**What is a random variable?**

In statistics, a random variable is a variable whose possible values depend on the outcomes of a certain random experiment. As random variables are quantifiable, they are always real numbers. A random variable can take on different real values, because a random event might have multiple outcomes. For example, number of heads in 5 tosses of coins. In 5 tosses we can get 0, 1, 2, 3, 4 or 5 heads. Since there could be any values , a random variable is represented using uppercase letters.More commonly, $X$ is used to denote a random variable. Some examples of a random variables include:

- Outcome of a dice roll
- Number of items sold at a grocery shop at a particular day
- Number of students in a class
- Weight of new born babies
- Height of all the students in a classroom.

Consequently $P(X = x)$ represents the probability of the random variable $X$ taking on any particular value $x$.

Random variables are classified into discrete and continuous variables. The main difference between the two categories is the type of possible values that each variable can take. While a discrete random variable can have an exact value, the value of a continuous random variable will lie within a specific range.

<u>**Discrete random variables**</u>

A ***discrete random variable*** **consisting of numerical values** that are **countable** and have a **finite number of possibilities**. Hence it can take on only a countable or finite number of distinct values like 0, 1, 2, 50, 1000, etc. Since discrete…. $P(X = x)$ represents the probability of the random variable $X$ taking on any particular value $x$.

Properties:

- Probability of each value between 0 and 1, $0 \leq P(X = x) \leq 1$.
- Sum of all probabilities $\sum P(x) = 1$
- Examples
  - as the number of defectives in a box of k items,
  - number of cars sold by a company in a day,
  - Vote count in an election.
  - Number of students in a class
  - Number of citizens of a country

Even if it would take a long time to count the citizens of a large country, it is still technically doable. Moreover, for all examples, the number of possibilities is **finite**. Whatever the number of children in a family, it will never be 3.58 or 7.912 so the number of possibilities is a finite number and thus countable.

**Continuous random variables**

On the other hand, **quantitative continuous** variables are variables for which the values are **not countable** and have an **infinite number of possibilities**. While a discrete variable, as mentioned earlier, only takes on countable whole numbers, a continuous variable can include values with infinite decimal places like 0.03, 1.2374553, etc

For example:

- Age
- Weight
- Height
- Cholesterol level.

For all measurements, we usually stop at a standard level of granularity, but nothing (except our measurement tools) prevents us from going deeper, leading to an **infinite number of potential values**. For example, a 28-year-old man could actually be 28 years, 7 months, 16 days, 3 hours, 4 minutes, 5 seconds, 31 milliseconds, 9 nanoseconds old.

Like discrete variable, continuous variables also posses certain properties:

- Probability of **of any one particular outcome** is 0
- The area under the curve (i.e. the indefinite integral) is 1.
- Examples like height, weight, and temperature

**What is a probability distributions ?**

The probability distribution for a random variable describes how the probabilities are distributed over the values of the random variable. In other words it tells us how likely it is for the random variable $X$ to take on different values of $x$. It lists all the possible values and likelihoods that a random variable $X$ can take within a given range. Probability distribution could be defined as the table or equations showing respective probabilities of different possible outcomes of a defined event or scenario**.** When we describe the values in the range of a random variable in terms of the probability of their occurrence, we are essentially talking about the probability distribution of the random variable. In other words, the probability distribution of a random variable can be determined by calculating the probability of occurrence of every value in the range of the random variable.

There are two different types of probability distribution depending on the two type of (random) variable we have discussed earlier The type of (random) variable implies the particular method of finding a probability distribution function.

## 1) Discrete Probability distribution

A discrete random variable is used to model a discrete probability distribution. This type of distribution lists the probability of each discrete outcome of a random variable.

*The probability distribution of a discrete random variable X is a list of each possible value of X together with the probability that X takes that value in one trial of the experiment*.

There are two main functions associated with such a random variable. These are the

i. probability mass function (pmf)
ii. probability distribution function / cumulative distribution function (CDF).

