Cauchy's Residue Theorem

Residue: If z_0 is an isolated singular point of $f(\mathrm{z})$ then the coefficient a_{-1} of $\dfrac{1}{z-z_0}$ in the Laurent series

$$f(\mathrm{z}) = \sum_{n=0}^{\infty} a_n (\mathrm{z}-\mathrm{z}_0)^n + \sum_{n=1}^{\infty} a_{-n} (\mathrm{z}-\mathrm{z}_0)^{-n}$$ is called the residue of at

Formulae for Finding Residues:

1. Residue at simple pole $z = z_0$: $\operatorname{Re}s(\mathrm{z}_0) = \lim\limits_{z\to z_0} (\mathrm{z}-\mathrm{z}_0)\,\mathrm{f}(\mathrm{z})$

2. Residue at pole of order *n* at $z = z_0$:

$$\operatorname{Re}s(\mathrm{z}_0) = \lim_{z\to z_0} \frac{1}{(\mathrm{n}-1)!} \frac{d^{n-1}}{dz^{n-1}} (\mathrm{z}-\mathrm{z}_0)^n\, \mathrm{f}(\mathrm{z})$$

Note: If $z = z_0$ is a point outside of C then $\mathrm{Res}(\mathrm{z}_0) = 0$.

Cauchy's Residue Theorem: Let $f(\mathrm{z})$ be analytic inside and on a simple closed curve C except at a finite number of singular points $z_1, z_2, \ldots, z_n$

$$\oint_C f(z) = 2\pi i \sum_{i=1}^{n} Res(z_i) = 2\pi i[Res(z_1) + Res(z_2) + \cdots + Res(z_n))$$

Cauchy's Residue Theorem

Example–1 : Find the residue of $f(z) = \dfrac{2z+1}{z^2 - z - 2}$.

Solution : Given that $f(z) = \dfrac{2z+1}{z^2 - z - 2} = \dfrac{2z+1}{(z-2)(z+1)}$

It is clear from here that $z = 2$ and $z = -1$ are two simple poles of $f(z)$

$\therefore$ Residue at $z = 2$ is

$$\text{Res}(2) = \lim_{z \to 2} (z-2)\, f(z)$$

$$= \lim_{z \to 2} (z-2) \cdot \frac{2z+1}{(z-2)(z+1)}$$

$$= \lim_{z \to 2} \frac{2z+1}{z+1}$$

$$= \frac{2.2 + 1}{2 + 1}$$

$$= \frac{5}{3}$$

Again, residue at $z = -1$ is

$$\text{Res}(-1) = \lim_{z \to -1} \{z - (-1)\}.\, f(z)$$

$$= \lim_{z \to -1} (z + 1) \frac{2z + 1}{(z - 2)(z + 1)}$$

$$= \lim_{z \to -1} \frac{2z + 1}{z - 2}$$

$$= \frac{2(-1) + 1}{-1 - 2}$$

$$= \frac{-1}{-3}$$

$$= \frac{1}{3}$$

Example–2 : Find the residue of $f(z) = \dfrac{z^2}{z^2 + a^2}$

Solution : Given that, $f(z) = \dfrac{z^2}{z^2 + a^2}$

$$= \frac{z^2}{z^2 - i^2a^2}$$

$$= \frac{z^2}{z^2 - (ia)^2} = \frac{z^2}{(z + ia)(z - ia)}$$

It is of the form $f(z) = \dfrac{\varphi(z)}{\psi(z)}$ where $\psi(z) = (z + ia)\ (z - ia) = 0$ for $z = ia$ and $z = -ia$ but $\varphi(z) = z^2 \neq 0$ for $z = ia$ and $z = -ia$

$\therefore$ The residue at $z = ia$ of $f(z) = \dfrac{\varphi(z)}{\psi(z)} = \dfrac{z^2}{z^2 + a^2}$ is

$$\text{Res}(ia) = \frac{\varphi(ia)}{\psi'(ia)} = \frac{(ia)^2}{2.ia} \quad [\because\ \psi'(z) = 2z \therefore \psi'(ia) = 2.ia]$$

$$\Rightarrow \text{Res}(ia) = \frac{ia \times ia}{2.ia} = \frac{ia}{2}$$

Again, the residue at $z = -ia$ is $\text{Res}(-ia) = \dfrac{\varphi(-ia)}{\psi'(-ia)}$

$$= \frac{(-ia)^2}{2(-ia)} \quad [\because\ \psi'(z) = 2z \therefore \psi'(-ia) = 2(-ia)]$$

$$= \frac{(-ia) \times (-ia)}{2(-ia)}$$

$$= \frac{-ia}{2}$$

Cauchy's Residue Theorem

Example–3 : Find Res (i) of $f(z) = \dfrac{e^{iz}}{(z^2+1)^4}$

Solution : Given that $f(z) = \dfrac{e^{iz}}{(z^2+1)^4} = \dfrac{e^{iz}}{(z^2-i^2)^4} = \dfrac{e^{iz}}{\{(z+i)(z-i)\}^4}$

$$\Rightarrow f(z) = \frac{e^{iz}}{(z-i)^4 (z+i)^4}$$

$z = i$ is a pole of order 4 of $f(z)$

Therefore by the theorem of residue at a pole of order 4 we have

$$\text{Res}(i) = \lim_{z\to i} \frac{1}{(4-1)!} \frac{d^{4-1}}{dz^{4-1}} [(z-i)^4 f(z)]$$

$$= \lim_{z\to i} \frac{1}{3!} \frac{d^3}{dz^3} \left[(z-i)^4 \cdot \frac{e^{iz}}{(z-i)^4 (z+i)^4}\right]$$

$$= \lim_{z\to i} \frac{1}{3\times 2\times 1} \frac{d^3}{dz^3} \left[\frac{e^{iz}}{(z+i)^4}\right]$$

$$= \frac{1}{6} \lim_{z\to i} \frac{d^3}{dz^3} [e^{iz}(z+i)^{-4}]$$

$$= \frac{1}{6} \lim_{z\to i} \left[(z+i)^{-4} \frac{d^3}{dz^3}(e^{iz}) + {}^3C_1 \frac{d^2}{dz^2}(e^{iz}) \frac{d}{dz}(z+i)^{-4} + {}^3C_2 \frac{d}{dz}(e^{iz}) \frac{d^2}{dz^2}(z+i)^{-4} + (e^{iz}) . \frac{d^3}{dz^3}(z+i)^{-4}\right]$$

$$= \frac{1}{6} \lim_{z \to i} \left[(z+i)^{-4}.i^3 e^{iz} + 3.i^2 e^{iz}(-4)(z+i)^{-5} + 3i\, e^{iz}.(-4)(-5)(z+i)^{-6} + e^{iz}(-4)(-5)(-6)(z+i)^{-7}\right]$$

