16.10.2023

CSE – 219

concrete Mathematics

Recurrent Problems :

Tower of Brahma
→ 3 niddle of diamond
→ gold plate

→ Fibonacci series

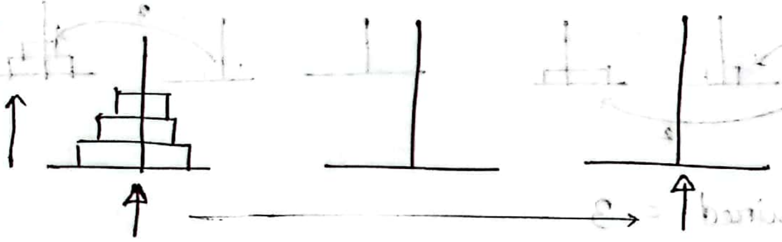
0	1	2	3
0	1	1	2

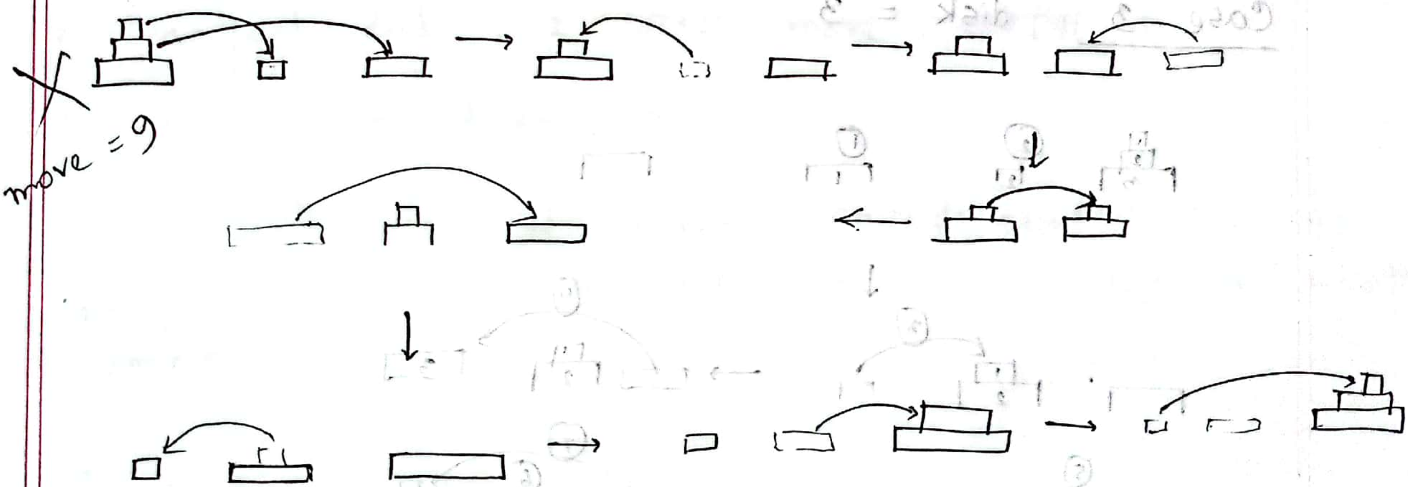
$F_n = F_{n-1} + F_{n-2}$

Tower of Hanoi : Lucas, 'Tower of Brahma' কে use করে build করে [disk = 8]

a) at a time একটা
b) ছোটর উপর বড় রাখা যাবে না

বড় disk এর উপরে ছোট disk

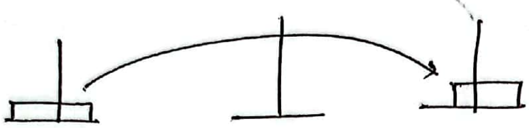




* move required
* how much time required

* When disk no. = 0 → moves req. = 0.

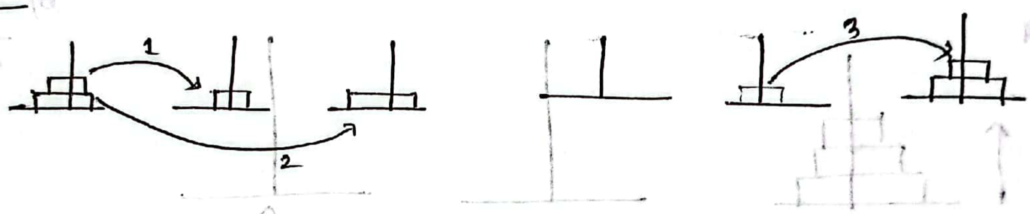
Case 1 : disk no. = 1



moves req. = 1

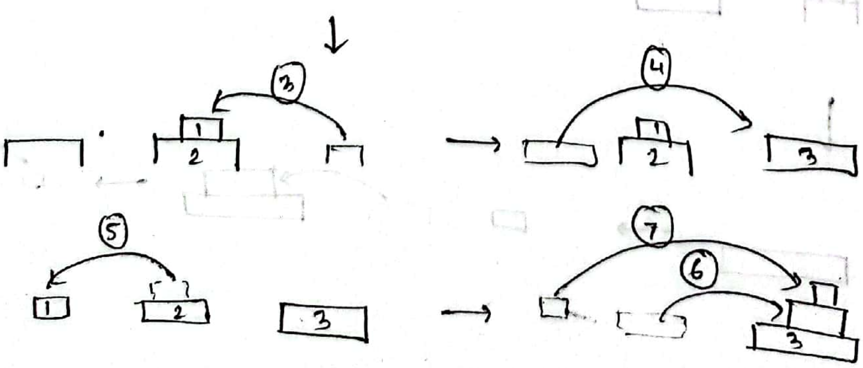
Time req. = 1 sec.

Case 2 : disk = 2



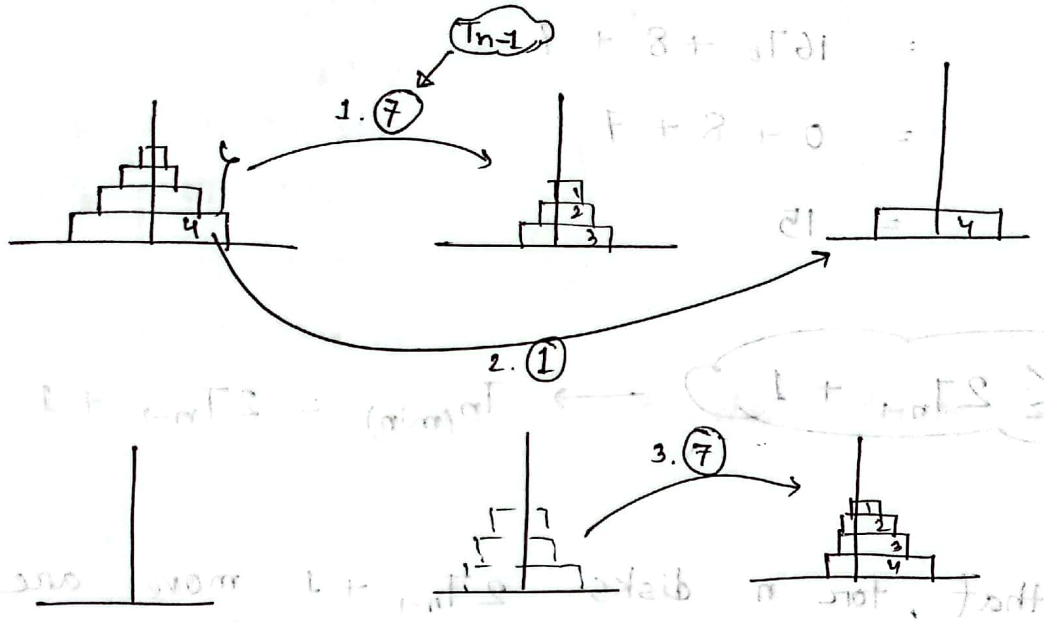
moves required = 3

Case 3 : disk = 3



Case 4 : disk $= 4$

moves $= 15$



$7 + 7 + 1$
$= 15$ moves

* largest disk , 1 বারই move করবো ।

* n no. of disks , 3 tower

T_n = no. of moves required that'll transfer n disks from one tower/peg to another

no. of moves (minimum)

$T_0 = 0$

$T_4 = 7 + 1 + 7 = 15$

$$\boxed{T_n = 2T_{n-1} + 1}$$

$T_4 = 2 \times T_3 + 1$
$= 2 \times (2T_2 + 1) + 1$
$= 4T_2 + 3$
$= 4(2T_1 + 1) + 3$

$= 8T_1 + 7$

$= 8(2T_0 + 1) + 7$

$= 16T_0 + 8 + 7$

$= 0 + 8 + 7$

$= 15$

$T_n \geq 2T_{n-1} + 1 \longrightarrow T_{n(min)} = 2T_{n-1} + 1$

Prove that, for n disks $2T_{n-1} + 1$ move are sufficient and necessary for making to the destination peg.

Closed Form :

$$
\left.
\begin{array}{l}
n=0 \rightarrow T_0 = 0 \\
n=1 \rightarrow T_1 = 1 \quad (2T_0+1) \\
n=2 \rightarrow T_2 = 3 \quad (2T_1+1) \\
n=3 \rightarrow T_3 = 7 \quad (2T_2+1) \\
n=4 \rightarrow T_4 = 15 \quad (2T_3+1) \\
n=5 \rightarrow T_5 = 31 \quad (2T_4+1) \\
n=6 \rightarrow T_6 = 63 \quad (2T_5+1)
\end{array}
\right\} 2^n - 1
$$

$\boxed{T_n = 2^n - 1} \longrightarrow$ Sol^n of closed form

Base $\longrightarrow T_0 = 2^0 - 1 = 0$.

hypothesis $\longrightarrow T_{n-1} = 2^{n-1} - 1$

Induction $\longrightarrow T_n = 2T_{n-1} + 1$

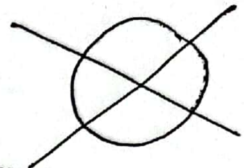
$= 2(2^{n-1} - 1) + 1$

$= 2^n - 1$

22.10.2023

Geometric problem :

How many slices of pizza can a person obtain by making n straight cut with a pizza knife ?



