Probability ⟶ is a branch of mathematics which deals with possibility of occurrence of something/event.

Event ⟶ fair coin ⟶ H ⟶ 0.5, T ⟶ 0.5 } for a biased coin, it's not true.

Axiom

* Prob. can never be neg or greater than 1.

  (0 - 1) $0 \le P(E) \le 1$ * $P(S) = 1$

* অবশ্যই ঘটবে $P = 1$

  " " না, $P = 0$ ⟶ সব possible outcome থাকতে হবে, exhaustive

No common element ⟵ * mutually exclusive, collectively exhaustive set

Sample space ⟶ S/Ω ⟶ all possible outcomes of an experiment

Ex: For a fair coin toss, $S = \{H, T\}$

for ~~a~~ ludo, $S = \{1, 2, 3, 4, 5, 6\}$

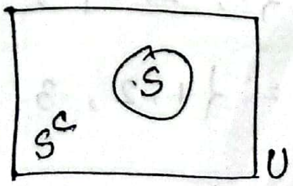
If there're 2 ~~dies~~ dice, $S = 36$ (6^2)

If " are 3 coins $S = 2^3 = 8$.

⟶ $P(4, 1) = \frac{1}{36}$

⟶ $P(HHH) = \frac{1}{8}$

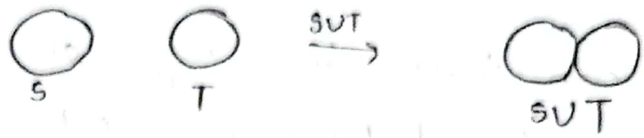
* Set ⟶ a collection of objects.
* If x is an element of S ⟶ $x \in S$
* If there's no element ⟶ NULL set (ϕ)
* Universal set (U)
* complement



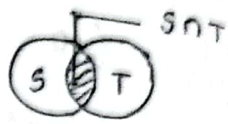
* The complement of a set S^c of S with respect to Ω are all outcomes which are not in S.

* $S \subseteq T$

* Union $\longrightarrow$



If there's some common part



* Intersection $\longrightarrow$ common section of sets.
* Disjoint set $\longrightarrow$ $S \cap T = \emptyset$.
* Event $\longrightarrow$ element of a sample space is an event

<u>Def^n</u> : Any subset E of the sample space.

<u>Ex</u> : $S = \{1, 2, 3, 4, 5, 6\}$

All even outcome (EVENT), $E = \{2, 4, 6\}$

* For any two events E and F of a sample spaces, we define a new event $E \cup F$, to consist all outcomes that're either in E or F or in both. The event $E \cup F$ occurs if either E or F occurs

Ex : $E = \{1, 2, 5\}$, $F = \{1, 2, 3\}$

$E \cup F = \{1, 2, 3, 5\}$

- Same for intersection ($E \cap F$ or EF)

$EF = \{1, 2\}$

** mutually exclusive

no common part

Ex : $S = \{H, T\}$, $E = \{H\}$, $F = \{T\}$, $EF = \{\varphi\}$

* two events ~~are~~ either can be dependent or independent

Axioms :

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

যতগুলো event (Union of all experiment)

প্রতিটা individual event এর sum.

* Non-negativity
* Normalization
* Addivity

$P(A \cup B) = P(A) + P(B) \longrightarrow A$ & B disjoint ($A \cap B = \phi$)

- E & E^c are always mutually exclusive.

$E \cup E^c = S$

S/Ω

E^c E

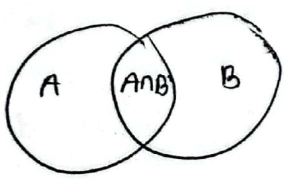
$P(S) = 1$

$\Rightarrow P(E \cup E^c) = 1$

$\Rightarrow P(E) + P(E^c) = 1$

$\Rightarrow \boxed{P(E^c) = 1 - P(E)}$

*



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\underbrace{P(A \cap B)}_{P(AB)} + P(A \cup B) = P(A) + P(B)$

Example 1.3 : Tossing 2 coins , $S = \{(H,H), (H,T), (T,H), (T,T)\}$

$E \rightarrow$ first coin falls head $= \{(H,H), (H,T)\}$

$F \rightarrow$ second " " head $= \{(H,H), (T,H)\}$

either the 1st or second coin falls head

$\mapsto P(E \cup F)$

$= P(E) + P(F) - P(E \cap F)$

$= \frac{2}{4} + \frac{2}{4} - \frac{1}{4}$

$= \frac{3}{4}$

Inclusion - exclusion identity :

$P(E \cup F \cup G)$

$= P(E \cup F) + P(G) - P((E \cup F)G)$ $\quad P(EG \cup FG)$

$= P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG)$
$+ P(EFG)$

$P(E_1 \cup E_2 \cdots \cup E_n) = \sum_i P(E_i) - \sum_{i<j} P(E_i E_j) + \cdots$
$+ (-1)^{n+1} P(E_1 E_2 \cdots E_n)$

$\mapsto$ Inclusion - exclusion identity)

(Prove it)

Conditional probability :

↳ If one event occurs, what's the prob. of another event.

Probability of getting 6, given that the output is even $\longrightarrow \frac{P\{6\}}{P\{2,4,6\}} = \frac{1}{3}$

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

A → identify, B → given

$P(A \cap B)$ → number of element in $A \cap B$

$P(B)$ → number of " " B

- $P(A \mid B) = \frac{P(AB)}{P(B)}$ where $P(B) > 0$

Example 1.4 :

$P(\text{getting } 10 \mid \text{at least } 5)$ ↳ $\{5, 6, 7, 8, 9, 10\}$

$= \frac{1}{6}$

the card will be both 10 and at least 5 if and only if it is number 10.

Note : $P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{P(E)}{P(F)}$

Example 1.7 :

7 black
5 white
12 balls

draw 2 balls from the box without replacement

E → First ball black

&

F → second " "

$$P(E) = \frac{7}{12}$$

$$P(F|E) = \frac{6}{11}$$

$$P(F|E) = \frac{P(FE)}{P(E)}$$

$$\Rightarrow P(FE) = P(E) \cdot P(F|E)$$

$$= \frac{7}{12} \cdot \frac{6}{11} = \frac{42}{132}$$

HHH, $\underset{\sim}{\text{HTH}}$, HHT, HTT

Example :

HTH, THT,

P(alternating H & T) | first toss head)

$$= \frac{1}{4} \qquad \left[\frac{1/8}{4/8} \right]$$

Example :

Karim $\longrightarrow$ gets bonus if & only if he sells more than 10 umbrellas.

1) If it's raining $\longrightarrow$ P(selling more than 10 umbrellas)

$= 0.8$

2) " " not " $\longrightarrow$ P(- - -) $= 0.25$

3) Probability that it rains tomorrow is 0.1.

P(doesn't rain & gets his bonus) = ?

X $\longrightarrow$ event that Karim sells more than 10 umb.

R $\longrightarrow$ It rains

B $\longrightarrow$ Karim gets bonus

same So $X = B$

$$P(X \mid R) = 0.8$$

$$P(X \mid R^c) = 0.25$$

$$P(R) = 0.1$$

$P(R^c \cap B)$ or $P(R^c B) = ?$

$$P(AB) = P(B) \cdot P(A \mid B)$$

$$P(BR^c) = P(R^c) \cdot P(B \mid R^c)$$

$$= \{1 - P(R)\} \cdot 0.25$$

$$= (1 - 0.1) \times 0.25$$

$$= 0.225$$

So, $P(B \cap R^c) = P(R^c \cap B) = P(R^c B) = 0.025$ (Ans).

*** **Independent probability** :

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

2 events A & B are independent. $P(A \mid B) = P(A)$.

event B doesn't have any effect on A.

$$P(A) = \frac{P(AB)}{P(B)}$$

$\Rightarrow \boxed{P(AB) = P(A) \cdot P(B)}$ [$\because$ A & B independent].

Example 1.8 :

Toss 2 fair dice

$E_1 \longrightarrow$ sum of dice is 6 (1, 5). (2, 4) (3, 3), (4, 2). (5, 1)

$F \longrightarrow$ first die equals four (4, 2), (4, 1), (4, 3), (4, 4), (4, 5) (4, 6).

$P(E_1 F) = \frac{1}{36}$

$P(E_1) = \frac{5}{36}$

$P(F) = \frac{1}{6}$ $\left(\frac{6}{36}\right)$

$P(E_1) \cdot P(F) = \frac{5}{216} \neq P(E_1 F)$

So, E_1 & F aren't independent

$E_2 \longrightarrow$ sum 7 (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)

$P(E_2 F) = \frac{1}{36}$ $P(E_2) \cdot P(F) = \frac{6}{36} \times \frac{1}{6} = \frac{1}{36} = P(E_2 F)$

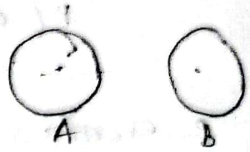
So, E_2 & F are independent.

(*) disjoint $\longrightarrow$ dependent

$$P(A \cap B) = \underbrace{P(A)}_{>0} \cdot \underbrace{P(B)}_{>0} \longrightarrow \text{independent}$$

+ve

So this cannot be zero. SO, THERE MUST BE COMMON ELEMENT.



A B

$P(A \cap B) = 0$

2 disjoint events can never be independent.

as $P(A \cap B) = 0$; but $P(A) \cdot P(B) > 0$.

*** Bayes' Theorem :

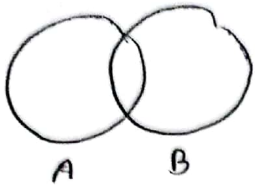
$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

Proof : $P(B|A) = \dfrac{P(BA)}{P(A)} = \dfrac{P(AB)}{P(A)}$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$\Rightarrow P(AB) = P(B) \cdot P(A|B)$$

$$\therefore \boxed{P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}}$$



$$A = A(B \cup B^c)$$

$$P(A) = P(AB) + P(AB^c)$$

$$= P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$$

$$\therefore \boxed{P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)}}$$

Home task :

You enter a chess tournament where your prob. of winning a game is 0.3 against half of the players (call them type 1), 0.4 against a quarter of " " (type 2), and 0.5 " the remaining quarter of the players (type 3). You played a game against a randomly chosen opponent and you win.

P (You had an opponent of type 1) = ?

$A_i \longrightarrow$ playing with an opponent of type i

$P(A_1) = \cancel{0.3}\ 0.5$

$P(A_2) = \cancel{0.4}\ 0.25$

$P(A_3) = 0.25$

$B \rightarrow$ event of winning

$P(B|A_1) = 0.3$

$P(B|A_2) = 0.4$

$P(B|A_3) = 0.5$

$P(A_1|B) = ?$

$$= \frac{P(B|A_1) \cdot P(A_1)}{P(B)}$$

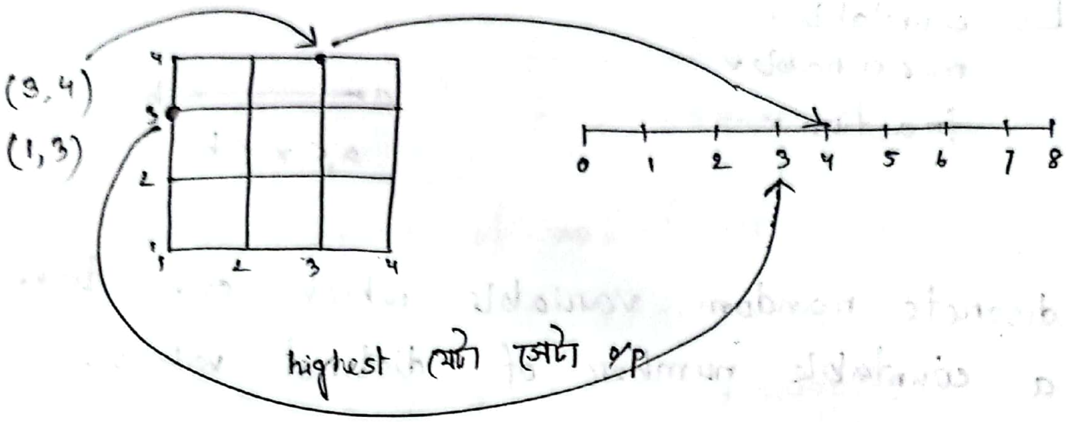
$$= \frac{P(B|A_1) \cdot P(A_1)}{P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) + P(B|A_3) \cdot P(A_3)}$$

$$= \frac{0.3 \times 0.5}{0.3 \times 0.5 + 0.4 \times 0.25 + \cancel{0.5}\ 0.5 \times 0.25}$$

$= 0.4$

Random variables :

Real valued function defined on the sample space



→ *** Read the example from ROS

* 2 dice → sum 2

$$P(x=2) = P\{(1,1)\} = \frac{1}{36}$$

$$P(x=3) = P\{(1,2),(2,1)\} = \frac{2}{36}$$

$$P(x=4) = P\{(1,3),(3,1),(2,2)\} = \frac{3}{36}$$

$\vdots$

$$P(x=12) = P\{(6,6)\} = \frac{1}{36}$$

* **No. of H appearing :**

(H,H), (H,T), (T,H), (T,T) $\quad H \to 0, 1, 2$

$\downarrow$ 2 H $\quad \downarrow$ 1 H $\quad \downarrow$ 1 H $\quad \downarrow$ 0 H

No of H appearing

$$P(X=0) = \frac{1}{4}$$

$$P(X=1) = \frac{1}{2} \quad \left(\frac{2}{4}\right)$$

$$P(X=2) = \frac{1}{4}$$

$\Big\}$ 1 $\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\right)$.

2H

- Discrete Random variables: $\longrightarrow$ countable ✓ measurable ✓ fraction ✗
- Continuous

  $\hookrightarrow$ countable ✗
  measurable ✓
  fraction ✓

a •———• b

$a \leq x \leq b$

* A discrete random variable which can take only a countable number of distinct values. (finite set of numbers).

* A continuous R.V is one which takes an infinite number of possible values.

<u>Probability Mass function (PMF)</u> :

PMF of discrete R.V is the list of probabilities associated with each of it's values. It is the list of all probabilities, when the discrete R.V takes all possible values.

$P(a),\ P_X(x),\ P(X = x)$

(X ↓ var, x ↓ value)

$$P(X = x) = \begin{cases} \frac{1}{4} ; & \text{when } x = 0, 2 \\ \frac{1}{2} ; & \text{when } x = 1 \\ \cancel{\frac{1}{4} ;} & \cancel{\text{when } x = 2} \\ 0 ; & \text{otherwise} \end{cases}$$

↳ Probability of no. of H appearing.

PMF $\longrightarrow \cancel{2} \cdot \cancel{\frac{1}{4}}_{2} + \frac{1}{2} = 1 \longrightarrow$ always.

Generic form $\longrightarrow$ $\boxed{\sum_{i=1}^{\infty} P(X_i) = 1}$

PMF : All possible possibilities of all possible values of discrete R.V.

X is R.V তার value a হওয়ার possibility হয়,

$P(a) = P_x(x) = P[X = a]$

value of random v.

$\sum_{i=1}^{\infty} P(X_i) = 1$

Cumulative distribution Func - CDF :

↳ All random variable have CDF

① collect all possible outcomes that gives rise to the event $\{X = x\}$.

② $P_X(x) = \begin{cases} \frac{1}{4}, & x = 2, 0 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$

~~may~~ there may be some ques. to generate PMF like this.

X = no. of heads obtained

P that least one H is obtained

$$\underbrace{P\{X = 1\}}_{1/2} + \underbrace{P\{x = 2\}}_{\frac{1}{4}}$$

$= \frac{3}{4}$

1* A basketball player shot 2 free throws, each equally likely either to be 'g' or 'b'. Each good shot worth 1 point. What is the PMF of X, where X is the number of points that he scored?

