22.08.2023
Lt Col Amir

Complex Var, harmonic function and statistics : $\boxed{\text{Lec 1}}$

Complex Number : The number of the form $z = a + ib$ is called complex number; where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$

Ex : $z_1 = 5 + 6i = 5 + 6i$

$z_2 = 5i = 0 + 5i$

$z_3 = 2 = 2 + i \cdot 0$

all are complex numbers.

Note : • $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$

set of
- $\mathbb{C}$: All complex numbers
- $\mathbb{R}$: All real numbers ($\mathbb{R}$ + pos & neg. fractions)
- $\mathbb{Z}$: All integers $(-\infty, \cdots 0, \cdots \infty)$
- $\mathbb{N}$: All positive numbers $(1, 2, \ldots \infty)$

• (1) $z = a + ib$ is usually written for number, where as $z = x + iy$ is written for complex variable.

(2) x & y are called the real & imaginary parts of z respectively and we usually write as.

$$R_e(z) = x \quad , \quad I_m(z) = y$$

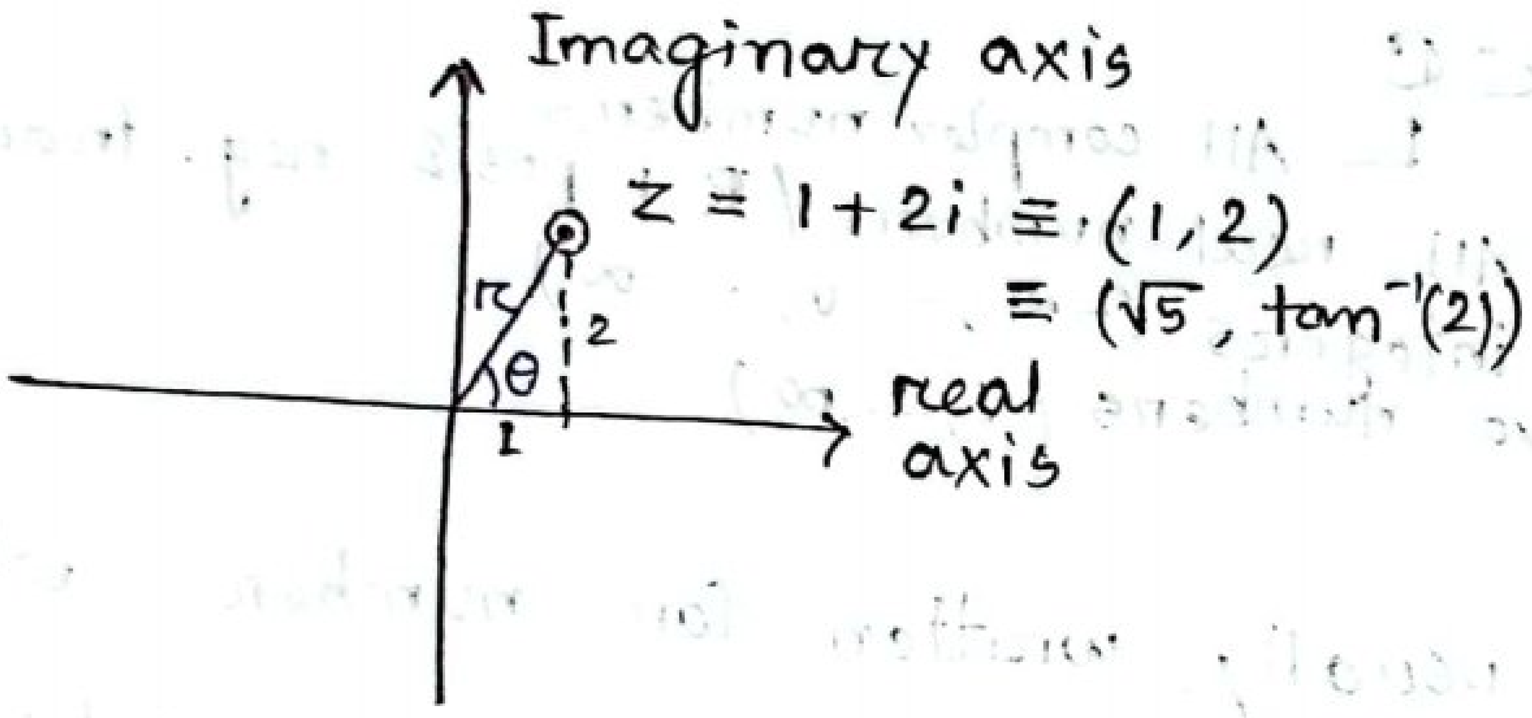
Conjugate of complex number / complex conjugate :

For the complex number $z = a + ib$, $\bar{z} = a - ib$ is called the conjugate and vice-versa.

Representation of Complex Number :

A. In Cartesian co-ordinate system, every complex number can be represented as ordered pair.

B. In polar co-ordinate system, it can be represented by (r, θ) ; where r & θ are called the modulus and argument of the complex number.



$$r = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1} 2$$

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Lec 2

Modulus & argument of complex numbers :

For the complex numbers, $z = a + ib$

$$|z| = \sqrt{a^2+b^2}$$

$$= \sqrt{\{R_e(z)\}^2 + \{I_m(z)\}^2}$$ is called modulus of z.

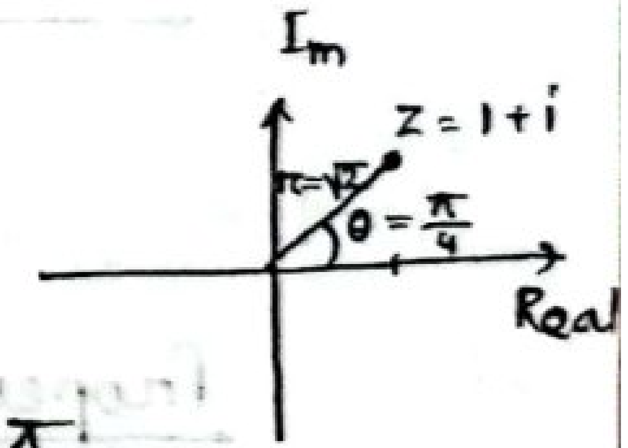
and, $\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left\{\frac{I_m(z)}{R_e(z)}\right\}$ is called the argument of z.