**The probability mass function** can be defined as a function that gives the probability of a discrete random variable, $X$, being exactly equal to some value, $x$. The formula is given as follows:

$$f(x) = P(X = x)$$

**probability distribution function also known as *cumulative distribution function*.** gives the probability when $X$ is less than or equal to $x$, for every value $x$.

$$F(x) = P(X \leq x)$$

For a discrete random variable, the cumulative distribution function is found by summing up the probabilities. Furthermore, if $(a,b]$ is a semi-closed interval, the probability distribution function is provided by the formula given below.

$$P(a < X \leq b) = F(b) - F(a)$$

A random variables probability distribution function is always between 0 and 1. It can never be negative.

For example, if you roll a die, the probability of obtaining a 1 or 2 or 3 or 4 or 5 or 6 is 16.667% (=1/6) individually.

The cumulative distribution function (CDF) of 1 is the probability that the next roll will take a value less than or equal to 1 i.e $P(X \leq 1)$. Since there is only one possible way to get a 1 so $P(X \leq 1)$ = $P(X = 1)$ which is equal to 16.67%

The cumulative distribution function (CDF) of 2 is the probability that the next roll will take a value less than or equal to 2 i.e $P(X \leq 2)$

Now since $P(X \leq 2) = P(X = 1) + P(X = 2)$ = 16.67% + 16.67% = 33.33%

The probability distribution of a discrete random variable can always be represented by a table

**EXAMPLE**

**1)** A fair coin is tossed twice. Let $X$ be the number of heads that are observed. **Now show the probability distribution of showing heads.**

The possible values that $X$ can take are 0, 1, and 2. Each of these numbers corresponds to an event in the sample space $S = \{hh, ht, th, tt\}$ of equally likely outcomes for this experiment:
When $X$ = 0 that means we get 2 tails $\{tt\}$ so out of 4 possible outcomes we get just 1
When $X$ = 1 that means we get $\{ht, th\}$ so out of 4 outcomes we get just 2
When $X$ = 2 that means we get $\{hh\}$ so out of 4 outcomes we get just 1
Now we can calculate the probabilities for each event using the formula $P(X) = \frac{number\ of\ outcomes}{number\ of\ total\ outcomes}$

| Number of Total Outcome | 4 | | |
|---|---|---|---|
| Number of heads = $x$ | 0 | 1 | 2 |
| Number of Outcomes | 1 | 2 | 1 |
| $P(X = x)$ | ¼ = 0.25 | ½ = 0.5 | ¼ = 0.25 |

This table is the probability distribution of $X$.

2) A pair of fair dice is rolled. Let $X$ denote the sum of the number of dots on the top faces.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| 2 | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| 3 | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| 4 | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| 5 | (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| 6 | (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

So we can see the number of total incomes i.e $n(Sample\ Space) = 36$
The possible values that $X$ can take are
$2 = \{1\ 1\}$
$3 = \{(1\ 2), (2\ 1)\}$

$4 = \{(1\ 3), (2\ 2), (3\ 1)\}$
$5 = \{(1\ 4), (2\ 3), (3\ 2), (4\ 1)\}$
$6 = \{(1\ 5), (2\ 4), (3\ 3), (4\ 2), (5\ 1)\}$
$7 = \{(1\ 6), (2\ 5), (3\ 4), (4\ 3), (5\ 2), (6\ 1)\}$
$8 = \{(2\ 6), (3\ 5), (4\ 4), (5\ 3), (6\ 2)\}$
$9 = \{(3\ 6), (4\ 5), (5\ 4), (6\ 3)\}$
$10 = \{(4\ 6), (5\ 5), (6\ 4)\}$
$11 = \{(5\ 6), (6\ 5)\}$
$12 = \{(6\ 6)\}$
Hence The probability of each of these events is

