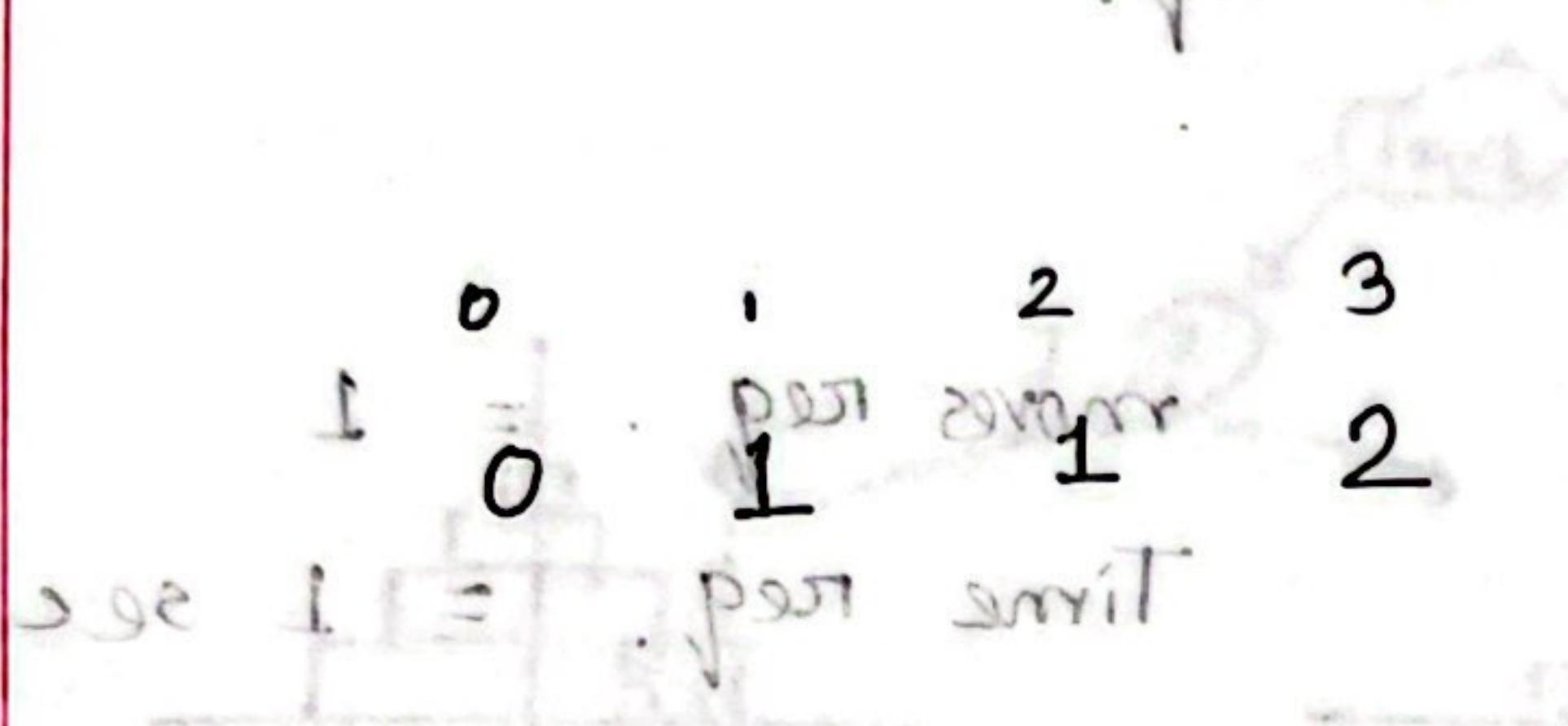


concrete Mathematics

Recurrent Problems

: beginning unit **Tower of Brahma**
 → 3 middle of diamond
 → gold plate

→ Fibonacci series



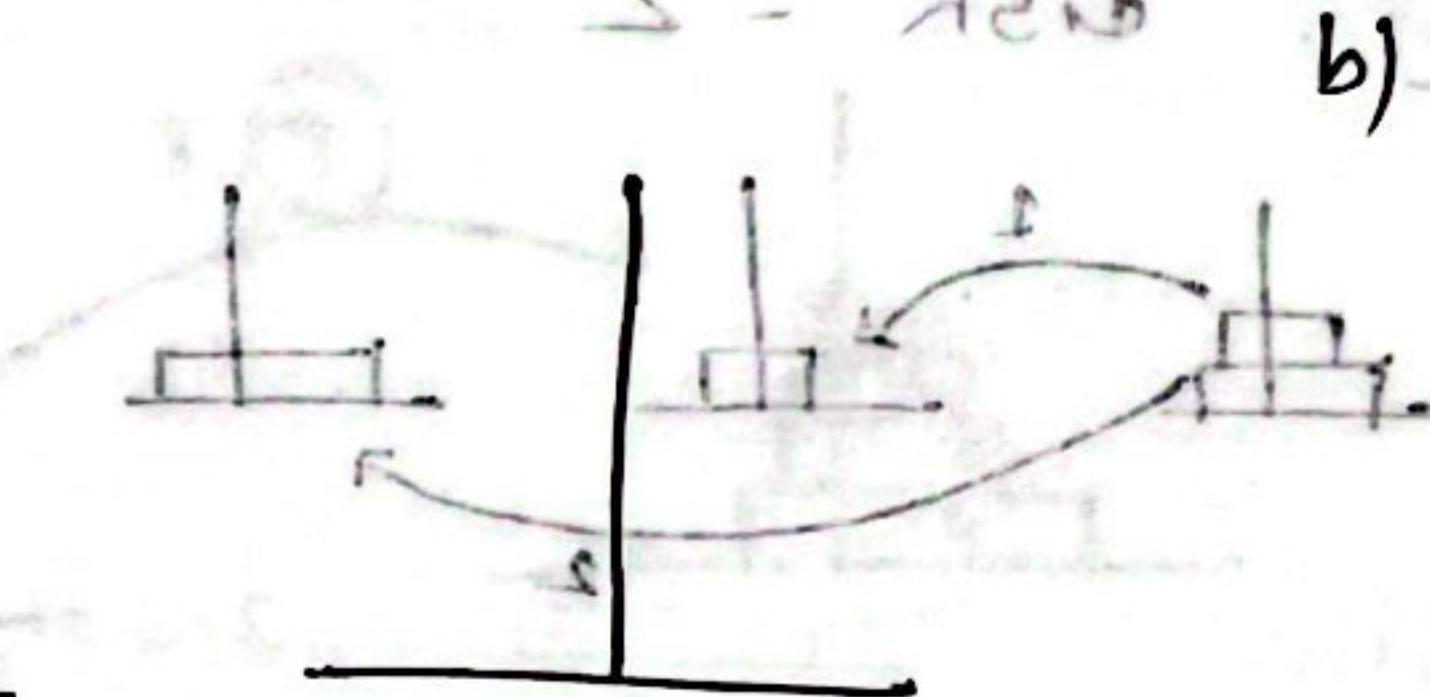
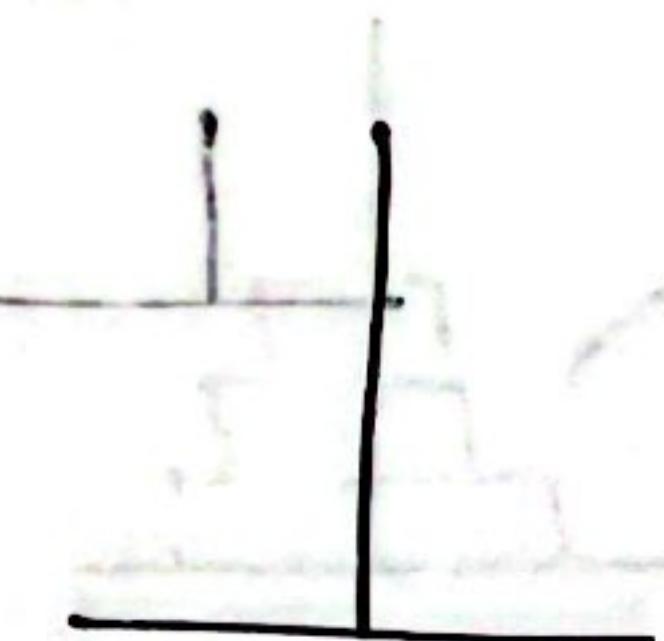
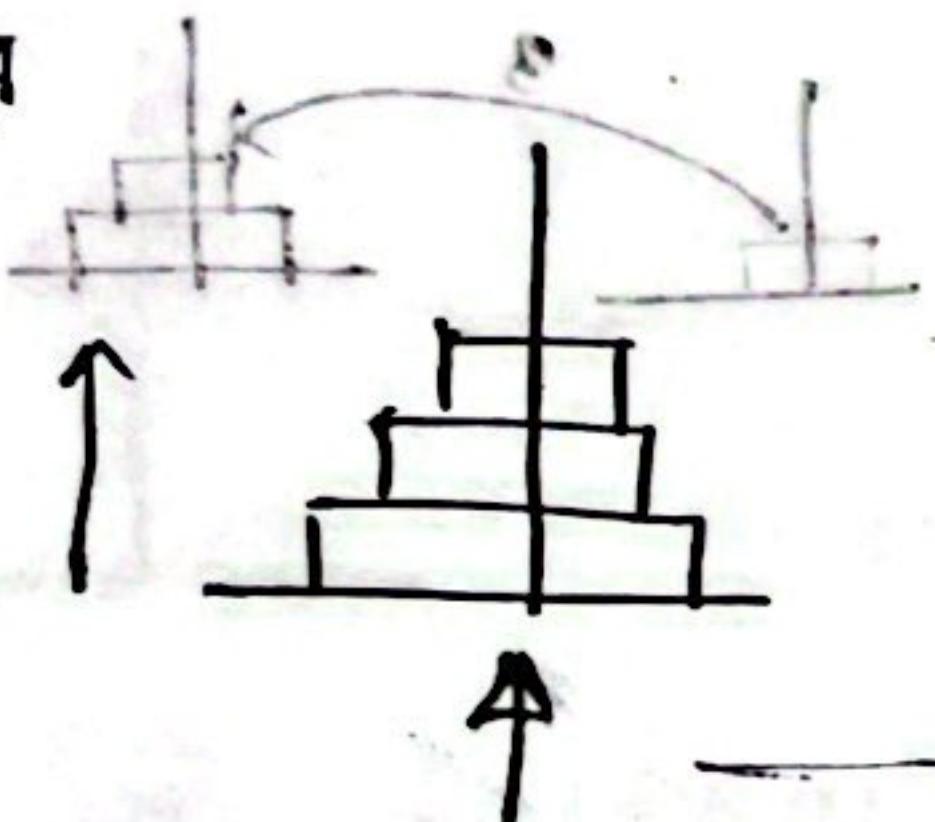
$$1 = 1^{\text{st}} \text{ unit} : 1 \text{ step}$$

$$F_n = F_{n-1} + F_{n-2}$$



Tower of Hanoi : Lucas, 'Tower of Brahma' का use করে
 build করে [disk = 8]

