1. 11. 2023

Chapter 2 : Sums

Notation

① $1+2+3+\cdots+n$ ✓

$1+2+\cdots+(n-1)+n$ ✓

$1+\cdots+n$ (not really clear)

* $a_1+a_2+a_3+\cdots+a_n$ → summation of terms

a_1 ↕ term

② Using 'Σ' : $\sum_{k=1}^{n} a_k$ → "sigma notation" (a_k → summand)

→ "delimited form" (limit 1 to n of sum)

or,

sigma notation (generalized)

→ $\sum_{\boxed{1\le k\le n}} a_k$ → generalized sigma form

$1\le k\le n$ → relation under sigma

(Ex) : $\sum_{\substack{1\le n\le 100 \\ k \text{ odd}}} k^2 \leftarrow 1^2+3^2+5^2+\cdots+99^2$

↓

$\sum_{k=0}^{49}(2k+1)^2$

- Sum of reciprocal of the prime number upto 100 :

$\hookrightarrow \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$

$\longrightarrow \sum_{\substack{1 \le n \le 100 \\ n\ \text{prime}}} \frac{1}{n}$ or, $\sum_{k=1}^{\pi(100)} \frac{1}{P_k}$

π একটা function যা 1 to 100 prime number generate করবে। $P_k \rightarrow k^{th}$ prime number

or, $\sum_{\substack{p\ \text{is prime} \\ p \le N}} \frac{1}{p}$ or, $\sum_{k=1}^{\pi(N)} \frac{1}{P_k}$

(upto N)

changing limits

- $\sum_{1 \le k \le n} a_k \longrightarrow$ we want to change limit ~~into~~ k into k+1 :

$\longrightarrow \sum_{1 \le k+1 \le n} a_{k+1}$

- $\sum_{k=1}^{n} a_k \longrightarrow$ k into k+1

$= \sum_{k=0}^{n-1} a_{k+1}$

Note :

→ problem statement represent → use delimited.

→ To manipulate sums → use relation under the Σ.

→ $\sum_{k=1}^{n} a_k (k-1)$

1 দিলে first term 0 → zero sum is not harmful ; we don't have to handle it.

$= \sum_{k=2}^{n} a_k (k-1)$

Kenneth Iversons conversion (KIC)

$$[p \text{ prime}] = \begin{cases} 0 & \longrightarrow p \text{ prime না হলে!} \\ 1 & \longrightarrow p \text{ prime হলে 1 o/p দিবে।} \end{cases}$$

$$\sum_{\substack{p \le N \\ p \text{ is prime}}} p \xrightarrow{K.I.C} \sum \underset{\substack{\downarrow \\ \text{that number } p}}{p}\, [p \text{ prime}]\, \underbrace{[p \le N]}$$

→ p ; N এর থেকে ছোট or সমান হলে 1 o/p দিবে।

Sums & Recurrence :

The sum, $S_n = \sum_{k=0}^{n} a_k$

$S_0 = 0$

$$S_n = \sum_{k=0}^{n-1} a_k + a_n$$

$$= S_{n-1} + a_n$$

So, the sum is equivalent to, $S_0 = a_0$

$S_n = S_{n-1} + a_n \; ; n > 0$ ← (1)

If, $a_n = \underbrace{\text{constant}}_{\beta} + \underbrace{\text{a multiple of } n}_{n \cdot \gamma}$, $a_0 = \alpha$

$$\left.\begin{aligned} R_0 &= \alpha \\ R_n &= R_{n-1} + (\beta + n\gamma) \end{aligned}\right\} \longleftarrow \text{(ii)}$$

$R_n = R_{n-1} + \beta + \gamma n$

$n = 1, 2, 3 \ldots$

$R_0 = \alpha$

$R_1 = R_0 + \beta + \gamma$
$= \alpha + \beta + \gamma$

$R_2 = R_1 + \beta + 2\gamma$
$= \alpha + 2\beta + 3\gamma$

$R_3 = R_2 + \beta + 3\gamma$
$= \alpha + 3\beta + 6\gamma$

generalised solution, $R_n = A(n)\cdot\alpha + B(n)\cdot\beta + C(n)\cdot\gamma$ — (iii)

$\boxed{\text{Set}, \; R_n = 1}$: $R_n = 1$
$R_{n-1} = 1$
$R_0 = 1 = \alpha$ (as, $R_0 = \alpha$)