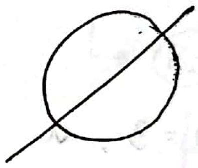
$n=2$

4 slices

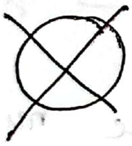
↓

What's the maximum number (L_n) of regions defined by n lines in the plane

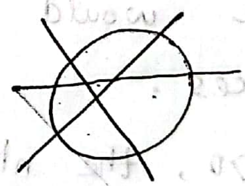
$n=0$, $L_n = 1$



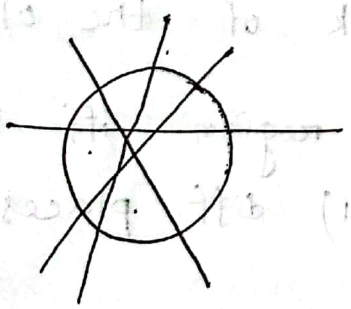
$n=1$, $L_n = 2$



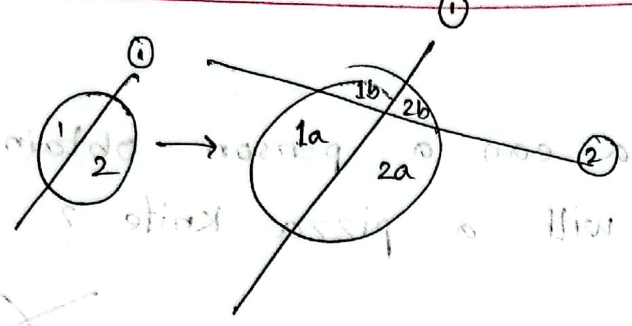
$n=2$, $L_n = 4$

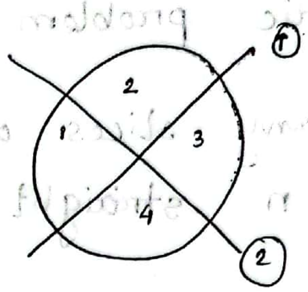


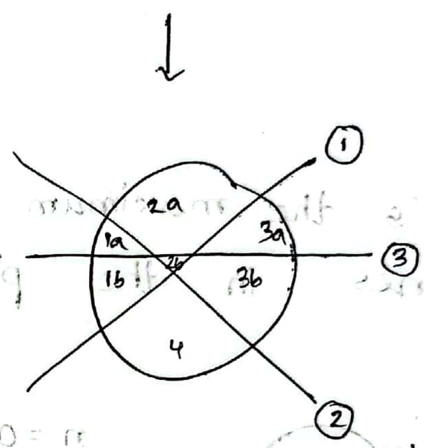
$n=3$, $L_n = 7$



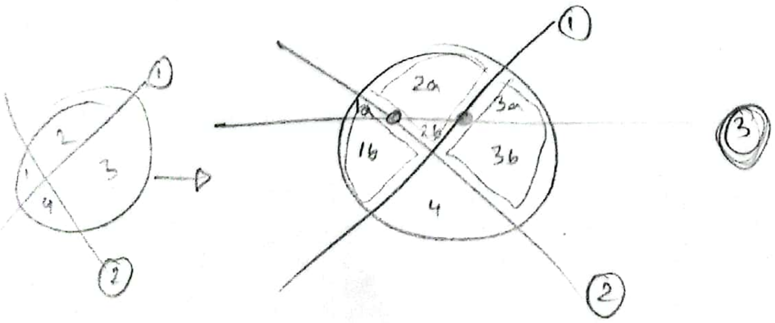
$n=4$, $L_n = 11$







1. For $n=3$, a line can split at most three old region
2. nth line would split an old regions at most two pieces.
3. For $n>0$, the nth line increases the number of regions by k iff it splits k of the old regions and it splits k old region iff it hits the previous lines in $(k-1)$ diff places.



3 টা region কে
2 টা point এ touch
করে split করেছে।

4. Two Lines can intersect in at most one point. Therefore the new line can intersect $(n-1)$ old lines in at most $(n-1)$ places.

* L_n = maximum # regions by n straight line

$n=0 \longrightarrow L_0 = 1$

$n=1 \longrightarrow L_1 = 1+1 = 2$

$n=2 \longrightarrow L_2 = 2+2 = 4$

$n=3 \longrightarrow L_3 = 4+3 = 7$

$n=4 \longrightarrow L_4 = 7+4 = 11$

$$\boxed{L_n = L_{n-1} + n}$$

↳ Recurrence sol^n

- $L_0 = 1$
- $L_n = L_{n-1} + n$ (for $n>0$)

Closed Form :

An expression for a quantity $f(n)$ is in closed form if we can compute it using at most a fixed number of well-known standard operations, independent of n.

$$L_n = L_{n-1} + n$$
$$= L_{n-2} + (n-1) + n$$
$$= L_{n-3} + (n-2) + (n-1) + n$$
$$= L_{n-4} + (n-3) + (n-2) + (n-1) + n$$

when $n = 4$, $L_4 = L_0 + 1 + 2 + 3 + 4$

For, $n \longrightarrow L_n = L_0 + \boxed{1 + 2 + 3 + \cdots + (n-3) + (n-2) + (n-1) + n}$

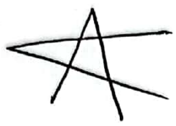
$$= L_0 + \frac{n(n+1)}{2}$$

$$\boxed{L_n = 1 + \frac{n(n+1)}{2}}$$

25. 10. 2023

self. Study : ☐ Straight line

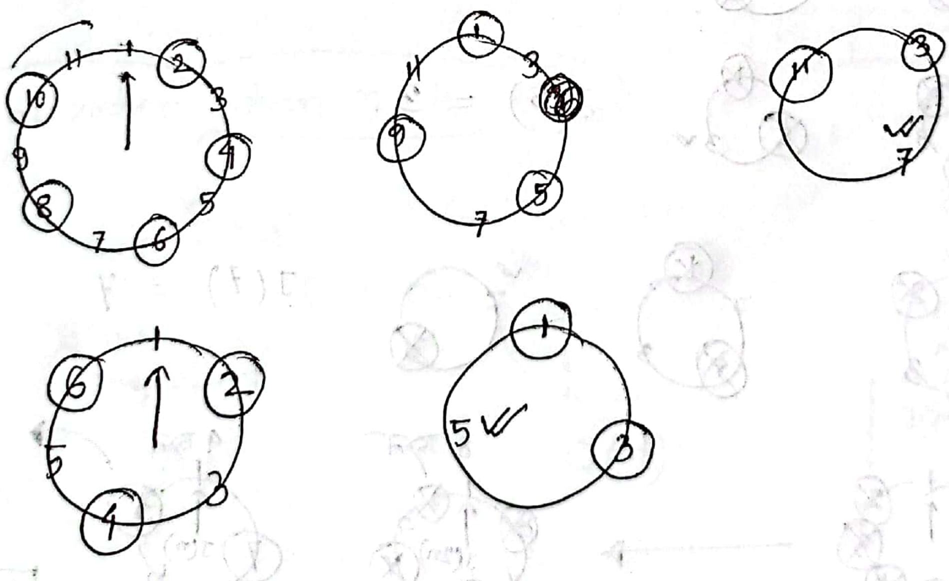
☐ If line isn't straight, line is bend



bend line ⟶ max no. of region determine, if n no. of *_* line is cut.

☐ Zigzag line

JOSEPHUS PROBLEM



$0 \leq l \leq 2^m - 1 = 3$

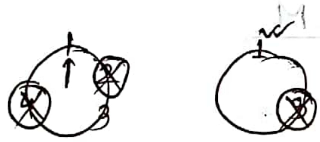
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5	7	9

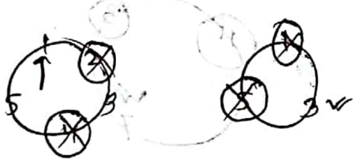
$n = 0$ ✗

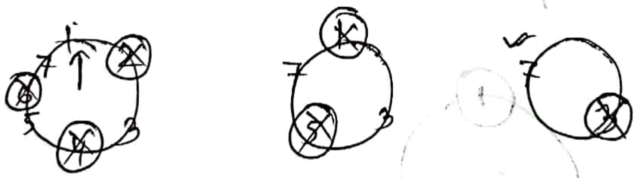
$n = 1$  1 survive থাকবে $\longrightarrow J(1) = 1$ (no of people; pos. no)

$n = 2$  $J(2) = 1$

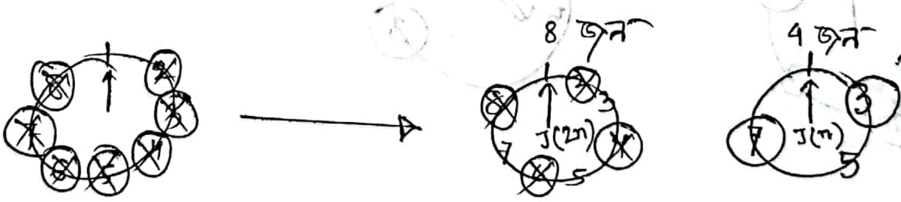
$n = 3$  $J(3) = 3$

$n = 4$  $J(4) = 1$

$n = 5$  $J(5) = 3$

$n = 7$  $J(7) = 7$

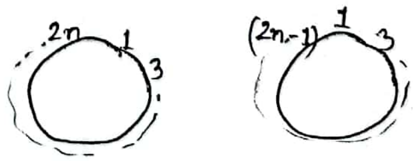
Visualization when n is even

$n = 8$ 

2nd round (pos)

$1 \to 1$ $(2*1-1)$
$3 \to 2$ $(2*2-1)$
$5 \to 3$ $(2*3-1)$
$7 \to 4$ $(2*4-1)$

$$\boxed{J(2n) = 2 * J(n) - 1}$$



Observations :

1. $J(n)$ is always odd (at least for $n = 1, 2, 3, 4, 5, 6, 7, 11$)
2. Even positions are eliminated at first round.

Note :

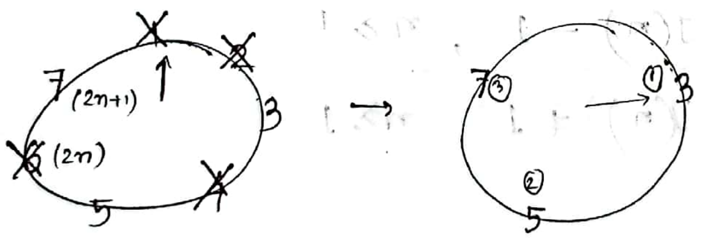
- $J(1) = 1$
- $J(2n) = 2 * J(n) - 1$

$\therefore J(10) = 2 * J(5) - 1 = 5$

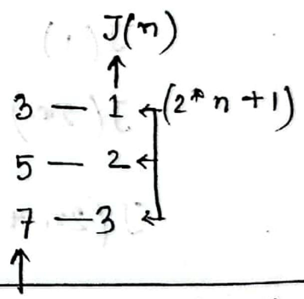
$J(20) = 2 * J(10) - 1 = 2 * 5 - 1 = 9$

$J(40) = 2 * J(10) - 1 = 9 * 2 - 1 = 17$

Visualization when n is odd



$J(2n+1)$ $\rightarrow$ $J(n)$



$$\boxed{J(2n+1) = 2 * J(n) + 1}$$

- $J(2n+1) = 2 * J(n) + 1$

$J(18) = 2\,J(9) - 1 = 2(2\,J(4) + 1) - 1 = 2(2 * 1 + 1) - 1$
$= 5$

3. For even no. of

Originally we have ~~2n~~ no. of people, after one round, it seems we have n no of people. except each ~~peop~~ person's position number has been doubled and decreased by 1.

4. For odd no, 2n+1 — — — — — — — — — n — and increased by 1.

$\boxed{\text{Recurrence problem}}$:

$$J(1) = 1$$

$$J(2n) = 2^{*} J(n) - 1 \quad , n \geq 1$$

$$J(2n+1) = 2^{*} J(n) + 1 \quad n \geq 1$$

$\boxed{\text{closed Form}}$:

1. when n is power of 2, $J(n) = 1$.
2. when $J(n)$ is 1, later $J(n+1)$, $J(n+2)$ are of consequtive odd numbers.