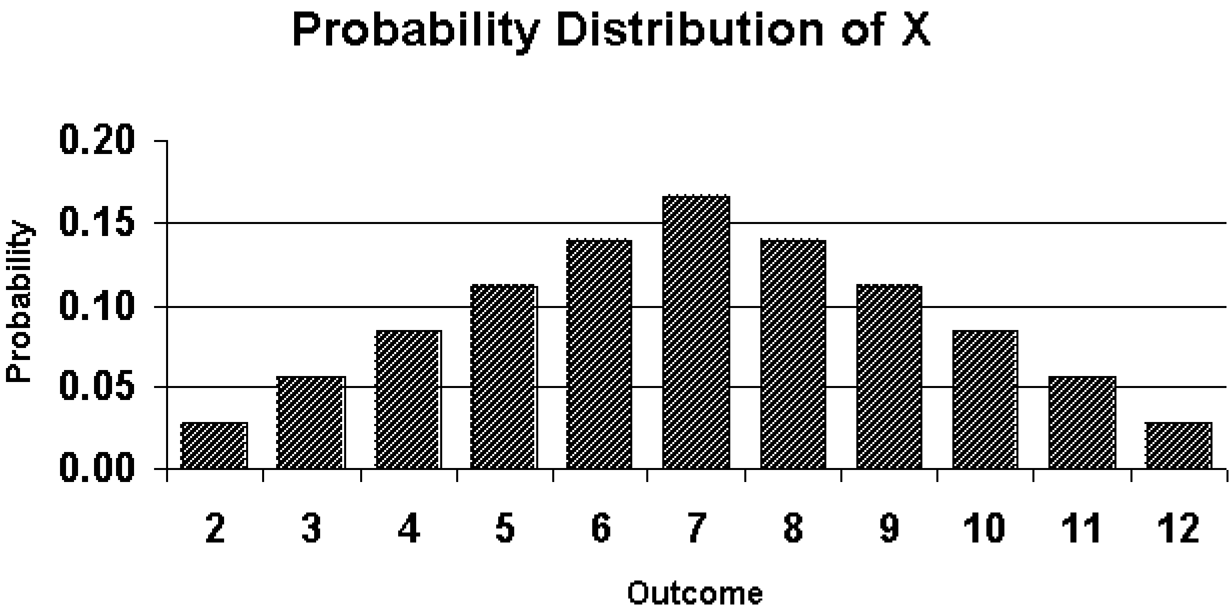
| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

This table is the probability distribution of $X$.

Find $P(X \geq 9) = P(X = 9) + P(X = 10) + P(X = 11) + P(X = 12) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}$
Find the probability that $X$ takes an even value

$$= P(X = 2) + P(X = 4) + P(X = 6) + P(X = 8) + P(X = 10) + P(X = 12)$$
$$= \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{18}{36} = \frac{1}{2}$$

To graph the probability distribution of a discrete random variable, construct a **probability histogram**.




3) Given a probability mass function $f(x) = bx^3$ for $x = 1, 2, 3$. Find the value of b.
**Solution:** According to the properties of probability mass function, $\sum f(x) = 1$
$\sum_{x=1}^{3} f(x) = 1$
$b(13 + 23 + 33) = 1$
$b\ (1 + 8 + 27) = 1$
$b = 1\ /\ 36$
$Answer: b = 1\ /\ 36$

4) Determine the constant $c$ so that the following p.m.f. of the random variable $Y$ is a valid probability mass function:

$$f(y) = c(\frac{1}{4})^y \text{ for } y = 1, 2, 3, \ldots$$

$\sum_{x=1}^{\infty} f(y) = 1 = c\left(\frac{1}{4}\right)^1 + c\left(\frac{1}{4}\right)^2 + c\left(\frac{1}{4}\right)^3 + \cdots. = c\ \left[\left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 +..\right]$

$$c\left[\frac{\frac{1}{4}}{1-\frac{1}{4}}\right] = 1$$

$c = 3$

**5)** The probability mass function table for a random variable X is given as follows:

| $\boldsymbol{x}$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\boldsymbol{P(X = x)}$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 |

Find the value of the CDF, $P(X \leq 2)$.
**Solution:** $P(X \leq 2)$, can be computed by using the pmf property P(X ∈ T) = ∑f(x).
$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$
$= 0 + 0.1 + 0.2$
$= 0.3$

6) Let X be a random variable, and P(X=x) is the PMF given by,

| $\boldsymbol{X}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{P(X = x)}$ | 0 | $k$ | $2k$ | $2k$ | $3k$ | $k^2$ | $2k^2$ | $7k^2 + k$ |

1. Determine the value of $k$
2. Find the probability (i) $P(X \leq 6)$, (ii) $P(3 < x \leq 6)$

**Solution :**

(1) We know that;
$\sum P(x_i)=1$
Therefore,
$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$
$10k^2 + 9k = 1$
$10k^2 + 9k - 1 = 0$
$10k^2 + 10k - k - 1 = 0$
$10k\,(k + 1) - 1(k + 1) = 0$
$(10k - 1)(k + 1) = 0$
$k = 0.1, -1$
$k = -1$ cannot be considered as all probabilities lie between 0 and 1.
Thus, $k = 0.1$
**Answer:** $k = 0.1$