$1 = 1 + (\beta + \gamma n)$
$\Rightarrow \beta + \gamma n = 0$
so, $\beta = \gamma = 0$

So, $\alpha = 1$, $\beta = 0$, $\gamma = 0$

So, eqn (iii), $1 = A(n)\cdot 1 + B(n)\cdot 0 + C(n)\cdot 0$

$\therefore \boxed{A(n) = 1}$

$\boxed{\text{Set } R_n = n}$: $R_0 = \alpha \quad \therefore R_0 = 0$
$R_n = R_{n-1} + \beta + \gamma n$
$\Rightarrow n = n - 1 + \beta + \gamma n$
$\Rightarrow \beta + \gamma n = 1$
$\Rightarrow \beta + \gamma n = 1 + 0.n$
$\Rightarrow \beta = 1, \gamma = 0$

So, from eq. (iii), $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$
$\Rightarrow n = A(n)\cdot 0 + B(n)\cdot 1 + C(n)\cdot 0$

$\therefore \boxed{B(n) = n}$

$\boxed{\text{Set } R_n = n^2}$

$R_0 = 0^2 = 0 = \alpha$

$\therefore \alpha = 0$

$\therefore n^2 = (n-1)^2 + \beta + \gamma n$

$\Rightarrow \cancel{n^2} = \cancel{n^2} - 2n + 1 + \beta + \gamma n$

$\Rightarrow -1 + 2n = \beta + \gamma n$

Repertive method

$\therefore \beta = -1, \gamma = 2$

From eqn (iii), $n^2 = A(n)\cdot 0 + B(n)\cdot(-1) + C(n)\cdot 2$

$\Rightarrow n^2 = -B(n) + 2\cdot C(n)$

$\Rightarrow C(n) = \frac{B(n) + n^2}{2}$

$= \frac{n + n^2}{2}$

So, from eqn (iii),

$$\boxed{R(n) = 1\cdot\alpha + n\cdot\beta + \frac{n+n^2}{2}\cdot\gamma} \quad \text{— (iv)}$$

closed form

$S_n = \sum_{k=0}^{n} (a + bk)$

when, $k = 0 \longrightarrow a + b\cdot 0$

$k = 1 \longrightarrow a + b\cdot 1$

$k = 2 \longrightarrow a + b\cdot 2$

$\vdots$

$k = n \longrightarrow a + bn$

$\longrightarrow \underbrace{\sum_{k=0}^{n-1}(a+bk)}_{R_{n-1}} + a + bn$

So, $\beta = a$

$\gamma = b$

and, $R_0 = \alpha = a$

$\therefore \alpha = a,\ \beta = a,\ \gamma = b$

$$\therefore R(n) = \alpha + n\beta + \frac{n+n^2}{2}\gamma \quad \text{(from eq}^n \text{ (iv))}$$

$$= a + na + \frac{n^2+n}{2} \cdot b$$

$$= a(n+1) + b \cdot \frac{n(n+1)}{2} \longrightarrow \text{sol}^n \text{ from } \sum_{k=0}^{n}(a+bk)$$

TOWER OF HANOI :

Recurrence sol^n $\longrightarrow T_0 = 0$

$\longrightarrow T_n = 2T_{n-1} + 1 \quad (\text{for } n>0)$ —— (2)

divide eq^n (2) by 2^n,

$$\frac{T_0}{2^0} = \frac{0}{2^0} = 0 \quad (n=0) \longrightarrow (S_0)$$

$$\frac{T_n}{2^n} = \frac{2T_{n-1} + 1}{2^n} \quad \text{for } n>0$$

$$\Rightarrow \frac{T_n}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n} \quad \text{for } n>0$$

assume, $\frac{T_n}{2^n} = S_n$

$\therefore S_0 = 0$

$$S_n = S_{n-1} + \frac{1}{2^n}$$

It follows that,

$$S_n = \sum_{K=1}^{n} 2^{-K} \qquad \frac{a(1-r^n)}{1-r}$$

$$S_n = 2^{-1} + 2^{-2} + 2^{-3} + \dots + 2^{-n}$$

$$= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \left(\frac{1}{2}\right)^n$$