$J(n) = 1$

$J(2^m) = 1$ here $n = 2^m$ $(2\times 0+1)$

$J(2^m+1) = 3$ $(2\times 1+1)$

$J(2^m+3) = 5$ $(2\times 2+1)$

$J(2^m+5) = 7$ $(2\times 3+1)$

when, $n = 2^m + l$

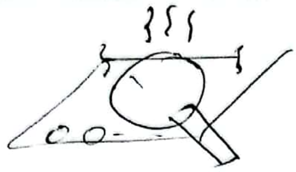
$J(n) = J(2^m + l) = 2 * l + 1$

$$\boxed{J(n) = J(2^m + l) = 2l + 1}$$

here, $2^m \leq n < 2^{m+1}$

$0 \leq l \leq 2^m - 1$

$J(100) = J(2^6 + 36) = 2 * 36 + 1 = 73$



Using the recurrence sol^n, prove by induction that, $J(\underbrace{2^m + l}) = 2l + 1$

এই part হয় $2n$ নতুবা $2n+1$

Basis : $m = 0$, $l = 0$

$$J(2^0 + 0) = J(1) = 1$$

Induction : l is even ~~/odd~~

If, $m > 0$ and $2^m + l = 2n$

$$J(2n) = 2J(n) - 1$$

$$J(2^m + l) = 2J\left(\frac{2^m}{2} + \frac{l}{2}\right) - 1$$

$$= 2J\left(2^{m-1} + \frac{l}{2}\right) - 1$$

$$= 2\left(\cancel{2} \cdot \frac{l}{\cancel{2}} + 1\right) - 1$$

$$= 2l + 2 - 1$$

$$= 2l + 1$$

when l is odd, $2^m + l = 2n + 1 \rightarrow n = \frac{2^m + l - 1}{2}$

$$J(2^m + l) = J(2n+1)$$

$$= 2J(n) + 1$$

$$= 2J\left(\frac{2^m + l - 1}{2}\right) + 1$$

$$= 2\left[\cancel{2} \cdot \frac{l-1}{\cancel{2}} + 1\right] + 1 = 2l + 1$$

The Josephus problem $\longrightarrow J(n) = J(2^m + l) = 2l + 1$

Examining the closed form

* power of 2^m played an important role.

if n can be represented in binary $\longrightarrow n$ & $J(n)$

n's binary ? $n = 2^m + l$

$$n = (b_m b_{m-1} \cdots b_1 b_0)_2$$

$\hookrightarrow$ each $b_i = \begin{cases} 0 \\ 1 \end{cases}$

leading bit, $b_m = 1$, so, $n = (1\, b_{m-1} b_{m-2} \cdots b_1 b_0)_2$ ($b_m = 1$)

$$l = (0\, b_{m-1} b_{m-2} \cdots b_1 b_0)_2$$

double: $$2l = (b_{m-1} b_{m-2} \cdots b_1 b_0 0)_2$$

adding 1: $$2l + 1 = (b_{m-1} b_{m-2} \cdots b_1 b_0 1)_2$$

$2 \to 10$
double
$4 \to 100$
double
$8 \to 1000$

$$\therefore J(n) = (b_{m-1} b_{m-2} \cdots b_1 b_0 b_m)_2$$

$$\therefore \boxed{J((b_m b_{m-1} \cdots b_1 b_0)_2) = (b_{m-1} b_{m-2} \cdots b_1 b_0 b_m)_2}$$

$\hookrightarrow$ 1 bit cyclic left shift

If, $n = 10$,

$J((1010_2)) = 0101_2 = 5$

If, $n = 100$,

$J((1100100)_2) = 1001001_2 = 73$

#/ $J(n) = n$; $J(n) = \frac{n}{2}$

n	1	3	7	15
$J(n)$	1	3	7	15

n	10
$J(n)$	5

- repeatedly Josephus problem apply :

$$J(J(J(13)))$$

$13 = 1101_2$

$J(1101_2) = 1011_2 = 11$

$J(1011_2) = 0111_2 = 7$ } একটা time এ fixed point এ চলে যাবে।

$J(\cancel{0}111_2) = 111\cancel{0}_2 = 7$

$\boxed{\text{when } J(n) \text{ will be } n}$?

↳ binary representation এ যখন সবগুলো bit হবে 1।

n কে represent করতে কয়টা '1' bit লাগে ↓ fix হবে $2^x - 1$, x = no of '1'

13 ⟶ binary representation এ 3 টা 1 (1101) 3 টা 1.

↳ fix হবে → $2^3 - 1 = 7$ এ।

$\therefore J(J(J(J(13)))) = 7$ (at least 2 বার ~~চালালে~~ fix হবে) repeat হবে 3 no. বারে।

$$J^4(J^3(J^2(J^1(21))))$$

$$= \underbrace{(21)}_{10101} \;\; 10101 \longrightarrow 3 \; 2 \; 1 \longrightarrow 2^3 - 1 = 7$$

$1 \rightarrow 01011$

$2 \rightarrow 0111 = 7$

$3 \rightarrow 111 = 7$

$4 \rightarrow 111 = 7$

$$J^6(J^5(J^4(J^3(J^2(J^1(\underbrace{101011110}))))))$$

$$\longrightarrow 2^6 - 1 = 63$$

$1 \rightarrow 010111101$

$2 \rightarrow 01111011$

$3 \rightarrow 1111011$

$4 \rightarrow 110111$

$5 \rightarrow 1011111$

$6 \rightarrow 111111$

$\boxed{\text{When } J(n) \text{ will be } n/2}$?

$$J(n) = 2l + 1 = \frac{2^m + l}{2} \qquad n = 2^m + l$$

$$\Rightarrow 4l + 2 = 2^m + 1$$

$$\Rightarrow 3l = 2^m - 2$$

$$\Rightarrow \boxed{l = \frac{1}{3}(2^m - 2)}$$

binary no. কে half করতে → right shift

$$n \qquad J(n)$$
$$(2)\ 10 \longrightarrow 01\ (1)$$
$$(10)\ 1010 \longrightarrow 0101\ (5)$$

n যখন $\underbrace{170}_{101010} \longrightarrow J(n) = \frac{170}{2} = 85$

* The pattern is presence of consequtive '10' in binary representation.

<u>Generalization</u> :

$$\left.\begin{aligned} J(1) &= \underline{1} \\ J(2n) &= 2J(n) - \underline{1} \\ J(2n+1) &= 2J(n) + \underline{1} \end{aligned}\right\}$$ constant values are represented like α, β, γ.

$$\therefore\ f(1) = \alpha$$
$$f(2n) = 2f(n) + \beta \qquad (\text{for } n \geq 1)$$
$$f(2n+1) = 2f(n) + \gamma \qquad (\text{for } n \geq 1)$$

According to the original eq^n.

$$\alpha = 1$$
$$\beta = -1$$
$$\gamma = 1$$

n	$f(n)$
1	α
2	$2\alpha+\beta$
3	$2\alpha + 0.\beta+\gamma$
4	$4\alpha + 3\beta + 0\cdot\gamma$
5	$4\alpha + 2\beta + \gamma$
6	$4\alpha + 1.\beta + 2\gamma$
7	$4\alpha + 0.\beta + 3\gamma$
8	$8\alpha + 7\beta + \gamma$
9	$8\alpha + 6\beta + 1\gamma$
10	$8\alpha + 5\beta + 2\gamma$

largest power of 2 — decrease 1 করে — increase 1 করে

$n=4 \rightarrow 2^2+0$

$l=0$
$C(n)=0$
$B(n)=3$
$A(n)=4$

$f(2) = 2f(1)+\beta$
$\quad = 2\alpha+\beta$

$f(3) = 2f(1)+\gamma$
$\quad = 2\alpha+\gamma$

$f(4) = 2f(2)+\beta$
$\quad = 2(2\alpha+\beta)+\beta$
$\quad = 4\alpha+3\beta$

$f(5) = 2f(2)+\gamma$
$\quad = 2(2\alpha+\beta)+\gamma$
$\quad = 4\alpha+2\beta+\gamma$

$f(6) = 2f(3)+\beta$
$\quad = 2(2\alpha+\gamma)+\beta$
$\quad = 4\alpha+\beta+2\gamma$

$f(7) = 2f(3)+\gamma$
$\quad = 2(2\alpha+\gamma)+\gamma$
$\quad = 4\alpha+3\gamma$

1. α's coefficient is n's largest power of 2
2. β's " " decreasing by 1 down to 0
3. γ's " " increasing by 1 up from 0

$$\boxed{f(n) = \underbrace{A(n)}_{2^m}\alpha + \underbrace{B(n)}_{2^m-1-l}\cdot\beta + \underbrace{C(n)}_{l}\gamma}$$

1. 11. 2023

Chapter 2 : Sums

Notation

① $1+2+3+\cdots+n$ ✓

$1+2+\cdots+(n-1)+n$ ✓

$1+\cdots+n$ (not really clear)

* $a_1+a_2+a_3+\cdots+a_n$ → summation of terms

a_1 ↓ term

② Using 'Σ' : $\sum_{k=1}^{n} a_k$ → "sigma notation" (a_k → summand)

→ "delimited form" (limit 1 to n ∃ sum)

or,

sigma notation (generalized)

→ $\sum_{\boxed{1\le k\le n}} a_k$ → generalized sigma form

$1\le k\le n$ → relation under sigma

Ex : $\sum_{\substack{1\le n\le 100,\\ k \text{ odd}}} k^2 \leftarrow 1^2+3^2+5^2+\cdots+99^2$

↓

$\sum_{k=0}^{49}(2k+1)^2$

- Sum of reciprocal of the prime number upto 100 :

$\hookrightarrow \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$

$\longrightarrow \sum_{\substack{1 \le n \le 100 \\ k\ \text{prime}}} \frac{1}{n}$ or, $\sum_{k=1}^{\pi(100)} \frac{1}{P_k}$

π একটা function যা 1 to 100 prime number generate করবে। $P_k \rightarrow k^{th}$ prime number

or, $\sum_{\substack{p\ \text{is prime} \\ p \le N}} \frac{1}{p}$ (upto N) or, $\sum_{k=1}^{\pi(N)} \frac{1}{P_k}$

Changing limits

- $\sum_{1 \le k \le n} a_k \longrightarrow$ we want to change limit ~~into~~ k into k+1 :

$\longrightarrow \sum_{1 \le k+1 \le n} a_{k+1}$

- $\sum_{k=1}^{n} a_k \longrightarrow$ k into k+1

$= \sum_{k=0}^{n-1} a_{k+1}$

Note :

→ problem statement represent → use delimited.

→ To manipulate sums → use relation under the Σ.

→ $\sum_{k=1}^{n} a_k (k-1)$

1 দিলে first term 0 → zero sum is not harmful ; we don't have to handle it.