Q) Determine the modulus & arg. of :

i) $z = 1+i$ ii) $z = 1-i$ iii) $z = -1+i$ iv) $z = -1-i$

v) $z = \frac{1-i}{1+i}$ vi) $\sqrt{3}+i$

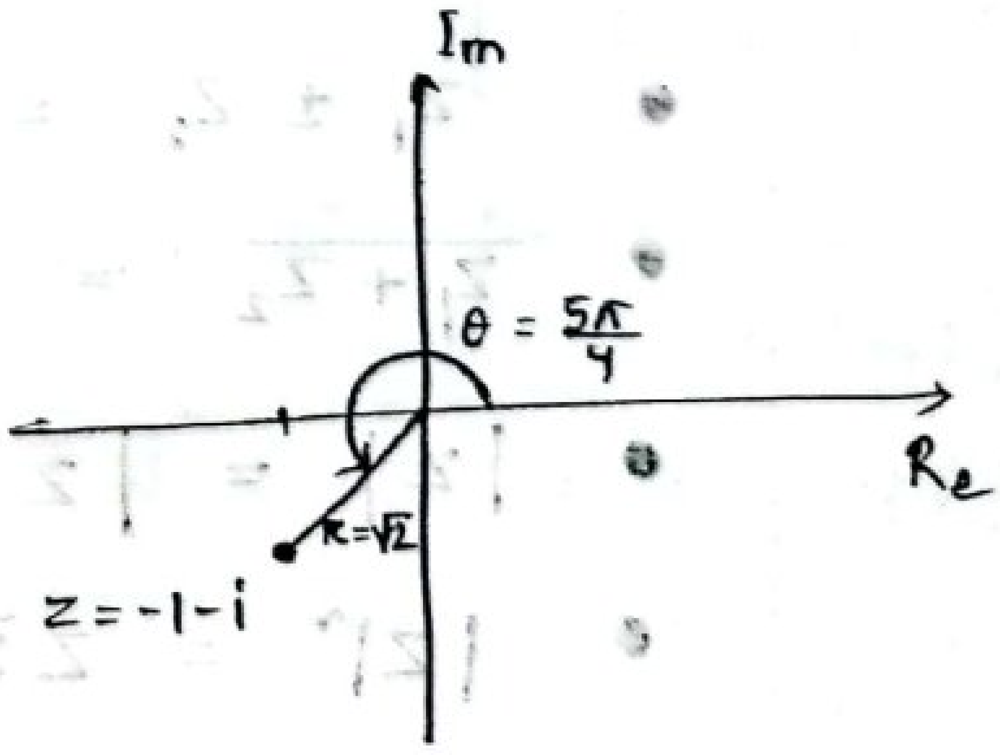
(i) $z = 1+i = 1 + i\cdot 1$



$\therefore |z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore \arg(z) = \theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

(iv) $z = -1-i = -1 + i(-1)$



$\therefore |z| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$

$\therefore \arg(z) = \theta = \pi + \tan^{-1}\left(\frac{-1}{-1}\right)$

$= \pi + \frac{\pi}{4}$

$= \frac{5\pi}{4}$

(v) $z = \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1-i-i+i^2}{1^2-i^2}$

$= \frac{-2i}{2}$

$= -i$

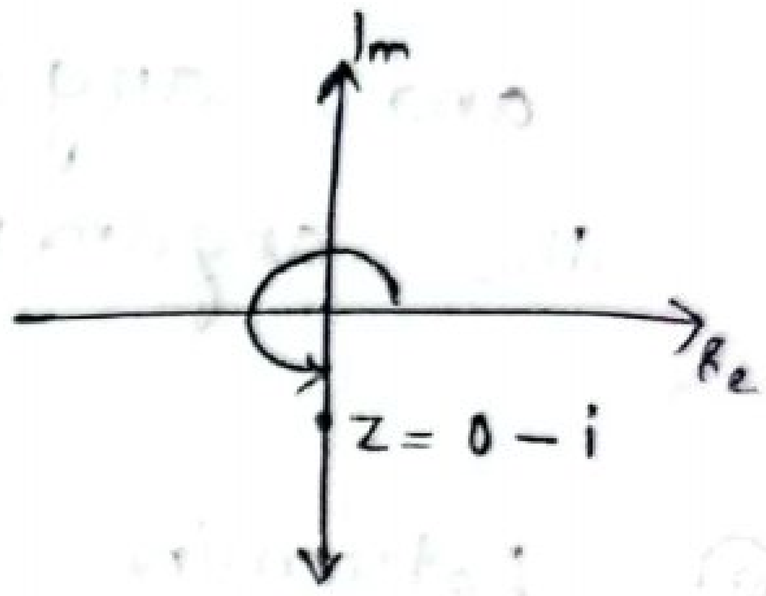
$= 0 + i(-1)$

$\therefore \; |z| = \sqrt{0^2+(-1)^2} = 1$

$\arg(z) = \tan^{-1}\left(\frac{-1}{0}\right) = \tan^{-1}(\infty)$

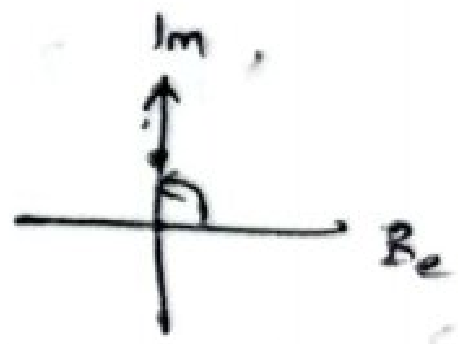
$= \tan^{-1}\left(\tan\left(\frac{3\pi}{2}\right)\right)$

$= \frac{3\pi}{2}$



<u>Note</u> : for, $z = \frac{1+i}{1-i} \xrightarrow{z=i} |z| = 1$

$\arg(z) = \frac{\pi}{2}$



<u>Properties of complex number</u> :

- For the complex numbers, $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$,
  $z_1 = z_2$ iff $a_1 = a_2$ and $b_1 = b_2$.
- $z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ $\quad (\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2})$
- $|z| = |\overline{z}|$
- $|z|^2 = z\overline{z}$ $\quad z = a + ib$, $|z|^2 = a^2 + b^2$ $\quad z\overline{z} = (a+ib)(a-ib) = a^2 + b^2$

$$z_2 \overline{z_1} = \overline{\overline{z_2}} \cdot \overline{z_1} = \overline{z_1 \overline{z_2}}$$

- $z + \overline{z} = 2R_e(z)$ $\qquad z + \overline{z} = a + ib + a - ib = 2a = 2R_e(z)$

- a) $R_e(z) \le |z|$; $a \le \sqrt{a^2+b^2}$

  b) $I_m(z) \le |z|$; $b \le \sqrt{a^2+b^2}$

Q) For the complex numbers $z_1, z_2, \ldots z_n$; Prove that.

