{9}{10}).(\frac{2}{5})$

13)Carolyn has 20 biscuits in a tin.She has

12 plain biscuits
5 chocolate biscuits
3 ginger biscuits

Carolyn takes at random two biscuits from the tin and eats it. Work out the probability that the two biscuits were not the same type.

$$P(not\ same\ type) = P(plain, chocolate) + P(chocolate, plain) + P(plain, ginger) + P(ginger, plain) + P(chocolate, ginger) + P(ginger, chocolate)$$

$$= (\tfrac{12}{20}).\left(\tfrac{5}{19}\right) + (\tfrac{5}{20}).\left(\tfrac{12}{19}\right) + (\tfrac{12}{20}).\left(\tfrac{3}{19}\right) + \left(\tfrac{3}{20}\right)(\tfrac{12}{19}) + \left(\tfrac{5}{20}\right)\left(\tfrac{3}{19}\right) + \left(\tfrac{3}{20}\right)\left(\tfrac{5}{19}\right)$$

$$= \left(\tfrac{60}{380}\right) + \left(\tfrac{60}{380}\right) + \left(\tfrac{36}{380}\right) + \left(\tfrac{36}{380}\right) + \left(\tfrac{15}{380}\right) + \left(\tfrac{15}{380}\right) = \tfrac{222}{380} = \tfrac{111}{190}$$

In another way

$$P(not\ same\ type) = 1 - P(same\ type)$$

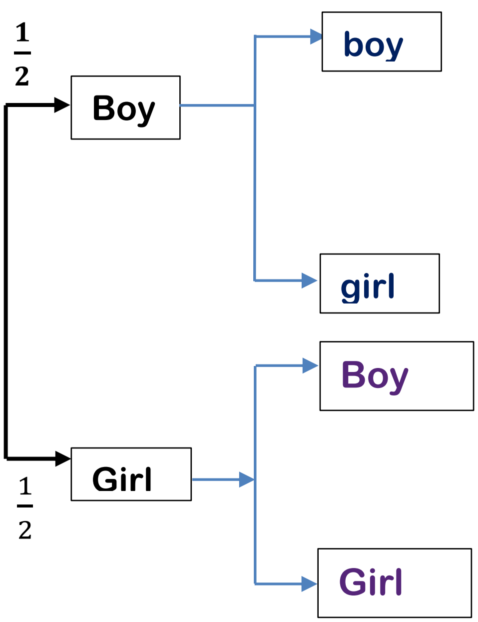
$$= 1 - [P(plain, plain) + P(choc, choc) + P(ginger, ginger)]$$

$$= 1 - [(\tfrac{12}{20}).\left(\tfrac{11}{19}\right) + (\tfrac{5}{20}).\left(\tfrac{4}{19}\right) + (\tfrac{3}{20}).\left(\tfrac{2}{19}\right)]$$

$$= \mathbf{1} - 158/380$$

Example 2

A family has two children. Assuming that boys and girls are equally likely, determine the probability that the family has



a) One boy and one girl GIVEN the first child is a boy

$$\text{P(second is girl | first is a boy)} = \frac{(B,G)}{(B,G)(B,B)} = \frac{\frac{1}{2}\,\frac{1}{2}}{\left(\frac{1}{2}\,\frac{1}{2}\right)+\left(\frac{1}{2}\,\frac{1}{2}\right)} = \frac{1}{2}$$