$= \sum_{k=2}^{n} a_k (k-1)$

* Kenneth Iverson's conversion (KIC)

$[p \text{ prime}] = \begin{cases} 0 & \longrightarrow p \text{ prime না হলে} \\ 1 & \longrightarrow p \text{ prime হলে } 1 \text{ o/p দিবে} \end{cases}$

$$\sum_{\substack{p \le N \\ p \text{ is prime}}} p \xrightarrow{K.I.C} \sum p\,[p \text{ prime}]\,[p \le N]$$

p → that number

$[p \le N]$ → p ; N এর থেকে ছোট or সমান হলে 1 o/p দিবে।

Sums & Recurrence :

The sum, $S_n = \sum_{k=0}^{n} a_k$

$S_0 = 0$

$S_n = \sum_{k=0}^{n-1} a_k + a_n$

$= S_{n-1} + a_n$

So, the sum is equivalent to, $S_0 = a_0$

$S_n = S_{n-1} + a_n \; ; n > 0$ —— (I)

If, $a_n = \underbrace{\text{constant}}_{\beta} + \underbrace{\text{a multiple of } n}_{n \cdot \gamma}$, $a_0 = \alpha$

$\therefore R_0 = \alpha$

$R_n = R_{n-1} + (\beta + n\gamma)$ —— (II)

$R_0 = 1$
$R_n = R_{n-1} + \beta + \gamma n$

$n = 1, 2, 3 \ldots$

$R_0 = \alpha$

$R_1 = R_0 + \beta + \gamma$
$= \alpha + \beta + \gamma$

$R_2 = R_1 + \beta + 2\gamma.$
$= \alpha + 2\beta + 3\gamma$

$R_3 = R_2 + \beta + 3\gamma$
$= \alpha + 3\beta + 6\gamma$

generalised solution, $R_n = A(n)\cdot\alpha + B(n)\cdot\beta + C(n)\cdot\gamma$ —— (iii)

$\boxed{\text{Set}, \ R_n = 1}$: $R_n = 1$
$R_{n-1} = 1$
$R_0 = 1 = \alpha$ (as, $R_0 = \alpha$)

$1 = 1 + (\beta + \gamma n)$
$\Rightarrow \beta + \gamma n = 0$
so, $\beta = \gamma = 0$

So, $\alpha = 1, \ \beta = 0, \ \gamma = 0$

So, eqn (iii), $1 = A(n)\cdot 1 + B(n)\cdot 0 + C(n)\cdot 0$

$\therefore \boxed{A(n) = 1}$

$\boxed{\text{Set } R_n = n}$: $R_0 = \alpha \quad \therefore R_0 = 0$
$R_n = R_{n-1} + \beta + \gamma n$
$\Rightarrow n = n-1 + \beta + \gamma n$
$\Rightarrow \beta + \gamma n = 1$
$\Rightarrow \beta + \gamma n = 1 + 0.n$
$\therefore \beta = 1, \ \gamma = 0.$

So, from eq (iii), $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$
$\Rightarrow n = A(n)\cdot 0 + B(n)\cdot 1 + C(n)\cdot 0$
$\therefore \boxed{B(n) = n}$

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$\boxed{\text{Set } R_n = n^2}$

$R_0 = 0^2 = 0 = \alpha$

$\therefore \alpha = 0$

Reperitive method

$\therefore n^2 = (n-1)^2 + \beta + \gamma n$

$\Rightarrow \cancel{n^2} = \cancel{n^2} - 2n + 1 + \beta + \gamma n$

$\Rightarrow -1 + 2n = \beta + \gamma n$

$\therefore \beta = -1, \ \gamma = 2$

From eqn (iii), $n^2 = A(n)\cdot 0 + B(n)\cdot(-1) + C(n)\cdot 2$

$\Rightarrow n^2 = -B(n) + 2\cdot C(n)$

$\Rightarrow C(n) = \dfrac{B(n) + n^2}{2}$

$= \dfrac{n + n^2}{2}$

So, from eqn (iii),

$$\boxed{R(n) = 1\cdot\alpha + n\cdot\beta + \frac{n+n^2}{2}\cdot\gamma} \text{ ——— (iv)}$$

$\hookrightarrow$ closed form

$S_n = \sum_{k=0}^{n} (a+bk)$

when, $k=0 \longrightarrow a + b\cdot 0$

$k=1 \longrightarrow a + b\cdot 1$

$k=2 \longrightarrow a + b\cdot 2$

$\vdots$

$k=n \longrightarrow a + bn$

$\hookrightarrow \underbrace{\sum_{k=0}^{n-1}(a+bk)}_{R_{n-1}} + \underbrace{a+bn}_{R_n}$

So, $\beta = a$, $\gamma = b$

and, $R_0 = \alpha = a$

$\therefore \alpha = a, \ \beta = a, \ \gamma = b$

$\therefore R(n) = \alpha + n\beta + \frac{n+n^2}{2}\gamma$ (from eq^n (IV))

$= a + na + \frac{n^2+n}{2} \cdot b$

$= a(n+1) + b \cdot \frac{n(n+1)}{2} \longrightarrow$ sol^n from $\sum_{k=0}^{n}(a+bk)$

TOWER OF HANOI :

Recurrence sol^n $\rightarrow$ $T_0 = 0$

$\rightarrow T_n = 2T_{n-1} + 1$ (for $n>0$) —— (2)

divide eq^n (2) by 2^n,

$\frac{T_0}{2^0} = \frac{0}{2^0} = 0 \quad (n=0) \longrightarrow (S_0)$

$\frac{T_n}{2^n} = \frac{2T_{n-1}+1}{2^n}$ for $n>0$

$\Rightarrow \frac{T_n}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$ for $n>0$

assume, $\frac{T_n}{2^n} = S_n$

$\therefore S_0 = 0$

$S_n = S_{n-1} + \frac{1}{2^n}$

It follows that,

$$\frac{a(1-r^n)}{1-r}$$

$$S_n = \sum_{k=1}^{n} 2^{-k}$$

$$S_n = 2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-n}$$

$$= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n$$

$$= \left[1 - \left(\frac{1}{2}\right)^n\right]$$

$a = \frac{1}{2}$

$r = \frac{1}{2}$

$$= 1 - \frac{1}{2^n}$$

$$\frac{\cancel{\frac{1}{2}}\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\cancel{\frac{1}{2}}\right)}$$

$$S_n = \frac{2^n - 1}{2^n}$$

$$\therefore \; 2^n S_n = 2^n - 1$$

$$\therefore \; \boxed{T_n = 2^n - 1}$$

$$a_n T_n = b_n T_{n-1} + c_n \quad \text{—— (3.1)}$$

$\downarrow \quad \downarrow \quad \downarrow$ } Summation factor

$$T_n = 2T_{n-1} + 1$$

let the summation factor is S_n

From (3.1), S_n multiply,

$$S_n \cdot a_n T_n = S_n b_n \cdot T_{n-1} + S_n \cdot c_n \quad \text{—— 3.2}$$

let $\boxed{S_n b_n = S_{n-1} \cdot a_{n-1}}$

$$S_n = S_n a_n T_n$$

* Recurrence → sol^n বের করতে হবে sum

1.3.2 → $S_n = s_{n-1}\, a_{n-1}\, T_{n-1} + s_n \cdot c_n$

$$S_n = S_{n-1} + s_n c_n \rightarrow S_{n-2} + S_{n-1} C_{n-1} + S_n C_n$$

Hence,

$$S_n = s_0 a_0 T_0 + \sum_{k=1}^{n} S_K C_K$$

$$= s_1 b_1 T_0 + \sum_{k=1}^{n} c_k S_K$$

$$S_n a_n T_n = s_1 b_1 T_0 + \sum_{k=1}^{n} c_k S_k$$

$$\Rightarrow \boxed{T_n = \frac{1}{s_n a_n}\left[s_1 b_1 T_0 + \sum_{k=1}^{n} c_k S_k \right]} \longrightarrow Sol^n.$$

$\boxed{n = 1} \rightarrow T_1 = \frac{1}{\cancel{s_1} a_1}\left[\cancel{s_1} b_1 T_0 + c_1 \cancel{s_1}\right]$

$$= \frac{1}{a_1}\left[b_1 T_0 + c_1\right]$$

$$s_n b_n = s_{n-1}\, a_{n-1}$$

$$s_n = \frac{s_{n-1}\, a_{n-1}}{b_n}$$

$$= \frac{s_{n-2} \cdot a_{n-2}}{b_{n-1}} \cdot \frac{a_{n-1}}{b_n}$$

$$S_n = \frac{a_{n-1}\, a_{n-2}\, a_{n-3} \cdots a_1}{b_n\, b_{n-1}\, b_{n-2} \cdots b_2}$$

$$a_n T_n = b_n T_{n-1} + c_n \qquad T_{n-1} = 2T_{n-2} + 1$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$1 \cdot T_n = 2 \cdot T_{n-1} + 1$$

$$\therefore \quad s_n = \frac{1 \cdot 1 \cdot 1 \cdots 1}{2 \cdot 2 \cdots 2}$$

$$\boxed{s_n = \frac{1}{2^{n-1}}} \longrightarrow \text{Summation factor}$$

Problem

In quicksort algorithm, one of the mostly used method for sorting data, data's need to compare with other element. The average number of comparision steps made by quicksort when it's applied to n items in random order which satisfies the following recurrence.

$$c_0 = 0$$

$$c_n = n+1 + \frac{2}{n} \sum_{k=0}^{n-1} c_k, \quad \text{for } n > 0$$

1. Form the above recurrence in $a_n T_n = b_n T_{n-1} + c_n$
2. Find the summation factor (s_n)
3. Prove that i) $c_n = 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1}$

ii) $c_n = 2(n+1)H_n - 2n$, where H_n harmonic sum

iii) Find closed form for c_n

iv) Find c_1, c_2 and c_3.

→

$$a_n T_n = b_n T_{n-1} + c_n$$

$$\underset{a_n}{n} C_n = \underset{b_n}{(n+1)} C_{n-1} + \underset{c_n}{2n} \qquad \underset{a_{n-1}}{(n-1)} C_{n-1} = \underset{b_{n-1}}{n} C_{n-2} + \underset{c_{n-1}}{2(n-1)}$$

$$S_n = \frac{a_{n-1} \, a_{n-2} \, a_{n-3} \cdots a_1}{b_n \cdot b_{n-1} \cdot b_{n-2} \cdots b_2}$$

$$= \frac{(n-1)(n-2)(n-3) \cdots 1}{(n+1)\, n \cdot (n-1) \cdots 3}$$

$$= \frac{\cancel{(n-1)!} \cdot 2}{(n+1)\, n \cdot \cancel{(n-1)!}} = \frac{2}{n(n+1)} \longrightarrow \text{summation factor}$$

Manipulation of Sums :

1. Distributive law. $\longrightarrow$ $\sum_{k\in K} \overset{\text{constant}}{\overset{\uparrow}{c}} a_k = c \sum_{k\in K} a_k$
2. Associative law $\longrightarrow$ $\sum_{k\in K} (a_k + b_k) = \sum_{k\in K} a_k + \sum_{k\in K} b_k$

3.

$$\sum_{k\in K} a_k = \sum_{p(k)\in K} a_{p(k)}$$

→ we can reorder the sum in any way

→ $P(K) \rightarrow$ any permutation of the set.

Ex : $\sum_{k=1,2,3} a_k = a_1 + a_2 + a_3$

↳ যেকোনো order এ লিখতে পারি।

- For comparing different set of ~~of~~ indices,

$\sum a_k [k\in K]$ kenelte iversion $\leftarrow$ $\sum_{k\in K} a_k + \sum_{k\in K'} a_k = \sum_{k\in K\cap K'} a_k + \sum_{k\in K\cup K'} a_k$

Ex : $\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{k=1}^{n} a_k$

↓ only $k=m$ common

↳ Perturbation method

→ Spliting a single term from a sum

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The operation of splitting of a term is the basis of perturbation method. It can be used to determine the closed form of any sum.

$$S_n = \sum_{0 \le k \le n} a_k$$

$$\Rightarrow S_n + a_{n+1} = \sum_{0 \le k \le n} a_k + a_{n+1}$$

$$= \sum_{0 \le k \le n+1} a_k$$

$$= a_0 + \sum_{1 \le k \le n+1} a_k$$

$$= a_0 + \sum_{0 \le k-1 \le n} a_k$$

$$= a_0 + \sum_{0 \le k \le n} a_{k+1}$$

Problem :

Find the sum of a general geometric progression using perturbation method, i.e.

$$S_n = \sum_{0 \le k \le n} a x^k$$

$$S_n = a x^0 + a x^1 + a x^2 + a x^3 + \dots + a x^n$$

$$S_0 = a$$

apply perturbation method,

$$S_n + ax^{n+1} = \sum_{0 \le k \le n} ax^k + ax^{n+1}$$

$$= ax^0 + \sum_{0 \le k \le n} ax^{k+1}$$

$$= ax^0 + \sum_{0 \le k \le n} ax^k \cdot x$$

$$= ax^0 + x \cdot \sum_{0 \le k \le n} ax^k$$

$$= ax^0 + x \cdot S_n$$

$$\Rightarrow S_n - xS_n = ax^0 - ax^{n+1}$$

$$\Rightarrow \cancel{(1-x)} S_n = \frac{a(1-x^{n+1})}{1-x}$$

$$\therefore \boxed{S_n = \frac{a(1-x^{n+1})}{1-x}}$$

Multiple of sums :

