CSE - 219

16.10.2023

concrete Mathematics

Recurrent Problems :

Tower of Brahma
→ 3 niddle of diamond
→ gold plate

→ Fibonacci series

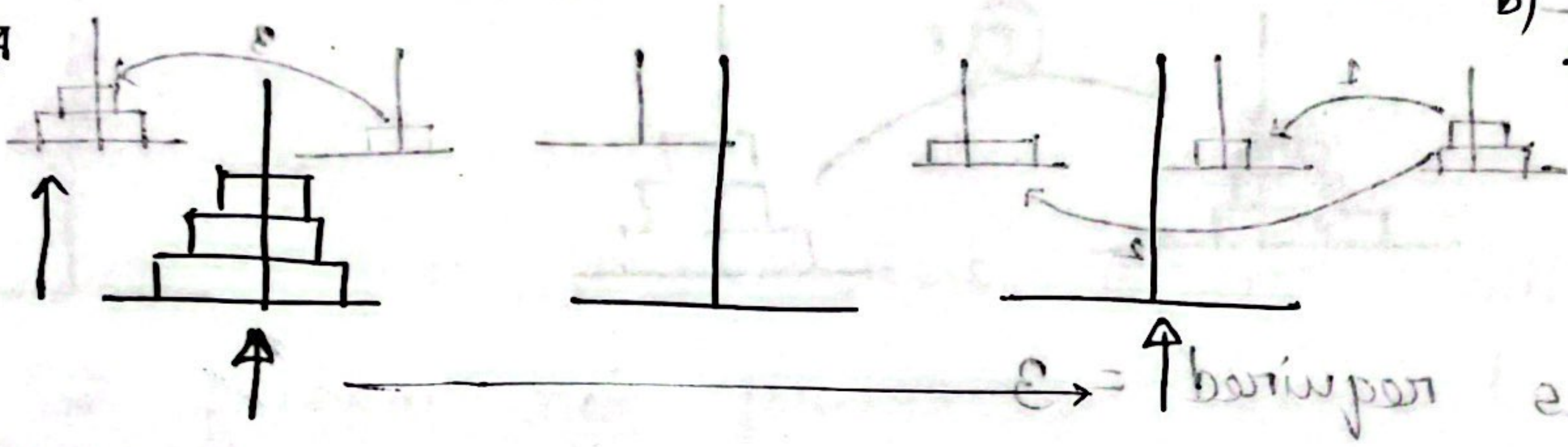
0	1	2	3
0	1	1	2

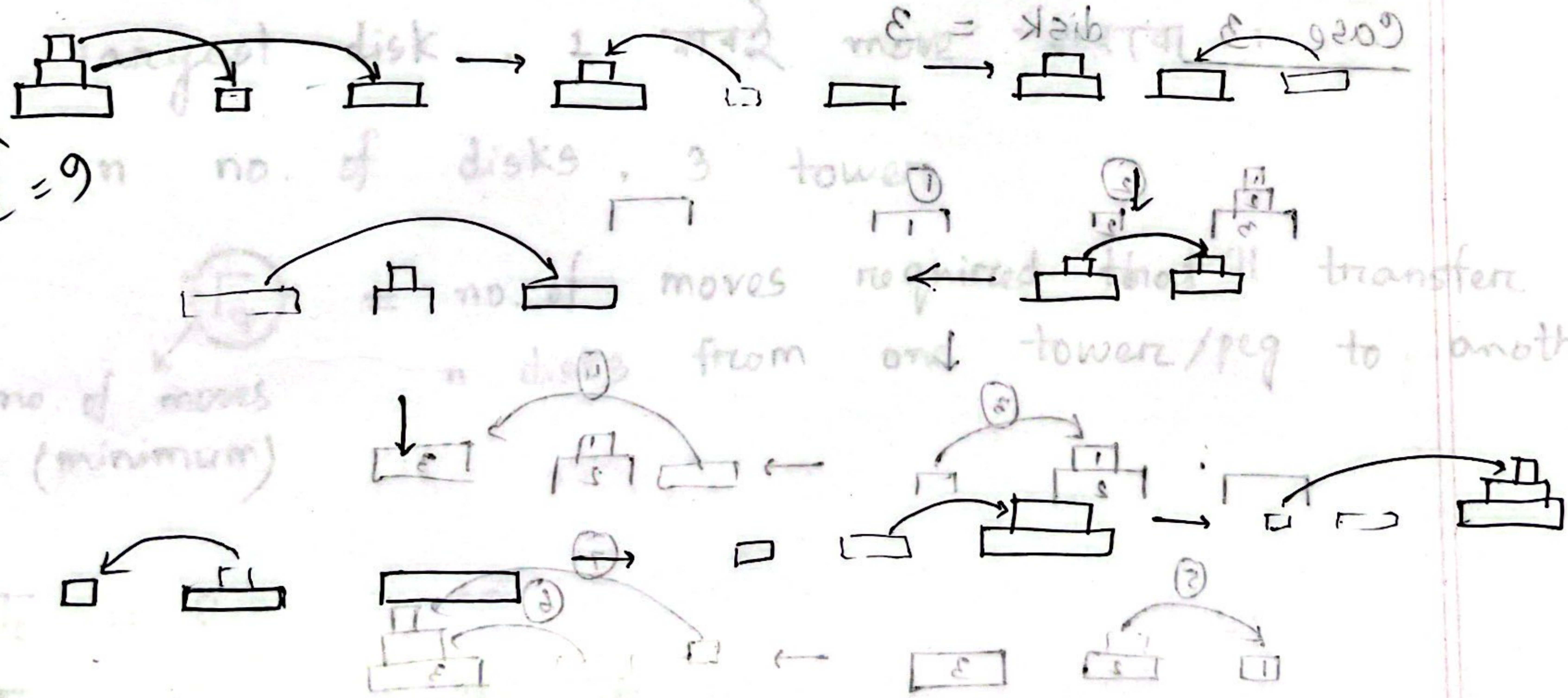
$$F_n = F_{n-1} + F_{n-2}$$

Tower of Hanoi : Lucas, 'Tower of Brahma' কে use করে build করে [disk = 8]

