Mathematical Analysis for Computer Architecture

Col Rahat Sir

Probability

Probability will never be negative and more than 1.

Sample Space: Set of all possible outcomes of an experiment. Denoted by S or Ω.

Coin toss: $S = \{H, T\}$ Dice $= \{1, 2, 3, 4, 5, 6\}$

If there's two dice, sample space $= 6 \times 6 = 36$.

So probability of getting (4,1) $P(4,1) = \frac{1}{36}$.

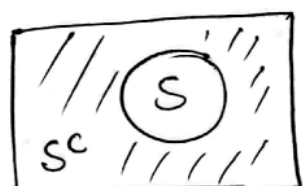
For determining probability we must have the sample space.

Event : Sample Space এর প্রত্যেকটা element ই event.

$x \in S$: x is a subset of S.

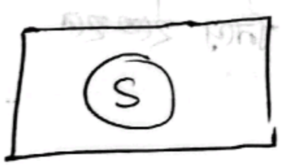
ϕ : Null set

S^c : $U - S$



U or S or Ω

$S \subseteq T$



S এর সব element T এর মধ্যে আছে
T এর সব " S " " নাই।

Union: $S \cup T$

$S \cap T$

disjoint set $S \cap T = \phi$

$S = \{1, 2, 3, 4, 5, 6\}$

অনেক event হতে পারে যেমন $\{1, 2\}$, $\{2, 4, 6\}$, $\{6\}$.

all subset, যতবার subset করা যাবে ততগুলোই event.

For any 2 events E and F of a sample space S we define a new event $E \cup F$ to consist all outcomes that are either in E or F or in both. The event $E \cup F$ occurs if either E or F occurs.

$S = \{1,2,3,4,5,6\}$ $E = \{1,3,5\}$ $F\{1,2,3\}$

$E \cup F = \{1,2,3,5\}$

$E \cap F = \{1,3\}$.

$E = \{H\}$ $E \cap F = \{\phi\}$
$F = \{T\}$

দুটোর Common Element না থাকলে Mutually Excalusive বলে।

Sample Space: Mutually exclusive
Collectively exhaustive (sample space সবগুলোকে নিয়ে হতে হবে)

23 AUG 23

Probability Axioms

1. $0 \le P(E) \le 1$ {0 থেকে 1 এর মধ্যে probability হবে}
2. $P(S) = 1$ ⟶ Sample Space এর total probability 1.
3. $P(A1 \cup A2 \cup A3 \cdots) = P(A) + P(B) + P(C) + \cdots$

$$P\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} P(E_n).$$ event

n = 10 হলে: $P\Big(\bigcup_{n=1}^{10} E_{10}\Big)$

~~Probability proof~~ : 1. Nonnegativity : $P(A) \geq 0$.

2. Normalization : entire $S = 1$,

3. Additivity : A, B disjoint হতে হবে, Then $P(A_1 \cup A_2 \cup A_3 \cdots) = P(A_1) + P(A_2) + P(A_3) \cdots$

if $A \cap B = \phi$ হলে (disjoint হতে পারলেই)

S or Ω

$E \cup E^c = S$

from axioms:

$P(S) = 1$.

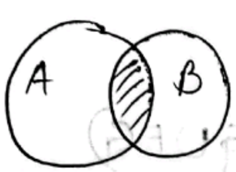
or, $1 = P(S) = P(E \cup E^c)$

or, $1 = P(E) + P(E^c)$.

$\therefore P(E^c) = 1 - P(E)$.

"disjoint না হলে Common part টা ২ বার count হবে, " তাই disjoint লাগবে।

** The probability of not occuring = 1 − probability of occuring



if not disjoint!

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or, $P(A \cup B) + P(A \cap B) = P(A) + P(B)$.

এই যেকোনো একটা করতে হবে

$P(EF) = P(E \cap F)$

Union → OR operation Intersection : AND ops.

$P(E \cup F) = P(E) + P(F) - P(EF)$

Ross pg 5 Example 1.3

we toss two coin, sample outcome: {(H,H), (H,T), (T,H), (T,T)}
all possiblity = 1/4.

Event is: the first coin always falls on head:

$E = \{HH, HT\}$

Second one always falls on head: $F = \{HH, TH\}$.

formula system

$$P(E \cup F) = P(E) + P(F) - P(EF)$$
$$= \frac{1}{2} + \frac{1}{2} - P(H,H)$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4}.$$
$$= \frac{3}{4}.$$

$P(E) = P\{(HH), (HT)\}$
$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(F)$

Counting system: $P(E \cup F) = P[\{(HH), (HT)\} \cup \{(H,H), (T,H)\}]$
$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$
$= \frac{3}{4}.$

Inclusion Exclusion Identity

$P(E \cup F \cup G)$

$\Rightarrow P(E \cup F) + P(G) - P((E \cup F) \cdot G)$

$\Rightarrow P(E) + P(F) - P(EF) + P(G) - P(EG \cup FG)$

$\Rightarrow P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$

** Inclusion Exclusion Identity Prove থাকলে এই পর্যন্ত লিখতে হবে।

২টা থাকলে: $P(E)+P(F)-P(EF)$
৩টা " : $P(E)+P(F)+P(G)-P(EF)-P(FG)-P(EG)+P(EFG)$.

n টা থাকলে:

$$P(E_1 \cup E_2 \cup E_3 \cdots E_n) = \sum_i P(E_i) - \sum_{i<j} P(E_iE_j) + \sum_{i<j<k} P(E_iE_jE_k) + \cdots + (-1)^{n+1} \cdot P(E_1E_2E_3 \cdots E_n)$$

General form or Generic form

Conditional Probability

অন্য ঘটনার probability থেকে ঘটনার partially probability বের করার procedure.

Rolling a dice: probability of getting a number $= \frac{1}{6}$.

Again Rolling a dice: probability of getting 6 (result is even) $= \frac{1}{3}$

cause even sample spaces are: 2, 4, 6.

$$P(A|B) = \frac{\text{No of elements in } A \cap B}{\text{No of elements in } B.}$$

$$P\left(\underbrace{\text{the outcome is } 6}_{\{6\}} \middle| \underbrace{\text{the outcome is even}}_{\{2,4,6\}}\right) = \frac{1}{3}$$

A → যেটার probability বের করব
B → যেটার " জানি। উপরে intersection তার মানে Known

$$\boxed{\frac{P(AB)}{P(B)} \text{ when } P(B) > 0.}$$

0 না হলে Condition Probability হবেনা,

Ex: 1·4, Page 7

10 cards are placed, mixed, the drawn card is at least 5; so whats the probability of getting 10.

$P(A|B) = \frac{P(A\cap B)}{P(B)}$. $P(\text{outcome is } 10) = \frac{1}{10}$; 10 outcomes since 1 to

$P(\text{outcome } 10, \text{ at least } 5) = \frac{1}{6} \longrightarrow \{5, 6, 7, 8, 9, 10\}$

$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)}$ [since $P(AB) = P(A)$ cause the number of the card is 10 and at least 5]

Example : 1·7 (Page 8)

an urn contains 7 black balls, 5 white balls. we draw 2 balls without replacement. what is the probability both the balls are black.

E = the drawn ball is black

F = the drawn (2nd) ball is black.

As without Replacement : after first draw : 6 black, 5 white. after second draw : 5 black 5 white.

$P(F|E) = \frac{6}{11}$

$P(E) = \frac{7}{12} \longrightarrow$ Getting Black at first.

$P(F|E) = \frac{P(FE)}{P(E)}$

$\therefore P(FE) = P(F|E) \cdot P(E)$

$= \frac{6}{11} \times \frac{7}{12} = \frac{42}{132}$.

Example: Tossing a fair coin three times, probability of getting a alternatively Head and tail, given the first toss is head.

A = Toss yields alternate H, T B = 'first' toss is head.

$S = \{HHH, HHT, HTH, THH, THT, HTT, TTT\}$

$A = \{HTH, THT\}$

$B = \{HHH, HHT, HTH, HTT\}$

Check: HTH alternate আছে, [illegible] : [illegible]
condition: First Head,

$P(B) = \frac{4}{8} = \frac{1}{2}$ $\therefore P(A \cap B) = \frac{1}{8}$.

$P(A) = \frac{1}{8}$. $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$.

— o —

Example
Karim works in a umbrella company. He gets a bonus if he sells more than 10 umbrellas a day.

① if its raining, that he sells more than 10 umbrellas is 0.8

② If its not raining, he sells more than 10 umbrellas is 0.25

③ The probability of raining tomorrow is 0.1

What is the probability that it does not rain tomorrow and he gets his bonus?

Solution: X = Karim sells more than 10 umbrellas
R = It rains
B = Karim gets bonus.

Then we can say $X = B$ as when Karim sells more than 10 umbrellas and gets the bonus.

Given that $P(X|R) = 0.8$
and $P(X|R^c) = 0.25$
$P(R) = 0.1$.

We need to find : $P(R^c \cap B) = ?$ Probability of not raining tomorrow <u>and</u> getting the bonus.

formulae : $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$\therefore P(A \cap B) = P(A|B) \cdot P(B)$

$$P(X|R^c) = \frac{P(X \cap R^c)}{P(R^c)}$$

or, $P(X \cap R^c) = P(X|R^c) \cdot P(R^c)$
$= P(X|R^c)\,(1 - P(R))$
$= (0.25) \times (1 - 0.1)$
$= 0.225$

Here $X = B$.

$\therefore P(B \cap R^c) = 0.225$

or $P(R^c \cap B) = 0.225$ Ans.

***
Exam

Independent Probability

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ Here in conditional probability ~~of~~ event B gives information about event A.

But if event B doesn't give any information about event A then it's called independent probability.

তাহলে B এর কোনো effect নাই। So: $P(A|B) = P(A)$ —— (1)

$\frac{P(A \cap B)}{P(B)} = P(A)$

or, $P(A \cap B) = P(A) \cdot P(B)$

Ross Example 1.8 (Page 9)

Two fair dice is thrown. The sum of the dice is 6. and the first dice is 4.

$\therefore E_1$ = Sum of the dice is six (6).
F = first dice 4.

$P(E_1 \cap F) = P(4,2)$
$= \frac{1}{36}$.

$P(E_1) \cdot P(F) = \frac{5}{36} \times \frac{1}{6}$
$= \frac{5}{216}$.

as $P(E_1 \cap F) \neq P(E_1) \cdot P(F)$
So the two events are not independent.

প্রথম Dice এ 4 আসলেও ২য় Dice এ 2 আসলেও condition full fill হবেনা, কেননা sum 6 হয় (৫ ভাবে হতে পারে।)

If the addition or sum was 7.

then: $P(E_2 F) = P(4,3)$
$= \frac{1}{36}$.

$P(E_2) \cdot P(F) = \frac{6}{36} \times \frac{1}{6}$
$= \frac{1}{36}$.

$P(E_2 F) = P(E_2) \cdot P(F)$
independent.

Venn Diagram এ Disjoint হলে dependent
Disjoint নয় হলে Independent.

যদি Positive Number A ও B এর মান (+) হলে

$P(A) > 0$ $\quad P(A \cap B) = P(A) \cdot P(B)$

$P(B) > 0$.

When $P(A) > 0$ and $P(B) > 0$, (two disjoint) can never be independent cause $A \cap B$ can never be zero.

$P(A \cap B)$ zero হলে disjoint $= P(A) \cdot P(B)$
↳ independent নয়,

$P(A \cap B)$ disjoint না হলে $P(A \cap B) = P(A) \cdot P(B)$ আর satisfy হবে।

* Bayes' Theorem (** exam)

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

prove:

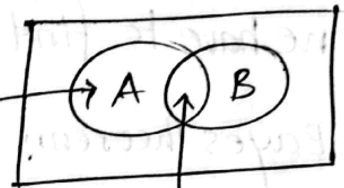
$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(AB)}{P(A)}$$

again $P(A|B) = \dfrac{P(AB)}{P(B)}$ or $P(AB) = P(A|B) \cdot P(B)$,

$\therefore P(B|A) = P(A\cdot$

Another format of Bayes

$A = AB \cup AB^c$



$AB^c = A \cap B^c$

$AB = A \cap B$

$P(A) = P(AB) + P(AB^c)$

$= P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)$

$$\boxed{P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)}}$$

You enter a chess tournament where your probability of winning a game is 0.3 against half of the players (type 1) 0.4 against a quarter of the players (type 2) and 0.5 against the remaining quarter of the players (type 3) against a randomly choosen opponent. You played a game against a randomly choosen opponent and you win. What is the probability that you won against type 1.

