

Chapter 2 : Sums

Notation

① $1 + 2 + 3 + \dots + n$ ✓

$$(1) + \underline{2} = (2) + 1 + 2 + \dots + (n-1) + n \quad \checkmark$$

$$q + (s) \uparrow - (e) \uparrow + n \quad (\text{not really clear})$$

$$9 + (9 + 25) \cdot 5 =$$

$$* \quad a_1 + a_2 + a_3 + \dots + a_n$$

$$(\mathbf{v})_{\text{BS}} = (\mathbf{v}) + \text{noise}$$

$$Q + \frac{1}{2} \leftarrow \mu = \alpha$$

\downarrow

$$Q = \frac{1}{2}$$

$$-\sigma \cdot (\alpha) + \alpha$$

$$\theta = (\sigma) \beta$$

$$\mu = (\alpha) \Lambda$$

\rightarrow summation of terms

$\delta_2 + \alpha_1 + \beta_1 = 0$

85 + 91 + 84 | 2

Sigma notation

$(a_k \rightarrow \text{summand})$

$\xrightarrow{\text{SL+}}$ "de Limited form"

(limit 1 to n) sum

generalized sigma form

der sigma

sigma notation (generalized)

$\rightarrow \sum_{k=0}^{\infty} a_k$ \rightarrow generalized sigma form

$\boxed{K \leq n}$ \rightarrow relation under sigma

Ex

Ex: $\sum k^2$ $1^2 + 3^2 + 5^2 + \dots + 99^2$

$$\sum_{\substack{1 \leq n \leq 100 \\ K \text{ odd}}}^{\wedge}$$

quadrilateral

Acceptance

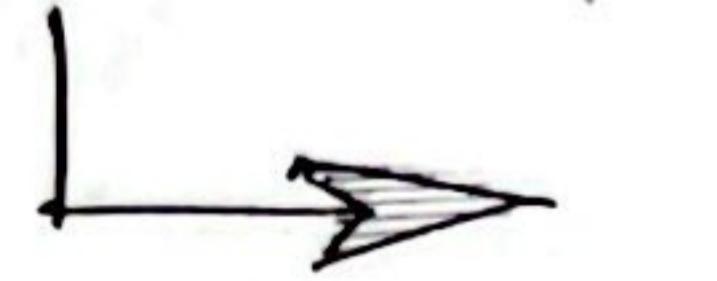
$$\sum_{k=0}^{49} (2k+1)^2$$

$$S(n) = q_1 \cdot \sum_{k=0}^{\infty} (2k+1) + q_0 \cdot A + B(n) = (n)^2$$

- Sum of reciprocal of the prime numbers upto 100:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots + \frac{1}{\text{prime}} \quad [n \geq 9]$$

EST 90 [using 9] ← 1



$$[n \geq 9] \quad [n \geq 9]$$

$$\sum_{1 \leq n \leq 100} \frac{1}{n} \quad \text{or,} \quad \sum_{k=1}^{\pi(100)} \frac{1}{P_k}$$

π grant
function
 $\sum 1$ to 100
prime number
generate

$P_k \rightarrow k^{\text{th}}$
prime
number

$$\text{or, } \sum_{\substack{p \text{ is prime} \\ p \leq N}} \frac{1}{p} \quad \text{or, } \sum_{k=1}^{\pi(N)} \frac{1}{P_k}$$

(upto N)

$$\sum_{k=1}^n \frac{1}{k} = n^2 \quad \text{more diff } O(n)$$

changing
limits

- $\sum_{1 \leq k \leq n} a_k \rightarrow$ we want to change limit ~~into~~ k into $k+1$:

$$\rightarrow \sum_{1 \leq k+1 \leq n} a_{k+1}$$

$$m^2 + \sum_{k=1}^{m-1} = m^2$$

$$m^2 + 1 - m^2 =$$

- $\sum_{k=1}^n a_k \rightarrow$ k into $k+1$

$$= \sum_{k=0}^{n-1} a_{k+1}$$

$\Rightarrow = 0 + \dots + \text{to algorithm } 0 + \text{last term} = m^2 + 1$

Note:

- problem statement represent → use delimited.
- To manipulate sums → use relation under the Σ .

$$\rightarrow \sum_{k=1}^n a_k (k-1)$$

1 first term 0 → zero sum is not harmful; we don't have to handle it.

$$= \sum_{k=2}^n a_k (k-1)$$

• 001 \Rightarrow kenneth silversons diff conversion w/ KIC

[p prime] = $\sum_{i=0}^{\lfloor \frac{N}{p} \rfloor} p^i$ \rightarrow p prime ता हले !

$$\sum_{\substack{P \leq N \\ P \text{ is prime}}} p^i \xrightarrow{\text{K.I.C}} \sum_{\substack{i=0 \\ i \leq N}} p^i [p \text{ prime}] [P \leq N]$$

↓
that number
 $N \geq p$

$p; N$ एवं अद्यते (इट)
or मान हले 1
o/p func,

Sums & Recurrence:

The sum, $S_n = \sum_{k=0}^n a_k$ (n of qm)