1) $|z_1 + z_2| \le |z_1| + |z_2|$

2) $|z_1 - z_2| \le |z_1| + |z_2|$

3) $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$

4) $|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$

Proof :

(1) We may write $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$ $\quad [\because |z|^2 = z\overline{z}]$

$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1\overline{z_1} + z_2\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_2}$$

$$= |z_1|^2 + z_1\overline{z_2} + z_2\overline{z_1} + |z_2|^2$$

$[\because \overline{\overline{z}} = z]$ $\quad = |z_1|^2 + z_1\overline{z_2} + \overline{\overline{z_2} z_1} + |z_2|^2$

$[z + \overline{z} = 2R_e(z)]$ $\quad = |z_1|^2 + 2R_e(z_1\overline{z_2}) + |z_2|^2$

$\Rightarrow |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2$ $\quad [\because R_e(z) \le |z|]$

$\Rightarrow |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2$ $\quad [\because |z_1 z_2| = |z_1||z_2|]$

$\Rightarrow |z_1 + z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ $\quad [\because |z| = |\overline{z}|]$

$\Rightarrow |z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2| \le |z_1| + |z_2|$

Proof :

(2) First prove, $|z_1 + z_2| \le |z_1| + |z_2|$ —— (1)

let us now replace z_2 by $(-z_2)$,

$\therefore |z_1 - z_2| \le |z_1| + |-z_2|$

$\Rightarrow |z_1 - z_2| \le |z_1| + |z_2|$

Proof (3) : First prove ; $|z_1 + z_2| \le |z_1| + |z_2|$ —— (1)

Now, $|z_1 + z_2 + z_3| = |(z_1 + z_2) + z_3|$

$|(z_1 + z_2) + z_3| \le |z_1 + z_2| + |z_3|$ [by (1)]

$\Rightarrow |z_1 + z_2 + z_3| \le |z_1 + z_2| + |z_3| \le |z_1| + |z_2| + |z_3|$

$\Rightarrow |z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$

Proof 4 :

First prove, $|z_1 + z_2| \le |z_1| + |z_2|$ —— (1)

$|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$ —— (11)

hence we see that the statement is true for 2 and 3

Let the statement is true for m $(m < n)$

$\therefore \ |z_1 + z_2 + \cdots + z_m| \le |z_1| + |z_2| + \cdots + |z_m|$ —— (iii)

Now,

$|z_1 + z_2 + \cdots + z_m + z_{m+1}|$

$= |(z_1 + z_2 + \cdots + z_m) + z_{m+1}| \le |z_1 + z_2 + \cdots + z_m| + |z_{m+1}|$

$\Rightarrow |z_1 + z_2 + \cdots + z_m + z_{m+1}| \le |z_1| + |z_2| + \cdots + |z_m| + |z_{m+1}|$ [by (iii)]

i.e the statement is true for $m+1$,

Therefore the statement is true for any positive int. n

$\therefore |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$

(9) Describe the following regions :

1) $|z + 2i| = 3$

2) $|z - 3| + |z + 3| = 4$

3) $R_e\left(\frac{1}{z}\right) = 1$

4) $I_m\left(\frac{1}{z}\right) = 1$

5) $2 \le |z + i| \le 4$

Ans: 1) $|z+2i| = 3 \Rightarrow |x + iy + 2i| = 3$

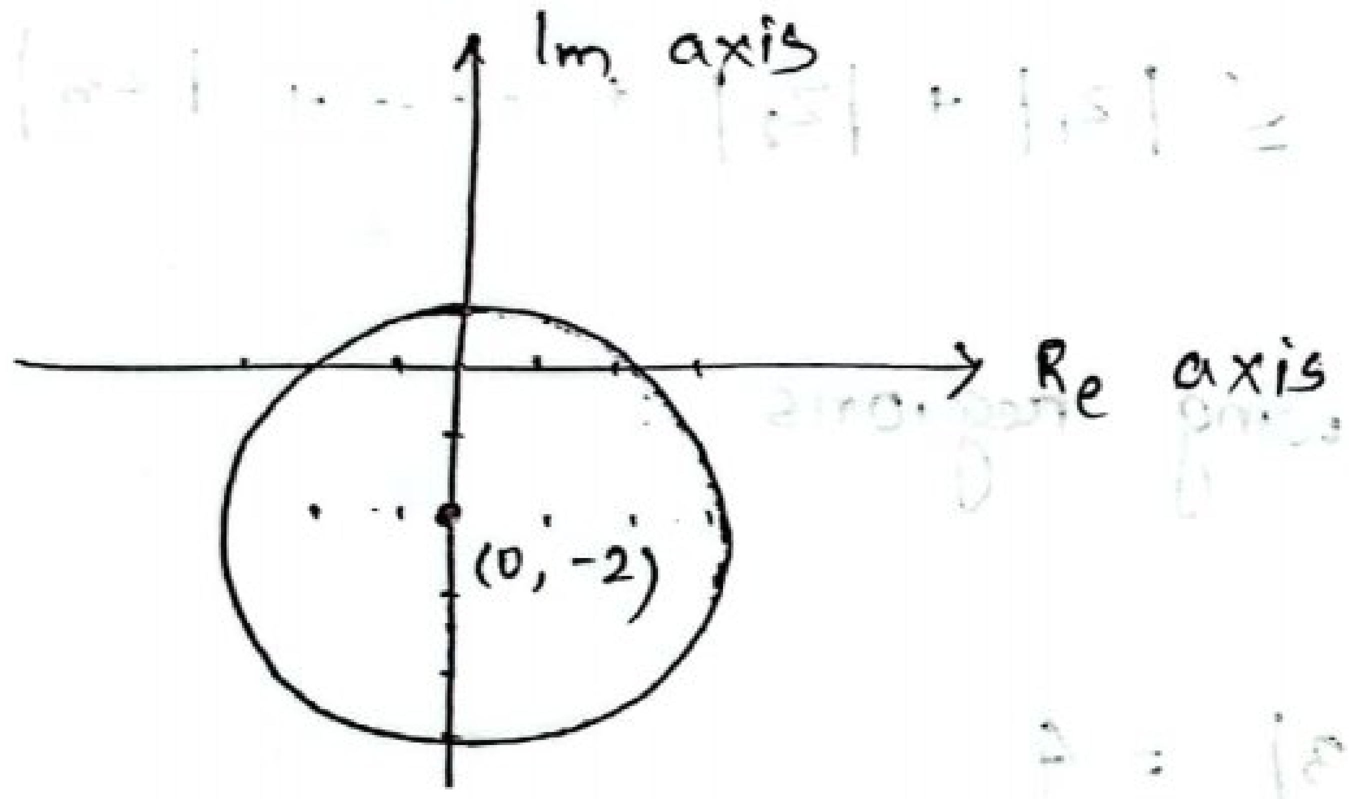
$\Rightarrow |x + i(y+2)| = 3 \qquad [\because z = x+iy]$

