Complex Variable, Harmonic Function & Statistics

Lt Col Amir Sir

Complex Number: The number in the form $z = a + ib$ is called complex number

where $a, b \in \mathbb{R}$, $i = \sqrt{-1}$.

Example: $Z_1 = 5 + 6i$, $Z_2 = 5i$, $Z_3 = 2$ all are complex no.

Z_2: $0 + 5i$; Z_3: $2 + i \cdot 0$.

$\{1, 2, 3 \cdots\}$; all positive negative ; $\{-\frac{3}{2}, -\frac{1}{2} \cdots 1, 2 \cdots\}$

every Real number is a complex Number.

अंक $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$

$\{-3, -2, 1, 0, 1, 2 \cdots\}$; Complex numbers.

Note: 1. $z = a + ib$ is usually written for numbers, whereas $z = x + iy$ is written for complex variables.

2. x and y are called real and imaginary part of z respectively and usually written as

$$Re(z) = x \; ; \; Im(z) = y.$$

Conjugate of Complex Number (Complex Conjugate)

For the complex number $z = a + ib$; $\bar{z} = a - ib$ is called complex conjugate and vice versa.

Representation of Complex Number:

A. In cartesian co ordinate system, every complex number can be represented as an ordered pair

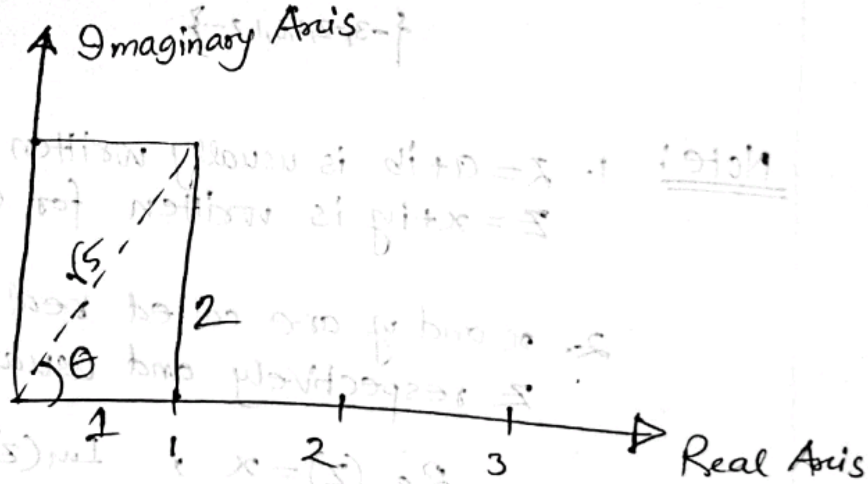
$$(1,2) \neq (2,1).$$

B. In polar system, it can be represented by (r,θ) where r and θ are called the modulous and argument of the complex number.

$z = 1+2i$

$\equiv (1,2)$

$\tan\theta = \frac{2}{1}$.



$\therefore (r,\theta) = (\sqrt{5}, \tan^{-1} 2)$.

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Modulus and Argument of Complex Number:

For the complex number $z = a + ib$;

$|z| = \sqrt{a^2+b^2} = \sqrt{Re(z)^2 + Im(z)^2}$ is called the modulus of z and

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$
$= \tan^{-1}\left\{\frac{Im(z)}{Re(z)}\right\}$ is called the argument of z.

Ques: Determine the modulus and argument of

(i) $z = 1+i$ (ii) $z = 1-i$ (iii) $z = -1+i$

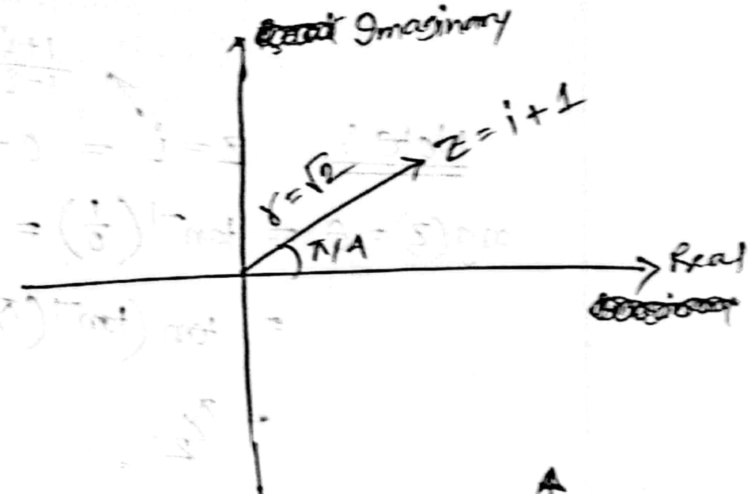
(iv) $z = -1-i$ (v) $z = \frac{1-i}{1+i}$ (vi) $z = \sqrt{3}+i$

Ans: (i) $z = 1+i$
$= 1 + i\cdot 1$.

modulus → $|z| = \sqrt{1^2+1^2} = \sqrt{2}$

and

$\arg(z) = \theta = \tan^{-1}\left(\frac{1}{1}\right)$
$= \frac{\pi}{4}$.

