Limit

If $f(z)$ approaches to a finite value 'l' when z tends to the number 'a' then l is called the limit of $f(z)$ at $z = a$. We write

$$\underset{z \to a}{\text{limit}}\, f(z) = l$$

In other words, If for a preassigned small positive number ε there corresponds another small positive number δ such that

$$|f(z) - l| < \varepsilon \text{ when } |z - a| < \delta$$

then l is called the limit of $f(z)$ at $z = a$.

We write $\lim\limits_{z \to a} f(z) = l$.

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Example: Previous one.

Limit, Continuity, Differentiability and Analyticity of Complex function

Harmonic#2
Lt Col Amir

Ans.6 (i) $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x - iy}{x+iy}$

Along the real axis $y = 0$ and $x \to 0$ when $z \to 0$

$\therefore \lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = \lim\limits_{x\to 0} \dfrac{x - i\cdot 0}{x + i\cdot 0} = \lim\limits_{x\to 0} \dfrac{x}{x} = \lim\limits_{x\to 0}(1) = 1$

Again, along the imaginary axis $x = 0$, $y \to 0$ when $z \to 0$

$\therefore \lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = \lim\limits_{y\to 0} \dfrac{0 - iy}{0 + iy} = -1$

since the limits in different paths are differen

the limit does not exist.

Qn: 6 (ii) Find the $\lim\limits_{z\to 0} \dfrac{z}{\bar{z}}$

Ans6.(ii): follow the same technique.

Continuity

The function $f(z)$ is called continuous at $z=a$ if

(i) $f(a)$ exists

(ii) $\lim_{z\to a} f(z)$ exists

(iii) $\lim_{z\to a} f(z) = f(a)$.

In other words, If for a preassigned small positive number $\varepsilon>0$ there corresponds another small positive number $\delta>0$ in such a way that

$$|f(z)-f(a)|<\varepsilon \text{ when } 0<|z-a|<\delta, \text{ then}$$

$f(z)$ is called continuous at $z=a$.

Differentiability

The function $f(z)$ is differentiable at $z=a$ if

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{exists.}$$

Q^n: show that $f(z)=\bar{z}$ is continuous at $z=0$ but not differentiable.

Ans: $|f(z)-f(0)| = |\bar{z}-0|$ as $f(0)=\bar{0}=0$

$\Rightarrow |f(z)-f(0)| = |\bar{z}| = |z| = |z-0|<\varepsilon$ when $|z-0|<\delta=\varepsilon$

Therefore, $f(z)$ is continuous at $z=0$.

Now. $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \frac{\bar{h}-0}{h}$

$= \lim_{h\to 0}\frac{a-ib}{a+ib}$ (say, $h=a+ib$)

Along the real axis $b=0$ and $a\to 0$

$\therefore f'(0) = \lim_{a\to 0}\frac{a-i\cdot 0}{a+i\cdot 0} = \lim_{a\to 0}\left(\frac{a}{a}\right) = 1.$

~~Along Imaginary axis $a=0$ and $b\to 0$~~

$f'(0) = \lim \quad a-ib \quad \ldots$

(*) $\therefore f(z)$ is not differentiable

Harmonic #4
Lt Col Amir

Analytic Function

The single valued function $f(z)$ is said to be analytic at $z=a$ if it is differentiable in the neighbourhood $|z-a|<\delta$ of a.

OR: $f(z) = u(x,y) + iv(x,y)$ is analytic if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$.

Note: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ called Cauchy-Riemann equations.

Question: Test whether the following functions are analytic or not.

(i) $f(z) = e^z$ [hints: $e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x \sin y$]

(ii) $f(z) = e^{z^2} = e^{(x+iy)^2} = e^{(x^2-y^2)+i2xy} = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$

P#62

(iii) $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2-y^2} = \frac{x}{x^2-y^2} + i\left(\frac{-y}{x^2-y^2}\right)$

(iv) $f(z) = \frac{1}{\bar{z}} = \frac{1}{x-iy} = \frac{x+iy}{x^2-y^2} = \frac{x}{x^2-y^2} + i\frac{y}{x^2-y^2}$.

(v) $f(z) = \sin 2z = \sin 2(x+iy) = \sin(2x+i2y) = \sin 2x\cos(i2y) + \cos 2x\sin(i2y)$

$= \sin 2x\cosh 2y + i\cos 2x\sinh 2y$ [$\because \cos i\theta = \cosh\theta$, $\sin i\theta = i\sinh\theta$]

Question: Differentiate the following function by using definition:

P#53 (i) $f(z) = 3z^2 - 2z + 4$, (ii) $f(z) = \frac{2z-i}{z+2i}$ at $z=-i$.