$P(A_1) = 0.5$ (half of the players) {type 1}

$P(A_2) = 0.25$ [Quarter] (type 2)

$P(A_3) = 0.25$ ["] (type 3).

B event of winning:

$P(B|A_1) = 0.3$ (against type 1)

$P(B|A_2) = 0.4$ (" " 2)

$P(B|A_3) = 0.5$ (" " 3).

we have to find $P(A_1|B) = ?$

As Bayes theorem:

$$P(A_1|B) = \frac{P(B|A_1) \cdot P(A_1)}{P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) + P(B|A_3) \cdot P(A_3)}$$

$$= \frac{0.3 \times 0.5}{(0.3 \times 0.5) + (0.4 \times 0.25) + (0.5 \times 0.25)}$$

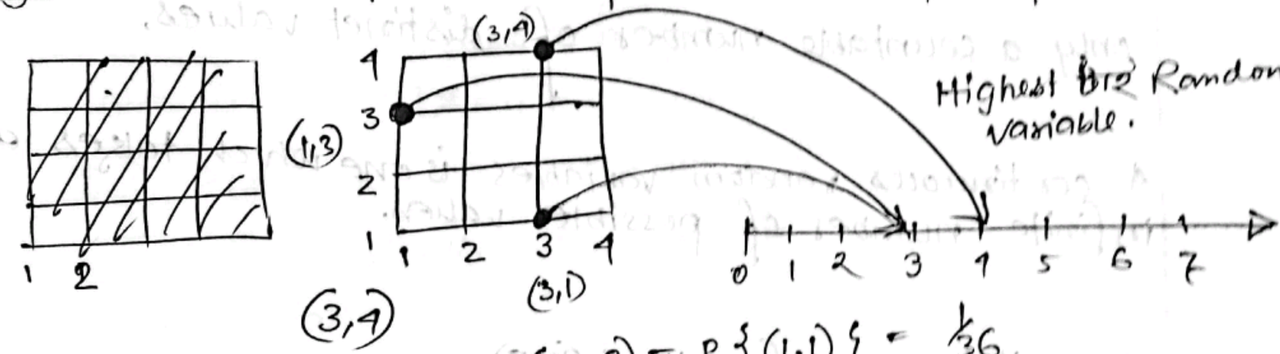
$$= 0.4$$

(Ans).

Random Variables

28 AUG 23

Defn: Real valued function defined on a sample space.



for two 6 sided dice: $p(x=2) = p\{(1,1)\} = \frac{1}{36}$

$p(x=3) = p\{(1,2),(2,1)\} = \frac{2}{36}$.

$p(x=4)\ p\{(1,3),(3,1)(2,2)\} = \frac{3}{36}$.

$\vdots$

$p(x=12) = p.\{(6,6)\} = \frac{1}{36}$;

Here x is random variable where its actually a function.

\# No of head appearing: $S = (H,H), (H,T), (T,H), (T,T)$

$\downarrow$ 2, $\downarrow$ 1 $\downarrow$ 1, $\downarrow$ 0.

possible value: $(0,1,2)$.

$p(X=0) = p\{(T,T)\} = \frac{1}{4}$.

$p(X=1) = p\{(H,T),(T,H)\} = \frac{2}{4} = \frac{1}{2}$

$p(x=2) = p\{(H,H)\} = \frac{1}{4}$

*** Total additional result of probability will be 1.

Random Variable : Discrete
Continuous.

Discrete: Countable

Continuous: (Hight, weight) measureable

A discrete random variable is one which may take only a countable number of distinct values.

A continuous random variables is one which takes an infinite number of possible values.

Probability Mass function: (PMF)

PMS of a discrete RV is the list of probabilities associated with each of its values. It is the list of all probabilities, when the discrete random variable takes all possible values.

$$P(a) \; , \; P_X(x) \; , \; P(X=x).$$

value

Variable

$$P(X=x) = \begin{cases} \frac{1}{4} , & x = 0,2 \\ \frac{1}{2} , & x = 1 \\ \frac{1}{4} , & x = 2 \\ 0 , & \text{otherwise}. \end{cases}$$

PMF

Generic form.

$$\boxed{\sum_{i=1}^{\alpha} P(x_i) = 1}$$

(30 AUG 23)

PMF — All possible probabilities of all possible values of discrete RV

$$P(a) = P_X(x) = P[X = a]$$

x → value of random var

$P[X=a]$ → Random Var এর value a হবার probability

$$\sum_{i=1}^{\infty} P(x_i) = 1$$ → সবগুলোর probability যোগ করলে 1 হবে,

Commulative Distribution Func → CDF

All random variable have CDF.

MF

ues

ধর

① Collect all possible outcomes that gives rise to the event $\{X = x\}$

② Add their probabilities to obtain $P_X(x)$

$$P_X(x) = \begin{cases} \frac{1}{4}, & X = 2, 0 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

story থেকে PMF বের করতে পারবে,

X = no of head obtained (head উঠার probability)

P that least one H is obtained

$$P\{X=1\} + P\{X=2\}$$

$$\Rightarrow \frac{3}{4}$$

A basketball player shot two free throw each equally likely either to be 'g' or 'b'. Each goal should ~~goal~~ worth 1 point what is the PMF of X is the number of point that he scored.

Sample space : $\{(g,g), (g,b), (b,g), (b,b)\}$

$P(X=0) = \{b,b\}$
$P(X=1) = \cancel{\quad} \{(g,b),(b,g)\}$
$P(X=2) = \{g,g\}$

$$P_X(x) = \begin{cases} \frac{1}{4}, & x=0 \\ \frac{1}{2}, & x=1 \\ \frac{1}{4}, & x=2 \\ 0, & \text{otherwise} \end{cases}$$

RV X has the following PMF

$$P_X(x) = \begin{cases} \frac{c}{x}, & x=1,2,3 \\ 0, & \text{otherwise.} \end{cases}$$

① Find out $c=?$
② $P[x=1]=?$
③ $P[x\geq 2]=?$
④ $P[x>3]=?$

① __Ans:__ $\sum_{x=1}^{3} P_X(x) = 1.$

$P_X(1)+P_X(2)+P_X(3)=1.$

or, $\frac{c}{1}+\frac{c}{2}+\frac{c}{3}=1.$

$\therefore \quad c = \frac{6}{11}.$

(2) Soln: We know $C = \frac{6}{11}$.

$\therefore \frac{C}{1} = \frac{\frac{6}{11}}{1} = \frac{6}{11}$

(3) Soln: $P_X(2) + P_X(3) = \frac{5}{11}$.

(4) Soln: 0 (Ans).

1st Type

Bernoulli RV : are the discrete variables that takes two values 1 and 0 depending on whether the outcome is a success or a failure.

যেসব exp এর outcome either 0 or 1, [illegible]

PMF : $P_X(x) = \begin{cases} p, & x = 1 \\ (1-p), & x = 0 \\ 0, & \text{otherwise} \end{cases}$

p = success

1. Only two possibilities : success or failure
2. Probability does not change from trial to trial
3. The trails are independent.

2nd Type

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \quad \text{(Binomial theorem)}$$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(a+b)^4 = \sum_{K=0}^{4} \binom{4}{K} a^{4-K} \cdot b^K$$

$$= \binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3$$

$=$

2nd Type

Binomial

independent হতে হবে, না হলে Binomial use করব না

n independent trials, each of which results in a "success" with probability "p" and in a failure with probability (1−p) are to be performed. X represents the number of successes that occur in n trials, then X is said to be Binomial RV with parameter (n, p).

4টা বললে বুঝতে পারবো 5, 7, 9, 3 হলে পারতাম না কেননা Result Biased আসবে, equally divideable হলে পারবো।

fair coin এ Head ও Tail এর probability same.

(my assumption) ২টির Head এর probability বের করলে ২টির Tail এর ~ same হবে।

4 fair coins are flipped. If the outcomes are independent what is the probability that two $\boxed{\text{heads}}$ and two $\boxed{\text{Tails}}$ are obtained.

Let X be the no of Heads (H). So X is a binomial RV with parameter $n = 4$, $p = \frac{1}{2}$.

$X = 2 \Rightarrow$ 2 heads

$$P(X=2) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{4-2}$$

$$= \frac{3}{8}.$$

$P(i) = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$ where $i = 1, 2, 3 \cdots n$.

A coin is tossed n times. At each toss the coin comes up H with probability p and a ~~total~~ T with probability $(1-p)$ independent of prior toss. Let X be the number of H in n toss sequence. We refer to X as a Binomial RV with parameter n and p.

at most 5 lekha থাকলে

$P\{X=0\} + P\{X=1\} + P\{X=2\} + P\{X=3\} + P\{X=4\} + P(X=5\}$

only 2 থাকলে শুধু 2 এর Term

at least two থাকলে নিলে 2 হতে নিচের সবগুলো

Any item procured by certain machine will be defative with probability 0.1, independent of other item. Whats the probability that in a sample of 3 items at most 1 item will be defactive.

$n = 3, \quad p = 0.1$.

Let X = the number of defective items in sample X is a Binomial RV with parameter $n=3$, $p=0.1$

So the desired Probability $= P\{X=0\} + P\{X=1\}$

$\Rightarrow \binom{3}{0}(0.1)^0(0.9)^3 + \binom{3}{1}(0.1)^1(0.9)^2$.

=

=

$= 0.972$ (Ans)

5th toss এ T আসবে তার P
প্রথম H না আসবে তখন Toss এর P.

independent হলে Applicable Geo RV

Type 3:
Geometric Random Variable:

Independent trials, each having probability p of being success, are performed untill success accured if X be the number of trials required untill the first success occurs then, X is said to be Geometric RV with parameter p.

PMF $\boxed{P_X(n) \text{ or } P\{X=n\} = (1-p)^{n-1} \cdot p}$

unfair probability $H = \frac{1}{3}$

Suppose that we toss a fair coin. What is the probability of appearing the first head on 5th toss.

→ Let X is the number of trials required untill the first head appears.

So X is Geometric RV with parameter.

Here, $X = 5$, $p = \frac{1}{2}$

$$P\{X=5\} = (1-p)^{n-1} \cdot p = \left(1-\frac{1}{2}\right)^{5-1} \cdot \frac{1}{2} = \frac{1}{32}$$

Solve

$$P(x=k) = (1-p)^{(k-1)} * p = \left(1-\frac{1}{3}\right)^{5-1} * \frac{1}{3} = \left(\frac{2}{3}\right)^4 * \frac{1}{3} = \frac{16}{243}$$

4th Type

Poisson RV: is a discrete probability distribution of the count of events that occur randomly in a given ~~material~~ interval of time.

Let X = the number of events in a given interval and if the mean number of events per interval is λ; So the p of observing i events in a given interval given by:

Common unit এ আনতে হবে। hr, min same রাখতে হবে।

$$P(i) \text{ or } P\{X=i\} = e^{-\lambda} \frac{\lambda^i}{i!}$$

(Validity of the equation)

$$\sum_{i=0}^{\infty} P(i) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \{ \underbrace{1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots}_{e^{\lambda}} \}$$

$$= e^{-\lambda} e^{\lambda}$$

$$= 1.$$

half hour এ 1.8
$\therefore$ hour এ $= 2 \times 1.8$
$= 3.6$

half থাকলে

hour এ convert করে নিব।

Birth in a hospital occur randomly at an average rate of 1.8 birth per hour. Whats the probability of observing 4 birth in a given hour.

Ans: $\lambda = 1.8$, Let X be the number of birth in a given hour.

$i = 4$

$$P\{X = i\} = e^{-\lambda} \frac{\lambda^i}{i!}$$

$$P(X=4) = e^{-1.8} \frac{(1.8)^4}{4!} = 0.0723$$

***
exam

The number of hits at a website in any time interval is possions RV. A site has on average, $\lambda = 2$ hits per sec.

(i) What is P there are no hits in an interval of 0.25 sec

(ii) What is the p that there are no more than 2 hits in an interval of 1 sec?

0.677

Ans:
0.607

(ii) $x=0$, $x=1$, $x=2$

$$\Rightarrow e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right)$$

$$= 0.677$$

(i) $e^{-0.5} \frac{\lambda^0}{0!}$

$\boxed{\text{1 sec এ 2, 0.25 sec এ } 2 \times .25 = 0.5}$

Poisson R.V. may be used to approximate a Binomial RV ~~where~~ when the Binomial Parameter n is large and p is small.