(2) $(i)$ $P(X \leq 6) = 1 - P(x > 6)$

$= 1 - (7k^2 + k)$

$= 1 - (7(1/10)^2 + (1/10))$

$= 1 - (7/100 + 1/10)$

$= 1 - (17/100)$

$= (100 - 17)/100$

$= 83/100$

Therefore , $P(X \leq 6) = 83/100$

(ii) $P(3 < x \leq 6) = P(x = 4) + P(x = 5) + P(X = 6)$

= $3k + k^2 + 2k^2$

= $(3/10) + (1/10)^2 + 2(1/10)^2$

= $3/10 + 1/100 + 2/100$

= $3/10 + 3/100$

= $(30 + 3)/100$

= $\frac{33}{100}$

7) Let $X$ be a discrete random variable with range $R_X = \{1, 2, 3, ...\}$. Suppose the PMF of $X$ is given by

$$R_X(k) = \frac{1}{2^k} \, for\ k = 1,2,3,...$$

a. Find the CDF of $X$

$P(X \leq 1) = P(X = 1) = ½$

$P(X \leq 2) = P(X = 1) + P(X = 2) = ½ + \frac{1}{4} = \frac{3}{4}$

$P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = ½ + \frac{1}{4} + \frac{1}{8} = \frac{3}{4}$

b. Find $P(2 < X \leq 5)$

$$P(2 < X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5) = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{7}{32}$$

c. Find $P(X > 4)$.

$P(X > 4) = 1 - P(X \leq 4) = 1 - \frac{15}{16} = \frac{1}{16}$.

## Discrete Probability Distribution Mean

The mean of a discrete probability distribution gives the weighted average of all possible values of the discrete random variable. It is also known as the expected value. The formula for the mean of a discrete random variable is given as follows:

$$E[X] = \sum x \, P(X = x)$$

## Discrete Probability Distribution Variance

The discrete probability distribution variance gives the dispersion of the distribution about the mean. It can be defined as the average of the squared differences of the distribution from the mean, μ. The formula is given below:

$$Var[X] = \sum (x - \mu)^2 \, P(X = x)$$

## END OF CHAPTER EXAMPLES

i. Let $X$ be a discrete random variable with the following $PMF$

$$PMF = \begin{cases} \frac{1}{4} & for\ x = -2 \\ \frac{1}{8} & for\ x = -1 \\ \frac{1}{8} & for\ x = 0 \\ \frac{1}{4} & for\ x = 1 \\ \frac{1}{4} & for\ x = 2 \\ 0 & otherwise \end{cases}$$

I define a new random variable $Y$ as $Y = (X+1)^2$

a. Find the range of $Y$
b. Find the PMF of $Y$.

If $x = -2$ then $Y = (-2+1)^2 = 1$

If $x = -1$ then $Y = (-1+1)^2 = 0$

If $x = 0$ then $Y = (0+1)^2 = 1$

If $x = 1$ then $Y = (1+1)^2 = 4$

If $x = 2$ then $Y = (2+1)^2 = 9$

Range of $x = \{-2, -1, 0, 1, 2\}$

Range of $y = \{1, 0, 1, 4, 9\} = \{0, 1, 4, 9\}$

**PMF of $Y$.**

$P(Y = 0)$ when $x = -1$ and $P(X = -1) = \frac{1}{8}$

$P(Y = 1)$ when $x = -2, 0$ and $P(X = -2) + P(X = 0) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

$P(Y = 4)$ when $x = 1$ and $P(X = 1) = \frac{1}{4}$

$P(Y = 9)$ when $x = 2$ and $P(X = 2) = \frac{1}{4}$

b. Suppose you play a game with a biased coin. You play each game by tossing the coin once. $P(heads) = \frac{2}{3}$ and $P(tails) = \frac{1}{3}$. If you toss a head, you pay $6. If you toss a tail, you win $10. Draw a distribution table

| $x$ | $\$ -6$ | $\$ 10$ |
|---|---|---|
| $P(X = x)$ | $\frac{2}{3}$ | $\frac{1}{3}$ |