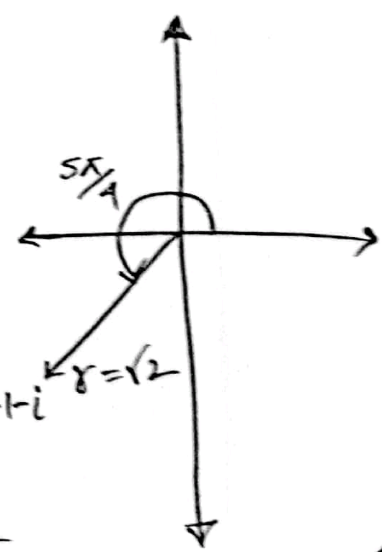


(iv) $z = -1-i$

or, $z = (-1) + i(-1)$

$\therefore |z| = \sqrt{(-1)^2 + (-1)^2}$
$= \sqrt{2}$

$\arg(z) =$
$\theta = \tan^{-1}\left(\frac{-1}{-1}\right)$
$= \tan\left(\tan^{-1}\left(\pi + \frac{\pi}{4}\right)\right)$
$= \frac{5\pi}{4}$.



π plus করতে হবে কারণ 3rd quadrant এ আছে।

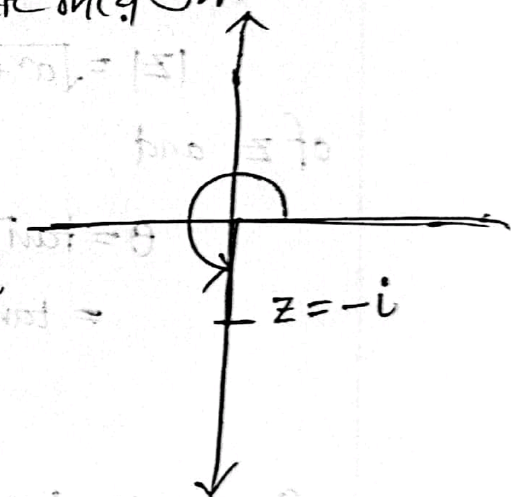
(v) $z = \dfrac{1-i}{1+i}$

Simplify করার easy নিয়ম হল নিচের complex Number এর conjugate দিয়ে গুণ।

$= \dfrac{(1-i)(1-i)}{(1+i)(1-i)}$

$= \dfrac{1^2 - 2i + i^2}{1^2 - i^2}$

$= \dfrac{1 - 2i - 1}{1+1} = \dfrac{-2i}{2} = -i$

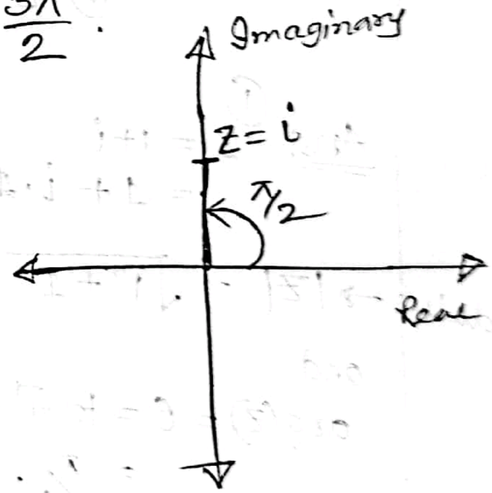


$\therefore z = 0 + i(-1)$

$|z| = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$

$\arg(z) = \theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \tan\left(\tan^{-1}\left\{\frac{3\pi}{2}\right\}\right)$

$= \dfrac{3\pi}{2}$.



$\frac{1+i}{1-i}$

Note: $z = i = 0 + i \cdot 1$.

$\arg(z) = \theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty)$

$= \tan\left(\tan^{-1}(\pi/2)\right)$

$= \pi/2$.

** সকল Real part এর সাথে Real Part এর হিসাব আর Imaginary Part এর সাথে Imaginary Part এর হিসাব।

Properties of Complex Number:

(i) For the complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

$z_1 = z_2$ iff $a_1 = a_2$ and $b_1 = b_2$.