Binomial RV $P(i) = \binom{n}{i} p^i \cdot (1-p)^{n-i}$

Let X is a Binomial RV with parameter n and p, such that

$$\lambda = np.$$

$$P\{x=i\} = \frac{n!}{(n-i)! \cdot i!} \cdot p^i (1-p)^{n-i} \qquad \boxed{\lambda = np} \therefore p = \frac{\lambda}{n}$$

$$= \frac{n(n-1)(n-2)\cdots\cdots(n-i+1)}{n^i} \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$\frac{n(n-1)(n-2)\cdots(n-i+1)\boxed{(n-i)(n-i-1)(n-i-2)}\cdots 3.2.1}{\{(n-i)(n-i-1)(n-i-2)\cdots 3.2.1\} \cdot i!}$$

$n \to$ big number

$$\Rightarrow \frac{(n(n-1)(n-2)\cdots(n-i+1))}{(n^i)_{\to\infty}} \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \cdots$$

$$\Rightarrow 1 \cdot \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$e^{-\lambda} = 1 - \frac{\lambda}{1!} + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!},$$

$$\frac{\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^i \to 1}$$

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y$$

$$= e^{-\lambda}$$

$$\left(1-\frac{\lambda}{n}\right)^n = \{{}^nC_0 \cdot 1^{n-0} \left(\frac{\lambda}{n}\right)^0 + {}^nC_1 1^{n-1} (+ {}^nC_2 1^{n-2} \left(\frac{\lambda}{n}\right)^2 + \cdots$$

$$P\{x=i\} = 1 \cdot \frac{(\lambda)^i}{i!} e^{-\lambda}$$

$$\Rightarrow 1 + n \cdot 1 - \frac{\lambda}{n} + \cdots$$

$$\Rightarrow 1 - \frac{\lambda}{1!} + \frac{\lambda^2}{2} + \cdots$$

Derivation full on next monday.

PMF → for discrete
PDF → for Continuous
CDF → Both

Continuous Random Variable

its (uncountable, we can measure)

interval of time থাকে most of the case এ,

PDF → Probability Diensity function
CDF → Cumulitive Distributive function.

যোগ করলে যদি addition 1 হয় তবে distribution.

5, 7, 3, 2 → not a distribution.

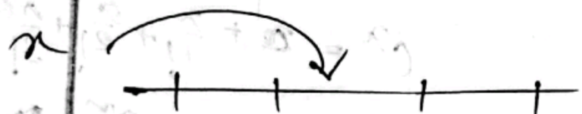
$\frac{5}{5+7+3+2} = \frac{5}{17} + \frac{7}{17} + \frac{3}{17} + \frac{2}{17}$ → is a distribution.

Station এ আসার time

$S_T = \{t \mid -5 \leq t \leq +5\}$ → এটার probability বেশি কারণ Range বড় তাই chance বেশি

$= \{t \mid -2 \leq t \leq +2\}$ → হবে:

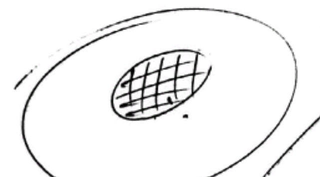
The amount of probability in an interval gets smaller and smaller as the interval shrinks.

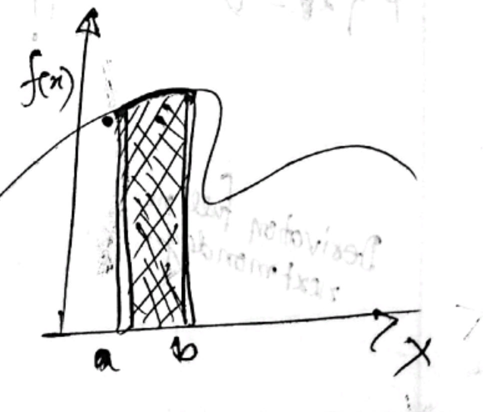


A random variable x is called continuous if there exists a non negative function $f(x)$ which is called probability density function (PDF) such that

$$P(X \in B) = \int_B f(x) \cdot dx$$ for every set of B in the real line.

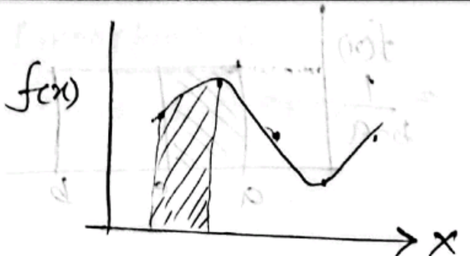
$$\text{CDF} \xrightarrow{\text{differentiation}} \text{PDF}$$
$$\text{PDF} \xrightarrow{\text{Integration}} \text{CDF}$$





পুরো Curve এর নিচে probability 1.

$a \leq x \leq b$
$a < x \leq b$
$a \leq x < b$
$a < x < b$
} for continuous varible inclusion exclusion doesn't matter.



CDF → area of the curve → probability
PDF → Shape of the curve.

$$P\{x \in (-\infty, \infty)\} \text{ OR } P\{-\infty \leq x \leq \infty\} = \int_{-\infty}^{\infty} f(x)\, dx$$

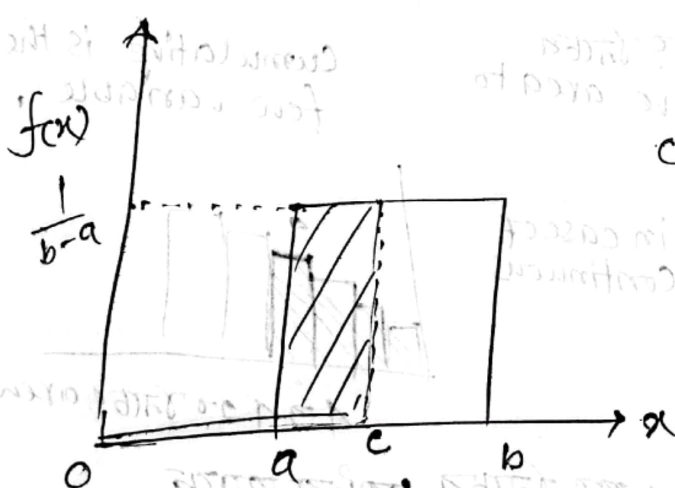
lower Limit না দেওয়া থাকলে $-\infty$ to Given limit
Upper " " " " ⇒ given Limit to $+\infty$.

CDF Cumulative Distributive Function

$F_X(x)$ → value denote করে।
X → name of variable
capital থাকলে CDF.

or $P(X \leq x)$

$$F_X(x) = P(X \leq x) = \begin{cases} \sum_{k \leq x} p(x) & \text{if } f_X \text{ is discrete} \\ \int_{-\infty}^{\infty} f(t)\, dt & \text{if " " continuous} \end{cases}$$



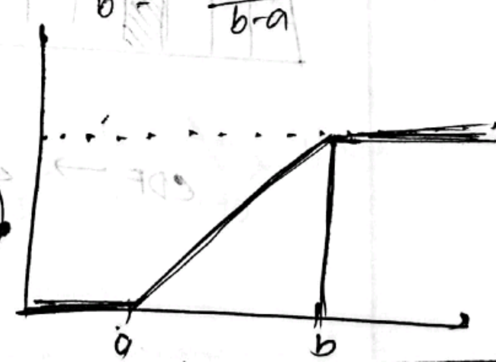
CDF = area under the curve

length × breadth $= (c-a) \times \frac{1}{(b-a)}$

$b = \frac{x-a}{b-a}$

so a এর নিচে থাকার probability 0.

CDF Graph

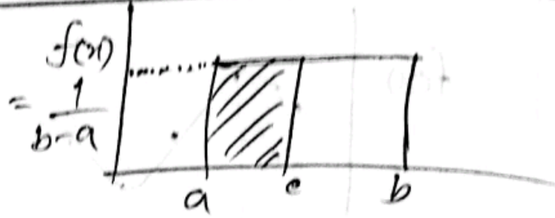


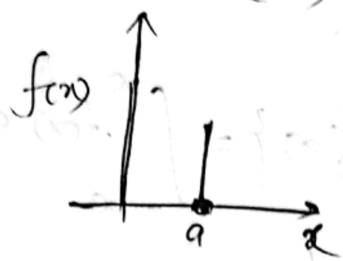
Discrete RV $\boxed{PMF}$ Mass → $\boxed{CDF}$ Distribution — Both

Continuous RV $\boxed{PDF}$ Density → $\boxed{CDF}$

$f(x) \cdot (b-a) = 1$

$\therefore f(x) = \frac{1}{(b-a)}$

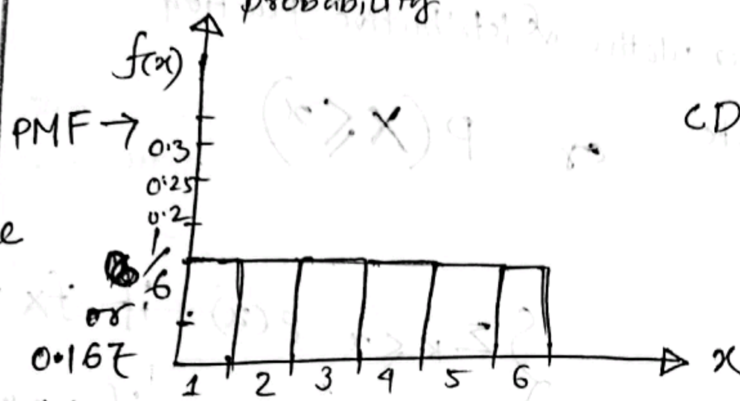




Limit
দুইটার point একই হলে line হয়। line এর area 0. So continuous Rules applicable নয়

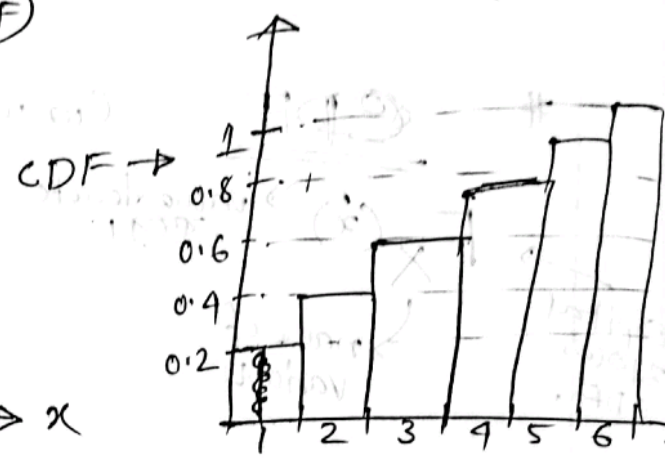
12 Sep 23

(Relation between PMF and CDF)



Fair Dice

outcome →

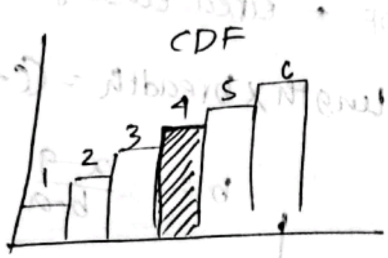


area under the curve is the distribution of probability

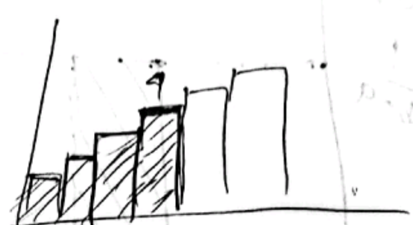
$P(X \leq 4)$ 4 এর নিচের curve area টা

Cumulative is the collection of few variable.

in case of discrete



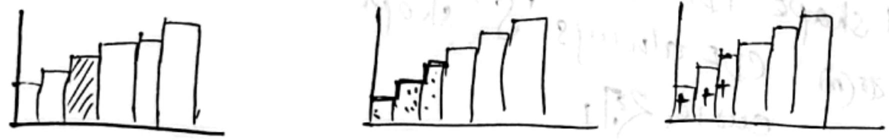
in case of Continuous



4 ও 4 এর নিচের area.