$$= \frac{1}{6} \lim_{z \to i} \left[\frac{e^{iz}}{(z+i)^4}\left\{-i + \frac{12}{z+i} + \frac{i60}{(z+i)^2} - \frac{120}{(z+i)^3}\right\}\right]$$

$$= \frac{1}{6}\, \frac{e^{i.i}}{(2i)^4}\left\{-i + \frac{12}{2i} + \frac{i60}{(2i)^2} - \frac{120}{(2i)^3}\right\}$$

$$= \frac{1}{6} \cdot \frac{e^{i^2}}{16i^4}\left\{-i + \frac{6}{i} + \frac{i60}{4i^2} - \frac{120}{8.i.i^2}\right\}$$

$$= \frac{1}{6}\, \frac{e^{-1}}{16}\left\{-i + \frac{6}{i} + \frac{15i}{-1} + \frac{15}{i}\right\} \quad [\because i^2 = -1]$$

$$= \frac{e^{-1}}{6 \times 16}\left\{-i - \frac{6i^2}{i} - 15i - \frac{15i^2}{i}\right\} \quad [\because i^2 = -1]$$

$$= \frac{e^{-1}}{6 \times 16}\left\{-i - 6i - 15i - 15i\right\}$$

$$= \frac{-37\, ie^{-1}}{96}$$

Example : 4 **Find the residue of the following functions.**

(a) $f(z) = \dfrac{z^4}{z^2 + a^2}$

(b) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2 + 4)}$

(c) $f(z) = \dfrac{z^2 + 2}{z - 1}$

(d) $f(z) = \dfrac{1}{z^2 (z - 1)}$

(e) $f(z) = \left(\dfrac{z+1}{z-1}\right)^2$

(f) $f(z) = \dfrac{z^2 + 16}{(z - i)^2 (z + 3)}$

(g) $f(z) = \dfrac{z^3}{(2z + 1)^3}$

(h) $f(z) = \dfrac{\sin z}{z^2}$

(i) $f(z) = \operatorname{sech} z$

(j) $f(z) = \dfrac{1 - \cos z}{z^2}$

(k) $f(z) = \dfrac{e^z}{z^2 + \pi^2}$

(l) $f(z) = \dfrac{\ln z}{(z^2 + 1)^2}$

Solution :

(a) Given that $f(z) = \dfrac{z^4}{z^2 + a^2} = \dfrac{z^4}{z^2 - i^2 a^2} = \dfrac{z^4}{z^2 - (ia)^2} = \dfrac{z^4}{(z + ia)(z - ia)}$

therefore $(z + ia)(z - ia) = 0$ will give the poles

$\therefore\ z + ia = 0, \qquad z - ia = 0$

$\Rightarrow z = -ia, \qquad \Rightarrow z = ia$

therefore the poles are $z = ia$ and $z = -ia$

$$\therefore \text{Res}(ia) = \lim_{z \to ia} (z - ia) f(z)$$

$$= \lim_{z \to ia} (z - ia) \frac{z^4}{(z + ia)(z - ia)}$$

$$= \lim_{z \to ia} \frac{z^4}{z + ia}$$

$$\Rightarrow \text{Res}(ia) = \frac{(ia)^4}{ia + ia} = \frac{i^4 a^4}{2ia} = \frac{i^3 a^3}{2} = \frac{-ia^3}{2} \quad [\because i^2 = -1]$$

$$\text{Again, Res}(-ia) = \lim_{z \to -ia} \{z - (-ia)\} f(z)$$

$$= \lim_{z \to -ia} (z + ia) \frac{z^4}{(z + ia)(z - ia)}$$

$$= \lim_{z \to -ia} \frac{z^4}{z - ia}$$

$$= \frac{(-ia)^4}{-ia - ia}$$

$$= \frac{i^4 a^4}{-2ia}$$

$$= \frac{-a^3}{2} . i^3$$

$$= \frac{-a^3}{2} (-i) \quad [\because i^2 = -1]$$

$$= \frac{ia^3}{2}$$

Solution : 4(b)

Given that

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} = \frac{z^2 - 2z}{(z+1)^2 (z^2 - i^2 2^2)} = \frac{z^2 - 2z}{(z+1)^2 (z^2 - (i\,2)^2)} = \frac{z^2 - 2z}{(z+1)^2 (z - i2)(z + i2)}$$

It is clear that $z = -1$ is a pole of order 2 and $z = i2$ and $z = -i2$ are simple poles

$$\therefore \quad \text{Res}(-1) = \lim_{z \to -1} \frac{1}{(2-1)!} \frac{d}{dz} [\{z - (-1)\}^2 f(z)] \quad \left[\because \text{Res}(a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)\right]$$

$$= \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} \left\{(z+1)^2 \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}\right\}$$

$$= \lim_{z \to -1} \frac{d}{dz} \left(\frac{z^2 - 2z}{z^2 + 4}\right)$$

$$= \lim_{z \to -1} \frac{(z^2+4) . (2z-2) - (z^2 - 2z) . 2z}{(z^2+4)^2}$$

$$= \frac{\{(-1)^2 + 4\} \{2(-1) - 2\} - \{(-1)^2 - 2(-1)\} . 2(-1)}{\{(-1)^2 + 4\}^2}$$

$$= \frac{(1+4)(-2-2) - (1+2)(-2)}{(1+4)^2}$$

$$= \frac{-20 + 6}{25} = \frac{-14}{25}$$

Now, $\text{Res}(2i) = \lim_{z \to 2i} [(z - 2i) f(z)]$

$$= \lim_{z \to 2i} \left[(z - 2i) . \frac{z^2 - 2z}{(z-2i)(z+2i)(z+1)^2}\right]$$