* R.V X has the following PMF,

$$P_X(x) = \begin{cases} \frac{c}{x}, & x = 1, 2, 3. \\ 0, & \text{otherwise} \end{cases}$$

1. Find out $c = ?$
2. $P[x = 1 = ?$
3. $P[x \geq 2] = ?$
4. $P[x > 3] = ?$

1

(gg), (gb), (bg), (b, b)

$P(X=0) = \{b, b\}$, $P(X=1) = \{(g, b)(b, g)\}$, $P(X=2) = \{gg\}$

$$P_X(x) = \begin{cases} \frac{1}{4}, & x = 0, 2 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

②

$$P_X(x) = \begin{cases} \frac{c}{x} & ; \quad x = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

$\therefore \quad \sum_{x=1}^{3} P_X(x) = 1 \Rightarrow P_X(1) + P_X(2) + P_X(3) = 1$

$$\frac{c}{1} + \frac{c}{2} + \frac{c}{3} = 1$$

$$\Rightarrow \frac{6c + 3c + 2c}{6} = 1$$

$$\Rightarrow 11c = 6$$

$$\Rightarrow c = \frac{6}{11}$$

$$P(X = 1) = \frac{6}{11}$$

$$P(X \geq 2) = P_X(2) + P_X(3) = \frac{\overset{3}{\cancel{6}}}{11 \times \cancel{2}} + \frac{\overset{2}{\cancel{6}}}{11 \times \cancel{3}} = \frac{5}{11}$$

$$P[X > 3] = ? = 0$$

Bernoulli R.V : are the discrete random variables that takes two values 1 and 0, depending on whether the outcome is a success or a failure.

PMF : $$\boxed{P_X(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \\ 0, & \text{otherwise} \end{cases}}$$

p = success

Features : only two possibilities - success or failure

1) Only two possibilities - success or failure
2) Probability doesn't change from trial to trial.
3) The trials are independent.

$$\boxed{(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} \cdot y^r}$$

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$(a+b)^4 = \sum_{k=0}^{4} \binom{4}{k} a^{4-k} \cdot b^k$$

$$= \binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3 + \binom{4}{4} a^0 b^4$$

$$=$$

Binomial R.V :

n independent trials each of which results in a success with probability p and in a failure with probability $(1-p)$ are to be performed. X represents the number of successes that occur in n trials, then X is said to be a binomial R.V with parameter (n, p).

$$\boxed{P(i) = \binom{n}{i} \cdot p^{i} \cdot (1-p)^{n-i}}$$ where $i = 1, 2, 3 \dots n$

p = success
$1-p$ = failure

A coin is tossed n times. At each toss the coin comes up H with probability p and a tail with probability $(1-p)$, independent of prior toss. Let X be the number of H in n toss sequence. We refer to X as a binomial R.V with parameter n & p.

Book ✓ (Q) 4 fair coins are flipped. If the outcomes are independent. What is the P that two heads and two tails are obtained.

→ fair coin → If first two are Heads then, it's obv. last two will be Tails.

at most 5

$\hookrightarrow P(X=0)+P(X=1)+P(X=2)+P\ldots + P(X=5)$

$\rightarrow$ Let X be the no. of H. So X is Binomial R.V with parameter, $n=4$, $p=\frac{1}{2}$.

$\boxed{X=2} \rightarrow$ no. of Heads

$\hookrightarrow$ Finding this Prob, is enough.

$$P[X=2] = \binom{4}{2}\cdot\left(\frac{1}{2}\right)^2\cdot\left(1-\frac{1}{2}\right)^{4-2} = \frac{3}{8}$$

Book X

9 Any item preoduced by certain machine will be defective with Prob 0.1, independent of other item. what's the prob that in a sample of 3 items, at most 1 will be defective.

$\rightarrow$ $n=3$, $p=0.1$

Let, X = the number of defective items in sample. X is a binomial R.V. with parameter $n=3$ and $p=0.1$.

So, the desired Probability $= P\{X=0\} + P\{X=1\}$

$= 0.972$

$$\binom{3}{0}(0.1)^0(0.9)^3 + \binom{3}{1}(0.1)^1(0.9)^2$$

$$= \frac{3!}{0!(3-0)!} \times 0.1 \times (0.9)^3 + \frac{3!}{1!(3-1)!}(0.1)^1\cdot(0.9)^2$$

Geometric R.V :

Independent trials, each having probability p of being success, are peformed until success occured. If X be the number of trials required until the first success occurs then X is said to be geometric R.V with parameter p.

PMF, $\boxed{P_X(n) \text{ or } P(X=n) = (1-P)^{n-1} \cdot P}$

(q) Suppose that we toss a fair coin. What is the prob. of appearing the first head on fifth toss?

→ Let, X be numbers of trial required until the first head appears. So X is geometric R.V. with here, $X = 5$. parameter $P = 1/2$.

$$P\{X = 5\} = (1-P)^{n-1} P$$

$$= \left(\frac{1}{2}\right)^{5-1} \left(\frac{1}{2}\right)^1$$

$$= \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

unfair ↓ Prob of $H = 1/3$ ↓ $P\{X=5\} = \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)$

Poisson R.V is a discrete prob. distribution of count of events that occurs randomly in a given interval of time.

Let X = the number of events in a given interval and if the mean number of events per interval is λ. So the P of observing i events in a given interval is given by,

$$\boxed{P(i) \text{ or } P\{X=i\} = e^{-\lambda} \left(\frac{\lambda^i}{i!}\right).}$$

validity of eq^n (from book)

$$\sum_{i=0}^{\alpha} P_{(i)} = \sum_{i=0}^{\alpha} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$= e^{-\lambda} \sum_{i=0}^{\alpha} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \left\{ 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right\}$$

$$= e^{-\lambda} \cdot e^{\lambda}$$

$$= 1$$

1.8 birth per half hour

$\hookrightarrow$ 2×1.8 " " hour (per hour e convert

(9) Birth in a Hospital occurs randomly at an average rate of 1.8 birth per hour. What is the P of observing 4 births in a given hour?

$\rightarrow$ $\lambda = 1.8$ in a given hour

Let X be the number of births

$$P\{X = i\} = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$$P\{X = 4\} = e^{-1.8} \cdot \left(\frac{(1.8)^4}{4!}\right)$$

$$= 0.0723$$

The number of hits of website in any time interval is Poisson R.V. A site has on average, $\lambda = 2$ hits per sec.

1) What is the P that where no hits in an interval of 0.25 sec?

2) What is the P that there are no more that 2 hits in an interval of 1 sec?

Ans : 1) 0.607

2) 0.677

Poisson RV may be used to approximate a binomial RV when the binomial parameter n is large and p is small.

Binomial RV $\longrightarrow$ $P(i) = \binom{n}{i} (p)^i (1-p)^{n-i}$

let, X be a binomial RV with parameter n & p such that,

$$\lambda = np$$

$$P\{X = i\} = \frac{n!}{(n-i)!\, i!} p^i (1-p)^{n-i}$$

$$\frac{n(n-1)(n-2)\cdots(n-i+1)\cancel{(n-i)\cdots 3\cdot 2\cdot 1}}{\cancel{(n-i)\cdots 3\cdot 2\cdot 1}\cdot i!}$$

$$= \underbrace{\frac{n(n-1)(n-2)\cdots(n-i+1)}{\cancel{i!}\; n^i}}_{1} \left(\frac{\lambda^i}{i!}\right) \underbrace{\left(1-\frac{\lambda}{n}\right)^{n-i}}_{\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}} \quad \text{---} (1)$$

$$\frac{(\text{very big no.})}{(\text{very big no.})} \approx 1$$

$$e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots$$

$$e^{-\lambda} = 1 - \frac{\lambda}{1!} + \frac{\lambda^2}{2!} - \cdots$$

$$(x+y)^n = \sum_{r=0}^{n} {}^nc_r (x)^{n-r} y^r$$

$$\left(1-\frac{\lambda}{n}\right)^n = \left\{1 + \left(-\frac{\lambda}{n}\right)\right\}^n$$

$$= {}^nc_0 (1)^{n-0} \cdot \left(-\frac{\lambda}{n}\right)^0 + {}^nc_1 (1)^{n-1} \left(-\frac{\lambda}{n}\right)^1 + {}^nc_2 (1)^{n-2} \cdot \left(-\frac{\lambda}{n}\right)^2 + \cdots$$

$$= 1 + \cancel{n} \cdot \left(-\frac{\lambda}{\cancel{n}}\right) + \frac{\lambda^2}{n^2} \cdot \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\, 2!}$$

$$= 1 - \frac{\lambda}{1!} + \frac{\lambda^2}{2!} - \cdots$$

$$= e^{-\lambda}$$

Assignment $\longrightarrow$ ross $\longrightarrow$ pg 27
Do this derivation in detail

$$\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i} \quad \begin{matrix} \longrightarrow e^{-\lambda} \\ \longrightarrow 1 \end{matrix}$$

[n is a really big no.; so, $\frac{1}{n}$ is really small]

so, $\left(1-\frac{\lambda}{n}\right)^i \approx 1^i = 1$

From ①,

$$P\{X=i\} = 1 \times \left(\frac{\lambda}{n}\right)^i \times \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= 1\left(\frac{\lambda^i}{i!}\right) \cdot e^{-\lambda}$$

PDF $\longrightarrow$

distribution $\longrightarrow$ sum is 1.

5.9.23

$S_T = \{ t \mid -5 \leq t \leq +5 \}$ ⟶ chances তাদের to be correct

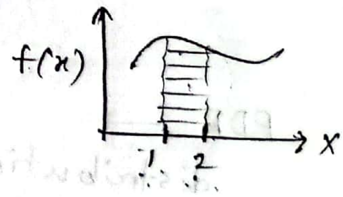
$= \{ t \mid -2 \leq t \leq +2 \}$

The amount of probability in an interval gets smaller and smaller as the interval shrinks.

A random variable X is called continuous if there exists a non-negative function $f(x)$, which is called Probability Density Function (PDF), such that

$$P(X \in B) = \int_B f(x) \cdot dx$$ for every set of B in the real line

random var. range



Note : $\text{CDF} \underset{\text{Int.}}{\overset{\text{Diff.}}{\rightleftarrows}} \text{PDF}$

CDF: area of an interval

PDF: denotes a curve

- $$\boxed{P\{X \in (-\alpha, \alpha)\} \text{ or } P\{-\alpha \leq x \leq \alpha\} = \int_{-\alpha}^{\alpha} f(x) \cdot dx}$$

↳ generic form of continuous R.V.

$a \le x \le b \xrightarrow{\text{CDF}} \int_a^b f(x)\cdot dx$

$a < x \le b$

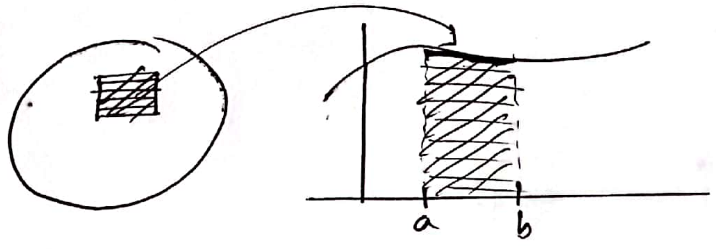
$a \le x < b$

$a < x < b$

<u>default</u>

→ lower limit $-\infty$

→ upper " $+\infty$



Total area will be 1

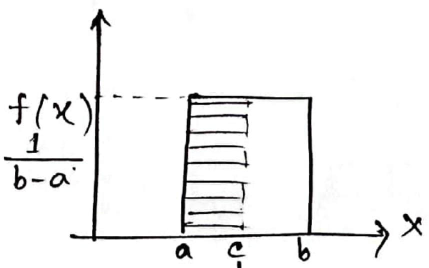
a to b is → CDF

CDF

$$\underset{\substack{\downarrow \\ \text{name of var.}}}{F_X}(x) = P(X \le x) = \begin{cases} \sum_{k \le x} P(k) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} f(t)\cdot dt & \text{" " " continuous} \end{cases}$$

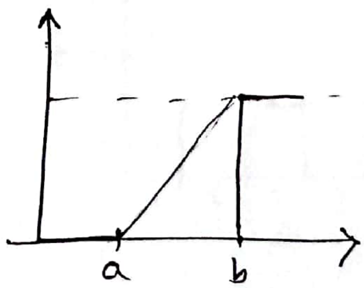
x → value X can take

read note $\Big\}$ $f(x) = \frac{1}{b-a}$



$\text{area} = (c-a)\cdot\frac{1}{(b-a)}$

$= \frac{c-a}{b-a}$



↳ CDF

<u>Note</u> :

$\underbrace{f(x)\cdot(b-a)}_{\text{area}} = 1$

$\Rightarrow f(x) = \frac{1}{b-a}$

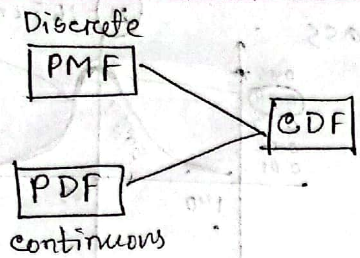
Upper limit = lower limit

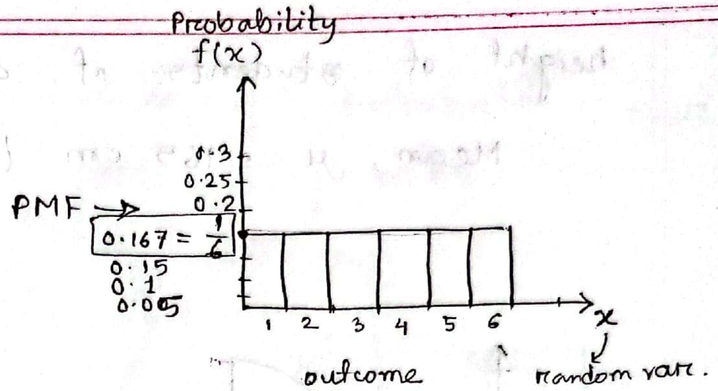
$\longmapsto$ area 0

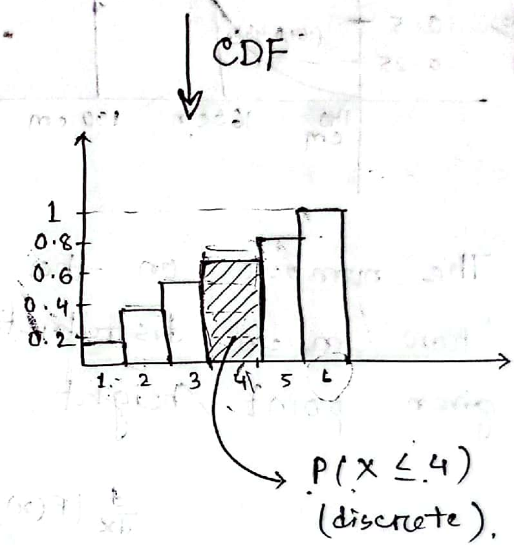
$\longrightarrow$ a single line

$\longrightarrow$ turns into Discrete R.V.

12.09.2023

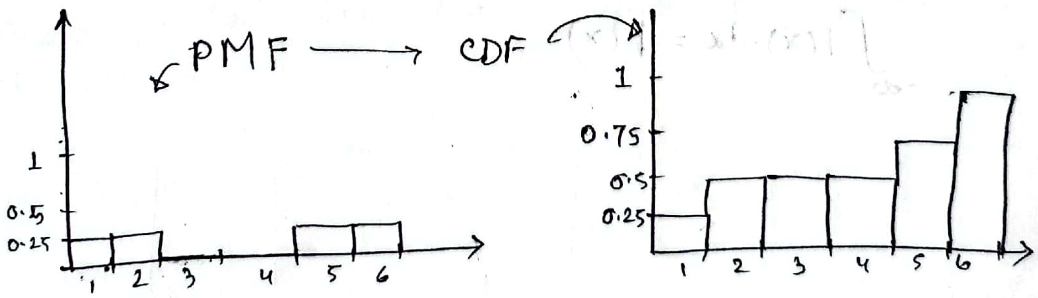






Imp property of CDF :

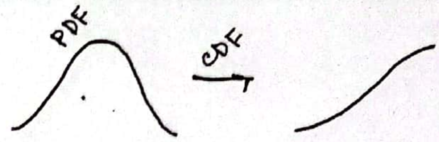
The final bar needs to be equal to 1.