b) Two girls GIVEN that the younger one is a girl

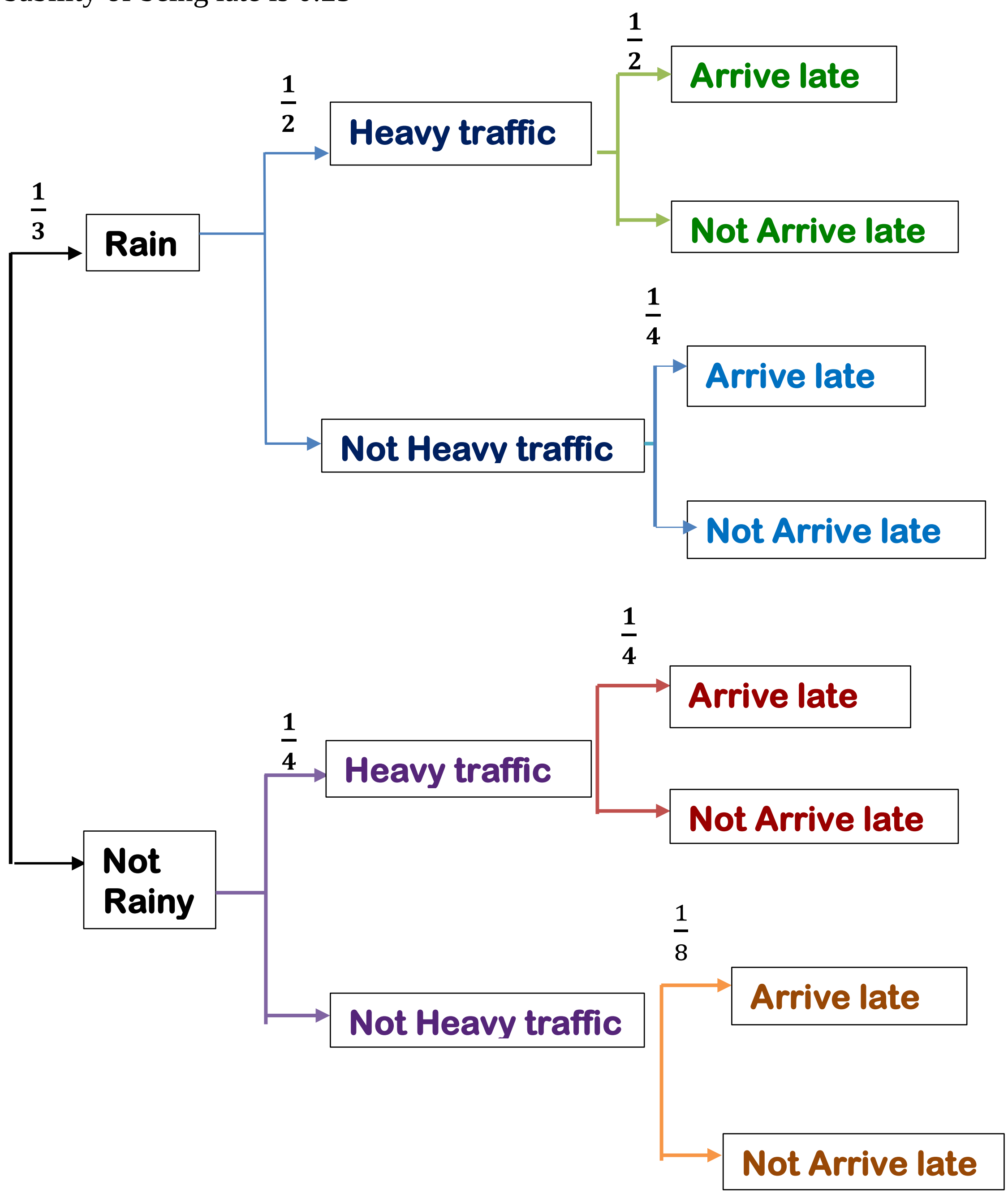
$$P(\text{2 girls} \mid \text{younger is a girl}) = \frac{(G,G)}{(B,G)(G,G)} = \frac{\frac{1}{2}\,\frac{1}{2}}{\left(\frac{1}{2}\,\frac{1}{2}\right)+\left(\frac{1}{2}\,\frac{1}{2}\right)} = \frac{1}{2}$$

c) Two girls GIVEN that at least one is a girl

$$P(\text{2 girls} \mid \text{atleast one is a girl}) = \frac{(G,G)}{(G,B)(B,G)(G,G)} = \frac{\frac{1}{2}\,\frac{1}{2}}{\left(\frac{1}{2}\,\frac{1}{2}\right)+\left(\frac{1}{2}\,\frac{1}{2}\right)+\left(\frac{1}{2}\,\frac{1}{2}\right)} = \frac{1}{3}$$

Example

In a town, it's rainy 1/3 of the days. Given that it is rainy, there will be heavy traffic with probability 1/2, and given that it is not rainy, there will be heavy traffic with probability 1/4. If it's rainy and there is heavy traffic, I arrive late for work with probability 1/2. On the other hand, the probability of being late is reduced to 1/8 if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic the probability of being late is 0.25



a) What is the probability that it's not raining and there is heavy traffic and I am not late?

$$P(not\ raining\ \cap\ heavy\ traffic\ \cap not\ late) = P(not\ raining).P(heavy\ traffic).P(not\ late)$$

$$= (1-\frac{1}{3}).\frac{1}{4}.(1-\frac{1}{4}) = \frac{2}{3}.\frac{1}{4}.\frac{3}{4} = \frac{1}{8}$$

b) The probability that I am late

$$P(late) = P(R,HT,L) + P(R,NHT,L) + P(NR,HT,L) + P(NR,NHT,L)$$

$$= \frac{1}{3}.\frac{1}{2}.\frac{1}{2} + \frac{1}{3}.(1-\frac{1}{2})\frac{1}{4} + (1-\frac{1}{3}).\frac{1}{4}.\frac{1}{4} + (1-\frac{1}{3}).(1-\frac{1}{4}).\frac{1}{8}$$

$$= \frac{1}{3}.\frac{1}{2}.\frac{1}{2} + \frac{1}{3}.\frac{1}{2}.\frac{1}{4} + \frac{2}{3}.\frac{1}{4}.\frac{1}{4} + \frac{2}{3}.\frac{3}{4}.\frac{1}{8} = \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} = \frac{11}{48}$$

c) Given that I arrived late at work, what is the probability that it rained that day?