$S_0 = 0$ \Rightarrow first step of know \leftarrow $\sum_{r \geq k \geq 1} r$

$$S_n = \sum_{k=0}^{n-1} a_k + a_n$$

$$= S_{n-1} + a_n$$

$\sum_{r \geq i+k \geq 1}$

So, the sum is equivalent to, $S_0 = a_0 \sum_{r=k}^n$

$$S_n = S_{n-1} + a_n ; n > 0$$

$\sum_{r=k}^n$ E 1

If, $a_n = \underbrace{\text{constant}}_{\beta} + \underbrace{\text{multiple of } n}_{n\gamma}$, $a_0 = \alpha$

$\therefore R_0 = \alpha$ \leftarrow first step of known \leftarrow trans state making \leftarrow

$R_n = R_{n-1} + (\beta + n\gamma)$ } 11 \leftarrow same step known of \leftarrow

$$(1-k) \sum_{r=k}^n$$

i) Inversed for all n < 0 \leftarrow a must trait \leftarrow 1
ii) absurd of word prob

$$(1-k) \sum_{r=k}^n$$

$$R_n = R_{n-1} + \beta + \gamma n$$

$n = 1, 2, 3, \dots$

$$\boxed{R_0 = \alpha}$$

$$R_0 = \alpha$$

$$\alpha = 0 = r_0 = \alpha$$

$$\begin{aligned} R_1 &= R_0 + \beta + \gamma \\ &= \alpha + \beta + \gamma \end{aligned}$$

$$(as. \alpha = \alpha) \quad \text{so, } \beta = \beta$$

$$\begin{aligned} R_2 &= R_1 + \beta + 2\gamma \\ &= \alpha + 2\beta + 3\gamma \end{aligned}$$

$$\begin{aligned} R_3 &= R_2 + \beta + 3\gamma \\ &= \alpha + 3\beta + 6\gamma \end{aligned}$$

$$\alpha + \beta + \gamma(1-n) = r_n$$

$$\alpha + \beta + (1+n\gamma - 1)\gamma = r_n$$

$$\alpha + \beta + n\gamma = r_n + (n-1)\gamma$$

$$\begin{aligned} R_3 &= R_2 + \beta + 3\gamma \\ &= \alpha + 3\beta + 6\gamma \end{aligned}$$

$$\beta = \gamma, 1 - \gamma = \gamma$$

$$\text{generalised solution, } R_n = A(n) \cdot \alpha + B(n) \cdot \beta + C(n) \cdot \gamma$$

③

Set, $R_n = 1$

$$\frac{\alpha + (n)\beta}{\gamma} = (n)\beta$$

$$\begin{aligned} R_n &= 1 \\ R_{n-1} &= 1 \\ R_0 &= 1 = \alpha \end{aligned}$$

$$\begin{cases} 1 = 1 + (\beta + \gamma n) \\ \Rightarrow \beta + \gamma n = 0 \\ \text{so, } \beta = \gamma = 0 \end{cases}$$

$$\text{So, } \alpha = 1, \beta = 0, \gamma = 0$$

$$\text{so, eqn ③, } 1 = A(n) \cdot 1 + B(n) \cdot 0 + C(n) \cdot 0$$

$$\therefore \boxed{A(n) = 1}$$

Set $R_n = n$

$$R_0 = \alpha \therefore R_0 = 0$$

$$R_n = R_{n-1} + \beta + \gamma n$$

$$\Rightarrow n = n-1 + \beta + \gamma n$$

$$\Rightarrow \beta + \gamma n = 1$$

$$\Rightarrow \beta + \gamma n = 1 + 0 \cdot n$$

$$\alpha + \beta = 1, \gamma = 0$$

$$(d+0) \sum_{n=1}^N = n\beta$$

$$0 \cdot d + 0 \leftarrow \beta = 1$$

$$1 \cdot d + 0 \leftarrow 1 = 1$$

$$0 \cdot d + 0 \leftarrow \beta = 1$$

$$m \cdot d + 0 \leftarrow m = 1$$

$$\text{so, from eq. ③, } R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$$\Rightarrow n\beta = A(n) \cdot 0 + B(n) \cdot 1 + C(n) \cdot 0$$

$$\alpha = 0 \quad \text{so, } \alpha = 0$$

$$d = k$$

$$\therefore \boxed{B(n) = n}$$

Set R_n = n^r

$$R_0 = 0^r = 0 = \alpha$$

$$\therefore \alpha = 0$$

$$\therefore n^r = (n-1)^r + \beta + \gamma n$$

$$\Rightarrow n^r = n^r - 2n + 1 + \beta + \gamma n$$

$$\Rightarrow -1 + 2n = \beta + \gamma n$$

$$\therefore \beta = -1, \gamma = 2$$

$$\text{From eqn } \textcircled{III}, n^r = A(n) \cdot 0 + B(n) \cdot (-1) + C(n) \cdot 2$$

$$\Rightarrow n^r = -B(n) + 2 \cdot C(n)$$

$$\Rightarrow C(n) = \frac{B(n) + n^r}{2}$$

$$\begin{cases} (\alpha\gamma + \beta) + 1 = 1 \\ 0 = n\gamma + \beta \\ 0 = \gamma = q, \text{ so} \end{cases} \quad \begin{cases} \gamma = \frac{n + n^r}{2} \\ (2 = 0, \text{ so}) \quad \gamma = \frac{2}{2} = 1 \end{cases} \quad : \quad \boxed{I = m\lambda + f\omega}$$

