Taylor and Laurent Series

Singular Point: The point at which the function $f(z)$ is not analytic but the neighborhood of that point contains at least one point where function is analytic is called the singular point. $z=1 \, and \, z=2$ are singular points for the function $f(z)=\dfrac{1}{(z-1)(z-2)}$

Note:1. Singular points are firstly of two types; (a) Isolated singular point and (b) Non-isolated singular point.

If there is no singular point in the neighborhood of the singular point, then this singular point is called the isolated singular point; otherwise it is the non-isolated singular point.

Note:2. Consider the Laurent series for $f(z)$

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}$$

The first series in $f(z)$ is called the regular part and second series is called the principal part of Laurent series.

Isolated singular points are of three types:

Removable Singular Point: If in the Laurent series for $f(\mathrm{z})$:

$$f(\mathrm{z})=\sum_{n=0}^{\infty}a_n(\mathrm{z}-\mathrm{z}_0)^n+\sum_{n=1}^{\infty}a_{-n}(\mathrm{z}-\mathrm{z}_0)^{-n}$$

the principal part is zero then the singular point $\mathrm{z}=z_0$ is called a removable singular point.

Example: For $f(\mathrm{z})=\dfrac{\sin z}{z}=\dfrac{1}{z}\{\mathrm{z}-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-...(-1)^{n-1}\dfrac{z^{2n-1}}{(2\mathrm{n}-1)!}\}$

$$=1-\frac{z^2}{3!}+\frac{z^4}{5!}-...(-1)^{n-1}\frac{z^{2n-2}}{(2\mathrm{n}-1)!}$$

$\mathrm{z}=0$ is a removable singular point.

Pole: If in the Laurent series for $f(\mathrm{z})$:

$$f(\mathrm{z})=\sum_{n=0}^{\infty}a_n(\mathrm{z}-z_0)^n+\sum_{n=1}^{\infty}a_{-n}(\mathrm{z}-z_0)^{-n}$$

the principal part contains a finite number of terms then the singular point $\mathrm{z}=z_0$ is called a pole.

Example: For

$$f(\mathrm{z})=\frac{\mathrm{z}-\sin(z-1)}{z-1}=\frac{1}{z-1}[\mathrm{z}-\{(\mathrm{z}-1)-\frac{(z-1)^3}{3!}+\frac{(z-1)^5}{5!}-\ldots(-1)^{n-1}\frac{(z-1)^{2n-1}}{(2\mathrm{n}-1)!}$$

$$=\frac{\mathrm{z}}{z-1}-1+\frac{(z-1)^2}{3!}-\frac{(z-1)^4}{5!}+\ldots$$

$\mathrm{z}=1$ is a pole.

Essential Singular Point: If in the Laurent series for $f(z)$:

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}$$

the principal part contains infinitely many terms then the singular point $z=z_0$ is called an essential singular point.

Example: For $f(z)=e^{\frac{1}{z}}=1+\dfrac{1}{z}+\dfrac{1}{2!z^2}+\dfrac{1}{3!z^3}+\ldots+$

$z=0$ is an essential singular point.

for $f(z)=\sin(\dfrac{1}{z-1})=\dfrac{1}{z-1}-\dfrac{1}{3!(z-1)^3}+\dfrac{1}{5!(z-1)^5}-\ldots$

$z=1$ is an essential singular point.

Taylor and Laurent Series

Question:1. Mention the singular points for the functions below and determine their type:

i $f(z)=\dfrac{e^{-z}}{(z-2)^4}$ AI: 106/15(a) $z=2$ is a pole.

ii $f(z)=\dfrac{1}{z(e^z-1)}$ AI: 107/15(b) $z=0$ and $z=2n\pi i$ are simple poles.

iii $f(z)=\dfrac{z^2}{(z+1)^2}\sin(\dfrac{1}{z-1})$ AI: 107/15(d) $z=1$ is an essential isolated singular point.

Question:2. Expand the following functions in Laurent series in the prescribed regions:

i $f(z)=\dfrac{z-1}{(z+2)(z+3)}$ in the regions: (a) $|z|<2$, (b) $2<|z|<3$,(c) $|z|>3$.

: 110/16(a).