বড় disk এবং
 ছোট disk



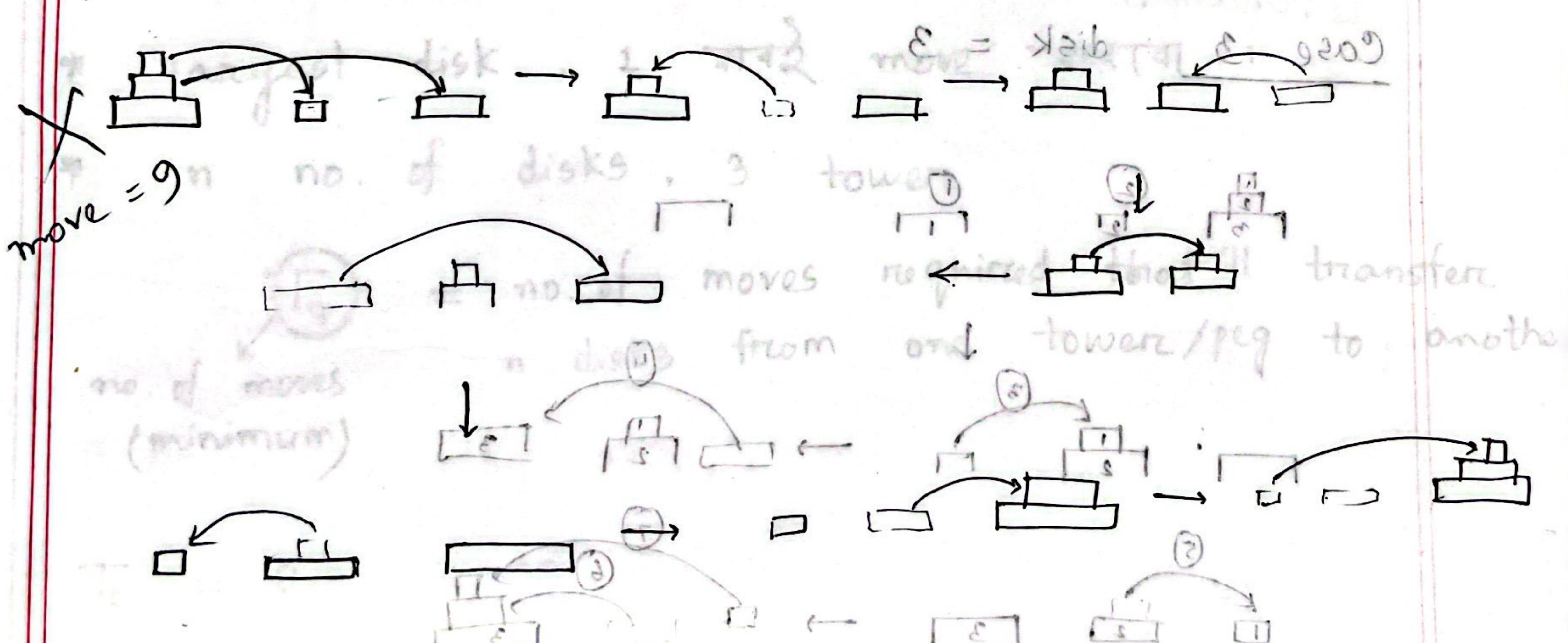
\rightarrow beginning move

a) at a time একটি

b) মাত্র উপর এক

বাধা যাবে না

move = 9



$$T_1 = 2T_2 + 1$$

$$\Rightarrow 2T_2 = T_1 - 1$$

$$\Rightarrow T_2 = \frac{T_1 - 1}{2}$$

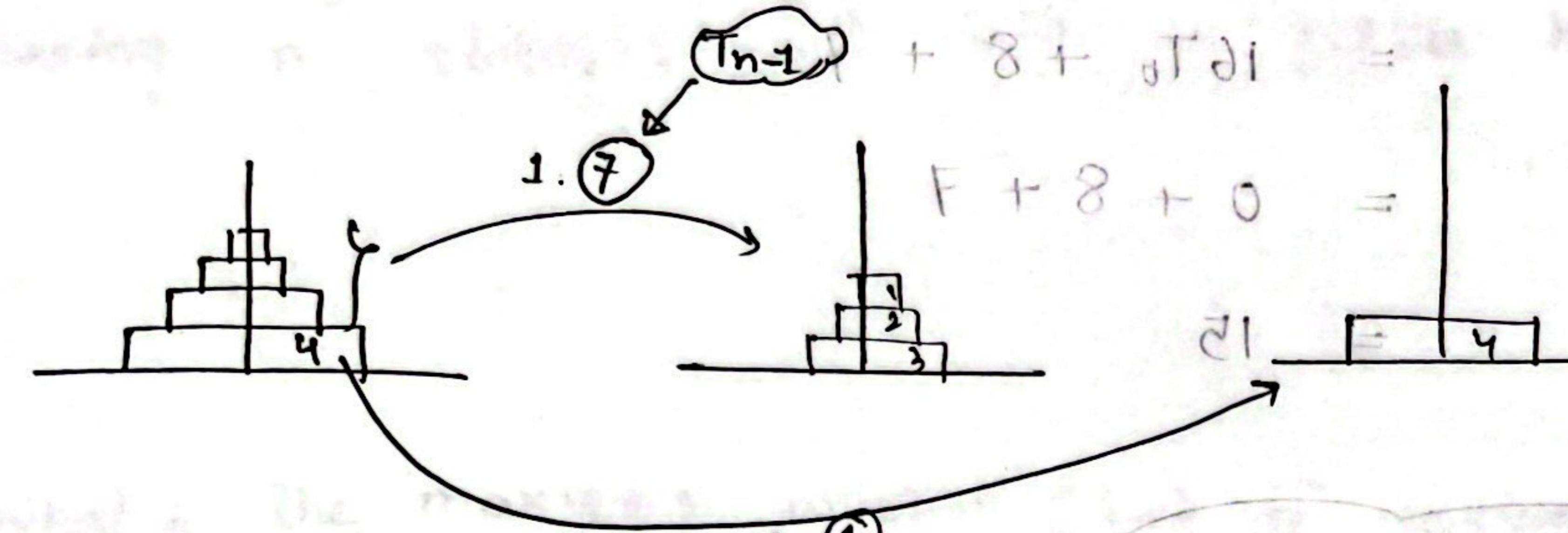
$$\Rightarrow 2T_2 = T_1 - 1 + 3$$

Case 4 : disk = 4

$$F + T_8 =$$

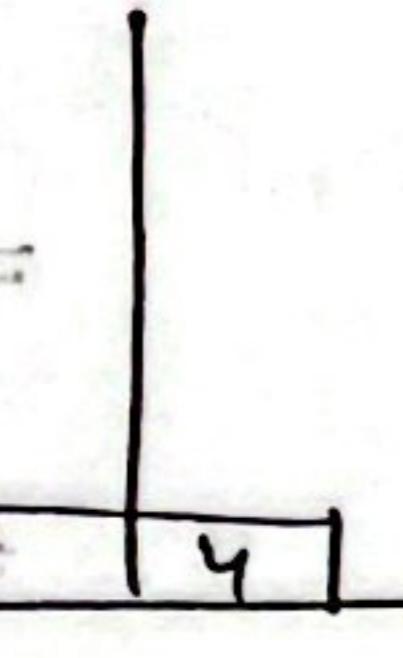
moves = 15

$$F + (1 + T_7)8$$



$$F + 8 + 0 =$$

21

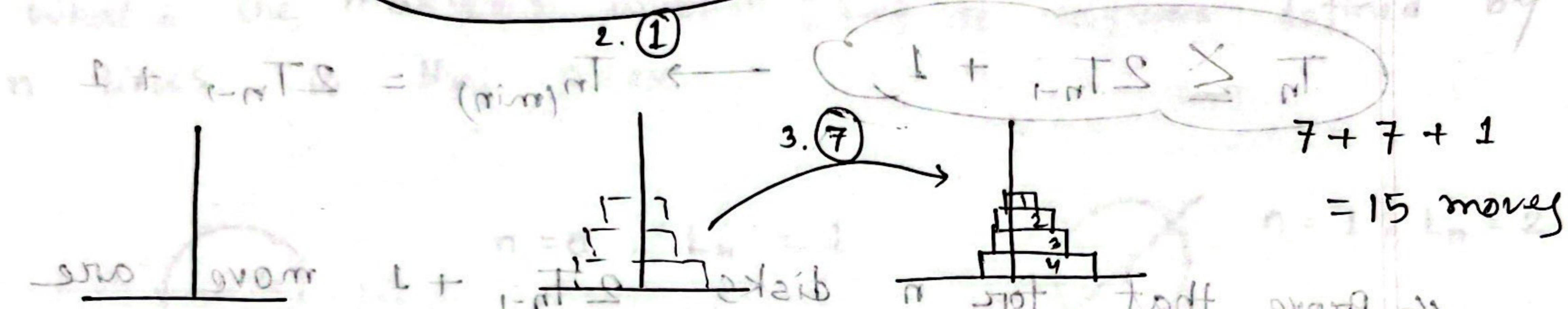


2. (1)

3. (7)

$$7 + 7 + 1$$

= 15 moves



* largest disk , 1 move

* n no. of disks , 3 tower

T_n
no. of moves
(minimum)

= no. of moves required that'll transfer
n disks from one tower/peg to another

$$T_0 = 0$$

$$T_4 = 7 + 1 + 7 = 15$$

$$T_n = 2T_{n-1} + 1$$

$$T_4 = 2 \times T_3 + 1$$

$$= 2 \times (2T_2 + 1) + 1$$

$$= 4T_2 + 3$$

$$= 4(2T_1 + 1) + 3$$

$$= 8T_1 + 7$$

$$\Delta = 42ib : \underline{P = 820}$$

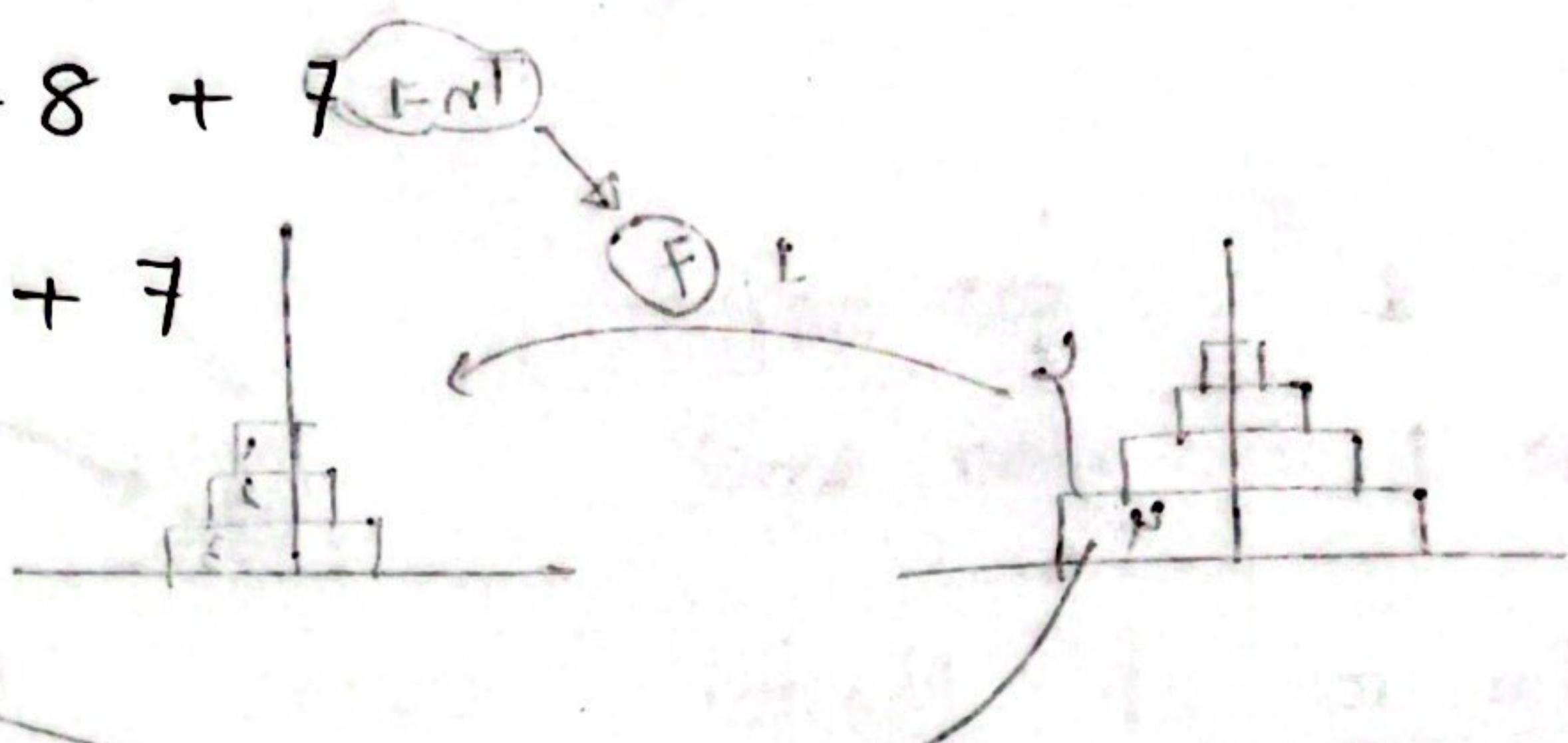
$$= 8(2T_0 + 1) + 7 \quad \text{EI} = 24000$$

$$= 16T_0 + 8 + 7$$

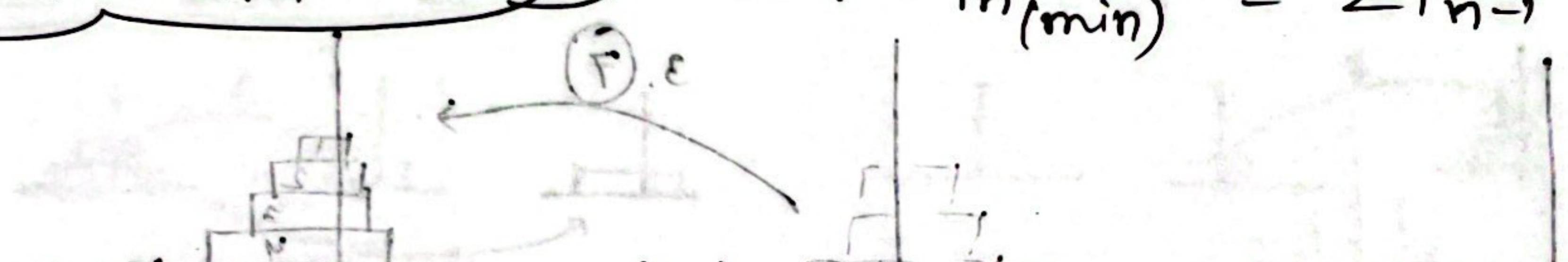
$$= 0 + 8 + 7$$

$$|P| = 15$$

$$F_{\text{ext}} = 15$$



$$T_n \geq 2T_{n-1} + 1 \rightarrow T_{n(\min)} = 2T_{n-1} + 1$$



Prove that, for n disks $2T_{n-1} + 1$ moves are sufficient and necessary for making to the destination peg.