$\Rightarrow \sqrt{x^2 + (y+2)^2} = 3$

$\Rightarrow \sqrt{(x-0)^2 + (y-(-2))^2} = 3$

$\Rightarrow (x-0)^2 + \{y-(-2)\}^2 = 3^2$;

represents a circle with radius 3 and center $(0,-2)$



2) $|z-3| + |z+3| = 4$

$\Rightarrow |x+iy-3| + |x+iy+3| = 4 \qquad [\because z = x+iy]$

$\Rightarrow \sqrt{(x-3)^2 + y^2} = 4 - \sqrt{(x+3)^2 + y^2}$

$\Rightarrow \cancel{x^2} - 6x + \cancel{9} + \cancel{y^2} = \left(4 - \sqrt{x^2+6x+9+y^2}\right)^2$

$= 16 - 8\sqrt{x^2+6x+9+y^2} + \cancel{x^2} + 6x + \cancel{9} + \cancel{y^2}$

$\Rightarrow -12x - 16 = -8\sqrt{x^2+6x+9+y^2}$

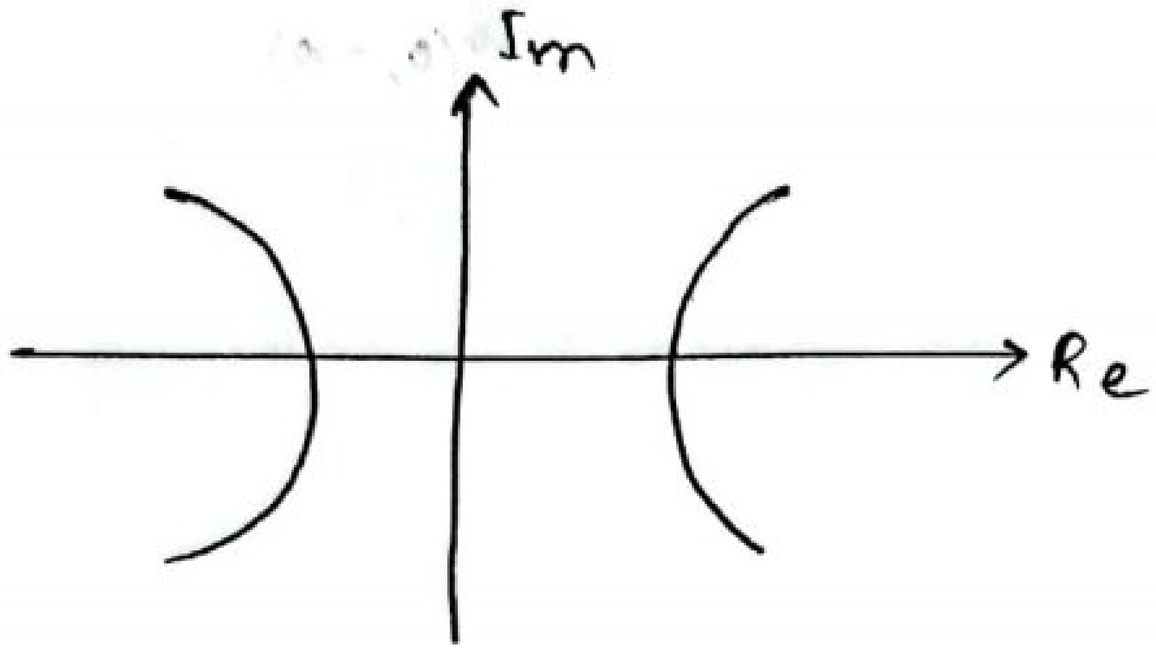
$\Rightarrow 3x + 4 = 2\sqrt{x^2+6x+9+y^2}$

$$\Rightarrow 9x^2 + 16 + 24x = 4(x^2 + 6x + 9 + y^2)$$

$$\Rightarrow 9x^2 + 16 + \cancel{24x} = 4x^2 + \cancel{24x} + 36 + 4y^2$$

$$\Rightarrow 5x^2 - 4y^2 = 20$$

$$\Rightarrow \frac{x^2}{2^2} - \frac{y^2}{(\sqrt{5})^2} = 1$$ represents hyperbola.



4) $\operatorname{Im}\left(\frac{1}{z}\right) = 1$

$$\Rightarrow \operatorname{Im}\left(\frac{1}{x+iy}\right) = 1$$

$$\Rightarrow \operatorname{Im}\left\{\frac{x-iy}{(x+iy)(x-iy)}\right\} = 1$$

$$\Rightarrow \operatorname{Im}\left\{\frac{x-iy}{x^2+y^2}\right\} = 1$$

$$\Rightarrow \operatorname{Im}\left\{\frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)\right\} = 1$$