a) at a time একটা
b) ছোটর উপর বড় রাখা যাবে না

বড় disk এর উপরে ছোট disk



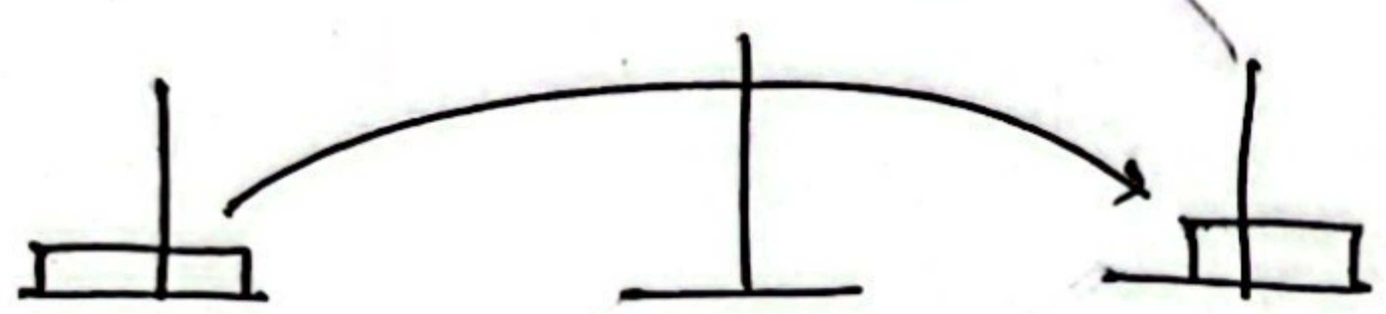


move = 9

* move required
* how much time required

* when disk no. = 0 $\longrightarrow$ moves req. = 0

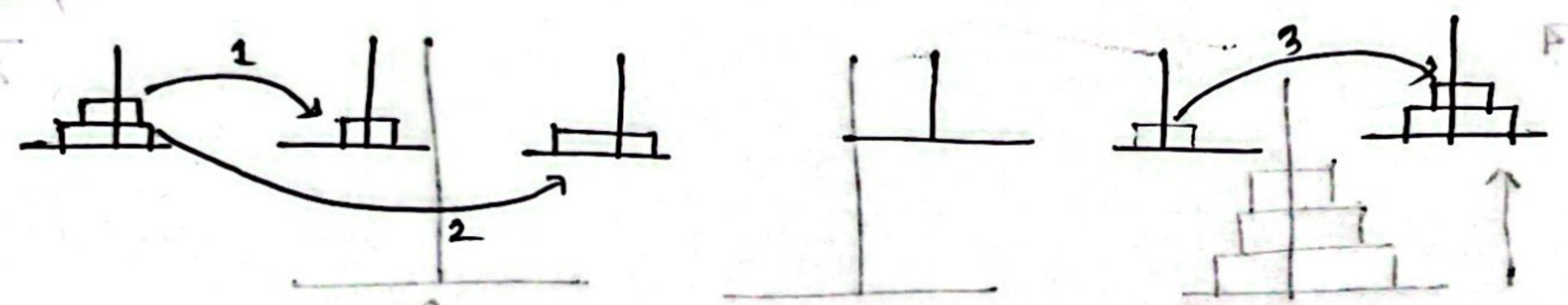
Case 1 : disk no. = 1



moves req. = 1

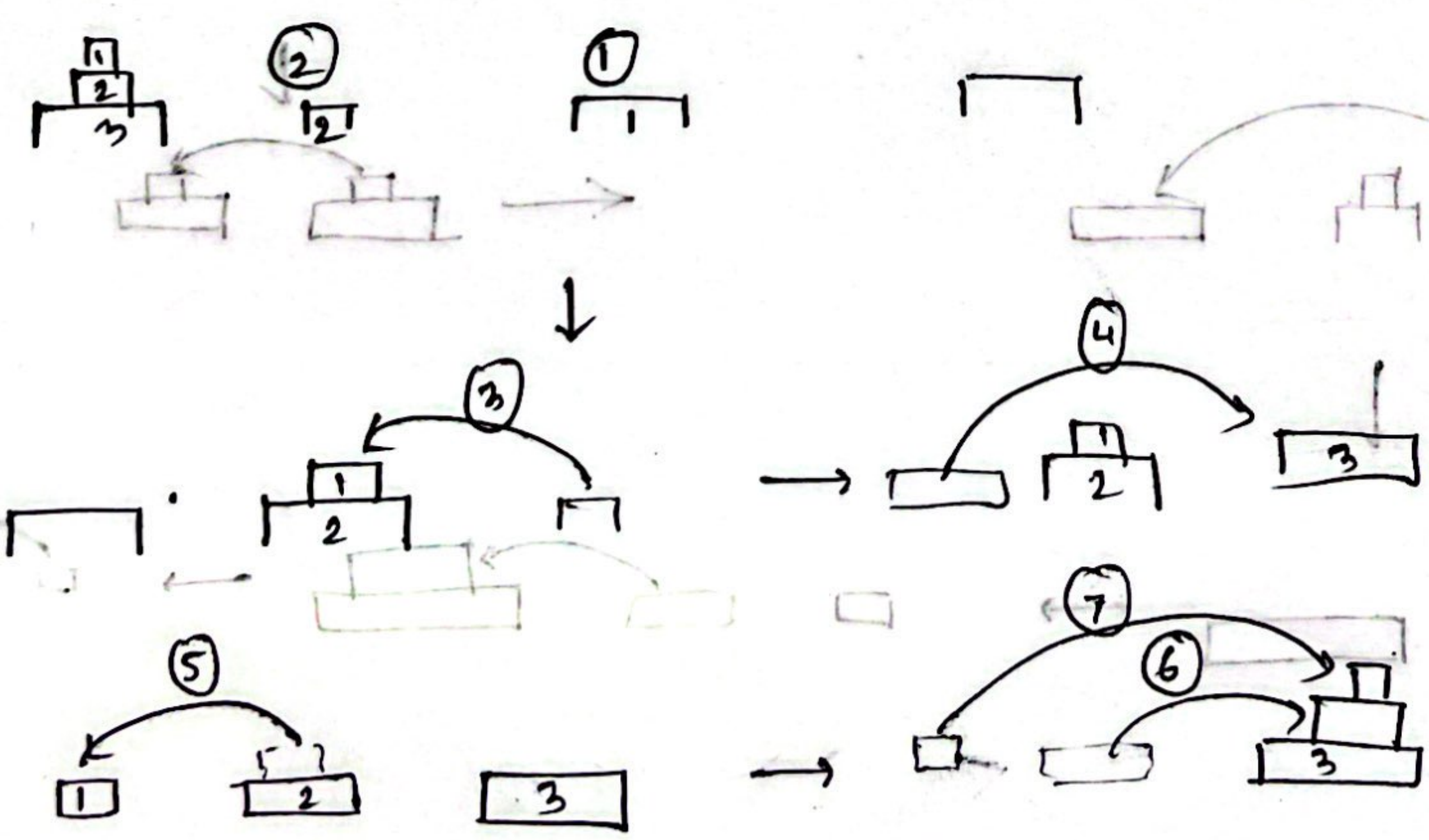
Time req. = 1 sec

Case 2 : disk = 2



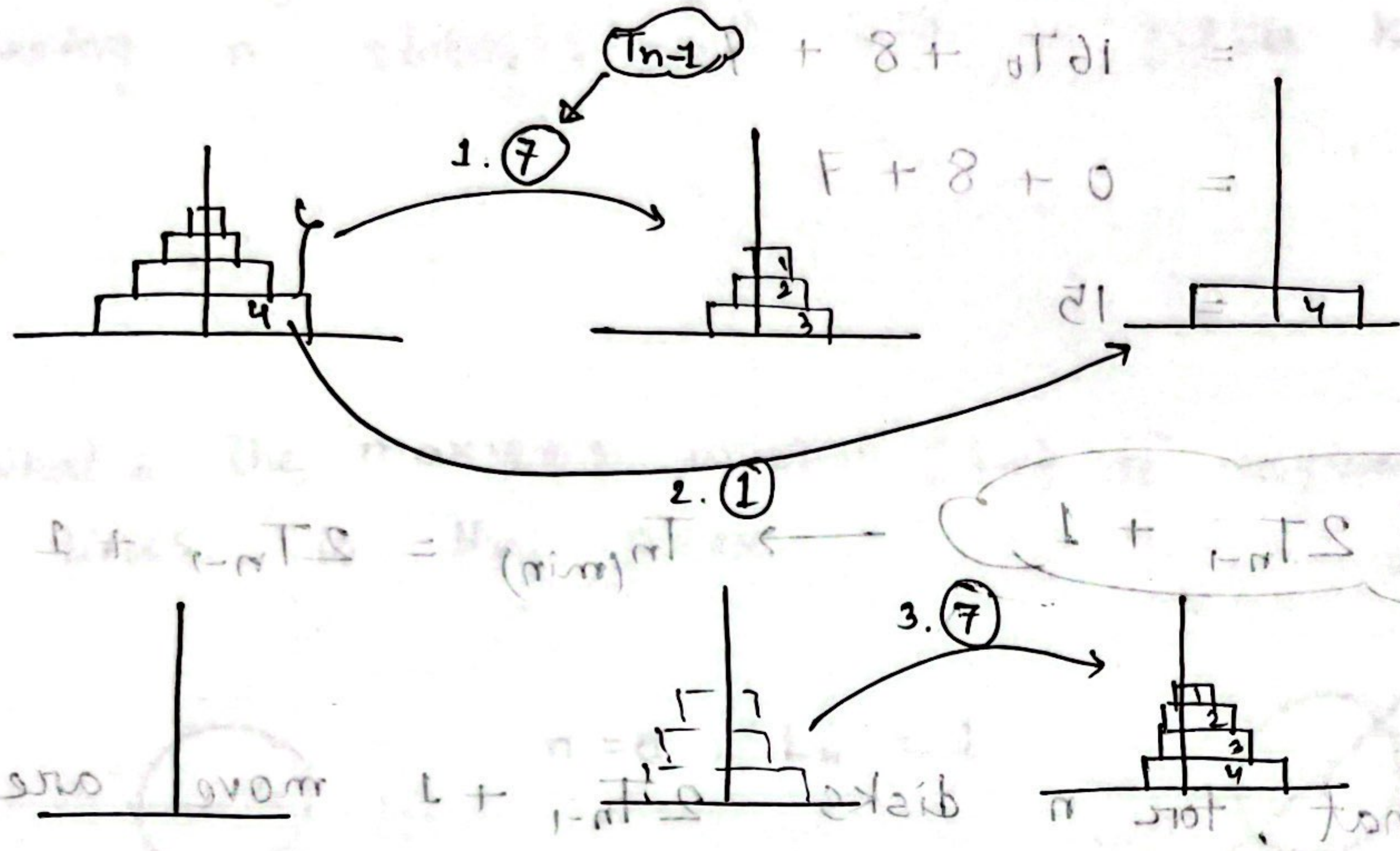
moves required = 3

Case 3 : disk = 3



Case 4 : disk = 4

moves = 15



7 + 7 + 1
= 15 moves

* largest disk , 1 বারই move করাবো ।

* n no. of disks , 3 tower

T_n = no. of moves required that'll transfer n disks from one tower/peg to another

no. of moves (minimum)