* $$\sum_{1 \leq j, k \leq 3} a_j b_k = a_1 b_1 + a_1 b_2 + a_1 b_3 + a_2 b_1 + a_2 b_2 + a_2 b_3 + a_3 b_1 + a_3 b_2 + a_3 b_3$$

* $$\sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k} \left[P(j,k) \right] \longrightarrow$$ kenette Iversion form

* $$\sum_{j} \sum_{k} a_{j,k} \left[P(j,k) \right]$$

- Law of interchanging the order of summation :

$$\sum_{j} \sum_{k} a_{j,k} \left[P(j,k) \right] = \sum_{P(j,k)} a_{j,k} = \sum_{k} \sum_{j} a_{j,k} \left[P(j,k) \right]$$

$$\sum_{1 \leq j, k \leq 3} a_j b_k = \sum_{j,k} a_j b_k \left[1 \leq j , k \leq 3 \right]$$

$$= \sum_{j,k} a_j b_k \left[1 \leq j \leq 3 \right] \left[1 \leq k \leq 3 \right]$$

$$= \sum_{j} \sum_{k} a_j b_k \left[1 \leq j \leq 3 \right] \left[1 \leq k \leq 3 \right]$$

$$= \left(\sum_{j} a_j \, [1 \le j \le 3]\right)\left(\sum_{k} b_k \, [1 \le k \le 3]\right)$$

$$= \left(\sum_{j=1}^{3} a_j\right)\left(\sum_{k=1}^{3} b_k\right)$$

General distributive law :

$$\sum_{\substack{j \in J \\ k \in K}} a_j b_k = \left(\sum_{j \in J} a_j\right)\left(\sum_{k \in K} b_k\right)$$

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2

* Find the closed form for the following,

$$2\sum_{1\le j,k\le n} a_k b_k$$

$$= 2\sum_{1\le j\le n} \cancel{a_k b_k} \sum_{1\le k\le n} a_k b_k$$

$$= 2\sum_{1\le k\le n}\left(\sum_{1\le j\le n} a_k b_k\right)$$

$$= 2\sum_{1\le k\le n}\left[a_k b_k \sum_{1\le j\le n}(1)\right]$$

$$= 2\sum_{1\le k\le n}\left[a_k b_k \cdot n\right]$$

$$= 2n\sum_{1\le k\le n} a_k b_k$$

$$\sum_{1\le j\le n} 1 = n$$

* Find closed form of the sum, $S_n = \sum_{1\le j<k\le n} \frac{1}{k-j}$

1) First summing on j,

$$S_n = \sum_{1\le j<k\le n} \frac{1}{k-j}$$

$$= \sum_{1\le k\le n}\sum_{1\le j<k} \frac{1}{k-j} \longrightarrow \text{replacing } j \text{ with } k-j$$

$1 \le k-j < k$
$\Rightarrow 1-k \le k-j-k < k-k$
$\Rightarrow 1-k \le -j < 0$
$\Rightarrow k-1 \ge j > 0$
$\Rightarrow 0 < j \le k-1$
$\Rightarrow \boxed{1 \le j \le k-1}$

$$= \sum_{1\le k\le n}\sum_{1\le k-j<k} \frac{1}{\cancel{k}-(\cancel{k}-j)}$$

$$= \sum_{1\le k\le n}\left(\sum_{1\le j\le k-1} \frac{1}{j}\right)$$

$j \rightarrow 1$ to $k-1$

$H_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k-1}$

↳ harmonic sum
here $n = k-1$

$$= \sum_{1\le k\le n} H_{k-1} = \sum_{0\le k<n} H_k$$

(Ans)

2) summing on k,

$$S_n = \sum_{1 \le j < k \le n} \frac{1}{k-j}$$

$$= \sum_{1 \le j \le n} \sum_{j < k \le n} \frac{1}{k-j}$$

$$= \sum_{1 \le j \le n} \sum_{j < k+j \le n} \frac{1}{k+\cancel{j}-\cancel{j}}$$ → replacing k with $k+j$

$$= \sum_{1 \le j \le n} \left(\sum_{j < k+j \le n} \frac{1}{k} \right)$$

$$= \sum_{1 \le j \le n} \left(\sum_{0 < k \le n-j} \frac{1}{k} \right)$$

$$= \sum_{1 \le j \le n} H_{n-j}$$

$$= \sum_{1 \le n-j \le n} H_{n-(n-j)}$$

$$= \sum_{1 \le n-j \le n} H_j$$

$$= \sum_{0 \le j \le n-1} H_j$$

$$= \sum_{0 \le j < n} H_j$$

$j < k+j \le n$

$\Rightarrow j-j < k+\cancel{j}-\cancel{j} \le n-j$

$\Rightarrow \boxed{0 < k \le n-j}$

$\to \boxed{1 \le k \le n-j}$

$1 \le n-j \le n$

$\Rightarrow 1-n \le \cancel{n}-j-\cancel{n} \le n-n$

$\Rightarrow 1-n \le -j \le 0$

$\Rightarrow n-1 \ge j \ge 0$

$\Rightarrow 0 \le j \le n-1$

$$\sum_{1\le k\le n} \frac{n}{k} - \sum_{1\le k\le n} 1$$

$$= nH_n - n$$

3) $$S_n = \sum_{1\le j<k\le n} \frac{1}{k-j}$$

$$= \sum_{1\le j<k+j\le n} \frac{1}{k}$$

[Replacing k by $k+j$ $\therefore k+\cancel{j}-\cancel{j} = k$]

$$\cancel{= \sum_{1\le k\le n}\left(\sum_{j\le k+j\le n} \frac{1}{k}\right)}$$

$j < k+j \le n$
$j-k < j \le n-k$
$1 \le j \le n-k$?

$$\cancel{= \sum_{1\le k\le n}\left(\sum_{0\le k} \frac{1}{k}\right)}$$

$$\cancel{= \sum_{1\le k\le n} \frac{1}{k} \cdot n}$$

$j < k+j$ $\quad$ $k+j\le n$
$0 < k$ $\quad$ $j \le n-k$
$1 \le k$

$$= \sum_{1\le k\le n} \sum_{1\le j\le n-k} \frac{1}{k}$$

$$= \sum_{1\le k\le n} \frac{n-k}{k}$$

$$= \sum_{1\le k\le n} \frac{n}{k} - \sum_{1\le k\le n} 1$$

$$= nH_n - n$$

* Find closed form for the sum of squares, i.e $\square_n$

$$S_n = \square_n = \sum_{0\le k\le n} k^2 = 0^2 + 1^2 + 2^2 + \dots + n^2$$

→ <u>Perturbation method</u> :

$$\square_n + (n+1)^2 = \sum_{0\le k\le n} k^2 + (n+1)^2$$

$$= \sum_{0\le k\le n+1} k^2 = \sum_{0\le k\le n} (k+1)^2$$

$$= \sum_{0\le k\le n} (k^2 + 2k + 1)$$

$$= \sum_{0\le k\le n} k^2 + \sum_{0\le k\le n} 2k + \sum_{0\le k\le n} 1$$

$$\Rightarrow \cancel{\square_n} + (n+1)^2 = \cancel{\square_n} + 2\sum_{0\le k\le n} k + n+1$$

$$\Rightarrow 2\sum_{0\le k\le n} k = (n+1)^2 - (n+1)$$

$$\Rightarrow \sum_{0\le k\le n} k = \frac{(n+1)(n+\cancel{1}-\cancel{1})}{2}$$

$$\therefore \boxed{\sum_{0\le k\le n} k = \frac{n(n+1)}{2}}$$

$$\boxdot_n = \sum_{0 \le k \le n} k^3$$

$$\cancel{\boxdot_n} + (n+1)^3 = \sum_{0 \le k \le n} k^3 + (n+1)^3$$

$$= \sum_{0 \le k \le n} (k+1)^3 = \sum_{0 \le k \le n} (k^3 + 3k^2 + 3k + 1)$$

$$= \sum_{0 \le k \le n} k^3 + 3\sum_{0 \le k \le n} k^2 + 3\sum_{0 \le k \le n} k + \sum_{0 \le k \le n} 1$$

$$= \cancel{\boxdot_n} + 3\sum_{0 \le k \le n} k^2 + 3\sum_{0 \le k \le n} k + \sum_{0 \le k \le n} 1$$

$$= 3\,\square_n + 3\,\frac{n(n+1)}{2} + (n+1)$$

$$\Rightarrow \quad \square_n = \frac{(n+1)^3}{3} - \frac{n(n+1)}{2} - \frac{n+1}{3}$$

$$= \frac{(n+1)\left\{ \begin{array}{c} 2n^2 + 4n + \cancel{2} - 3n - \cancel{2} \\ \cancel{2n^2 + 4n + 2 - 3n - 3n - 2n - 2} \end{array} \right\}}{6}$$

$$= \frac{(n+1)\,(\cancel{-n^2 - n})\,(2n^2 + n)}{6}$$

$$= \frac{n\,(n+1)\,(2n+1)}{6}$$

$1^2+2^2+3^2+\cdots+n^2$

$\sum_0 2+3k$

$2(3)+3 \frac{2(3)}{2}$

$=15$

$=15$

Repertoire method :

$S_n = \sum k^2$

$S_n = S_{n-1} + n^2$

$R_0 = \alpha$ $\quad a+bk$

$R_n = R_{n-1} + \beta + \gamma n$

$R_0 = \alpha$

$R_n = R_{n-1} + \beta + \gamma n + \delta n^2$

$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$

$1^2+2^2+\cdots$

$A(n) = 1$

$B(n) = n \quad (1+1+\cdots n)$

$C(n) = \frac{n^2+n}{2} \quad (1+2+3+\cdots n)$

Set,

$R(n) = n^3$

$R_0 = 0^3 = 0 \longrightarrow \boxed{\alpha = 0}$

$R_n = R_{n-1} + \beta + n\gamma + n^2\delta$

$\Rightarrow n^3 = (n-1)^3 + \beta + n\gamma + n^2\delta$

$\Rightarrow \beta + n\gamma + n^2\delta = n^3 - (n-1)^3$

$= n^3 - (n^3 - 3n^2 + 3n - 1)$

$= 3n^2 - 3n + 1$

$\Rightarrow \beta + n\gamma + n^2\delta = 3n^2 - 3n + 1$

$\alpha = 0, \quad \beta = 1, \quad \gamma = -3, \quad \delta = 3$

$R(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\cdot\delta$

$\Rightarrow n^3 = 1\cdot 0 + n\cdot 1 + \frac{n^2+n}{2}\cdot(-3) + 3D(n)$

$\Rightarrow \cancel{3}D(n) = 2n^3 + 3(n^2+n) - 2n$

$= \frac{1}{3}(2n^3 + 3n^2 + n)$

$\frac{n(n+1)(2n+1)}{6}$

12 . 11 . 2023

Chapter 4

NUMBER THEORY

Divisibility :

We say that, m divides n or n is divisible by m.

n/m $\quad \hookrightarrow m \backslash n$

if $m>0$ then, $m \backslash n$ is an integer

$m \backslash n \longrightarrow m>0$ and $n = mk$, for some integer k.