$$\therefore \frac{-y}{x^2+y^2} = 1$$

$$\Rightarrow x^2 + y^2 = -y$$

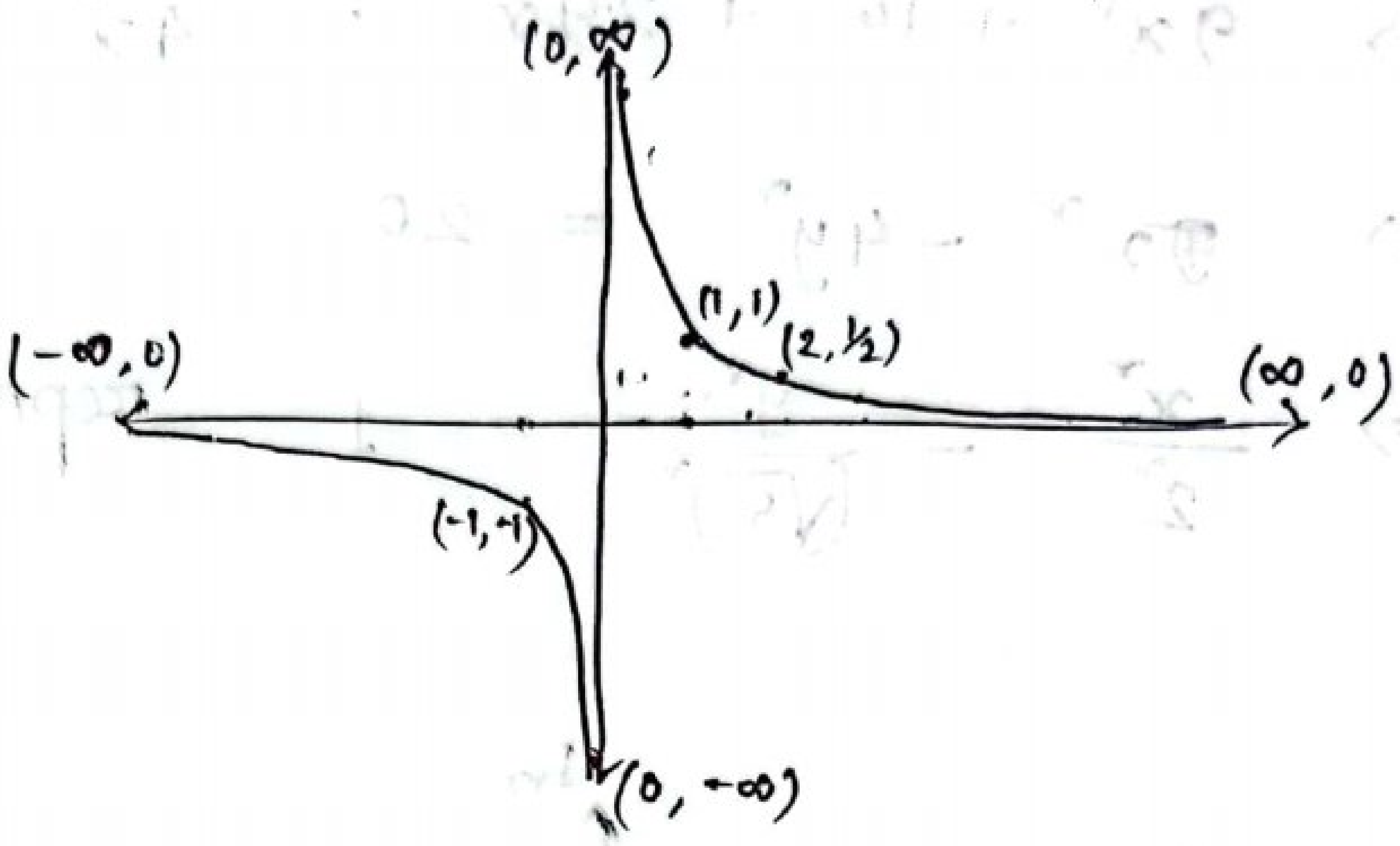
$$\Rightarrow x^2 + y^2 + y = 0$$

$$\Rightarrow x^2 + y^2 + 2 \cdot y \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow (x-0)^2 + \left(y + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$ represents a circle with radius $\frac{1}{2}$ and center $(0, -\frac{1}{2})$

Note: $xy = 1 \longrightarrow$ rectangular hyperbola.

$\Rightarrow y = \frac{1}{x}$



5. 9. 2023

Euler's Formula :

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

Q. State and prove DeMoivre's Theorem

→ **DeMoivre's Theorem :**

For any positive integer n : $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof :

For $n = 1$, we have, $(\cos\theta + i\sin \cancel{n}\theta) = \cos\theta + i\sin\theta$

i.e. the theorem is true for $n = 1$.

Again, for $n = 2$, we have, $(\cos\theta + i\sin\theta)^2 = \cos^2\theta + i^2\sin\theta + 2i\sin\theta\cos\theta$

$= \cos^2\theta - \sin^2\theta + i\cdot 2\sin\theta\cos\theta$

$= \cos 2\theta + i\sin 2\theta$

i.e. the theorem is also true for $n = 2$.

Let, the theorem is true for m $(m<n)$

$\therefore (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$

Now, $(\cos\theta + i\sin\theta)^{m+1}$

$$= (\cos\theta + i\sin\theta)^m (\cos\theta + i\sin\theta)$$

$$= (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$$

$$= \cos m\theta \cdot \cos\theta + i^2 \sin m\theta \sin\theta + i(\sin m\theta \cos\theta + \cos m\theta \cdot \sin\theta)$$

$$= \cos m\theta \cdot \cos\theta - \sin m\theta \cdot \sin\theta + i\{\sin(m+1)\theta\}$$

$$= \cos(m+1)\theta + i\{\sin(m+1)\theta\}$$

i.e. the theorem is true for $m+1$ also.

therefore, the theorem is true for all positive integer n.

$\cos 3\theta$
$\cos 4\theta$
$\cos 5\theta$

Q) Determine the values of $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$ respectively.

→ We know that,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\therefore (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\therefore \cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta + 3\cos^2\theta \cdot i\sin\theta + 3\cos\theta (i\sin\theta)^2 + (i\sin\theta)^3$$

or, $(a+b)^n = a^n + {}^nc_1 a^{n-1} b^1 + {}^nc_2 a^{n-2} b^2 + \cdots + b^n$

$= \cos^3\theta + 3i \sin\theta \cos^2\theta - 3\sin^2\theta \cos\theta - i\sin^3\theta$

$= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\cos^2\theta \sin\theta - \sin^3\theta)$

$\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta$ —— (I)

$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$ —— (II)

Now, from (1),

$\cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

(Ans).