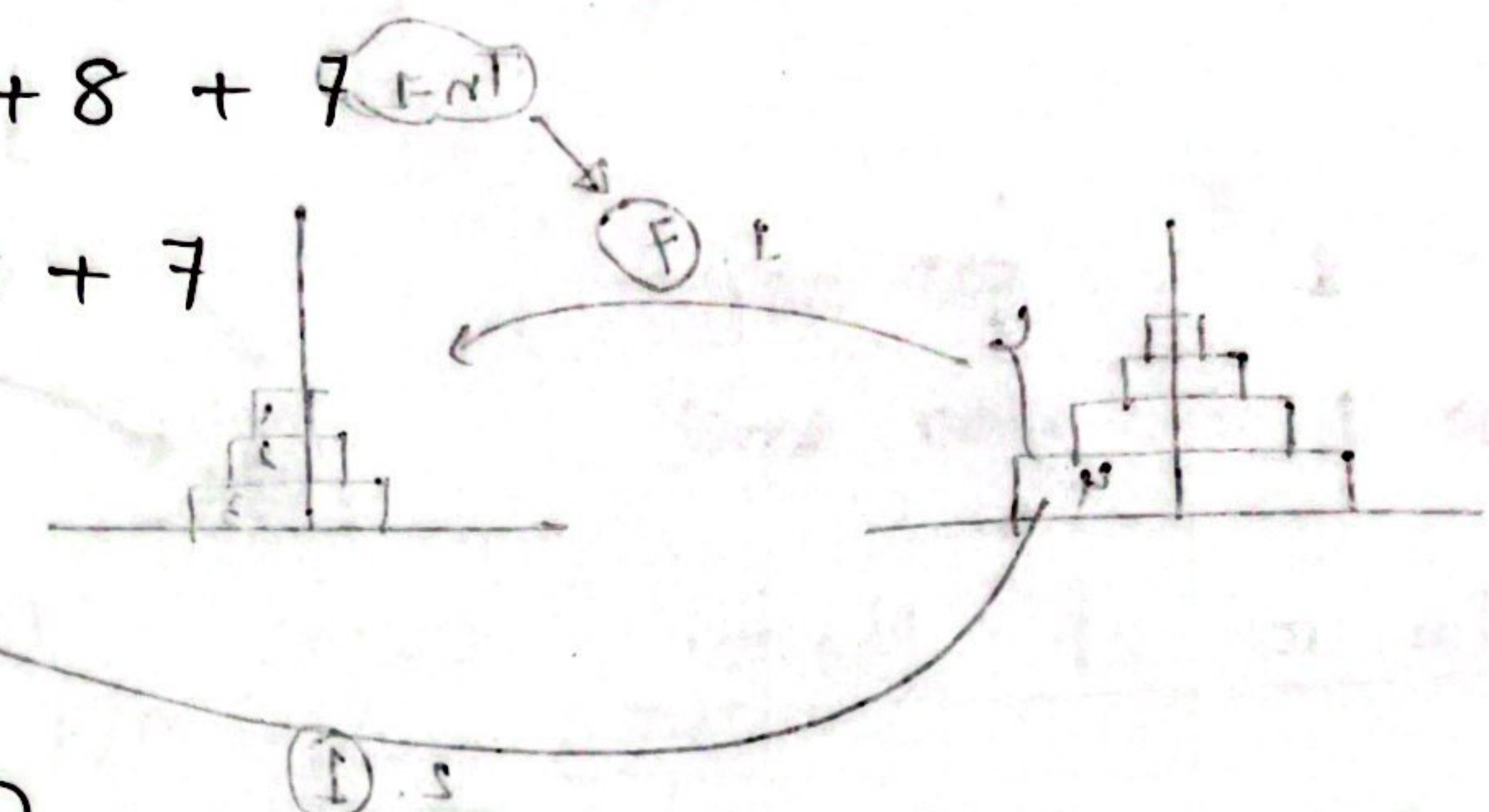
$T_0 = 0$

$T_4 = 7 + 1 + 7 = 15$

$\boxed{T_n = 2T_{n-1} + 1}$

$$
\begin{aligned}
T_4 &= 2 \times T_3 + 1 \\
&= 2 \times (2T_2 + 1) + 1 \\
&= 4T_2 + 3 \\
&= 4(2T_1 + 1) + 3
\end{aligned}
$$

$$= 8T_1 + 7$$
$$= 8(2T_0 + 1) + 7$$
$$= 16T_0 + 8 + 7$$
$$= 0 + 8 + 7$$
$$= 15$$

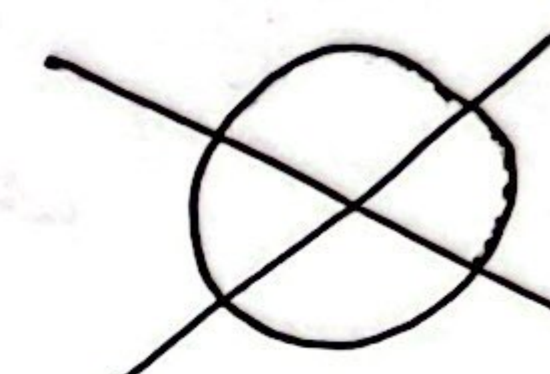


$$T_n \geq 2T_{n-1} + 1 \longrightarrow T_{n(min)} = 2T_{n-1} + 1$$

\# Prove that, for n disks $2T_{n-1} + 1$ move are sufficient and necessary for making to the destination peg.

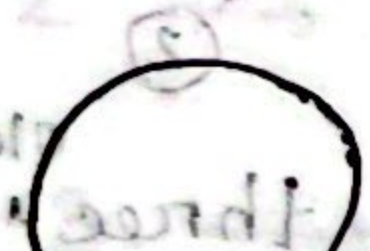
Geometric problem :

How many slices of pizza can a person obtain by making n straight cut with a pizza knife ?