GCD & LCM :

- $\gcd(m,n) = \max\{k \mid k \backslash m \text{ and } k \backslash n\}$
- $\text{LCM}(m,n) = \min\{k \mid m \backslash k \text{ and } n \backslash k,\ k>0\}$

* $\gcd(5, 14)$ using Euclid's algo :

$$
\begin{array}{r|l|l}
5 & 14 & 2 \\
 & 10 & \\
\hline
4 & 5 & 1 \\
 & 4 & \\
\hline
1 & 4 & 4 \\
 & 4 & \\
\hline
 & 0 &
\end{array}
$$

$\gcd = 1$

Euclid's algorithm :

$$\gcd(m,n) = \gcd(n \bmod m, m)$$

$\hookrightarrow$ here, $m<n$

$$\gcd(0,n) = n$$

So, $\gcd(0, n) = n$

$\gcd(m, n) = \gcd(n \bmod m, m)$ for $m > 0$

$$\begin{array}{r|r|l} 4 & 14 & 3 \\ & 12 & \\ \hline \gcd = 2 \longleftarrow 2 & 4 & 2 \\ & 4 & \\ \hline & 0 & \end{array}$$

<u>Euclid's algo (another)</u> :

$\gcd(m, n) = m'm + n'n$

Ex : $\gcd(4, 14) = 2 = \underset{m'}{(-3)} \cdot \underset{m}{4} + \underset{n'}{(1)} \cdot \underset{n}{14}$

$\gcd(\underset{m}{0}, \underset{n}{2}) = 2 = m'm + n'n$

$= 0 + \underset{n'}{1} \cdot 2$

<u>determining m' & n'</u> :

base case: if $m = 0$, take, $m' = 0$, $n' = 1$

$\gcd(m, n) = 0 \cdot m + 1 \cdot n = n$

$\therefore \gcd(0, n) = n$

Second case:

remainder, $r = n \bmod m$

$\gcd(m, n) = \gcd(n \bmod m, m)$

$= \gcd(r, m)$

$= \gcd(m \bmod r, r)$

$= \gcd(s, r)$

$$\gcd(r, m) = \bar{r}'r + \bar{m}'m$$

If $r = n \bmod m$, recursively apply r & m in place of m and n, computing r' and m', such that,

$$\bar{r}'r + \bar{m}'m = \gcd(r, m)$$

here, $r = n \bmod m$,

$$= n - \left\lfloor \frac{n}{m} \right\rfloor * m$$

$$\begin{array}{c} m \quad n \\ \uparrow \quad \uparrow \\ 3 \overline{)\,14\,(} 4 \\ \underline{12} \\ 2 \end{array}$$

$14 - (3 \times 4)$

$14 - 3 \times \left\lfloor \frac{14}{3} \right\rfloor$

$\downarrow \quad \downarrow$

$n - m \cdot \left\lfloor \frac{n}{m} \right\rfloor$

$$\gcd(m, n) = \gcd(r, m)$$

$$\Rightarrow m'm + n'n = \bar{r}'r + \bar{m}'m$$

$$= \bar{r}\left(n - \left\lfloor \frac{n}{m} \right\rfloor \cdot m\right) + \bar{m}\,m$$

$$= \bar{r}n - \left\lfloor \frac{n}{m} \right\rfloor m\bar{r} + \bar{m}m$$

$$= \bar{r}n + m\left(\bar{m} - \left\lfloor \frac{n}{m} \right\rfloor \bar{r}\right)$$

equating co-efficients of m & n,

$$\boxed{\begin{aligned} m' &= \bar{m} - \left\lfloor \frac{n}{m} \right\rfloor \bar{r} \\ n' &= \bar{r} \end{aligned}}$$

$m' = \bar{m} - \left\lfloor \frac{n}{m} \right\rfloor \bar{n}$

$n' = \bar{n}$

Ex : $\gcd(12, 18) = (-1)\cdot 12 + 1 \cdot 18$

$= \gcd(6, 12) = 1 \cdot 6 + 0 \cdot 12$

$= \gcd(0, 6) = m'm + n'n$

$12 \overline{)18(} 1$
$\quad 12$
$6 \overline{)12(} 2$
$\quad 12$
$\quad 0$

$\gcd(6,12)$: $m' = 1 - \left\lfloor \frac{12}{6} \right\rfloor \cdot 0 = 1$

$n' = \bar{n} = 0$

$\gcd(12,18)$: $m' = 0 - \left\lfloor \frac{18}{12} \right\rfloor \cdot 1 = -1$

$n' = \bar{n} = 0$

15.11.2023

Zinia ma'am

concrete mathematics
~~Statement~~ 4.2

4.2 : Primes :

A positive integer P is called prime if it has just two divisors - 1 & p itself.

1 is not prime.

So, prime sequence is 2, 3, 5, 7, 11, ...

composite number, $15 = 1 \times 15$
$= 3 \times 5$

A number can be either prime or composite.

- Prove that, any positive integer (n) can be written as a product of primes and it is unique.

mathematical representation → $$n = P_1 P_2 P_3 \dots P_m = \prod_{i=1}^{n} P_i \quad \text{——} \quad (4.1)$$

where, $P_1 \le P_2 \le P_3 \le \dots \le P_m$

Note
* n is a product of primes
* product π
  sum Σ
* $18 = 2 \times 3 \times 3$ → increasing order

Part 1 :

Product of prime

$n = P_1 \cdot P_2 \cdot \dots P_m$

if $m = 0$, $n = 1$ [empty product] ; which implies $n = 1$.

└→ $\boxed{\text{Note}}$: কোনো একটা product থাকলেই 1 এর চেয়ে বড় হবে।

if $n > 1$ and, it is not prime ; then there's some divisor n_1, such that, $n_1 < n$

$n = n_1 \times n_2$ ↳ $1 < n_1 < n$

↳ $n_1' \times n_2'$

n_1 and n_2 can be rewritten as products of prime otherwise n_1 is prime.

So, n can be always represented as product of primes.

Ex : $36 = 4 \times 9$ $\quad n_1 = 2 \times 2 \quad n_2 = 3 \times 3$

$\underset{n}{36} = \underset{n_1}{4} \times \underset{n_2}{9}$

$36 = 2 \times 2 \times 3 \times 3$

Part II : Product of primes is unique.

There is only one way to write n as a product of primes in non-decreasing order.

Proof : when $n = 1$, there's only one possibility since the product must be empty.

let's assume, $n > 1$, and we have two factorization, where p's and q's are all prime.

$$n = p_1 p_2 p_3 \cdots p_m = q_1 q_2 q_3 \cdots q_k$$

where, $p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_m = q_1 \leq q_2 \leq q_3 \leq \cdots \leq q_k$

We'll prove, $p_1 = q_1$

if $p_1 \neq q_1$, then we can assume that. $q_1 < p_1$ making q_1 smaller than p_1,

As, p_1 and q_1 is prime,

$$\gcd(p_1, q_1) = 1$$

According to the euclid's self certifying algorithm.

$$\gcd(p_1, q_1) = ap_1 + bq_1 \quad [\text{where } a \text{ and } b \text{ are integers}]$$

$$\Rightarrow ap_1 + bq_1 = 1$$

$$\Rightarrow ap_1(p_2 p_3 \cdots p_m) + bq_1(p_2 p_3 \cdots p_m) = 1 \cdot (p_2 p_3 \cdots p_m)$$

$$\therefore \underbrace{an}_{\substack{\text{1st} \\ \text{term}}} + \underbrace{bq_1 \cdot p_2 p_3 \cdots p_m}_{\substack{\text{2nd} \\ \text{term}}} = \underbrace{p_2 p_3 \cdots p_m}_{\text{R.H.S.}}$$

L.S $\longrightarrow q_1$ দিয়ে divide করা যায়

but, R.S $\longrightarrow q_1$ term নেই ;

In eq^n 4.2, q_1 divides the R.H.S. i.e $p_2 p_3 \cdots p_m$

Thus, $p_2 p_3 \cdots p_m / q_1$ will be an integer and $p_2 p_3 \cdots p_m$ has a prime factorization q_1 which contradicts our assumption.

So, p_1 must be equal to q_1.
Thus, we can prove for all $p_2 \cdots p_m$ are equal to $q_2 \cdots q_k$. So, the product of primes is unique.

[proved].

Prime exponent representation :

18
$= 2 \times 9$
$= 2 \times 3 \times 3$
$= 2 \times 3^2 \qquad \langle 1, 2, 0, 0, \cdots \rangle$

power
$2, 3, 5, 7 \cdots$
$\downarrow \ \downarrow$
$1 \ \ 2$

$20 = 2 \times 2 \times 5 = 2^2 \times 5 = \langle 2, 0, 1, 0 \rangle$
$2 \to 2,\ 3 \to 0,\ 5 \to 1,\ 7$

$$n = \prod_p p^{n_p} \quad \text{where } n_p \geq 0$$

1) $\gcd(m, n) \longrightarrow ?$

2) $\text{lcm}(m, n) \longrightarrow ?$

$$n_p \quad 18 \longrightarrow \langle \overset{n_2}{1}, \overset{n_3}{2}, \overset{n_5}{0}, \overset{n_7}{0} \ldots \rangle \longrightarrow 2 \times 3^2$$

$$m_p \quad 16 \longrightarrow \langle 4, 0, 0, 0 \ldots \rangle \longrightarrow 2^4$$

$$\gcd \longrightarrow \min(n_p, m_p)$$

$$\text{lcm} \longrightarrow \max(n_p, m_p)$$

4.4 factorial's factors :

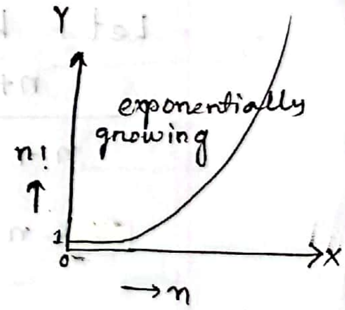
$$n! = 1 \cdot 2 \cdot 3 \cdots n = \prod_{k=1}^{n} k, \quad n \geq 0$$

for $n=0$, empty product. ie. $0! = 1$

$$n! = n(n-1)!$$

$$= n(n-1)(n-2)!$$

n	0	1	2	3	4	5	6	7	8
$n!$	1	1	2	6	24	120	720	5040	40320



Prove that, $n!$ increases exponentially using Gauss Trick or, $n!$ is very big

$$n! = 1\cdot 2\cdot 3 \cdots n$$

$$(n!)^2 = (1\cdot 2\cdot 3\cdots n)(1\cdot 2\cdot 3\cdots n)$$

$$= (1\cdot 2\cdot 3\cdots n)(n\cdot(n-1)\cdots 1)$$

$$= 1\cdot n\cdot 2(n-1)\cdot 3(n-2)\cdots n\cdot 1$$

$$= \underbrace{1(n-0)}\cdot\underbrace{2(n-1)}\cdot\underbrace{3(n-2)}\cdots\underbrace{(n-0)\cdot 1}$$

$1(n-0) = 1(n+1-1)$

$2\cdot(n-1) = 2(n+1-2)$

$3(n-2) = 3(n+1-3)$

$\downarrow$

$k(n+1-k)$

or, $(i+1)(n-i)$

$$(n!)^2 = \prod_{k=1}^{n} k(n+1-k) \quad \text{—— (4.4.1)}$$

Let, $k = a$

$n+1-k = b$

$n+1 = a+b$

$2k-n-1 = a-b$

From 4.4.1, $k(n+1-k) = ab$

$$= \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

$$= \left(\frac{n+1}{2}\right)^2 - \left(\frac{2k-n-1}{2}\right)^2$$

$$= \frac{(n+1)^2}{4} - \frac{\{2k-(n+1)\}^2}{4}$$

$k \to 1$ to n

$$k(n+1-k) = \frac{1}{4}(n+1)^2 - \underbrace{\left(k - \frac{n+1}{2}\right)^2}$$ —— 4.4.2