(ii) $z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$

(iii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$. {conjugate নিয়ে যোগ করলে যোগফল যোগফলের conjugate এর সমান}

(iv) $|z| = |\bar{z}|$; $\left[\begin{array}{ll} z = a + ib, & \bar{z} = a - ib = a + i(-b) \\ |z| = \sqrt{a^2+b^2} & |\bar{z}| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2} \\ \therefore |z| = |\bar{z}| & \end{array} \right.$

(v) $|z|^2 = z \cdot \bar{z}$; $\left[\begin{array}{l} z = a + ib, \; |z|^2 = a^2 + b^2 \\ z \cdot \bar{z} = (a+ib)(a-ib) \\ \quad = a^2 - iab + iab - i^2b^2 \\ \quad = a^2 + b^2 \end{array} \right]$

(vi) $z + \bar{z} = 2\,Re(z)$ $\left[z + \bar{z} = a + ib + a - ib = 2a \right]$ (a হল Real Part)

(vii) (a) $Re(z) \leq |z|$; $a \leq \sqrt{a^2+b^2}$

(b) $Im(z) \leq |z|$; $b \leq \sqrt{a^2+b^2}$

Question: For the complex number $z_1, z_2, \cdots\cdots z_n$ prove that

(i) $|z_1+z_2| \leq |z_1|+|z_2|$

(ii) $|z_1-z_2| \leq |z_1|+|z_2|$

(iii) $|z_1+z_2+z_3\cdots z_n| \leq |z_1|+|z_2|+\cdots\cdots |z_n|$.

Proof (i) We may write $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$ $\left\{ \text{আমরা জানি } |z|^2 = z.\bar{z} \right.$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$ $\left[\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2\right.$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2.\bar{z}_2$.

$= |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2$ $\{ \bar{\bar{z}} = z \}$.

$|z_1+z_2|^2 = |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$.

$\left[z+\bar{z} = 2Re(z) \right]$

কোনো complex number এর Real part এর ছোট বা সমান part absolute

$\Rightarrow |z_1+z_2|^2 \leq |z_1|^2 + 2|z_1.\bar{z}_2| + |z_2|^2$ $\left[Re(z) \leq |z| \right]$

$\therefore |z_1+z_2|^2 \leq |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$ $\left[|z_1\bar{z}_2| = |z_1||\bar{z}_2| \right]$

or, $|z_1+z_2|^2 \leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ $\left[|z| = |\bar{z}| \right]$

or, $|z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$

$\therefore |z_1+z_2| \leq |z_1|+|z_2|$

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proof (ii) minus সরণের জন্য $|z_1+z_2| \leq |z_1|+|z_2|$ পর্যন্ত prove করব।

$\therefore |z_1+z_2| \leq |z_1|+|z_2|$ —— (i)

now lets replace z_2 by $(-z_2)$.

$\therefore |z_1+(-z_2)| \leq |z_1|+|-z_2|$.

$\therefore |z_1-z_2| \leq |z_1|+|z_2|$ $[\because |z_1| = |-z_1|]$.

$|z_1+z_2+z_3| \leq |z_1|+|z_2|+|z_3|$

prove firstone upto $|z_1+z_2| \leq |z_1|+|z_2|$ —— (i)

Now $|z_1+z_2+z_3| = |(z_1+z_2)+z_3| \leq |z_1+z_2|+|z_3|$ by (i)

$|z_1+z_2+z_3| \leq |z_1+z_2|+|z_3| \leq |z_1|+|z_2|+|z_3|$.

$\therefore$ ~~[illegible]~~ $|z_1+z_2+z_3| \leq |z_1|+|z_2|+|z_3|$.

proof (ii) First prove: $|z_1+z_2| \leq |z_1|+|z_2|$ —— (i)

$|z_1+z_2+z_3| \leq |z_1|+|z_2|+|z_3|$ —— (ii)

Hence we see that the statement is true for 2 and 3.

$\therefore$ Let the statement is true for m $[\because m<n]$.

$|z_1+z_2+z_3 \cdots z_m| \leq |z_1|+|z_2|+|z_3|+\cdots |z_m|$ —— (iii)

Now $|z_1+z_2+\cdots+z_m+z_{m+1}| = |(z_1+z_2+\cdots z_m)|+|z_{m+1}|$

$\leq |z_1+z_2+\cdots+z_m|+|z_{m+1}|$.

$\Rightarrow |z_1+z_2+\cdots z_m+z_{m+1}| \leq |z_1|+|z_2|+\cdots |z_m|+|z_{m+1}|.$

i.e. the statement is true for $m+1$.

Therefore, the statement is true for any positive integer n.

$\therefore |z_1+z_2+\cdots z_n| \leq |z_1|+|z_2|+\cdots\cdots z_n|$

— o —

Q7: Describe the following regions:

(i) $|z+2i|=3$.

(ii) $|z-3|+|z+3|=4$

(iii) $Re\left(\frac{1}{z}\right)=1$.