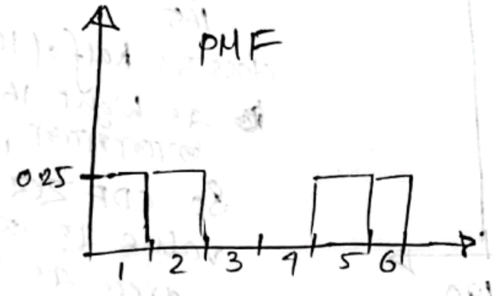
CDF → 4 ও 4 এর নিচের যতটুকু আছে তার যোগফল (cumiliative)

PMF ③ PDF ③

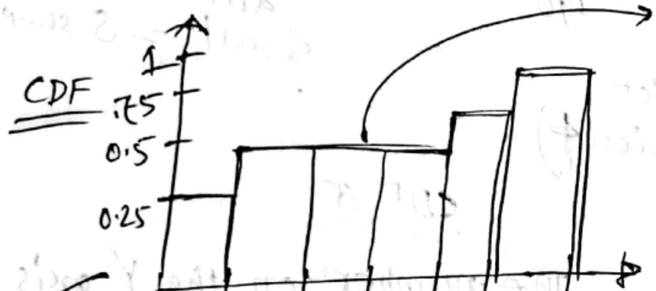


Important Property of CDF:

The final bar needs to be equal 1



এমন dice যেখানে 3,4 ওঠেনা (modified)
বাকি 4টা ওঠার probability same
$\therefore 1,2,5,6 \rightarrow \frac{1}{4} = 0.25$

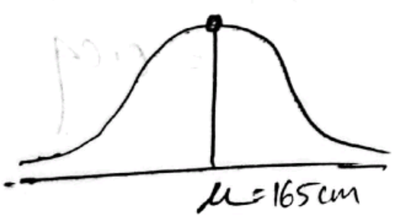


0 হবেনা PMF এর মত cause cumulitive.
$P(X \leq 4)$ বা $P(X \leq 3)$ বললে আগের সবগুলোর cumulitive নিতে হবে।

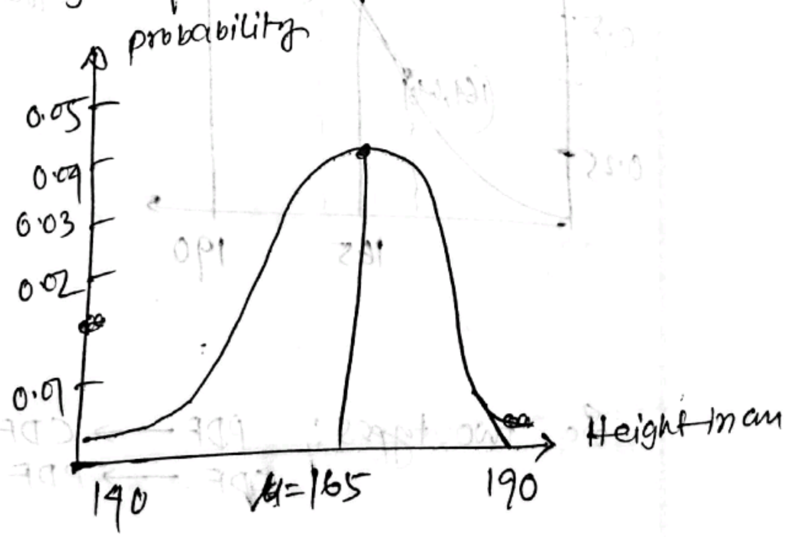
$$\therefore P(X \leq 4) = P(X = 1) + P(X = 2) + \underset{0}{P(X=3)} + \underset{0}{P(X=4)}$$
$$= P(X \leq 2).$$

The flatness in CDF indicates that there is no "mass" around 3 and 4.

Lets take the average height of this class: 165 cm $= \mu$

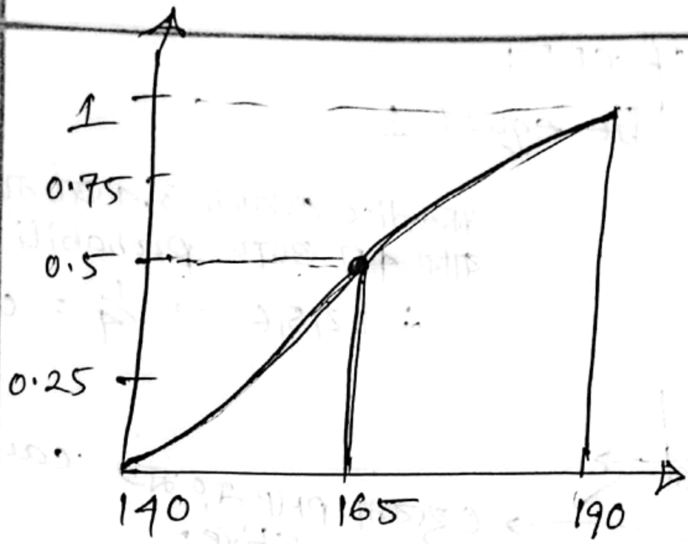


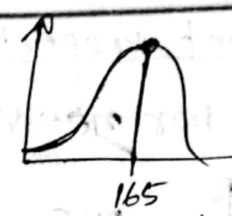
Highest height = 190cm
lowest " = 140 cm
avg density $\mu = 165$cm



PDF

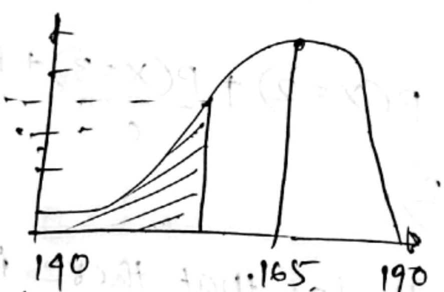
Bell shape PDF curve থাকলে CDF always 'S' shape curve হয়।





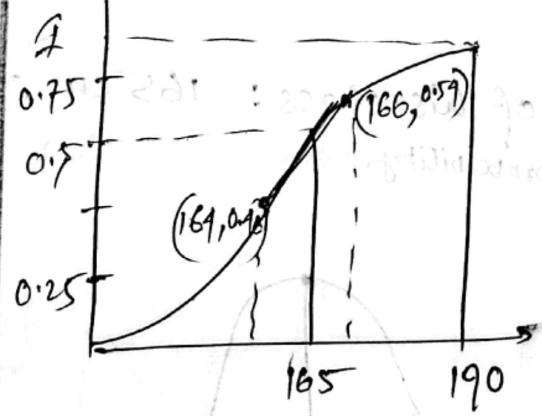
class এর half of the student এর hight 165 এর আশেপাশে।
So CDF হবে mean value এর আগের cumulative area এর Result = S shape Curve

PDF এর Y axis এ যে value plot করি তাই CDF এর curve (Gradient) slope



CDF কি

The number on the Y-axis actually telling us "How much distribution is on the left side of the given point/hight,



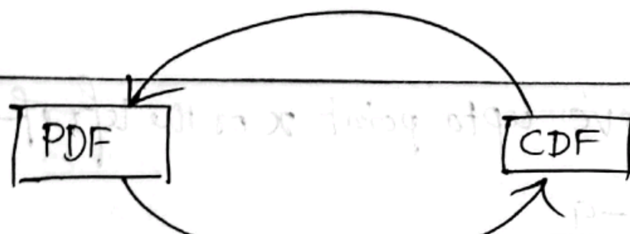
$$\text{Gradient} = \frac{\text{Rise}}{\text{Run}} = \frac{0.08}{2} = 0.04$$

So, two types: PDF $\longrightarrow$ CDF

CDF $\longrightarrow$ PDF

$$\frac{d\,F(x)}{dx} = f(x)$$ → CDF

$f(x)$ = PDF
$F(x)$ = CDF



$$\int_{-\infty}^{\infty} f(x)\cdot dx = F(x)$$ → PDF

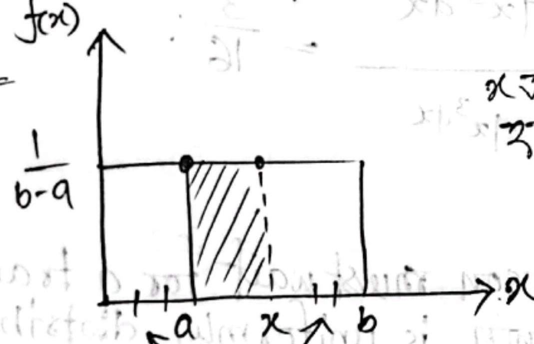
Uniform RV

(18 Sep 23)

A RV is said to be unformly distributed over interval (0,1) if its PDF is given by:

$$f(x) = \begin{cases} 1; & 0<x<1 \\ 0; & \text{otherwise} \end{cases}$$

PDF

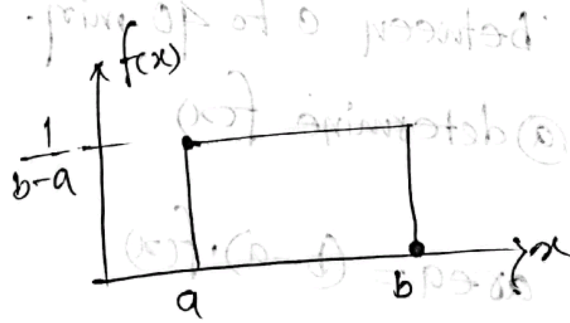


x এর value 0-1 পর্যন্ত যেকোনো বিন্দু হবার probability Same.

Length Same নয় হলে probability same নয়.

এই দুইটা Segment এর Length same হলে দুটোতেই আছে Probability Same.

CDF area under the curve.



Area = B × H
$= (b-a)\cdot f(x)$

$1 = (b-a)\cdot f(x)$

$\therefore f(x) = \frac{1}{b-a}$

CDF $\int_{-\infty}^{\infty} f(x)\,dx$ → PDF

for uniform RV: $\int_0^1 1\,dx$
$= 1.$

$$p(a \le x \le b) = \int_a^b f(x)\,dx$$

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha \le x \le \beta \\ 0, & \text{otherwise.} \end{cases}$$

CDF: Area under the curve upto point x or the left of x.

$$\frac{(x-a)}{(b-a)} \qquad \therefore CDF \longrightarrow \frac{x-a}{b-a}.$$

1) # Let x be a continuous RV with PDF

$$f_x(X) = \begin{cases} 4x^3 & , 0 \leq x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

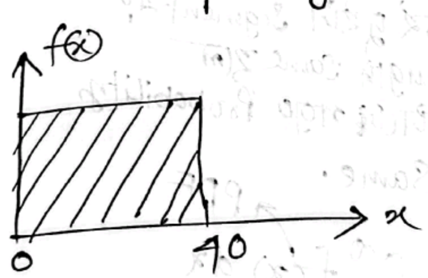
Find out $P(x \leq \frac{2}{3} \mid x \geq 1/3)$

$1/3 \qquad 2/3$

$$\frac{P(1/3 \leq x \leq 2/3)}{P(x \geq 1/3)} = \frac{\int_{1/3}^{2/3} 4x^3 \, dx}{\int_{1/3}^{1} 4x^3 \, dx} = \frac{3}{16}.$$

2) # The amount of time a person must wait for a train to arrive in a certain town is uniformly distributed between 0 to 40 min.

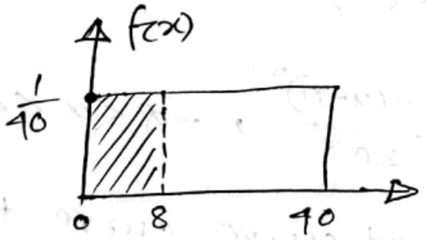
(a) determine $f(x)$



$$\text{area} = (b-a) \cdot f(x)$$

$$\therefore f(x) = \frac{1}{b-a} \qquad \therefore f(x) = \begin{cases} \frac{1}{40} & , 0 < x < 40 \\ 0 & , \text{otherwise} \end{cases}$$

ⓑ What is the probability that a person must wait less than 8 min.



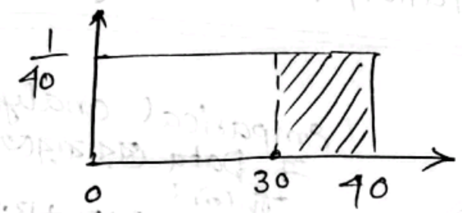
$P(x<8) = (b-a)\cdot f(x)$

$= 8 \times \frac{1}{40}$

$= \frac{1}{5} \times 100\%$

$= 20\%$

ⓒ What is the Probability a person must wait more than 30 min?