The flatness in CDF indicates that there's no 'mass' around 3 and 4.

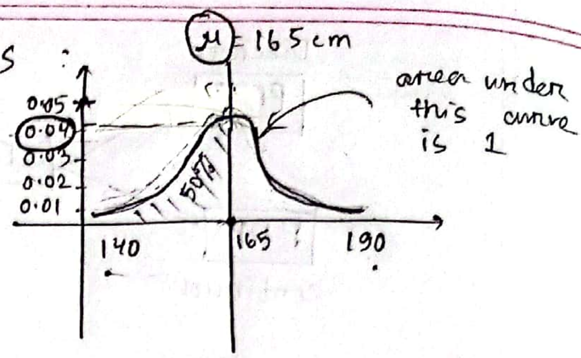
* 3, 4 উঠবে না

$$P(X \le 4) = P(X \le 2)$$
$$= P(X=1) + P(X=2) + \underbrace{P(X=3)}_{0} + \underbrace{P(X=4)}_{0}$$
$$= P(X \le 2)$$



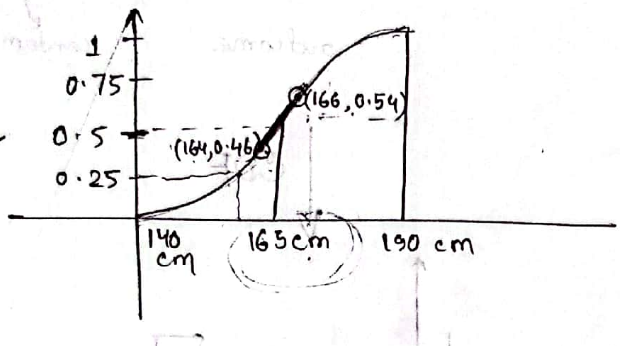
height of students of a class :

Mean, μ = 165 cm (let)



PDF

$\frac{rise}{run} = \frac{0.54-0.46}{166-164} = \frac{0.08}{2} = \boxed{0.04}$



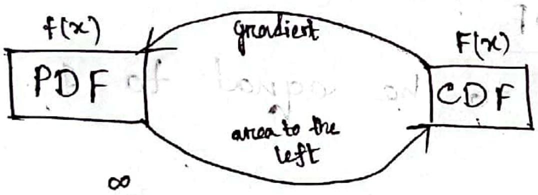
Gradient $= \frac{0.08}{2} = 0.04$

CDF -এর grad = PDF এর y axis এর value.

The number on the Y-axis actually telling us 'how much distribution is on the left of the given point/height.

$$\frac{d}{dx}(F(x)) = f(x)$$

PDF → f(x)
CDF → F(x)

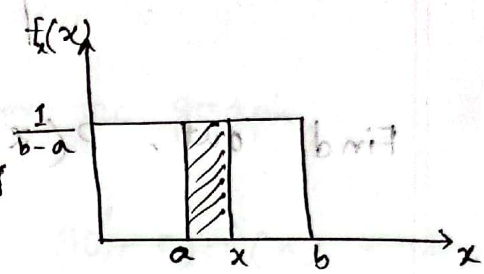


$$\int_{-\infty}^{\infty} f(x) \cdot dx = F(x)$$

Uniform R.V. :

A R.V. is said to be uniformly distributed over interval (0, 1) if it's PDF is given by,

$$f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$



CDF : $\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{PDF}}\, dx \xrightarrow{\text{for Uniform R.V}} \int_{0}^{1} 1 \cdot dx$

$= 1$

$A = B \times H$

$= (b-a) \times f(x)$

$\Rightarrow 1 = (b-a) \times f(x)$

$\Rightarrow \boxed{f(x) = \frac{1}{b-a}}$

$P(a \le x \le b)$

$= \int_{a}^{b} f(x) \cdot dx$

$= \int_{a}^{b} \frac{1}{b-a}\, dx = 1 \cancel{b-a}$?

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$

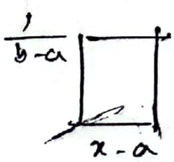
$f(x) = \frac{1}{b-a}$; $a<x<b$

$\mu = \frac{a+b}{2}$

$\sigma = \frac{b-a}{\sqrt{12}}$

$P(a \le x \le b) = \int_{a}^{b} \frac{1}{b-a} dx = 1$

(Q) Area under the curve upto the point x or to the left of x.

$\rightarrow$ 

Area $= (x-a) \cdot \frac{1}{b-a}$

$= \frac{x-a}{b-a}$

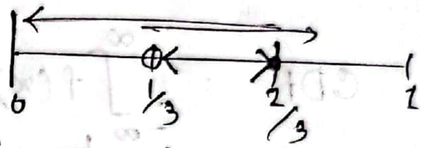
continuous R.V এর PDF $\longrightarrow$ shape of the curve denote করে

(9) Let x be continuous R.V, with PDF

$$f_x(x) = \begin{cases} 4x^3 , & 0 \le x \le 1 \\ 0 , & \text{otherwise} \end{cases}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Find out, $P(x \le \frac{2}{3} \mid x > \frac{1}{3})$



$\frac{1}{3} < x \le \frac{2}{3}$

$$\rightarrow \frac{P(\frac{1}{3} < x \le \frac{2}{3})}{P(X > \frac{1}{3})}$$

$$= \frac{\int_{1/3}^{2/3} 4x^3 \, dx}{\int_{1/3}^{1} 4x^3 \, dx}$$

$$= \frac{[x^4]_{1/3}^{2/3}}{[x^4]_{1/3}^{1}}$$

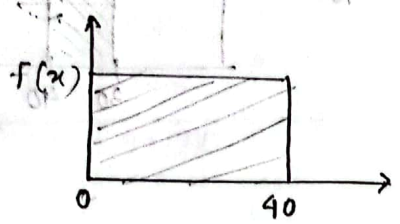
$$= \frac{(\frac{2}{3})^4 - (\frac{1}{3})^4}{1 - (\frac{1}{3})^4}$$

$$= \frac{3}{16}$$

(9) The amount of time a person must wait for a train to arrive in a certain town is uniformly distributed between 0 to 40 minutes.

a) Determine $f(x)$. b) Draw the graph.

→



area $= (40-0) \cdot f(x) = 1$

$\Rightarrow f(x) = \frac{1}{40}$

$$f(x) = \begin{cases} \frac{1}{40} & ; \ 0 < x < 40 \\ 0 & ; \ \text{otherwise} \end{cases}$$

you can include 0 and 40 if you want! Doesn't matter!

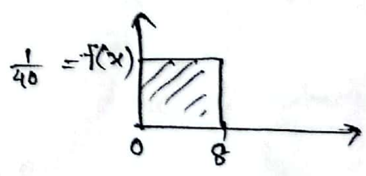
(Ans)

c) What is the probability that a person wait less than 8 minute.

→ $P(x < 8) = (b-a) \times f(x)$

$= (8-0) \times \frac{1}{40}$

$= \frac{1}{5} = 0.20 = 20\%$

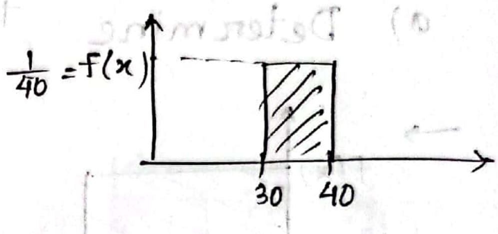


CamScanner

d) What is p that a person must wait more than 30 min ?

$\longrightarrow P(x > 30)$ or $P(30 < x < 40)$

$= (40 - 30) \cdot \frac{1}{40}$



$= \frac{1}{4}$

$= 25\%$

e) Calculate $P(10 < x < 26)$, $P(X = 20)$ and $P(X > 45)$

$(b-a) \cdot f(x)$

$(26 - 10) \cdot \frac{1}{40}$

$= \frac{2}{5} = 40\%$

0

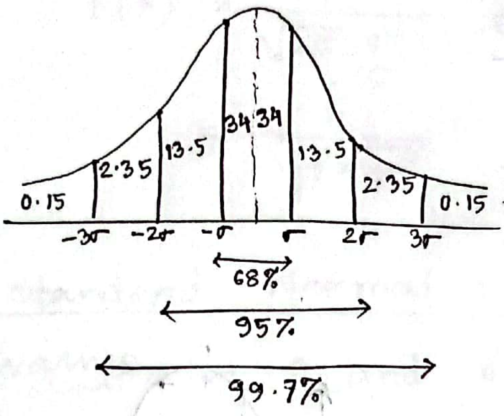
(No area at one point)

0

(between 0 to 40)

20.09.2023

Normal R.V. : Gaussian R.V. $\longrightarrow$ Bell shaped curve.



* x is a Normal R.V. with parameter μ and σ^2 if the density of x is given by,

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

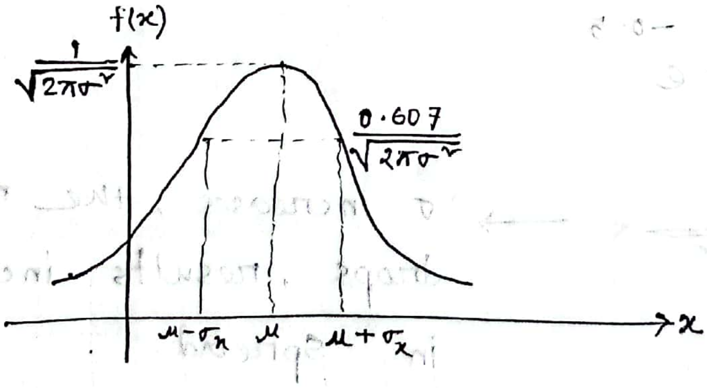
The density function is bell shaped curve that is symmetric around μ.

μ = Mean

σ = std. deviation

σ^2 = variance

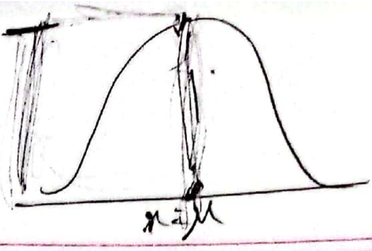
* Notation : $X \sim N(\mu, \sigma^2)$



$\sigma_x^2 \rightarrow$ always a positive number

$\mu \longrightarrow$ can be any no. between $-\infty$ to $+\infty$.

Spread of the distribution depends on σ.



PDF :

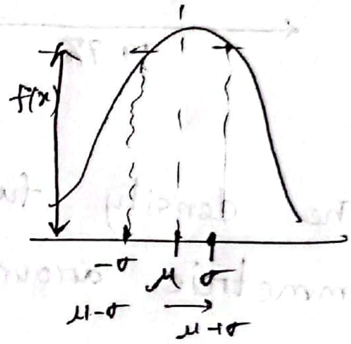
$\boxed{x = \mu}$,

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{\frac{-(\mu-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma}\, e^{0}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}$$



$\boxed{x = \mu - \sigma}$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(\cancel{\mu}-\sigma-\cancel{\mu})^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-0.5}$$

$$= \frac{0.607}{\sqrt{2\pi\sigma^2}}$$

$\longrightarrow$ σ increases, the magnitude drops, results increase in "spread.

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

area under the curve using integral of $f(x)$.

To make it 1,

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

$\boxed{x = \mu + \sigma}$,

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{\frac{-(\mu+\sigma-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-0.5} = \frac{0.607}{\sqrt{2\pi\sigma^2}}$$

Standard Normal : Normal distribution with parameter values $\mu = 0$ and $\sigma^2 = 1$.

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Notation : $Z \sim N(0,1)$

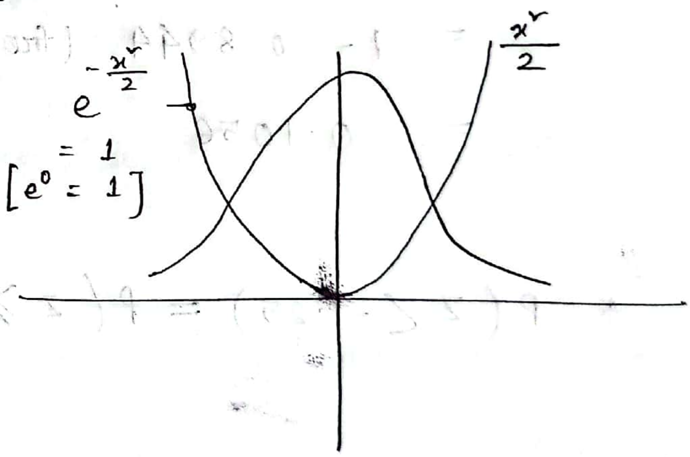
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

PDF

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

$e^{-\frac{x^2}{2}} = 1$

$[e^0 = 1]$



$$P(a \le x \le b) = \int_a^b f(x)\cdot dx$$

it's denoted by Z

$Z \sim N(0,1)$

* Pg 76

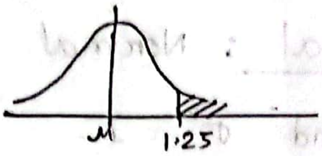
$\phi(1.25)$

CDF :

special notation is used (z)

$$F(z) = \phi(z) = P(Z < z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, dx$$

* $P(Z \geq 1.25)$



$\rightarrow 1 - P(Z < 1.25)$

$= 1 - \phi(1.25)$

$= 1 - 0.8944$ (from table)

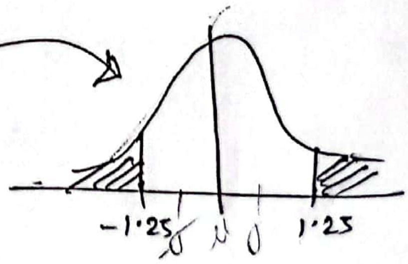
$= 0.1056$

Homework

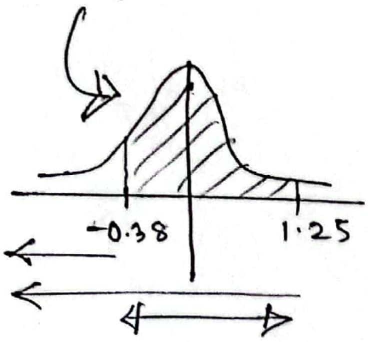
or,

$$\int_{1.25}^{\infty} \underbrace{f(x)}_{\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}} \cdot dx$$

* $P(Z \leq -1.25) = P(Z \geq 1.25)$



* $P(-0.38 \leq Z \leq 1.25) = P(Z \leq 1.25) - P(Z < -0.38)$

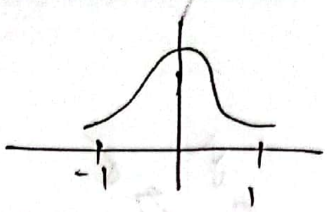


$= \phi(1.25) - \phi(-0.38)$

$= \underbrace{\phi(1.25)}_{0.8944} - 1 + \underbrace{\phi(0.38)}_{0.6480}$

$= 0.5424$

Note : $\phi(-z) = 1 - \phi(z)$



$-1 \le z \le 1 \rightarrow 68\%$
$-2 \le z \le 2 \rightarrow 95\%$

* $P(-1 \le z \le 1)$

$$= P(z \le 1) - P(z \le -1)$$
$$= \varphi(1) - \varphi(-1)$$
$$= \varphi(1) - 1 + \varphi(1)$$
$$= (0.8413 \times 2) - 1$$
$$= 0.6826$$

Standard Normal (CDF) :

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \, dx$$

* If X is a Gaussian (μ, σ) R.V. the CDF of X is,

$$F_X(x) = \varphi\left(\frac{x-\mu}{\sigma}\right)$$

* The probability that X is in interval (a, b) is,

$$P(a \le x \le b) = \varphi\left(\frac{b-x}{\sigma}\right) - \varphi\left(\frac{a-x}{\sigma}\right)$$

Using these formula, we can convert values of Gaussian R.V. X to equivalent values of the standard random variable, Z.