$$0 = \gamma, 0 = q, 1 = \gamma, \text{ so}$$

So, from eqn \textcircled{III} ,

$$R(n) = 1 \cdot \alpha + n \cdot \beta + \frac{n + n^r}{2} \cdot \gamma \quad \text{--- } \textcircled{IV}$$

$$S_n = \sum_{k=0}^n (a + bk)$$

$$\text{when, } k=0 \rightarrow a + b \cdot 0$$

$$k=1 \rightarrow a + b \cdot 1$$

$$k=2 \rightarrow a + b \cdot 2$$

$$k=n \rightarrow a + bn$$

$$S(n) = a + q(n)q + b(n)K$$

$$a \cdot (n) + L \cdot (n)\theta + a \cdot R(n)A = R_{n+1} = + \cdot R_n \text{ mod } .02$$

$$\text{and } aR_0 = \alpha (=) a$$

$$\text{so, } \beta = \alpha \\ \gamma = b$$

Recursive
method

... E, S, L = 11

$$\alpha = 0$$

$$\gamma + q + \alpha = 1$$

$$\gamma + q + \alpha =$$

$$\gamma \alpha + q + \alpha =$$

$$(E + q) \alpha + \alpha =$$

$$\gamma E + q + \alpha = \alpha$$

$$\alpha + q \alpha + \alpha =$$

$$\alpha + q \alpha$$

$$\therefore \alpha = a, \beta = b, \gamma = b \quad \rightarrow \text{from equation 1}$$

$$\begin{aligned} \therefore R(n) &= \alpha + n\beta + \frac{n+n}{2}\gamma \quad (\text{from eqn 1}) \\ &= a + na + \frac{n+n}{2}b \\ &= a(n+1) + b \cdot \frac{n(n+1)}{2} \end{aligned}$$

(from eqn 1)

TOWER OF HANOI :

$$\begin{array}{l} \xrightarrow{\text{Recurrence soln}} T_0 = 0 \\ \xrightarrow{T_n = 2T_{n-1} + 1 \quad (\text{for } n > 0)} \end{array} \quad \text{--- (2)}$$

divide eqn (2) by 2^n ,

$$\frac{T_0}{2^0} = \frac{0}{2^0} = 0 \quad (n=0) \quad \rightarrow (S_0)$$

$$\frac{T_n}{2^n} = \frac{2T_{n-1} + 1}{2^n} \quad \text{for } n > 0 \quad \text{not ad} = aT + b$$

not admissible

$$\Rightarrow \frac{T_n}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n} \quad \text{for } n > 0, TS = aT$$

not admissible

$$\text{assume, } \frac{T_n}{2^n} = S_n \quad \text{equation 2, i.e. most}$$

$$\therefore S_0 = 0 + b = aT + b$$

$$S_n = S_{n-1} + \frac{1}{2^n}$$

$$(aT + b) + \frac{1}{2^n} = aT + b$$

$$aT + b = 2$$

It follows that,

$$d = b, \quad a = q \quad \frac{a(1-r^n)}{1-r}$$

$$S_n = \sum_{k=0}^n 2^{-k}$$

$$(i) \text{ (by result)} \quad S_n = \frac{n+1}{2} + qn + 1 = (n+1) \cdot 2$$

$$S_n = 2^{-1} + 2^{-2} + 2^{-3} + \dots + 2^{-n} + 1 =$$

$$\text{Ansatz} = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \left(\frac{1}{2}\right)^n =$$

$$= [1 - \left(\frac{1}{2}\right)^n]$$

$$a = \frac{1}{2}, \quad r = \frac{1}{2}$$

$$= 1 - \frac{1}{2^n}$$

ANSWER

$$\frac{2(1 - (\frac{1}{2})^n)}{(1 - \frac{1}{2})}$$

$$S_n = \frac{2^n - 1}{2^n}$$

$$0 = \alpha T \quad \leftarrow \text{Resultant R.P.}$$

$$\therefore 2^n S_n = 2^n - 1$$

$$\therefore \boxed{T_n = 2^n - 1}$$

pd (i) by subvib

So, from

$$(0^2) \leftarrow (0 = \alpha) \quad 0 = -\frac{\alpha}{\alpha^2} = -\frac{\alpha T}{\alpha^2}$$

$$a_n T_n = b_n T_{n-1} + c_n \quad (ii) \quad \frac{1 + \dots + \alpha^{n-1}}{\alpha^n} = \frac{1}{\alpha^n}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \left\{ \begin{array}{l} \text{Summation factor} \\ \frac{1}{\alpha^n} + \frac{\alpha}{\alpha^n} + \dots + \frac{\alpha^{n-1}}{\alpha^n} = \frac{1}{\alpha^n} \end{array} \right.$$

let the summation factor is S_n

from (ii), S_n multiply, $\alpha^2 = \frac{\alpha T}{\alpha^2} = \alpha^2$

$$S_n \cdot a_n T_n = S_n b_n \cdot T_{n-1} + S_n c_n \quad 3:2$$

$$\text{let. } \boxed{S_n b_n = S_{n-1} \cdot a_{n-1}}$$

$$S_n = S_n a_n T_n$$

$$\frac{1}{\alpha^n} + 1 - \alpha^2 = \alpha^2$$

$$+ 3 \cdot 2 \rightarrow S_n = s_{n-1} b_{n-1} T_{n-1} + s_n c_n$$

$$S_n = S_{n-1} + s_n c_n \rightarrow S_{n-2} + S_{n-1} c_{n-1} + s_n c_n$$

Hence,

$$S_n = s_0 a_0 T_0 + \sum_{k=1}^n s_k c_k$$

$$= s_0 b_0 T_0 + \sum_{k=1}^n c_k s_k$$

$$S_n a_n T_n = s_0 b_0 T_0 + \sum_{k=1}^n c_k s_k$$

$$\Rightarrow T_n = \frac{1}{s_n a_n} \left[s_0 b_0 T_0 + \sum_{k=1}^n c_k s_k \right] \rightarrow \text{Sol}^n$$