(iv) $Im\left(\frac{1}{z}\right)=1$

(v) $2 \leq |z+i| \leq 4$.

(Polar এ থাকলে তার Cartesian এ Convert করবা।)

① $\left|\frac{z-1}{z+1}\right|=2$ $\quad \left(x+\frac{5}{3}\right)^2+(y-0)^2=\left(\frac{4}{3}\right)^2$

② $|z-1|=|z+i|$

Ans: $x-y=0,\ y=0$

③ $\left|\frac{z-3}{z+3}\right|=2$ $\quad (x+5)^2+(y-0)^2=4^2$

equation convert $z=x+iy$

Ans (i) $\Rightarrow |z+2i|=3$

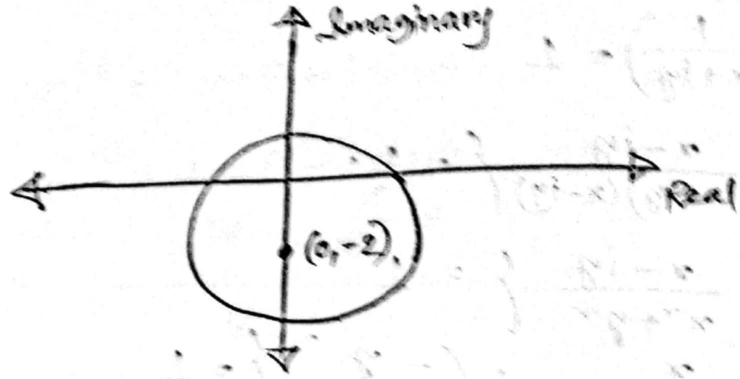
$\Rightarrow |x+iy+2i|=3.$

$\Rightarrow |x+i(2+y)|=3. \quad [\because z=x+iy]$

$\Rightarrow \sqrt{x^2+(y+2)^2}=3$

$\Rightarrow (x-0)^2+\{y-(-2)\}^2=3^2$

∴ Represents a circle whose radius is 3, and center is $(0,-2)$.



(11) $|z-3|+|z+3|=4$.

$\Rightarrow |x+iy-3|+|x+iy+3|=4 \quad [z=x+iy]$

$\Rightarrow \sqrt{(x-3)^2+y^2}=4-\sqrt{(x+3)^2+y^2}$

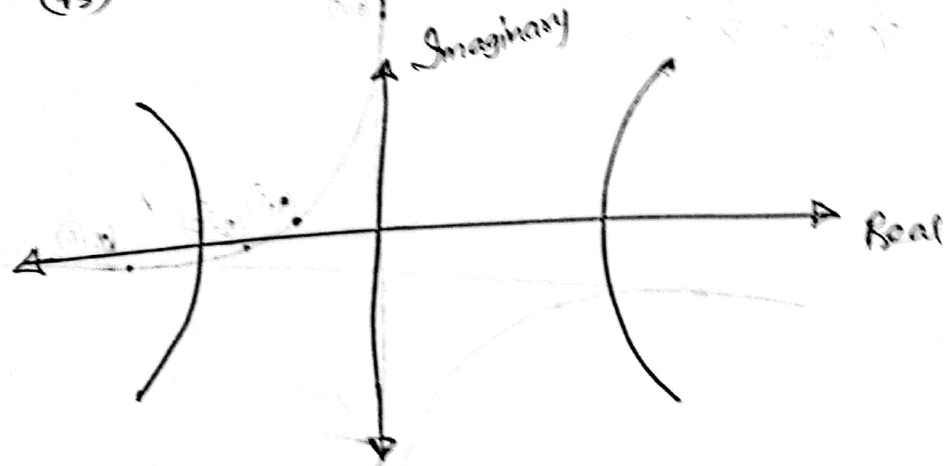
$\Rightarrow x^2-6x+9+y^2=16-8\sqrt{x^2+6x+9+y^2}+x^2+6x+9+y^2$

$\Rightarrow 8\sqrt{x^2+6x+9+y^2}=12x+16$

$\Rightarrow 4(x^2+6x+9+y^2)=9x^2+24x+16$.

$\Rightarrow 5x^2-4y^2=20$.

$\Rightarrow \frac{x^2}{2^2}-\frac{y^2}{(\sqrt{5})^2}=1$ represents hyperbola.



(iv) $\operatorname{Im}\left(\frac{1}{z}\right) = 1$:

$\Rightarrow \operatorname{Im}\left(\frac{1}{x+iy}\right) = 1$

$\Rightarrow \operatorname{Im}\left\{\frac{x-iy}{(x+iy)(x-iy)}\right\} = 1$.