Taylor and Laurent Series

ii. $f(z) = \dfrac{1}{(z+1)(z+3)}$ in the regions: (a) $1 < |z| < 3$, (b) $|z| > 3$,

(c) $0 < |z+1| < 2$, (d) $|z| < 1$. : 112/16(b).

iii. $f(z) = \dfrac{z^2}{(z-1)(z-2)}$ in the regions: (a) $1 < |z| < 2$, (b) $|z| < 2$,

(c) $0 < |z| < 1$: 114/16(c).

iv. $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in the regions: (a) $2 < |z| < 3$, (b) $|z| < 2$,(c) $|z| > 3$.

115/16(d).

v. $f(z) = \dfrac{3z-3}{(2z-1)(z-2)}$ in the regions: (a) $|z| < 1$, (b) $|z| > 1$: 116/16(e).

vi. $f(z) = \dfrac{z^2+1}{(z+1)(z-2)}$ in the regions: (a) $1 < |z| < 2$, (b) $0 < |z| < 1$:117(f).

Taylor and Laurent Series

Note: Let us recall the following series:

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}\,; |z| < \infty$

2. $sinz = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots (-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}\,; |z| < \infty$

3. $cosz = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots (-1)^{n-1}\frac{z^{2n-2}}{(2n-2)!}\,; |z| < \infty$

4. $\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots (-1)^{n-1}\frac{z^n}{n}\,; |z| < 1$

5. $tan^{-1}z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots (-1)^{n-1}\frac{z^{2n-1}}{(2n-1)}; |z| < 1$

6. $(1 + z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \cdots + \frac{p(p-1)\ldots(p-n+1)}{n!}z^n$

Example–1 **Find the singular points of the following functions and determine their nature**

(a) $f(z) = \dfrac{e^{-z}}{(z-2)^4}$

(b) $f(z) = \dfrac{1}{z(e^z - 1)}$

(c) $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$

(d) $f(z) = \dfrac{z^2}{(z+1)^2} \sin\left(\dfrac{1}{z-1}\right)$

(e) $f(z) = \dfrac{z - \sin(z-1)}{z-1}$

Solution :

(a) Given that $f(z) = \dfrac{e^{-z}}{(z-2)^4} = \dfrac{e^2 . e^{-2} . e^{-z}}{(z-2)^4}$

$$\Rightarrow f(z) = \frac{e^{-2} e^{-z+2}}{(z-2)^4} = e^{-2} \frac{e^{-(z-2)}}{(z-2)^4}$$

$$= \frac{e^{-2}}{(z-2)^4}\left[1 - \frac{(z-2)}{!!} + \frac{(z-2)^2}{2!} - \frac{(z-2)^3}{3!} + \frac{(z-2)^4}{4!} - \frac{(z-2)^5}{5!} + \ldots\ldots\right]$$

$$= e^{-2}\left[\frac{1}{(z-2)^4} - \frac{1}{(z-2)^3} + \frac{1}{2!(z-2)^2} - \frac{1}{3!(z-2)} + \frac{1}{4!} - \frac{(z-2)}{5!} + \ldots\ldots\right]$$

We see that $f(z)$ has a finite number of terms of negative powers of $(z-2)$. Therefore $z = 2$ is a singular point and it is a pole . And also we see that $(z-2)$ has power 4. Therefore $z = 2$ is pole of order 4.

(b) Given that $f(z) = \dfrac{1}{z(e^z - 1)}$ .... (i)

$\therefore$ Singular points of $f(z)$ can be obtained by considering $z(e^z - 1) = 0$

$\therefore z(e^z - 1) = 0$

$\Rightarrow z = 0$ and $e^z - 1 = 0$

$\Rightarrow e^z = 1 = \cos 0 + i \sin 0$

$\Rightarrow e^z = \cos 2n\pi + i \sin 2n\pi$

$\Rightarrow e^z = e^{i2n\pi}$; $n = 0, \pm 1, \pm 2,$ ....

$\therefore z = 2n\pi i$, $n = 0, \pm 1, \pm 2,$ ....

Therefore $z = 0$ and $z = 2n\pi i$ are simple poles.

(c) Given that $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$.

If we consider $(z-1)^3 (3z+2)^2 = 0$ then we get singular points of $f(z)$.

$\therefore (z-1)^3 (3z+2)^2 = 0$

$\Rightarrow (z-1)^3 = 0$ and $(3z+2)^2 = 0$

$\Rightarrow z = 1 \quad \Rightarrow 3z + 2 = 0 \Rightarrow z = -\dfrac{2}{3}$

$\therefore z = 1$ is a pole of order 3 and $z = \dfrac{-2}{3}$ is a pole of order 2.