Q. Determine 1) $(i)^{1/4}$

2) $(1+i)^{1/4}$

2) $(1+i)^{1/4} = (r\cos\theta + ir\sin\theta)^{1/4}$

$= r^{1/4}(\cos\theta + i\sin\theta)^{1/4}$

$= (\sqrt{2})^{1/4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/4}$

$= (\sqrt{2})^{1/4}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}^{1/4}$

$[\underbrace{n = 0, 1, 2, 3}_{\text{fourth root ; so 4 টা}} \ldots]$

$1 + i = 1 + i * 1$

let,

$1 = r\cos\theta$

$1 = r\sin\theta$

$r = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right)$

$= \frac{\pi}{4}$

$= (\sqrt{2})^{1/4}\left\{\cos(8n+1)\frac{\pi}{4} + i\sin(8n+1)\frac{\pi}{4}\right\}^{1/4}$

$= (\sqrt{2})^{1/4}\left\{\cos\cdot\frac{1}{4}\cdot(8n+1)\frac{\pi}{4} + i\sin\cdot\frac{1}{4}(8n+1)\frac{\pi}{4}\right.$

[by De Moivre's Theorem

$= (\sqrt{2})^{1/4}\left\{\cos(8n+1)\frac{\pi}{16} + i\sin(8n+1)\frac{\pi}{16}\right\};$

$n = 0, 1, 2, 3.$

Limit, Continuity, Differentiability, Analyticity and Harmonicity:

Continuity : i) $f(a)$ exists

ii) $\lim_{x \to a} f(x)$ exists

iii) $\lim_{x \to a} f(x) = f(a)$

(limiting val = func. value)

derivative $\rightarrow$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{or, } f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(x)}{h}$$ at $x = a$ exists then differentiable

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- Analytic function & harmonic function

Analyticity :

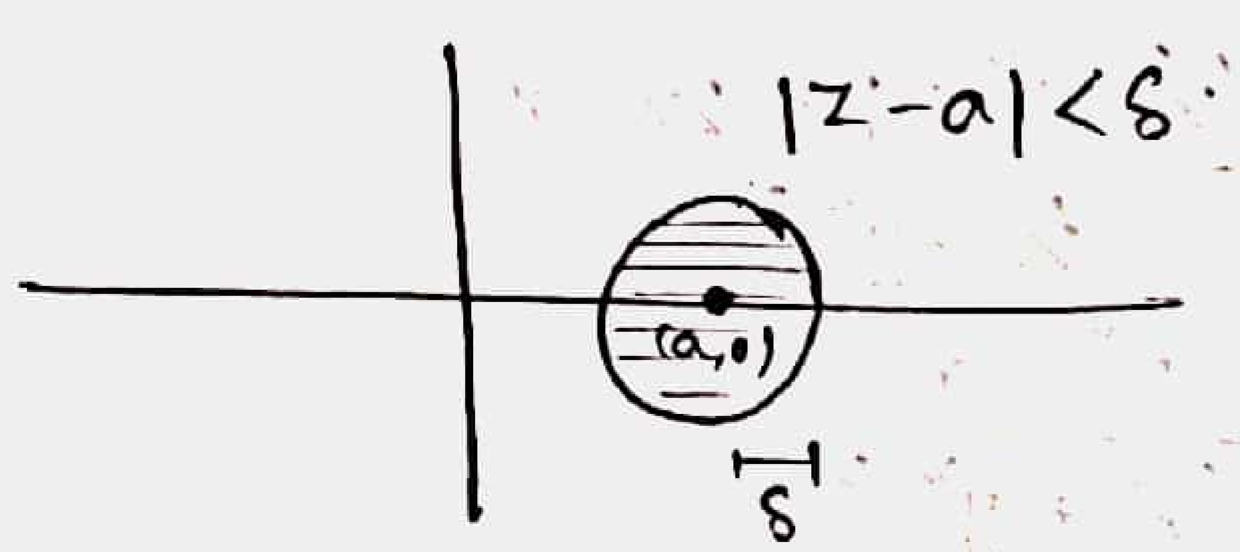
The complex function, $f(z) = u(x,y) + iv(x,y)$ is said to be analytic at $z = a$ if it is differentiable in the neighbourhood $|z-a| < \delta$ of a.

Note :

$$|z - a| = \delta$$

$$\Rightarrow |x + iy - a| = \delta$$

$$\Rightarrow (x-a)^2 + y^2 = \delta^2$$



$\mapsto$ a এর neighbourhood

or, The complex function $f(z) = u(x,y) + iv(x,y)$ is analytic if it satisfies the Cauchy - Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Test whether the functions are analytic or not :

i) $f(z) = e^z$

ii) $f(z) = \frac{1}{\bar{z}}$

Ans : $f(z) = \frac{1}{\bar{z}} = \frac{1}{x - iy}$

$= \frac{x + iy}{x^2 + y^2}$

$= \frac{x}{x^2 + y^2} + i \cdot \frac{y}{x^2 + y^2}$

$= u(x,y) + iv(x,y)$ where

$u(x,y) = \frac{x}{x^2 + y^2}$

$v(x,y) = \frac{y}{x^2 + y^2}$

$$\frac{\partial u}{\partial x} = \frac{(x^2 + y^2) \cdot 1 - 2x \cdot x}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{x^2 + y^2)}$$

Note :

$$\lim_{x \to 0} \{ f_1(x) + f_2(x) \}$$
$$= \lim_{x \to 0} f_1(x) + \lim_{x \to 0} f_2(x)$$

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(8) State and establish the necessary conditions for the function $f(z) = u(x,y) + iv(x,y)$ to be analytic.

→ Necessary conditions for $f(z) = u(x,y) + iv(x,y)$ to be analytic are $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ must exist and satisfy the cauchy - Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof :

Since $f(z)$ is analytic.