$$= \lim_{z \to 2i} \frac{z^2 - 2z}{(z+2i)(z+1)^2}$$

$$= \frac{(2i)^2 - 2.2i}{(2i+2i)(2i+1)^2}$$

$$= \frac{4.i^2 - 4i}{4i(4i^2 + 2.2i.1 + 1)}$$

$$= \frac{4i(i-1)}{4i(4(-1) + 4i + 1)} \quad [\because i^2 = -1]$$

$$= \frac{i-1}{4i-3}$$

$$= \frac{(i-1)(4i+3)}{(4i-3)(4i+3)}$$

$$= \frac{4i^2+3i-4i-3}{(4i)^2-3^2}$$

$$= \frac{4(-1)-i-3}{16.i^2-9}$$

$$= \frac{-7-i}{-16-9}$$

$$= \frac{7+i}{25}$$

Again $\operatorname{Res}(-2i) = \lim\limits_{z \to -2i} [\{z-(-2i)\} f(z)]$

$$= \lim_{z \to -2i} (z+2i) \frac{z^2-2z}{(z-2i)(z+2i)(z+1)^2}$$

$$= \frac{(-2i)^2-2.(-2i)}{(-2i-2i)(-2i+1)^2}$$

$$= \frac{4i^2+4i}{-4i((-2i)^2+2.(-2i)1+1^2)}$$

$$= \frac{4i(i+1)}{-4i(4i^2-4i+1)}$$

$$= \frac{i+1}{-(-4-4i+1)} \quad [\because i^2 = -1]$$

$$= \frac{(i+1)(4i-3)}{(-)(-1)(4i+3)(4i-3)}$$

$$= \frac{4i^2-3i+4i-3}{4^2i^2-3^2}$$

$$= \frac{-4+i-3}{-16-9} \quad [\because i^2 = -1]$$

$$= \frac{-(7-i)}{-25}$$

$$= \frac{7-i}{25}$$

Solution : 4(c)

Given that $f(z) = \dfrac{z^2+2}{z-1}$

Here $z = 1$ is a simple pole, and $f(z) = \dfrac{\varphi(z)}{\psi(z)}$; $\varphi(1) \neq 0$, where $\varphi(z) = z^2+2$, $\psi(z) = z-1$.

Therefore $\operatorname{Res}(1) = \dfrac{\varphi(1)}{\psi'(1)}$

$$= \frac{1^2+2}{1} \quad [\because \psi'(z) = 1 \Rightarrow \psi'(1) = 1]$$

Solution : 4(d)

Given that $f(z) = \frac{1}{z^2(z-1)}$; where z = 0 is a pole of order 2 and z = 1 is a simple pole.

Therefore $\text{Res}(0) = \lim_{z \to 0} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} (z-0)^2 f(z)$

$$= \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz} z^2 \cdot \frac{1}{z^2(z-1)}$$

$$= \lim_{z \to 0} \frac{d}{dz}\left(\frac{1}{z-1}\right)$$

$$= \lim_{z \to 0} \left[\frac{-1}{(z-1)^2}\right]$$

$$= \frac{-1}{(0-1)^2} = \frac{-1}{1} = -1$$

Again $\text{Res}(1) = \lim_{z \to 1} (z-1) f(z)$

$$= \lim_{z \to 1} (z-1) \frac{1}{z^2(z-1)}$$

$$= \lim_{z \to 1} \frac{1}{z^2}$$

$$= \frac{1}{1^2} = 1$$

(e) Given that $f(z) = \left(\frac{z+1}{z-1}\right)^2 = \frac{(z+1)^2}{(z-1)^2}$; where z = 1 is a pole of order 2. Therefore

$\text{Res}(1) = \lim_{z \to 1} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} (z-1)^2 f(z)$

$$= \lim_{z \to 1} \frac{1}{1!} \frac{d}{dz} (z-1)^2 \frac{(z+1)^2}{(z-1)^2}$$

$$= \lim_{z \to 1} \frac{d}{dz} (z+1)^2$$

$$= \lim_{z \to 1} 2(z+1)$$

$$= 2(1+1)$$

$$= 4$$

(f) Given that $f(z) = \frac{z^2+16}{(z-i)^2(z+3)}$ where z = i is a pole of order 2 (Double Pole) and z = −3 is a simple pole

$\therefore$ $\text{Res}(i) = \lim_{z \to i} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \{(z-i)^2 f(z)\}$

$$= \lim_{z \to i} \frac{1}{1!} \frac{d}{dz} \{(z-i)^2 \frac{z^2+16}{(z-i)^2(z+3)}\}$$

$$= \lim_{z \to i} \frac{(z+3)\frac{d}{dz}(z^2+16) - (z^2+16)\frac{d}{dz}(z+3)}{(z+3)^2}$$

Solution : 4(f)

$$= \lim_{z\to i} \frac{(z+3).2z-(z^2+16).1}{(z+3)^2}$$

$$= \lim_{z\to i} \frac{2z^2+6z-z^2-16}{(z+3)^2}$$

$$= \lim_{z\to i} \frac{z^2+6z-16}{(z+3)^2}$$

$$= \frac{i^2+6i-16}{(i+3)^2}$$

$$= \frac{-1+6i-16}{i^2+2.i.3+3^2}$$

$$= \frac{-17+6i}{-1+6i+9}$$

$$= \frac{(6i-17)(6i-8)}{(6i+8)(6i-8)}$$

$$= \frac{36i^2-102i-48i+136}{36i^2-64}$$

$$= \frac{-36-150i+136}{-36-64} \quad [\because i^2=-1]$$

$$= \frac{100-150i}{-100}$$

$$= -1+\frac{3}{2}i$$

Again Res $(-3) = \lim_{z\to -3} \{z-(-3)\} f(z)$

$$= \lim_{z\to -3} (z+3)\frac{z^2+16}{(z-i)^2(z+3)}$$

$$= \frac{(-3)^2+16}{(-3-i)^2} = \frac{9+16}{(3+i)^2}$$

$$= \frac{9+16}{9+6i+i^2}$$

$$= \frac{25}{8+6i} \quad [\because i^2=-1]$$

$$= \frac{25(8-6i)}{(8+6i)(8-6i)}$$

$$= \frac{200-150i}{64-36i^2}$$

$$= \frac{200-150i}{64+36}$$

$$= \frac{200-150i}{100}$$

$$= 2-\frac{3}{2}i.$$

Solution : 4(g)

Given that $f(z) = \frac{z^3}{(2z+1)^3}$

For pole $(2z+1)^3 = 0$

$\Rightarrow z = \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}$

$\therefore z = \frac{-1}{2}$ is a pole of order 3

$$\therefore \text{Res}\left(\frac{-1}{2}\right) = \lim_{z\to -1/2} \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}}[\{z-(-1/2)\}^3.f(z)]$$

$$= \lim_{z\to -1/2} \frac{1}{2!}\frac{d^2}{dz^2}\left[\left(z+\frac{1}{2}\right)^3 \cdot \frac{z^3}{(2z+1)^3}\right]$$

$$= \lim_{z\to -1/2} \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{(2z+1)^3}{2^3} \cdot \frac{z^3}{(2z+1)^3}\right]$$

$$= \frac{1}{2}\cdot\frac{1}{8}\lim_{z\to -1/2} \frac{d^2}{dz^2}(z^3)$$

$$= \frac{1}{16}\lim_{z\to -1/2} 6z$$

$$= \frac{6}{16}\left(\frac{-1}{2}\right)$$

$$= \frac{-3}{16}$$

(h) Given that $f(z) = \frac{\sin z}{z^2}$ where $z = 0$, is a pole of order 2.