$\mu = [-6 \times \frac{2}{3}] + [10 \times \frac{1}{3}] = -\frac{12}{3} + \frac{10}{3} = -\frac{2}{3}$

c. You pay $1 to flip three fair coins. If the result contains three heads, you win $4. If the result is two heads, you win $1. Otherwise there is no prize. We are interested in the net amount of money gained or lost in one game.
a. Define the random variable and the values it can have.
b. Write the PDF for the amount gained or lost in one game.
c. Find the expected value for this game (Expected NET GAIN OR LOSS)
d. Find the expected total net gain or loss if you play this game 50 times

| $x$ | $\$ 4 - 1 = \$3$ | $\$ 10$ |
|---|---|---|
| $P(X = x)$ | $\frac{1}{2}$ | $\frac{1}{3}$ |

8) A service organization in a large town organizes a raffle each month. One thousand raffle tickets are sold for $1 each. Each has an equal chance of winning. First prize is $300, second prize is $200, and third prize is $100. Let $X$ denote the net gain from the purchase of one ticket.

So for first prize $300 the net gain will be $300 - $1 (ticket price) = $ 299
So for second prize $200 the net gain will be $200 - $1 (ticket price) = $ 199
So for third prize $100 the net gain will be $100 - $1 (ticket price) = $ 99

| $x$ | $ 299 | $ 199 | $ 99 | $ -1 |
|---|---|---|---|---|
| $P(X = x)$ | $\frac{1}{1000}$= 0.001 | $\frac{1}{1000}$= 0.001 | $\frac{1}{1000}$= 0.001 | 1 – (0.003)<br>0.997 |

a) Find the probability of winning any money in the purchase of one ticket.
$P(win) = P(X = \$299) + P(X = \$199) + P(X = \$99)$
$= 0.001 + 0.001 + 0.001 = 0.003$

b) Find the expected value of $X$, and interpret its meaning.
$\mu = [\ \$299 \times 0.001] + [\$199 \times 0.001] + [\$99 \times 0.001] + [\$ - 1 \times 0.997] = -0.4$

9) A discrete random variable $X$ has the following probability distribution:

| $x$ | $-1$ | 0 | 1 | 4 |
|---|---|---|---|---|
| $P(X = x)$ | 0.2 | 0.5 | $a$ | 0.1 |

Compute each of the following quantities.

1. $a$.
   $a = 1 - (0.2 + 0.5 + 0.1) = 0.2$
2. $P(0)$
   Directly from the table, $P(0) = 0.5$
3. $P(X > 0)$.
   From the table, $P(X > 0) = P(1) + P(4) = 0.2 + 0.1 = 0.3$.
4. $P(X \geq 0)$.
   From the table, $P(X \geq 0) = P(0) + P(1) + P(4) = 0.5 + 0.2 + 0.1 = 0.8$.
5. $P(X \leq -2)$.
   Since none of the numbers listed as possible values for $X$ is less than or equal to $-2$, the event $X \leq -2$ is impossible, so $P(X \leq -2) = 0$.
6. $The\ mean\ \mu\ of\ X$
   $\mu = \Sigma x\, P(x) = (-1) \cdot 0.2 + 0 \cdot 0.5 + 1 \cdot 0.2 + 4 \cdot 0.1 = 0.4$
7. $The\ variance\ \sigma^2\ of\ X$.
   $\sigma 2 = \Sigma (x - \mu)^2 P(x)$
   $= (-1 - 0.4)^2 0.2 + (0 - 0.4)^2 \cdot 0.5 + (1 - 0.4)^2 \cdot 0.2 + (4 - 0.4)^2 \cdot 0.1$
   $= 1.84$
8. $The\ standard\ deviation\ \sigma\ of\ X$
   $\sigma = \sqrt{1.84} = 1.3565$

**More practice problems**

1) Let $X$ denote the number of boys in a randomly selected three-child family. Assuming that boys and girls are equally likely, construct the probability distribution of $X$.

2) Let $X$ denote the number of times a fair coin lands heads in three tosses. Construct the probability distribution of $X$.

3) Five thousand lottery tickets are sold for \$1 each. One ticket will win \$1,000, two tickets will win \$500 each, and ten tickets will win \$100 each. Let $X$ denote the net gain from the purchase of a randomly selected ticket.