Question: state and establish the polar form of Cauchy-Riemann equations.

P#48

Harmonic#6
Lt Col Amir

Answer. The polar form of Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Proof We know that the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ —(i) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ —(ii)

Again, we know that $x = r\cos\theta \therefore \frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta$

$$y = r\sin\theta \therefore \frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Now $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta \quad \text{—(i)}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y} \quad \text{—— (ii)}$$

Again, $$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cdot\cos\theta + \frac{\partial v}{\partial y}\cdot\sin\theta$$

$$\Rightarrow \frac{\partial v}{\partial r} = \left(-\frac{\partial u}{\partial y}\right)\cos\theta + \frac{\partial u}{\partial x}\sin\theta \quad \left[\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$$

$$\therefore -\frac{\partial v}{\partial r} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y} \quad \text{—— (iii)}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = -\sin\theta\cdot\left(-\frac{\partial u}{\partial y}\right) + \cos\theta\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \quad \text{—(iv)}$$

from (i) and (iv) we have $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$

from (ii) and (iii) we have $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$

$$\therefore \boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

Question State the necessary conditions for the function $f(z) = u(x,y) + i v(x,y)$ to be analytic.

Harmonic #5
Lt Col Amir

Answer. The single valued function $f(z) = u(x,y) + i v(x,y)$ is analytic in the region R if the derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

P#47

Proof. Let $f(z)$ is analytic. Then $f(z)$ is differentiable in the region R.

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists and unique.}$$

Now
$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y)\} - \{u(x,y) + i v(x,y)\}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}$$

Along the real axis, $\Delta x \to 0$ and $\Delta y = 0$

$$\therefore f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{—— (ii)}$$

Again, along the imaginary axis, $\Delta x = 0$ and $\Delta y \to 0$

$$\therefore f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$= \frac{i}{i^2} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right) \quad \text{—— (iii)}$$

From (ii) and (iii) we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}, \text{ proved}$$

Question. If $f(z) = u(x,y) + iv(x,y)$ is analytic in R and $u(x,y)$, $v(x,y)$ has continuous second partial derivatives then $u(x,y)$ and $v(x,y)$ are harmonic in R. $\boxed{\text{Harmonic\#7 Lt Col Amir}}$

[Note: The real variable function $u(x,y)$ is harmonic if it satisfies the Laplacian equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$]

Answer. Since $f(z) = u(x,y) + iv(x,y)$ is analytic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (i)} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (ii)}$$

Now $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ --- (A)

Again, $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$

$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$ --- (B) [$\because$ $v(x,y)$ has continuous second partial derivatives, $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$]

$\therefore$ (A) + (B) $\Rightarrow$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, shows $u(x,y)$ is harmonic

Now $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}$ --- (C)

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) \Rightarrow \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$

$\Rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y \partial x}$ --- (D) [$\because$ $u(x,y)$ has continuous second partial derivative, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$]

$\therefore$ (C) + (D) $\Rightarrow$ $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, shows $v(x,y)$ is harmonic.

FINDING HARMONIC CONJUGATE & $f(z)$ [Harmonic # 08, IT Col Amir]

A If $u(x,y)$ of $f(z) = u(x,y) + i v(x,y)$ is given then

* $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \left(-\frac{\partial u}{\partial y}\right) dx + \frac{\partial u}{\partial x} dy \quad [\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}]$

$= M dx + N dy$ TYPE

$\Rightarrow v = \int M dx + \int (x\text{-excluded part of } N) dy + c$

$\boxed{* \; f'(z) = u_1(z,0) - i u_2(z,0)}$

B If $v(x,y)$ of $f(z) = u(x,y) + i v(x,y)$ is given then

* $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = M dx + N dy$ TYPE

$\Rightarrow u = \int (y\text{-excluded part of } M) dx + \int N dy + c$

$\boxed{* \; f'(z) = i[v_1(z,0) - i v_2(z,0)]}$

COMPLEX POTENTIAL

The analytic function $f(z) = u(x,y) + i v(x,y)$ in two dimensional irrotational incompressible fluid pattern is called complex potential where $u(x,y)$ is called the velocity potential (potential function) and $v(x,y)$ is called stream function (flux function).

QUESTION: Test whether the functions are harmonic or not:

(i) $u(x,y) = x^3 + 6x^2y - 3xy^2 - 2y^3$

(ii) $u(x,y) = x^2 - y^2 - 2xy - 2x + 3y$

(iii) $u(x,y) = xe^x \cos y - ye^x \sin y$

(iv) $u(x,y) = e^{-x}(x \sin y - y \cos y)$ [KHANNA P# 62] [Schaum's P# 3.13]

QUESTION: Find the relation between a, b, c and d so that $u(x,y) = ax^3 - bx^2y - cxy^2 + dy^3$ is harmonic.