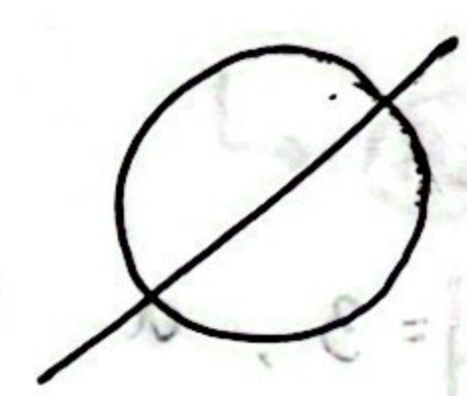


$n=2$
4 slices

What's the maximum number (L_n) of regions defined by n lines in the plane

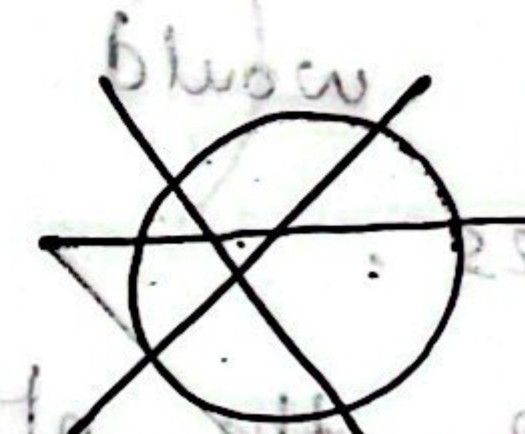


$n=0$, $L_n = 1$

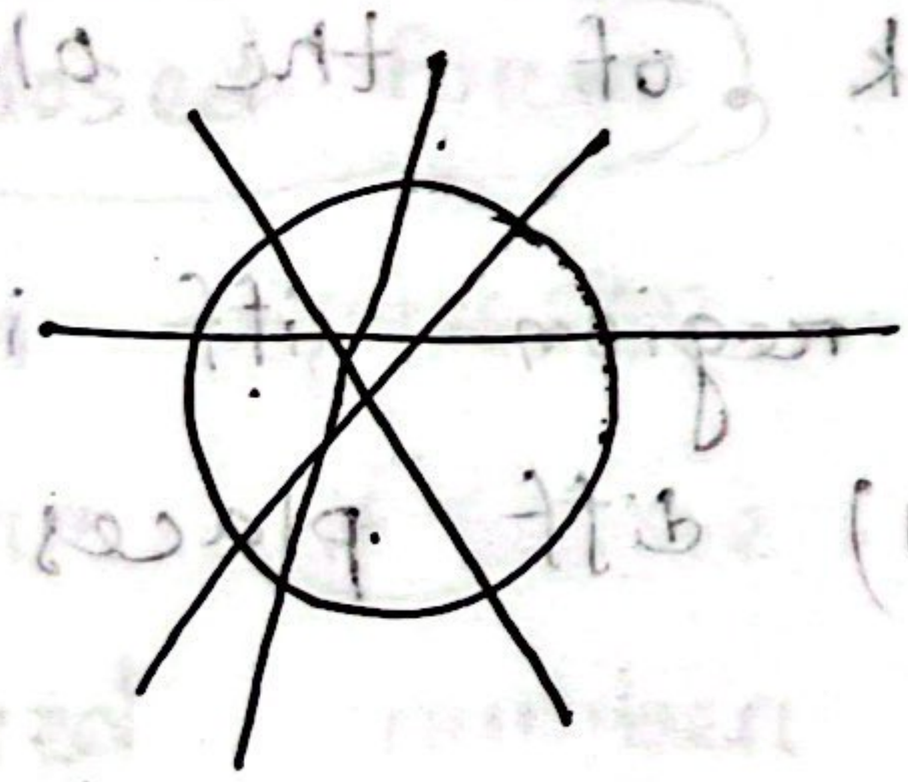


$n=1$, $L_n = 2$

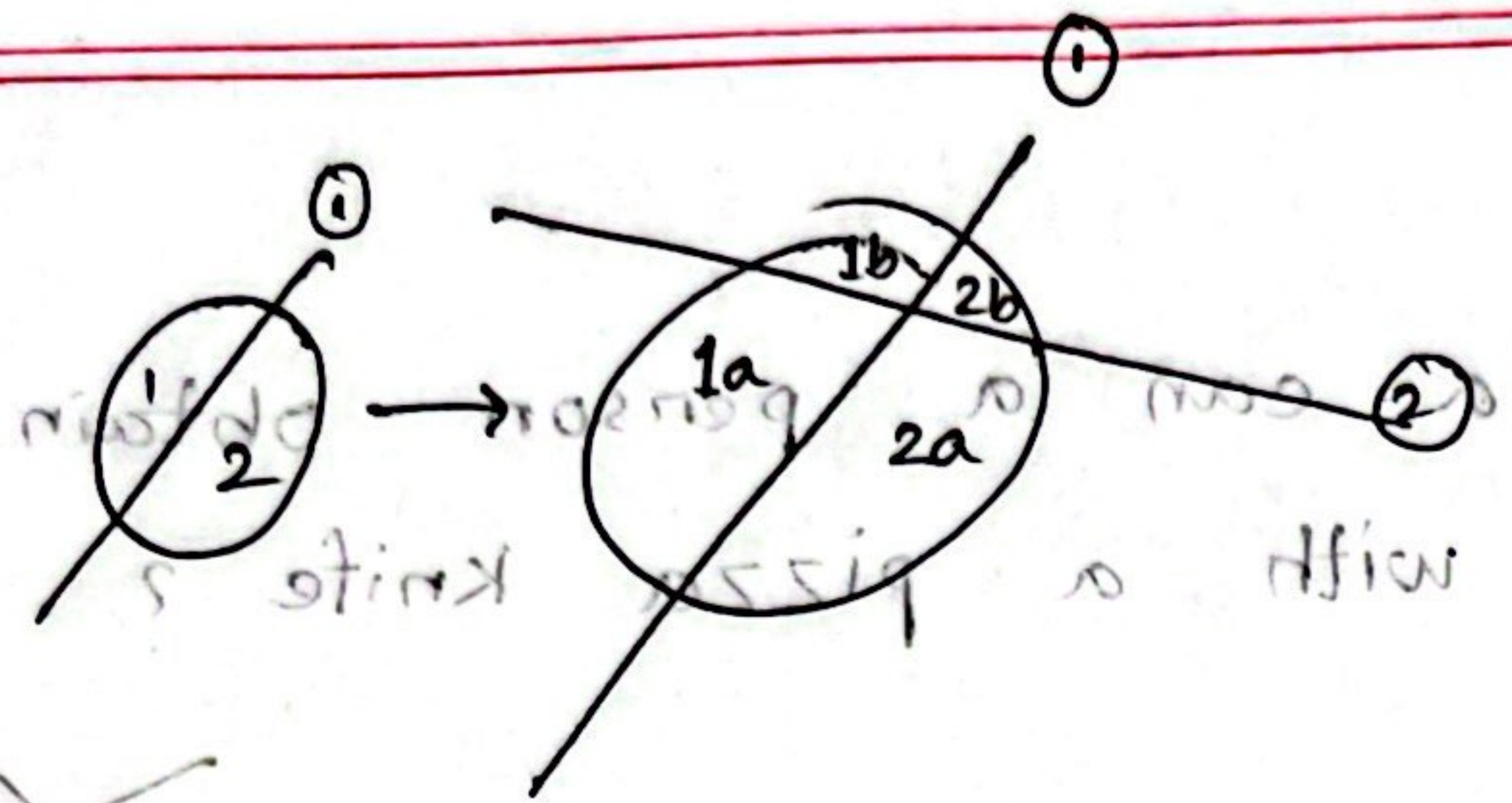
$n=2$, $L_n = 4$

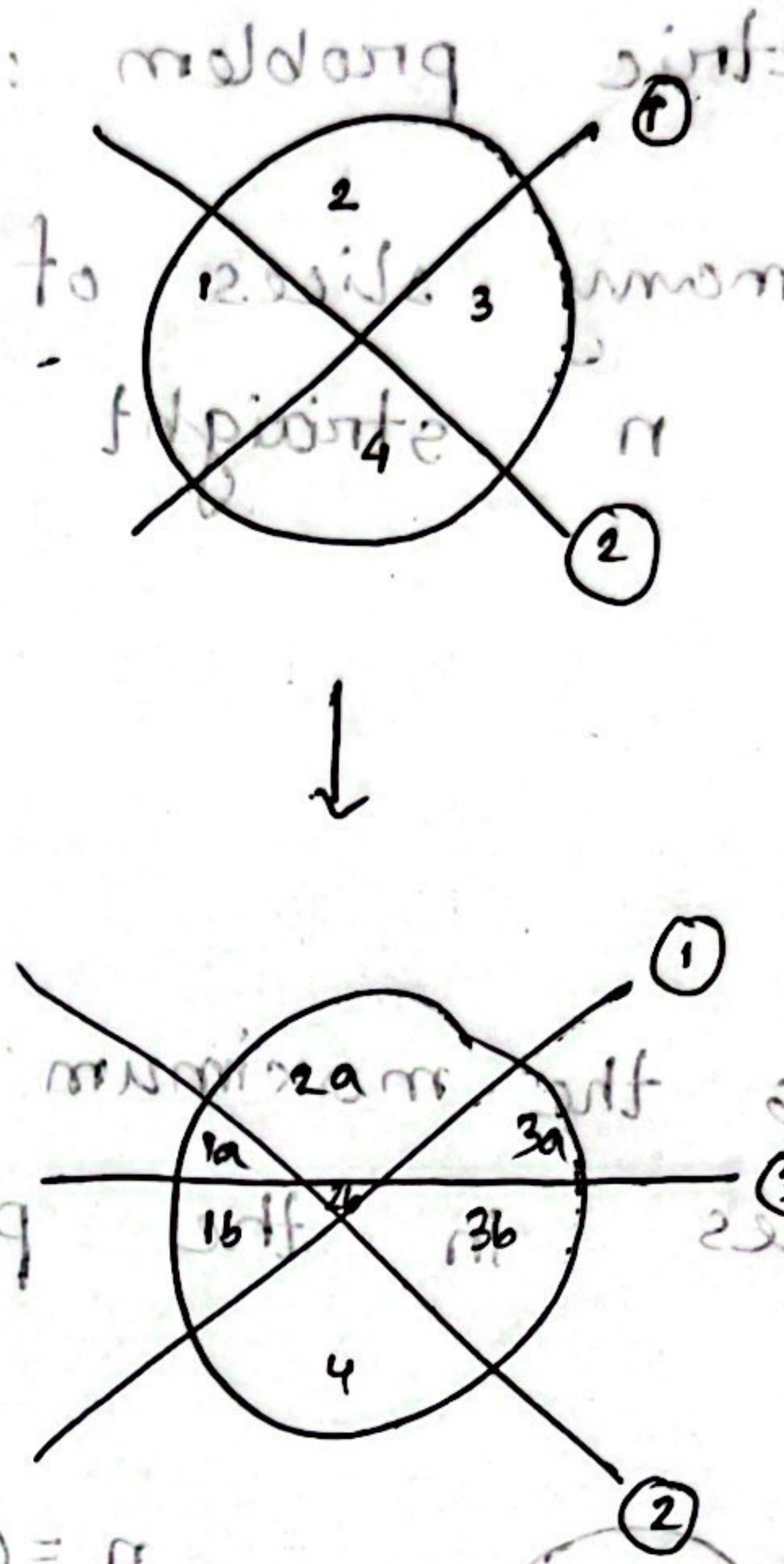


$n=3$, $L_n = 7$

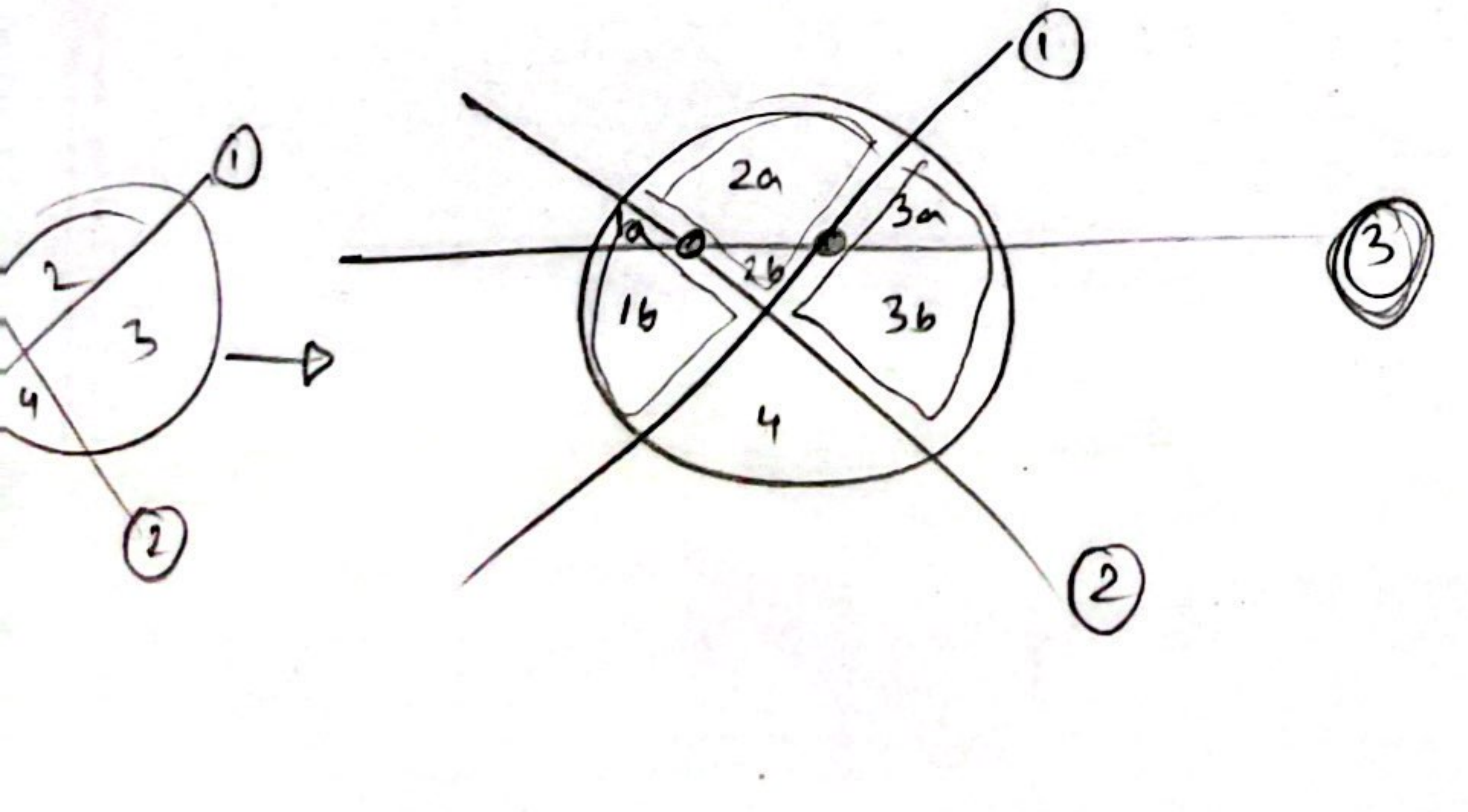


$n=4$, $L_n = 11$





1. For $n = 3$, a line can split at most three old region
2. nth line would split an old regions at most two pieces.
3. For $n > 0$, the nth line increases the number of regions by k iff it splits k of the old regions and it splits k old region iff it hits the previous lines in $(k-1)$ diff places.



3 টা region কে
2 টা point এ touch
করে split করেছে।

4. Two lines can intersect in at most one point. Therefore the new line can intersect $(n-1)$ old lines in at most $(n-1)$ places.

* L_n = maximum (#) regions by n straight line

$n=0 \longrightarrow L_0 = 1$

$n=1 \longrightarrow L_1 = 1 + \underset{\sim}{1} = 2$