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$ must exist and be unique.

$$\therefore f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y} \quad \text{--- (1)}$$

Now along the real axis, $\Delta x \to 0$, $\Delta y = 0$.

$$\therefore f'(z) = \lim_{\Delta x \to 0} \frac{\{u(x+\Delta x, y) + iv(x+\Delta x, y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \quad \text{--- (A)}$$

$* \frac{1}{i} = \frac{i}{i^2} = -i$

Again, along the imaginary axis, $\Delta x = 0$, $\Delta y \to 0$,

from (1),

$$f'(z) = \lim_{\Delta y \to 0} \frac{\{u(x, y+\Delta y) + i\,v(x, y+\Delta y)\} - \{u(x,y) + i\,(v(x,y)\}}{i\,\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\,\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= -i \cdot \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \frac{\partial v}{\partial y}$$

$$= -i \cdot \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right) \quad \text{——— (B)}$$

Now from (A) and (B), we have,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}.$$

(Proved)

** See the polar form of cauchy - Riemann eq^n from given lecture notes **

Harmonic Function : The real valued function $u(x,y)$ is said to be harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(q) If $f(z) = u(x,y) + iv(x,y)$ is analytic then $u(x,y)$ and $v(x,y)$ will be harmonic provided they have continuous second partial derivatives.

Note :

$\frac{\partial u}{\partial x} \longrightarrow$ first partial derivative w.r.t. x

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \longrightarrow$ second p.d. w.r.t x then y

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \longrightarrow$ " p.d. w.r.t

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} \longrightarrow$ " " " y then x

For continuous 2nd partial derivative,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \cdot \partial y}$$

(9) Test whether $u(x,y) = 2x(1-y)$ is harmonic or not. If it's harmonic, find the harmonic conjugate $v(x,y)$ so that $f(z) = u(x,y) + iv(x,y)$ is analytic. Also, determine $f(z)$ in terms of z.

Cauchy Reimann মানে

$\rightarrow$ $\frac{\partial u}{\partial x} = 2 - 2y$, $\frac{\partial u}{\partial y} = -2x$

$\frac{\partial^2 u}{\partial x^2} = 0$, $\frac{\partial^2 u}{\partial y^2} = 0$

i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ $u(x,y)$ is harmonic.

Now, $dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$ [by def^n of total derivative]

$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(-\frac{\partial u}{\partial x}\right) dy$ $\left[\because \frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$

$= -(-2x)dx + (2-2y)dy$

$dv = 2x\,dx + (2-2y)\,dy$

$= M(x,y)\,dx + N(x,y)\,dy$ —— (1)

where $M(x,y) = 2x$

$N(x,y) = 2-2y$

$\boxed{\text{Note}}$: $M dx + N \cdot dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The R.H.S. of ① is exact because $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$$\therefore \int dv = \int M\,dx + \int (\text{Part of } N \text{ excluding terms of } x)\,dy + c$$

$$\Rightarrow v = \int 2x\,dx + \int (2 - 2y)\,dy + c$$

$$\boxed{v = x^2 + 2y - y^2 + c}$$

Now, $\underbrace{f'(z) = u_1(z,0) - i u_2(z,0)}$ where, $u_1(x,y) = \frac{\partial u}{\partial x}$

[Milne - Thomson property]

$u_2(x,y) = \frac{\partial u}{\partial y}$

$\downarrow$

replacing x by z and y by 0.

$u_1(x,y) = 2 - 2y$

$\therefore u_1(z,0) = 2$

$u_2(x,y) = -2x$

$\therefore u_2(z,0) = -2z$

$$\therefore f'(z) = 2 - i(-2z) = 2 + 2iz$$

$$\therefore f(z) = 2z + \cancel{2}i \cdot \frac{z^2}{\cancel{2}} + c$$

$$\boxed{f(z) = 2z + iz^2 + c}$$

(Q) Test whether, $u(x,y) = e^{x} \cos y$ is harmonic or not - - -

→ Part one : Solve yourself. (See pdf)

Part two : " " $(v = e^{x} \sin y + c)$

Now, $f(z) = u(x,y) + iv(x,y)$

$$= e^{x} \cos y + i(e^{x} \cdot \sin y + c)$$

$$= e^{x} \cos y + i e^{x} \sin y + ic$$

$$= e^{x}(e^{iy}) + k$$

$$= e^{x+iy} + k$$

$$= e^{z} + k$$

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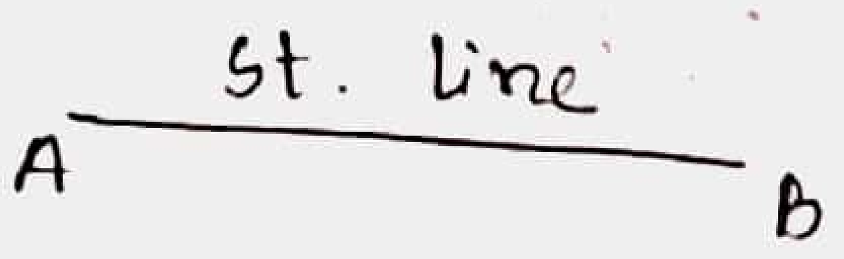
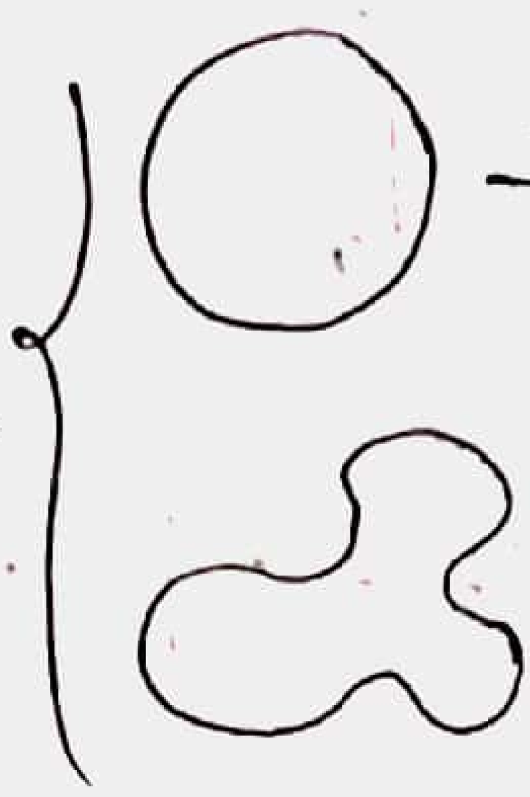
complex integration :