From step 1, $42ib$ transfer

now ϵ , $42ib$ to on n

$\frac{1}{2}T$
zero to on
(minimum)



$$0 = \frac{1}{2}T$$

$$\text{EI} = F + L + F = \frac{1}{2}T$$

$$1 + \epsilon^2 \times \frac{L}{2} = \frac{1}{2}T$$

$$1 + (1 + \frac{1}{2}T) \times \frac{L}{2} =$$

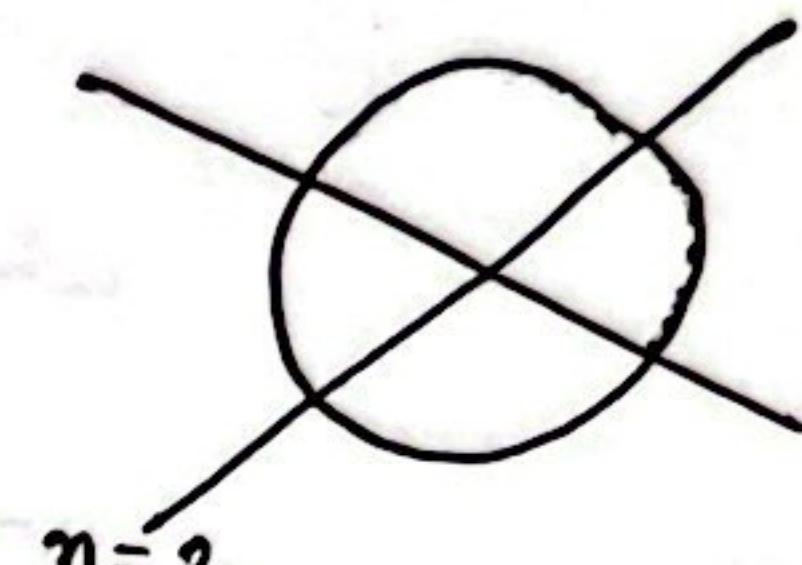
$$\epsilon + \frac{1}{2}T \frac{L}{2} =$$

$$1 + \frac{1}{2}T \frac{L}{2} = \frac{1}{2}T$$

Geometric problem:

How many slices of pizza can a person obtain by making n straight cut with a pizza knife?

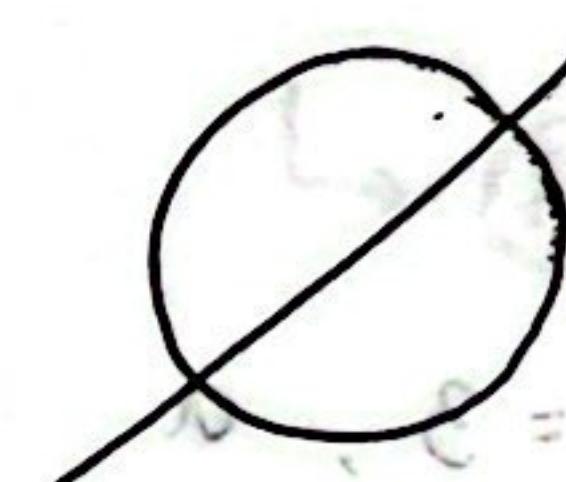
L_n = maximum regions by n straight lines



What's the maximum number (L_n) of regions defined by n lines in the plane

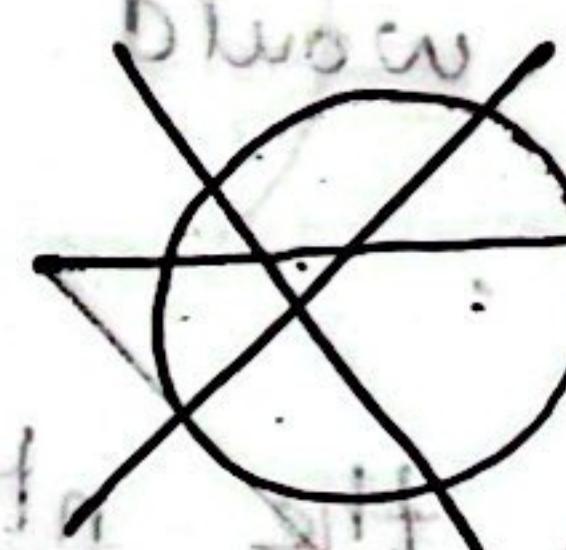
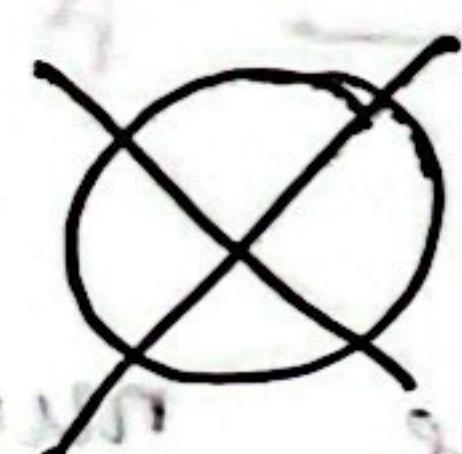
$$n=0, L_n = 1$$

$$n=1, L_n = 2$$

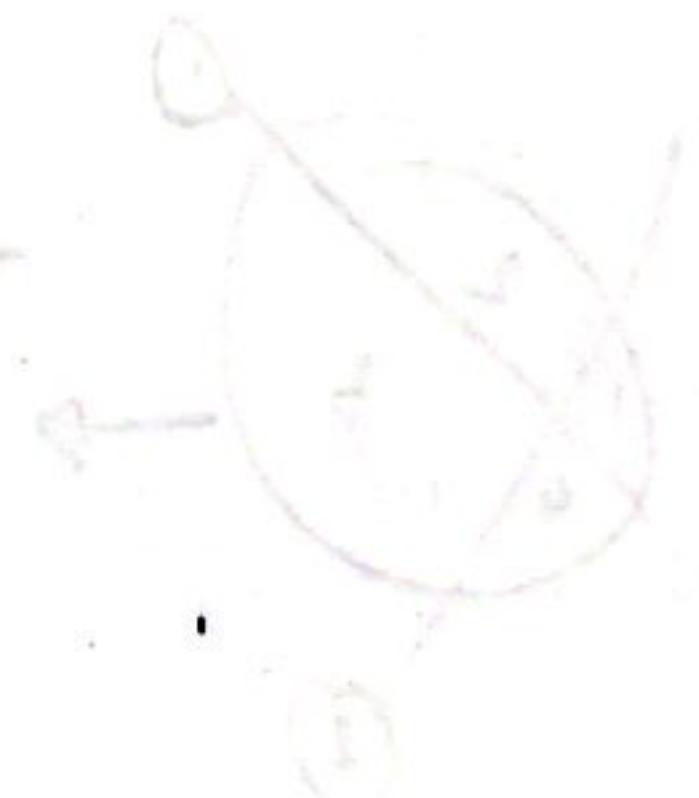
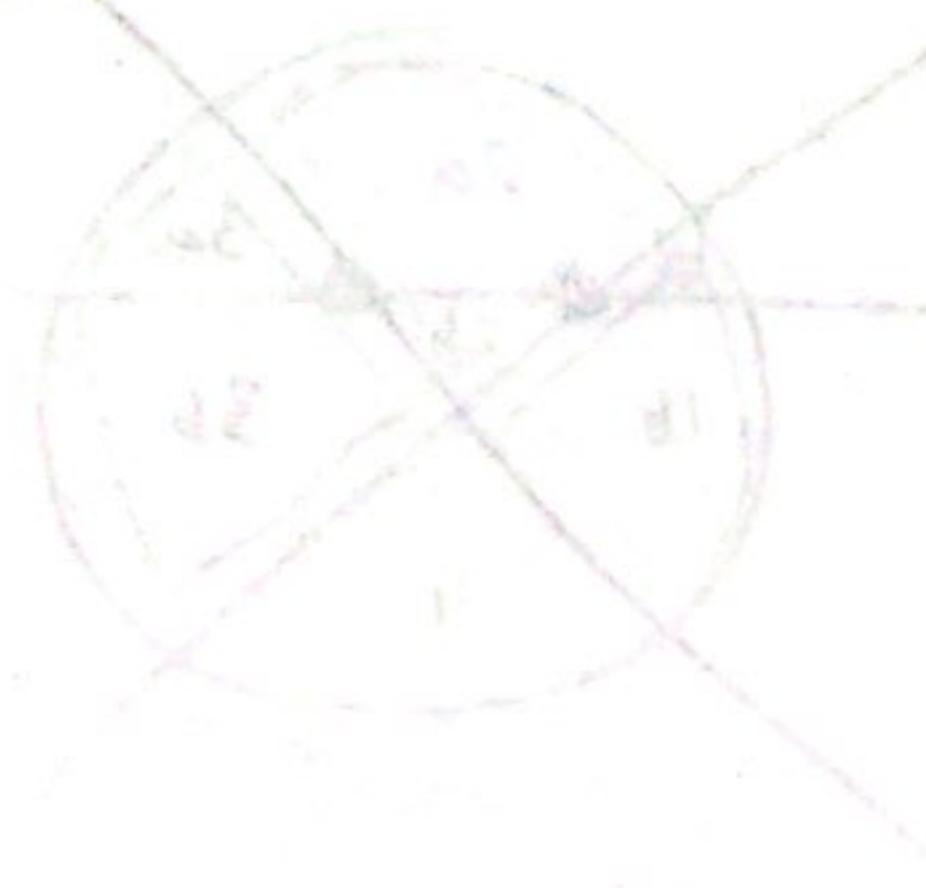
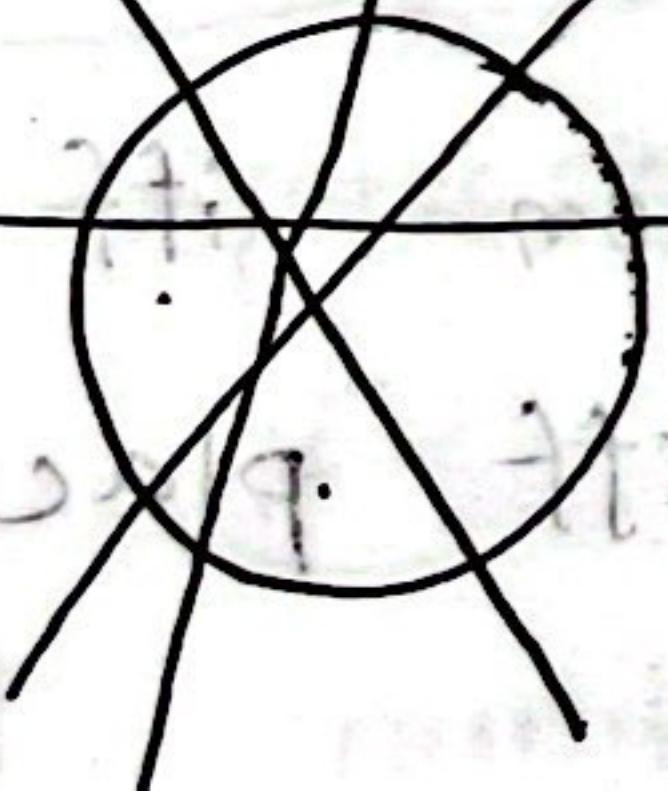