$P(x>30)$ or $P(30<x<40)$

$= (b-a)\, f(x)$

$= 10 \times \frac{1}{40} \times 100\%$

$= 25\%$

ⓓ Calculate $P(10<x<26)$; $P(x=20)$ and $P(x>45)$

$P(10<x<26) = (b-a) \times f(x)$

$= 16 \times \frac{1}{40}$

$= \frac{2}{5} \times 100\%$

$= 40\%$

$P(x=20) = 0$ (out 1)

$P(x>45) = 0$

(20 Sep 23)

Normal RV: Gaussian RV → Bell shaped curve

X is a normal RV with parameter μ and σ^2 if the density of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

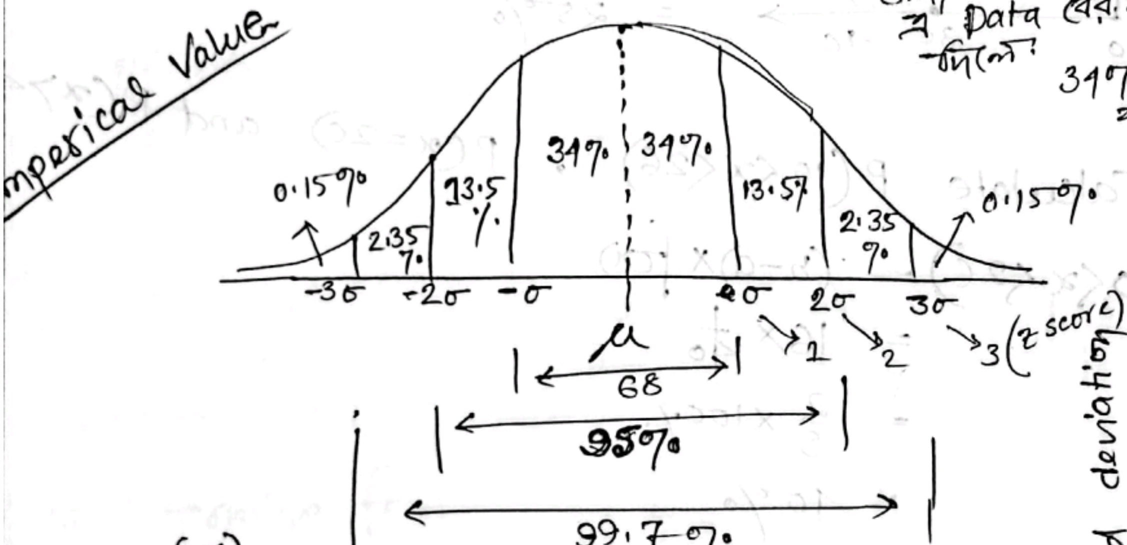
The density function is bell shaped curve curve that is symmetric around μ.

μ = Mean
σ = Standard deviation
σ^2 = Variance

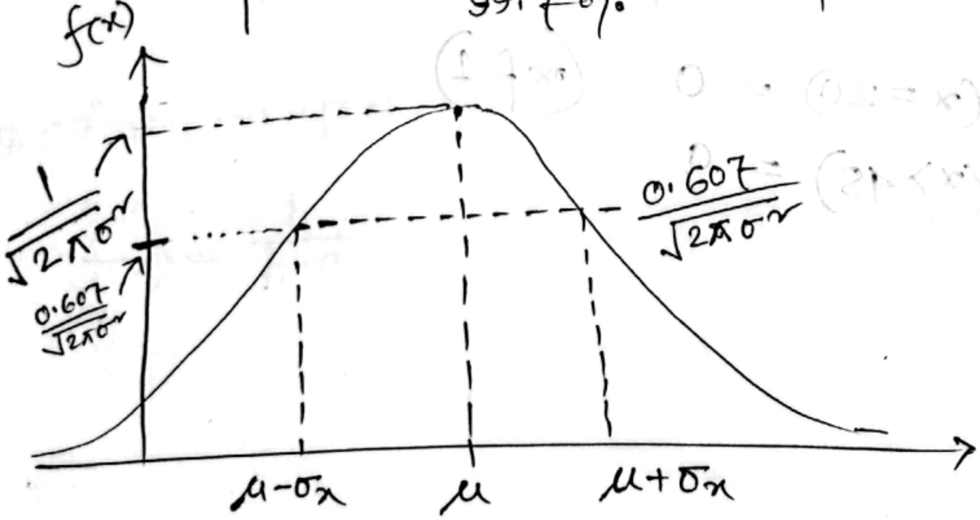
$\boxed{\text{Notation : } X \sim N(\mu, \sigma^2)}$

Emparical analysis এ Data বের করতে চাইলে?

$34\% + 13.5\% = 47.5\%$

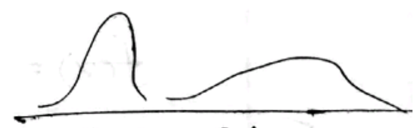


Z score : standard deviation থেকে কতটুকু দূরে।



PDF always Gives the shape of the Curve

σ_x^2 = Always a positive number
μ = Can be any number between $-\infty$ to $+\infty$.
SPREAD ~~...~~ of the distribution depends on σ.



***SPREAD : কতটুকু ছড়ানো থাকবে Bell shape Curve. depends on σ.

PDF : $\boxed{x = \mu}$

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-(\mu-\mu)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{0}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}$$

$(\mu-\sigma)$: 1 standard deviation towards left.

Deviation : ... থাকবে, mean : avg

$\boxed{x = \mu - \sigma}$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(\mu-\sigma-\mu)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-1/2}$$

$$= \frac{0.607}{\sqrt{2\pi\sigma^2}}$$ $\longrightarrow$ σ increases the magnitude drops result increase in SPREAD.

(আগে পরে Same value আসবে, $\frac{0.607}{\sqrt{2\pi\sigma^2}}$

$\boxed{x = \mu + \sigma}$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(\mu+\sigma-\mu)^2/2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{\sigma^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-1/2}$$

$$= \frac{0.607}{\sqrt{2\pi\sigma^2}}$$

CDF এ Z score লাগবে।

Standard Normal :

Normal distribution with parameter values $\mu = 0$ and $\sigma = 1$.

PDF

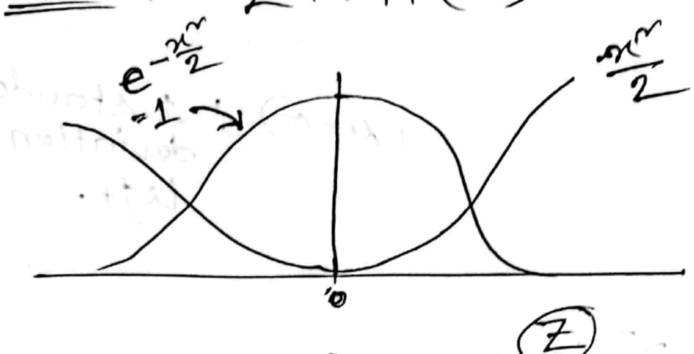
$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{(x-0)^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma^2} e^{-\frac{(x)^2}{2}} \longrightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}}$$

Constant value কোথা থেকে এলো? from where it came?

CDF : $Z \sim N(0,1)$



$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

হতে হবে 1.
so we multiplied this with $\frac{1}{\sqrt{2\pi}}$ to make 1.

CDF

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx$$

*** (pg 176, the table) probability Models.

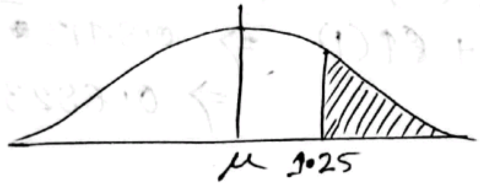
	0.01	0.02	0.03	0.04	0.05
1.0					
1.1					
1.2					

$\therefore$ 1.25 এর value :

CDF: $\longrightarrow$ Special notation is used Z.

$$P(z) = \phi(z) = P(Z < z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$P(z \geqslant 1.25)$



CDF দেয় বামের অংশ,
∴ ডানের অংশ বের করতে হলে (1 − ...)

$\Rightarrow 1 - P(z < 1.25)$

$\Rightarrow 1 - \phi(1.25)$

$= 1 - 0.8944$

$= 0.1056.$

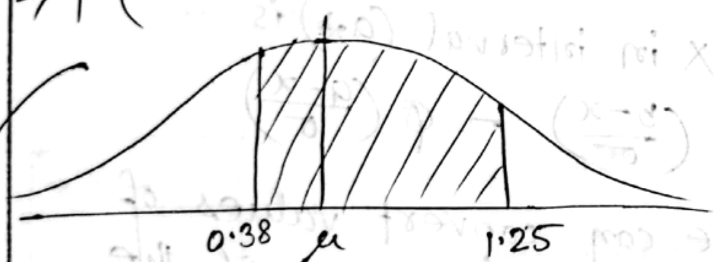
$\int_{1.25}^{\infty} f(x)\, dx$ HW

integrate করে Same Val আসবে, এটা দেখাবে,

$P(z \leqslant -1.25) = P(z \geqslant 1.25)$

Symmetric : তাই একটা থাকলে অন্যটা

$\Rightarrow P(-0.38 \leqslant z \leqslant 1.25)$



$P(z \leqslant 1.25) - P(z < -0.38)$

$\phi(-z) = 1 - \phi(z)$

$\Rightarrow \phi(1.25) - \phi(-0.38)$

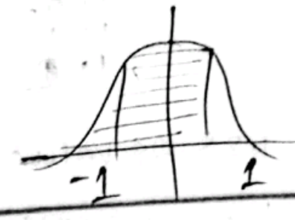
Table will be

$\Rightarrow \phi(1.25) - 1 + \phi(0.38)$

$\Rightarrow 0.8944 - 1 + 0.6480$

Table থেকে বের করতে হবে।

$= 0.5424$



$P(-1 \leq Z \leq +1) = ?$

$\Rightarrow P(Z \leq 1) - P(Z < -1)$

$\Rightarrow \phi(1) - \phi(-1)$

$\Rightarrow \phi(1) - 1 + \phi(1) \Rightarrow 0.8413 - 1 + 0.8413$

$\Rightarrow 0.6823$

Standard Normal (CDF)

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$$

If X is a Gaussian(μ, σ) RV the CDF of x is

$$F_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

The probability that X in interval (a,b) is

$$P(a \leq x \leq b) = \phi\left(\frac{b-x}{\sigma}\right) - \phi\left(\frac{a-x}{\sigma}\right)$$

using these formula we can convert values of the Gaussian RV X to equivalent values of standard random variable, Z

→ a sample value x of the RV X, the corresponding sample value of Z is

$$Z = \frac{x-\mu}{\sigma}$$

If X is the Gaussian RV $(61, 10)$ what is $P(x \leq 46)$?

$$F_X(x) = \varphi\left(\frac{x-\mu}{\sigma}\right)$$

$$F_X(46) = \varphi\left(\frac{46-61}{10}\right)$$

$$= \varphi\left(\frac{-15}{10}\right)$$

$$= \varphi\left(-\frac{3}{2}\right)$$

$$\therefore \varphi(-1.5)$$

$$\varphi(-z) = 1 - \varphi(z)$$

$$\varphi(-1.5) = 1 - \varphi(1.5)$$

$$= 1 - 0.933$$

$$= 0.067$$

$$= 6.7\%.$$

— o —

Exponential RV

A continuous RV X is said to be exponential RV with parameter λ, for $\lambda > 0$, if its PDF is given by

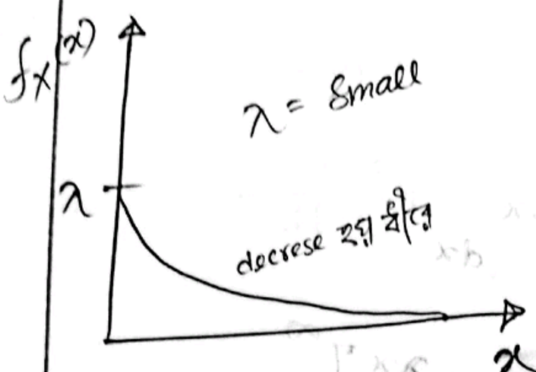
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; \text{if } x \geq 0. \\ 0 & ; \text{otherwise} \end{cases}$$

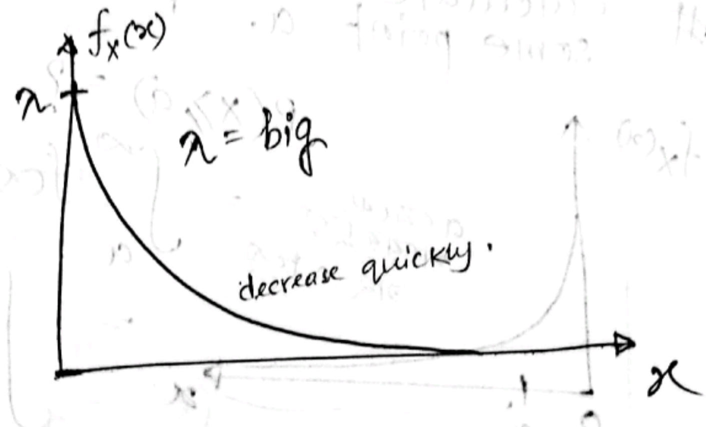
for any negative value $x < 0$, p is 0.

$$\boxed{\lambda = \frac{1}{\mu}}$$

For any $(-)$ve value of x, PDF, $f(x) = 0$

→ Its Probability is Zero.





$$f(x) = \lambda e^{-\lambda x} \quad [\text{যখন } x = 0]$$

$$= \lambda e^{-\lambda \cdot 0}$$

$$f(x) = \lambda$$

$\therefore$

$\nearrow$ PDF

$F_X(x) = P(X \le x)$

$F_X(a) = P(x \le a) = \int_{-\infty}^{a} f_X(x)\,dx \longrightarrow$ CDF

$= \int_0^a \lambda \cdot e^{-\lambda x}\,dx$

$= \int_0^a \lambda e^{-\lambda x}\,dx$

$= \lambda \int_0^a e^{-\lambda x}\,dx$

$= \left[\lambda \cdot \frac{e^{-\lambda x}}{-\lambda}\right]_0^a$

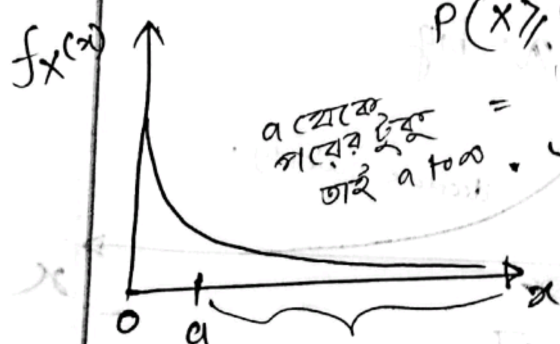
$= \left[-e^{-\lambda x}\right]_0^a$

We can use the formula $e^{-\lambda x \infty} + 1$

CDF $\rightarrow = \left(1 - e^{-\lambda a}\right)$

$$F_X(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 1 - e^{-\lambda x} & ; x \ge 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Calculate the P of x taking a value greater than some point a.