$\downarrow$ $0 \to k(n+1-k)$ max

$k = \cancel{max}\ 1 \to k(n+1-k)$ min (lowest)

In 4.42 has largest value, when $\left(k - \frac{n+1}{2}\right)^2 = 0$.

i.e $k = \frac{n+1}{2}$

and has smallest value when $k = 1$ $[1 \le k \le n]$

when, $k = \frac{n+1}{2}$, then $k(n+1-k) = \frac{1}{4}(n+1)^2 - 0 = \frac{(n+1)^2}{4}$ (largest val)

when, $k = 1$, then, $k(n+1-k) = \frac{1}{4}(n+1)^2 - \left(1 - \frac{n+1}{2}\right)^2$

$$= \frac{1}{4}(n+1)^2 - \left(\frac{2-n-1}{2}\right)^2$$

$$= \frac{1}{4}(n+1)^2 - \frac{(n-1)^2}{4}$$

$$= \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2$$

$$= n \cdot 1$$

$$= n \quad \text{(smallest val)}$$

So, $n \le k(n+1-k) \le \frac{1}{4}(n+1)^2$

$$\Rightarrow \prod_{k=1}^{n} n \le \prod_{k=1}^{n} k(n+1-k) \le \prod_{k=1}^{n} \frac{1}{4}(n+1)^2$$

$$\Rightarrow n^n \leq (n!)^2 \leq \left\{\frac{1}{4}(n+1)^2\right\}^n$$

$$\Rightarrow n^n \leq (n!)^2 \leq \left\{\left(\frac{n+1}{2}\right)^2\right\}^n$$

$$\Rightarrow \boxed{n^{n/2} \leq n! \leq \left(\frac{n+1}{2}\right)^n}$$

$\hookrightarrow$ exponentially growing

From this relation, we conclude that, the factorial function grows exponentially.

19. 11. 23

Chapter 5

"Binomial Co-efficient"

Symbol : $\binom{n}{k}$ read as n choose k (or, ${}^{n}C_{r}$).

Ex : $\{1, 2, 3, 4\} \longrightarrow$ choose two element $\longrightarrow \binom{4}{2} = 6$

$\{1,2\}, \{2,3\}, \{3,4\}, \{1,3\}, \{1,4\}, \{2,4\}$

If I have to choose k things from n things :

$$\underset{k}{\underline{\;n\;}} \times \underset{k-1}{\underline{\;n-1\;}}$$

$$\boxed{\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1}} \longrightarrow \text{when } k = 2, \quad \binom{n}{2} = \frac{n(n-1)}{2}$$

For real number $\longrightarrow \binom{r}{k} = \binom{-1}{2} = \frac{(-1)(-1-1)}{2(2-1)}$

$$\binom{1/2}{2} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2(2-1)}$$

Note :

$$\binom{\circ}{\circ}$$

whole number or real number

can't be negative

$$= \frac{r(r-1)(r-2)\cdots(r-k+1)(r-k)\cdots 1}{k!\,(r-k)\cdots 1}$$

$$= \frac{r!}{k!\,(r-k)!} \longrightarrow \binom{r}{k}$$

- For real number

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)(r-2)\cdots(r-k+1)}{k(k-1)(k-2)\cdots 1} & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$$

$$\frac{r^{\underline{k}}}{k!}$$

$r^{\underline{m}} \longrightarrow r$ to the power falling factor of m

$$= r(r-1)(r-2)\cdots(r-m+1)$$

- $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$; n integer, $0 \leq k \leq n$

Pascal's Triangle :

* symmetry identity

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	
0	1	0	0	0	0	0	$(k>n)$
1	1	1	0	0	0	0	
2	1	2	1	0	0	0	
3	1	3	3	1	0	0	
4	1	4	6	4	1	0	
5	1	5	10	10	5	1	

Symmetry Identity :

From table → $\binom{3}{0} = \binom{3}{3}$, $\binom{5}{0} = \binom{5}{5}$

$\binom{3}{1} = \binom{3}{2}$, $\binom{5}{1} = \binom{5}{4}$

$\binom{5}{2} = \binom{5}{3}$

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

So, $\binom{n}{k} = \binom{n}{n-k}$ where $n \geq 0$, integer k. → this formula is not for real numbers

Q why n cannot be negative in symmetry identity ?

→ Let $n = -1$, $\binom{-1}{k} = \binom{-1}{-1-k}$ a.c.t symmetry rule

$= \binom{-1}{-(1+k)}$ → $k<0$ here

$= 0$

L.S. $= \binom{-1}{k}$ Not equal

if $k = 0$, $\binom{-1}{0} = \frac{(-1)!}{0!(-1-0)!} = \frac{1}{0!} = 1$

So, n cannot be negative. as, $\binom{-1}{k} \neq \binom{-1}{-1-k}$

Formally :

$$\binom{-1}{k} = \frac{(-1)(-2)(-3)\cdots(-k)}{k!} = \frac{(-1)^k \cdot k!}{k!} = (-1)^k$$

→ +1 or −1

$$\binom{-1}{-1-k} = \binom{-1}{-(1+k)} = 0$$

$-(1+k) < 0$

So, as $\binom{-1}{k} \neq \binom{-1}{-1-k}$, n cannot be negative.

Absorption identity :

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$$

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1} = \binom{r-1}{k-1}\cdot\frac{r}{k}$$

(9) Prove that, $(n-k)\cdot\binom{n}{k} = n\binom{n-1}{k}$ using symmetry & absorption identity :

$\rightarrow$

$$(n-k)\cdot\binom{n}{k} = (n-k)\binom{n}{n-k} \quad \text{[by symmetry]}$$

$$= \cancel{(n-k)}\cdot\frac{n}{\cancel{(n-k)}}\binom{n-1}{n-k-1} \quad \text{[by absorption]}$$

$$= n\binom{n-1}{n-1-k}$$

$$= n\binom{n-1}{k} \quad \text{[by symmetry]}$$

Formula :

- Addition Formula :

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1} \text{, integer } k.$$

$$\binom{4}{2} = \binom{3}{2}\binom{3}{1}$$

Note :
* recursive

Home Study :

Prove this addition formula.

Proof must

Addition formula :

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Example :

Option 1
last element নিয়ে ভাঙো

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{3}{1}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{2}{0}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0} + \underbrace{\binom{1}{-1}}_{k<0,\ \text{so},\ 0.}$$

$$\Rightarrow \quad a_1 \qquad a_2 \qquad a_3 \qquad a_4$$

$$= \sum_{k=1}^{4} a_k$$

$$\binom{\overset{1+1}{2}+3}{3} = \binom{1+3}{3} + \binom{1+2}{2} + \binom{1+1}{1} + \binom{1+0}{0}$$

$$\binom{1+1+3}{3} = \binom{1+3}{3} + \binom{1+2}{2} + \binom{1+1}{1} + \binom{1+0}{0}$$

$$= \binom{x+k}{k} + \binom{x+(k-1)}{k-1} + \dots + \binom{x+0}{0}$$

Note :
→ x constant term

$$\binom{x+n+1}{n} = \sum_{k=0}^{n \to 3} \binom{x+k}{k}$$

$$\boxed{\binom{x+n+1}{n} = \sum_{k=0}^{n} \binom{x+k}{k}}$$

Closed form

Option 2
1st element fn(2)

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$
$$= \binom{3}{3} + \binom{3}{2} + \binom{4}{2}$$
$$= \binom{2}{3} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$
$$= \underbrace{\binom{1}{3}}_{1<3} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

pattern

$$= \underbrace{\binom{0}{3}}_{0} + \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

pattern

$$\binom{4+1}{2+1} = \sum_{k=0}^{4} \binom{k}{2}$$

n, m

$$\Rightarrow \boxed{\binom{n+1}{m+1} = \sum_{k=0}^{n} \binom{k}{m}}$$

closed form for the sum

* Prove it, $\sum_{k=0} \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$

Table 174
formulas

HOME STUDY :

Σ of something ব্যাপারটা $= \sum_{k\leq n} \binom{r}{k} x^k \cdot y^{r-k}$

1. Find $(x+y)^r$ in terms of sum and binomial co-efficient

2. Find 2^n when $x=1$ and $y=1$, $r=n$

3. Find 0^n , $x=-1$ and $y=1$, $r=n$

4. Prove that, $\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$

5. Table 174

Chapter 6

Special Numbers

- **Stirling Number :**
  - → First kind
  - → Second kind

Second kind : $\left\{ {n \atop k} \right\}$ — # ways to partition a set of n element into k non-empty subset.

Example :

$\left\{ {4 \atop 2} \right\}$

4 → 4 টা element

2 → 2 টা non-empty subset

$\{1, 2, 3, 4\}$ ——

1. $\{1, 2, 3\}, \{4\}$
2. $\{1, 2\}, \{3, 4\}$
3. $\{1, 3\}, \{2, 4\}$
4. $\{2, 3\}, \{1, 4\}$
5. $\{1\}, \{2, 3, 4\}$
6. $\{2\}, \{1, 3, 4\}$
7. $\{3\}, \{1, 2, 4\}$

$$\left\{ {4 \atop 2} \right\} = 7$$

$$\left\{ {n \atop k} \right\} = ?$$

$\left\{ {n \atop n} \right\} = 1$

In $\left\{ {n \atop k} \right\}$, if $k = 1$. i.e. 1 non-empty subset

$\left\{ {n \atop 1} \right\} = 1$

$\left\{ {0 \atop 1} \right\} = 0$

When $k = 0$ i.e. 0 non-empty subset.

$\left\{ {n \atop 0} \right\} = 0$

$\left\{ {0 \atop 0} \right\} = 1 \longrightarrow \{\}$

When $k = 2$ i.e. 2 non-empty subsets,

$\left\{ {n \atop 2} \right\} = ?$

$\left\{ {0 \atop 2} \right\} = 0$, if $n > 1$ & $k = 2$

↳ if, $n = 0$

└ 2 parts
- one of these part contains the last element
- and some subset of the first $(n-1)$ elements

4 এর আলাদা করে (4 → 1 element)

subsets of $(n-1)$ elements

$\{1,2,3\}, \{4\}$

$n = 4$
$n-1 = 3$

$P(\{1,2,3\}) = \{1,2,3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1\}, \{2\}, \{3\}, \{\}$

$\{2,3\} \rightarrow \{2,3,4\}, \{1\}$ ✓

$\{1,2\} \rightarrow \{1,2,4\}, \{3\}$ ✓

$\{1,2,3\}$ ✗ $\rightarrow \{1,2,3,4\}, \{\} \rightarrow$ not possible

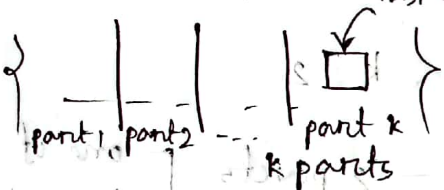
$$\boxed{\left\{ {n \atop 2} \right\} = 2^{n-1} - 1}$$

Generalization of $\left\{ {n \atop k} \right\}$:

- Given a set of $n > 0$ elements/objects/things
- to be partition into k non-empty subsets
- # ways

either —
- last object is in a class/part by itself
- last object is with some non-empty subset of the first $(n-1)$ objects.