$$n=2, L_n = 4$$

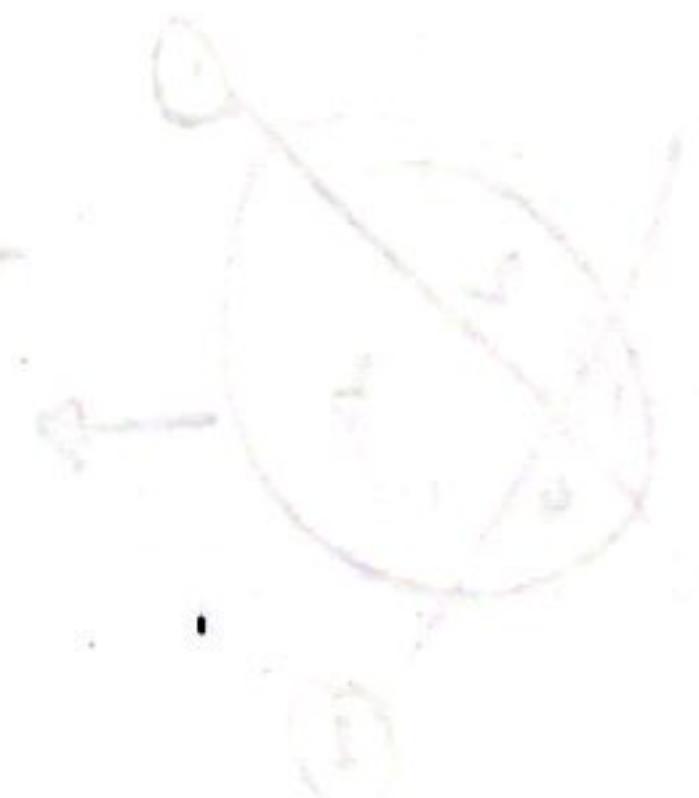
$$n=3, L_n = 7$$

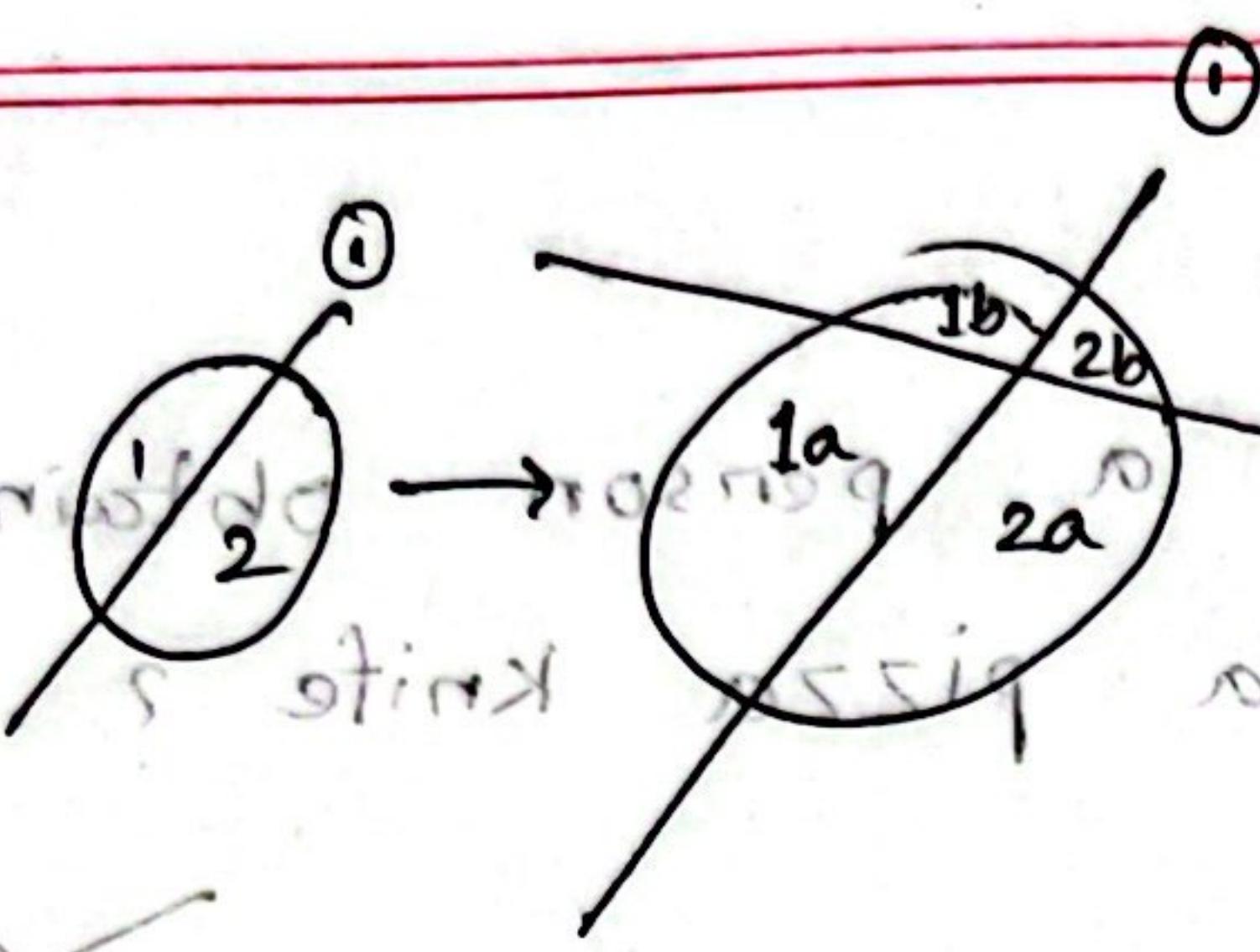


$$n=4, L_n = 11$$

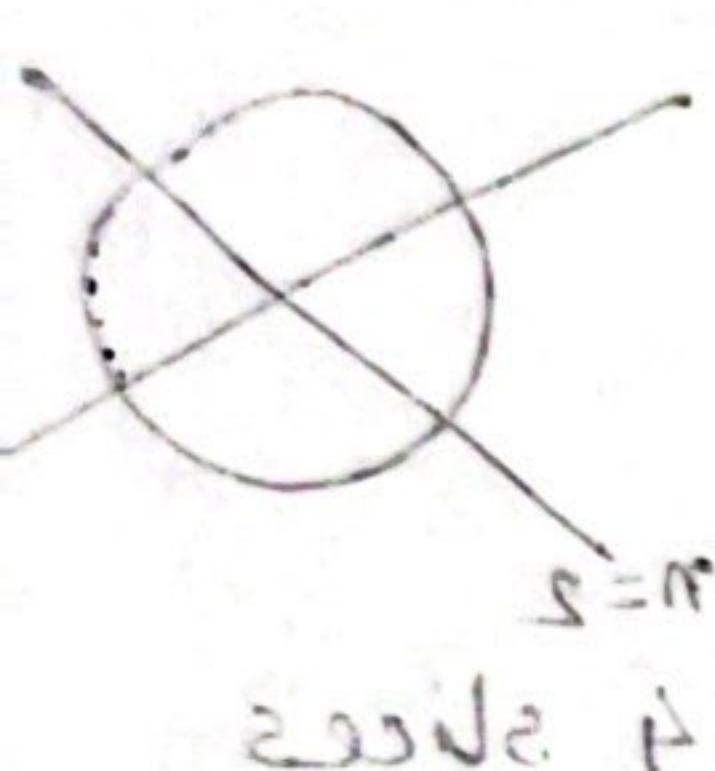


independently of the order of the lines
does not require a diagram
to prove this



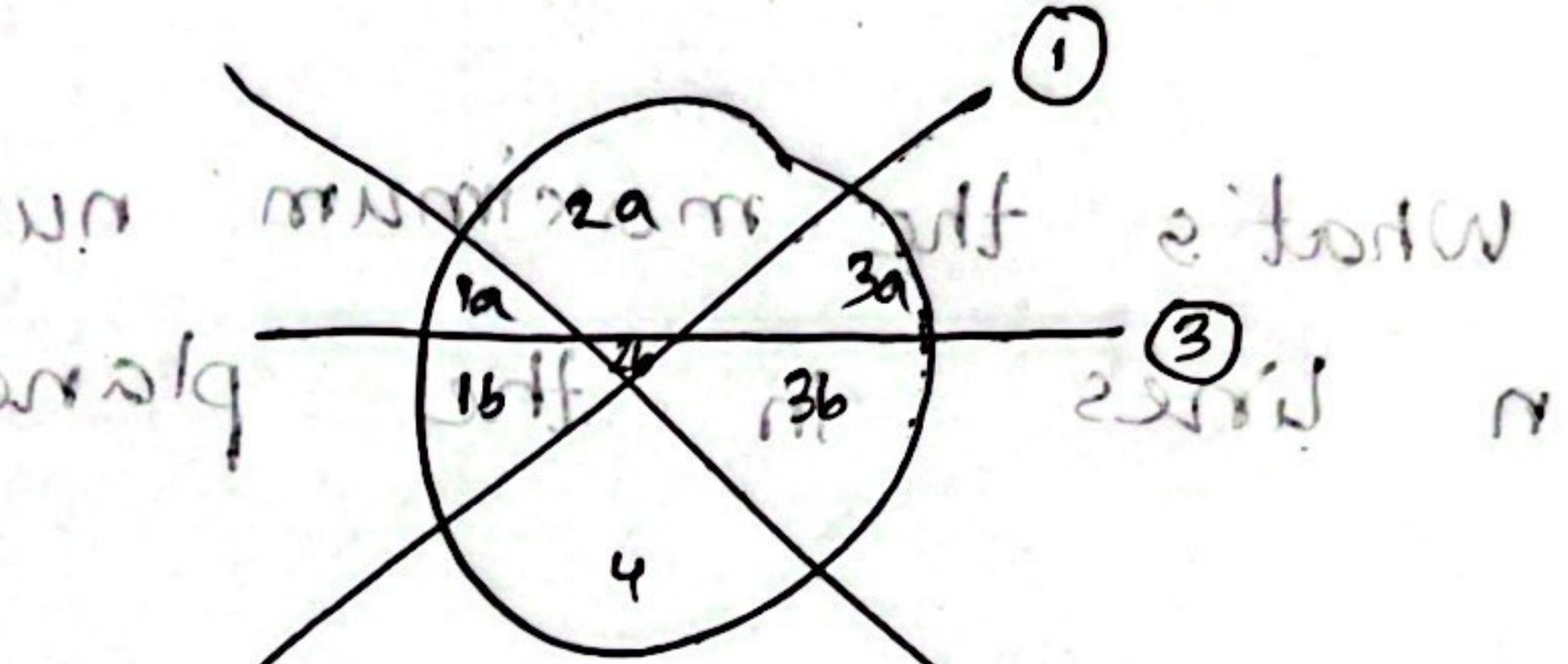


The diagram shows a circle divided into four quadrants by a horizontal and vertical axis. A large 'X' is drawn through the circle. The top-left quadrant contains the number '2'. The top-right quadrant contains the number '1'. The bottom-left quadrant contains the number '3'. The bottom-right quadrant contains the number '4'. The entire circle is enclosed in a dashed oval.



二九三

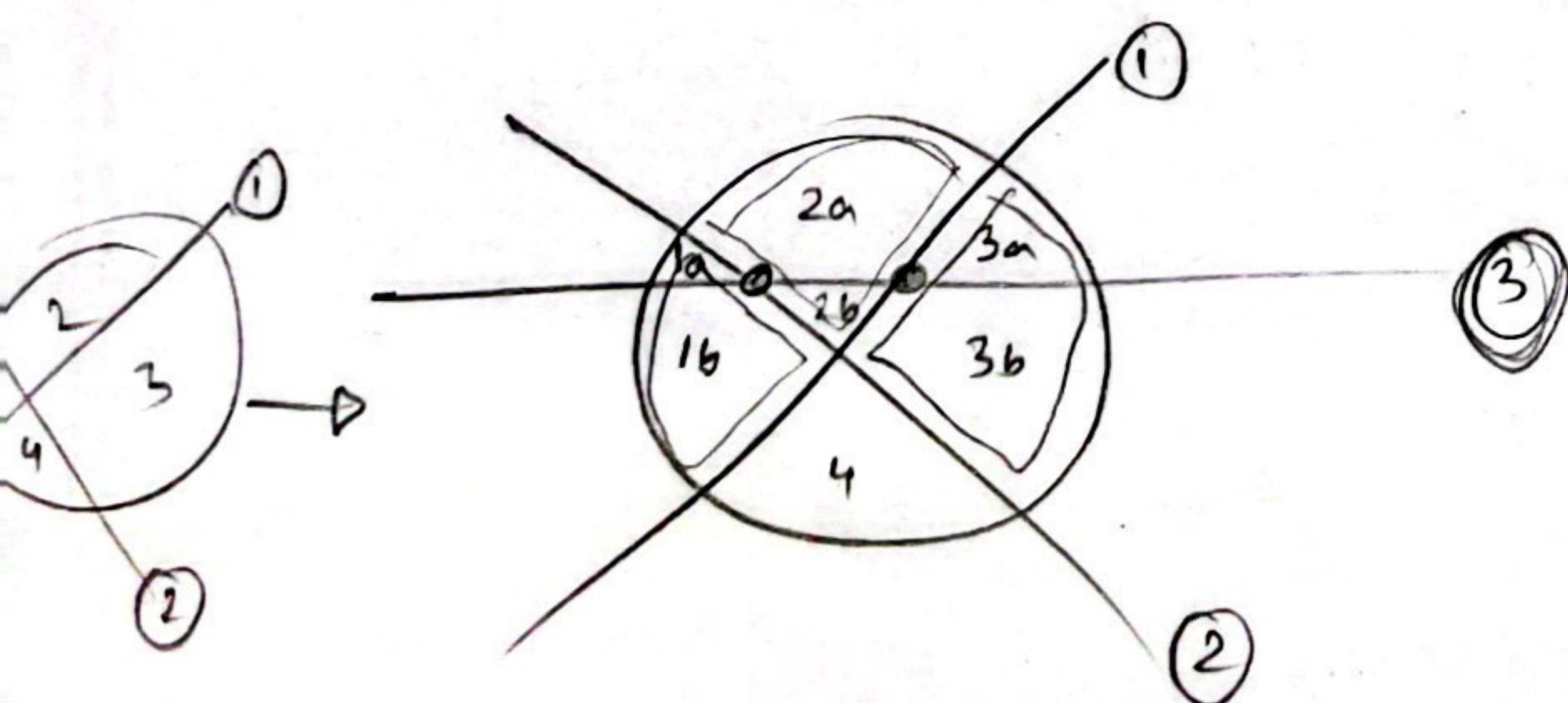
and benefits assigned to (n.)