QUESTION: Test whether the functions are harmonic or not If they are harmonic, find their conjugate so that $f(z) = u(x,y) + i\,v(x,y)$ is analytic. Determine $f(z)$ in terms of z:

Harmonic#09
Lt Col Amir

* (i) $u(x,y) = 2x(1-y)$: Ans $v(x,y) = x^2 + 2y - y^2 + c$, $f(z) = 2z + iz^2 + c$

(ii) $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$, Ans $f(z) = z^3 + 3z^2 + c$

(iii) $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

Ans: $v(x,y) = -x^3 + 3xy^2 + 4xy + c$, $f(z) = 2z^2 - iz^3 + c$

(iv) $u(x,y) = y^3 - 3x^2y$ Ans $f(z) = iz^3 + c$.

(v) $u(x,y) = \frac{1}{2}\ln(x^2+y^2)$

Ans: $v(x,y) = -\tan^{-1}(x/y) + c$, $f(z) = \ln z + c$

(vi) $u(x,y) = -x^3 + 3xy^2 + 2y + 1$

Ans: $v(x,y) = -3x^2y - 2x + y^3 + c$, $f(z) = -z^3 - 2iz + c$

* (vii) $u(x,y) = e^x \cos y$ [Schaum's P#3.17, 2nd Edition]

Ans: $v(x,y) = e^x \sin y + c$, $f(z) = e^z + c$.

KHANNA P#533

(viii) $u(x,y) = x^3 - 3xy^2$ Ans $v(x,y) = 3x^2y - y^3 + c$

$f(z) = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3) + ic = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + i$

$= (x+iy)^3 + ic = z^3 + ic = z^3 + k.$

$\boxed{\text{Harmonic\#10, Lt Col Amir}}$

QUESTION: In a two dimensional irrotational incompressible fluid flow the stream functions $v(x,y)$ are given below. Test whether it is harmonic or not. If it is harmonic, find the velocity potential $u(x,y)$ so that the complex potential $f(z) = u(x,y) + iv(x,y)$ is analytic. Determine $f(z)$ in terms of z:

(i) $v(x,y) = \ln[(x-1)^2 + (y-2)^2]$ $\boxed{P\# 69}$

(ii) $v(x,y) = \frac{1}{2}\ln(x^2+y^2)$ $\boxed{P\# 70}$, schaum's P# 3.14 2nd Edition

$= \ln\sqrt{x^2+y^2}$

(iii) $v(x,y) = \dfrac{-y}{x^2+y^2}$ [Schum's P# 3.15, 2nd Edition]

Harmonic# 10, Lt Col Amir

QUESTION: In a two dimensional irrotational incompressible fluid flow the stream functions $v(x,y)$ are given below. Test whether it is harmonic or not. If it is harmonic, find the velocity potential $u(x,y)$ so that the complex potential $f(z) = u(x,y) + iv(x,y)$ is analytic. Determine $f(z)$ in terms of z : * $f(z) =$

(i) $v(x,y) = \ln[(x-1)^2 + (y-2)^2]$ $\boxed{P\# 69}$ *

$\boxed{\text{Ans}}$ $u(x,y) = 2\tan^{-1}\left(\frac{y-2}{x-1}\right)$. $f(z) = -2\tan^{-1}\left(\frac{z-1}{2}\right) + i\ln\{(z-1)^2+4\} + c$.

(ii) $v(x,y) = \frac{1}{2}\ln(x^2+y^2)$ $\boxed{P\# 70}$, Schaum's P# 3.14 2nd Edition

$= \ln\sqrt{x^2+y^2}$ $\boxed{\text{Ans}}$ $u(x,y) = \tan^{-1}(y/x)$, $f(z) = i\ln z + c$

(iii) $v(x,y) = \frac{-y}{x^2+y^2}$ [Schum's P# 3.15, 2nd Edition]

$\boxed{\text{Ans:}}$ $u(x,y) = \frac{x}{x^2+y^2} + c$, $f(z) = \frac{1}{z} + c$

Hints $f(z) = u(x,y) + iv(x,y)$

$$\Rightarrow \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \left[\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$$

$$\Rightarrow \frac{df}{dz} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

Replacing x by z and y by 0 on R.H.S we get

$$\frac{df}{dz} = \frac{-z^2}{z^4} + i\cdot 0 = \frac{-1}{z^2}$$

$$\Rightarrow \boxed{f(z) = \frac{1}{z} + c}$$