→ open curve

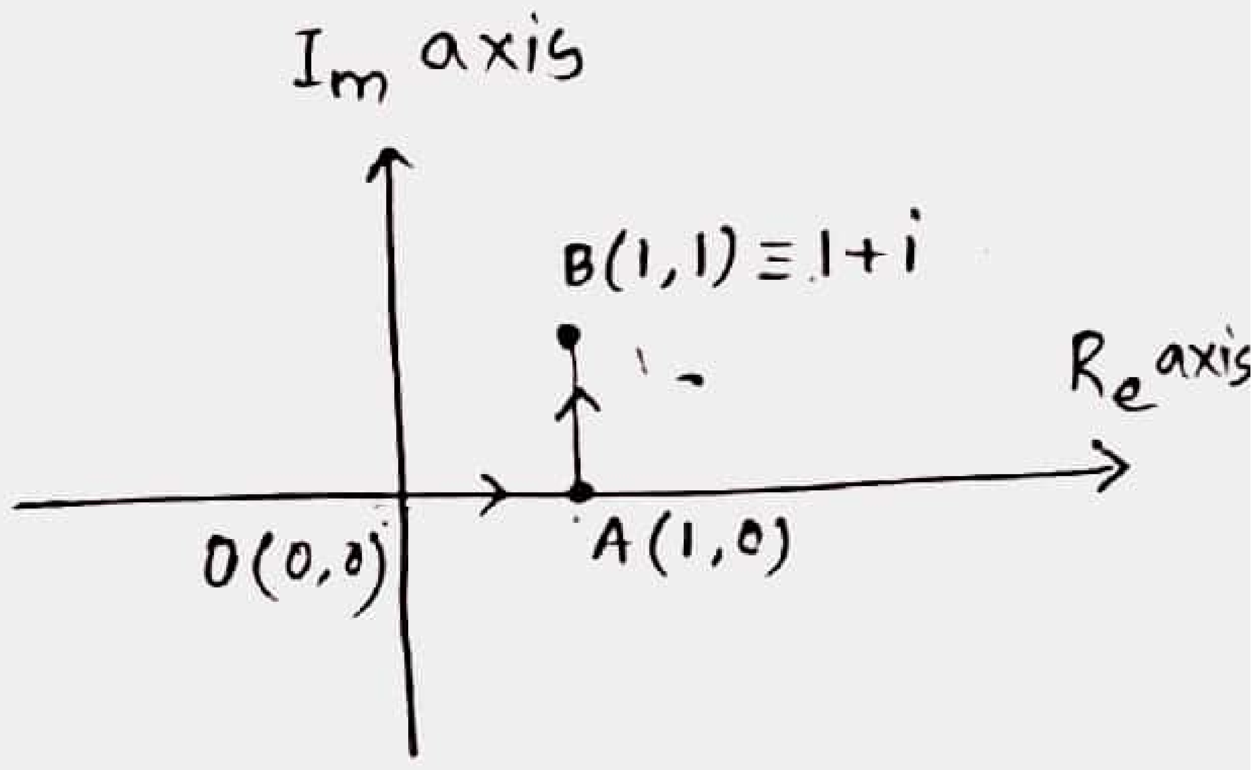
closed curve



→ simple closed curve / contour

→ non-simple closed curve

Q) Evaluate $\int_0^{1+i} z^2\,dz$



→ Integration from 0 to $1+i$

i.e. $O(0,0)$ to $B(1,1)$ we have to move along OA to AB.

We know that, $z = x + iy$

Now, along OA, $z = x + i\cdot 0 = x$

and, $dz = dx$

Now, along AB, $z = 1 + i\cdot y$

$\therefore dz = i\cdot dy$

Note :

OA $\begin{cases} O(0,0) \\ A(1,0) \end{cases}$ → x var., $y=0$

AB $\begin{cases} A(1,0) \\ B(1,1) \end{cases}$ → $x=1$, y var.

$$\therefore \int_0^{1+i} z^2\,dz = \int_{OA} z^2\,dz + \int_{AB} z^2\,dz$$

$$= \int_0^1 x^2\,dx + \int_0^1 (1+iy)^2 \cdot i\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \int_0^1 (1+2iy-y^2)\,i\,dy$$

$$= \left[\tfrac{1}{3} - 0\right] + i\left[y + \cancel{2}i\cdot\frac{y^2}{\cancel{2}} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{3} + i\left(1 + i - \tfrac{1}{3}\right)$$

$$= \frac{1}{3} + i\left(i + \frac{2}{3}\right)$$

$$= \frac{1}{3} + i^2 + \frac{2}{3}i$$

$$= -\frac{2}{3} + i\cdot\frac{2}{3}$$

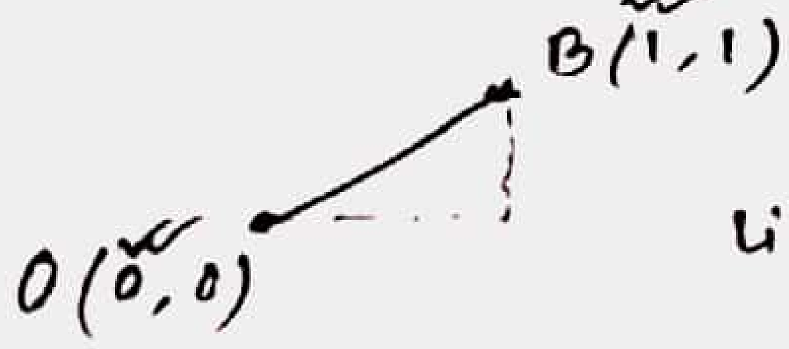
Or, integration from $O(0,0)$ to $1+i = B(1,1)$:

Eq^n of st. line OB is,

$$\frac{x-0}{0-1} = \frac{y-0}{0-1}$$

$$\Rightarrow y = x$$

$$\therefore dy = dx$$



lim of $x \to 0$ to 1

$$\therefore \int_0^{1+i} z^2\,dz = \int_0^{1+i} (x+iy)^2 \cdot (dx + i\,dy)$$

$\because z = x + iy$

$dz = dx + i \cdot dy$

$$= \int_0^1 (x + ix)^2 \cdot (dx + i\,dx)$$

$$= \int_0^1 x^2 (1+i)^2 (1+i)\,dx$$

$$= \cancel{\int_0^1} (1+i)^3 \cdot \left[\frac{x^3}{3}\right]_0^1$$

$$= \left(1 + 3\cdot 1^2 \cdot i + 3 \cdot 1 \cdot i^2 + i^3\right)\left(\frac{1}{3} - 0\right)$$

$$= (-2 + 2i) \cdot \frac{1}{3}$$

$$= -\frac{2}{3} + i\frac{2}{3}$$