1. For $n=3$, a line can split at most three regions
2. n -th line would split an old regions at most

2. n -th line would split an old region at most into $n+1$ pieces.

3. For $n > 0$, the n th line increases the number of regions by k iff it splits k of the old regions and it splits k old region iff it hits the previous lines in $(k-1)$ diff places.



3 BT region too
2 ♂ point 7 touch
কম্বেল স্প্লিট কম্বেল,

4. Two lines can intersect in at most one point.

Therefore the new line can intersect $(n-1)$ old lines in at most $(n-1)$ places.

* L_n = maximum regions by n straight line

Zigzag line

$$n=0 \rightarrow L_0 = 1$$

$$n=1 \rightarrow L_1 = 1 + 1 = 2$$

$$n=2 \rightarrow L_2 = 2 + 2 = 4$$

$$n=3 \rightarrow L_3 = 4 + 3 = 7$$

$$n=4 \rightarrow L_4 = 7 + 4 = 11$$

$$L_n = L_{n-1} + n$$

↳ Recurrence solⁿ

$$\rightarrow L_0 = 1$$

$$\rightarrow L_n = L_{n-1} + n$$

(for $n > 0$)

Closed Form :

An expression for a quantity $f(n)$ is in closed form if we can compute it using at most a fixed number of well-known standard operations, independent of n .

log erro L_n from L_{n-1} ~~base case~~ mod 2 will give A

$$\begin{aligned}
 (i-n) \text{ base case} &= L_{n-1} + n \\
 &= L_{n-2} + (n-1) + n \quad \text{add } n \text{ to both sides} \\
 &\text{adding } (i-n) \text{ from to } n \text{ will give} \\
 &= L_{n-3} + (n-2) + (n-1) + n
 \end{aligned}$$

and similarly $= L_{n-4} + (n-3) + (n-2) + (n-1) + n = \dots$

When $n=4$, $L_4 = L_0 + 1 + 2 + 3 + 4$

For $i=n \rightarrow L_n = L_0 + [1+2+3+\dots+(n-3)+(n-2) + (n-1)+n]$

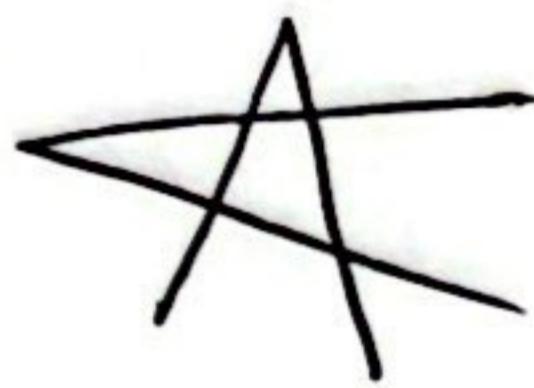
$\boxed{L_n = L_0 + \frac{n(n+1)}{2}}$

ask mi ei (n)? fitting or not missing ex. A
 com to prius fi strings mod sw fi most
 usage bumblebee want how to reduce box
 & to transcode

$$\frac{m}{c} \geq 1 \geq 0$$

Self. Study : Straight line

If line isn't straight, line is bend



bend line \rightarrow max no. of region determining.

Zigzag line



Z_2



if 'n' no. of \uparrow line is cut.

$$J(2n) = 2^n J(n) + 1$$

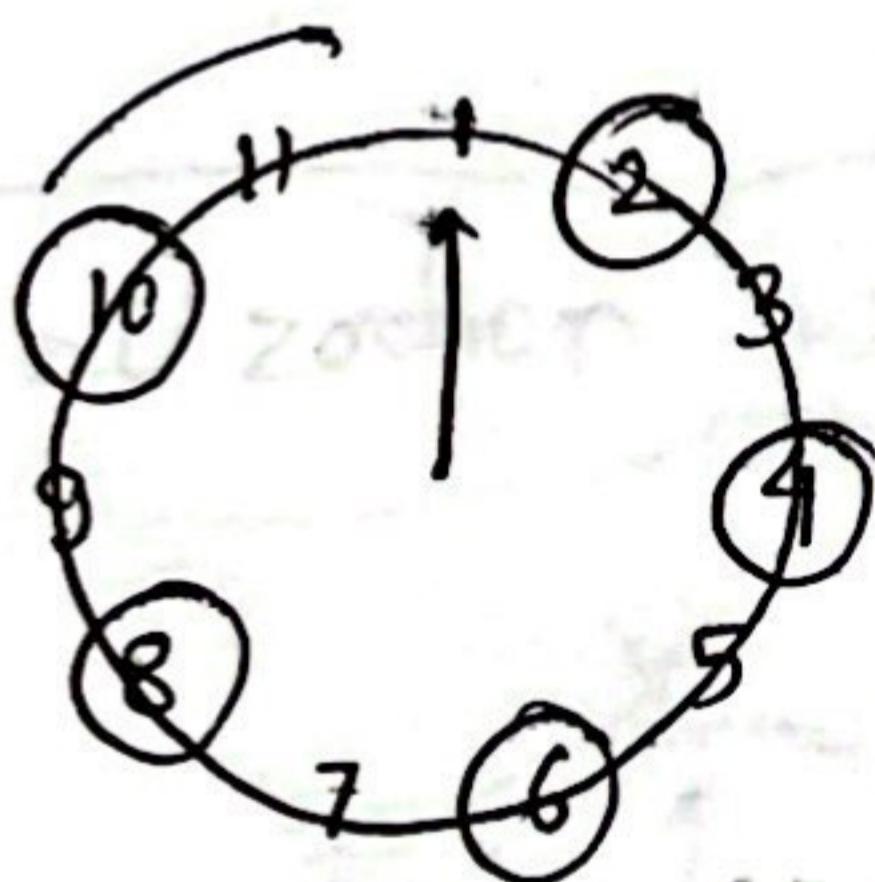
$$L = (S)t$$

$$J(n) = 2^n J(1) - 1 = 1$$

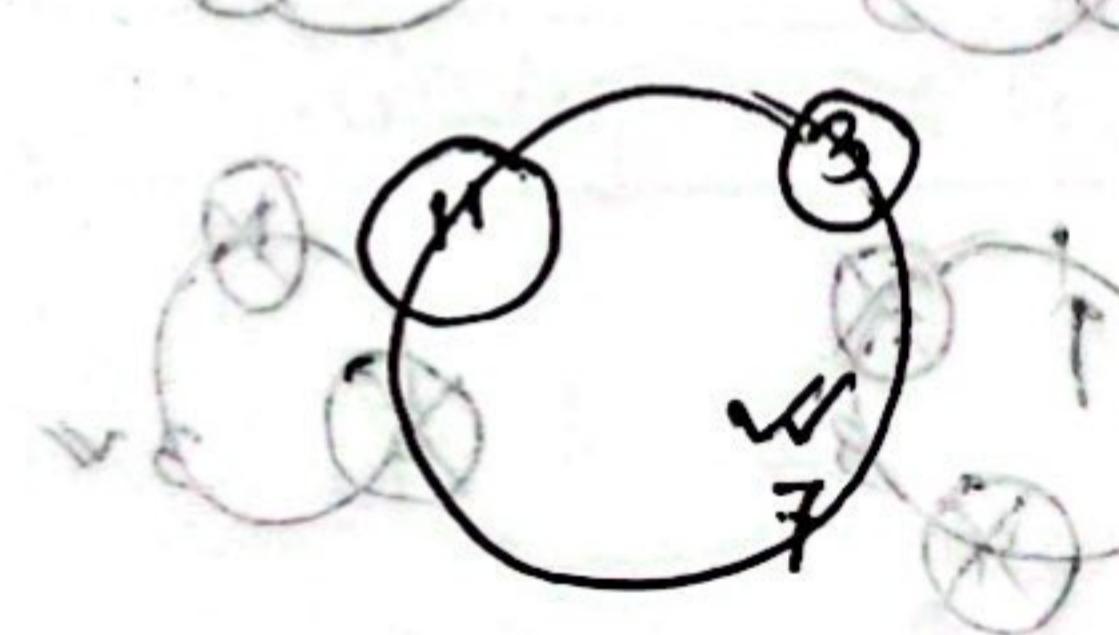
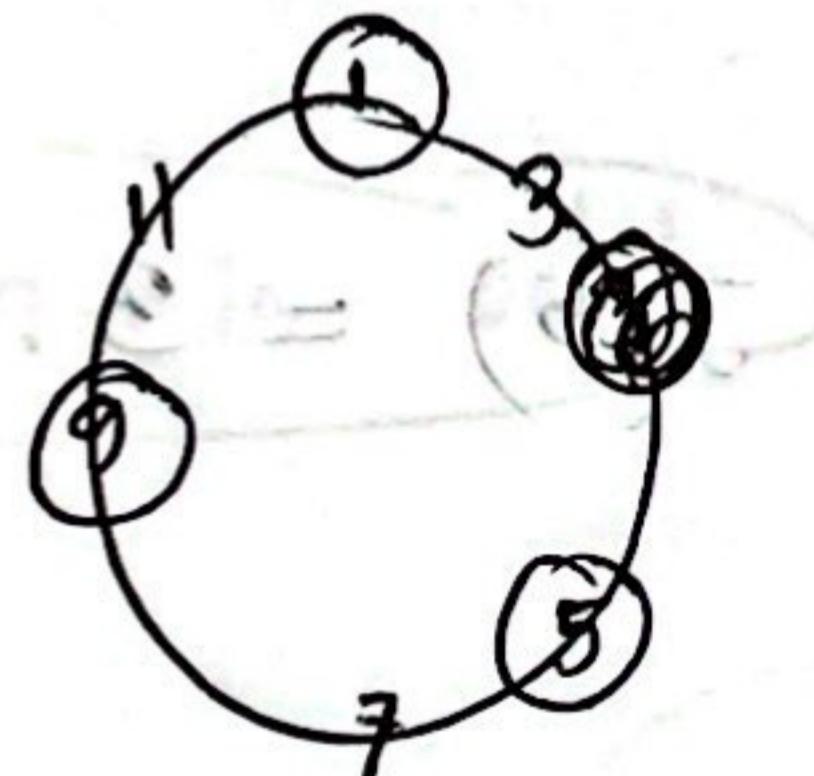
$$S = (E)t$$