$$\lim_{z\to 0} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}} \{(z-0)^2.f(z)\}$$

$$\therefore \text{Res}(0) = \lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\{z^2.\frac{\sin z}{z^2}\}$$

$$= \lim_{z\to 0}\frac{d}{dz}(\sin z)$$

$$= \lim_{z\to 0}(\cos z)$$

$$= \cos 0 = 1$$

(i) Given that $f(z) = \text{sech} z = \frac{1}{\cosh z}$. It is clear that $\cosh z = 0$ will give the pc

$\therefore \cosh z = 0$

$\Rightarrow \frac{1}{2}(e^z + e^{-z}) = 0$

$\Rightarrow e^z + \frac{1}{e^z} = 0$

$\Rightarrow \frac{e^z.e^z+1}{e^z} = 0$

$\Rightarrow e^{2z} + 1 = 0$

$\Rightarrow e^{2z} = -1 = -1 + i.0 = \cos\pi + i\sin\pi$

$= \cos(2n\pi + \pi) + i\sin(2n\pi+\pi) = \cos(2n+1)\pi + i\sin(2n+1)\pi$

$= e^{i(2n+1)\pi}$

$\Rightarrow \quad 2z = i\,(2n+1)\pi$

$\Rightarrow \quad z = i(2n+1)\frac{\pi}{2}\,;\ n = 0 \pm 1, \pm 2, \pm 3, \ldots\ldots\ldots\ldots$

Therefore $z = i(2n+1)\frac{\pi}{2}$ is a simple pole for $f(z)$

$$\therefore \quad \text{Res}\,[i\,(2n+1)\tfrac{\pi}{2}] = \lim_{z \to i(2n+1)^{\pi}/_{2}} \{z - i(2n+1)\tfrac{\pi}{2}\}.f(z)$$

$$= \lim_{z \to i(2n+1)^{\pi}/_{2}} [z - i(2n+1)\tfrac{\pi}{2}] \cdot \frac{1}{\cosh z} \qquad [\tfrac{0}{0}\text{ form}]$$

$$= \lim_{z \to i(2n+1)^{\pi}/_{2}} \frac{1-0}{\sinh z} \qquad [\text{by L' Hospital rule}]$$

$$= \frac{1}{\sinh\,[i\,(2n+1)\frac{\pi}{2}]}$$

$$= \frac{1}{i\sinh\,(2n+1)\frac{\pi}{2}}\,;\ n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots\ldots\ldots$$

Solution : 4(j)

Given that $f(z) = \frac{1-\cos z}{z^2}$; where $z = 0$ is a double pole

$$\therefore \quad \text{Res}\,(0) = \lim_{z \to 0} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\{(z-0)^2 . f(z)\}$$

$$= \lim_{z \to 0} \frac{1}{1!}\frac{d}{dz}\left(z^2 . \frac{1-\cos z}{z^2}\right)$$

$$= \lim_{z \to 0} \frac{d}{dz}(1-\cos z)$$

$$= \lim_{z \to 0} (0 + \sin z)$$

$$= \sin 0$$

$$= 0$$

(k) Given that $f(z) = \frac{e^z}{z^2+\pi^2} = \frac{e^z}{z^2 - i^2\pi^2} = \frac{e^z}{(z+i\pi)\,(z-i\pi)}$

$\therefore \quad z = i\pi$ and $z = -\,i\pi$ are two poles of $f(z)$.

$$\therefore \quad \text{Res}\,(i\pi) = \lim_{z \to i\pi} (z - i\pi)\,f(z)$$

$$= \lim_{z \to i\pi} (z - i\pi)\frac{e^z}{(z-i\pi)\,(z+i\pi)}$$

$$= \lim_{z \to i\pi} \frac{e^z}{(z+i\pi)}$$

$$= \frac{e^{i\pi}}{i\pi + i\pi}$$

$$= \frac{e^{i\pi}}{2i\pi} = \frac{\cos\pi + i\sin\pi}{2\,i\pi} = \frac{-1+i.0}{2i\pi} = \frac{i^2}{2\pi i} = \frac{i}{2\pi} \quad [\because i^2 = -1]$$

$$\text{Again Res}(-i\pi) = \lim_{z\to -i\pi} \{z-(-i\pi)\} f(z)$$

$$= \lim_{z\to -i\pi} (z+i\pi)\frac{e^z}{(z+i\pi)(z-i\pi)}$$

$$= \frac{e^{-i\pi}}{-i\pi - i\pi}$$

$$= \frac{e^{-i\pi}}{-2i\pi} = \frac{e^{i(-\pi)}}{-2\pi i}$$

$$= \frac{\cos(-\pi) + i\sin(-\pi)}{-2i\pi}$$

$$= \frac{\cos\pi - i\sin\pi}{-2i\pi}$$

$$= \frac{-1 - i.0}{-2i\pi} = \frac{-1}{-2i\pi}$$

$$= \frac{1}{2i\pi} = \frac{-i^2}{2i\pi} = \frac{-i}{2\pi}$$

Solution : 4(l)

$$\text{Given } f(z) = \frac{lnz}{(z^2+1)^2} = \frac{lnz}{(z^2-i^2)^2} = \frac{lnz}{(z+i)^2(z-i)^2}$$

$\therefore$ $z = i$ and $z = -i$ are two poles of order 2.

$$\therefore \text{Res}(i) = \lim_{z\to i} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\{(z-i)^2 f(z)\}$$

$$= \lim_{z\to i} \frac{1}{1!}\frac{d}{dz}\left\{(z-i)^2\frac{lnz}{(z+i)^2(z-i)^2}\right\}$$

$$= \lim_{z\to i} \frac{(z+i)^2\frac{d}{dz}lnz - lnz\frac{d}{dz}(z+i)^2}{((z+i)^2)^2}$$

$$= \lim_{z\to i} \frac{(z+i)^2 . \frac{1}{z} . - lnz.2(z+i).1}{(z+i)^4}$$

$$= \frac{(2i)^2 . \frac{1}{i} - 2.(i+i)\, lni}{(i+i)^4}$$

$$= \frac{4i - 4i\, lni}{16(i^4)}$$

$$= \frac{4i - 4i ln(0+i.1)}{16}$$

$$= \frac{4i - 4i ln\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{16}$$

$$= \frac{4i - 4i\, ln.e^{\frac{i\pi}{2}}}{16}$$

$$= \frac{4i - 4i \times i\frac{\pi}{2}}{16}$$

$$= \frac{4i - 2i^2\pi}{16}$$

$$= \frac{4i + 2\pi}{16}$$

$$= \frac{i}{4} + \frac{\pi}{8}$$

Solution : 4(l)