basis, pf 2010 auf zu 200, multiplikativ traktieren mit

$$n=1 \rightarrow T_1 = \frac{1}{s_1 a_1} [s_0 b_0 T_0 + c_1]$$

$$\text{e' f' r' d' r' d' i' n' g' e' q' u' e' r' g' o' n' g'}$$

$$s_n b_n = s_{n-1} a_{n-1}$$

$$s_n = \frac{s_{n-1} a_{n-1}}{b_n}$$

$$= \frac{s_{n-2} \cdot a_{n-2}}{b_{n-1}} \frac{a_{n-1}}{b_n} + \dots = \dots$$

$$s_n = \frac{(a_{n-1} a_{n-2} a_{n-3} \dots a_1)}{b_n b_{n-1} b_{n-2} \dots b_2}$$

$$a_n T_n = b_n T_{n-1} + c_n \quad \text{or} \quad T_n = b_n T_{n-1} + c_n$$

↓

$$T_n = 2 \cdot T_{n-1} + 1$$

$$\therefore S_n = \frac{1 \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{2 \cdot 2 \cdot \dots \cdot 2} + T_0 D_n = n^2$$

$$S_n = \frac{1}{2^{n-1}} \quad \text{Summation factor}$$

Problem

In quicksort algorithm, one of the mostly used method for sorting data, data's need to compare with other element. The average number of comparison steps made by quicksort when it's applied to n items in random order which satisfies the following recurrence.

$$c_0 = 0$$

$$c_n = n+1 + \frac{2}{n} \sum_{k=0}^{n-1} c_k, \text{ for } n > 0$$

1. Form the above recurrence in $a_n T_n = b_n T_{n-1} + c_n$

2. Find the summation factor (S_n)

3. Prove that i) $c_n = 2(n+1) \sum_{k=1}^n \frac{1}{k+1}$

ii) $c_n = 2(n+1)H_n - 2n$, where H_n

$\sum_{k=1}^n \frac{1}{k}$ is called harmonic sum

iii) Find closed form for c_n

$$\sum_{k=1}^n k + \sum_{k=1}^n \frac{1}{k} = (n+1) \sum_{k=1}^n \frac{1}{k}$$

iv) Find c_1, c_2 and c_3 .

$$a_n T_n = b_n T_{n-1} + c_n \quad (\text{I}) \quad \sum_{k=1}^n a_k T_k = \sum_{k=1}^n b_k T_{k-1} + \sum_{k=1}^n c_k$$

$$a_n c_n = (n+1) c_{n-1} + 2n$$

↑
for a_n to make terms b_n cancel c_{n-1}

$$b_n c_{n-1} = n c_{n-2} + 2(n-1)$$

↑
 b_{n-1} ↑
 c_{n-1}

$$S_n = \frac{a_{n-1} a_{n-2} a_{n-3} \dots a_1}{b_n \cdot b_{n-1} \cdot b_{n-2} \dots b_2}$$

$$= \frac{(n-1)(n-2)(n-3)\dots 1}{(n+1) n \cdot (n-1) \dots 3}$$

↓
progression

$$= \frac{\cancel{(n-1)! 2!}}{(n+1) n \cdot \cancel{(n-1)!}} = \frac{\cancel{2}}{n(n+1)}$$

→ summation factor

$$\sum_{k=1}^n + \sum_{m=1}^n = \sum_{m=1}^n + \sum_{k=1}^m$$

↓
number of terms, $m = n$

bottom most spring \rightarrow middle \rightarrow
most most spring \rightarrow middle \rightarrow
middle

Manipulation of Sums:

1. Distributive law. $\rightarrow \sum_{k \in K} c a_k = c \sum_{k \in K} a_k$ (ii)
 2. Associative law $\rightarrow \sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$ (iii) (iv) (v)
 3. $\sum_{k \in K} a_k = \sum_{P(k) \in K} a_{P(k)}$ (vi) (vii) (viii) (ix) (x)
- ↑
we can reorder the sum in any way
↑ $P(k) \rightarrow$ any permutation of the set.