→ a sample value x of the R.V. X, the corresponding value of Z is,

$$z = \frac{x-\mu}{\sigma}$$

If X is Gaussian R.V. $(\overset{\mu}{61}, \overset{\sigma}{10})$, what is $P(X \le 46)$?

$\rightarrow \quad F_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$

$F_X(46) = \phi\left(\frac{46-61}{10}\right) = \phi\left(-\frac{15}{10}\right) = \phi(-3/2)$

$= \phi(-1.5)$

$\phi(-z) = 1 - \phi(z)$

$\therefore \phi(-1.5) = 1 - \phi(1.5)$

$= 1 - 0.933$

$= 0.067 = 6.7\%$

Exponential R.V. :

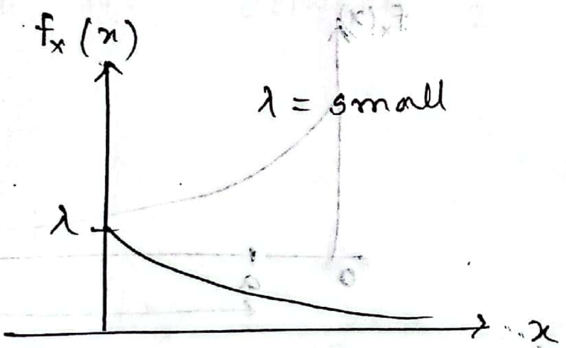
A continuous R.V. is said to be exponential R.V. with parameter λ, for ~~$\lambda <$~~ $\lambda > 0$, if it's PDF is given by.

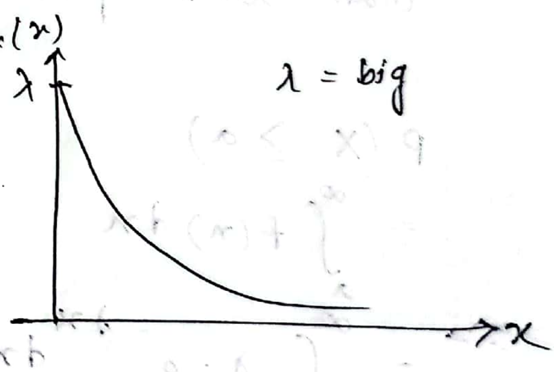
$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{\lambda = \frac{1}{\mu}}$$

* For -ve value of x, PDF, $f(x) = 0$.

↳ It's probability is zero.





$$f(x) = \lambda \cdot e^{-\lambda x} = \lambda e^{-\lambda \cdot 0} = \lambda$$

$$F_x(x) = P(X \le x)$$

$$F_X(a) = P(X \le a) = \int_{-\infty}^{a} f_x(x)\,dx$$

$$= \int_{0}^{a} f_X(x)\,dx \leftarrow \text{For exponential R.V.}$$

$$= \int_{0}^{a} \lambda \cdot e^{-\lambda x}\,dx$$

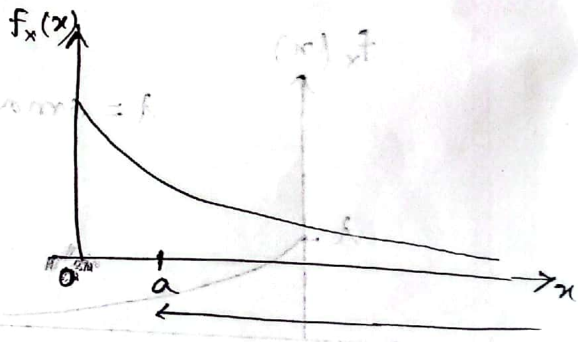
$$= \cancel{\lambda} \cdot \left[\frac{e^{-\lambda x}}{-\cancel{\lambda}} \right]_0^a$$

$$= -\left(e^{-\lambda a} - 1\right)$$

$$= 1 - e^{-\lambda a}$$

$$\boxed{F_X(x) = \int_{-\infty}^{x} f(y) \cdot dy = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}}$$

Q. Calculate the P of x taking a value greater than some point a.



$$P(X \geq a)$$

$$= \int_a^{\infty} f(x)\, dx$$

$$= \int_a^{\infty} \lambda \cdot e^{-\lambda x}\, dx$$

$$= \cancel{\lambda} \cdot \left[\frac{e^{-\lambda x}}{-\cancel{\lambda}} \right]_a^{\infty}$$

$$= -e^{-\infty} + e^{-\lambda a}$$

$$= e^{-\lambda a}$$

$$\boxed{P(X \geq a) = e^{-\lambda a}}$$

when $a = 0$, $P(X \geq a) = 1$

1 (Q) If job arrives 15 sec on an average, then $\lambda = 4$ per min, what is the P of waiting less than a equal to 30 sec? $P(t \leq 0.5)$

2 (Q) The P that a telephone call lasts no more than t min is often modeled as an exponential CDF as,

$$F_T(t) = \begin{cases} 1 - e^{-t/3}, & t \geq 0 \\ 0, & \end{cases}$$

What is PDF of the duration in min of a telephone conversation? What is P that a conversati will last between 2 to 4 min?

1 (A)

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & x \geq 0 \\ 0, & \end{cases}$$

time is always continuous

$* \; 1 - e^{-\lambda a}$

$= 1 - e^{-4 \times 0.5}$

$1 - e^{-4 \cdot 0.5}$

$1 - e^{-2}$

$T = 30$ sec $= 0.5$ min

$$P(T \leq 0.5) = \int_0^{0.5} 4 \cdot e^{-4t} \, dt$$

$$= \left[\frac{\cancel{4} \cdot e^{-4t}}{-\cancel{4}} \right]_0^{0.5}$$

$$= -e^{-4 \times 0.5} + e^{-4 \cdot 0}$$

$$= 1 - e^{-2} = 0.86466$$ (Ans)

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2 A

$$f_T(t) = \frac{d}{dt} F_T(t)$$

$$= \frac{d}{dt}\left(1 - e^{-t/3}\right)$$

$$= \frac{d}{dt}(1) - \frac{d}{dt}\left(e^{-t/3}\right)$$

$$= -\left(-\frac{1}{3} e^{-t/3}\right) = \frac{1}{3} e^{-t/3}$$

PDF $\rightarrow f_T(t) = \begin{cases} \frac{1}{3} \cdot e^{-t/3}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$ $\qquad \lambda = \frac{1}{3}$

We know, $f_x(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

$$P(2 \leq x \leq 4) = \int_2^4 f(t) \cdot dt$$

$$= \int_2^4 \frac{1}{3} \cdot e^{-t/3} \, dt$$

or,

$$P(2 \leq x \leq 4) = F_T(4) - F_T(2)$$

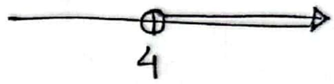
$$= (1 - e^{-\lambda 4}) - (1 - e^{-\lambda 2}).$$

$(1 - e^{-\lambda \cdot 4}) - (1 - e^{-\lambda \cdot 2})$

25.09.2023

(19) The time (in hour) required to repair a car is an exponentially distributed R.V. with parameter $\lambda = \frac{1}{2}$. What is the probability that repair time exceeds 4 hr? If it exceeds 4 hr then what is the probability that it will exceed 8 hour?

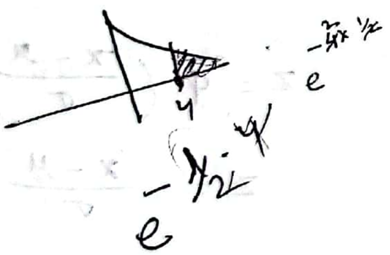
→ $f_T(t) = \lambda \cdot e^{-\lambda t}$



$\lambda = \frac{1}{2}$

$$P(T>4) = \int_4^\infty f(t) \cdot dt = \lambda \int_4^\infty e^{-\lambda t} dt$$

$$= \cancel{\lambda} \left[\frac{e^{-\lambda t}}{-\cancel{\lambda}} \right]_4^\infty$$

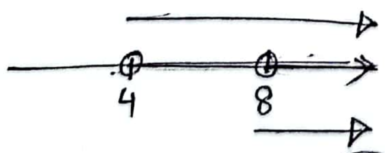


$$= -\left[e^{-\lambda t} \right]_4^\infty$$

$$= -\left(0 - e^{-4\lambda} \right)$$

$$= e^{-4 \times \frac{1}{2}} = e^{-2}$$

$$= 0.1353352$$



$$P(T>8 \mid T>4)$$

$$= \frac{P(T>8)}{P(T>4)} = \frac{e^{-\frac{1}{2} \cdot 8}}{e^{-\frac{1}{2} \cdot 4}}$$

$-\infty \longrightarrow 4 \longrightarrow 8 \longrightarrow \infty$ → common

* $P(A|B) = \dfrac{P(AB)}{P(B)}$

* $P(X \geq a) = e^{-\lambda a}$

common part

$$P(T>8 \mid T>4) = \frac{P(T>8)}{P(T>4)}$$

$$= \frac{e^{-\cancel{\frac{1}{2}} \times \cancel{8}^{4}}}{e^{-\cancel{\frac{1}{2}} \times \cancel{4}^{2}}}$$

$$= e^{-4+2} = e^{-2}$$

(29) Give that, $X \sim N(\overset{\mu}{50}, \overset{\sigma^2}{10})$; what is the value of mean & standard deviation? What value of X has a Z-score of 1.4? What is the Z-score that corresponds to $X = 30$?

→ $\mu \rightarrow 50$ (Mean)

$\sigma = \sqrt{10}$ (std. d.)

$z = \phi\left(\dfrac{x-\mu}{\sigma}\right)$

$z = \dfrac{x-\mu}{\sigma}$

default :
σ^2 দেবো,
define করে ভাবে =>
দিলে ,
দেবো

$$Z = \frac{x-\mu}{\sigma}$$

$$\boxed{x = z\sigma + \mu}$$

$$= 50 + (1.4 \times \sqrt{10})$$

$$Z = \frac{x-\mu}{\sigma} = \frac{30-50}{\sqrt{10}}$$

(39) The continuous R.V. X has PDF $f(x)$ which is given by,

$$f(x) = \begin{cases} k(x^2 - 2x + 2) & ; \; 0 < x \le 3 \\ 3k & ; \; 3 < x \le 4 \\ 0 & , \text{ otherwise} \end{cases}$$

1) Find out value of k :

Total, $P = 1 = \int_{-\infty}^{\infty} f(x)\, dx$

$$\int_0^3 k(x^2 - 2x + 2)\, dx \qquad \Bigg| \qquad \int_3^4 3k\, dx$$

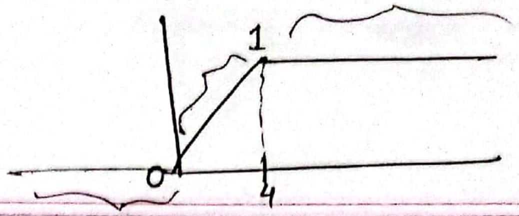
$$= k\left[\frac{x^3}{3} - \cancel{2}\cdot\frac{x^2}{\cancel{2}} + 2x\right]_0^3 \qquad \Bigg| \qquad = 3k\cdot[x]_3^4$$

$$= k\cdot(9 - 9 + 6) = 6k \qquad \Bigg| \qquad = 3k(4-3)$$

$$= 3k$$

$\therefore \; 6k + 3k = 1$

$\Rightarrow \boxed{k = 1/9}$



2) Find out the CDF ?

$$F(x) = \int \frac{1}{9}(x^2 - 2x + 2)\, dx$$

$$= \frac{1}{9}\left(\frac{x^3}{3} - 2 \cdot \frac{x^2}{2} + 2x + c \right)$$

So, $F(0) = 0$

$$\boxed{F(x) = \frac{1}{9}\left(\frac{x^3}{3} - x^2 + 2x \right)}$$

$k = 1/9$

$$F(x) = \int 3k\, dx = \int \frac{1}{3}\, dx = \frac{1}{3}x + D$$

$$F(4) = \frac{1}{3} \times 4 + D = 1$$

$$\Rightarrow D = -1/3$$

$$\boxed{F(x) = \frac{1}{3}x - 1/3}$$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{9}\left(\frac{1}{3}x^3 - x^2 + 2x\right), & 0 < x \leq 3 \\ \frac{1}{3}x - 1/3, & 3 < x \leq 4 \\ 1, & x > 4 \end{cases}$$

variance বের করতে বললে Expectation এর formula ব
করতে হবে।

3) Find the mean of X or $E[X]$

<u>Expectation of X :</u>

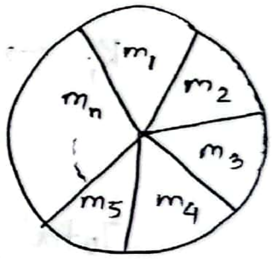
→ Weighted Avg of the possible values of x.

→ Expected value of a R.V. is the theoretical Mean of the R.V.

→ In Simple term, expected value or mean is a measure of central tendency. of a probability distribution.

Total money received

$$= m_1K_1 + m_2K_2 + m_3K_3 + \cdots + m_iK_i$$



$$avg = \frac{m_1K_1 + m_2K_2 + \cdots + m_iK_i}{K}$$

↑ K times roll

$K_1 \rightarrow m_1$

$K_2 \rightarrow m_2$

...

$$P_i = \frac{K_i}{K}$$

$$= m_1P_1 + m_2P_2 + m_3P_3 + \cdots$$

27.9.2023

Expectation of R.V :

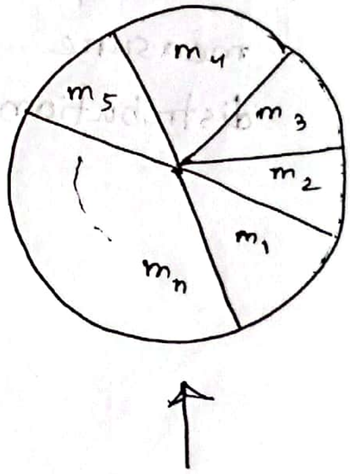
→ Expectation of x → weighted avg

→ Theoretical mean of x

→ A measure of central tendency

→ It represents that the avg value that a R.V is likely to take on.

K → number of times the wheel is spinned



K_i → is the number of time that the o/p is m_i

Total received amount (Per spin)

$$= \frac{m_1 K_1 + m_2 K_2 + m_3 K_3 + \cdots + m_n K_n}{K}$$

$$P_i = \frac{K_i}{K}$$

m_i আসবে K_i times

$$= m_1 \cdot P_1 + m_2 P_2 + m_3 P_3 + \cdots + m_n P_n$$

$$\boxed{E[X] = \sum_x x \cdot P_{X}(x)}$$

output o/p এর Prob.