Again, $\text{Res}\,(-1) = \lim\limits_{z \to -i} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} [\{z - (-i)\}^2 f(z)]$

$$= \lim_{z \to -i} \frac{1}{1!} \frac{d}{dz} \{(z+i)^2 \cdot \frac{lnz}{(z-i)^2 (z+i)^2}\}$$

$$= \lim_{z \to -i} \frac{d}{dz} \{\frac{lnz}{(z-i)^2}\}$$

$$= \lim_{z \to -i} \frac{(z-i)^2 \cdot \frac{1}{z} - lnz \,.\, 2(z-i).1}{(z-i)^4}$$

$$= \lim_{z \to -i} \frac{\frac{(z-i)^2}{z} - 2\,(z-i)\,.\,lnz}{(z-i)^4}$$

$$= \frac{\frac{(-2i)^2}{-i} - 2(-2i)\,ln(-i)}{(-2i)^4}$$

$$= \frac{-4i + 4i\,ln\,\{0 - i.1\}}{16}$$

$$= \frac{-4i + 4i\,ln\,\{\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\}}{16}$$

$$= \frac{-4i + 4i\,ln\,e^{\frac{-i\pi}{2}}}{16}$$

$$= \frac{-4i + 4i\left(\frac{-i\pi}{2}\right)}{16}$$

$$= \frac{-4i - 2i^2\pi}{16}$$

$$= \frac{-4i + 2\pi}{16}$$

$$= \frac{\pi}{8} - \frac{i}{4}$$

Example : 5 Find the residue of $f(z) = \dfrac{z^3}{z^2-1}$ at $z = \infty$.

Solution : Given that $f(z) = \dfrac{z^3}{z^2-1} = \dfrac{z^3}{(z+1)(z-1)}$.

Which shows that $z = 1$ and $z = -1$ are two poles of $f(z)$.

Now, $\text{Res}(1) = \lim\limits_{z \to 1} (z-1) f(z)$

$$= \lim_{z \to 1} (z-1) \frac{z^3}{(z+1)(z-1)}$$

$$= \lim_{z \to 1} \frac{z^3}{z+1}$$

$$= \frac{1^3}{1+1} = \frac{1}{2}$$

Again, $\text{Res}(-1) = \lim\limits_{z \to -1} [z-(-1)] f(z)$

$$= \lim_{z \to -1} (z+1) \frac{z^3}{(z+1)(z-1)}$$

$$= \lim_{z \to -1} \frac{z^3}{z-1}$$

$$= \frac{(-1)^3}{-1-1} = \frac{-1}{-2} = \frac{1}{2}$$

Now, $\text{Res}(1) + \text{Res}(-1) + \text{Res}(\infty) = 0$ [$\because$ Summation of all residues = 0]

$\Rightarrow \frac{1}{2} + \frac{1}{2} + \text{Res}(\infty) = 0$

$\Rightarrow \text{Res}(\infty) = -1$

Example : 6 : **By using Cauchy's residue theorem evaluate the following integrals.**

(a) $\oint_C \dfrac{e^{-iz}}{(z+3)(z-i)^2} dz$; $C = \{z : z = 1 + 2e^{i\theta}, 0 \le \theta \le 2\pi\}$

(b) $\oint_C \dfrac{1}{z^2+9} dz$; **C is the square whose sides are** $x = \pm 2, y = \pm 2$

(c) $\oint_C \dfrac{1}{z(z^2+9)} dz$; **C is the square whose sides are** $x = \pm 2, y = \pm 2$,

(d) $\oint_C \dfrac{1}{(z^2+1)(z^2+9)} dz$; **C is the square whose sides are** $x = \pm 2, y = \pm 2$,

(e) $\oint_C \dfrac{e^{-z}}{(z-1)^2} dz$; $C = \{z : |z| = 3\}$

(f) $\oint_C z^2 e^{\frac{1}{z}} dz$; $C = \{z : |z| = 3\}$

(g) $\oint_C \dfrac{\sin 3z}{(z-\frac{\pi}{4})^4} dz$; $C = \{(x, y) : |x| \le 2, |y| \le 2\}$ **positively oriented.**

(h) $\oint_c \frac{2z^2 - z + 1}{(2z-1)(z+1)^2}\,dz$; $C = \{(r, \theta) : r = 2\cos\theta, 0 \le \theta \le 2\pi\}$.

(i) $\oint_c \frac{1}{z(2z-5)(z-4)}\,dz$; $C = \{z : |z+2| + |z-2| = 6\}$

(j) $\oint_c \frac{e^{3z}}{z - \pi i}\,dz$; $C = \{z : |z-2| + |z+2| = 6\}$

Solution :

Solution : 6(a)

Given that $\oint_c \frac{e^{-iz}\,dz}{(z+3)(z-i)^2} = I$ [Say] .... (i)

Here C is $z = 1 + 2e^{i\theta}$; $0 \le \theta \le 2\pi$

$\Rightarrow z - 1 = 2e^{i\theta}$

$\Rightarrow |z-1| = |2e^{i\theta}|$

$\Rightarrow |x + iy - 1| = |2|\,|\cos\theta + i\sin\theta|$ $[\because e^{i\theta} = \cos\theta + i\sin\theta]$

$\Rightarrow \sqrt{(x-1)^2 + y^2)} = 2\sqrt{\cos^2\theta + \sin^2\theta}$ $[\because |x+iy| = \sqrt{x^2+y^2}]$

$\Rightarrow (x-1)^2 + (y-0)^2 = 2^2$ which is a circle centred at $(1, 0)$ and with radius 2

From (i) it is clear that

$z = -3$ is a simple pole and $z = i$ is a pole of order two.