$n=2 \longrightarrow L_2 = 2 + \underset{\sim}{2} = 4$

$n=3 \longrightarrow L_3 = 4 + \underset{\sim}{3} = 7$

$n=4 \longrightarrow L_4 = 7 + \underset{\sim}{4} = 11$

$$\boxed{L_n = L_{n-1} + n}$$

↳ Recurrence sol^n

- $L_0 = 1$
- $L_n = L_{n-1} + n$ (for $n>0$)

Closed Form :

An expression for a quantity $f(n)$ is in closed form if we can compute it using at most a fixed number of well-known standard operations, independent of n.

$$L_n = L_{n-1} + n$$

$$= L_{n-2} + (n-1) + n$$

$$= L_{n-3} + (n-2) + (n-1) + n$$

$$= L_{n-4} + (n-3) + (n-2) + (n-1) + n$$

When $n = 4$, $L_4 = L_0 + 1 + 2 + 3 + 4$

For, $n \longrightarrow$ $L_n = L_0 + [1 + 2 + 3 + \dots + (n-3) + (n-2) + (n-1) + n]$

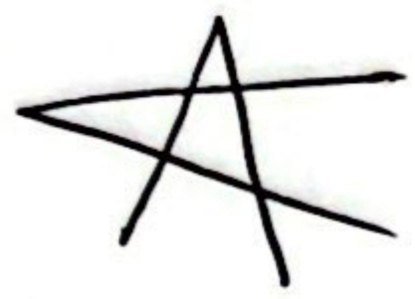
$$= L_0 + \frac{n(n+1)}{2}$$

$$\boxed{L_n = 1 + \frac{n(n+1)}{2}}$$

23. 10. 2025

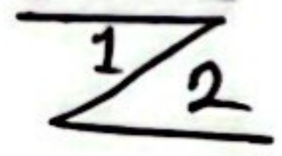
Self. Study : Straight line

If line isn't straight, line is bend

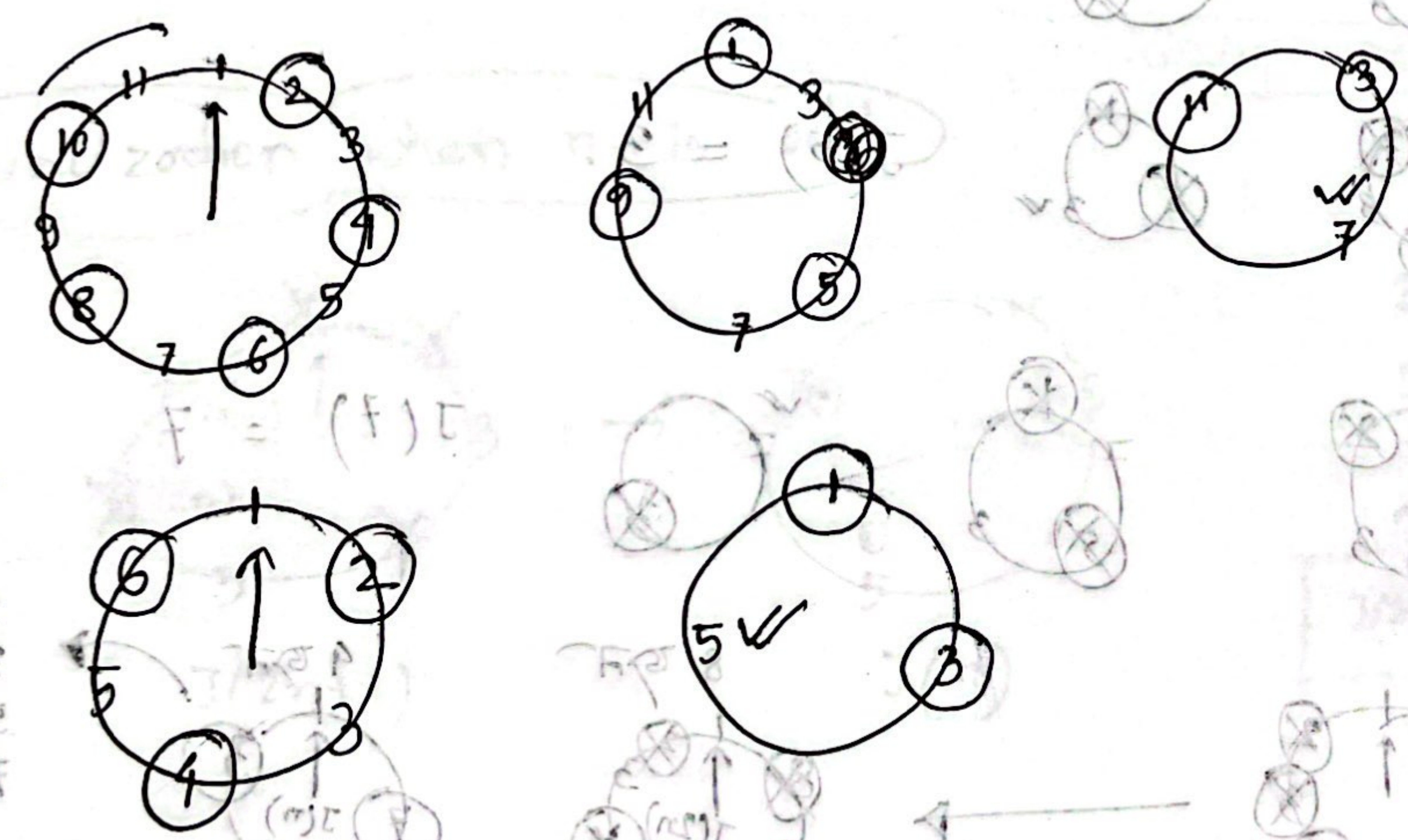


bend line ⟶ max no. of region determine, if n no. of line is cut.

