Analytic or not :

1) $f(z) = e^z$

$= e^{x+iy}$

$= e^x \cdot e^{iy}$

$= e^x \cdot (\cos y + i \sin y)$

$= \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$

$\frac{\delta u}{\delta x} = e^x \cos y$ | $\frac{\delta v}{\delta x} = e^x \sin y$

$\frac{\delta u}{\delta y} = -e^x \sin y$ | $\frac{\delta v}{\delta y} = e^x \cos y$

equal

$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

So, e^z is analytic

2) $f(z) = e^{z^2} = e^{(x+iy)^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2} \cdot e^{i \cdot 2xy}$

$= e^{x^2-y^2}(\cos 2xy + i\sin 2xy)$

$= e^{x^2-y^2} \cdot \cos 2xy + i \cdot e^{x^2-y^2} \cdot \sin 2xy$

equal

$\frac{\delta u}{\delta x} = 2x \cdot e^{x^2-y^2} \cdot \cos 2xy - 2y e^{x^2-y^2} \cdot \sin 2xy$

$\frac{\delta u}{\delta y} = -2y \cos 2xy \cdot e^{x^2-y^2} - 2x e^{x^2-y^2} \cdot \sin 2xy$

(-)

$\frac{\delta v}{\delta x} = 2x \sin 2xy \cdot e^{x^2-y^2} + 2y e^{x^2-y^2} \cos 2xy$

$\frac{\delta v}{\delta y} = -2y \cdot \sin 2xy \cdot e^{x^2-y^2} + 2x e^{x^2-y^2} \cos 2xy$

$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

So, e^{z^2} is analytic.

* $(a+ib)(a-ib) = a^2+b^2$

iii) $f(z) = \frac{1}{z}$

$$= \frac{1}{x+iy}$$

$$= \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \underbrace{\frac{x}{x^2+y^2}}_{u} + i\underbrace{\left(\frac{-y}{x^2+y^2}\right)}_{v}$$

$$\frac{\delta u}{\delta x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\delta u}{\delta y} = \frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\delta v}{\delta x} = \frac{(x^2+y^2)\cdot 0 - (-y)\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\delta v}{\delta y} = \frac{(x^2+y^2)\cdot(-1) - (-y)\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

So, $\frac{1}{z}$ is analytic.

(v) $f(z) = \sin 2z$

$= \sin 2(x+iy)$

$= \sin(2x + 2iy)$

$= \sin 2x \cos i\cdot 2y + \sin 2iy \cdot \cos 2x$

$= \underbrace{\sin 2x \cdot \cos 2hy}_{u} + i \cdot \underbrace{\cos 2x \cdot \sin 2hy}_{v}$

$\frac{\delta u}{\delta x} = 2\cos 2x \cdot \cos 2hy$ | $\frac{\delta v}{\delta x} = -2\sin 2x \sin 2hy$

$\frac{\delta u}{\delta y} = \sin 2x \cdot \sin 2hy$ | $\frac{\delta v}{\delta y} = 2\cos 2x \cos 2hy$

equal

(-)

$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and

$\frac{\delta u}{\delta y} = +\frac{\delta v}{\delta x}$

So, $\sin 2z$ is analytic

(i) $f(z) = 3z^2 - 2z + 4$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3(z+\Delta z)^2 - 2(z+\Delta z) + 4 - (3z^2 - 2z + 4)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{3z^2} + 6z\cdot\Delta z + 3\Delta z^2 - \cancel{2z} - 2\Delta z + \cancel{4} - \cancel{3z^2} + \cancel{2z} - \cancel{4}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (6z + 3\Delta z - 2)$$

$$\boxed{f'(z) = 6z - 2}$$

(ii) $f(z) = \dfrac{2z - i}{z + 2i}$ at $z = -i$

$$f'(-i) = \lim_{h \to 0} \frac{f(h-i) - f(-i)}{h}$$

$$f(-i) = \frac{-2i - i}{-i + 2i} = \frac{-3\cancel{i}}{\cancel{i}} = -3$$

$$f(-i+h) = \frac{2(-i+h) - i}{-i + h + 2i} = \frac{-2i + 2h - i}{i + h} = \frac{2h - 3i}{i + h}$$

$$f'(-i) = \lim_{h\to 0} \frac{f(-i+h) - f(-i)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2h-3i}{i+h} - (-3)}{h}$$

$$= \lim_{h\to 0} \frac{2h - \cancel{3i} + \cancel{3i} + 3h}{h\,(i+h)}$$

$$= \lim_{h\to 0} \left(\frac{5}{i+h}\right)$$

$$= \frac{5}{i}$$

$$= \frac{5i}{i^2}$$

$$\boxed{f'(-i) = -5i}$$

Harmonic or not :

1) $u = x^3 + 6x^2y - 3xy^2 - 2y^3$

$\frac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2$ | $\frac{\partial u}{\partial y} = 6x^2 - 6xy - 6y^2$

$\frac{\partial^2 u}{\partial x^2} = 6x + 12y$ | $\frac{\partial^2 u}{\partial y^2} = -6x - 12y$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 12y - 6x - 12y = 0$ ✓ Harmonic

2) $u = x^2 + y^2 - 2xy - 2x + 3y$

$\frac{\partial u}{\partial x} = 2x - 2y - 2$ | $\frac{\partial u}{\partial y} = -2y - 2x + 3$

$\frac{\partial^2 u}{\partial x^2} = 2$ | $\frac{\partial^2 u}{\partial y^2} = -2$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ ✓ harmonic