But only $z = i$ lies inside C

Therefore by Cauchy's residue theorem we have

$I = \oint_c \frac{e^{-iz}\,dz}{(z+3)(z-i)^2} = 2\pi i \times$ sum of the residues .... (ii)

Now, $\text{Res}(i) = \lim_{z\to i} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \{(z-i)^2 f(z)\}$

$= \lim_{z\to i} \frac{1}{1!} \frac{d}{dz} \{(z-i)^2 . \frac{e^{-iz}}{(z+3)(z-i)^2}\}$

$= \lim_{z\to i} \frac{d}{dz}\left[\frac{e^{-iz}}{(z+3)}\right]$

$= \lim_{z\to i} \frac{(z+3).(-ie^{-iz}) - e^{-iz}.1}{(z+3)^2}$

$= \frac{(i+3).(-i).e^{-i.i} - e^{-i.i}}{(i+3)^2}$

$= \frac{(i+3)(-i)e^{-i^2} - e^{-i^2}}{(i+3)^2}$

$= \frac{(i+3).(-i).e - e}{i^2 + 6i + 9}$ $[\because i^2 = -1]$

$= \frac{e(-i^2 - 3i - 1)}{-1 + 6i + 9}$

$= \frac{e(1 - 3i - 1)}{6i + 8}$

$= \frac{-3ie(8 - 6i)}{(8 + 6i)(8 - 6i)}$

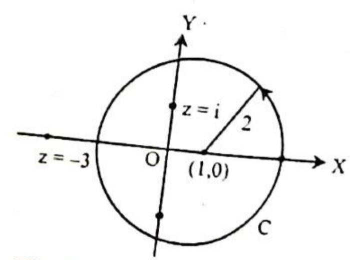

Fig : 3 : $(x-1)^2 + (y-0)^2 = 2^2$

$$= \frac{-3ie(8-6i)}{64-36i^2}$$

$$= \frac{-3ie(8-6i)}{100} \quad [\because i^2 = -1]$$

Now by (ii)

$$I = 2\pi i \times \frac{-3ie(8-6i)}{100}$$

$$= \frac{-6\pi i^2 e(8-6i)}{100}$$

$$= \frac{6\pi e.2(4-3i)}{100}$$

$$= \frac{3\pi e(4-3i)}{25}$$

Solution : 6(b)

Let $I = \oint_c \frac{1}{z^2+9} dz$

$$= \oint_c \frac{1}{z^2 - i^2 3^2} dz$$

$$= \oint_c \frac{1}{(z-i3)(z+i3)} dz$$

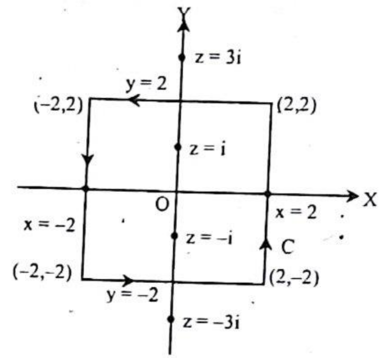

Fig : 4

Here C is the square sides $x = \pm 2$, $y = \pm 2$

$z = 3i$ and $z = -3i$ are two poles of $f(z) = \frac{1}{z^2+9} = \frac{1}{(z+3i)(z-i3)}$ both of them lie outside C. therefore

Res (3i) = 0 and Res (−3i) = 0

Now by Cauchy's Residue theorem

$$I = \oint_c \frac{1}{z^2+9} dz$$

$$= 2\pi i\,[\text{Res}(3i) + \text{Res}(-3i)]$$

$$= 2\pi i\,[0+0] = 2\pi i \times 0 = 0$$

$$\Rightarrow \oint_c \frac{1}{z^2+9} dz = 0$$

(c) Given that $\oint_c \frac{1}{z(z^2+9)} dz = \oint_c \frac{1}{z(z-3i)(z+3i)} dz$ .... (i)

Whose poles are $z = 0$, $z = 3i$ and $z = -3i$, among them only $z = 0$ lies inside C. [Fig: 4]. Therefore

$$\text{Res}(3i) = 0,\ \text{Res}(-3i) = 0 \text{ and } \text{Res}(0) = \lim_{z\to 0} (z-0)\cdot\frac{1}{z(z^2+9)}$$

$$= \lim_{z\to 0} \frac{1}{z^2+9}$$

$$= \frac{1}{9}$$

Then by Cauchy's residue theorem we have

$$\oint_c \frac{1}{z(z^2+9)} dz = 2\pi i\,[\text{Res}(3i) + \text{Res}(-3i) + \text{Res}(0)]$$

$$= 2\pi i\left[0+0+\frac{1}{9}\right] = \frac{2\pi i}{9}$$

Solution : 6(d)

(d) Given that $\oint_c \frac{1}{(z^2+1)(z^2+9)} dz = \oint_c \frac{1}{(z^2-i^2)(z^2-i^2.3^2)} dz = \oint_c \frac{1}{(z+i)(z-i)(z-3i)(z+3i)} dz$

Whose poles are $z = i$, $z = -i$, $z = 3i$ and $z = -3i$; among them only $z = i$ and $z = -i$ lies inside C (Fig :4)

Therefore Res $(3i) = 0$, Res $(-3i) = 0$

$$\text{Res}(i) = \lim_{z\to i} (z-i)\frac{1}{(z-i)(z+i)(z-3i)(z+3i)}$$
$$= \lim_{z\to i} \frac{1}{(z+i)(z^2+9)}$$
$$= \frac{1}{(i+i)(i^2+9)}$$
$$= \frac{1}{2i\times 8} \quad [\because i^2 = -1]$$
$$= \frac{-i^2}{16\,i}$$
$$= \frac{-i}{16}$$

Similarly, $\text{Res}(-i) = \frac{i}{16}$

Therefore by Cauchy's residue theorem we have

$$\oint_c \frac{1}{(z^2+1)(z^2+9)} dz = 2\pi i\,[\text{Res}(3i) + \text{Res}(-3i) + \text{Res}(i) + \text{Res}(-i)]$$
$$= 2\pi i\,[0 + 0 - \frac{i}{16} + \frac{i}{16}]$$
$$= 2\pi i \times 0$$
$$\Rightarrow \oint_c \frac{1}{(z^2+1)(z^2+9)} dz = 0$$

(e) Given that $\oint_c \frac{e^{-z}}{(z-1)^2} dz$; $C = \{z : |z| = 3\}$

Let $I = \oint_c f(z)\, dz$; where $f(z) = \frac{e^{-z}}{(z-1)^2}$

Here $z = 1$ is a pole of order 2

But in C , $|z| = 3$

$\Rightarrow |x + iy| = 3$

$\Rightarrow \sqrt{x^2+y^2} = 3$

$\Rightarrow (x-0)^2 + (y-0)^2 = 3^2$

Now $\text{Res}(1) = \lim_{z\to 1} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\{(z-1)^2 f(z)\}$

$$= \lim_{z\to 1} \frac{1}{1!}\frac{d}{dz}\{(z-1)^2\frac{e^{-z}}{(z-1)^2}\}$$
$$= \lim_{z\to 1}(-e^{-z})$$
$$= -e^{-1}$$
$$= \frac{-1}{e}$$

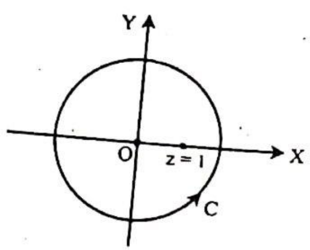

Fig : 5

Therefore by Cauchy's residue theorem we have

$$I = \oint_c \frac{e^{-z}}{(z-1)^2}\,dz = 2\pi i \text{ Res } (1)$$

$$= 2\pi i \left(\frac{-1}{e}\right)$$

$$\therefore \oint_c \frac{e^{-z}}{(z-1)^2}\,dz = \frac{-2\pi i}{e}$$

Solution : 6(f)