(d) Given that $f(z) = \dfrac{z^2}{(z+1)^2} \sin\left(\dfrac{1}{z-1}\right)$

$f(z)$ is not analytic if $(z+1)^2 = 0$ and $z - 1 = 0$

$\therefore z = -1$ and $z = 1$ are singulsr points. Here $z = -1$ is a pole of order 2 clearly.

Again $\sin\dfrac{1}{z-1} = \dfrac{1}{z-1} - \dfrac{1}{3!(z-1)^3} + \dfrac{1}{5!(z-1)^5} - \dots\dots\dots\dots$

We see that there are infinite number of terms $(z - 1)$ in the Laurent expansion of $\sin\dfrac{1}{z-1}$ with negative powers. therefore $z = 1$ is an isolated essential singular point.

(e) Given that $f(z) = \dfrac{z - \sin(z-1)}{(z-1)}$

$$\Rightarrow f(z) = \frac{1}{(z-1)}\left[z - \left\{(z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} - \dots\right\}\right]$$

$$= \frac{z}{z-1} - \frac{z-1}{z-1} + \frac{(z-1)^2}{3!} - \frac{(z-1)^4}{5!} + \dots\dots\dots$$

$$= \frac{z-1+1}{z-1} - 1 + \frac{(z-1)^2}{3!} - \frac{(z-1)^4}{5!} + \dots\dots\dots$$

$$= 1 + \frac{1}{z-1} - 1 + \frac{(z-1)^2}{3!} - \frac{(z-1)^4}{5!} + \dots\dots\dots$$

$$= \frac{1}{z-1} + \frac{(z-1)^2}{3!} - \frac{(z-1)^4}{5!} + \dots\dots\dots$$

We see that $f(z)$ has one (finite number of) term of negative power of $(z - 1)$. Therefore $z = 1$ is a pole of order one.

Example–2 Find the Laurent expansion of the following series in the indicated regions.

(a) $f(z) = \dfrac{z-1}{(z+2)(z+3)}$; (i) $|z| < 2$, (ii) $2 < |z| < 3$, (iii) $|z| > 3$.

(b) $f(z) = \dfrac{1}{(z+1)(z+3)}$; (i) $1 < |z| < 3$, (ii) $|z| > 3$, (iii) $0 < |z+1| < 2$, (iv) $|z| < 1$.

(c) $f(z) = \dfrac{z^2}{(z-1)(z-2)}$; (i) $1 < |z| < 2$, (ii) $0 < |z| < 1$.

(d) $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$; (i) $2 < |z| < 3$, (ii) $|z| < 2$, (iii) $|z| > 3$.

(e) $f(z) = \dfrac{3z-3}{(2z-1)(z-2)}$; (i) $|z| < 1$, (ii) $|z| > 1$.

(f) $f(z) = \dfrac{z^2+1}{(z+1)(z-2)}$; (i) $1 < |z| < 2$, (ii) $0 < |z| < 1$.

Solution :

(a) Given that $f(z) = \dfrac{z-1}{(z+2)(z+3)}$

$$\Rightarrow f(z) = \frac{z+2-3}{(z+2)(z+3)} = \frac{z+2}{(z+2)(z+3)} - \frac{3}{(z+2)(z+3)}$$

$$= \frac{1}{z+3} - 3\left[\frac{1}{(z+2)(-2+3)} + \frac{1}{(z+3)(-3+2)}\right]$$

$$= \frac{1}{z+3} - \frac{3}{z+2} - \frac{3}{(z+3)(-1)}$$

$$= \frac{1}{z+3} - \frac{3}{z+2} + \frac{3}{z+3}$$

$$= \frac{4}{z+3} - \frac{3}{z+2} \quad \text{.........................} \quad (A)$$

(i) Given region is

$|z| < 2$ $\qquad\qquad$ $\therefore |z| < 3$

$\Rightarrow \dfrac{|z|}{2} < 1$ $\qquad\qquad$ $\Rightarrow \dfrac{|z|}{3} < 1$

$\Rightarrow \left|\dfrac{z}{2}\right| < 1$ $\qquad\qquad$ $\Rightarrow \left|\dfrac{z}{3}\right| < 1$