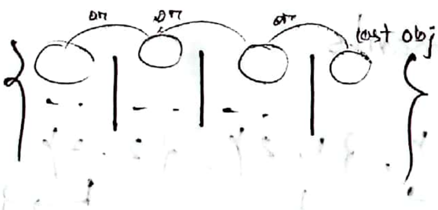
either ① →



rest parts $= k-1$

elements left $= n-1$ $\Big\rangle \left\{ {n-1 \atop k-1} \right\}$

either ② →



→ # element $= n-1$

parts $= k$ $\Big\rangle\ k \left\{ {n-1 \atop k} \right\}$

$$\boxed{\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}}$$

→ recurrence

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ → recurrence

26. 11. 2023

Next class → First kind

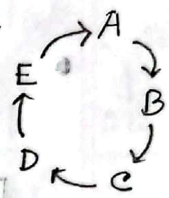
First kind of the stirling number :

$\begin{bmatrix} n \\ k \end{bmatrix}$ → counts #nways to arrange n elements/ objects into k cycle

→ n cycle k..

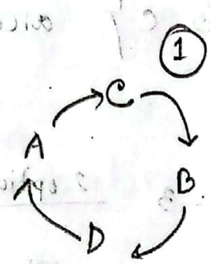
→ cyclic arrangement like necklace.

[ABCDE] $\xrightarrow[\text{1 cycle}]{\text{5 elements}}$



$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\xrightarrow[\text{1 cycle}]{\text{4 elements}}$

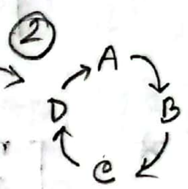
1



$\boxed{[CBDA]}$

$\boxed{[ABCD]}$
[BCDA]
[CDAB]
[DABC]
↓
same

2



$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\xrightarrow[\text{2 cycle}]{\text{4 elements}}$

3 | 1		2 | 2	
[ABC]	[D]	[AB]	[CD]
[CBA]	[D]	[BC]	[AD]
[ABD]	[C]	[BD]	[AC]
[DBA]	[C]		
[BCD]	[A]		
[DCB]	[A]		
[ACD]	[B]		
[DCA]	[B]		

→ 11 options.

$\{A\}$ or, $\{B\}$ . . .

- 1 element cycle or single tone (1-cycle) is equivalent to 1-set (1 element set)

$\{B,C\}$ or $\{C,A\}$ . . .

- 2 " " or double tone (2-cycle) is equivalent to 2-set.

- 3 element cycle (3 cycle) has several options.

[A, B, C] & $\{A, B, C\}$ are not equal.

★ $\begin{bmatrix} n \\ 1 \end{bmatrix} \longrightarrow$ when $n = 3 \xrightarrow{\text{2 options}}$ [A B C]

[B A C]

ABC
CAB
BCA

$\underbrace{\qquad}$ 3 element ↳ 3 টা way

[B C A] বাদ পড়েছে

[C A B]

বাদ

[ACB]

[CBA]

n কে সাজাতে পারি n! উপায়ে

So, $\frac{3!}{3} = 2$

$$\boxed{\begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{n!}{n} = (n-1)! \qquad \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = 1}$$

$\begin{bmatrix} n \\ 1 \end{bmatrix} \geqslant \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\}$

$\boxed{\text{When } k = n}$,

$$\left[{n \atop n}\right] = 1$$

$$\left\{{n \atop n}\right\} = 1$$

$$\left[{n \atop n}\right] = \left\{{n \atop n}\right\}$$

$\boxed{\text{When , } k = n - 1}$

$$\left[{n \atop n-1}\right] = \binom{n}{2}$$

$$\left\{{n \atop n-1}\right\} = \binom{n}{2}$$

A B C D

2 elements

[A] [B] [CD]

[AD] [B] [C]

[A] [BD] [C]

[AC] [B] [D]

The number of ways to arrange n objects into (n-1) cycles or subset is the number of ways to choose the two objects that will be in the same cycle or subset.

Generalizing $\left[{n \atop k}\right]$:

- $n > 0$
- # ways to choose k cycles from n elements

either —
- Put the last object into a cycle by itself.
  ($\left[{n-1 \atop k-1}\right]$ ← last obj এরপর)
- insert the last object into one of the $\left[{n-1 \atop k}\right]$ cycle arrangement of the first $(n-1)$ objects and there are $(n-1)$ different options of insertions.

$\left[{5 \atop 2}\right]$

[A B C] [D]

E কে either [D] or [A B C] তে insert করা possible

$\left[{n-1 \atop k}\right]$ → D কে বাদ দিয়ে

example :

$[A_\uparrow, B_\uparrow, C_\uparrow] \longrightarrow$ D insert করলে

$(n-1)$ options

$$\boxed{\left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right]}$$

29.11.2023
Zinia ma'am

Eulerian Number :

Eulerian
- represented by $\left\langle {n \atop k} \right\rangle$ ⤻ n ascent k
- n ascent k
- Angle bracket suggests less than, greater than

$\left\langle {n \atop k} \right\rangle$ - is the # permutations $\pi_1 \pi_2 \pi_3 \ldots \pi_n$ of $\{n_1, n_2, n_3, \ldots n\}$ elements that have k ascents :

$\left\langle {3 \atop 2} \right\rangle \rightarrow$
$1<2<3$
↳ যে permutation এ ২টি ascent

Example : 4 element $\{1, 2, 3, 4\} \rightarrow \left\langle {4 \atop 2} \right\rangle = ?$

permutation $= 4! = 24$

$\left\langle {4 \atop 2} \right\rangle \rightarrow$

1. $1<2<4>3$
2. $1<3>2<4$

3 element $\longrightarrow 3! = 6$

π_1	π_2	π_3	#< or number of ascents	
1	2	3	2	$(1<2<3)$
1	3	2	1	$(1<3>2)$
2	1	3	1	$(2>1<3)$
2	3	1	1	$(2<3>1)$
3	1	2	1	$(3>1<2)$
3	2	1	0	$(3>2>1)$

$\left\langle {3 \atop 2} \right\rangle = 1$

$\left\langle {3 \atop 1} \right\rangle = 4$

$\left\langle {3 \atop 0} \right\rangle = 1$

$\left\langle {3 \atop 3} \right\rangle = 0$

Note : $\left\langle {n \atop k} \right\rangle$

- If $k = n$ or $k > n$, then $\left\langle {n \atop k} \right\rangle = 0$

Table - 268 : Eulerian Triangle

n	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\left\langle {n \atop 5} \right\rangle$
0	~~0~~ 1					
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	

$\left\langle {2 \atop 0} \right\rangle = \left\langle {2 \atop 1} \right\rangle$ → $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$

$n-1$ $1-0 = 1$

$\left\langle {3 \atop 0} \right\rangle = \left\langle {3 \atop 2} \right\rangle$

$\left\langle {4 \atop 0} \right\rangle = \left\langle {4 \atop 3} \right\rangle$

$\left\langle {4 \atop 1} \right\rangle = \left\langle {4 \atop 2} \right\rangle$

$\left\langle {5 \atop 0} \right\rangle = \left\langle {5 \atop 4} \right\rangle$

$\left\langle {5 \atop 1} \right\rangle = \left\langle {5 \atop 3} \right\rangle$

Observations :

- There can be $(n-1)$ ~~of~~ positions for measuring ascent or not ascent.
- at most $(n-1)$ ascents possible.
- There is symmetry between left & right part

So, symmetry Law, $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$

The permutation $\pi_1 \pi_2 \pi_3 \dots \pi_n$ has $n-1-k$ ascents if it's reflection $\pi_n \pi_{n-1} \dots \pi_1$ has k ascents.

$$1 < 2 < 3 \xrightarrow{\text{reflect}} 3 > 2 > 1$$

$$\left\langle {3 \atop 2} \right\rangle = 1 \qquad \left\langle {3 \atop 0} \right\rangle = 1$$

General Sol^n of $\left\langle {n \atop k} \right\rangle$:

Each permutation $\rho = \rho_1 \rho_2 \dots \rho_{n-1}$ of elements $\{1, 2, 3 \dots n-1\}$ leads to n permutations of $\{1, 2, 3, \dots n\}$ if we insert the new element n in all possible way i.e, $n = 5$ $\{1, 2, 3, 4, 5\}$

$$\uparrow\ 1 \ \uparrow < 2 \ \uparrow < 3 \ \uparrow < 4 \ \uparrow$$

5 can be placed: or or or or — ③ ③ ③ ③ ④ ← #<

$$1 < 2 < 4 \overset{\langle 5 \rangle}{} > 3$$

$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
#< = 2 same	#< = 2 Same	#< = 2 Same	increment by 1 #< = 3	#< = 3

first

ascent বাড়বে না last element বসালে

$\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4$ এ বসালে always increment by 1

last position

or

মাঝের position এ ascent থাকলে no change

" " " descent " increment by 1.

Two case

- the permutation has k ascents
- the " " " $k-1$ "

If we put n in position j,

we get the permutation, $\pi = \rho_1 \rho_2 \cdots \rho_{j-1}\, n\, \rho_j \cdots \rho_{n-1}$

ascents in π in same as $\rho = \rho_1 \rho_2 \cdots \rho_{n-1}$

if
- n inserted at the first position
- or middle of any ascent positions

there are k ascents

ascent $\rightarrow k$

not ascent $\rightarrow \underbrace{n-1}_{\text{comparison}} - k$

ascents in π increased by 1 from ρ,

- n inserted at the last position
- or middle of any not ascent position
  - there are $(n-1-k)$ not ascent position

$$\rightarrow \underbrace{\overset{\text{first}}{\left\langle {n-1 \atop k} \right\rangle} + k\overset{\text{ascent}}{\left\langle {n-1 \atop k} \right\rangle}} + \underbrace{\overset{\text{last}}{\left\langle {n-1 \atop k-1} \right\rangle} + (n-1-k)\overset{\text{not ascent}}{\left\langle {n-1 \atop k-1} \right\rangle}} = \left\langle {n \atop k} \right\rangle$$

$$\Rightarrow \left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle$$

base cond^n : $\left\langle {0 \atop 0} \right\rangle = 1$

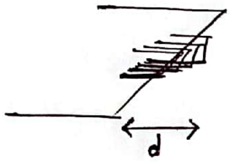
√ 8

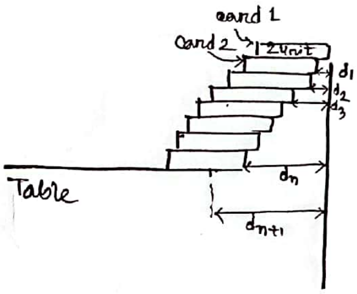
3. 12. 2023

Harmonic Number (H_n) :

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

(Q) 52 টা card একের উপর এক table এর এক কোনায় রাখলে কত দূর পর্যন্ত extend করা যাবে ?





$d_{n+1} = ?$

(Q) Worm on the rubber band :

w → প্রতি sec এ 1 cm

rubber → 100 cm

w 1 cm গেলে → rubber strech হলে doubled হয়ে যায়

	1%	99%
1 sec →	1 →	100
2 sec →	2 →	200
3 sec →	3 →	300

$$\frac{H_n}{100} = \frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \cdots = \frac{1}{100}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)$$

CSE-219 → Sec A

chap-(1-6)

compulsory 1 & 6

previous question

exercise → related problem-১৭

Self study

Sec A – CSE-213

chap-(1-7)

1, 5 full, 6 partial, 7 selfstudy

বাদ দিলেও হবে,

compulsory 2 & 3

*| Maths