   1. Construct the probability distribution of $X$.
   2. Compute the expected value $E(X)$ of $X$. Interpret its meaning.
   3. Compute the standard deviation $\sigma$ of $X$.

4) An insurance company will sell a \$90,000 one-year term life insurance policy to an individual in a particular risk group for a premium of \$478. Find the expected value to

the company of a single policy if a person in this risk group has a 99.62% chance of surviving one year.

5) An insurance company estimates that the probability that an individual in a particular risk group will survive one year is 0.9825. Such a person wishes to buy a $150,000 one-year term life insurance policy. Let $C$ denote how much the insurance company charges such a person for such a policy.

1. Construct the probability distribution of $X$. (Two entries in the table will contain $C$.)
2. Compute the expected value $E(X)$ of $X$.
3. Determine the value $C$ must have in order for the company to break even on all such policies (that is, to average a net gain of zero per policy on such policies).
4. Determine the value $C$ must have in order for the company to average a net gain of $250 per policy on all such policies.

6) A roulette wheel has 38 slots. Thirty-six slots are numbered from 1 to 36; half of them are red and half are black. The remaining two slots are numbered 0 and 00 and are green. In a $1 bet on red, the bettor pays $1 to play. If the ball lands in a red slot, he receives back the dollar he bet plus an additional dollar. If the ball does not land on red he loses his dollar. Let $X$ denote the net gain to the bettor on one play of the game.

1. Construct the probability distribution of $X$.
2. Compute the expected value $E(X)$ of $X$, and interpret its meaning in the context of the problem.
3. Compute the standard deviation of $X$.

## Continuous Probability distribution

A continuous probability distribution is a type of probability distribution that deals with random variables that can take on any value in an interval on the real number line or in a collection of intervals. Just like discrete there are two main functions associated with such a continuous random variable. These are the

i. probability density function (pmf)
ii. probability distribution function / cumulative distribution function (CDF).

Lets say probability density function (PDF) is given by $f(x)$ and
cumulative distribution function (CDF) given by $F(x)$

If we differentiate the cumulative distribution function of a continuous random variable it results in the probability density function. Then the formula for the probability density function, f(x), is given as follows:

$$f(x) = \frac{dF(x)}{dx} = F'(x)$$

Conversely, on integrating the probability density function we get the cumulative distribution function.

$$F(x) = \int_a^b f(x)\,dx$$

## probability density function (PDF)

For discrete random variables, we use the probability mass function which is analogous to the probability density function. The graph of a probability density function is in the form of a bell curve. The area that lies between any two specified values gives the probability of the outcome of the designated observation. To determine this probability, we solve the integral of the probability density function between two specified points.

Let $x$ be the continuous random variable with density function $f(x)$, that takes some value between certain limits, say $a$ and $b$, the PDF is calculated by finding the area under its curve. Thus, the PDF is given by

$$P(x) = \int_a^b f(x)\, dx$$

- PDF is non-negative for all the possible values, i.e. $\boldsymbol{f(x) \geq 0}$, for all $x$.
- The area between the density curve and horizontal X-axis is equal to 1, i.e. $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- Due to the property of continuous random variables, the density function curve is **continued** for all over the given range. Also, this defines itself over a range of continuous values or the domain of the variable.

### Example

we have a continuous random variable whose probability density function is given by $f(x) = x + 2$, when $0 < x \leq 2$.
Find $P(0.5 < X < 1)$.

$$P(x) = \int_{0.50}^{1} x + 2\, dx = \frac{x^2}{2} + 2x$$

This gives us 1.375. Thus, the probability that the continuous random variable lies between 0.5 and 1 is 1.375.

### Continuous Probability Distribution Mean

If $f(x)$ is the probability density function of the random variable $X$, then mean is given by the following formula:

$$E[X] = \mu = \int_{-\infty}^{\infty} x\, f(x) dx$$

**Continuous Probability Distribution Variance**

The formula is given below:

$$Var[X] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x).dx$$

**Example**

Let $X$ be a random variable with PDF given by

$$f(x) = \begin{cases} cx^2, & |x| \le 1 \\ 0, & otherwise \end{cases}$$

a. Find the constant $c$
b. Find $E[X]$ and $Var(X)$
c. Find $P(X \geq ½)$.