Given that $\oint_c z^2 e^{\frac{1}{z}} dz$; $C = \{z : |z| = 3\}$

Let $I = \oint_c f(z)\,dz$ ....(i)

where $f(z) = z^2 e^{\frac{1}{z}}$

Now $f(z) = z^2 e^{\frac{1}{z}} = z^2 \left[1 + \frac{\frac{1}{z}}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^4}{4!} + \ldots\ldots\ldots\right]$ $\left[\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\right]$

$$\Rightarrow f(z) = z^2 \left[1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \frac{1}{24z^4} + \ldots\ldots\ldots\right]$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6}\cdot\frac{1}{z} + \frac{1}{24z^2} + \ldots\ldots\ldots\ldots$$

Where, $\frac{1}{6}$ is the residue at the pole $z = 0$. [since $\frac{1}{6}$ is the co-efficient of $\frac{1}{z}$]

Therefore, by Cauchy's residue theorem we have

$$I = \oint_c f(z)\,dz = \oint_c z^2 e^{\frac{1}{z}}\,dz = 2\pi i \text{ Res } (0)$$

$$\therefore \oint_c z^2 e^{\frac{1}{z}}\,dz = 2\pi i \cdot \frac{1}{6} = \frac{\pi i}{3}$$

(g) Given that $\oint_c \frac{\sin 3z}{(z - \frac{\pi}{4})^4}\,dz$ and $C = \{(x, y) : |x| \le 2, |y| \le 2\} \Rightarrow C = \{(x, y) : -2 \le x \le 2, -2 \le y \le 2\}$

Here $z = \frac{\pi}{4}$ is a pole of order 4 inside C.

$$\text{Therefore, Res}\left(\frac{\pi}{4}\right) = \lim_{z \to \frac{\pi}{4}} \frac{1}{(4-1)!} \frac{d^{4-1}}{dz^{4-1}} \left\{(z - \frac{\pi}{4})^4 \cdot \frac{\sin 3z}{(z - \frac{\pi}{4})^4}\right\}$$

$$= \lim_{z \to \frac{\pi}{4}} \frac{1}{3!} \frac{d^3}{dz^3} (\sin 3z)$$

$$= \lim_{z \to \frac{\pi}{4}} \frac{1}{6} \frac{d^2}{dz^2} (3 \cos 3z)$$

$$= \lim_{z \to \frac{\pi}{4}} \frac{1}{6} \frac{d}{dz} (-9 \sin 3z)$$

$$= \lim_{z \to \frac{\pi}{4}} \left(\frac{-27}{6} \cos 3z\right)$$

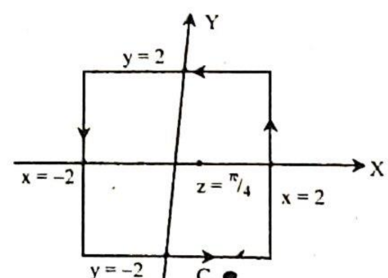

Fig : 6

$$= \frac{-9}{2} \cos \frac{3\pi}{4}$$
$$= \frac{-9}{2} \cos \left(\pi - \frac{\pi}{4}\right)$$
$$= \frac{-9}{2}\left(-\cos \frac{\pi}{4}\right)$$
$$= \frac{-9}{2} \times \frac{-1}{\sqrt{2}}$$
$$= \frac{9}{2\sqrt{2}}$$

Therefore, by Cauchy's residue theorem we have

$$\therefore \oint_c \frac{\sin 3z}{\left(z-\frac{\pi}{4}\right)^4} dz = 2\pi i \times \frac{9}{2\sqrt{2}} = \frac{9\pi i}{\sqrt{2}} = \frac{9\sqrt{2}\pi i}{\sqrt{2}\,.\sqrt{2}} = \frac{9\sqrt{2}\,\pi i}{2}$$

Solution : 6(h)

Given that $\oint_c \frac{2z^2 - z + 1}{(2z-1)(z+1)^2} dz = \oint_c \frac{2z^2 - z + 1}{2\left(z - \frac{1}{2}\right)(z+1)^2} dz$

Where, $z = \frac{1}{2}$ is a simple pole and $z = -1$ is a pole of order 2.

Here only $z = \frac{1}{2}$ lies inside

$C = \{(r, \theta) : r = 2\cos\theta\} = \{(r, \theta) : r^2 = 2r\cos\theta\}$
$= \{(r, \theta) : x^2 + y^2 = 2x\}$
$= \{(x, y) : x^2 - 2x + 1 + y^2 = 1\}$
$= \{(x, y) : (x-1)^2 + (y-0)^2 = 1^2\}$

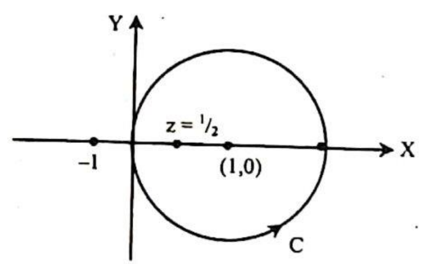

Fig : 7

$$\text{Now Res}\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) . \frac{2z^2 - z + 1}{2\left(z-\frac{1}{2}\right)(z+1)^2}$$
$$= \lim_{z\to\frac{1}{2}} \frac{2z^2 - z + 1}{2(z+1)^2}$$
$$= \frac{2 . \frac{1}{4} - \frac{1}{2} + 1}{2\left(\frac{1}{2}+1\right)^2} = \frac{1}{2\left(\frac{3}{2}\right)^2} = \frac{4}{2 \times 9} = \frac{2}{9}$$

Therefore by Cauchy's residue theorem we have $\oint_c \frac{2z^2 - z + 1}{(2z-1)(z+1)^2} dz = 2\pi i \times \frac{2}{9} = \frac{4\pi i}{9}$

(i) Given that $\oint_c \frac{1}{z(2z-5)(z-4)} dz = \oint_c \frac{1}{2z\left(z-\frac{5}{2}\right)(z-4)} dz$