$$J(20) = 2^{10} J(1) - 1 = 1023$$

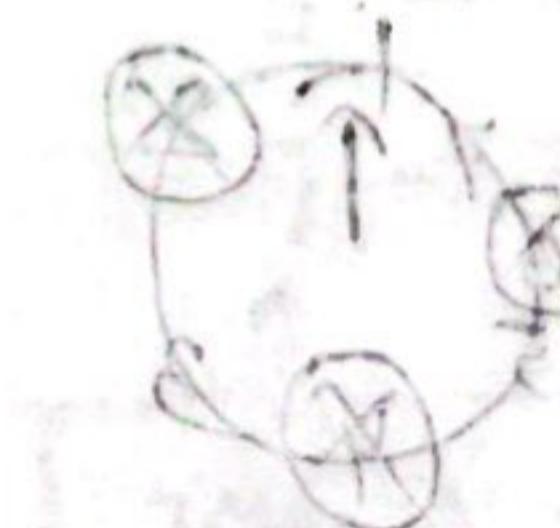
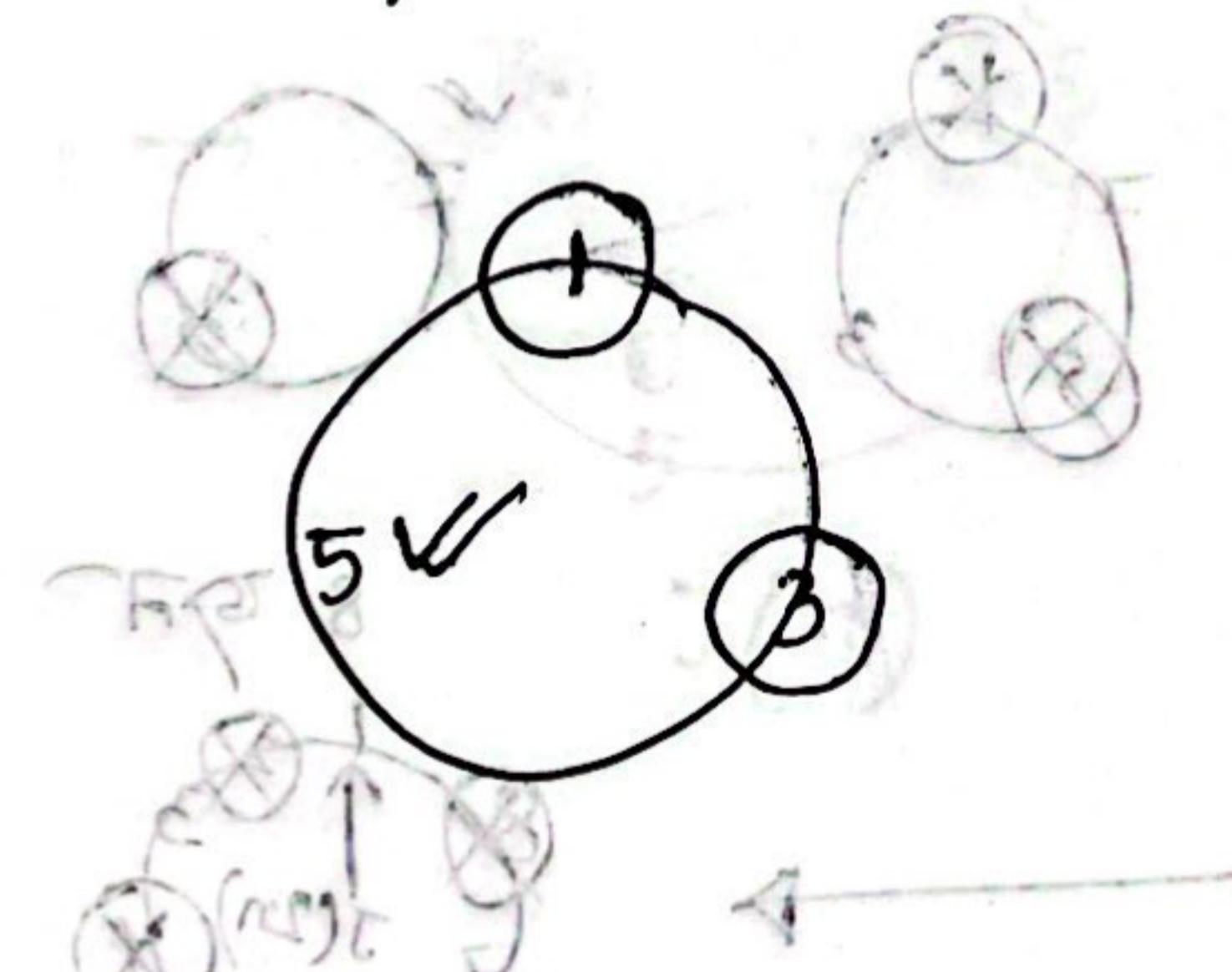
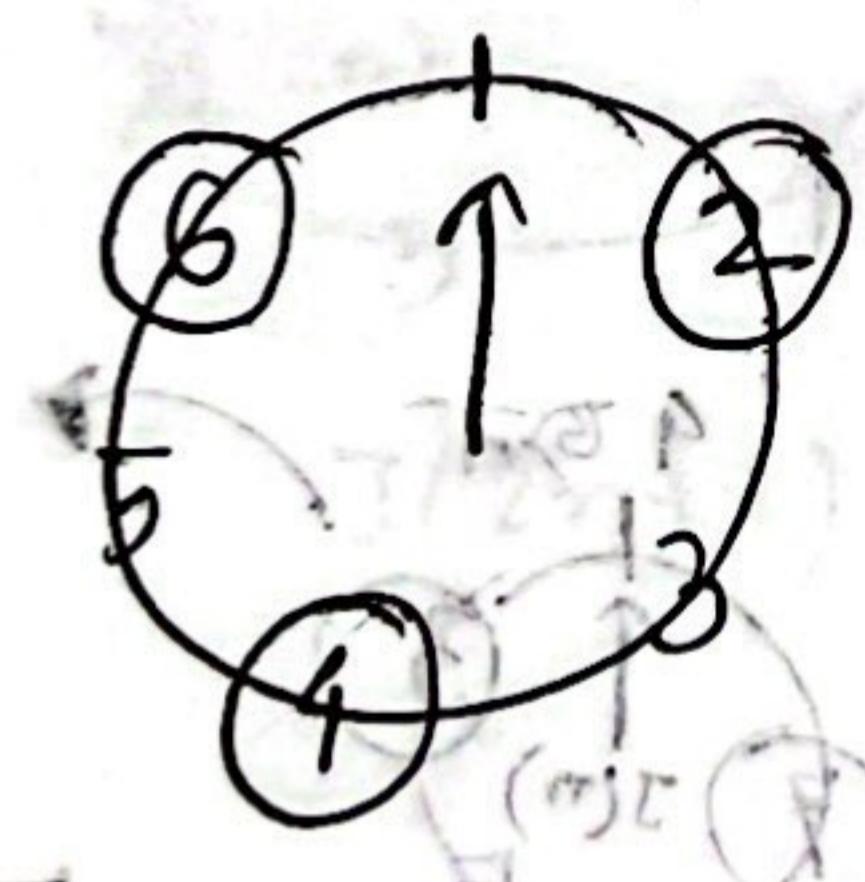
JOSEPHUS PROBLEM



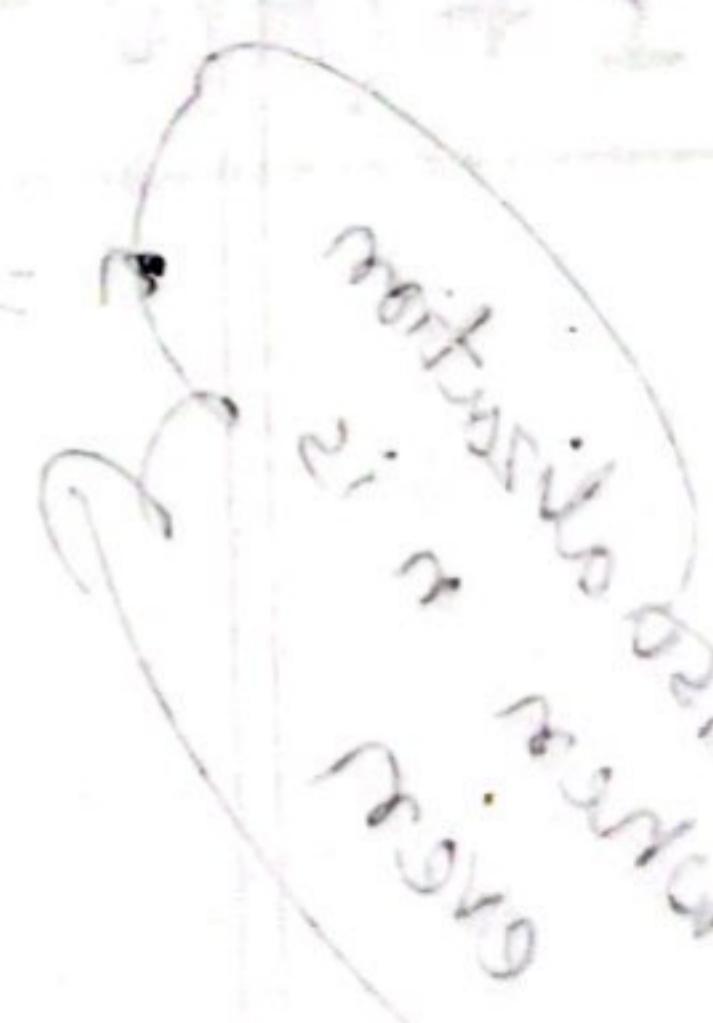
$$F = (F)t$$



$$Z = n$$



$$F = n$$



$$Z = n$$

$$\begin{aligned} F &= (F)t \\ S &= (E)t \\ E &= (S)t \\ P &= F \\ S = (nS)t \end{aligned}$$

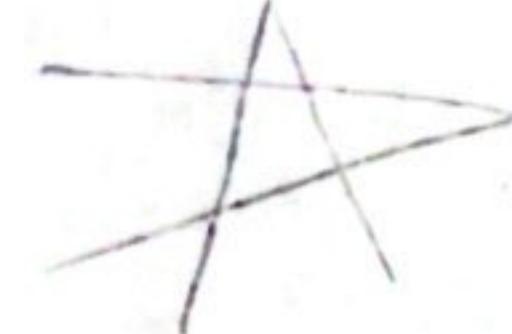
$$\begin{aligned} J(2n+1) &= 2^n J(n) + 1 \\ J(2n) &= 2^n J(n) \end{aligned}$$

$$Z = n$$

$$0 \leq \lambda \leq \frac{2}{3} - \frac{1}{n}$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5	7	9