$\therefore$ (A) $\Rightarrow f(z) = 4.\frac{1}{z+3} - \frac{3}{z+2}$

$$= 4.\frac{1}{3\left(1+\frac{z}{3}\right)} - 3\frac{1}{2\left(1+\frac{z}{2}\right)}$$

$$= \frac{4}{3}\left(1+\frac{z}{3}\right)^{-1} - \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{4}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \ldots\ldots\right] - \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \ldots\ldots\right]$$

[$\because$ (1 + x)–1 = 1 – x + x2 – x3 + ...]

$$= \left(\frac{4}{3} - \frac{3}{2}\right) + \left(\frac{3}{4} - \frac{4}{9}\right) z + \left(\frac{4}{27} - \frac{3}{8}\right) z^2 + \ldots\ldots$$

$$= \frac{-1}{6} + \frac{11}{36} z - \frac{49}{216} z^2 + \ldots\ldots$$

Which is the Laurent expansion valid for $|z| < 2$ without principal part. This is actually the Taylor's series.

(ii) Given region is $2 < |z| < 3$

$\Rightarrow 2 < |z|$ and $|z| < 3$

$\Rightarrow \frac{2}{|z|} < 1$ $\quad$ $\Rightarrow \frac{|z|}{3} < 1$

$\Rightarrow \left|\frac{2}{z}\right| < 1$ $\quad$ $\Rightarrow \left|\frac{z}{3}\right| < 1$

(A) $\Rightarrow f(z) = 4.\frac{1}{z+3} - 3.\frac{1}{z+2}$

$$= 4\frac{1}{3\left(1+\frac{z}{3}\right)} - 3.\frac{1}{z\left(1+\frac{2}{z}\right)}$$

$$= \frac{4}{3}\left(1+\frac{z}{3}\right)^{-1} - \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{4}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \ldots\ldots\right] - \frac{3}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \ldots\ldots\right]$$

$$= \left[\frac{4}{3} - \frac{4z}{9} + \frac{4z^2}{27} - \frac{4z^3}{81} + \ldots\ldots\right] - \left[\frac{3}{z} - \frac{6}{z^2} + \frac{12}{z^3} - \frac{24}{z^4} + \ldots\ldots\right]$$

$$= \ldots\ldots + \frac{24}{z^4} - \frac{12}{z^3} + \frac{6}{z^2} - \frac{3}{z} + \frac{4}{3} - \frac{4z}{9} + \frac{4z^2}{81} - \ldots\ldots$$

Which is the Laurent expansion valid for $2 < |z| < 3$.

(iii) Given region $|z| > 3$ $\quad$ $\therefore |z| > 2$

$\Rightarrow 1 > \frac{3}{|z|}$ $\quad$ $\Rightarrow 1 > \frac{2}{|z|}$

$\Rightarrow \left|\frac{3}{z}\right| < 1$ $\quad$ $\Rightarrow \left|\frac{2}{z}\right| < 1$

$\therefore$ (A) $\Rightarrow f(z) = \frac{4}{z+3} - \frac{3}{z+2}$

$$= 4.\frac{2}{z\left(1+\frac{3}{z}\right)} - 3.\frac{1}{z\left(1+\frac{2}{z}\right)}$$

$$= \frac{4}{z}\left(1+\frac{3}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{4}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\ldots\ldots\ldots\right) - \frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\ldots\ldots\right)$$

$$= \frac{4}{z}-\frac{12}{z^2}+\frac{36}{z^3}-\ldots\ldots\ldots\ldots - \left(\frac{3}{z}-\frac{6}{z^2}+\frac{12}{z^3}-\ldots\ldots\ldots\right)$$

$$= \left(\frac{4}{z}-\frac{3}{z}\right)+\left(\frac{6}{z^2}-\frac{12}{z^2}\right)+\left(\frac{36}{z^3}-\frac{12}{z^3}\right)-\ldots\ldots\ldots$$

$$= \frac{1}{z}-\frac{6}{z^2}+\frac{24}{z^3}-\ldots\ldots\ldots$$ which is the Laurent expansion valid for $|z| > 3$.