$\Rightarrow \operatorname{Im}\left\{\frac{x-iy}{x^2+y^2}\right\} = 1$

$\Rightarrow \operatorname{Im}\left\{\frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)\right\} = 1$

$\therefore \frac{-y}{x^2+y^2} = 1$

$\Rightarrow x^2+y^2 = -y$

$\Rightarrow x^2+y^2+y = 0$.

$\Rightarrow x^2+y^2+2.y.\frac{1}{2}+\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

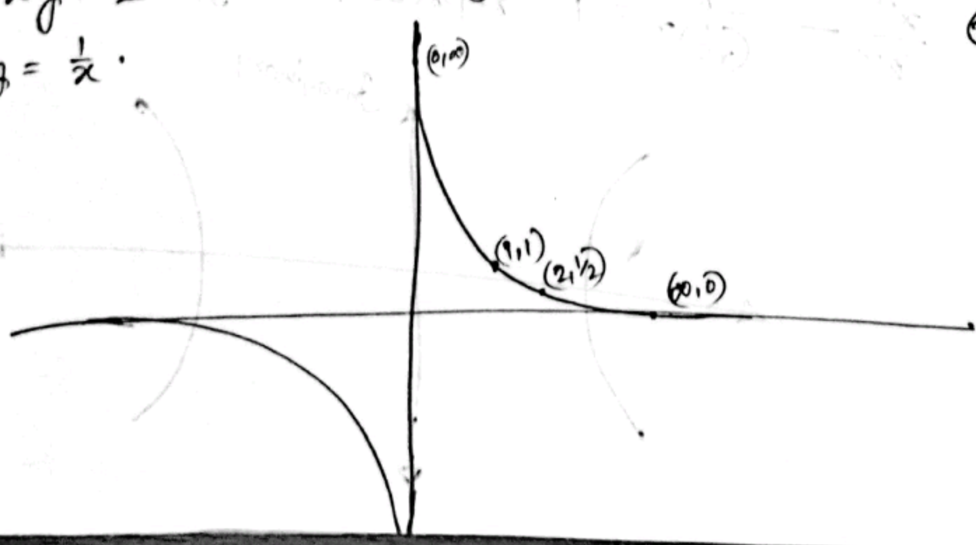
$\Rightarrow (x-0)^2+\left(y+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$ represents a circle with radius $\frac{1}{2}$ and entre $\left(0,-\frac{1}{2}\right)$.

$\operatorname{Im}(z^2) = 4 \Rightarrow \operatorname{Im}\left((x+iy)^2\right) = 4 \Rightarrow \operatorname{Im}(x^2+2.x.iy+i^2y^2) = 4$
$\Rightarrow \operatorname{Im}(x^2-y^2+2ixy) = 4 \quad \therefore \operatorname{Im}(z^2) = 2xy = 4$

<u>Note:</u> $xy = 1$ আমাদের Rectangular hyperbola. represent করে।

$xy = 2$

$\Rightarrow y = \frac{1}{x}$.



(05 Sep 23)

Eulers Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

Q^n: State and prove De Moivers th^m

Proof: For any positive integer n $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

For $n=1$, $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$
$= \cos 1\theta + i\sin 1\theta$.

$\boxed{i^2 = -1}$

the theorem is proved for $n=1$.

for $n=2$, $(\cos\theta + i\sin\theta)^2 = (\cos\theta)^2 + 2.\cos\theta.\, i\sin\theta + (i\sin\theta)^2$
$= \cos^2\theta - \sin^2\theta + i\, 2\sin\theta\cdot\cos\theta$
$= \cos 2\theta + i\sin 2\theta$

thus the theorem is also true for $n=2$.

Let the theorem is true for m. $[m < n]$.

$\therefore (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$.

Now, $(\cos\theta + i\sin\theta)^{m+1} = (\cos\theta + i\sin\theta)^m \cdot (\cos\theta + i\sin\theta)^1$

$= (\cos m\theta + i\sin m\theta)(\cos\theta + i\sin\theta)$

$= \cos m\theta \cdot \cos\theta + i^2 \sin m\theta \cdot \sin\theta$
$\quad + i(\sin m\theta \cdot \cos\theta + \cos m\theta \cdot \sin\theta)$

$= \cos m\theta \cdot \cos\theta - \sin m\theta \cdot \sin\theta + i\sin(m\theta + \theta)$

$[\sin A \cos B + \cos A \sin B = \sin(A+B)$

$= \cos(m\theta + \theta) + i\sin(m+1)\theta$.

$= \cos(m+1)\theta + i\sin(m+1)\theta$.

$\therefore$ The the theorem is proved for $(m+1)$.

Therefore it's proved for all positive integer.

Binomial

$$[a+b]^n = a^n + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \dots \quad b^n.$$

Determine the values of cos3θ and sin 3θ in terms of cosθ and sinθ respectively.