$P(X \ge a) = ?$

a থেকে পরের টুকু তাই a to ∞ $= \int_a^{\infty} f(x) \cdot dx$

$= \int_a^{\infty} \lambda \cdot e^{-\lambda x}\,dx$

$\Rightarrow \lambda \left[\frac{1}{\lambda} \cdot e^{-\lambda x}\right]_0^{\infty}$

$\Rightarrow e^{-\lambda \infty} + e^{-\lambda a}$

CDF $P(X \ge a) \Rightarrow e^{-\lambda a}$

★ Time কখনোই Discrete নয়, Time always Continuous.

If job arrives 15 sec on an average, the $\lambda = 4$ per minute what is the probability of waiting less than or equal to 30 sec?

Soln: $f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, \\ 0, & \text{otherwise} \end{cases}$

$T = 30\text{sec} = 0.5$ min

$$P(T \leq 0.5) = \int_0^{0.5} 4 \cdot e^{-4t} \, dt$$

$$= \left[\frac{4e^{-4t}}{-4}\right]_0^{0.5}$$

$$= \left[-e^{-4t}\right]_0^{0.5}$$

$$\Rightarrow -e^{-4 \times 0.5} + e^{-4 \cdot 0}$$

$$\Rightarrow e^0 - e^{-2}$$

$$\Rightarrow 1 - e^{-2}$$

$$\Rightarrow 0.86466 \text{ (Ans).}$$

DF দিয়ে করলে

$1 - e^{-\lambda a}$

$\Rightarrow 1 - e^{-4 \times 0.5}$

$= 0.86466$

The Probability that a telephone call lasts ~~no more~~ than 4 min is after modeled as an exponential CDF as:

$$F_T(t) = \begin{cases} 1 - e^{-t/3}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

① What is PDF of the duration of a telephone of conversation?

② What is the Probability that a conversation will last between 2 to 4 min

$f_T(t) = \frac{d}{dt} F_T(t) = \frac{d}{dt}(1 - e^{-t/3})$

$$= \frac{d}{dt}(1) - \frac{d}{dt}(e^{-t/3})$$

$$= -\left(-\frac{1}{3} e^{-t/3}\right)$$

$$= \frac{1}{3} e^{-t/3}.$$

$$\text{PDF} \Rightarrow f_T(t) = \begin{cases} \frac{1}{3} e^{t/3}; & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(Ans) ①

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & , \text{otherwise.} \end{cases}$$

$$f_T(t) = f_X(x) \qquad \therefore \lambda = 1/3 .$$

Ans

(11) $\Rightarrow P(2 \leq x \leq 4) = \int_2^4 f(t)\,dt$

$$= \int_2^4 \frac{1}{3} \cdot e^{-4/3}\,dt$$

$$P(2 \leq x \leq 4) = F_T(4) - f$$

$$\therefore P(2 \leq x \leq 4)$$

$$\Rightarrow F_T(4) - F_T(2)$$

$$\Rightarrow (1 - e^{-\lambda \times 4}) - (1 - e^{-\lambda \times 2})$$

(25 Sep 23)

The time (in hr) required to repair a car in an exponentially distributed RV with parameter $\lambda = \frac{1}{2}$, what is the probability that the repair time exceeds 4 hr. If it exceeds 4 hr than what is the probability it will exceed 8 hr.

$\Rightarrow f_T(t) = \lambda e^{-\lambda t}$

$\lambda = \frac{1}{2}$

এখানে P>8 বের করো → $P(T>8 \mid T>4) =$

$P(T>4) = \int_4^{\infty} f(t) \cdot dt$

$= \int_4^{\infty} \lambda \cdot e^{-\lambda t} dt$

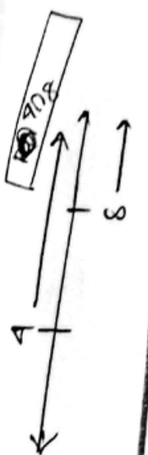
$= \lambda \left[\frac{e^{-\lambda t}}{-\lambda} \right]_4^{\infty}$

$= -e^{-\infty \lambda} + e^{-4\lambda}$

$= 0 + e^{-4 \cdot \frac{1}{2}}$

$= e^{-2}$

$= 0.1353352$



$P(A|B) = \frac{P(AB)}{P(B)}$

shortcut

$P(x>4)$

$= e^{-\lambda^2 a^2}$

$= e^{-\frac{1}{2} \cdot 4}$

$= e^{-2}$

$= 0.1353352$

Given that $X \sim N(\overset{\mu,\,\sigma^2}{50, 10})$ what is the value of mean and standard deviation? what value of x has a z-score of 1.4? what is the z-score that corresponds to $X = 30$?

(Soln) Mean $\mu = 50$

Standard Deviation, $\sigma = \sqrt{10}$.

$Z = \frac{30 - 50}{\sqrt{10}}$ → σ ধরে করব again

$Z = \frac{x - \mu}{\sigma}$

$\therefore x = \mu + Z\sigma$

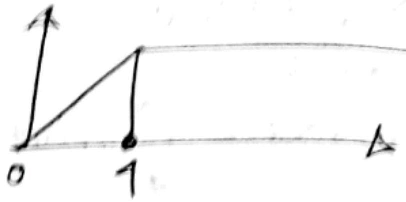
$\therefore x = 50 + (1.4 \times \sqrt{10})$

$=$

μ, σ^2	μ, σ
কিছু বলা না থাকলে σ^2 ধরব	sir বলে দিলে σ. গুণ এ

The continuous RV X has PDF f(x) which is given by

$$f(x) = \begin{cases} K(x^2-2x+2), & 0<x\le 3 \\ 3K, & 3<x\le 4 \\ 0, & \text{otherwise.} \end{cases}$$



(1) find out the value of K.

Total $P = 1 = \int_{-\infty}^{\infty} f(x)\,dx$

$\int_0^3 K(x^2-2x+2)\,dx$

$K\left[\frac{1}{3}x^3 - \frac{2}{2}x^2 + 2x\right]_0^3$

$= 6K$

$\int_3^4 3K\,dx$

$\Rightarrow 3K[x]_3^4$

$\Rightarrow 3K$

$\therefore 6K+3K = 1$

$\therefore K = \frac{1}{9}$

(2) Find out the CDF

$F(x) = \int \frac{1}{9}(x^2-2x+2)\,dx$

$= \frac{1}{9}\left(\frac{1}{3}x^3 - 2\frac{x^2}{2} + 2x\right) + C$

So $F(0) = 0$.

$F(x) = \frac{1}{9}\left(\frac{1}{3}\cdot x^3 - x^2 + 2x\right)$

$F(x) = \int 3K\,dx = \int \frac{1}{3}dx = \frac{1}{3}x + D$

$F(4) = \frac{1}{3}\cdot 4 + D = 1$

$\therefore D = -\frac{1}{3}$. $\quad \therefore \frac{1}{3}x + \left(-\frac{1}{3}\right)$

$$\therefore F(x) = \begin{cases} 0, & x\le 0 \\ \frac{1}{9}\left(\frac{1}{3}x^3 - x^2 + 2x\right), & 0<x\le 3 \\ \frac{1}{3}x - \frac{1}{3}, & 3<x\le 4 \\ 1, & x>4 \end{cases}$$

we will do the math afterwards

(Expectation)

③ Find the mean of X or $E[X]$

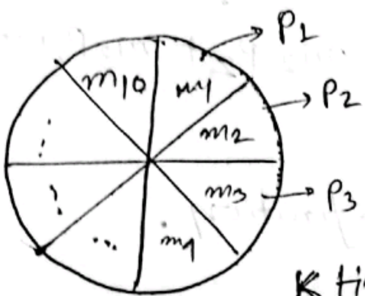
(Theory of $E[X]$ calculation)

Expectation of X

→ weighted avg of the possible values of x

→ Expected value of a RV is the theoritical mean of RV

→ In simple term → Expected value or mean is a measure of central tendency of a probability distribution.



K times Rolling/ spin

$\frac{K_i}{K}$

$K_1 = m_1$ কতবার উঠে

$K_2 = m_2$ " "

$K_i = m_i$ "

$K_n = M_n$ " "

Total money Received $= (m_1 K_1 + m_2 K_2 + m_3 K_3 + \dots m_i K_i)$

$$Avg = \frac{m_1 K_1 + m_2 K_2 + \dots m_i K_i}{K}$$

$$= m_1 P_1 + m_2 P_2 + m_3 P_3 + \dots$$

its a Kind of Random Variable

Expectation of RV

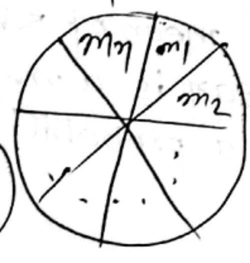
(27 Sep 23)

Expectation of $x \to$ weighted Avg

→ Theoritical mean of RV

→ A measure of central Tendency

→ It ~~xxxxxx~~ represents the avg value that a RV is likely to take on.



K times

$K_i \longrightarrow$ The number of time that the output is m_i

$P_i = \frac{K_i}{K}$

Total Amount ~~of~~ Received Per spin $= \dfrac{m_1K_1 + m_2K_2 + m_3K_3 + \cdots m_nK_n}{K}$

$\Rightarrow m_1P_1 + m_2P_2 + m_3P_3 + \cdots m_nP_n$

$$\therefore \boxed{E[X] = \Sigma_x \, x \cdot P_X(x)}$$

formal Definition.

$P(1) = 1/2$

$P(2) = 1/2$ # $\therefore E[X] = 1 \times \frac{1}{2} + 2 \times \frac{1}{2}$

$= \frac{3}{2}$

Expectation can be more than one

$$E[X] = \Sigma_{x : P(x) > 0} \, x \cdot P(x)$$

The expected value of X is a weighted average of the possible values that X can take on each value being weighted by its probability that X assumes that value.

Playing a game
while I can
get 100tk with $p: 1/6$
$200 \to p = 1/2$
$400 \to p = 1/3$

$$\left(\frac{1}{6}\times 100\right)+\left(\frac{1}{2}\times 200\right)+\left(\frac{1}{3}\times 400\right)$$
$$= 250 \longrightarrow E[X].$$

Expectation of Bernoulli R.V:

$$P_X(x) = \begin{cases} p & ; \text{ if } x=1 \\ (1-p) & ; \text{ if } x=0. \end{cases}$$

$P(0) = (1-p)$
$P(1) = p$

$$\therefore E[X] = \sum_x x \cdot P(x)$$
$$= 0\times(1-p) + 1\times p$$
$$= p.$$

(Possible Question)
Show that the expected number of success in a single trial is just the probability that the trial will be a success

Find out $E[X]$ where x is the outcome when a fair Die is tossed.

$\Rightarrow$ The PMF is $P_X(x) = \begin{cases} 1/6 & , x = 1,2,3,4,5,6 \\ 0 & , \text{otherwise} \end{cases}$

$$E[X] = \sum_{x=1}^{6} x \cdot P(x)$$

$$= 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot 1/6 + 6\cdot\frac{1}{6}.$$

$$= \frac{7}{2} \text{ (Ans)}.$$

3.5 পাওয়া discrete case possible নয়। Expectation is more like getting avg. So it can be.