Ex :

$$\sum_{k=1,2,3} a_k = a_1 + a_2 + a_3$$

\hookrightarrow (123) order \rightarrow (321) order

- For comparing different set of indices,

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cup K'} a_k$$

~~(1-m) · n (1-n)~~

$$\text{Ex : } \sum_{k=1}^m a_k + \sum_{k=m}^n a_k = a_m + \sum_{k=1}^n a_k$$

↓
only $k=m$ common

\hookrightarrow Perturbation method

\rightarrow Splitting a single term from a sum

if we want to find the solution of the system of equations

$$S_n + \alpha x = \sum_{0 \leq k \leq n} \alpha x^k + \alpha x^{n+1}$$

$$= \alpha x^0 + \sum_{0 \leq k \leq n} \alpha x^{k+1} = \alpha x^0 + \sum_{m \geq 1} \alpha x^m$$

$$= \alpha x^0 + x \cdot \sum_{0 \leq k \leq n} \alpha x^k$$

$$= \alpha x^0 + x \cdot S_n$$

$$\Rightarrow S_n - x S_n = \alpha x^0 - \alpha x$$

$$\Rightarrow (\cancel{1-x}) S_n = \frac{\alpha (1 - x^{n+1})}{1-x}$$

Definition:

$$S_n = \frac{\alpha (1 - x^{n+1})}{1-x}$$

$$x^0 + \dots + x^0 + x^0 + x^0 + x^0 = \alpha^2 \leftarrow$$

$$n = \alpha^2$$

Problem : Apply

$$+ \epsilon^{d_1} x^{d_1} + \epsilon^{d_2} x^{d_2} + \dots + \epsilon^{d_n} x^{d_n}$$
$$+ \epsilon^{d_{n+1}} x^{d_{n+1}} + \epsilon^{d_{n+2}} x^{d_{n+2}}$$
$$\dots$$
$$\epsilon^{d_{\ell-1}} x^{d_{\ell-1}} + \epsilon^{d_{\ell}} x^{d_{\ell}}$$

General recursive form

$$\text{otherwise } \rightarrow [(\epsilon, i)^q] x^{d_i} \sum_{j=1}^q = x^{d_i} \sum_{j=1}^q (\epsilon, i)^q$$

$$[(\epsilon, i)^q] x^{d_i} \sum_{j=1}^q$$

returning to rules with grammatical form

$$[(\epsilon, i)^q] x^{d_i} \sum_{j=1}^q = x^{d_i} \sum_{j=1}^q = [(\epsilon, i)^q] x^{d_i} \sum_{j=1}^q$$

$$[\epsilon \geq x \cdot i \geq 1] x^{d_i} \sum_{j=1}^q = x^{d_i} \sum_{j=1}^q$$

$$[\epsilon \geq x \geq 1] [\epsilon \geq i \geq 1] x^{d_i} \sum_{j=1}^q =$$

$$[\epsilon \geq x \geq 1] [\epsilon \geq i \geq 1] x^{d_i} \sum_{j=1}^q$$

Multiple of sums :

(S99A) : msJ don't

$$* \sum_{1 \leq j, k \leq 3} a_j b_k = a_1 b_1 + a_1 b_2 + a_1 b_3 + \\ a_2 b_1 + a_2 b_2 + a_2 b_3 + \\ a_3 b_1 + a_3 b_2 + a_3 b_3$$

$$* \sum_{P(j,k)} a_{j,k} = \sum_{j,k} a_{j,k} [P(j,k)] \rightarrow \text{Kronecker}\ \text{Inversion}\ \text{form}$$

$$* \sum_j \sum_k a_{j,k} [P(j,k)]$$

• Law of interchanging the order of summation.

$$\sum_j \sum_k a_{j,k} [P(j,k)] = \sum_{P(j,k)} a_{j,k} = \sum_k \sum_j a_{j,k} [P(j,k)]$$