Ans) We know that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

or, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$.

or, $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

or, $\cos 3\theta + i\sin 3\theta = (\cos\theta)^3 + {}^3C_1 (\cos\theta)^{3-1} (i\sin\theta)^1 + {}^3C_2 (\cos\theta)^{3-2} (i\sin\theta)^2 + {}^3C_3 (\cos\theta)^0 (i\sin\theta)^3$.

$$= \cos^3\theta + 3\times\cos^2\theta \cdot i\sin\theta + 3\cos\theta \cdot i^2\sin^2\theta + i \cdot i^2 \sin^3\theta$$

Real Part এবং imaginary part আলাদা করে নেও

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i\,(3\cos^2\theta \cdot \sin\theta - \sin^3\theta)$$

$\therefore$ $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ —— (i)

& $\sin 3\theta = (3\cos^2\theta \cdot \sin\theta - \sin^3\theta)$ —— (ii)

Now: $\cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$$= \cos^3\theta - 3\cos\theta + 3\cdot\cos^3\theta$$

$$= 4\cos^3\theta - 3\cos\theta.$$

$\sin 3\theta = \sin\theta \cdot (3\cos^2\theta - \sin^2\theta)$

$$= \sin\theta\,(3 - 3\sin^2\theta - \sin^2\theta)$$

$$= 3\sin\theta - 4\sin^3\theta.$$

$x = 1 = r\cos\theta$
$y = 1 = r\sin\theta$.
$r^2 + 1^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2 = 2 \quad r = \sqrt{2}$

Q7: Determine: (1) $(i)^{1/4}$ (2) $(1+i)^{1/4}$.

Ans: $(1+i)^{1/4}$

$= (r\cos\theta + i\, r\sin\theta)^{1/4}$ where $1 = r\cos\theta$, $i = r\sin\theta$.

$\therefore r = \sqrt{2}$ $\qquad \tan\theta =$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$.

$\therefore r^{1/4}(\cos\theta + i\sin\theta)^{1/4}$

$\Rightarrow (\sqrt{2})^{1/4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{\frac{1}{4}}$.

$\Rightarrow (\sqrt{2})^{\frac{1}{4}}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}^{\frac{1}{4}}$; $n = 0,1,2,3$.

$\Rightarrow (\sqrt{2})^{1/4}\left\{\cos(8n+1)\frac{\pi}{4} + i\sin(8n+1)\frac{\pi}{4}\right\}^{1/4}$

$\Rightarrow (\sqrt{2})^{\frac{1}{4}}\left\{\cos\frac{1}{4}\cdot(8n+1)\frac{\pi}{4} + i\sin\frac{1}{4}\cdot(8n+1)\frac{\pi}{4}\right\}$ by De Moivors thm.

$\therefore (1+i)^{\frac{1}{4}} = (\sqrt{2})^{1/4}\left\{\cos(8n+1)\cdot\frac{\pi}{16} + i\sin(8n+1)\frac{\pi}{16}\right\}$

$[n = 0,1,2,3]$

$(i)^{1/4}$ $\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}$

$\Rightarrow n = 1,2,3,4$ বসাতে।

$(1)^{\frac{1}{7}}, 2n\pi$
$n = 0,1,2,3,4,5,6$

অবিচ্ছিন্নতা

Continuity, Differentiability, Analyticity and Harmonicity:

(12 Sep 23)

To Review: slide given on whats app, Limit

Continuity:

ex $f(x) = \frac{x^2+5}{x-1}$

$f(1)$ doesn't exist
$f(2)$ exist

(i) $f(a)$ exists (functional value)

(ii) $\lim_{x \to a} f(x)$ exists

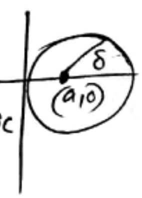
(iii) $\lim_{x \to a} f(x) = f(a)$.

Differentiability: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ if $f(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exist করলে differentiable

Analyticity:

The complex function $f(z) = u(x,y) + iv(x,y)$ is said to be anlytic ~~in the region R~~ at $z=a$ if it is differentiable in the neighbourhood $|z-a| < \delta$ of a.

এই বৃত্তের ভেতরের সব বিন্দুকে satify করলে analytic



$|z-a| = \delta$
$|x+iy-a| = \delta$
$\sqrt{(x-a)^2+y^2} = \delta$
$(x-a)^2+(y-0)^2 = \delta^2$

(or)

A complex function $f(z) = u(x,y) + iv(x,y)$ is analytic if it satisfies the Cauchy-Riemann equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Qn Test whether the function are analytic or not:

(i) $f(z) = e^z$

(ii) $f(z) = \frac{1}{z}$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(Ans):

(ii) $f(z) = \frac{1}{z} = \frac{1}{x+iy}$

$$= \frac{(x+iy)}{(x-iy)(x+iy)} = \frac{x+iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

$= u(x,y) + iv(x,y)$; where

$u(x,y) = \frac{x}{x^2+y^2}$, $v(x,y)$

$\Rightarrow$

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Analytic
Proof

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Qn: State and establish the necessary conditions for the function $f(z) = u(x,y) + iv(x,y)$ to be analytic.