$P(i) = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$ when $i = 0, 1, 2, \ldots$

and $\binom{n}{i} = \dfrac{n!}{(n-i)!\, i!}$

$$E[X] = \sum_{i=0}^{n} i \cdot P(i)$$

$$= \sum_{i=0}^{n} i \cdot \binom{n}{i} \cdot p^i \cdot (1-p)$$

$$= \sum_{i=0}^{n} i \cdot \frac{n!}{(n-i)! \times i!} \cdot p^i \cdot (1-p)^{n-i}$$

$$E[X] \cdot \sum_{i=0}^{n} \frac{n!}{(n-i)!\,(i-1)!} \times p^i \times (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{n\,(n-1)!}{(n-i)!\,(i-1)!} \times p \cdot p^{(i-1)} \times (1-p)^{n-i}$$

$$= n \cdot p \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} \times p^{(i-1)} \times (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{((n-1) - (i-1))!\,(i-1)!} \times p^{(i-1)} \times (1-p)^{n-i}$$

Let $\Rightarrow K = i-1$

So when, $i = 1$, $K = 0$.
$i = n$, $K = n-1$.

$$E[X] = np \sum_{K=0}^{n-1} \frac{(n-1)!}{((n-1)-K)! \times K!}\, p^K\, (1-p)^{n-1}$$

$$= np \sum_{K=0}^{n-1} \frac{(n-1)!}{((n-1)-K)! \cdot K!}\, p^K \cdot (1-p)^{n-1+1-}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{((n-1)-k)!\cdot k!} p^k \cdot (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k \cdot (1-p)^{(n-1)-k}$$

$$\boxed{\begin{aligned} \Sigma_{i=0}^{\infty} P(i) &= 1 \\ &= (p + (1-0))^n \end{aligned}} \longrightarrow \text{Generic Term}$$

$$\Rightarrow \sum_{i=0}^{n} \binom{n}{i} p^i \cdot (1-p)^{n-i} = (p + (1-p))^n$$

$$E[X] = n \cdot p \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot p^k \cdot (1-p)^{(n-1)-k}$$

$$= np\ (p + (1-p))^{(n-1)-k}$$

$$= np \cdot (1)^{n-1}.$$

$$= np$$

Prove that

The expected number of success in n independent trial is n multiplied by the probability that the trial result in success.

Question

Qn: The expected number of independent trials we need to perform untill we attain our first success is equal to the reciprocal of the probability that any one trial result in success.

Expectation of Geometric RV

From definition of Geometric RV

$$P_X(n) = P\{X=n\} = (1-p)^{n-1} \cdot p$$

from definition, $E[X] = \sum_{x=1}^{\infty} x \cdot P(x)$

$$\Rightarrow \sum_{n=1}^{\infty} np(1-p)^{n-1}$$

For Geometric Series

$\sum_{n=1} ar^n$

Infinite

$\sum_{n=1}^{\infty} ar^n$

Finite case

$\sum_{k=0}^{n-1} ar^k$

$S = ar^0 + ar + ar^2 + \dots + ar^{n-1}$ — (1)

$Sr = ar^1 + ar^2 + ar^3 + ar^4 + \dots ar^{n-1} + ar^n$ — (2)

(1) − (11)

$S - Sr = ar^0 - ar^n$

$S(1-r) = a(r^0 - r^n)$

$S = \dfrac{a(1-r^n)}{1-r}$

So, for a known range (known range $(n-1)$ in this case)

$$\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{(1-r)} \quad (3)$$

For infinite case; $\sum_{n=0}^{\infty} ar^n = ?$ $\quad n = \infty$

$$\sum_{k=0}^{n=1} ar^k = \frac{a(1-r^n)}{(1-r)}$$

$= r^{\infty}$ will be very very small if r is a fraction

So, $\sum_{K=0}^{\infty} \cdot a r^{K} = \frac{a}{(1-r)}$. $\longrightarrow$ (4)

$$E[X] = \sum_{n=1}^{\infty} n \cdot p\,(1-p)^{n-1}$$

Let $q = 1-p$

$$E[X] = p \sum_{n=1}^{\infty} n \cdot q^{n-1}$$

p and q all are small (Fraction).

$$E[X] = p \sum_{n=1}^{\infty} \cdot \frac{d}{dq} \cdot q^{n} \quad \left[\frac{d}{dq} \cdot x^{n} = n \cdot x^{n-1} \right]$$

$$\Rightarrow p \cdot \frac{d}{dq} \left(\sum_{n=0}^{\infty} q^{n} \right) - q^{0} .$$

$$\Rightarrow p \frac{d}{dq} \cdot \left[\left(\sum_{n=0}^{\infty} 1 \cdot q^{n} \right) - 1 \right]$$

$$\Rightarrow p \frac{d}{dq} \left[1 \left(\frac{1}{1-q} - 1 \right) \right] .$$ from (4)

$$\Rightarrow p \frac{d}{dq} \left[\frac{q}{1-q} \right] \cdot \quad u = q, \; v = 1-q$$

$$p \frac{(1-q) \cdot \frac{d}{dq} \cdot q - q \frac{d}{dq}(1-q)}{(1-q)^2}$$

$$= p \frac{(1-q) - q(0-1)}{(1-q)^2}$$

$$= p \frac{1}{(1-q)^2}$$

$$\therefore E[X] = p \cdot \frac{1}{(1-q)^2} = p \cdot \frac{1}{p^2} \Rightarrow \frac{1}{p}$$

(Ans)

Expectation of Continuous Random Variable:

If a continuous RV having the PDF $f(x)$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

Expectation of Uniform RV

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & ; \text{ if } \alpha \leq x \leq \beta \\ 0 & ; \text{ otherwise.} \end{cases}$$

$$E[x] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha} \cdot dx$$

$$= \left[\frac{x^2}{2(\beta-\alpha)} \right]_{\alpha}^{\beta}$$

$$= \left[\frac{\beta^2-\alpha^2}{2(\beta-\alpha)} \right] \quad \therefore \frac{\beta+\alpha}{2}$$

Expected value of a RV uniformly distributed over interval (α, β) is just the midpoint of the interval.

Expectation of Exponential RV

If x is a RV which is exponentially distributed with parameter λ

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & ; \text{ if } x \geq 0 \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\boxed{\int u\,dv = uv - \int v\cdot du}$$

formulae

$$E[x] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} x \cdot \lambda \cdot e^{-\lambda x}\,dx$$

here,
let $x = u$
$\frac{\partial x}{\partial x} = \frac{\partial u}{\partial x}$
$du = \partial x$
and $v = -e^{-\lambda x}$

$\partial v = \lambda e^{-\lambda x} dx$

$v = \lambda \frac{e^{-\lambda x}}{-\lambda}$

$= -e^{-\lambda x}$

$$= \int u \cdot dv - \int v\,du$$

$$= x \cdot (-e^{-\lambda x}) \Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda x} dx$$

$$= \left[-x \cdot e^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

$$= \left[-\infty \cdot e^{-\lambda \infty} + 0 \cdot e^{-\lambda \cdot 0}\right] + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty}$$

full part $= 0$

∞^x

$$= \left[\frac{e^{-\lambda\infty}}{-\lambda} + \frac{e^0}{-\lambda}\right]$$

$$= \frac{1}{\lambda}.$$

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-(x-\mu)^2 / 2\sigma^2} \quad [-\infty < x < +\infty)$$

$$\Rightarrow E[x] = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$x = (x-\mu) + \mu$

$$E[x] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} ((x-\mu) + \mu) \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} (x-\mu) \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$+ \mu \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Let: $y = x - \mu$

$$E[x] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} y \cdot e^{\frac{-y^2}{2\sigma^2}}\, dy + \mu \int f(x)\, dx$$

$y^2 = x$

$\frac{2y}{dx} = \frac{dx}{dx}$

$\Rightarrow 2y = dx$

$y = \frac{1}{2} dx$

1st Portion

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} y \cdot e^{-\frac{y^2}{2\sigma}}\, dy$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int e^{-\frac{x}{2\sigma}} \frac{1}{2}\, dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{+\infty}^{+\infty} e^{-\frac{x}{2\sigma}} \cdot \frac{1}{2}\, dx$$

$= 0$

2nd Portion

$$\mu \int_{-\infty}^{\infty} \underbrace{f(x)\, dx}_{1} = \mu \times 1$$

$= \mu$ (Ans).

Prob #

Consider two independent coin tosses, each with $\frac{3}{4}$ probability of a head and let X be the number of heads obtained this Binomial RV with parameter $n = 2$ and $p = \frac{3}{4}$. $E[X] = ?$

shortcut $np = \frac{3}{4} \times 2 = \frac{3}{2}$

$$P(0) = \binom{2}{0} \left(\frac{3}{4}\right)^0 \left(1 - \frac{3}{4}\right)^{2-0}$$

$$= \left(\frac{1}{4}\right)^2$$

$$P(1) = \binom{2}{1} \left(\frac{3}{4}\right)^1 \cdot \left(1 - \frac{3}{4}\right)^{2-1}$$

$$= 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$$

$$P(2) = \binom{2}{2} \cdot \left(\frac{3}{4}\right)^2 \cdot \left(1 - \frac{3}{4}\right)^{2-2}$$

$$= \left(\frac{3}{4}\right)^2$$

So the PMF will be:

$$P_X(K) = \begin{cases} \left(\frac{1}{4}\right)^2, & \text{when } K = 0 \\ 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right); & K = 1 \\ \left(\frac{3}{4}\right)^2; & K = 2 \end{cases}$$

$$\therefore E[X] = 0 \cdot \left(\frac{1}{4}\right)^2 + 1 \cdot \left(2 \cdot \frac{3}{4} \cdot \frac{1}{4}\right) + 2\left(\frac{3}{2}\right)^2 = \frac{3}{2} \text{ (Ans)}.$$

(Qn). Suppose that X has following PMF. $P(0)=0.2$, $P(1)=0.5$, $P(2)=0.3$
Calculate $E[X^2]$. not $E[X]$.

ex COLONEL RAHAT SIR

Expectation of a function of a RV

(04 OCT 23)

(Solution) $E[X] := \sum_x x \cdot p_X(x)$

$E[X] = (0\times 0^2) + (1\times 0.5) + (2\times 0.3)$
$= 1.1$

$Y := E[X^2]$ $\quad$ Y is a RV that can take the values of $\cdot Y = 0^2, 1^2, 2^2$,
$= 0, 1, 4$.

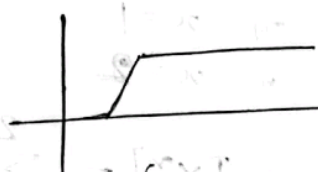
$p_Y(0) = 0.2$
$p_Y(1) = 0.5$
$p_Y(4) = 0.3$

$\therefore E[X^2] = (0\times 0.2) + (1\times 0.5) + (4\times 0.3) = 1.7$

$\therefore 1.7 = E[X^2] \neq \{E[X]\}^2 = 1.21$

(*) If X is uniformly distributed over $(0,1)$ then calculate $E[X^3]$.
(example of continuous case)

formula: $E[X] = \int_{-\infty}^{\infty} x\cdot(f(x))\,dx$



$Y = X^3$

$0 \leqslant a \leqslant 1$

CDF $F_Y(a) = P\{Y \leqslant a\}$
$= P\{X^3 \leqslant a\}$
$= P\{X \leqslant a^{1/3}\}$

$F_Y(a) = a^{1/3}$

$f_Y(a) = \frac{d}{da} F_Y(a)$ $\quad$ formula (∞ [illegible]
$= \frac{d}{da}(a^{1/3})$
$= \frac{1}{3} a^{1/3 - 1}$
$= \frac{1}{3} a^{-2/3} \rightarrow$ pdf.