(b) The given function is $f(z) = \dfrac{1}{(z+1)(z+3)}$

$$= \frac{1}{(z+1)(-1+3)} + \frac{1}{(-3+1)(z+3)} \quad \text{[by cover up rule]}$$

$$= \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2}\cdot\frac{1}{z+3} \quad \ldots\ldots\ldots\ldots\ldots \text{(A)}$$

(i) Given region is

$1 < |z| < 3$

$\Rightarrow 1 < |z|$ and $\quad |z| < 3$

$\Rightarrow \dfrac{1}{|z|} < 1 \quad\quad \Rightarrow \dfrac{|z|}{3} < 1$

$\Rightarrow |\dfrac{1}{z}| < 1 \quad\quad \Rightarrow |\dfrac{z}{3}| < 1$

$$\therefore f(z) = \frac{1}{2}\,\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\,\frac{1}{3\left(1+\frac{z}{3}\right)} = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\ldots\ldots\right] - \frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\ldots\ldots\ldots\right]$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\ldots\ldots\right) - \left(\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\ldots\ldots\right)$$

$$= \ldots\ldots\ldots + \frac{1}{2z^5}-\frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}-\ldots\ldots\ldots$$

(ii) Given region is

$|z| > 3 \quad\quad \therefore |z| > 1$

$\Rightarrow 1 > \dfrac{3}{|z|} \quad\quad \Rightarrow \dfrac{1}{|z|} < 1$

$\Rightarrow |\dfrac{3}{z}| < 1 \quad\quad \Rightarrow |\dfrac{1}{z}| < 1$

$\Rightarrow |\dfrac{3}{z}| < 1$

$$\therefore f(z) = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\,\frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left[\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{3}{z}\right)^{-1}\right]$$

$$= \frac{1}{2z}\left[\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\ldots\right) - \left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\ldots\ldots\ldots\right)\right]$$

$$= \frac{1}{2z}\left[\left(\frac{3}{z}-\frac{1}{z}\right)+\left(\frac{1}{z^2}-\frac{9}{z^2}\right)+\left(\frac{27}{z^3}-\frac{1}{z^3}\right)-\ldots\ldots\right]$$
$$= \frac{1}{2z}\left[\frac{2}{z}-\frac{8}{z^2}+\frac{26}{z^3}-\ldots\ldots\right]$$

(iii) Given region is $0 < |z+1| < 2$

$\Rightarrow 0 < |z+1|$ and $|z+1| < 2$

$$\Rightarrow \frac{|z+1|}{2} < 1$$

$$\Rightarrow \left|\frac{z+1}{2}\right| < 1$$

The given function is

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{(z+1)(z+1+2)}$$

$$\Rightarrow f(z) = \frac{1}{2(z+1)\left[\frac{z+1}{2}+1\right]}$$

$$= \frac{1}{2(z+1)}\left(1+\frac{z+1}{2}\right)^{-1}$$

$$= \frac{1}{2(z+1)}\left\{1-\frac{z+1}{2}+\left(\frac{z+1}{2}\right)^2-\left(\frac{z+1}{2}\right)^3+\ldots\ldots\right\}$$

$$= \frac{1}{2(z+1)}-\frac{1}{4}+\frac{z+1}{8}-\frac{(z+1)^2}{16}+\ldots\ldots$$

(iv) Given region is $|z| < 1 \therefore |z| < 3$

$$\Rightarrow \frac{|z|}{3} < 1$$

$$\Rightarrow \left|\frac{z}{3}\right| < 1$$

$$\therefore (A) \Rightarrow f(z) = \frac{1}{2}\cdot\frac{1}{1+z}-\frac{1}{2}\,\frac{1}{z+3}$$

$$= \frac{1}{2}(1+z)^{-1}-\frac{1}{2}\cdot\frac{1}{3(1+\frac{z}{3})}$$

$$= \frac{1}{2}(1+z)^{-1}-\frac{1}{6}\cdot\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[1-z+z^2-z^3+z^4-\ldots\ldots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\ldots\ldots\right]$$

$$= \left[\frac{1}{2}-\frac{z}{2}+\frac{z^2}{2}-\frac{z^3}{2}+\ldots\ldots\right]+\left[-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}-\ldots\ldots\right]$$

$$= \left(\frac{1}{2}-\frac{1}{6}\right)+\left(\frac{z}{18}-\frac{z}{2}\right)+\left(\frac{z^2}{2}-\frac{z^2}{54}\right)+\left(\frac{z^3}{162}-\frac{z^3}{2}\right)+\ldots\ldots$$

$$= \frac{3-1}{6}+\frac{z-9z}{18}+\frac{27z^2-z^2}{54}+\frac{z^3-81z^3}{162}+\ldots\ldots$$

$$= \frac{2}{6}-\frac{8z}{18}+\frac{26z^2}{54}-\frac{80}{162}z^3+\ldots\ldots$$

$$= \frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2-\frac{40}{81}z^3+\ldots\ldots$$