To. out xam ← serial based
mengurut



$n=0$ x



$n=1$

1 survive जीवा → $J(1) = 1$

$n=2$



$$J(2) = 1$$

$n=3$



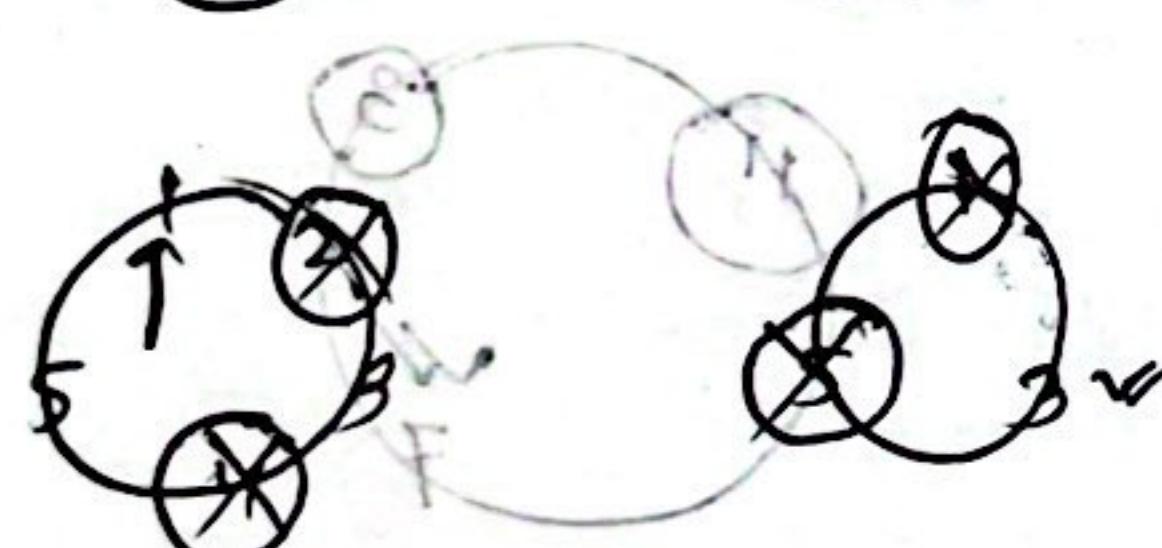
$$J(3) = 3$$

$n=4$



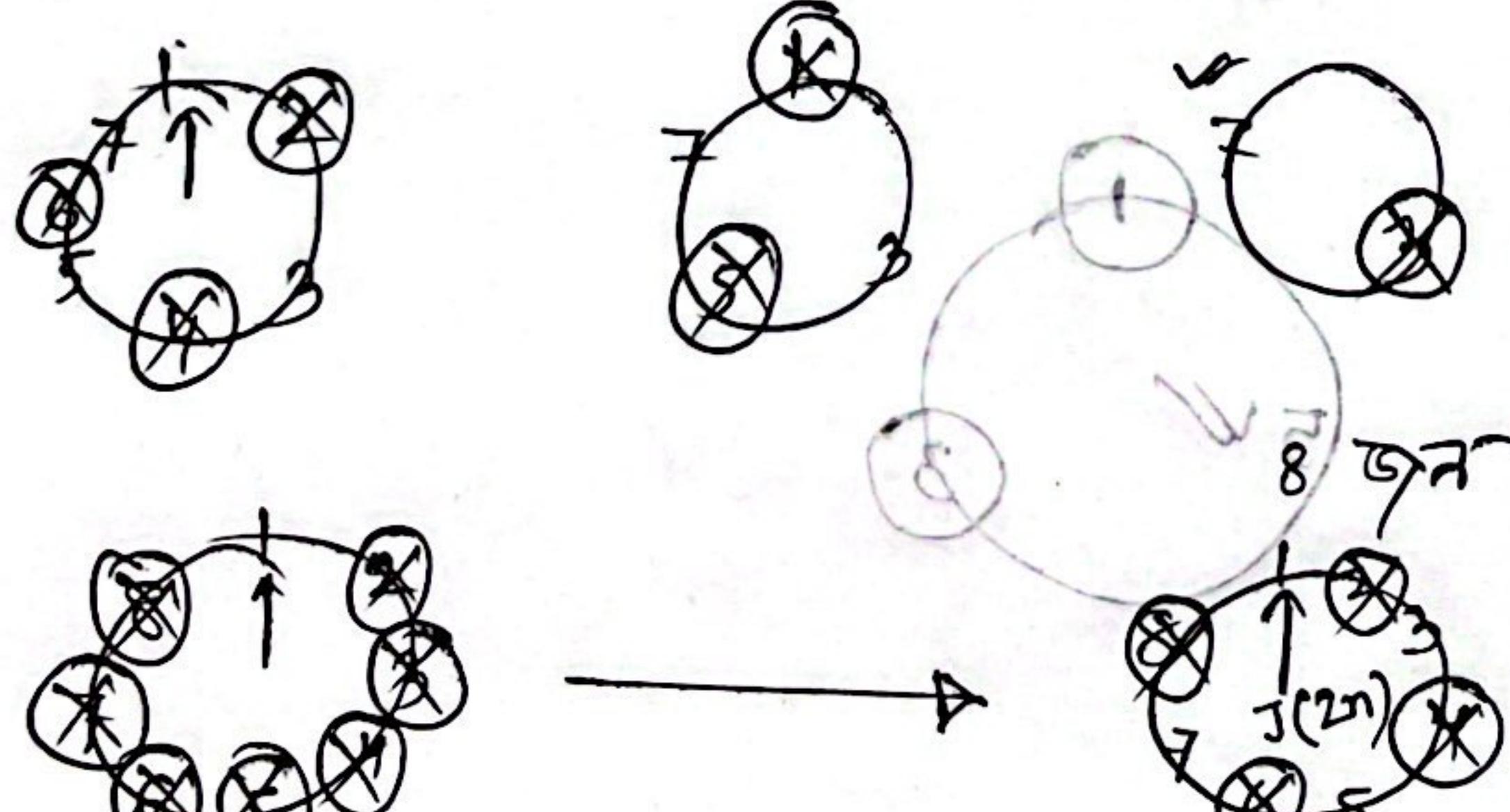
$$J(4) = 1$$

$n=5$



$$J(5) = 3$$

$n=7$



$$J(7) = 7$$

2nd round

$$1 \rightarrow 1 (2^{*1}-1)$$

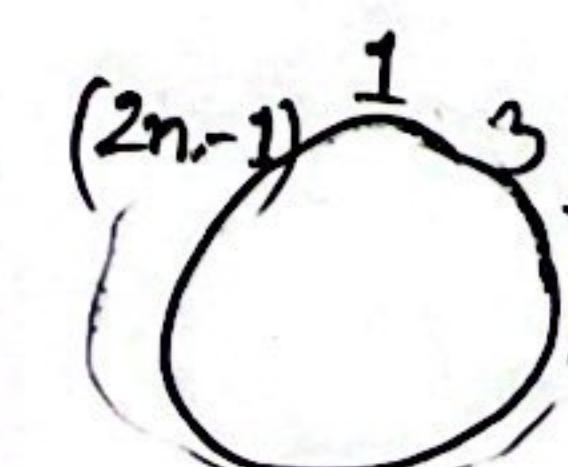
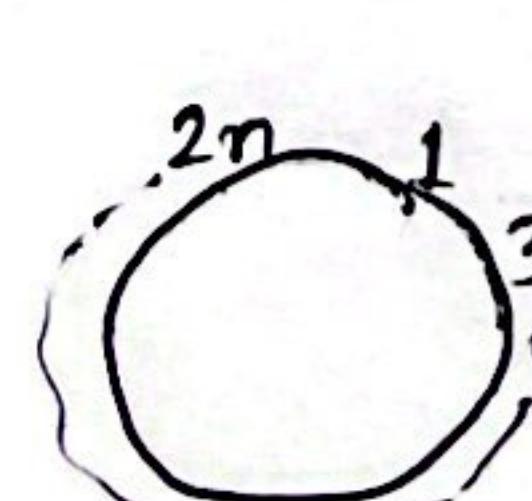
$$3 \rightarrow 2 (2^{*2}-1)$$

$$5 \rightarrow 3 (2^{*3}-1)$$

$$7 \rightarrow 1 (2^{*4}-1)$$

$$J(2n) = 2^n J(n) -$$

Visualization when n is even



Observations :

1. $J(n)$ is always odd. (at least for $n = 1, 2, 3, 4, 5, 6, 7, 11$)
2. Even positions are eliminated at first round.

Note: Every odd number has digits adding to

- $J(1) = 1$ (odd and even)

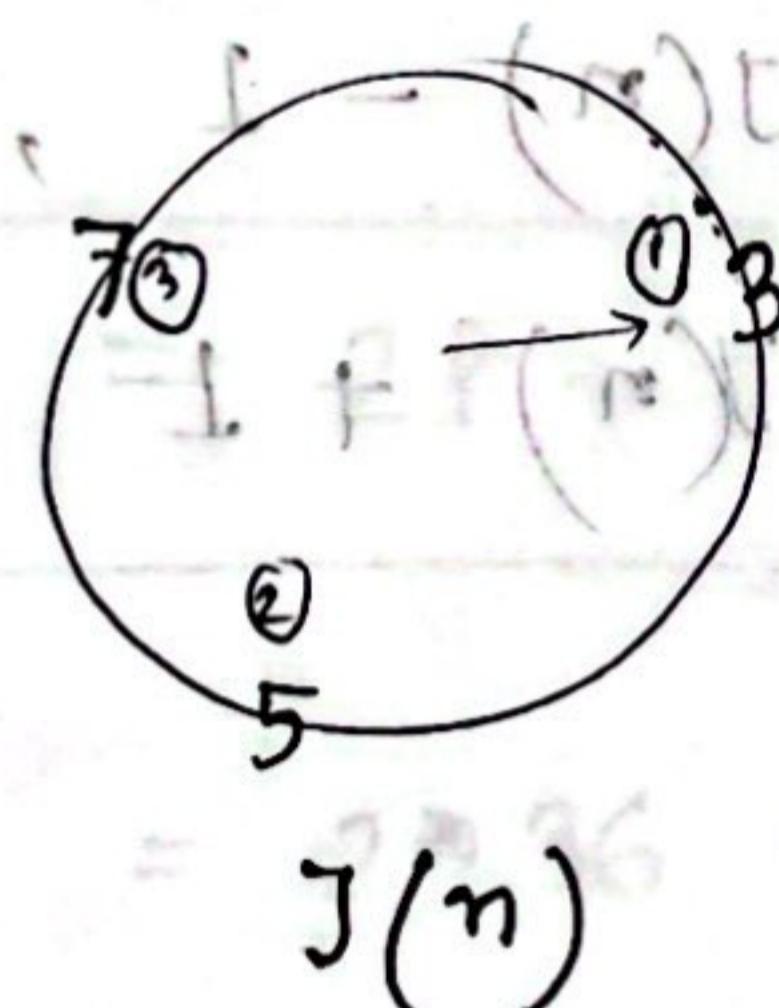
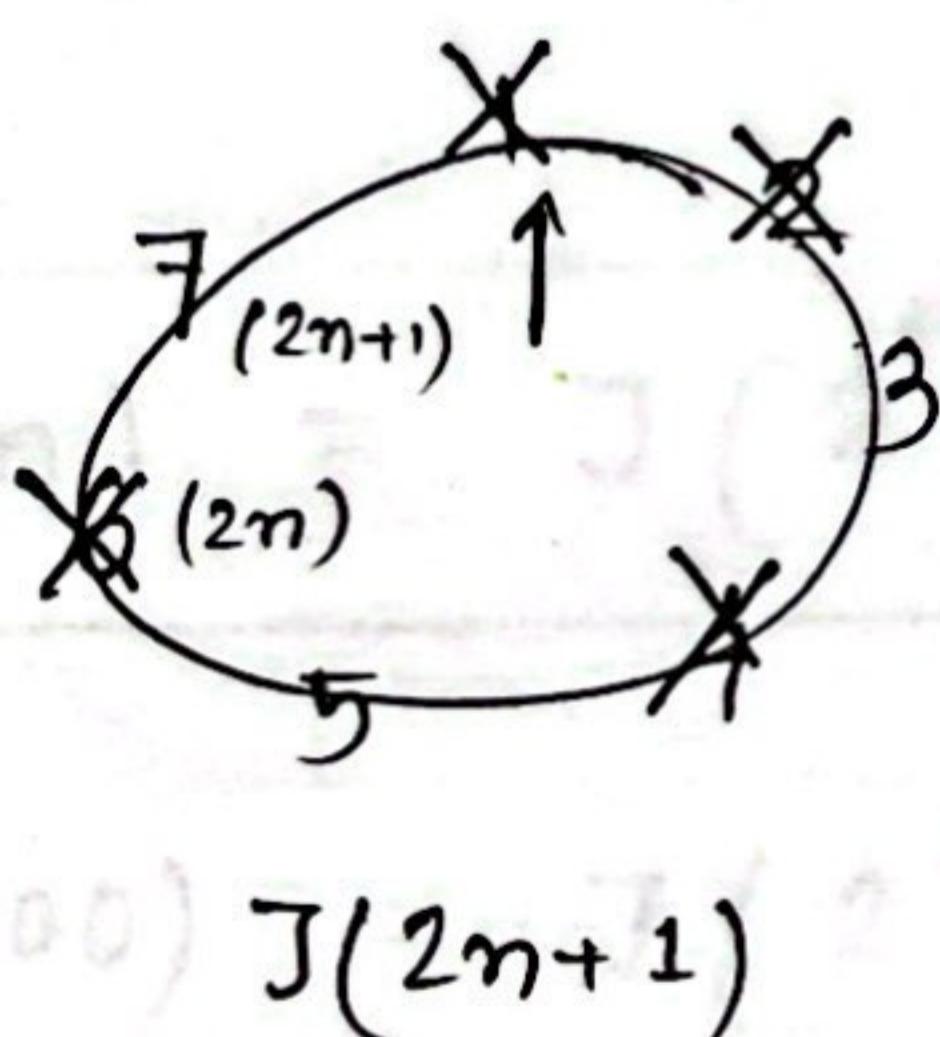
- $J(2n) = 2^* J(n) - 1$ (even)

$$\therefore J(10) = 2^* J(5) - 1 = 5$$

$$J(20) = 2^* J(10) - 1 = 2^* 5 - 1 = 9$$

$$J(40) = 2^* J(20) - 1 = 9^* 2 - 1 = 17$$

Visualization when n is odd



$$\begin{aligned} J(n) &= (1) \\ (2^* n + 1) &\leftarrow \text{ } \uparrow \\ (3 \rightarrow 1) &\leftarrow \text{ } \uparrow \\ 5 - 2 &\leftarrow \text{ } \uparrow \\ 7 - 3 &\leftarrow \text{ } \uparrow \\ \boxed{J(2n+1) = 2^* J(n) + 1} \end{aligned}$$

- $J(2n+1) = 2^* J(n) + 1$

$$J(18) = 2 J(9) + 1 = 2(2 J(4) + 1) + 1 = 2(2^* 2 + 1) + 1 = 5$$

3. For even no. of

F.O.P.E.S.C or not f2o (to). bbo apwlo si $(m)C$

Originally we have 2^n no. of people, after one round, it seems we have n no of people except each person's position number has been doubled and ~~decreased by 1.~~

$$1 + (m)C * S = (mS)C$$

4. For odd no, $2n+1$

$$n \overset{S}{\rightarrow} \underset{1 - (S)C * S}{\overbrace{\dots}} = (01)C$$

$$\underset{1 - (01)C * S}{\overbrace{\dots}} = (0S)C$$

and increased by 1.

$$F1 = 1 + S * C = 1 + (01)C * S = (0N)C$$

Recurrence problem:

$$(m)J(1) = 1$$

$$(1+mS)J(2n) = 2^* J(n) - 1, n \geq 1$$

$$J(2n+1) = 2^* J(n) + 1 \quad n \geq 1$$

$$+ (m)C * S = (1+mS)C$$

bbo si m molar ratios iloveiv

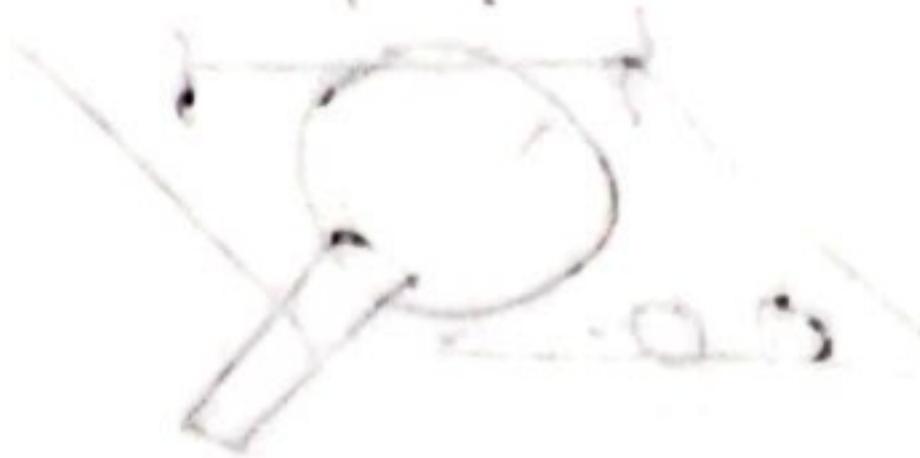
$$(m)C$$

$$(1+mS)C$$

$$1 + (m)C * S = (1+mS)C$$

$$1 + (1+mS)C = 1 + (1+(1)C S)C = 1 + (0)C S = (81)C$$

closed form :