$x \rightarrow$ value
$X \rightarrow$ R.V.

x	P
1	$\frac{1}{2}$
2	$\frac{1}{2}$

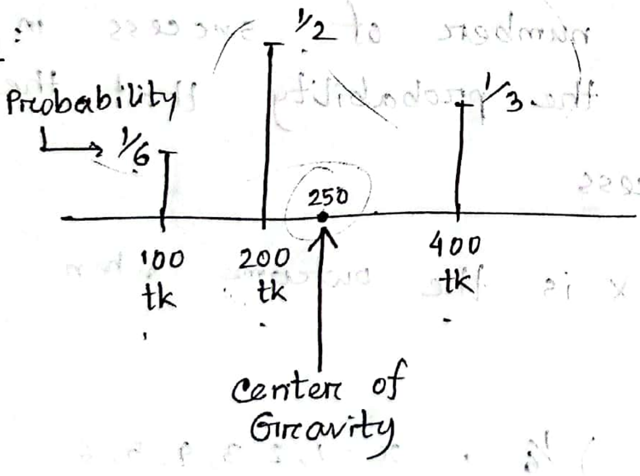
$$E[X] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}$$

$$= \frac{3}{2} \longrightarrow$$ Expectation can be more than 1.

$$\boxed{E[X] = \sum_{x:\, p(x)>0} x \cdot P(x)}$$

Expected value of X is a weighted avg of the possible values that X can take on, each value being weighted by it's probability that X assumes that value.

Ex :



$$100 \cdot \frac{1}{6} + 200 \cdot \frac{1}{2} + 400 \cdot \frac{1}{3}$$

$$= 250 \longrightarrow E[X]$$

Expectation of Bernoulli R.V :

$$P_X(x) = \begin{cases} P, & \text{if } x = 1 \\ 1-P, & \text{if } x = 0 \end{cases}$$ ← PMF for bernoulli R.V.

$P(0) = 1-P \rightarrow$ failure

$P(1) = P \rightarrow$ success

$E[X] = \sum_x x \cdot P(x) = 0 \cdot (1-P) + 1 \cdot P = P$

$\boxed{\text{For Bernoulli R.V, } E[X] = P}$

Show that, the expected number of success in a single trial is just the probability that the trial will be a success.

Q) Find out $E[X]$ where X is the outcome when fair die is tossed?

→ The PMF is, $P_X(x) = \begin{cases} 1/6, & x = 1,2,3,4,5,6 \\ 0, & \text{otherwise} \end{cases}$

$$E[X] = \sum_{x=1}^{6} x \cdot P(x) = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right)$$

$$= \frac{7}{2}$$ (Ans)

$\frac{21}{6}$

$E[X] = np$

12

Expectation of binomial R.V :

$$P(i) = \binom{n}{i} \cdot p^{i} \cdot (1-p)^{n-i} \quad \text{when } i = 0, 1, 2, 3 \ldots, n$$

$$\binom{n}{i} = \frac{n!}{(n-i)! \cdot i!}$$

$$E[X] = \sum_{i=1}^{n} i \cdot P(i) = \sum_{i=1}^{n} i \cdot \binom{n}{i} \cdot p^{i} \cdot (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i \cdot \frac{n!}{(n-i)!\, i!} \cdot p^{i} \cdot (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \cancel{i} \frac{n!}{(n-i)!\, (i-1)!\cancel{i}} \cdot p^{i} \cdot (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{n!}{(n-i)!\, (i-1)!} \cdot p^{i} \cdot (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{n \cdot (n-1)!}{(n-i)!\, (i-1)!} \cdot p \cdot (p)^{i-1} \cdot (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\, (i-1)!} \cdot p^{i-1} \cdot (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{\{(n-1)-(i-1)\}!\, (i-1)!} \, p^{i-1} \cdot (1-p)^{n-i}$$

$n-i = (n-1)-(i-1)$

let, $K = i-1$,

So, when, $i = 1$, $K = 0$

$i = n$, $K = n-1$

$$E[X] = np\sum_{k=0}^{n-1} \frac{(n-1)!}{\{(n-1)-k\}!\,k!}\cdot p^{k}(1-p)^{n-i}$$

$$= np\cdot\sum_{k=0}^{n-1} \frac{(n-1)!}{\{(n-1)-k\}!\,k!}\cdot p^{k}\cdot(1-p)^{n-i+1-1}$$

$$= np\cdot\sum_{k=0}^{n-1} \frac{(n-1)!}{\{(n-1)-k\}!\,k!}\cdot p^{k}\cdot(1-p)^{(n-1)-k}$$

$$\boxed{E[X] = np\cdot\sum_{k=0}^{n-1} \binom{n-1}{k}\cdot p^{k}\cdot(1-p)^{(n-1)-k}}$$

$$\sum_{i=0}^{\infty} P(i) = 1 = (p+(1-p))^{n}$$

$$\sum_{i=0}^{n} \binom{n}{i} p^{i}(1-p)^{n-i} = (p+(1-p))^{n} = 1$$

$$E[X] = np(p+(1-p))^{(n-1)}$$

$$= np\cdot(1)^{n-1} = np \longrightarrow E[X]$$

trials success

Prove that, The expected number of success in n independent trials is n multiplied by the Prob. that the trial result in success.

<u>Expectation of geometric R.V :</u>

From def^n of geometric R.V,

$$P_X(n) = P\{X = n\} = (1-p)^{n-1} \cdot p^1$$

From definition, $$E[X] = \sum_{x=1}^{\infty} x \cdot P(x)$$

$$= \sum_{n=1}^{\infty} n \cdot p(1-p)^{n-1}$$

For geometric series, $\sum_{n=1} ar^n$

Infinite case : $\sum_{n=1}^{\infty} ar^n$

Finite case : $\sum_{k=0}^{n-1} ar^k$

$$S = ar^0 + ar^1 + ar^2 + \cdots + ar^{n-1} \quad \text{--- (1)}$$

$$S \cdot r = ar^1 + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \quad \text{--- (2)}$$

(1) − (II)

$$S - Sr = ar^0 - ar^n$$

$$\Rightarrow S(1-r) = a(1-r^n)$$

$$\Rightarrow S = \frac{a(1-r^n)}{1-r}$$

So, for a known range ($n-1$ in this case),

$$\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r} \quad \text{--- (3)}$$

For infinite case,

$$\sum_{n=0}^{\infty} ar^n = ? \qquad n = \infty$$

$$\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}$$

r^∞ will be very very small if r is fraction,

So, $$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \text{ ——— (iv)}$$

$$E[X] = \sum_{n=1}^{\infty} n \cdot P(1-P)^{n-1}$$

let, $\boxed{q = 1-P}$

$$\therefore E[X] = P \sum_{n=1}^{\infty} n \cdot q^{n-1}$$

P & q all are small (fraction)

$$\therefore E[X] = P \sum_{n=1}^{\infty} \frac{d}{dq}(q^n)$$

$$= P \cdot \frac{d}{dq}\left[\left(\sum_{n=0}^{\infty} q^n\right) - q^0\right]$$

$$= P \cdot \frac{d}{dq}\left(\frac{1}{1-q} - 1\right)$$

$$= P \cdot \frac{d}{dq}\left(\frac{\cancel{1} - \cancel{1} + q}{1-q}\right)$$

$$= P \cdot \frac{d}{dq}\left(\frac{q}{1-q}\right)$$

$$\sum_{k=0}^{\infty} ar^k$$
$$\sum_{n=0}^{\infty} 1 \cdot q^n$$
$$= \frac{1}{1-q}$$
a
r

$$= p \cdot \frac{(1-q)\cdot 1 - q\cdot(-1)}{(1-q)^2}$$

$$= p \cdot \frac{1-\cancel{q}+\cancel{q}}{(1-q)^2}$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \longrightarrow E[X]$$

The expected number of independent trials we need to perform until we attain our first success is equal to the reciprocal of the probability that any one trial results in success.

If a continuous R.V having the PDF f(x).

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

Expectation of Uniform R.V. :

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} ; & \text{if } \alpha \le x \le \beta \\ 0 \end{cases}$$

$$E[X] = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta - \alpha} dx$$

$$= \left[\frac{x^2}{2(\beta - \alpha)} \right]_{\alpha}^{\beta}$$

$$= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} = \frac{\beta + \alpha}{2}$$

The expected value of a R.V uniformly distributed over interval (α, β) is just the midpoint of the interval.

E[X] of Exponential R. V. :

If X is a RV which exponentially distributed with parameter λ.

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & \text{if } x \ge 0 \\ 0 , & \text{otherwise.} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \int_{0}^{\infty} \underbrace{x}_{u} \cdot \underbrace{\lambda \cdot e^{-\lambda x} \, dx}_{dv}$$

$$= x\left(-e^{-\lambda x}\right)\Big|_0^{\infty} - \int -\left(e^{-\lambda x}\right) \cdot dx$$

$$= \left[-x \cdot e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} \cdot dx$$

$$= \underbrace{\left[-\infty \cdot e^{-\lambda \infty} + 0 \cdot e^{-\lambda \cdot 0}\right]}_{0} + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty}$$

$$= -\left(\frac{e^{0}}{-\lambda}\right)$$

$$= \frac{1}{\lambda}$$

$\int u \, dv = uv - \int v \, du$

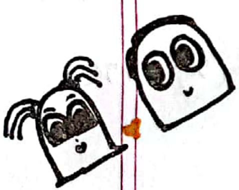
$x = u$

$\Rightarrow \frac{dx}{dx} = \frac{du}{dx}$

$\Rightarrow du = dx$

$v = -e^{-\lambda x}$

$dv = \lambda e^{-\lambda x} dx$



By def, PDF of Normal R.V.

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (-\infty < x < \infty)$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sigma} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$x = (x-\mu) + \mu$,

$$E[X] = \frac{1}{\sqrt{2\pi} \cdot \sigma} \int_{-\infty}^{\infty} ((x-\mu)+\mu) \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sigma} \int_{-\infty}^{\infty} (x-\mu) \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$+ \mu \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

let, $y = x - \mu$,

$$E[X] = \frac{1}{\sqrt{2\pi} \cdot \sigma} \int_{-\infty}^{\infty} y \cdot e^{\frac{-y^2}{2\sigma^2}} dy + \mu \int_{-\infty} f(x) \cdot dx$$

$$\frac{1}{\sqrt{2\pi} \cdot \sigma} \int_{-\infty}^{\infty} y \cdot e^{-\frac{y^2}{2\sigma^2}} dy$$

$y^2 = x$

$\Rightarrow 2y \cdot dy = dx$

$\Rightarrow y \, dy = \frac{1}{2} dx$

$$= \frac{1}{\sqrt{2\pi} \cdot \sigma} \int_{+\infty}^{+\infty} e^{-\frac{x}{2\sigma^2}} \cdot \frac{1}{2} dx$$

$$= 0$$

Binomial $\longrightarrow \binom{n}{i}(p^i)(1-p)^{n-i} = P(i)$

$$E[X] = 0 + \mu \underbrace{\int_{-\infty}^{\infty} f(x) \cdot dx}_{1}$$

(area under the curve f(x))

$$= \mu \text{ (Ans)}.$$

(Q) consider two independent coin tosses, each with $3/4$ probability of a head and let X be the number of heads obtained. This binomial R.V with parameter $n = 2$, and $p = \frac{3}{4}$, $E[X] = ?$

$$P(0) = \binom{2}{0}\left(\frac{3}{4}\right)^0 \cdot \left(1 - \frac{3}{4}\right)^{2-0} = \left(\frac{1}{4}\right)^2$$

$$P(1) = \binom{2}{1}\left(\frac{3}{4}\right)^1 \left(1 - \frac{3}{4}\right)^{2-1} = 2 \cdot \frac{3}{4} \cdot \frac{1}{4}$$

$$P(2) = \binom{2}{2}\left(\frac{3}{4}\right)^2 \left(1 - \frac{3}{4}\right)^{2-2} = \left(\frac{3}{4}\right)^2$$

So, the PMF will be,

$$P_X(k) = \begin{cases} (1/4)^2, & \text{when } k = 0 \\ 2 \cdot \frac{3}{4} \cdot \left(\frac{1}{4}\right) & \text{when } k = 1 \\ (3/4)^2, & \text{when } k = 2 \end{cases}$$

$$E[X] = 0\left(\frac{1}{4}\right)^2 + 1 \cdot \left(2 \cdot 3/4 \cdot \frac{1}{4}\right) + 2\left(\frac{3}{4}\right)^2$$

$$= \frac{3}{2} \text{ (Ans)}$$

4.10.2023

Suppose X has the following PMF, $P(0) = 0.2$, $P(1) = 0.5$, $P(2) = 0.3$.

Expectation of a function of a R.V. :

$E[X] = \sum_x x \cdot P_X(x)$

$E[X] = (0 \times 0.2) + (1 \times 0.5) + (2 \times 0.3) = 1.1$

$Y = E[X^2] \longrightarrow$ Y is a R.V. that can take the values of ,

$Y = 0^2, 1^2, 2^2$

$= 0, 1, 4$

$P_Y(0) = 0.2$

$P_Y(1) = 0.5$

$P_Y(4) = 0.3$

$\sum_y y \cdot P_Y(y)$

$E[X^2] = (0 \times 0.2) + (1 \times 0.5) + (4 \times 0.3) = 1.7$

$\therefore 1.7 = E[X^2] \neq \{E[X]\}^2 = 1.21$

$P(X) = P(\ldots) = 0.3$

Q (continuous case)

If X is a uniformly distributed over 0, 1 then calculate $E[X^3] = ?$

Poisson $\xrightarrow{\ \frac{\lambda^i}{i!} e^{-\lambda}\ }$ $var = \lambda$, $E[x] = \lambda$

$Y = X^3$

$0 \le a \le 1$ (Uniform) is true then,

$$F_Y(a) = P\{Y \le a\}$$
$$= P\{X^3 \le a\}$$
$$= P\{X \le a^{1/3}\}$$

$$F_Y(a) \rightarrow a^{1/3}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$f_y(a) = \frac{d}{da} F_Y(a) = \frac{d}{da}(a^{1/3}) = \frac{1}{3} a^{-2/3}$$

$$E[Y] = E[X^3] \frown \int_{-\infty}^{\infty} a \cdot f_Y(y)\, da$$

$$= \int_0^1 a \cdot \frac{1}{3} \cdot a^{-2/3}\, da$$

$$= \frac{1}{3}\int_0^1 a^{1/3}\, da = \frac{1}{3} \cdot \left[\frac{a^{4/3}}{4/3}\right]_0^1$$

$$\cancel{\frac{1}{3}} \cdot \frac{\cancel{3}}{4}(1-0)$$

$$= \frac{1}{4}$$

Property 1 : Expectation of function :

If X is a discrete R.V. with PMF $P_X(x)$ for any real valued function, or in other words, the derived R.V, $Y = g(x)$, the expected value of Y (or $g(x)$) is given by,

$$E[Y] = E[g(x)] = \sum_{x : P(x) > 0} g(x) \cdot P(x)$$

$$= \sum_{i=1}^{n} g(x) \cdot P(x) \longrightarrow \text{discrete case}$$

$$E[Y] = E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) \cdot dx$$

$\longrightarrow$ continuous case

* $$P_x(x) = \begin{cases} 0.2 & \text{when } x = 0 \\ 0.5 & \text{when } x = 1 \\ 0.3 & \text{''} \quad x = 2 \end{cases}$$

let, $g(x) = x^2$

$$\therefore E[X^2] = \sum_{x=0}^{2} g(x) \cdot P(x)$$

$$= (0^2 \times 0.2) + (1^2 \times 0.5) + (2^2 \times 0.3)$$

$$= 1.7$$

For continuous case,

$$Y = x^3 = g(x) \qquad \int_0^1 g(x)\cdot dx \quad \underbrace{\int_0 f(x)\,dx}_{1}$$

$$E[X^3] = \int_0^1 g(x)\underbrace{f(x)}_{1}dx$$

$$= \int_0^1 x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}.$$

(9) Given PMF is, $f_X(x) = \begin{cases} 2/3, & x = 1 \\ 1/3, & x = 2 \end{cases}$

calculate $E[x^2+1]$

→ let, $Y = X^2 + 1$

PMF of $Y \Rightarrow f_Y(y) = \begin{cases} 2/3, & y = 2 \\ 1/3, & y = 5 \end{cases}$

$$E[Y] = \left(2\times\frac{2}{3}\right) + \left(5\times\frac{1}{3}\right) = 3$$

$$E[X] = \left(1\cdot\frac{2}{3}\right) + \left(2\times\frac{1}{3}\right) = \frac{4}{3}$$

$$E[Y] = \int_{-\infty}^{\infty} \underbrace{g(x)\cdot f(x)}_{\#}\cdot dx$$

$$E[X] = \int_{-\infty}^{\infty} x\cdot f(x)\cdot dx$$

remember, $E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y) \cdot dy$

Advantage of ① is we do not need to find out the PDF of Y.