$a = \frac{1}{2}$

$r = \frac{1}{2}$

$$= \left[1 - \left(\frac{1}{2}\right)^n\right]$$

$$= 1 - \frac{1}{2^n} \qquad \frac{\cancel{\frac{1}{2}}\left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \cancel{\frac{1}{2}}\right)}$$

$$S_n = \frac{2^n - 1}{2^n}$$

$$\therefore \ 2^n S_n = 2^n - 1$$

$$\therefore \boxed{T_n = 2^n - 1}$$

$$a_n T_n = b_n T_{n-1} + c_n \text{ --- } (31)$$

$$T_n = 2T_{n-1} + 1$$

Summation factor

let the summation factor is s_n

From (31), s_n multiply,

$$s_n \cdot a_n T_n = s_n b_n \cdot T_{n-1} + s_n \cdot c_n \text{ --- } 3.2$$

let. $\boxed{s_n b_n = s_{n-1} \cdot a_{n-1}}$

$$S_n = s_n a_n T_n$$

sum

$1.3.2 \longrightarrow S_n = S_{n-1} a_{n-1} T_{n-1} + S_n \cdot C_n$

$$S_n = S_{n-1} + s_n c_n \rightarrow S_{n-2} + S_{n-1} C_{n-1} + S_n C_n$$

Hence,

$$S_n = s_0 a_0 T_0 + \sum_{k=1}^{n} S_K C_K$$

$$= s_1 b_1 T_0 + \sum_{k=1}^{n} c_K S_K$$

$$S_n a_n T_n = s_1 b_1 T_0 + \sum_{k=1}^{n} c_k S_k$$

$$\Rightarrow \boxed{T_n = \frac{1}{s_n a_n}\left[s_1 b_1 T_0 + \sum_{k=1}^{n} c_K S_K\right]} \longrightarrow \text{Sol}^n.$$

$$\boxed{n=1} \longrightarrow T_1 = \frac{1}{\cancel{s_1} a_1}\left[\cancel{s_1} b_1 T_0 + c_1 \cancel{s_1}\right]$$

$$= \frac{1}{a_1}\left[b_1 T_0 + c_1\right]$$

$$S_n b_n = S_{n-1} a_{n-1}$$

$$S_n = \frac{S_{n-1} a_{n-1}}{b_n}$$

$$= \frac{S_{n-2} \cdot a_{n-2}}{b_{n-1}} \frac{a_{n-1}}{b_n}$$

$$S_n = \frac{a_{n-1} a_{n-2} a_{n-3} \cdots a_1}{b_n b_{n-1} b_{n-2} \cdots b_2}$$

$$a_n T_n = b_n T_{n-1} + c_n \qquad T_{n-1} = 2T_{n-2} + 1$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$1 \cdot T_n = 2 \cdot T_{n-1} + 1$$

$$\therefore \quad s_n = \frac{1 \cdot 1 \cdot 1 \cdot}{2 \cdot 2 \cdots 2}$$

$$\boxed{s_n = \frac{1}{2^{n-1}}} \longrightarrow \text{Summation factor}$$

Problem

In quicksort algorithm, one of the mostly used method for sorting data, data's need to compare with other element. The average number of comparision steps made by quicksort when it's applied to n items in random order which satisfies the following recurrence.

$$c_0 = 0$$

$$c_n = n+1 + \frac{2}{n} \sum_{k=0}^{n-1} c_k, \text{ for } n>0$$

1. Form the above recurrence in $a_n T_n = b_n T_{n-1} + c_n$
2. Find the summation factor (s_n)
3. Prove that i) $c_n = 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1}$

ii) $c_n = 2(n+1)H_n - 2n$, where H_n harmonic sum

iii) Find closed form for c_n

iv) Find c_1, c_2 and c_3.