To find $c$, we can use $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$1 = \int_{-1}^{1} cx^2\, dx + 0$$

$$= c\,\frac{x^3}{3} \mid = c\left[\frac{1}{3} - \frac{-1}{3}\right] = c[\frac{2}{3}]$$

$$\frac{1}{\frac{2}{3}} = c = \frac{3}{2}$$

b) $E[X] = \int_{-1}^{1} x.\left(\frac{3}{2}\, x^2\right) dx = \frac{3}{2}\,\frac{x^4}{4} \mid = 0$

$Var[X] = \int_{-1}^{1} (x - 0)^2\, \frac{3}{2}\, x^2.dx = \int_{-1}^{1} \frac{3}{2}\, x^4.dx = \frac{3}{2}\,\frac{x^5}{5} \mid = \frac{3}{10} - \left(-\frac{3}{10}\right) = \frac{6}{10}$

c) $P(x) = \int_{½}^{1} \frac{3}{2}\, x^2\, dx = \frac{7}{16}$

**Example**

Let $X$ be a random variable with PDF given by

$$f(x) = \begin{cases} 4x^3, & 0 < x \le 1 \\ 0, & otherwise \end{cases}$$

Find $P(X \le \frac{2}{3} \mid X > \frac{1}{3})$

$$P(X \le \frac{2}{3} \mid X > \frac{1}{3}) = \frac{P(\frac{1}{3} < X \le \frac{2}{3})}{\mathrm{P}(X > \frac{1}{3})} = \frac{\int_{\frac{1}{3}}^{\frac{2}{3}} 4x^3\,dx}{\int_{\frac{1}{3}}^{1} 4x^3\,dx} = \frac{3}{16}$$

**Example**

Let $X$ be a random variable with PDF given by

$$f(x) = \begin{cases} \frac{2}{75}x & 0 \le x \le 5 \\ \frac{2}{15} & 5 < x \le 10 \\ 0 & otherwise \end{cases}$$

a. Find $P(X > 4)$.

$$P(X > 4) = 1 - P(X \le 4) = 1 - \int_0^4 \frac{2}{75}x\,dx = 1 - \frac{2}{75}\frac{x^2}{2}| = 1 - \frac{1}{75}x^2|$$

$$= 1 - \frac{16}{75} = \frac{59}{75}$$

**Example**

Let $X$ be a random variable with PDF given by

$$f(x) = \begin{cases} \frac{2}{21}x & 0 \le x \le k \\ \frac{2}{15}(6-x) & k < x \le 6 \\ 0 & otherwise \end{cases}$$

Find $P(X < \frac{1}{3}k \mid X < k)$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$1 = \int_0^k \frac{2}{21}x\,dx + \int_k^6 \frac{2}{15}(6-x)dx + 0$$

$$1 = \frac{2}{21} . \frac{x^2}{2} | + \frac{2}{15}\left(6x - \frac{x^2}{2}\right)| = \frac{1}{21} . k^2 + [\frac{2}{15}(36 - 18) - \frac{2}{15}\left(6k - \frac{k^2}{2}\right)]$$

$$1 = \frac{k^2}{21} + [\frac{36}{15} - \frac{4k}{5} + \frac{k^2}{15}] = \frac{12k^2}{105} + \frac{36}{15} - \frac{4k}{5}$$

$$0 = \frac{12k^2}{105} + \frac{7}{5} - \frac{4k}{5} = 12k^2 - 84k + 147 = k = \frac{7}{2}$$

$P(X < \frac{1}{3}k \mid X < k) = P(X < \frac{1}{3}. \frac{7}{2} \mid X < \frac{7}{2}) = P(X < \frac{7}{6} \mid X < \frac{7}{2})$

$P\left(X < \frac{7}{2}\right) = \int_0^{\frac{7}{2}} \frac{2}{21} x \, dx = \frac{7}{12}$

$P(X < \frac{7}{6}) = \int_0^{\frac{7}{6}} \frac{2}{21} x \, dx = \frac{7}{108}$

Hence $P(X < \frac{7}{6} \mid X < \frac{7}{2}) = \frac{\frac{7}{108}}{\frac{7}{12}} = \frac{1}{9}$