<u>Property 2</u> : n^{th} moment of X :

If X is a R.V. then expected value of R.V. is referred to as the mean or the first moment of X.

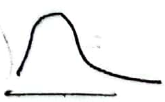
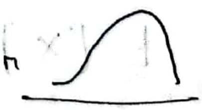
$E[X] = \mu$ = the first moment of X.

$$E[X^n] = \begin{cases} \sum_{x: P(x)>0} x^n \cdot P(x) & \text{if X is discrete.} \\ \int_{-\infty}^{\infty} x^n \cdot f(x) \cdot dx & \text{if X is continuous.} \end{cases}$$

0^{th} moment $\longrightarrow$ total prob.

1^{st} " $\longrightarrow$ mean

2^{nd} " $\longrightarrow$ variance

3^{rd} " $\longrightarrow$ skewness  or 

4^{th} " $\longrightarrow$ Kurtosis  কিত পরিমাণে data

$E[\overset{\text{constant}}{2}] = 2$

Linearity :

$E[X]$ is linear, if a and b are constant,

$E[ax+b] = aE[X] + b$

Discrete

$$E[g(x)] = \sum_{x:P(x)>0} g(x) \cdot P(x)$$

$$= \sum_{x:P(x)>0} (ax+b) \cdot P(x)$$

$$= a \sum_{x:P(x)>0} x \cdot P(x) + b \underbrace{\sum_{x:P(x)>0} P(x)}_{1}$$

$$= aE[X] + b$$

Continuous :

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) \cdot dx$$

$$= \int_{-\infty}^{\infty} (ax+b) f(x) \cdot dx$$

$$= a \underbrace{\int_{-\infty}^{\infty} x \cdot f(x) \cdot dx}_{E[X]} + b \underbrace{\int_{-\infty}^{\infty} f(x) \cdot dx}_{1}$$

$$= aE[X] + b$$

Variance :

$$\boxed{Var[X] = E\{X - E[X]\}^2}$$

$$Var[X] = \int_{-\infty}^{\infty} \{X - E[X]\}^2 f(x) \cdot dx$$

$$= \int_{-\infty}^{\infty} [x^2 - 2x \cdot E[X] + E[X]^2] f(x) \cdot dx$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f(x) dx - 2 E[X] \underbrace{\int_{-\infty}^{\infty} x \cdot f(x) dx}_{E[X]}$$

$$+ E[X]^2 \underbrace{\int_{-\infty}^{\infty} f(x) = dx}_{1}$$

$$\boxed{Var[X] = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx - E[X]^2}$$

Property of independence :

$$E[X + Y] = E[X] + E[Y]$$

$$E[XY] = E[X] \cdot E[Y]$$

• Show that the variance of a R.V. equals the expected value of the square of the R.V minus the square of the expected value of the R.V.

$$Var[X] = E[X^2] - E[X]^2$$

Example : Calculate Var [X] when X represents the outcome when a fair die is rolled ?

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

$$E[X] = 1 \cdot \tfrac{1}{6} + 2 \times \tfrac{1}{6} + 3 \times \tfrac{1}{6} + 4 \cdot \tfrac{1}{6} + 5 \cdot \tfrac{1}{6} + 6 \cdot \tfrac{1}{6}$$

$$= \frac{7}{2}$$

$$E[X^2] = 1^2 \times \tfrac{1}{6} + 2^2 \times \tfrac{1}{6} + 3^2 \times \tfrac{1}{6} + 4^2 \times \tfrac{1}{6} + 5^2 \times \tfrac{1}{6} + 6^2 \cdot \tfrac{1}{6}$$

$$= \frac{91}{6}$$

$$\therefore Var[X] = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = 2.91.$$

(Ans).

⑨ 1) what will be PMF of R ?

$\rightarrow$ $P(R=0) = \frac{1}{8} + \frac{1}{8} = 1/4$.

$P(R=2) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{4}$

PMF of R is $P_R(r) = \begin{cases} 1/4, & \text{when } r = 0 \\ 3/4, & \text{when } r = 2. \end{cases}$

2) what will be expected value, $E[X]$?

$\rightarrow$ $E[R] = \sum x \cdot P_x(x)$

$= (0 \times 1/4 + 2 \times 3/4) = 3/2$

3) what will be the expected value of the function $V = g(R) = 4R + 7 = ?$

$\rightarrow Y = 4R + 7$

$E[Y] = E[g(R)] = \sum_{x: P(x) > 0} (4R + 7) \cdot P(r)$

$= \{(4 \times 0 + 7) \times \frac{1}{4}\} + \{(4 \times 1 + 7) \times \frac{3}{4}\}$

$= 7 \cdot \frac{1}{4} + 15 \cdot \frac{3}{4}$

$= 13$

number of pages X of a FAX

Or,

$E[aX+b] = a\,E[X]+b$

$\Rightarrow E[4R+7] = 4 \times \frac{3}{2} + 7 \quad \longleftarrow \quad a=4,\ b=7$

$= 13$

(Ans).

Example (expected value):

1) $Y = g(x) = \begin{cases} 10.5x - 0.5x^2 & ;\ 1 \le x \le 5 \\ 50, & 6 \le x \le 10 \end{cases}$ in cents

Page 1 $\to$ 10 cent
" 2 " 19 "
" 3 " 27 "
" 4 " 34 "
" 5 " 40 "
" 6 " 50 "
" 7 " 50 "
" 8 " 50 "
" 9 " 50 "
" 10 " 50

2) $P_X(x) = \begin{cases} 1/4, & x = 1,2,3,4 \\ 0, & \text{otherwise} \end{cases}$

$$P_Y(y) = \begin{cases} 1/4, & y = 10, 19, 27, 34 \\ 0, & \text{otherwise} \end{cases}$$

$$E[Y] = \left(\tfrac{1}{4} \times 10\right) + \left(\tfrac{1}{4} \times 19\right) + \left(\tfrac{1}{4} \times 27\right) + \left(\tfrac{1}{4} \times 34\right)$$

$$= 22.4 \text{ cents}$$

3) The P of the cost of the FAX to be 0.10\$ (when no. of P is 1) = 0.15

$$P_Y(50) = P(6) + P(7) + P(8) = 0.1 + 0.1 + 0.1$$

$$= 0.3$$

$$P_Y(y) = \begin{cases} 0.15, & y = 10, 19, 27, 34 \\ 0.1, & y = 40 \\ 0.3, & y = 50 \\ 0, & \text{otherwise} \end{cases}$$

$$E[Y] = (10 + 19 + 27 \times 34) \times 0.15 + (0.1 \times 40) + (0.3 \times 50)$$

$$= 32.5$$

decision নিবো $\longrightarrow$ expected value use করে

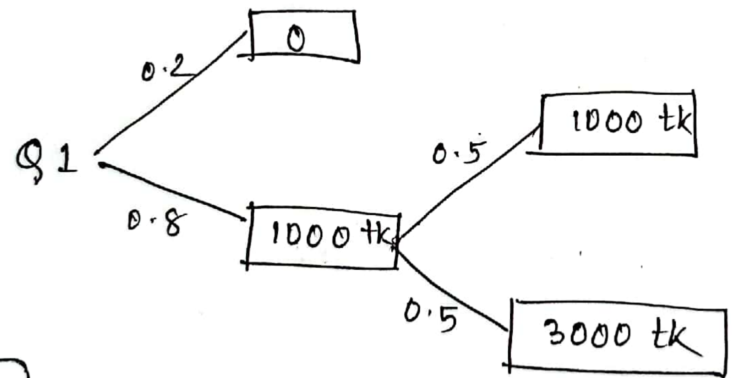
quiz $\longrightarrow$ 2 questions

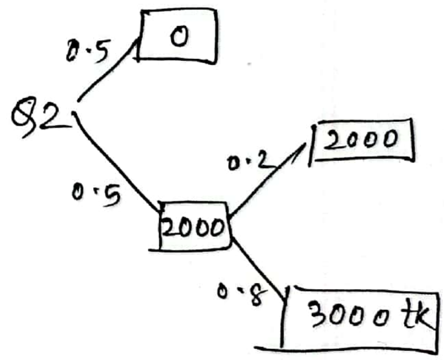
must decide which ques. to answer first.

$Q1 \xrightarrow[\text{answered)}]{P(\text{correctly}} 0.8 \xrightarrow{\text{money}} 1000 \text{ tk}$

$Q2 \xrightarrow[\text{answered)}]{P(\text{correctly}} 0.5 \xrightarrow{\text{"}} 2000 \text{ tk}$

If the first question (eith 1 or 2) is wrongly answered the quiz terminates.





$\boxed{Q1}$

$P_x(0) = 0.2$

$P_x(1000) = 0.8 \times 0.5 = 0.4$

$P_x(3000) = 0.8 \times 0.5 = 0.4$

$E[X] = (0 \times 0.2) + (1000 \times 0.4) + (3000 \times 0.4)$

$= 1600 \text{ tk}$

$\boxed{Q2}$

$E[X] = (0 \times 0.5)$

$+ (2000 \times 0.1)$ $\quad 0.2 \times 0.5$

$+ (3000 \times 0.4)$ $\quad 0.5 \times 0.8$

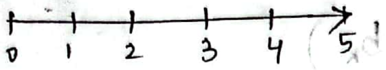
$= 1400 \text{ tk.}$

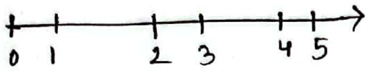
So, Q1 আগে answer করব।

R

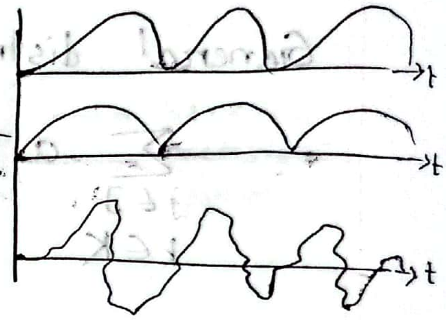
↳ Study of random function of time

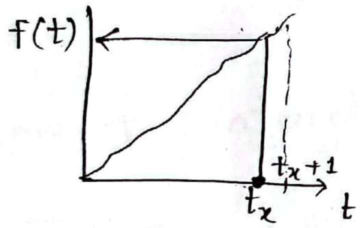
STOCHASTIC PROCESS : Time series analysis এর জন্য

Regular 

Irregular 

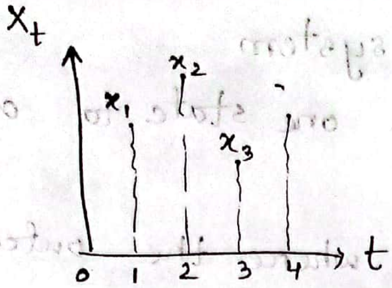
3 components :
1) Trend →
2) Seasonal $\xrightarrow{\text{pattern same}}$
3) Residual
↳ কোনো pattern নেই

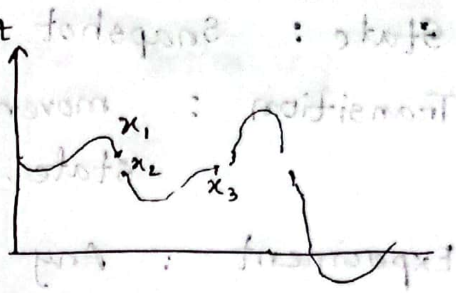




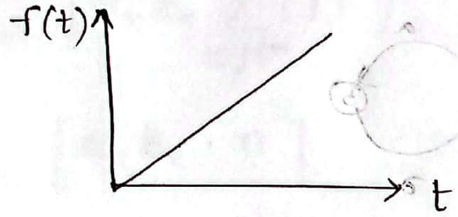
A stochastic process is a collection of random variables indexed by time.

A stochastic process $X(t)$ consists of an experiment with probability measure $P(\cdot)$ defined on a sample space S and a function that assigns a time function $X(t, s)$ to each outcome s in the sample space of the experiment



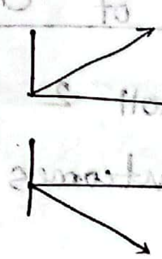


1) $f(t) = t$ with probability 1

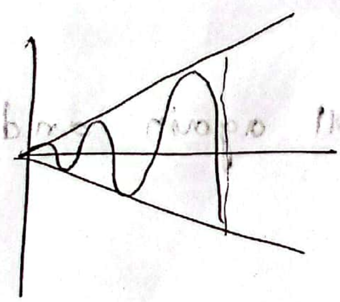


2) $F(t) = t \ \forall_t$ with prob $\frac{1}{2}$

$f(t) = -t \ \forall_t$ $\frac{1}{2}$



3) for each t, $f(t) = \begin{cases} t & \text{with } p \ \frac{1}{2} \\ -t & \text{with } p \ \frac{1}{2} \end{cases}$



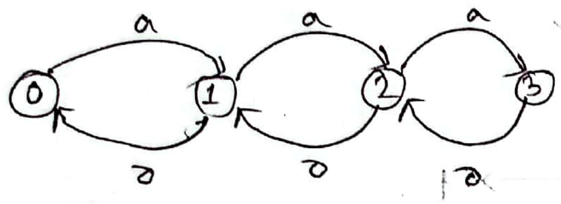
State : Snapshot of a system

Transition : movement from one state to another state.

Experiment : Any situation where the outcome is uncertain

Sample Space

Event : Any collection of outcomes in sample space.



Game of Craps :

→ Roll 2 dice

→ outcomes : win 7 or 11
lose 2, 3, 12
point 4, 5, 6, 8, 9, 10

→ If point, then roll again
→ win if point
→ lose if 7
→ otherwise roll again and so on.