→ $a_n T_n = b_n T_{n-1} + c_n$

$$\underset{a_n}{n c_n} = \underset{b_n}{(n+1)} c_{n-1} + \underset{c_n}{2n} \qquad \underset{a_{n-1}}{(n-1)} c_{n-1} = \underset{b_{n-1}}{n} c_{n-2} + \underset{c_{n-1}}{2(n-1)}$$

$$s_n = \frac{a_{n-1}\, a_{n-2}\, a_{n-3} \cdots a_1}{b_n \cdot b_{n-1} \cdot b_{n-2} \cdots b_2}$$

$$= \frac{(n-1)(n-2)(n-3)\cdots 1}{(n+1)\, n \cdot (n-1) \cdots 3}$$

$$= \frac{\cancel{(n-1)!}\cdot 2}{(n+1)\, n \cdot \cancel{(n-1)!}} = \frac{2}{n(n+1)} \longrightarrow \text{summation factor}$$

Manipulation of Sums :

1. Distributive law. $\longrightarrow \sum_{k\in K} c a_k = c \sum_{k \in K} a_k$ (constant: c)
2. Associative law $\longrightarrow \sum_{k\in K} (a_k + b_k) = \sum_{k\in K} a_k + \sum_{k \in K} b_k$

3.

$$\sum_{k\in K} a_k = \sum_{P(k)\in K} a_{P(k)}$$

→ we can reorder the sum in any way

→ $P(K)$ → any permutation of the set.

Ex : $\sum_{k=1,2,3} a_k = a_1 + a_2 + a_3$

↳ যেকোনো order এ লিখতে পারি।

- For comparing different set of ~~of~~ indices,

$$\sum_{k\in K} a_k + \sum_{k\in K'} a_k = \sum_{k\in K\cap K'} a_k + \sum_{k\in K\cup K'} a_k$$

$\sum a_k [k\in K]$ kenette iversion

Ex : $$\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{k=1} a_k$$

↓ only $k=m$ common

↳ Perturbation method

→ Spliting a single term from a sum

The operation of splitting of a term is the basis of perturbation method. It can be used to determine the closed form of any sum.

$$S_n = \sum_{0 \le k \le n} a_k$$

$$\Rightarrow S_n + a_{n+1} = \sum_{0 \le k \le n} a_k + a_{n+1}$$

$$= \sum_{0 \le k \le n+1} a_k$$

$$= a_0 + \sum_{1 \le k \le n+1} a_k$$

$$= a_0 + \sum_{0 \le k-1 \le n} a_k$$

$$= a_0 + \sum_{0 \le k \le n} a_{k+1}$$

Problem :

Find the sum of a general geometric progression using perturbation method, i.e.

$$S_n = \sum_{0 \le k \le n} ax^k$$

$$S_n = ax^0 + ax^1 + ax^2 + ax^3 + \dots + ax^n$$

$$S_0 = a$$

apply perturbation method,

$$S_n + ax^{n+1} = \sum_{0 \le k \le n} ax + ax^{n+1}$$

$$= ax^0 + \sum_{0 \le k \le n} ax^{k+1}$$

$$= ax^0 + \sum_{0 \le k \le n} ax^k \cdot x$$

$$= ax^0 + x \cdot \sum_{0 \le k \le n} ax^k$$

$$= ax^0 + x \cdot S_n$$

$$\Rightarrow \quad S_n - xS_n = ax^0 - ax^{n+1}$$

$$\Rightarrow \quad \cancel{(1-x)}\, S_n = \frac{a(1 - x^{n+1})}{1-x}$$

$$\therefore \quad \boxed{S_n = \frac{a(1 - x^{n+1})}{1-x}}$$

Problem : Apply

Multiple of sums :

* $\sum_{1 \le j,k \le 3} a_j b_k = a_1 b_1 + a_1 b_2 + a_1 b_3 +$
  $a_2 b_1 + a_2 b_2 + a_2 b_3 +$
  $a_3 b_1 + a_3 b_2 + a_3 b_3$

* $\sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k} \left[P(j,k) \right] \longrightarrow$ kenette Iversion form.

* $\sum_{j} \sum_{k} a_{j,k} \left[P(j,k) \right]$

• Law of interchanging the order of summation:

$$\sum_j \sum_k a_{j,k} \left[P(j,k) \right] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k} \left[P(j,k) \right]$$

$$\sum_{1 \le j,\, k \le 3} a_j b_k = \sum_{j,k} a_j b_k \left[1 \le j ,\ k \le 3 \right]$$

$$= \sum_{j,k} a_j b_k \left[1 \le j \le 3 \right] \left[1 \le k \le 3 \right]$$

$$= \sum_j \sum_k a_j b_k \left[1 \le j \le 3 \right] \left[1 \le k \le 3 \right]$$

$$= \left(\sum_{j} a_j \,[1 \le j \le 3]\right)\left(\sum_{k} b_k \,[1 \le k \le 3]\right)$$

$$= \left(\sum_{j=1}^{3} a_j\right)\left(\sum_{k=1}^{3} b_k\right)$$

General distributive law :

$$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j\right)\left(\sum_{k \in K} b_k\right)$$