$$\sum_{1 \leq j, k \leq 3} a_j b_k = \sum_{j, k} a_j b_k [1 \leq j, k \leq 3]$$

$$= \sum_{j, k} a_j b_k [1 \leq j \leq 3] [1 \leq k \leq 3]$$

$$= \sum_j \sum_k a_j b_k [1 \leq j \leq 3] [1 \leq k \leq 3]$$

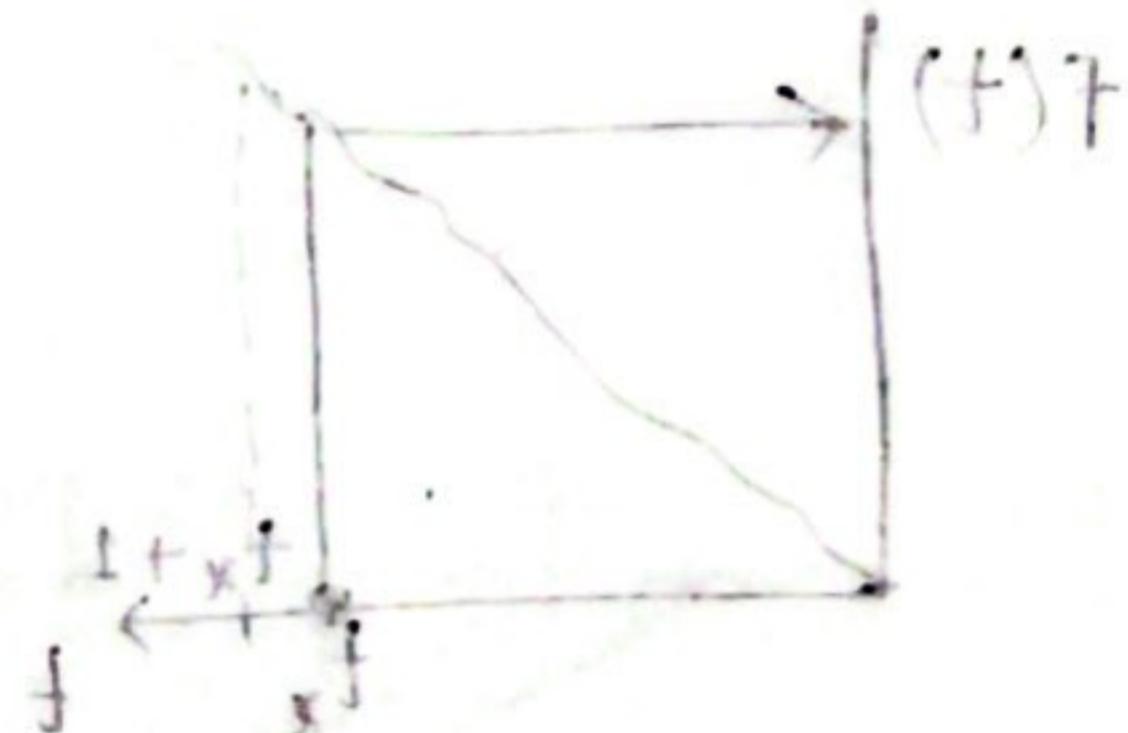
smith to mukundan mabmoi) o pente

$$\begin{aligned} &= \left(\sum_j a_j [1 \leq j \leq 3] \right) \left(\sum_k b_k [1 \leq k \leq 3] \right) \\ &= \left(\sum_{j=1}^3 a_j \right) \left(\sum_{k=1}^3 b_k \right) \end{aligned}$$

General distributive law : braint (i) : etnaregma 8

$$\sum_{j \in J} a_j b_k = \left(\sum_{j \in J} a_j \right) \left(\sum_{k \in K} b_k \right)$$

armer mabmoi
mukundan bmoi
mukundan bmoi



mabmoi to mukundan ei sange sifondote A
smith yd baxbri zedlova

as to etnaregma $(t)x$ sange sifondote A

braint (i) q: dimension mukundan affine thymiqo
toft mukundan a base sange elqmoi is no

base of $(x,t)x$ mukundan smith a empiezo

smith to sange elqmoi int in'e emtne

fruminteqs

* Find the closed form for the following,

$$2 \sum_{\substack{i-n \geq j, k \leq n \\ i-k \geq 0}} a_k b_k$$

$$\frac{1}{i-k} \sum_{\substack{n \geq i \geq 1 \\ i-k \geq 0}} =$$

$$= 2 \sum_{\substack{i-n \geq j \leq n \\ i-k \geq 0}} a_k b_k$$

$$\sum_{\substack{n \geq i \geq 1 \\ i-k \geq 0}} = \left| \sum_{1 \leq j \leq n} 1 \right| = n$$

$$= 2 \sum_{1 \leq k \leq n} \left(\sum_{k \leq j \leq n} a_k b_k \right)$$

$$\frac{1}{i-k} \sum_{\substack{n \geq i \geq 1 \\ i-k \geq 0}} =$$

$$= 2 \sum_{1 \leq k \leq n} \left[a_k b_k \sum_{1 \leq j \leq n} 1 \right]$$

$$\left(\frac{1}{i-k} \sum_{\substack{n \geq i \geq 1 \\ i-k \geq 0}} \right) \sum_{1 \leq j \leq n} =$$

$$= 2 \sum_{1 \leq k \leq n} [a_k b_k \cdot n]$$

$$\left(\frac{1}{i-k} \sum_{\substack{n \geq i \geq 1 \\ i-k \geq 0}} \right) \sum_{1 \leq j \leq n} =$$

$$= 2n \sum_{1 \leq k \leq n} a_k b_k$$

* Find closed form of the sum, $S_n = \sum_{1 \leq j \leq k \leq n} \frac{1}{k-j}$

i) First summing on j ,

$$S_n = \sum_{1 \leq j \leq k \leq n} \frac{1}{k-j}$$

$$= \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq k} \frac{1}{k-j}$$

$$= \sum_{1 \leq k \leq n} \sum_{1 \leq k-j < k} \frac{1}{k-(k-j)}$$

$$= \sum_{1 \leq k \leq n} \left(\sum_{1 \leq j \leq k-1} \frac{1}{j} \right)$$

$$= \sum_{1 \leq k \leq n} H_{k-1} = \sum_{0 \leq k \leq n} H_k$$

(Ans).

$$\begin{aligned} & 1 \leq k-j < k \\ & \Rightarrow 1-k \leq k-j-k < k-k \\ & \Rightarrow 1-k \leq -j < 0 \\ & \Rightarrow k-1 \geq j > 0 \end{aligned}$$

$$\begin{aligned} & 0 < j \leq k-1 \\ & \Rightarrow \boxed{1 \leq j \leq k-1} \quad \text{with } j \text{ replaced by } k-j \end{aligned}$$

$$\begin{aligned} & j \rightarrow 1 \text{ to } k-1 \\ & H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k-1} \end{aligned}$$