$E[Y] = E[X^3]$

$= \int_{-\infty}^{\infty} a\cdot f_Y(a)\,da$

$= \int_0^1 a\cdot\frac{1}{3}\cdot a^{-\frac{2}{3}}\,da$

$= \frac{1}{3}\int_0^1 a^{\frac{1}{3}}\,da$

$= \frac{1}{3}\left[\frac{a^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right]_0^1 = \frac{1}{4}$ (Ans)

PDF → CDF : Integration

আমরা যদি $F_Y(x)$ কে Convert (y এর Respect এ Convert) করে নাহ
বলেছি এবং নিচে property

Property 1:
Expectation of Function: If X is a discrete RV with PMF $P_X(x)$ for any real value function or in other words, the derived RV $Y = g(x)$ the expected value of Y (or $g(x)$) is given by:

$$E[Y] = E[g(x)] = \sum_{x: P(x) > 0} g(x) \cdot P(x) \Rightarrow \sum_{i=1}^{n} g(x) \cdot P(x) \rightarrow \text{discrete}$$

$$E[Y] = E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) \, dx \rightarrow \text{Continuous}$$

Qn: firstone

$$P_X(x) = \begin{cases} 0.2 & \text{when } x = 0 \\ 0.5 & \text{" } x = 1 \\ 0.3 & \text{" } x = 2 \end{cases}$$

$$g(x) = X^2 \qquad E[X^2] = \sum_{x=0}^{2} g(x) \cdot p(x)$$

$$= (0^2 \times 0.2) + (1^2 \times 0.5) + (2^2 \times 0.3)$$

$$= 1.7$$

for Continuous case:

second one

$$Y = X^3 = g(x)$$

value 1 [cause area under the curve $0 \to 1$ is 1]

$$E[X^3] = \int_0^1 g(x) \cdot f(x) \, dx$$

$$= \int_0^1 g(x) \, dx$$

$$= \int_0^1 x^3 \, dx$$

$$\left[\frac{x^{3+1}}{3+1}\right]_0^1$$

$$= \frac{1}{4}$$

(Ans)

Show that the variance of a random variable equals the expected value of the square of the random variable minus the square of the Expected Value of the Random variable

Prob → Given PMF is $f_X(x) = \begin{cases} 2/3, & x=1 \\ 1/3, & x=2 \end{cases}$ calculate $E[X^2+1]$

Soln Let $Y = X^2 + 1$

PMF of $Y \rightarrow f_Y(Y) = \begin{cases} \frac{2}{3}, & y=2 \\ \frac{1}{3}, & y=5 \end{cases}$

$E[Y] = (2 \cdot \frac{2}{3}) + (5 \cdot \frac{1}{3}) = 3.$

$E[X] = (1 \cdot \frac{2}{3}) + (2 \cdot \frac{1}{3}) = \frac{4}{3}.$

$E[Y] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$ —— ①

$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$

Remember:

$E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y) \cdot dy$

adv of ① is we do not need to find out the PDF of Y.

Property 2 n^{th} moment of X.

If X is a RV then expected value of RV X is reffered to as the ~~ea~~ mean or the first moment of X.

$E[X] = \mu =$ the first moment of X.

$$E[X^n] = \begin{cases} \sum_{x \cdot P(x) \geq 0} x^n \cdot P(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x^n \cdot f(x) \cdot dx. & \text{if } x \text{ " continuous} \end{cases}$$

0^{th} moment : total Probability

$E[X]$ | 1^{st} " : mean (μ)

$E[X^2]$ | 2^{nd} " : Varience

3^{rd} moment : Skewness $E[X^3]$.

4^{th} moment : Kurtosis

সেই parameter যে নির্দেশ করে Tail এ কতটুকু data আছে

Property 3 (Linearity).

$E[X]$ is linear, if a and b are constant,

$$E[aX+b] = a\,E[X] + b.$$

Discrete $E[g(x)] = \sum_{x \cdot p(x) > 0} g(x) \cdot p(x)$

$$\Rightarrow \sum_{x \cdot p(x) > 0} (ax+b)\, p(x)$$

$$\Rightarrow a \sum_{x \cdot p(x) > 0} x \cdot p(x) + b \underbrace{\sum_{x \cdot p(x) > 0} p(x)}$$

total Probability equals 1

$$= a\,E[X] + b \cdot 1$$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x) \cdot dx$$

$$= \int_{-\infty}^{\infty} (ax+b)\, f(x) \cdot dx.$$

$$= a\int_{-\infty}^{\infty} x \cdot f(x)\, dx + b \underbrace{\int_{-\infty}^{\infty} f(x) \cdot dx}_{1}.$$

$$= a\,E[X] + b$$

property 4 এর derivation আসবেনা

$x \to RV$

$E[X] \to$ not a RV, it's a value

$(X - E[x]) \to RV$

$(X - E[X])^2 \to RV$. এর expectation বের করব।

Property 4: Variance

$x - E[X]$

$$Var[X] = E[X - E[X]]^2$$

$$Var[X] = \int_{-\infty}^{\infty} \{x - E[X]\}^2 \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} [x^2 - 2x E[X] + E[X]^2]\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - 2E[X] \underbrace{\int x \cdot f(x)\, dx} + E[X]^2 \int_{-\infty}^{\infty} f(x)\, dx$$

$$\boxed{= \int_{-\infty}^{\infty} x^2 f(x)\, dx - E[X]^2}$$ shortcut for Property 4

Q: Calculate variance $Var[X]$ when X represents the when a fair dice is Rolled.

$P(1) = \frac{1}{6} \ldots\ldots P(6) = \frac{1}{6}$,

$$E[X] = \sum_{x=1}^{6} x \cdot P(x)$$

$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}.$$

$$= \frac{7}{2}.$$

$$E[X^2] = \left(1^2 \cdot \frac{1}{6}\right) + \left(2^2 \cdot \frac{1}{6}\right) + \left(3^2 \cdot \frac{1}{6}\right) + \left(4^2 \cdot \frac{1}{6}\right) + \left(5^2 \cdot \frac{1}{6}\right) + \left(6^2 \cdot \frac{1}{6}\right)$$

$$= \frac{91}{6}.$$

$$\therefore Var[X] = \frac{91}{6} - \left(\frac{7}{2}\right)^2$$

$$= 2.91$$

(proof আমার নাই)

Property 5: (Independence)

$$E[X+Y] = E[X]+E[Y] \text{ and } E[XY] = E[X] \cdot E[Y]$$

Show that the variance of a RV equals the expected value of the sequence of the RV minus the square of the Expected value of the R.V.

$$Var[X] = E[X^2] - E[X]^2.$$

Example

Suppose we observe at telephone switch voice call (v), data call (d), X = no of voice calls
Y = " " data "
R function R = X·Y.

outcom	ddd	ddv
P[·]	1/8	1/8
X	0	1
Y	3	2
R	0	2

(a) PMF of R

(b) expected value E[R]?

(c) expected value of the function V = g(R) = 4R+7 = ?

(Ans)

(a) $P(R=0) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$P(R=2) = \left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) = \frac{3}{4}$ or $\left(1-\frac{1}{4}\right) = 3/4$.

$\therefore$ PMF of R is $P_R(r) = \begin{cases} \frac{1}{4}, & \text{when } r=0 \\ 3/4, & \text{when } r=2 \\ 0, & \text{otherwise.} \end{cases}$

(b): $E[R] = \sum R_i \, P_R(r)$

$= \left(0 \times \frac{1}{4}\right) + \left(2 \times \frac{3}{4}\right) = \frac{3}{2}.$

(c) $E[Y] = E[g(R)]$

$= \sum_{x \cdot P(x)>0} (4R+7) \cdot P(r)$

$= \{(4\times0)+7\} \cdot \frac{1}{4} + \{(4\times2)+7\} \frac{3}{4}$

$= 7 \cdot \frac{1}{4} + 15 \cdot \frac{3}{4}$

$= 13$ (Ans).

Shortcut

$E[aX+b] = a\,E[X]+b$

$E[4R+7] = 4 \times \frac{3}{2} + 7 \Rightarrow 13$

$a = 4, \quad b = 7,$

Qn: 1 · $P_X(x)$ · : phone company charge:

1st pg : 0.10 \$
2nd " : 0.09 \$
...
5th pg : 0.06 \$

But : Fax from 6th to 10th pg: 0.50 \$
no longer than 10 pg.

① $Y = g(x)$ · for the change in cent for sending one fax
② suppose, 1, 2, 3, or 4 with equal probability. Find the PMF, expected val of Y, charge of Fax.

③ Suppose the probability model for the number of pages X of a Fax is :

$$P_X(x) = \begin{cases} 0.15, & \text{when } x = 1,2,3,4 \\ 0.1, & x = 5,6,7,8 \\ 0, & \text{otherwise.} \end{cases}$$

Ans! ① $Y = g(x) = \begin{cases} 10.5x - 0.5x^2, & 1 \leq x \leq 5 \\ 50, & 6 \leq x \leq 10 \end{cases}$

$x = 1$ হলে $10.5 \times 1 - 0.5 \times 1^2$
$= 10$ cent
$x = 2$ হলে $= 19$ cent
$x = 3$, $10.5 \times 3 - 0.5 \times 3^2$
$= 27$

$x = 6$, 50 cent
$x = 7$, 50 cent

1 pg হলে 10
2 " " : 19
3 " " : 27
4 " " : 36.
5 " " : 40
6 " " : 50
7 " " : 50
8 " " : 50

(11) ~~1 pg 10, 2 pg 19, 3 pg 27~~

$$P_X(x) = \begin{cases} \frac{1}{4}, & x = 1,2,3,4 \\ 0, & \text{otherwise} \end{cases} \quad \text{(PMF)}$$

$$P_Y(y) = \begin{cases} \frac{1}{4}, & y = 10, 19, 27, 34 \\ 0, & \text{otherwise.} \end{cases} \quad [y \text{ [illegible] cost}]$$

$$E[Y] = \left(\tfrac{1}{4}\times 10\right) + \left(\tfrac{1}{4}\times 19\right) + \left(\tfrac{1}{4}\times 27\right) + \left(\tfrac{1}{4}\times 34\right)$$
$$= 22.4 \text{ cents.}$$

— o —

(3) find the PMF and expected value of Y, the cost of FAX.

The P of the cost of the FAX to be 0.10 \$ or 10 cent (when the no of page is 1) $\Rightarrow$ 0.15

$$P_Y(50) = P_Y(6) + P_Y(7) + P_Y(8)$$
$$= 0.1 + 0.1 + 0.1 = 0.3$$

$$\therefore P_Y(y) = \begin{cases} 0.15, & y = 10, 19, 27, 34 \\ 0.1, & y = 40 \\ 0.3, & y = 50. \\ 0, & \text{otherwise} \end{cases}$$

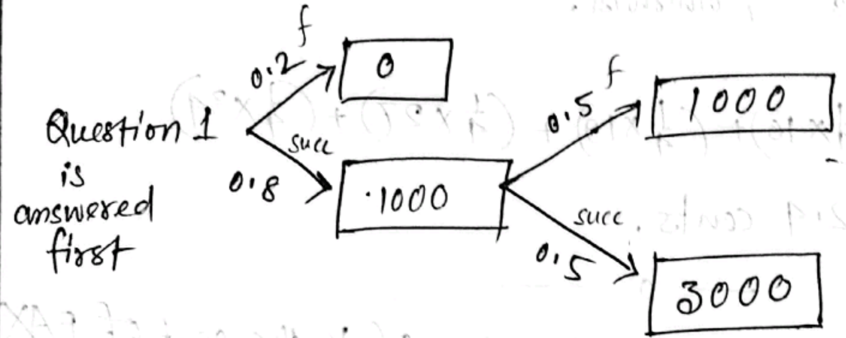
$$E[Y] = (10 + 19 + 27 + 34) \times 0.15 + (0.1 \times 40) + (0.3 \times 50).$$

$$= 32.5 \text{ cents.}$$

Q1 or Q2 can be selected by the examinee.

\# Q1 : P of being correct : 0.8 reward : 1000 tk
Q2 : P of being " : 0.5 reward : 2000 tk

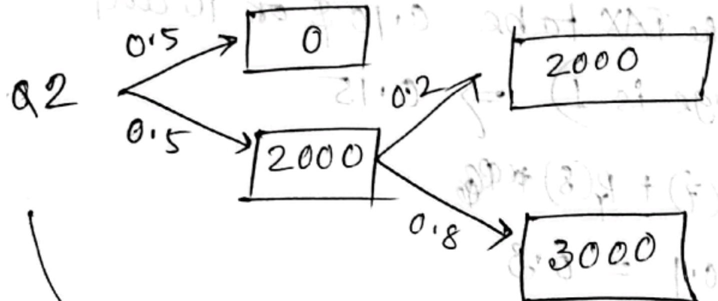
If the question is wrong at first : exam terminates
if the qsn is correct at first : examinee can attend the second question.



$P_X(0) = 0.2$
$P_X(1000) = 0.8 \times 0.5$
$= 0.4$

$P_X(3000) = 0.8 \times 0.5 = 0.$

$E[X] = (0 \times 0.2) + (1000 \times$
$+ (3000 \times 0.4)$
$= 1600$ tk



$(0 \times 0.5) + (2000 \times 0.1) + (3000 \times 0.4)$

$= 1400$ tk