(c) Given function is $f(z) = \frac{z^2}{(z-1)(z-2)}$

$$\Rightarrow f(z) = \frac{z^2}{(z-1)(z-2)} = \frac{z^2-1+1}{(z-1)(z-2)} = \frac{z^2-1}{(z-1)(z-2)} + \frac{1}{(z-1)(z-2)}$$

$$= \frac{(z+1)(z-1)}{(z-1)(z-2)} + \left[\frac{1}{(z-1)(1-2)} + \frac{1}{(2-1)(z-2)}\right]$$

$$= \frac{z+1}{z-2} - \frac{1}{z-1} + \frac{1}{z-2}$$

$$= \frac{z-2+3}{z-2} - \frac{1}{z-1} + \frac{1}{z-2}$$

$$= \frac{z-2}{z-2} + \frac{3}{z-2} - \frac{1}{z-1} + \frac{1}{z-2}$$

$$\Rightarrow f(z) = 1 + \frac{4}{z-2} - \frac{1}{z-1} \text{ .... (A)}$$

(i) Given region is $1 < |z| < 2$

$\Rightarrow 1 < |z|$ and $|z| < 2$

$\Rightarrow \frac{1}{|z|} < 1$ $\Rightarrow \frac{|z|}{2} < 1$

$\Rightarrow \left|\frac{1}{z}\right| < 1$ $\Rightarrow \left|\frac{z}{2}\right| < 1$

$$A \Rightarrow f(z) = 1 + \frac{4}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= 1 - 2\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= 1 - 2\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \ldots\ldots\right]$$

$$= 1 - 2 - z - \frac{z^2}{2} - \frac{z^3}{4} - \ldots\ldots\ldots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \ldots\ldots$$

$$= \ldots\ldots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - 1 - z - \frac{z^2}{2} - \frac{z^3}{4} - \ldots\ldots$$

(ii) Given region is $0 < |z| < 1$

$\Rightarrow |z| < 1$ $\therefore |z| < 2$

$\Rightarrow \frac{|z|}{2} < 1$

$\Rightarrow \left|\frac{z}{2}\right| < 1$

$$\therefore (A) \Rightarrow f(z) = 1 + \frac{4}{z-2} - \frac{1}{z-1}$$

$$= 1 + \frac{4}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{-(1-z)}$$

$$= 1 - 2\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$= 1 - 2\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots\ldots\right] + (1 + z + z^2 + z^3 + \ldots\ldots)$$

$$= 1 - 2 - z - \frac{z^2}{2} - \frac{z^3}{4} - \ldots\ldots + (1 + z + z^2 + z^3 + \ldots\ldots)$$

$$= \left(z^2 - \frac{z^2}{2}\right) + \left(z^3 - \frac{z^3}{4}\right) + \ldots\ldots$$

$$= \frac{z^2}{2} + \frac{3z^3}{4} + \ldots\ldots$$

(d) Given that $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)} = \dfrac{z^2 - 4 + 3}{(z+2)(z+3)}$

$$\Rightarrow f(z) = \frac{z^2 - 2^2}{(z+2)(z+3)} + \frac{3}{(z+2)(z+3)}$$

$$= \frac{(z+2)(z-2)}{(z+2)(z+3)} + 3\left[\frac{1}{(z+2)(-2+3)} + \frac{1}{(-3+2)(z+3)}\right]$$

$$= \frac{z-2}{z+3} + 3\left[\frac{1}{z+2} - \frac{3}{z+3}\right]$$

$$= \frac{z+3-5}{z+3} + \frac{3}{z+2} - \frac{3}{z+3}$$

$$= \frac{z+3}{z+3} - \frac{5}{z+3} + \frac{3}{z+2} - \frac{3}{z+3}$$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3} \quad \ldots\ldots\ldots\ldots \text{(A)}$$

(i) Given region is $2 < |z| < 3$

$\Rightarrow 2 < |z|$ and $|z| < 3$

$\Rightarrow \dfrac{2}{|z|} < 1$ $\quad \Rightarrow \dfrac{|z|}{3} < 1$

$\Rightarrow \left|\dfrac{2}{z}\right| < 1$ $\quad \Rightarrow \left|\dfrac{z}{3}\right| < 1$