↳ harmonic sum
here $n = k-1$

2) summing on k , so not bold soft limit

$$S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k-j}$$

$$\begin{aligned} j < k+j \leq n \\ \Rightarrow j-j < k+j-j \leq n-j \\ \Rightarrow 0 < k \leq n-j \end{aligned}$$

$$= \sum_{1 \leq j \leq n} \sum_{j < k \leq n} \frac{1}{k-j}$$

$$\sum_{n \geq k \geq 1}$$

$$\boxed{1 \leq k \leq n-j}$$

$$= \sum_{1 \leq j \leq n} \sum_{j < k \leq n} \frac{1}{k+j-j}$$

(\cancel{k}) \rightarrow replacing k with $k+j$

$$= \sum_{1 \leq j \leq n} \left(\sum_{j < k+j \leq n} \frac{1}{k} \right)$$

$$(1) \sum_{n \geq k \geq 1} \sum_{n \geq k \geq 1}$$

$$= \sum_{1 \leq j \leq n} \left(\sum_{0 < k \leq n-j} \frac{1}{k} \right)$$

$$(2) \sum_{n \geq k \geq 1} \sum_{n \geq k \geq 1}$$

$$= \sum_{1 \leq j \leq n} H_{n-j}$$

$$\cancel{j} \rightarrow 1 \leq n-j \leq n$$

$$\Rightarrow 1-n \leq n-j-n \leq n-n$$

$$= \sum_{1 \leq n-j \leq n} H_{n-(n-j)}$$

$$\Rightarrow 1-n \leq -j \leq 0$$

$$\Rightarrow n-1 \geq j \geq 0$$

$$\Rightarrow 0 \leq j \leq n-1$$

$$= \sum_{1 \leq n-j \leq n} H_j$$

$$\frac{1}{i-k} \sum_{n \geq k \geq 1}$$

$$= \sum_{0 \leq j \leq n-1} H_j$$

$$\frac{1}{i-k} \sum_{k \geq 1} \sum_{n \geq k \geq 1}$$

$$= \sum_{0 \leq j < n} H_j$$

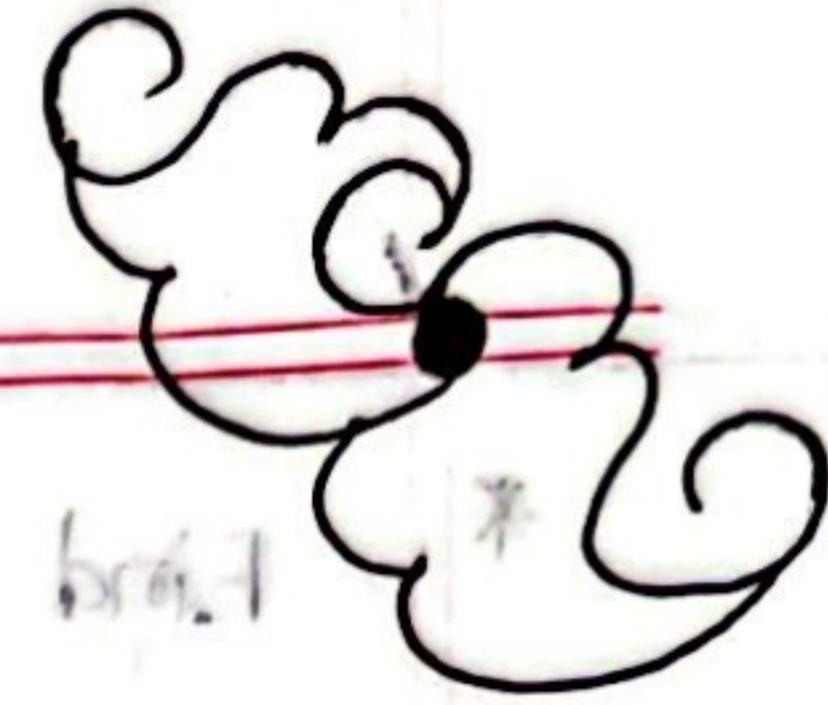
$$\frac{1}{(i-k)} \sum_{k \geq 1} \sum_{n \geq k \geq 1}$$

$$\left(\frac{1}{i} \sum_{k \geq 1} \right) \sum_{n \geq k \geq 1}$$

$$H \sum_{n \geq k \geq 0} = \frac{1}{i} H \sum_{n \geq k \geq 1}$$

$$\sum_{1 \leq k \leq n} \frac{n}{k} - \sum_{1 \leq k \leq n} 1$$

$$= nH_n - n$$



3) $S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k-j}$ not now below but
 $= \sum_{1 \leq j < k \leq n} \frac{1}{k}$ [replacing k by $k+j$]
 $= \sum_{1 \leq j < k} \left(\sum_{1 \leq k \leq n} \frac{1}{k} \right)$: bottom part $\therefore k+j-j = k$
 $= \sum_{1 \leq j < k} \left(\sum_{n \geq k \geq 0} \frac{1}{k} \right)$
 $= (1 + \sum_{1 \leq k \leq n} \sum_{n \geq k \geq 0} \frac{1}{k})$
 $= (1 + \sum_{1 \leq k \leq n} \sum_{n \geq k \geq 0} \frac{1}{k})$
 $= (1 + \sum_{1 \leq k \leq n} \sum_{n \geq k \geq 0} \frac{1}{k})$

$$\sum_{n \geq k \geq 0} + \sum_{n \geq k \geq 0} + \sum_{n \geq k \geq 0} =$$

$$(1+n) + \sum_{n \geq k \geq 0} + \sum_{n \geq k \geq 0} = (1+n) + \sum_{n \geq k \geq 0}$$

$$(1+n) - (1+n) = \sum_{n \geq k \geq 0}$$

$$\frac{(n-n)(1+n)}{2} = \sum_{n \geq k \geq 0}$$

$$\boxed{\frac{(1+n)n}{2} = \sum_{n \geq k \geq 0}}$$

* Find closed form for the sum of squares, i.e D_n

$$S_n = D_n = \sum_{0 \leq k \leq n} k^2 = 0^2 + 1^2 + 2^2 + \dots + n^2$$

→ Perturbation method:

$$\begin{aligned} D_n + (n+1)^2 &= \sum_{0 \leq k \leq n} k^2 + (n+1)^2 \\ &= \sum_{0 \leq k \leq n+1} k^2 = \sum_{0 \leq k \leq n} (k+1)^2 \\ &= \sum_{0 \leq k \leq n} (k^2 + 2k + 1) \end{aligned}$$

$$= \sum_{0 \leq k \leq n} k^2 + \sum_{0 \leq k \leq n} 2k + \sum_{0 \leq k \leq n} 1$$

$$\Rightarrow D_n + (n+1)^2 = D_n + 2 \sum_{0 \leq k \leq n} k + n+1$$

$$\Rightarrow 2 \sum_{0 \leq k \leq n} k = (n+1)^2 - (n+1)$$

$$\Rightarrow \sum_{0 \leq k \leq n} k = \frac{(n+1)(n+1-1)}{2}$$

$$\therefore \boxed{\sum_{0 \leq k \leq n} k = \frac{n(n+1)}{2}}$$

$$\sum_{k=1}^n k^3 = \sum_{0 \leq k \leq n} k^3$$

Rechteckspolyeder

$$V = \frac{1}{3}$$

$$\square_n + (n+1)^3 = \sum_{0 \leq k \leq n} k^3 + (n+1)^3 + q + r = \frac{1}{3}$$

$$= \sum_{0 \leq k \leq n} ((k+1)^3 + q + r) = \sum_{0 \leq k \leq n} (k^3 + 3k^2 + 3k + 1)$$

$$= \sum_{0 \leq k \leq n} k^3 + 3 \sum_{0 \leq k \leq n} k^2 + 3 \sum_{0 \leq k \leq n} k + \sum_{0 \leq k \leq n} 1$$

$$= \square_n + 3 \sum_{0 \leq k \leq n} k^2 + 3 \sum_{0 \leq k \leq n} k + \sum_{0 \leq k \leq n} 1$$

$$= 3 \square_n + 3 \frac{n(n+1)}{2} + (n+1)$$

$$V = \frac{1}{3} \left(\frac{n+1}{2} \right)^3 + \frac{n+1}{3} + q + r = \frac{1}{3}$$

$$\Rightarrow \square_n = \frac{(n+1)^3}{3} - \frac{n(n+1)}{2} + \frac{n+1}{3} + q + r = \frac{1}{3}$$

$$= (n+1) \left\{ \frac{2n^3 + 4n^2 + 2}{6} - \frac{3n^2 + 2}{3} \right\}$$

$$= \frac{(n+1)(-n^2 - n + 6)(2n^2 + n)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$c = 3, e = 8, t = 4, \theta = 0$$

$$3 \cdot (n)c + 8 \cdot (n)t + 4 \cdot (n)\theta + 0 \cdot (n)A = (n)s$$

$$(n)c + (e-t) \cdot \frac{n}{2} + t \cdot n + 0 \cdot \frac{n}{3} = n$$

$$ns = (n+1)e + \frac{n}{2} = (n)s$$

$$(n+1)e + \frac{n}{2} = s$$

Reperatoire method:

$$R_0 = \alpha$$

$$R_n = R_{n-1} + \beta + \gamma n + \delta n^2 + \sum_{k \geq 0} \epsilon_k = (1+\alpha + \beta + \gamma n + \delta n^2) + \sum_{k \geq 0} \epsilon_k$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\cdot S + \sum_{k \geq 0} \epsilon_k$$

$$\sum_{k \geq 0} \epsilon_k = S_n = S_{n-1} + n^2$$

$$R_0 = \alpha$$

$$R_n = R_{n-1} + \beta + \gamma n + S_n$$

$$A(n) = 1$$

$$B(n) = n$$

$$C(n) = \frac{n^2+n}{2}$$

Set,

$$R(n) = n^3 \epsilon + n^2 \epsilon =$$

$$R_0 = 0^3 = 0 \rightarrow \alpha = 0$$

$$R_n = R_{n-1} + \beta + n\gamma + n^2\delta - \frac{\epsilon(1+\alpha)}{\epsilon} = \frac{\epsilon(1+\alpha)}{\epsilon} = \frac{\epsilon(1+\alpha)}{\epsilon}$$

$$\Rightarrow n^3 = (n-1)^3 + \beta + n\gamma + n^2\delta$$

$$\Rightarrow \beta + n\gamma + n^2\delta = n^3 - (n-1)^3$$

$$(n+1)^3 = n^3 - (n^3 - 3n^2 + 3n - 1)$$

$$= 3n^2 - 3n + 1$$

$$\Rightarrow \beta + n\gamma + n^2\delta = (3n^2 - 3n + 1)$$

$$\alpha = 0, \beta = 1, \gamma = -3, \delta = 3$$

$$R(n) = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\cdot S$$

$$\Rightarrow n^3 = 1 \cdot 0 + n \cdot 1 + \frac{n^2+n}{2} \cdot (-3) + 3D(n)$$

$$\Rightarrow 3D(n) = 2n^3 + 3(n^2+n) - 2n$$

$$= \frac{1}{3}(2n^3 + 3n^2 + n)$$