Ans: Necessary conditions for $f(z) = u(x,y) + iv(x,y)$ to be analytic are

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ must exist and satisfy the

Cauchy-Rieman equations $\rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Proof: ~~Since~~ Let $f(z)$ is analytic.

$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ must exist and unique.

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y} \quad \text{——— (1)}$$

Now along real axis $\Delta x \to 0$, $\Delta y = 0$ (Real axis এ y এর পরিবর্তন নাই)

$$\therefore f'(z) = \lim_{x \to 0} \frac{\{u(x+\Delta x, y) + iv(x+\Delta x, y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i \cdot 0}$$

$$= \lim_{x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{——— (A)}}$$

Now along Imaginary Axis: $\Delta y \to 0$, $\Delta x = 0$. {Im. axis এ x এর increment নাই}

① : $f'(z) = \lim_{y \to 0} \frac{\{u(x, y+\Delta y) + iv(x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{i\Delta y}$

complex number এর সাথে i রাখলাম

$= \lim_{y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$

$= -i \lim_{y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \frac{\partial v}{\partial y}$

$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$f'(z) = \frac{\partial v}{\partial y} - i\left(\frac{\partial u}{\partial y}\right)$ ——— (B)

Now from (A) and (B) we have :

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

(proved)

Analytic + Cauchy Rieman এর polar form slide থেকে পড়তে হবে।

Harmonic Function:

The real valued function $u(x,y)$ is said to be harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(Q) If $f(z) = u(x,y) + iv(x,y)$ is analytic then $u(x,y)$ and $v(x,y)$ will be harmonic provided they have continuous second partial derivation.

Note: $\frac{\partial u}{\partial x} \rightarrow$ first partial derivative w.r.t x. , $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \cdot \partial x}$ second partial derivative w.r.t x and y

$\frac{\partial u}{\partial y}$, $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \rightarrow \frac{\partial^2 u}{\partial x \cdot \partial y}$ [আগে y এর পরে x এর derivative]

continuous Second partial derivative : $\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x \cdot \partial y}$ [x, y আগে বা পরে Same Result হবে]

$\times$ $\frac{\delta u}{\delta x} \longrightarrow$ increment

$\checkmark$ $\frac{\partial u}{\partial x} \longrightarrow$ derivative

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Qn: Test whether $u(x,y) = 2x(1-y)$ is harmonic or not. If it is harmonic, find the harmonic conjugate $v(x,y)$ so that $f(z) = u(x,y) + iv(x,y)$ is analytic. Also determine $f(z)$ in terms of z.

Solution

$$\frac{\partial u}{\partial x} = 2-2y \qquad \frac{\partial u}{\partial y} = -2x$$

$$\frac{\partial^2 u}{\partial x^2} = 0, \qquad \frac{\partial^2 u}{\partial y^2} = 0.$$

1st Part

i.e $= \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 x} = 0+0=0$ $\therefore$ $u(x,y)$ is harmonic

$$dv = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} dy$$ [by definition of total derivative]

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$ [cauchy Riemen $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$]

$$= (-)(-2x)\,dx + (2-2y)\,dy$$

$$dv = 2x\,dx + (2-2y)\,dy \longrightarrow (1)$$

$$= M(x,y)\,dx + N(x,y)\,dy$$

$M\,dx + N\,dy = 0$ is exact if $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

where $M(x,y) = 2x$
$N(x,y) = 2-2y$

The RHS of (1) is exact because

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

$$\int dv = \int M\,dx + \int (\text{Part of } N \text{ excluding the terms of } x)\,dy + c$$

$$v(x,y) = \int 2x\,dx + \int (2-2y)\,dy + c$$

$$= x^2 + 2y - y^2 + c$$

2nd Part

$$v(x,y) = x^2 + 2y - y^2 + c$$

Milne ~~Milne~~ Thompson property

Now $f'(z) = u_1(z,0) - i\,u_2(z,0)$; where $u_1(x,y) = \frac{\partial u}{\partial x}$, $u_2(x,y) = \frac{\partial u}{\partial y}$.

[Milne Thompson property]

$$u_1(x,y) = \frac{\partial u}{\partial x} = 2-2y \qquad u_1(z,0) = 2$$

$$u_2(x,y) = \frac{\partial u}{\partial y} = -2x \qquad u_2(z,0) = -2z.$$

$\Rightarrow f'(z) = 2 - i(-2z)$.

$f'(z) = 2 + i2z$.