Now (A) $\Rightarrow f(z) = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

$$= 1 + \frac{3}{z\left(1 + \frac{2}{z}\right)} - \frac{8}{3\left(\frac{z}{3} + 1\right)}$$

$$= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \ldots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \ldots\right]$$

$$= \left[1 + \frac{3}{z} - \frac{6}{z^2} + \frac{12}{z^3} - \frac{24}{z^4} + \ldots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \ldots\right]$$

(ii) Given region is $|z| < 2 \therefore |z| < 3$

$\dfrac{|z|}{2} < 1$ $\quad \Rightarrow \dfrac{|z|}{3} < 1$

$\Rightarrow \left|\dfrac{z}{2}\right| < 1$ $\quad \Rightarrow \left|\dfrac{z}{3}\right| < 1$

Now (A) $\Rightarrow f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

$$= 1 + \frac{3}{2\left(\frac{z}{2}+1\right)} - \frac{8}{3\left(\frac{z}{3}+1\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \ldots\ldots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \ldots\ldots\right]$$

$$= 1 + \frac{3}{2} - \frac{8}{3} + \left(\frac{8}{9}z - \frac{3}{4}z\right) + \frac{81z^2 - 64z^2}{216} + \ldots\ldots\ldots$$

$$= \frac{-1}{6} + \frac{5}{36}z + \frac{17}{216}z^2 + \ldots\ldots\ldots$$

(iii) Given region is $|z| > 3$ $\qquad\qquad \therefore |z| > 2$

$$\Rightarrow 1 > \frac{3}{|z|} \qquad\qquad \Rightarrow 1 > \frac{2}{|z|}$$

$$\Rightarrow \left|\frac{3}{z}\right| < 1 \qquad\qquad \Rightarrow \left|\frac{2}{z}\right| < 1$$

Now (A) $\Rightarrow f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$\Rightarrow f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \ldots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \ldots\right]$$

$$= 1 + \frac{3}{z} - \frac{6}{z^2} + \frac{12}{z^3} - \frac{24}{z^4} + \ldots\ldots\ldots - \frac{8}{z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \ldots\ldots\right]$$

$$= 1 + \left(\frac{3}{z} - \frac{8}{z}\right) + \left(\frac{-6}{z^2} + \frac{24}{z^2}\right) + \left(\frac{12}{z^3} - \frac{72}{z^3}\right) + \ldots\ldots\ldots$$

$$= 1 - \frac{5}{z} + \frac{18}{z^2} - \frac{60}{z^3} + \ldots\ldots\ldots$$

(e) Given function is $f(z) = \frac{3z-3}{(2z-1)(z-2)} = \frac{3}{2}\frac{(z-1)}{\left(z-\frac{1}{2}\right)(z-2)}$

$$\Rightarrow f(z) = \frac{3}{2}\left[\frac{\frac{1}{2}-1}{\left(z-\frac{1}{2}\right)\left(\frac{1}{2}-2\right)} + \frac{2-1}{\left(2-\frac{1}{2}\right)(z-2)}\right]$$

$$= \frac{3}{2}\left[\frac{\frac{-1}{2}}{-\frac{3}{2}\left(z-\frac{1}{2}\right)} + \frac{2-1}{\frac{3}{2}(z-2)}\right]$$

$$=\frac{3}{2}\left[\frac{1}{3\left(z-\frac{1}{2}\right)}+\frac{2}{3}\cdot\frac{1}{(z-2)}\right]$$

$$=\frac{1}{2}\cdot\frac{1}{z-\frac{1}{2}}+\frac{1}{z-2}$$

$$=\frac{1}{2}\cdot\frac{2}{2z-1}+\frac{1}{z-2}$$

$$=\frac{1}{2z-1}+\frac{1}{z-2} \text{ .... (A)}$$

(i) Given region is $|z| < 1$ $\quad\quad \therefore |z| < 2 \Rightarrow |\frac{z}{2}| < 1$

$$(A) \Rightarrow f(z) = \frac{1}{2z-1}+\frac{1}{z-2}$$

$$=\frac{1}{2z-1}+\frac{1}{z}(1-\frac{2}{z})^{-1}$$

$$=\frac{1}{2z-1}+\frac{1}{z}\left\{1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\text{..............}\right\}$$

$$=\frac{1}{2z-1}+\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\text{.............}$$