Rahal Sin 219

09.11.23

→ Roll 2 dice

→ Outcome – Win => 7 or 11

– Loose => 2, 3, 11

– Point => 4, 5, 6, 8, 9, 10

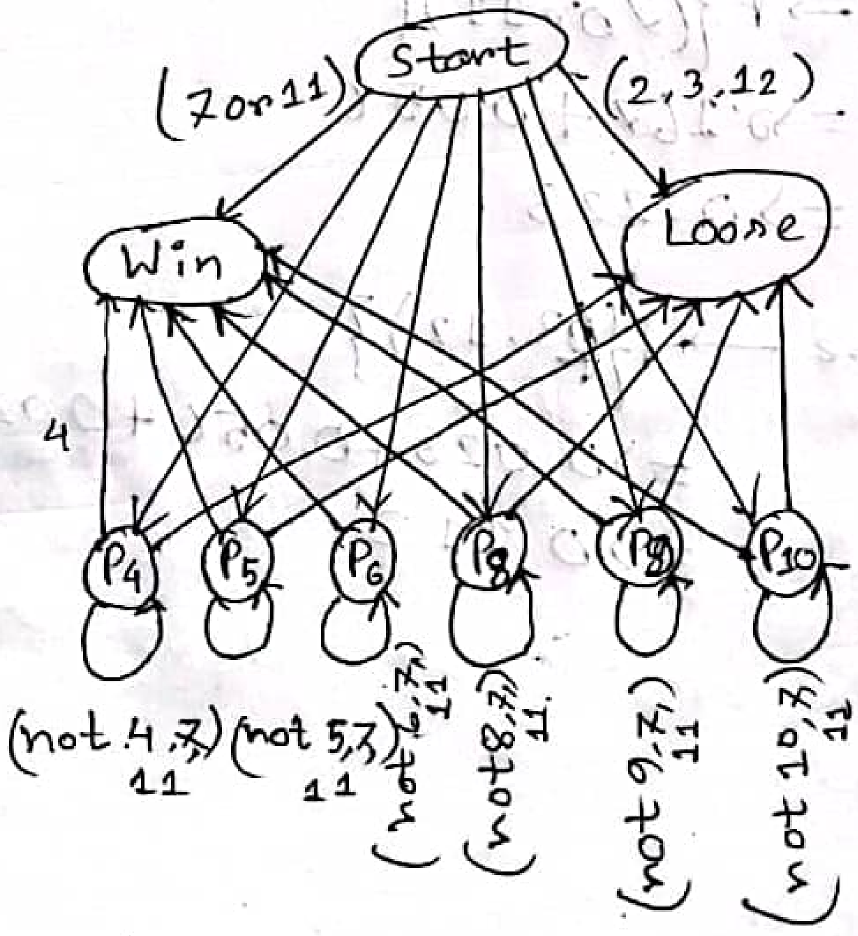
→ If point, then roll again

- win if same point

- loose if 7 or 11

- otherwise roll again. so on

State transition Diagram:



$2 \longrightarrow (1,1) = 1/36$

$3 \longrightarrow (1,2).(2,1) = \frac{2}{36}$

$4 \longrightarrow (2,2)(1,3)(3,1) = \frac{3}{36}$

$5 \longrightarrow (1,4)(4,1)(2,3).(3,2) = \frac{4}{36}$

$6 \longrightarrow (1,5)(5,1)(2,4),(4,2),(3,3) = \frac{5}{36}$

$7 \longrightarrow (1,6)(6,1)(2,5)(2,5)(5,2)(3,4)(4,3) = \frac{6}{36}$

$8 \longrightarrow \frac{5}{36}$

$9 \longrightarrow \frac{4}{36}$

$10 \longrightarrow 3/36$

$11 \longrightarrow \frac{3}{36}$

Prob of win $\rightarrow P\{(7 \text{ or } 11)\}$

$\Rightarrow 0.167 + 0.056$

$\Rightarrow 0.223$

Prob of loose $\longrightarrow P\{(2,3,12)\}$

$\Rightarrow 0.028 + 0.056 + 0.028$

$\Rightarrow 0.112$

প্রতি line - এর যোগফল 1

Transition Matrix:

	Start	Win	Loose	P_4	P_5	P_6	P_8	P_9	P_{10}
Start	0	0.223	0.111	0.083	0.111	0.139	0.139	0.111	0.083
Win	0	1	0	0	0	0	0	0	0
Loose	0	0	1	0	0	0	0	0	0
P_4	0	0.083	~~0.167~~ 0.223	$1-(0.083+0.223)$					
P_5	0								
P_6	0								
P_8	0								
P_9	0								
P_{10}	0								

Markrov Process: It is a type of stochastic process. Let X_n where $n = 0, 1, 2, 3 \cdots$ be stochastic process that takes on a finite or countable numbers of possible values.

If $X_n = i$ — X is a random variable at time n with value i

$X_n = i$ হলে, $X_{n+1} = j$

Probability → P_{ij}

বইয়ের 1st 2 pg (42)

P_{ij} মানে probability of X going from i to j.

এই state এর উপর P depended.

$$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i, X_0 = i_0\} = P_{ij}$$

n সময়ে value i হলে n+1 সময়ে value j হবে

generic form of the transition matrix:

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & & & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \end{bmatrix} \quad P_{ij} \leq 1$$

$$\sum_{j=0}^{\alpha} P_{ij} = 1$$

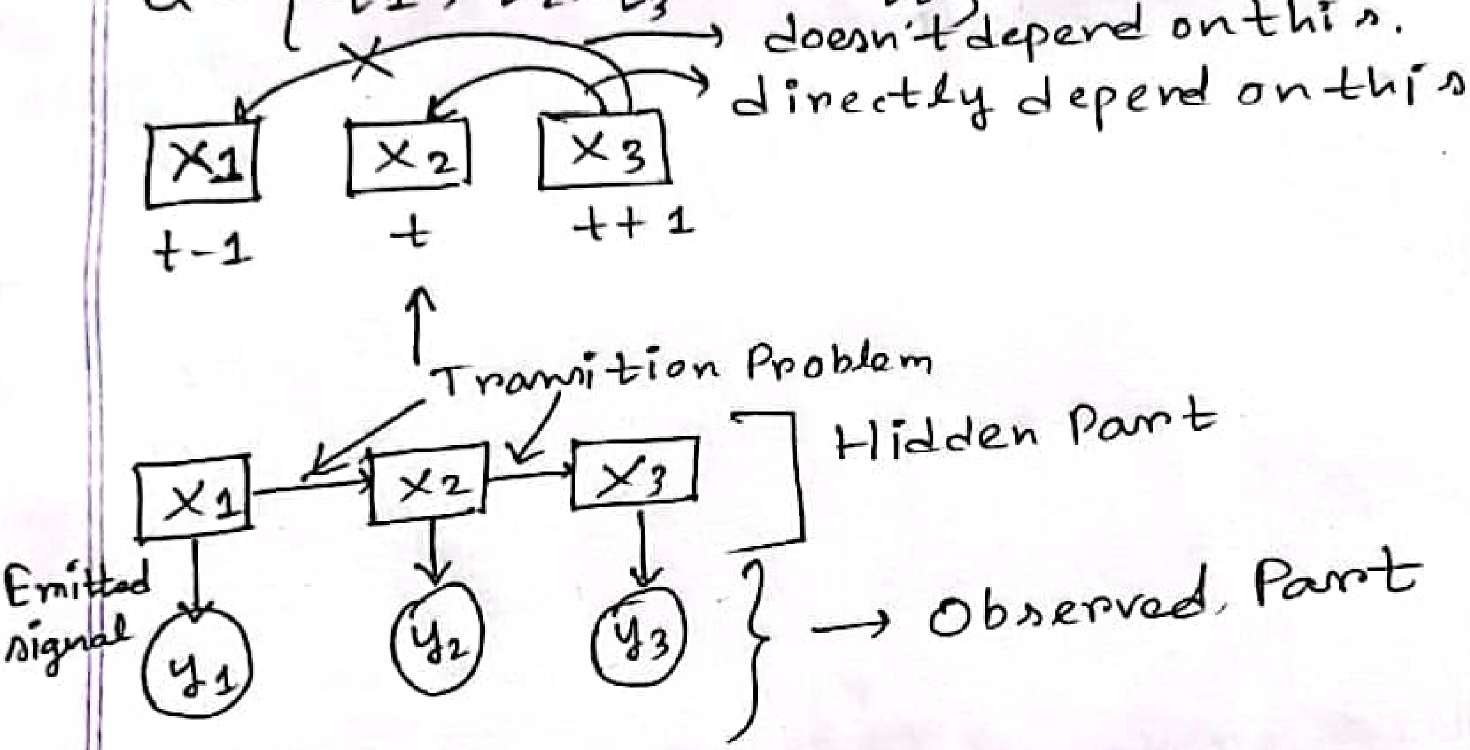
Hidden Markov Model → It is a stochastic process where the states of the model are hidden. Here each state emits an output which is observed.

We'll have an observation sequence.

$$O = \{o_1, o_2, o_3 \cdots o_t\}$$

we will have an unknown sequence.

$$Q = \{q_1, q_2, q_3 \cdots q_t\}$$



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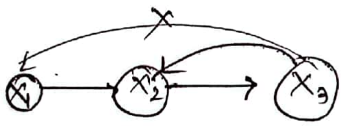
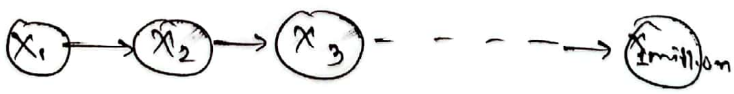
Property of Markov chain :

Markov property : Given that the present state is known, the conditional probability of the next state is independent of the state prior to the present state. → Given the present, the past is irrelevent in determining the future. The of the system at time $t+1$, depends only on the state of the system of time t. (and not on the state prior to time t. In fact, the effects of all previous states are embaded already on the state at time t).



Stationary assumption :

The transitional probabilities are the same at all time. That is the transitional probabilities are independent of time.



- $P(X_1 = R) = 0$
- $P(X_1 X_2) = P(X_2|X_1) \cdot P(X_1)$

* X_4 এর ক্ষেত্রে, X_3 parent, $(X_1, X_2) \rightarrow$ non-descendent

---

$$P(X_2 = S) = P(X_2 = S | X_1 = S) + \underbrace{P(X_2 = S | X_1 = R)}_{0}$$

$$\therefore P(X_2 = S) = P(X_2 = S | X_1 = S)$$

or,

Using formula,

$$P(X_2 = S) = P(X_2 = S | X_1 = S) \cdot P(X_1 = S) + P(X_2 = S | X_1 = R) \cdot \underbrace{P(X_1 = R)}_{0}$$

$$\therefore \boxed{P(X_2 = S) = P(X_2 = S | X_1 = S) \cdot P(X_1 = S)}$$

$$= 0.9 * 1 + 0$$

$$= 0.9 \leftarrow P(X_1 = S) = 1$$

Next দিন sunny হওয়ার Prob কমলো।

$$\boxed{P(X_2 = R) = 0.1}$$

(9) $\boxed{P(X_3 = S) = ?}$

$$\rightarrow P(X_3 = S) = P(X_3 | X_2) \cdot P(X_2)$$

$$\therefore P(X_3 = S) = P(X_3 = S | X_2 = S) \cdot P(X_2 = S) + P(X_3 = S | X_2 = R) \cdot P(X_2 = R)$$

From Transition Table

$$= (0.9 * 0.9) + (0.3 * 0.1)$$

$$= 0.84$$

$$P(X_3 = R) = 0.16$$

Another way :

$$P(X_1) = \begin{array}{c} S \\ R \end{array}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{Transpose}} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$P(X_2) = \begin{array}{c} S \\ R \end{array}\begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}$$

$$P(X_3) = \begin{array}{c} S \\ R \end{array}\begin{bmatrix} 0.84 \\ 0.16 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.9, & 0.1 \end{bmatrix}\begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.84 \\ 0.16 \end{bmatrix}$$

(at one time যেখানে এটা fixed হয়ে যাবে)

$$P(X_4) = \begin{bmatrix} 0.84 & 0.16 \end{bmatrix}\begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

again:

$$P(X_5) = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix}\begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} \rightarrow \text{same}$$

(stationary distribution)

Note : * at one point, the probability will be stationary.

* যদি $x_1 = R$ হয়, then do the process and find when the process is going to give the same probability

* Interpretation : • It doesn't effect on the initial state
• depends on the transition table only.

* $\boxed{\text{Stationary distribution}}$ is a distribution to which a Markov model converges after running a long time. When it wil depend on the transition probability. It does not depend on the initial value.

For most chains,

→ Influence of the initial distribution gets less and less over time.

→ The distribution we end up with, is independent of initial distribution.

→ For stationary distribution → the distribution we end up with is called the stationary distribution of the chain.

(Q) On a given day Samara is either cheerful (C), so-so (S), or gloomy (G). If she is cheerful today, she will be C, S, G tomorrow with respective Prob 0.5, 0.4, 0.1. If she is feeling S today, she will be C, S, G tomorrow with Prob, 0.3, 0.4, 0.3. If she she is G today, she will be C, S, G tomorrow with prob: 0.2, 0.3, 0.5. If she is C today. find out the prob. that she will be G day after tomorrow.

→ let X_n be the mood of Samara on nth day. Then $\{X_n, n \geq 0\}$ is a three state Markov chain, say state 0 = c, state 1 = S, state 2 = G.

Transition Prob. Matrix.

$$\begin{array}{c|ccc} & C & S & G \\ \hline C & 0.5 & 0.4 & 0.1 \\ S & 0.3 & 0.4 & 0.3 \\ G & 0.2 & 0.3 & 0.5 \end{array}$$

She is cheerful today $\longrightarrow P(X_1 = c) = 1$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = P(X_1) \xrightarrow{\text{Transpose}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$P(X_2) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0.4 \\ 0.1 \end{bmatrix} \xleftarrow{\text{Transpose}}$$

$$P(X_3) = \begin{bmatrix} 0.5 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

$$= \begin{array}{ccc} c & S & G \\ \left[0.39 \right. & 0.39 & \left. 0.22 \right] \end{array}$$

If she is c today, the prob. that she will be G today is 0.22. [Ans]

P_0 = today → ①

Q

Day 1 (tomorrow)

$$P^1 = \begin{array}{c|ccc} & C & S & G \\ \hline C & 0.5 & 0.4 & 0.1 \\ S & 0.3 & 0.4 & 0.3 \\ G & 0.2 & 0.3 & 0.5 \end{array} \longrightarrow ②$$

~~Third~~ Day 2 (day after tomorrow)

$$P^2 = \begin{array}{c|ccc} & C & S & G \\ \hline C & 0.39 & 0.39 & 0.22 \\ & 0.33 & 0.37 & 0.3 \\ & 0.29 & 0.35 & 0.36 \end{array} \longrightarrow ③$$

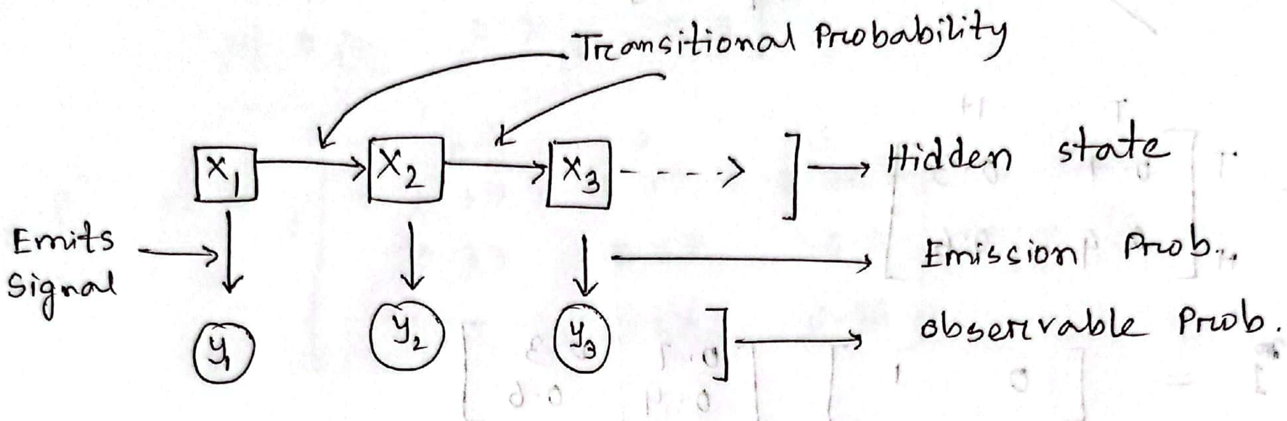
Q Given that a person's last purchase was a Toyota. There's 30% chance that the next car will be a Honda and 70% chance that he will buy a toyota again. If his last car was a Honda, there's 60% chance that his car purchase will also be a Honda.
Given that presently the person is a Honda user, what is the probability that he will purchase a Toyota from two purchases from now.