3) $u = xe^x \cos y - ye^x \sin y$

$\frac{\partial u}{\partial x} = x\cos y \cdot e^x + e^x \cdot \cos y - ye^x \sin y$

$\frac{\partial^2 u}{\partial x^2} = x\cos y \cdot e^x + 2e^x \cos y - ye^x \sin y$

$\frac{\partial u}{\partial y} = -xe^x \sin y - ye^x \cos y - e^x \sin y$

$\frac{\partial^2 u}{\partial y^2} = -xe^x \cdot \cos y + ye^x \sin y - 2e^x \cos y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ✓ harmonic

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iv) $u = e^{-x}(x\sin y - y\cos y)$

$= xe^{-x}\cdot\sin y - y\cdot e^{-x}\cos y$

$$\frac{\partial u}{\partial x} = -xe^{-x}\sin y + e^{-x}\sin y + ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = +xe^{-x}\sin y - e^{-x}\sin y - e^{-x}\sin y - ye^{-x}\cos y$$

$$\frac{\partial u}{\partial y} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + e^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ ✓ harmonic

(Q) $u = ax^3 - bx^2y - cxy^2 + dy^3$

$\frac{\partial u}{\partial x} = 3ax^2 - 2bxy - cy^2$

$\frac{\partial^2 u}{\partial x^2} = 6ax - 2by$

$\frac{\partial u}{\partial y} = -bx^2 - 2cxy + 3dy^2$

$\frac{\partial^2 u}{\partial y^2} = -2cx + 6dy$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6ax - 2by - 2cx + 6dy = 0$$

$$\Rightarrow (6a - 2c)x + y(-2b + 6d) = 0$$

$6a - 2c = 0$

$\Rightarrow \boxed{c = 3a}$

$-2b + 6d = 0$

$\Rightarrow \boxed{b = 3d}$

~~Ans~~ Pg 9

1) $u(x,y) = 2x(1-y) = 2x - 2xy$

$\frac{\partial u}{\partial x} = 2 - 2y$ | $\frac{\partial u}{\partial y} = -2x$

$\frac{\partial^2 u}{\partial x^2} = 0$ | $\frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow$ harmonic ✓

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$= \underbrace{\left(2x\right)}_{M} dx + \underbrace{\left(2-2y\right)}_{N} dy$$

$\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so eqn is exact.

$$\int dv = \int 2x \cdot dx + \int (2-2y)\, dy$$

$$= \cancel{2} \cdot \frac{x^2}{\cancel{2}} + 2y - \cancel{2} \cdot \frac{y^2}{\cancel{2}} + c$$

$$\boxed{v = x^2 - y^2 + 2y + c}$$

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$$f'(z) = u_1(z,0) - i u_2(z,0)$$

$$u_1(x,y) = \frac{\partial u}{\partial x} = 2 - 2y \qquad u_2(x,y) = -2x$$

$$\Rightarrow u_1(z,0) = 2 \qquad \Rightarrow u_2(z,0) = -2z$$

$$f'(z) = 2 + i \cdot 2z$$

$$f(z) = 2z + i \cdot \cancel{2} \frac{z^2}{\cancel{2}} + c$$

$$\boxed{f(z) = 2z + i z^2 + c}$$

(v) $u = \frac{1}{2}\ln(x^2+y^2)$

$$\frac{\delta u}{\delta x} = \cancel{\frac{1}{2}\ln(x} \quad \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{(x^2+y^2)} = \frac{x}{x^2+y^2}$$

$$\frac{\delta u}{\delta y} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}y}{(x^2+y^2)} = \frac{y}{x^2+y^2}$$

$$dv = \frac{\delta v}{\delta x}\cdot dx + \frac{\delta v}{\delta y}dy$$

$$= \left(-\frac{\delta u}{\delta y}\right)dx + \left(\frac{\delta u}{\delta x}\right)dy$$

$$= \underbrace{\left(\frac{-y}{x^2+y^2}\right)}_{M}dx + \underbrace{\left(\frac{x}{x^2+y^2}\right)}_{N}dy$$

$$\int dv = -\int\left(\frac{y}{x^2+y^2}\right)dx + \int\left(\frac{x}{x^2+y^2} - \frac{x}{x^2+y^2}\right)dy$$

$$= -y\int\frac{1}{x^2+y^2}dx + \int 0\, dy$$

$$= -\cancel{y}\cdot\frac{1}{\cancel{y}}\tan^{-1}\frac{x}{y} + c \quad \left[\because \int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$\boxed{v = -\tan^{-1}\left(\frac{x}{y}\right) + c}$$

(viii) $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow \text{harmonic}$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$= \underbrace{(e^x \sin y)}_{M} dx + \underbrace{e^x \cos y}_{N} dy$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= e^x \cos y \\ \frac{\partial N}{\partial x} &= e^x \cos y \end{aligned}\right\} \text{equal so exact.}$$

$$\therefore \int dv = \int e^x \sin y \, dx + \int (e^x \cos y - e^x \cos y) \, dy$$

$$\boxed{v = e^x \sin y + c}$$

$$f'(z) = u_1(z,0) - i u_2(z,0)$$

$$\therefore u_1(z,0) = e^z \cdot \cos 0 = e^z$$

$$u_2(z,0) = -e^x \cdot \sin 0 = 0$$

$$\therefore\ f'(z) = e^{z} - i\cdot 0$$

$$= e^{z}$$

$$\Rightarrow \boxed{f(z) = e^{z} + c}$$