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* Find the closed form for the following,

$$2\sum_{1\le j,k\le n} a_k b_k$$

$$= 2\sum_{1\le j\le n} \cancel{a_k b_k} \sum_{1\le k\le n} a_k b_k$$

$$= 2\sum_{1\le k\le n} \left(\sum_{1\le j\le n} a_k b_k\right)$$

$$= 2\sum_{1\le k\le n} \left[a_k b_k \sum_{1\le j\le n} (1)\right]$$

$$= 2\sum_{1\le k\le n} \left[a_k b_k \cdot n\right]$$

$$= 2n\sum_{1\le k\le n} a_k b_k$$

$$\sum_{1\le j\le n} 1 = n$$

* Find closed form of the sum, $S_n = \sum_{1\le j<k\le n} \frac{1}{k-j}$

1) First summing on j,

$$S_n = \sum_{1\le j<k\le n} \frac{1}{k-j}$$

$$= \sum_{1\le k\le n} \sum_{1\le j<k} \frac{1}{k-j} \longrightarrow \text{replacing } j \text{ with } k-j$$

$1 \le k-j < k$

$\Rightarrow 1-k \le k-j-k < k-k$

$\Rightarrow 1-k \le -j < 0$

$\Rightarrow k-1 \ge j > 0$

$\Rightarrow 0 < j \le k-1$

$\Rightarrow \boxed{1 \le j \le k-1}$

$$= \sum_{1\le k\le n} \sum_{1\le k-j<k} \frac{1}{\cancel{k}-(\cancel{k}-j)}$$

$$= \sum_{1\le k\le n} \left(\sum_{1\le j<k-1} \frac{1}{j}\right)$$

$j \rightarrow 1 \text{ to } k-1$

$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$

↳ harmonic sum here $n = k-1$

$$= \sum_{1\le k\le n} H_{k-1} = \sum_{0\le k<n} H_k$$

(Ans)

Scanned with CamScanner

2) summing on k,

$$S_n = \sum_{1 \le j < k \le n} \frac{1}{k-j}$$

$$= \sum_{1 \le j \le n} \sum_{j < k \le n} \frac{1}{k-j}$$

$$= \sum_{1 \le j \le n} \sum_{j < k+j \le n} \frac{1}{k+\cancel{j}-\cancel{j}} \longrightarrow \text{replacing } k \text{ with } k+j$$

$$j < k+j \le n$$
$$\Rightarrow j-j < k+\cancel{j}-\cancel{j} \le n-j$$
$$\Rightarrow \boxed{0 < k \le n-j} \to \boxed{1 \le k \le n-j}$$

$$= \sum_{1 \le j \le n} \left(\sum_{j < k+j \le n} \frac{1}{k} \right)$$

$$= \sum_{1 \le j \le n} \left(\sum_{0 < k \le n-j} \frac{1}{k} \right)$$

$$= \sum_{1 \le j \le n} H_{n-j}$$

$$= \sum_{1 \le n-j \le n} H_{n-(n-j)}$$

$$= \sum_{1 \le n-j \le n} H_j$$

$$1 \le n-j \le n$$
$$\Rightarrow 1-n \le \cancel{n}-j-\cancel{n} \le n-n$$
$$\Rightarrow 1-n \le -j \le 0$$
$$\Rightarrow n-1 \geqslant j \geqslant 0$$
$$\Rightarrow 0 \le j \le n-1$$

$$= \sum_{0 \le j \le n-1} H_j$$

$$= \sum_{0 \le j < n} H_j$$

$$\sum_{1\le k\le n} \frac{n}{k} - \sum_{1\le k\le n} 1$$

$$= nH_n - n$$

3) $$S_n = \sum_{1\le j<k\le n} \frac{1}{k-j}$$

$$= \sum_{1\le j<k\le n} \frac{1}{k}$$

[replacing k by $k+j$ $\therefore k+\cancel{j}-\cancel{j} = k$]