$\longrightarrow$ $\begin{array}{ccc} & H & \cancel{T} \\ H & \left[\begin{matrix} 1 \\ \end{matrix}\right. & \left.\begin{matrix} 0 \\ \end{matrix}\right] \\ T & & \end{array}$ $\begin{array}{cc} T & \left[\begin{matrix} 0 \\ 1 \end{matrix}\right] \\ H & \end{array}$

$$P = \begin{array}{c|c} & \begin{matrix} T & H \end{matrix} \\ \begin{matrix} T \\ H \end{matrix} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$$P1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$P2 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.52 \\ 0.48 \end{bmatrix}$$



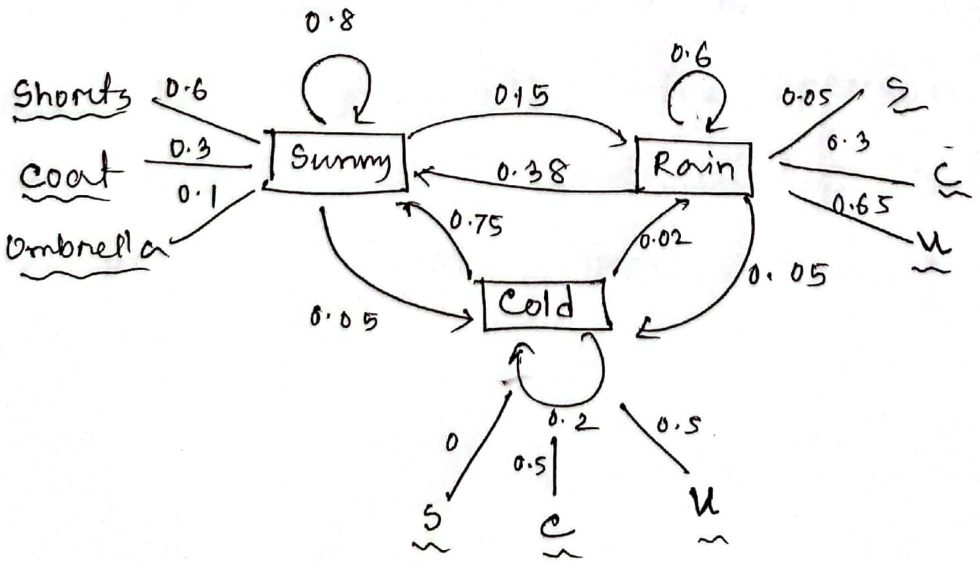
HMM (hidden Markov Model)

States (which kind of state)

1. Hidden state
2. Observable state

Probabilities (which kind of Prob)

1. Transition P
2. Emission P.
3. Initial P.



Note : ~ → emission p.

Trans. Prob. Matrix

$$
\begin{array}{c|ccc}
 & Su & Ra & Cold \\
\hline
Su & 0.8 & 0.15 & 0.05 \\
R & 0.38 & 0.6 & 0.02 \\
C & 0.75 & 0.05 & 0.2
\end{array}
$$

Emission P.

$$
\begin{array}{c|ccc}
 & Su & C & U \\
\hline
Su & 0.6 & 0.3 & 0.1 \\
Ra & 0.05 & 0.3 & 0.65 \\
C & 0 & 0.5 & 0.5
\end{array}
$$

So, HMM is governed by following 3 parameters,

$\lambda = \{A, B, \pi\}$ where,

. B = Emission / observation / state conditional Prob.

A = State transition Prob. Matrix.

π = Initial or Prior state prob.

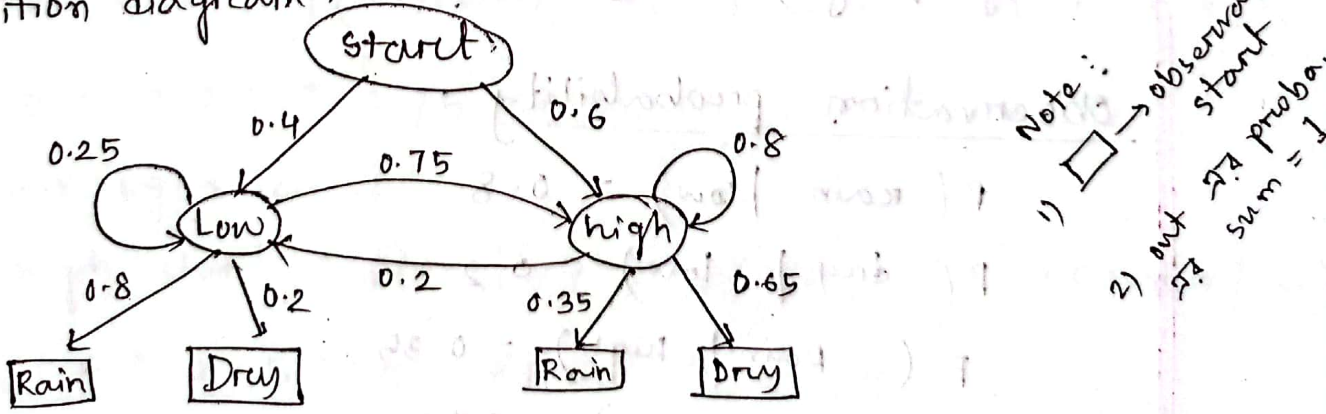
14.11.2023

For forecasting weather, it is observed that low atmospheric pressure causes a chance of 80% rain, otherwise dry. On the other hand, 65% chance of dryness happens if high atmospheric pressure prevails. Again there is 75% probability of changing from low to high atmospheric pressure. but high to low changing occurs in a propability of 20%. The initial probability of high atmospheric pressure is 60%. Answer the following.

1) Draw the hidden Markov model

2) If sunny day prevails today, take a decision whether the cricket match can be held tomorrow or not.

→ Transition diagram:



state trans.
Initial
emission

Initial probability :

$P(\text{high} \mid \text{start}) = 0.6$

$P(\text{low} \mid \text{start}) = 0.4$.

Transition probability :

→ where I want to go (next states)

Present ↓	low	high
low	0.25	0.75
high	0.2	0.8

$P(\underbrace{\text{low}}_{\text{want to go}} \mid \underbrace{\text{low}}_{\text{Present}}) = 0.25$

$P(\text{low} \mid \text{high}) = 0.2$

$P(\text{high} \mid \text{low}) = 0.75$

$P(\text{high} \mid \text{high}) = 0.8$

Observation probability :

$P(\text{rain} \mid \text{low}) = 0.8$

$P(\text{dry} \mid \text{low}) = 0.2$

$P(\text{Rain} \mid \text{high}) = 0.35$

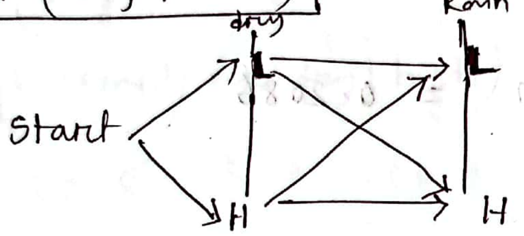
$P(\text{Dry} \mid \text{high}) = 0.65$

we need to find out,

$P(\text{dry} \mid \text{rain}) =$

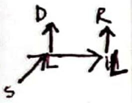
$P(\underset{\text{Present}}{\text{dry}} \mid \underset{\text{Next}}{\text{dry}}) =$

$\boxed{\text{For } P(\text{dry} \mid \text{rain})}$



* Present → dry
  L → dry
  H → dry
* Next → Rain
  L → rain
  H → rain

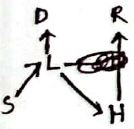
<u>hidden state sequence</u>:



Start → low → low

$\therefore P(\text{low} \mid \text{start}) * P(\text{dry} \mid \text{low}) * P(\text{low} \mid \text{low}) * P(\text{Rain} \mid \text{low})$

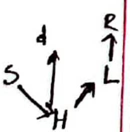
$= 0.4 * 0.2 * 0.25 * 0.8 = 0.016$



Start → low → high

$\therefore P(\text{low} \mid \text{start}) * P(\text{dry} \mid \text{low}) * P(\text{high} \mid \text{low}) * P(\text{rain} \mid \text{high})$

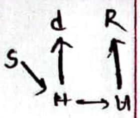
$= 0.4 * 0.2 * 0.75 * 0.35 = 0.021$



Start → high → low

$\therefore P(\text{high} \mid \text{start}) * P(\text{dry} \mid \text{high}) * P(\text{low} \mid \text{high}) * P(\text{rain} \mid \text{low})$

$= 0.6 * 0.65 * 0.2 * 0.8 = 0.0624$



Start → high → high

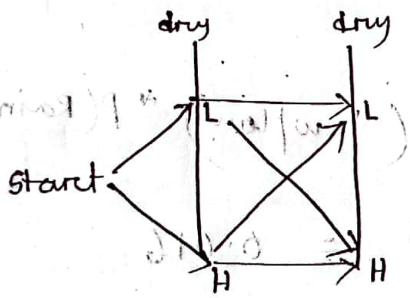
P(high | start) * P(dry | high) * P(high | high) * P(rain | high)

= 0.6 * 0.65 * 0.8 * 0.3

= 0.1092

Probability of Rain = 0.2086

$\boxed{\text{For } P(\text{dry} \mid \text{dry})}$



Start → low → low

P(~~start~~ low | ~~low~~ start) * P(dry → low) * P(low | low) * P(dry | low)

= 0.004

Start → low → high

P(low | start) * P (dry | low) * P(high | low) * P(dry | high)

= 0.039

Start $\rightarrow$ high $\rightarrow$ low

$P(\text{high} \rightarrow \text{start}) * P(\text{dry} | \text{high}) * P(\text{low} | \text{high}) * P(\text{dry} | \cancel{\text{high}}\ \text{low})$

$= 0.0156$

Start $\rightarrow$ high $\rightarrow$ high

$P(\text{High} | \text{start}) * P(\text{dry} | \text{high}) * P(\text{high} | \text{high}) * P(\text{dry} | \text{high})$

$= 0.2028$

Probability of dry $= 0.2614$ ✓

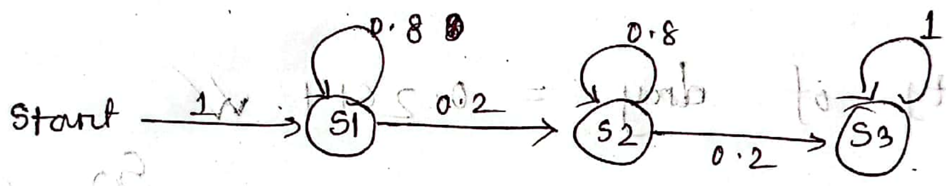
So. cricket খেলা হবে।

od

Suppose that after character image segmentation the following sequence of island number is 4 slices was observed {1, 3, 2, 1}

What is more likely to generate this observation sequence, HMM for "A", or HMM for "B".

→ Given, structure of hidden state is,



Transition Probability :

$$\begin{array}{c|c} & \begin{array}{ccc} S1 & S2 & S3 \end{array} \\ \begin{array}{c} S1 \\ S2 \\ S3 \end{array} & \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.8 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

observation → no. of islands in theoretical slice

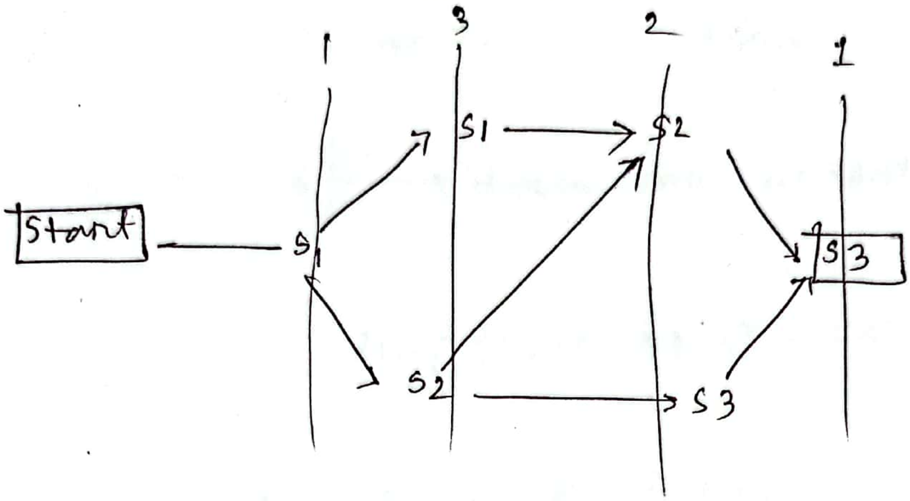
Observation Prob. A

$$\begin{array}{c|c} & \begin{array}{ccc} S1 & S2 & S3 \end{array} \\ \begin{array}{c} S1 \\ S2 \\ S3 \end{array} & \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0.9 & 0.1 & 0 \end{bmatrix} \end{array}$$

Observation Prob. B

$$\begin{array}{c|c} & \begin{array}{ccc} S1 & S2 & S3 \end{array} \\ \begin{array}{c} S1 \\ S2 \\ S3 \end{array} & \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.2 & 0.8 \\ 0.6 & 0.4 & 0 \end{bmatrix} \end{array}$$

Answer



Start → S1 → S1 → S2 → S3 → 0

" → S1 → S2 → S2 → S3 → 0.00208

" → S1 → S2 → S3 → S3 → 0. 000324

→ 0.0023976

B → 0.0096768

28.11.2023

Rahat Sir

* Queuing Theory :

o o o o [o] →

Queue server

L = 5
$L_Q = 4$
W
W_Q

* cost equations :

avg rate at which system earns

= λ_a × avg amount an entering customer pays

* Heuristic Proof of eqn 8.1

avg rate at which the system earns

= λ_a × avg. amount an entering customer pays

⇒ $\underbrace{\text{Avg. no. of customer in the system}}_{L}$ × $\underbrace{\text{amount paid per unit time}}_{1}$

= λ_a × $\underbrace{\text{avg amount of time customer spends in the system}}_{W}$ × $\underbrace{\text{amount paid per unit time}}_{1}$

$$\boxed{L = \lambda_a W}$$

avg. no. of customers in service = $\lambda_a \times \underbrace{E[S]}_{\text{avg no. of time customer spends in service}}$

8

B

Example 8.1
Pg 484

$\lambda \to$ arrival rate
$\mu \to$ departure rate
M/M/1 $\to$ related maths (7–8 ET)

Proposition 8.1 (Proof).

Poisson process

Example 8.2 :

$$\lambda P_0 = \mu P_1 \Rightarrow P_1 = \frac{\lambda}{\mu} P_0$$

$$P_{n+1} = \frac{\lambda}{n} \cdot P_n + \left(P_n - \frac{\lambda}{n} P_{n-1}\right)$$

$P_0 = P_0$

$P_1 = \frac{\lambda}{\mu} P_0$

$P_2 = \frac{\lambda}{n} P_1 + \left(P_1 - \frac{\lambda}{n} P_0\right)$

$= \frac{\lambda}{n} P_1 + \left(\frac{\lambda}{\mu} P_0 - \frac{\lambda}{\mu} P_0\right) = \left(\frac{\lambda}{\mu}\right)^2 P_0$

$P_3 = \ldots = \left(\frac{\lambda}{\mu}\right)^3 P_0$

$$P_{n+1} = \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1}\right) = \boxed{\frac{\lambda}{\mu} P_n}$$

$$\boxed{P_{n+1} = \left(\frac{\lambda}{\mu}\right)^{n+1} P_0} \quad (1)$$

$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$

$$1 = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \cdot P_0$$

$$= \frac{P_0}{1 - \frac{\lambda}{\mu}}$$

$$\Rightarrow P_0 = 1 - \frac{\lambda}{\mu}$$

CamScanner

λ] μ exponential

From ①,

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot P_0$$

$$\Rightarrow \boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \left(1 - \frac{\lambda}{\mu}\right)}$$

state 0 $\longrightarrow \lambda P_0 = \mu P_1$

" 1 $\longrightarrow (\lambda + \mu) P_1 = \lambda P_0 + \mu P_2$

$$\downarrow$$

$$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$$

$$\Rightarrow P_{n+1} = \frac{(\lambda + \mu)}{\lambda} P_n - \frac{\lambda}{\mu} P_{n-1}$$

$$= \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1}\right)$$

$\frac{\lambda}{\mu} < 1 \longrightarrow \lambda < \mu$

read from the slides of this class