Zigzag line



JOSEPHUS PROBLEM



$0 \le 1 \le 2^m - 1 = 3$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5	7	9

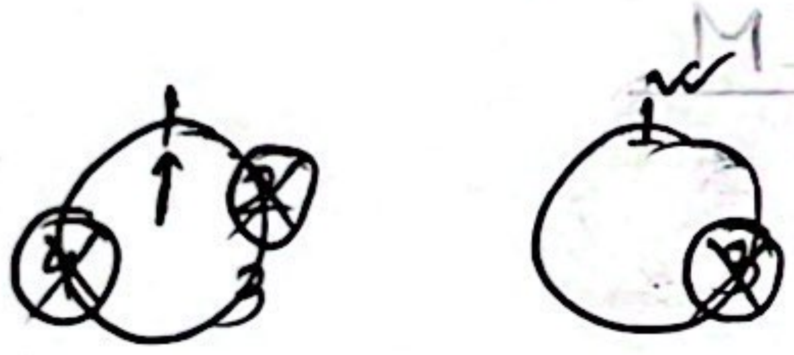
n = 0 x

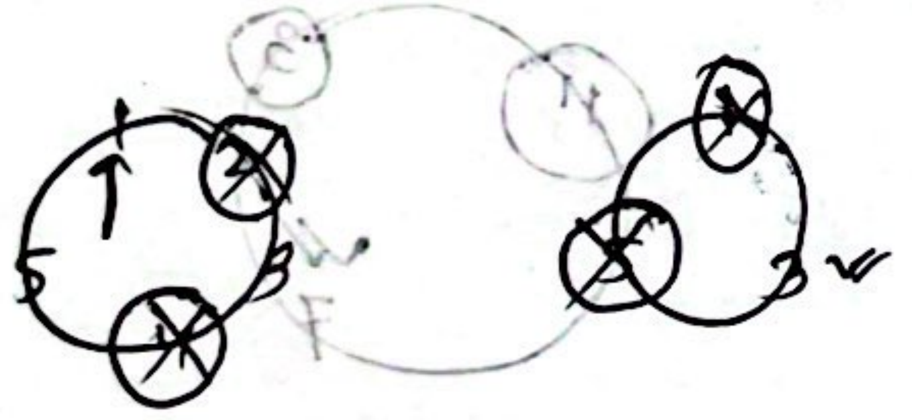
n = 1  1 survive করলো → J(1) = 1

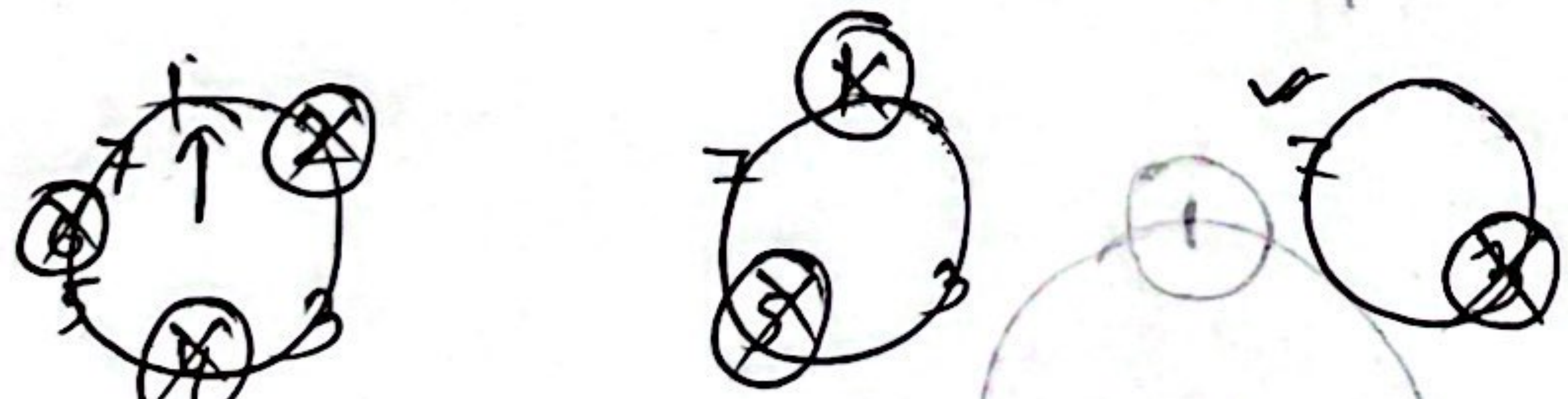
no of people, pos. no

n = 2  J(2) = 1

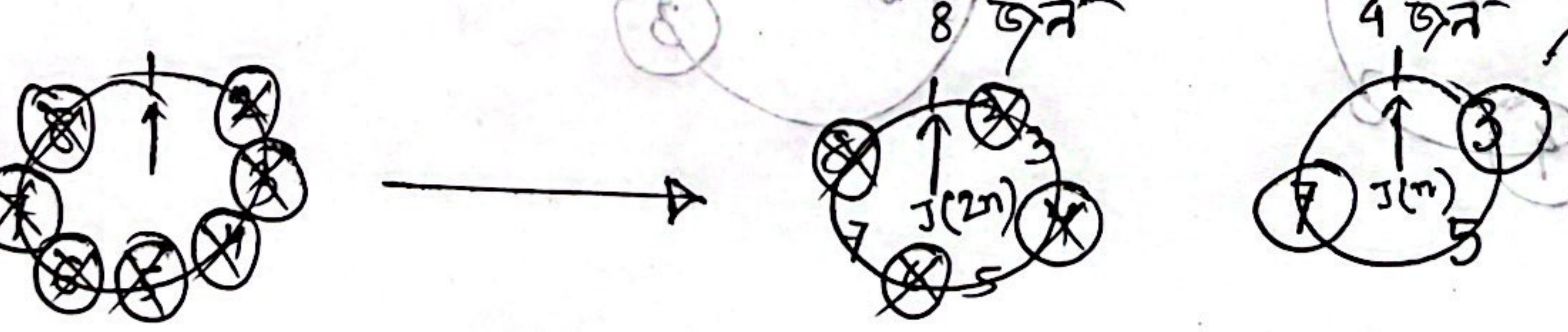
n = 3  J(3) = 3

n = 4  J(4) = 1

n = 5  J(5) = 3

n = 7  J(7) = 7

Visualization when n is even

n = 8 

2nd round

pos

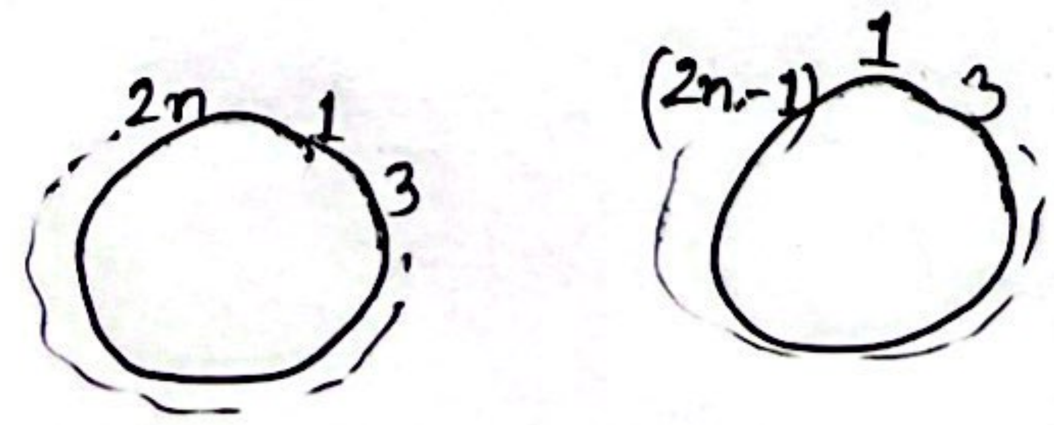
1 → 1 (2*1−1)

3 → 2 (2*2−1)

5 → 3 (2*3−1)

7 → 4 (2*4−1)

$\boxed{J(2n) = 2 * J(n) - 1}$



Observations :

1. $J(n)$ is always odd (at least for $n = 1, 2, 3, 4, 5, 6, 7, 11$)
2. Even positions are eliminated at first round.