$\therefore \boxed{f(z) = 2z + iz^2 + c}$ [by integrating both side]

3rd Part

Qn Test. $u(x,y) = e^x \cos y$ is harmonic or not.......

.....

Same as previous Question

without applying Milne Thomson property

Part One

Part Two

Self

$v(x,y) = e^x \sin y + C$

$f(z) = u(x,y) + iv(x,y)$

$= (e^x \cos y) + i(e^x \sin y + c)$.

$= e^x \cos y + i e^x \sin y + ic$

$\therefore f(z) = e^x \cos y + i \sin y) + K$

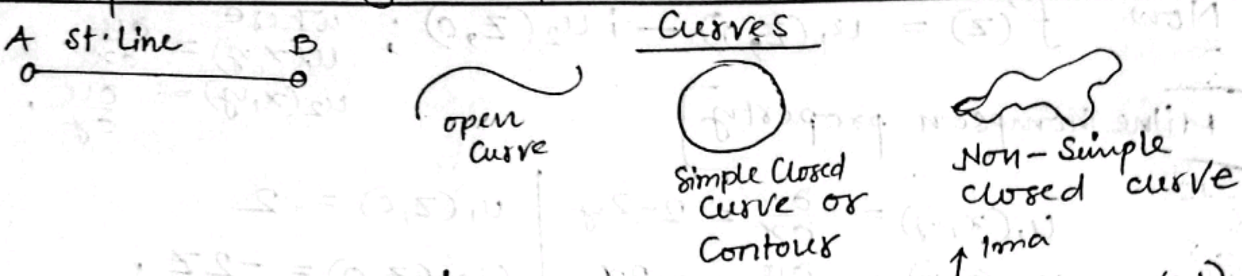
$= e^x (\cos y + i \sin y)$ [Eulers thm]

$= e^x \cdot e^{iy} + K$

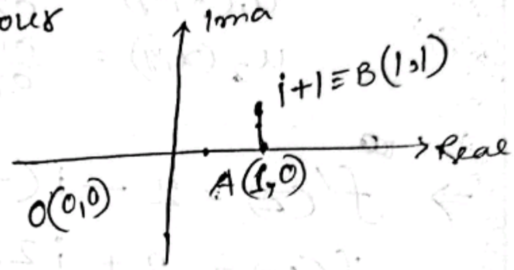
$f(z) = e^{x+iy} = e^z + K$.

Complex Integration :

(03 OCT 2023)



Q^n : Evaluate $\int_0^{1+i} z^2 \, dz$



Ans: Integration from 0 to $(1+i)$ i.e. $O(0,0)$ to $B(1,1)$ we have to move along OA to AB.

We know, $z = x + iy$.

Now along OA : $z = x + i0$ (OA বরাবর x বরাবর x এর value vary করে (z) y এর value vary করে নাই)

$= x$.

and $dz = dx$.

Along AB : $z = 1 + iy$ (AB বরাবর x এর value constant থাকে, $x=1$)

$dz = i\,dy$

$$\int_0^{1+i} z^2 dz = \int_{OA} z^2 dz + \int_{AB} z^2 dz$$

$$= \int_{OA} x^2 dx + \int_{AB} (1+iy)^2 \, dy$$

$$= \int_0^1 x^2 dx + \int_{y=0}^{1} (1+iy)^2 \, dy$$

$$= \frac{1}{3} -$$

$\left[\frac{x^3}{3}\right]_0^1 \; (1+iy)$

$1^2 + 2iy + i^2y^2$

$= 1 + 2iy -$

$= 1 + 2i\left(\frac{1}{2}\right) -$

$= 1 + i - \frac{1}{3}$

$= \frac{2}{3} +$

x এর Limit → Lower → সবচেয়ে Left বিন্দু
Upper → 〃 Right 〃

(OR) (Shortcut) Integration from $O(0,0)$ to $1+i = B(1,1)$ Eqn of st line OB is $= \frac{x-0}{0-1} = \frac{y-0}{0-1} \Rightarrow y = x$.

$$z = x+iy$$
$$dz = dx + i\,dy$$

(Single Integral)

$$\therefore \int_0^{1+i} z^2\,dz = \int_0^{1+i} (x+iy)^2 (dx+i\,dy)$$

$$= \int_0^1 (x+ix)^2 (dx + i\,dx)$$

$$= \int_0^1 x^2 (1+i)^2 (1+i)\,dx$$

(Line Integral)

$$= (1+i)^3 \left[\frac{x^3}{3}\right]_0^1 \Rightarrow (1+3\cdot1^2 i + 3\cdot1\cdot i^2 + i^3)\left(\frac{1}{3}-0\right)$$

$$\Rightarrow (-2+2i)$$

$$= -\frac{2}{3}(1-i)$$