The poles are $z = 0$, $z = \frac{5}{3}$ and $z = 4$, But only $z = 0$ and $z = \frac{5}{2}$ lies inside $C = \{z : |z+2| + |z-2| = 6\}$

$$\text{Now, Res}(0) = \lim_{z\to 0} z \frac{1}{2z\left(z-\frac{5}{2}\right)(z-4)} = \lim_{z\to 0} \frac{1}{2\left(z-\frac{5}{2}\right)(z-4)}$$

$$= \frac{1}{2\left(0-\frac{5}{2}\right)(0-4)} = \frac{1}{20}$$

$$\text{Res}\left(\frac{5}{2}\right) = \lim_{z\to\frac{5}{2}} \left(z-\frac{5}{2}\right)\frac{1}{2z\left(z-\frac{5}{2}\right)(z-4)}$$

$$= \lim_{z\to\frac{5}{2}} \frac{1}{2z(z-4)}$$

$$= \frac{1}{2.\frac{5}{2}\left(\frac{5}{2}-4\right)} = \frac{1}{5.\frac{5-8}{2}} = \frac{2}{5(-3)} = \frac{-2}{15}$$

And Res (4) = 0

∴ By Cauchy's residue theorem we have

$$\oint_c \frac{1}{z(2z-5)(z-4)}dz = 2\pi i\left(\text{Res}(0) + \text{Res}\left(\frac{5}{2}\right) + \text{Res}(4)\right]$$

$$= 2\pi i.\left[\frac{1}{20} + \frac{-2}{15} + 0\right] = 2\pi i\,\frac{3-8}{60} = \frac{-10\pi i}{60} = \frac{-\pi i}{6}$$

Solution : 6(j)

Given that $\oint_c \frac{e^{3z}}{z-\pi i}dz$

whose pole is $z = \pi i$ which lies outside

$C = \{z : |z-2| + |z+2| = 6\}$;

$|z-2| + |z+2| = 6$

$\Rightarrow |x+iy-2| + |x+iy+2| = 6$

$\Rightarrow |(x-2)+iy) + |(x+2)+iy| = 6$

$\Rightarrow \sqrt{(x-2)^2+y^2} + \sqrt{(x+2)^2+y^2} = 6$

$\Rightarrow \sqrt{x^2-4x+4+y^2} = 6 - \sqrt{x^2+4x+4+y^2}$

$\Rightarrow x^2-4x+4+y^2 = 36 - 12\sqrt{x^2+4x+4+y^2} + x^2+4x+4+y^2$

$\Rightarrow -8x-36 = -12\sqrt{x^2+4x+4+y^2}$

$\Rightarrow 2x+9 = 3\sqrt{x^2+4x+4+y^2}$

$\Rightarrow 4x^2+36x+81 = 9(x^2+4x+4+y^2)$

$\Rightarrow 4x^2+36x+81 = 9x^2+36x+36+9y^2$

$\Rightarrow 5x^2+9y^2 = 45$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$

$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1^2$

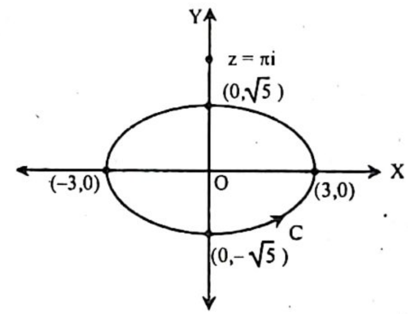

Fig : 8

∴ Res $(\pi i) = 0$

Therefore by Cauchy's residue theorem we have

$$\oint_c \frac{e^{3z}}{z-\pi i}dz = 2\pi i \times \text{Res}(\pi i) = 2\pi i \times 0$$

$$\therefore \oint_c \frac{e^{3z}}{z-\pi i}dz = 0$$

1. **Find poles and rasidue of the following functions** [নিম্নের ফাংশনগুলোর অবশেষ নির্ণয় কর]

(i) $f(z) = \dfrac{\ln(z+2)}{2z+1}$

(ii) $f(z) = \dfrac{2z+3}{z^2 - 5z + 6}$

(iii) $f(z) = \dfrac{1}{z^4 + 1}$

(iv) $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$

(v) $f(z) = \dfrac{e^{iz}}{(z^2+1)^4}$

(vi) $f(z) = \dfrac{z^3}{(z-1)(z-2)(z-3)}$

(vii) $f(z) = \dfrac{1}{z^2(z-1)}$

(viii) $f(z) = \dfrac{z^{1/4}}{z+1}$

(ix) $f(z) = \dfrac{\ln z}{(z^2+1)^2}$

(x) $f(z) = \dfrac{e^z}{z^2 - 3z + 2}$

(xi) $f(z) = \dfrac{e^z}{(z^2+\pi^2)^2}$

(xii) $f(z) = \dfrac{ze^z}{(z^2-1)}$

(xiii) $f(z) = \dfrac{e^z}{z(z^2-1)^2}$

2. **Evaluate the following integrats by cauchy's residue theorem in the indicated region** [নির্দেশিত অঞ্চলে কচির অবশেষ উপপাদ্যের মাধ্যমে নিম্নলিখিত সমাকলনগুলোর মান বের কর]

(i) $\oint_c \dfrac{e^{tz}}{(z^2+1)^2}\,dz;\ t > 0,\ C = \{z : |z| = 3\}$

(ii) $\oint_c \dfrac{e^{-z}}{z^2}\,dz;\ C = \{z : |z| = 3\}$

(iii) $\oint_c \dfrac{e^{tz}}{z^2(z^2+2z+2)}\,dz;\ C = \{z : |z| = 3\}$

(iv) $\oint_c \dfrac{z+1}{z^2-2z}\,dz;\ C = \{z : |z| = 3\}$

(v) $\oint_c \dfrac{-z^5}{z^3-1}\,dz;\ C = \{z : |z| = 3\}$

(vi) $\oint_c \dfrac{e^z}{(z^2+\pi^2)^2}\,dz;\ C = \{z : |z| = 4\}$

(vii) $\oint_c \dfrac{ze^z}{(z^2-1)}\,dz;\ C = \{z : |z| = 2\}$

(viii) $\oint_c \dfrac{e^z}{z(z-1)^2}\,dz;\ C = \{z : |z| = 2\}$

(ix) $\oint_c \dfrac{zdz}{(z^2+1)(z-3)^2}\,dz;\ C = \{z : |z| = 2\}$

(x) $\oint_c \dfrac{\cosh z}{z^3}\,dz$; C is the square with vertices at $2 + 2i$, $2 - 2i$, $-2 + 2i$ and $-2 - 2i$

(xi) $\oint_c \dfrac{e^{tz}}{z(z^2+1)}\,dz;\ t > 0$, C is the square with vertices at $\pm 1 + i$.

(xii) $\oint_c \dfrac{3z^2+2}{(z-1)(z^2+9)}\,dz;\ C = \{z : |z-2| = 2\}$

(xiii) $\oint_c \dfrac{2z^2+5}{z^2(z+2)^3(z^2+4)}\,dz;\ C = \{z : |z-2i| = 6\}$