Note :

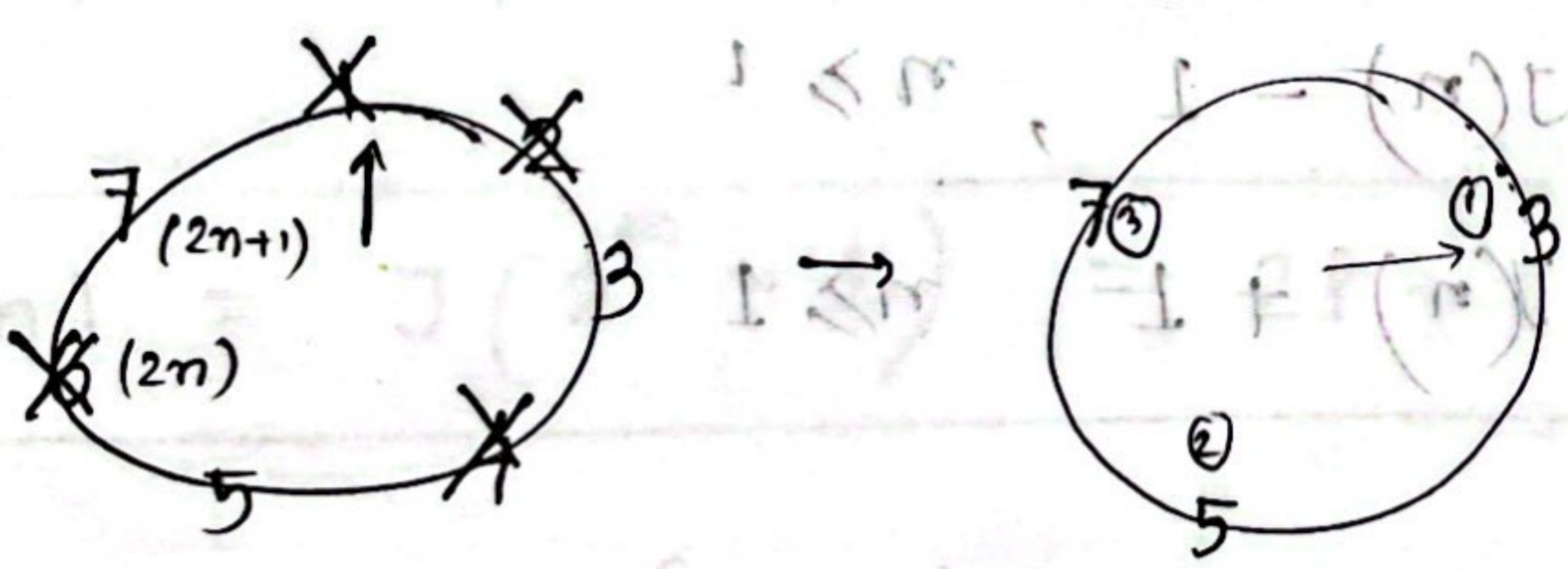
- $J(1) = 1$
- $J(2n) = 2 * J(n) - 1$

$\therefore J(10) = 2 * J(5) - 1 = 5$

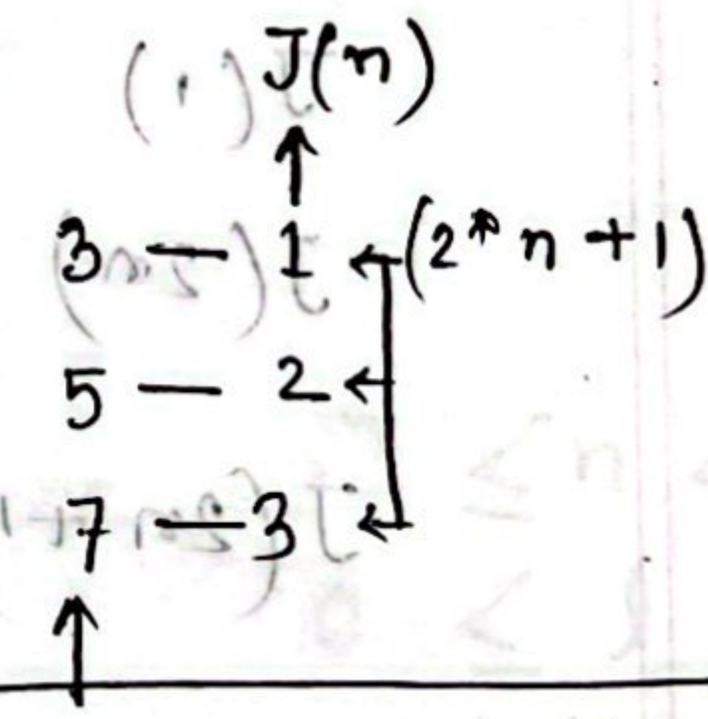
$J(20) = 2 * J(10) - 1 = 2 * 5 - 1 = 9$

$J(40) = 2 * J(10) - 1 = 9 * 2 - 1 = 17$

Visualization when n is odd



$J(2n+1)$ $\qquad\qquad$ $J(n)$



$$\boxed{J(2n+1) = 2 * J(n) + 1}$$

- $J(2n+1) = 2 * J(n) + 1$

$J(18) = 2\,J(9) - 1 = 2\left(2\,J(4) + 1\right) - 1 = 2\left(2 * 1 + 1\right) - 1$

$= 5$

3. For even no. of

Originally we have ~~2n~~ no. of people, after one round, it seems we have n no of people. except each ~~peop~~ person's position number has been doubled and (decreased by 1.

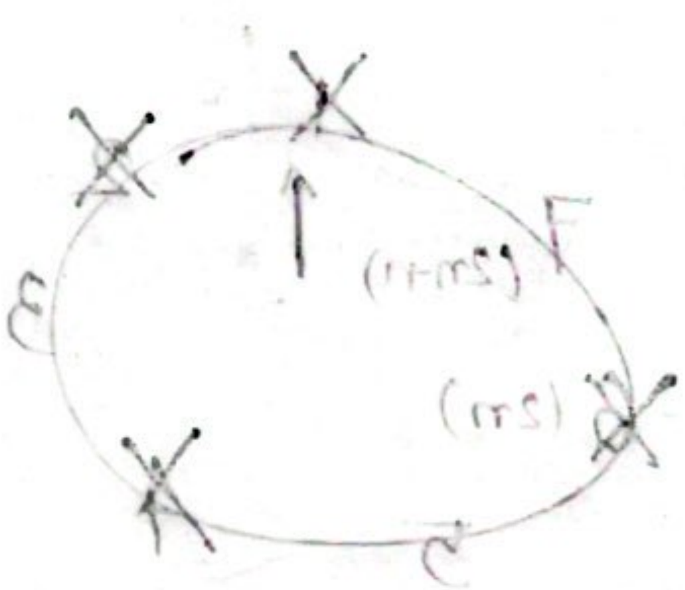
4. For odd no, 2n+1 — — — — — — — — n — — — — — — — — — — — — — — — — and increased by 1.

Recurrence problem:

$J(1) = 1$

$J(2n) = 2^* J(n) - 1, \quad n \geq 1$

$J(2n+1) = 2^* J(n) + 1 \quad n \geq 1$



$\boxed{\text{closed Form}}$:

1. when n is power of 2, $J(n) = 1$

2. when $J(n)$ is 1, later $J(n+1)$, $J(n+2)$ are of consequtive odd numbers.