(ii) Given region is $|z| > 1 \Rightarrow \frac{1}{|z|} < 1 \Rightarrow |\frac{1}{z}| < 1$

$$\therefore (A) \Rightarrow f(z) = \frac{1}{2z-1}+\frac{1}{z-2}$$

$$=\frac{1}{2z\left(1-\frac{1}{2z}\right)}+\frac{1}{z-2}$$

$$=\frac{1}{2z}\left(1-\frac{1}{2z}\right)^{-1}+\frac{1}{z-2}$$

$$=\frac{1}{2z}\left\{1+\frac{1}{2z}+\frac{1}{4z^2}+\frac{1}{8z^3}+\text{......}\right\}+\frac{1}{z-2}$$

$$=\frac{1}{z-2}+\frac{1}{2z}+\frac{1}{4z^2}+\frac{1}{8z^3}+\frac{1}{16z^4}+\text{..................}$$

(f) Given that $f(z) = \frac{z^2+1}{(z+1)(z-2)} = \frac{z^2-1+2}{(z+1)(z-2)} = \frac{(z+1)(z-1)}{(z+1)(z-2)}+\frac{2}{(z+1)(z-2)}$

$$\Rightarrow f(z) = \frac{z-1}{z-2}+\left[\frac{2}{(z+1)(-1-2)}+\frac{2}{(2+1)(z-2)}\right]$$

$$=\frac{z-2+1}{z-2}-\frac{2}{3}\cdot\frac{1}{z+1}+\frac{2}{3}\,\frac{1}{z-2}$$

$$=\frac{z-2}{z-2}+\frac{1}{z-2}-\frac{2}{3}\,\frac{1}{z+1}+\frac{2}{3}\cdot\frac{1}{z-2}$$

$$=1-\frac{2}{3}\cdot\frac{1}{z+1}+\frac{5}{3}\cdot\frac{1}{z-2} \text{ .... (A)}$$

(i) Given region is $1 < |z| < 2 \Rightarrow 1 < |z|$ and $|z| < 2$

$$\Rightarrow \frac{1}{|z|} < 1 \qquad \Rightarrow \frac{|z|}{2} < 1$$

$$\Rightarrow \left|\frac{1}{z}\right| < 1 \qquad \Rightarrow \left|\frac{z}{2}\right| < 1$$

$$\text{Now (A)} \Rightarrow f(z) = 1 - \frac{2}{3}\frac{1}{z\left(1+\frac{1}{z}\right)} + \frac{5}{3}\cdot\frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= 1 - \frac{2}{3z}\left(1+\frac{1}{z}\right)^{-1} - \frac{5}{6}\left(1-\frac{z}{2}\right)^{-1}$$

$$= 1 - \frac{2}{3z}\left\{1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right\} - \frac{5}{6}\left\{1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right\}$$

$$= \left(\frac{-2}{3z} + \frac{2}{3z^2} - \frac{2}{3z^3} + \frac{2}{3z^4} - \ldots\right) + \left\{\frac{1}{6} - \frac{5z}{12} - \frac{5z^2}{24} - \frac{5z^3}{48} - \ldots\right\}$$

(ii) Given region is $0 < |z| < 1 \Rightarrow |z| < 1$,

$$\therefore \quad |z| < 2$$

$$\Rightarrow \quad \frac{|z|}{2} < 1$$

$$\Rightarrow \quad \left|\frac{z}{2}\right| < 1$$

$$\text{Now (A)} \Rightarrow f(z) = 1 - \frac{2}{3}\frac{1}{1+z} + \frac{5}{3}\cdot\frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= 1 - \frac{2}{3}(1+z)^{-1} - \frac{5}{6}\cdot\left(1-\frac{z}{2}\right)^{-1}$$

$$= 1 - \frac{2}{3}\left\{1 - z + z^2 - z^3 + \ldots\right\} - \frac{5}{6}\left\{1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right\}$$

$$= 1 - \frac{2}{3} - \frac{5}{6} + \left(\frac{2}{3} - \frac{5}{12}\right)z + \left(\frac{-2}{3} - \frac{5}{24}\right)z^2 + \left(\frac{2}{3} - \frac{5}{48}\right)z^3 - \ldots$$

$$= \frac{6-4-5}{6} + \frac{8-5}{12}z + \frac{-16-5}{24}z^2 + \frac{32-5}{48}z^3 - \ldots$$

$$= \frac{-1}{2} + \frac{1}{4}z - \frac{7}{8}z^2 + \frac{9}{16}z^3 - \ldots$$