$$P(\text{Rainy} \mid \text{Late}) = \frac{P(\text{rainy} \cap \text{late})}{P(\text{late})}$$

$$P(\text{rainy} \cap \text{late}) = P(R,HT,L) + P(R,NHT,L)$$

$$= \frac{1}{3}.\frac{1}{2}.\frac{1}{2} + \frac{1}{3}.\left(1-\frac{1}{2}\right)\frac{1}{4} = \frac{1}{3}.\frac{1}{2}.\frac{1}{2} + \frac{1}{3}.\frac{1}{2}.\frac{1}{4} = \frac{1}{12} + \frac{1}{24} = \frac{3}{24} = \frac{1}{8}$$

$$P(\text{late}) = \frac{11}{48}$$

$$P(\text{Rainy} \mid \text{Late}) = \frac{P(\text{rainy} \cap \text{late})}{P(\text{late})} = \frac{\frac{1}{8}}{\frac{11}{48}} = \frac{6}{11}$$

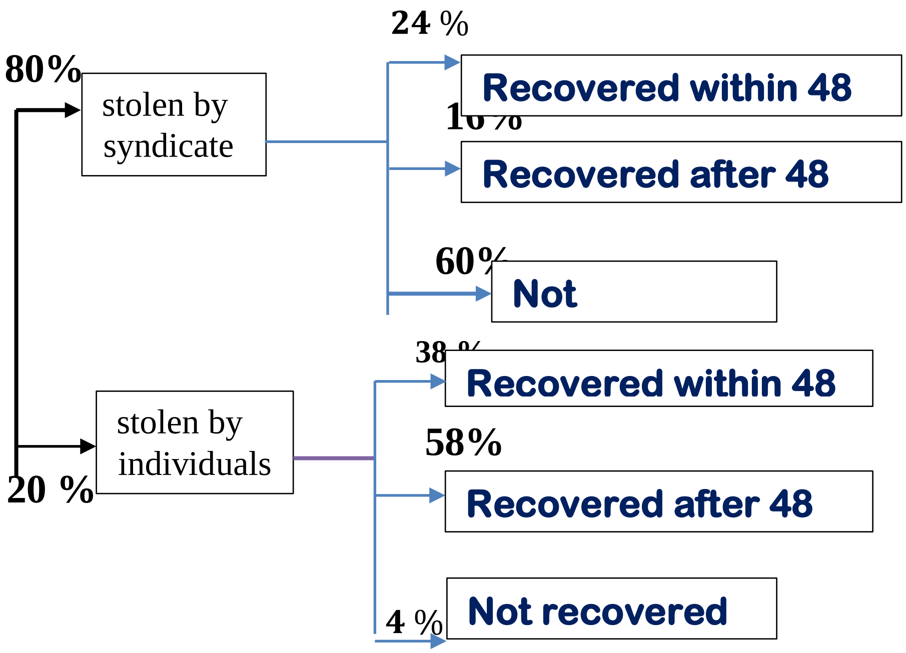
<u>Example</u>

1. Figures obtained from a city’s police department seem to indicate that of all the motor vehicles reported stolen, 80% were stolen by syndicates to be sold off and 20% were stolen by individuals for their own use

   Of those vehicles presumed stolen by syndicates:
   - 24% were recovered within 48 hours
   - 16% were recovered after 48 hours
   - 60% were never recovered

   Of those vehicles presumed stolen by individuals for their own use
   - 38% were recovered within 48 hours
   - 58% were recovered after 48 hours
   - 4% were never recovered

   a) Draw a tree diagram representing the above information
   b) Calculate the probability that if a vehicle were stolen in the city it would be stolen by a syndicate and recovered within 48 hours
   c) Calculate the probability that a vehicle stolen in this city will not be recovered



a) $P(sydicate\ and\ within\ 48\ hours)\ =\ 80\ \% \times 24\ \% = 19.2\%$

b) $P(not\ recovered) = (80\ \% \times 60\ \%) + (20\% \ \times 4\%) = 48.8\%$