$J(n) = 1$

$J(2^m) = 1$ here $n = 2^m$ $(2\times 0+1)$

$J(2^m + 1) = 3$ $(2\times 1+1)$

$J(2^m + 3) = 5$ $(2\times 2+1)$

$J(2^m + 5) = 7$ $(2\times 3+1)$

when, $n = 2^m + l$

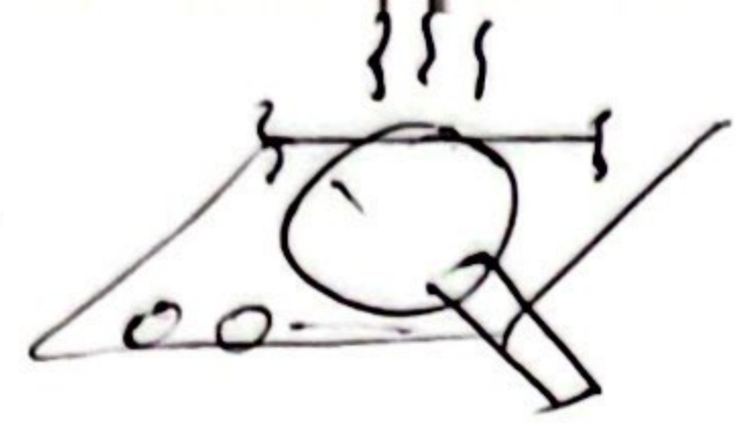
$J(n) = J(2^m + l) = 2 * l + 1$

$$\boxed{J(n) = J(2^m + l) = 2l + 1}$$

here, $2^m \leq n < 2^{m+1}$

$0 \leq l \leq 2^m - 1$

$J(100) = J(2^6 + 36) = 2 * 36 + 1 = 73$



Using the recurrence sol^n , $\boxed{\text{closed form}}$

prove by induction that $J(\underbrace{2^m+l}) = 2l+1$

এই part হয় 2n নতুবা 2n+1

Basis : $m=0,\ l=0$

$$J(2^0+0) = J(1) = 1$$

Induction : l is even ~~odd~~

If $m>0$ and $2^m+l = 2n$

$$J(2n) = 2J(n) - 1$$

$$J(2^m+l) = 2J\left(\frac{2^m}{2}+\frac{l}{2}\right) - 1$$

$$= 2J\left(2^{m-1}+\frac{l}{2}\right) - 1$$

$$= 2\left(\cancel{2}\cdot\frac{l}{\cancel{2}}+1\right) - 1$$

$$= 2l+2-1$$

$$= 2l+1$$

when l is odd, $2^m+l = 2n+1 \rightarrow n = \frac{2^m+l-1}{2}$

$$J(2^m+l) = J(2n+1)$$

$$= 2J(n)+1$$

$$= 2J\left(\frac{2^m+l-1}{2}\right)+1$$

$$= 2\left[\cancel{2}\cdot\frac{l-1}{\cancel{2}}+1\right]+1 = 2l+1$$

The Josephus problem $\rightarrow J(n) = J(2^m + l) = 2l + 1$

Examining the closed form :

power of 2^m played an important role.

if n can be represented in binary $\rightarrow n$ & $J(n)$

n's binary ? $n = 2^m + l$

$n = (b_m b_{m-1} \cdots b_1 b_0)_2$

$\hookrightarrow$ each $b_i = \begin{cases} 0 \\ 1 \end{cases}$

leading bit , $b_m = 1$, so, $n = (\underset{\substack{\downarrow \\ b_m = 1}}{1} b_{m-1} b_{m-2} \cdots b_1 b_0)_2$

$l = (0\, b_{m-1} b_{m-2} \cdots b_1 b_0)_2$

double $\quad 2l = (b_{m-1} b_{m-2} \cdots b_1 b_0 0)_2$

adding 1 $\quad 2l + 1 = (b_{m-1} b_{m-2} \cdots b_1 b_0 1)_2$

$2 \rightarrow 10$
$\downarrow$ double
$4 \rightarrow 100$
$\downarrow$ double
$8 \rightarrow 1000$

$\therefore J(n) = (b_{m-1} b_{m-2} \cdots b_1 b_0 b_m)_2$

$\therefore \boxed{J((b_m b_{m-1} \cdots b_1 b_0)_2) = (b_{m-1} b_{m-2} \cdots b_1 b_0 b_m)_2}$

$\hookrightarrow$ 1 bit cyclic left shift

If, $n = 10$,

$J((1010_2)) = 0101_2 = 5$

If, $n = 100$,

$J((1100100)_2) = 1001001_2 = 73$

$J(n) = n$ ... $J(n) = \frac{n}{2}$

n	1	3	7	15
$J(n)$	1	3	7	15

n	10
$J(n)$	5

- repeatedly Josephus problem apply :

$$J(J(J(13)))$$

$13 = 1101_2$

$J(1101_2) = 1011_2 = 11$

$J(1011_2) = 0111_2 = 7$ } একটা time এ fixed point এ চলে যাব ।

$J(\cancel{0}111_2) = 111_2 = 7$

$\boxed{\text{when } J(n) \text{ will be } n}$?

↳ binary representation এ যখন সবগুলো bit ই 1 ।

n কে represent করতে কয়টা '1' bit লাগে → fix হবে $2^x - 1$, x = no of '1'

13 → binary representation এ 3 টা 1 (1101) 3 টা 1.

↳ fix হবে → $2^3 - 1 = 7$ এ ।

$\therefore J(J(J(J(13)))) = 7$ (at least 2 বার চালালে fix হবে) repeat হবে 3 no. বারে ।

$$J^4(J^3(J^2(J^1(21))))$$

$$= \underbrace{10101}_{21} \longrightarrow \underset{\sim}{3}\ 2\ 1 \longrightarrow 2^3 - 1 = 7$$

$1 \rightarrow 01011$

$2 \rightarrow 0111 = 7$

$3 \rightarrow 111 = 7$

$4 \rightarrow 111 = 7$

$$J^6(J^5(J^4(J^3(J^2(J^1(\underbrace{101011110}))))))$$

$$\longrightarrow 2^6 - 1 = 63$$

$1 \rightarrow 010111101$

$2 \rightarrow 01111011$

$3 \rightarrow 1110111$

$4 \rightarrow 110111$

$5 \rightarrow 1011111$

$6 \rightarrow 111111$

$\boxed{\text{When } J(n) \text{ will be } n/2}$?

$$J(n) = 2l + 1 = \frac{2^m + l}{2} \qquad n = 2^m + l$$

$$\Rightarrow 4l + 2 = 2^m + l$$

$$\Rightarrow 3l = 2^m - 2$$

$$\Rightarrow \boxed{l = \frac{1}{3}(2^m - 2)}$$

binary no. কে half করতে → right shift

n	$J(n)$
(2) $10 \longrightarrow$	01 (1)
(10) $1010 \longrightarrow$	0101 (5)

n যখন $\underbrace{170}_{101010} \longrightarrow J(n) = \frac{170}{2} = 85$

* The pattern is presence of consequtive '10' in binary representation.

Generalization :

$$J(1) = \underset{\sim}{1}$$
$$J(2n) = 2J(n) - \underset{\sim}{1}$$
$$J(2n+1) = 2J(n) + \underset{\sim}{1}$$

constant values are represented like α, β, γ.

$\therefore f(1) = \alpha$

$f(2n) = 2f(n) + \beta$ (for $n \geq 1$)

$f(2n+1) = 2f(n) + \gamma$ (for $n \geq 1$)

According to the original eq^n,

$\alpha = 1$

$\beta = -1$

$\gamma = 1$

n	$f(n)$
1	α
2	$2\alpha+\beta$
3	$2\alpha+0\beta+\gamma$
4	$4\alpha+3\beta+0\cdot\gamma$
5	$4\alpha+2\beta+\gamma$
6	$4\alpha+1\beta+2\gamma$
7	$4\alpha+0\cdot\beta+3\gamma$
8	$8\alpha+7\beta+\gamma$
9	$8\alpha+6\beta+1\gamma$
10	$8\alpha+5\beta+2\gamma$

$\downarrow$ largest power of 2 $\qquad \downarrow$ decrease 1 করে $\qquad \downarrow$ increase 1 করে

$n=4 \rightarrow 2^2+0$

$l=0$
$C(n)=0$
$B(n)=3$
$A(n)=4$

$f(2) = 2f(1)+\beta$
$= 2\alpha+\beta$

$f(3) = 2f(1)+\gamma$
$= 2\alpha+\gamma$

$f(4) = 2f(2)+\beta$
$= 2(2\alpha+\beta)+\beta$
$= 4\alpha+3\beta$

$f(5) = 2f(2)+\gamma$
$= 2(2\alpha+\beta)+\gamma$
$= 4\alpha+2\beta+\gamma$

$f(6) = 2f(3)+\beta$
$= 2(2\alpha+\gamma)+\beta$
$= 4\alpha+\beta+2\gamma$

$f(7) = 2f(3)+\gamma$
$= 2(2\alpha+\gamma)+\gamma$
$= 4\alpha+3\gamma$

① α's coefficient is n's largest power of 2

② β's " " decreasing by 1 down to 0

③ γ's " " increasing by 1 up from 0

$$\boxed{f(n) = \underbrace{A(n)}_{2^m}\alpha + \underbrace{B(n)}_{2^m-1-l}\cdot\beta + \underbrace{C(n)}_{l}\gamma}$$