$$= \sum_{1\le j<k} \left(\sum_{1\le k\le n} \frac{1}{k} \right)$$

$$= \sum_{1\le k\le n} \left(\sum_{1\le j\le n} \frac{1}{k} \right)$$

$$= \sum_{1\le k\le n} \frac{1}{k} \cdot n$$

* Find closed form for the sum of squares, i.e $\Box_n$

$$S_n = \Box_n = \sum_{0\le k\le n} k^2 = 0^2 + 1^2 + 2^2 + \cdots + n^2$$

→ <u>Perturbation method</u> :

$$\Box_n + (n+1)^2 = \sum_{0\le k\le n} k^2 + (n+1)^2$$

$$= \sum_{0\le k\le n+1} k^2 = \sum_{0\le k\le n} (k+1)^2$$

$$= \sum_{0\le k\le n} (k^2 + 2k + 1)$$

$$= \sum_{0\le k\le n} k^2 + \sum_{0\le k\le n} 2k + \sum_{0\le k\le n} 1$$

$$\Rightarrow \cancel{\Box_n} + (n+1)^2 = \cancel{\Box_n} + 2\sum_{0\le k\le n} k + n + 1$$

$$\Rightarrow 2\sum_{0\le k\le n} k = (n+1)^2 - (n+1)$$

$$\Rightarrow \sum_{0\le k\le n} k = \frac{(n+1)(n+\cancel{1}-\cancel{1})}{2}$$

$$\therefore \boxed{\sum_{0\le k\le n} k = \frac{n(n+1)}{2}}$$

$$\boxdot_n = \sum_{0 \le k \le n} k^3$$

$$\cancel{\boxdot_n} + (n+1)^3 = \sum_{0 \le k \le n} k^3 + (n+1)^3$$

$$= \sum_{0 \le k \le n} (k+1)^3 = \sum_{0 \le k \le n} (k^3 + 3k^2 + 3k + 1)$$

$$= \sum_{0 \le k \le n} k^3 + 3\sum_{0 \le k \le n} k^2 + 3\sum_{0 \le k \le n} k + \sum_{0 \le k \le n} 1$$

$$= \cancel{\boxdot_n} + 3\sum_{0 \le k \le n} k^2 + 3\sum_{0 \le k \le n} k + \sum_{0 \le k \le n} 1$$

$$= 3\,\square_n + 3\,\frac{n(n+1)}{2} + (n+1)$$

$$\Rightarrow \quad \square_n = \frac{(n+1)^3}{3} - \frac{n(n+1)}{2} - \frac{n+1}{3}$$

$$= \frac{(n+1)\left\{ \overset{2n^2 + 4n + \cancel{2} - 3n - \cancel{2}}{\cancel{2n^2 + 4n + 2 - 3n - 3n - 2n - 2}} \right\}}{6}$$

$$= \frac{(n+1)(\cancel{-n^2 - n})(2n^2 + n)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Repertoire method :

$S_n = \sum k^2$

$S_n = S_{n-1} + n^2$

$R_0 = \alpha$

$R_n = R_{n-1} + \beta + \gamma n$

$R_0 = \alpha$

$R_n = R_{n-1} + \beta + \gamma n + \delta n^2$

$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$
$+ D(n) \cdot \delta$

$A(n) = 1$

$B(n) = n$

$C(n) = \frac{n^2+n}{2}$

Set,

$R(n) = n^3$

$R_0 = 0^3 = 0 \longrightarrow \boxed{\alpha = 0}$

$R_n = R_{n-1} + \beta + n\gamma + n^2\delta$

$\Rightarrow n^3 = (n-1)^3 + \beta + n\gamma + n^2\delta$

$\Rightarrow \beta + n\gamma + n^2\delta = n^3 - (n-1)^3$

$= n^3 - (n^3 - 3n^2 + 3n - 1)$

$= 3n^2 - 3n + 1$

$\Rightarrow \beta + n\gamma + n^2\delta = 3n^2 - 3n + 1$

$\alpha = 0, \quad \beta = 1, \quad \gamma = -3, \quad \delta = 3$

$R(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\cdot\delta$

$\Rightarrow n^3 = 1 \cdot 0 + n \cdot 1 + \frac{n^2+n}{2} \cdot (-3) + 3D(n)$

$\Rightarrow \cancel{3}D(n) = 2n^3 + 3(n^2+n) - 2n$

$= \frac{1}{3}(2n^3 + 3n^2 + n)$