1. When $(n+1)$ is power of 2, $J(n)$ is 1 and among

2. When $J(n)$ is 1, later $J(n+1)$, $J(n+2)$ are of consecutive odd numbers.

$$J(n) = 1 \quad \text{if } n = 2^m \text{ where } l + n = 2^m \text{ and } J(n+1), J(n+2) \text{ are of consecutive odd numbers.}$$

$$J(2^m) = 1 \quad \text{here } l + n = 2^m \quad (2 \times 0 + 1) \text{ if}$$

$$J(2^m + 1) = 3 \quad \text{if } l + n = 2^m + 1 \quad (2 \times 1 + 1)$$

$$J(2^m + 3) = 5 \quad \text{if } l + n = 2^m + 3 \quad (2 \times 2 + 1)$$

$$J(2^m + 5) = 7 \quad \text{if } l + n = 2^m + 5 \quad (2 \times 3 + 1)$$

$$\text{when, } n = 2^m + l \quad (l + n = 2^m + l)$$

$$J(n) = J(2^m + l) = 2 \times l + 1$$

$$J(n) = J(2^m + 1) = 2l + 1$$

here, $2^m \leq n < 2^{m+1}$
 $0 \leq l \leq 2^m - 1$

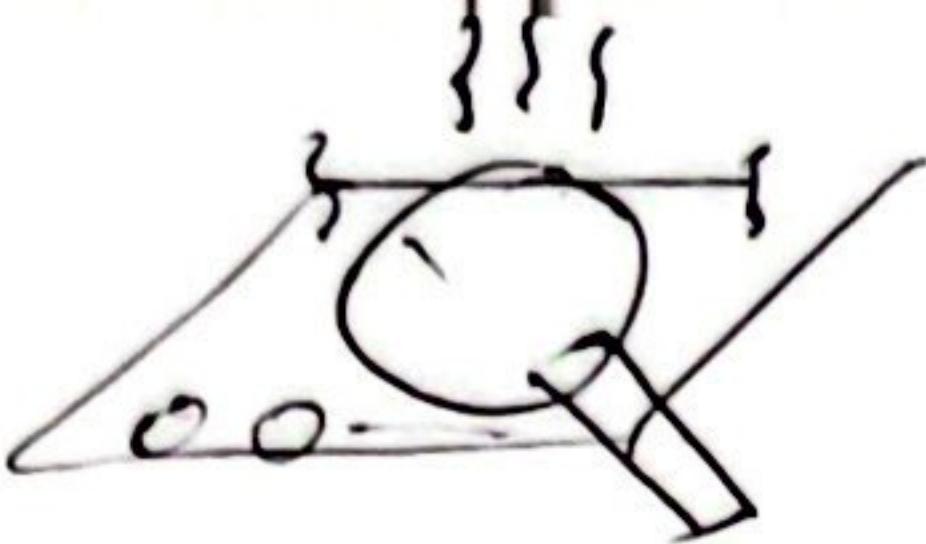
$$J(100) = J(2^6 + 36) = 2 \times 36 + 1 = 73$$

$$(l + n) \in (l + 2^m) \in$$

$$l + (r) \in$$

$$l + \left(l + \frac{1-2^{-m}}{2} \right) \in$$

$$l + \left[l + \frac{1-2^{-m}}{2} \right] \in$$



Using the recurrence solⁿ, [method below]
prove by induction s that $J(2^m + l) \equiv 2l + 1 \pmod{2}$.

Basis: $m=0, l=0$ $\Rightarrow J(2^0 + 0) = J(1) = 1$ $\pmod{2}$

$$J(2^0 + 0) = J(1) = 1 \quad \text{mod } 2$$

Induction: l is even ~~odd~~ $\Rightarrow J(l) = (l) \Gamma$

If $(1+m) \geq 0$ and $2^m + l \leq 2n$, $J = (2^m) \Gamma$

$$J(2n) = 2 J(n) - 1 \quad \begin{matrix} (1+1 \times S) \\ (1+S \times S) \end{matrix} \quad \begin{matrix} S = (1+^m S) \Gamma \\ C = (S+^m S) \Gamma \end{matrix}$$

$$J(2^m + l) = (1+2 \times J\left(\frac{2^m}{2} + \frac{l}{2}\right)) - 1 \quad F = (C+^m S) \Gamma$$

$$= 2 J\left(2^{m-1} + \frac{l}{2}\right) - 1 \quad k + S = n \pmod{2}$$

$$= 2 \left(2 \cdot \frac{l}{2} + (l) + \frac{m-1}{2}\right) - 1 \Gamma = (n) \Gamma$$

$$= 2l + 2 - 1$$

$$= \boxed{2l + 1} \quad \begin{matrix} 1+2S = (1+^m S) \Gamma = (n) \Gamma \\ 2l + 1 \end{matrix}$$

when l is odd, $2^m + l = 2n + 1 \rightarrow n = \frac{2^m + l - 1}{2}$

$$J(2^m + l) = J(2n + 1)$$

$$= 2 J(n) + 1$$

$$= 2 J\left(\frac{2^m + l - 1}{2}\right) + 1$$

$$= 2 \left[2 \cdot \frac{l-1}{2} + 1\right] + 1 = 2l + 1$$

The Josephus problem $\rightarrow J(n) = J(2^m + l) (=) 2l + 1$

Examining the closed form:

e_1	e_2	e_3	e_4	e_5
e_1	e_2	e_3	e_4	e_5

power of 2^m played an important role.

if n can be represented in binary $\rightarrow n \& J(n)$

$$\underline{n \text{ is binary?}} \quad n = 2^m + l$$

$$n = (b_m b_{m-1} \dots b_1 b_0)_2$$

$$\hookrightarrow \text{each } b_i = \begin{cases} 0 \\ 1 \end{cases}$$

$$(((e_1) \cup) \cup) \cup$$

$$1011 = e_1$$

$$\text{leading bit, } b_m = 1, \text{ so, } n = (1 b_{m-1} b_{m-2} \dots b_1 b_0)_2$$

$$l = (0 b_{m-1} b_{m-2} \dots b_1 b_0)_2$$

2 → 10
↓ double
4 → 100.
↓ double
8 → 1000

$$\text{double } \hookrightarrow 2l = (b_{m-1} b_{m-2} \dots b_1 b_0 0)_2 \quad (\text{now } (n) \cup \text{ needed})$$

$$\text{adding 1} \hookrightarrow 2l + 1 = (b_{m-1} b_{m-2} \dots b_1 b_0 1)_2$$

$$\therefore J(n) = (b_{m-1} b_{m-2} \dots b_1 b_0 b_m)_2$$

$$J((b_m b_{m-1} \dots b_1 b_0)_2) = (b_{m-1} b_{m-2} \dots b_1 b_0 b_m)_2$$

↳ 1 bit cyclic left shift

$$\text{If, } n = 10, \text{ it is equal to } F = (((((e_1) \cup) \cup) \cup) \cup) \cup : 10 \text{ to } m = 3$$

$$J((1010)_2) = 0101_2 = 5$$

$$\text{If, } n = 100,$$

$$J((1100100)_2) = 1001001_2 = 73$$

$$J(n) \leftarrow J\left(\frac{n}{2}\right)$$

n	1	3	7	15
$J(n)$	1	3	7	15

cannot divide 10 with 5
also known as binary representation

(n) $\leftarrow n \leftarrow$ process in descending order of n if

- repeatedly Josephus problem apply:

$$J(J(J(13)))$$

$$s(\text{odd}, \text{odd}, \dots, \text{odd}, \text{odd}) = n$$

$$13 = 1101_2$$

$$J(1101_2) = 1011_2 \stackrel{\text{odd}}{=} 11$$

$$J(1011_2) = 0111_2 = 7 \quad \begin{array}{l} \text{fixed point} \\ \text{একটি সময়ে } \end{array}$$

$$J(0111_2) = 1111_2 \stackrel{\text{odd}}{=} 7 \quad \begin{array}{l} \text{এখন যাব} \\ s(\text{odd}, \text{odd}, \dots, \text{odd}, \text{odd}) = 7 \end{array}$$

when $J(n)$ will be n ?

↪ binary representation \rightarrow এখন কোনো bit

$$\begin{array}{ccccccccc} & & & & & & & & \\ \cdot & \overbrace{1} & 1 & 1 & 1 & 1 & 1 & 1 & \\ & & & & & & & & \end{array} \stackrel{s(\text{odd}, \text{odd}, \dots, \text{odd}, \text{odd}) = (n))}{=} (n))$$

13 → binary representation \rightarrow 3 7 1 (1101)
 $s(\text{odd}, \text{odd}, \dots, \text{odd}, \text{odd}) = (s(\text{odd}, \text{odd}, \dots, \text{odd}, \text{odd}))$

$$\rightarrow \text{fix } 1 \rightarrow 2^3 - 1 = 7 \rightarrow 1$$

∴ $J(J(J(J(13)))) = 7$ (at least 2 বার ছান্দোলে fix হবে)
 \rightarrow repeat শব্দ = 3 no. করে ।
 $\rightarrow 1010 = ((0101))$

$$EF = 1001001 = (0010011)C$$

$$J(3(J(J(J(J(2)))))) \text{ that } J(n) \text{ is pyramid}$$

$$= 10101 \rightarrow \underbrace{3}_{(r)t} \underbrace{7}_{(l)t} \underbrace{1}_{(c)t} \xrightarrow{(l)t} 2^3 - 1 = 7$$

$$1 \rightarrow 01011$$

$$2 \rightarrow 0111 = 7$$

$$3 \rightarrow 111 = 7$$

$$4 \rightarrow 111 = 7 \text{ visiting nodes, in reading left to right}$$

$$J(3(J(J(J(J(J(10101110)))))))$$

$$\xrightarrow{(l)t} 2^6 - 1 = 63$$

$$1 \rightarrow 01011101$$

$$2 \rightarrow 01111011$$

$$3 \rightarrow 1110111$$

$$4 \rightarrow 1101111$$

$$5 \rightarrow 1011111$$

$$6 \rightarrow 111111$$

$$1 = (l)t$$

$$1 + (r)t = (rs)t$$

$$1 + (r)t + (rs)t = (1+rs)t$$

$$30 = (l)t$$

$$30 + (r)t + (rs)t = (rs)t$$

When $J(n)$ will be $\frac{n}{2} + 1$?

$$J(n) = 2l + 1 = \frac{2^m + l}{2} \text{ increasing by } 3 \text{ up from } n = 2^m + l \text{ fibonnacci}$$

$$\Rightarrow 4l + 2 = 2^m + 1$$

$$\Rightarrow 3l = 2^m - 2$$

$$\Rightarrow l = \frac{1}{3}(2^m - 2)$$

binary no. for half \rightarrow right shift

$$\begin{array}{l} n \\ (2) \ 10 \rightarrow 01(1) \\ (10) \ 1010 \rightarrow 0101(5) \end{array}$$

$$n \text{ मध्यन } \frac{170}{101010} \rightarrow J(n) = \frac{170}{2} = 85$$

* The pattern is consecutive '10' in binary representation.

Generalization:

$$J(1) = 1$$

$$J(2n) = 2J(n) - 1$$

$$J(2n+1) = 2J(n) + 1$$

constant values are represented like α, β, γ .

$$\therefore f(1) = \alpha$$

$$f(2n) = 2f(n) + \beta \quad (\text{for } n \geq 1)$$

$$f(2n+1) = 2f(n) + \gamma \quad (\text{for } n \geq 1)$$

According to the original eqns, $s + 1s = (n)5$

$$\alpha = 1$$

$$\beta = -1$$

$$\gamma = 1$$

$$\boxed{(s + 1s) \cdot \frac{1}{5} = 1}$$

Ergebnis: S mitforden

n	$f(n)$
1	α
2	$2\alpha + \beta$
3	$2\alpha + 3\beta + \gamma$
4	$4\alpha + 3\beta + 0 \cdot \gamma$
5	$4\alpha + 2\beta + \gamma$
6	$4\alpha + 1\beta + 2\gamma$
7	$4\alpha + 0\beta + 3\gamma$
8	$8\alpha + 7\beta + \gamma$
9	$8\alpha + 6\beta + 1\gamma$
10	$8\alpha + 5\beta + 2\gamma$

largest power of 2

decrease by 1

increase by 1

(besonders) mitforden

simples rechnen mitforden

$$\begin{aligned}
 f(2) &= 2f(1) + \beta \\
 &= 2\alpha + \beta \\
 f(3) &= 2f(1) + \gamma \\
 &= 2\alpha + \gamma \\
 f(4) &= 2f(2) + \beta \\
 &= 2(2\alpha + \beta) + \beta \\
 &= 4\alpha + 3\beta \\
 f(5) &= 2f(2) + \gamma \\
 &= 2(2\alpha + \beta) + \gamma \\
 &= 4\alpha + 2\beta + \gamma \\
 f(6) &= 2f(3) + \beta \\
 &= 2(2\alpha + \gamma) + \beta \\
 &= 4\alpha + \beta + 2\gamma \\
 f(7) &= 2f(3) + \gamma \\
 &= 2(2\alpha + \gamma) + \gamma \\
 &= 4\alpha + 3\gamma
 \end{aligned}$$

- ① α 's coefficient is n 's largest power of 2
- ② β 's " decreasing by $\frac{1}{2}$ down to 0
- ③ γ 's " increasing by 1 up from 0

$$f(n) = \underbrace{A(n)\alpha}_{2^m} + \underbrace{B(n)\beta}_{2^{m-1}} + \underbrace{C(n)\gamma}_l$$