Lec-1 Zakaria Sir

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* Fourier Analysis :

① Fourier series — ① 6C ↘
② Fourier integral — ① ⟶ 4C Compulsory (Application)
③ Fourier Transform — ① ↗ 8C

↓

* Signal Processing
* Image processing
* Thermal Analysis
* Control system

$\boxed{\text{Periodic Function (complicated)}}$

$\downarrow$ F.S.

$\boxed{\text{Simple Periodic function sine/cosine}}$

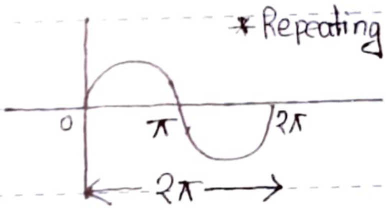
* Periodic Function :

The function $f(x)$ of a real variable x is said to be periodic if there exists a non-zero number T, independent of x, such that $f(x) = f(x+T)$ holds for all real values of x. The least value of $T>0$ is called the period of $f(x)$.

Exp: Let $f(x) = \sin x$

$$f(x+2\pi) = \sin(x+2\pi)$$
$$= \sin x$$
$$= f(x)$$

* Repeating Cycle * Constant interval



* Fourier series and Fourier coefficients :

Let $f(x)$ be a periodic function defined in an interval 2π say $(-\pi,\pi)$ and $f(x)$ is sectionally continuous on that interval. Then the fourier series of $f(x)$ is given by

$(-\pi,\pi) \to$ every subinterval এ continuous হবে

$(-\pi,0), (0,\pi/2), (\pi/2,\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

[where the constants a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

(II)

Then the series is called fourier series of $f(x)$. The constants a_0, a_n, b_n in given equation (11) are called fourier coefficients.

* Fourier Analysis with Application to BVP - (Murray & Spiegel)
(*) Mathematical Methods - Vol-1, Vol-2 ; (Abdur Rahman)

Some Important Integral

1. $\int_{-\pi}^{\pi} \sin nx \, dx = 0$ 2) $\int_{-\pi}^{\pi} \cos nx \, dx = 0$ 3) $\int_{-\pi}^{\pi} \sin nx . \cos mx \, dx = 0$

4) $\int_{-\pi}^{\pi} \sin nx . \sin mx . \, dx = 0$ if $n \neq m$ 5) $\int_{-\pi}^{\pi} \cos nx . \cos mx \, dx = 0$ if $n \neq m$

6) $\int_{-\pi}^{\pi} \sin^2 mx = \pi$ $m, n \in \mathbb{N}$ 7) $\int_{-\pi}^{\pi} \cos^2 nx = \pi$

$$\frac{1}{n}\left[\sin nx\right]_{-\pi}^{\pi}$$
$$= \frac{1}{n}\left[\sin n\pi + \sin n\pi\right]$$
$$= 0$$

Continuous Function

~~In function~~ A continuous function is a function that does not have discontinuities that means any unexpected changes in value. A function is continuous if we can ensure arbitrarily small changes by restricting enough minor changes to input.

$$f(x) = \begin{cases} x+2 ; x \leq 1 \\ x-2 ; x > 1 \end{cases}$$

~~Given~~

At $x=1$: $f(x) = x+2$

L.H. $\lim_{x \to 1^-} x+2 = 3$

R.H. $\lim_{x \to 1^+} f(x) = x-2 = -1$

not continuous at $x = 1$

Case 1: when $x < 1$

$f(x) = x+2$

Consider $c < 1$ & $f(c) = c+2$

$\lim_{x \to c} f(x) = \lim_{x \to c} x+2$
$= c+2$

continuous at all real numbers less than 1

2) When $x > 1$

$f(x) = x-2$

Consider $c > 1$ and $f(c) = c-2$

$\lim_{x \to c} f(x) = \lim_{x \to c} x-2 = c-2$

Lec-2 Zakaria sir

* Process of determining the coefficients a_0, a_n, b_n

By the defition of fourier series ~~ao~~ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ — ①

① To determine a_0 integrating both sides of ① w.r.to x from $-\pi$ to π, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2} [\pi + \pi] \Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

a_n, b_n বের করার সময় $\cos nx$ দিয়ে multiple. $b_n \times \sin nx$

(2) To determine a_n, we multiply both sides of (1) with $\cos mx$ {where m is fixed positive integer} and the integrate with r. to x from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right]\cos mx\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx\,dx+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\cdot\cos mx\,dx+b_n\int_{-\pi}^{\pi}\sin nx\cdot\cos mx\,dx\right]$$

Here $\int_{-\pi}^{\pi}\cos mx\,dx = 0$ $\quad \int_{-\pi}^{\pi}\sin nx\,\cancel{\cos nx}\cdot\cos mx = 0$

$\therefore$ Now $\int_{-\pi}^{\pi}\cos nx\cdot\cos mx\,dx$

n = 1 থেকে ∞
m = fixed ~~match করতে পা~~
fixed না হলে 0 বসাতে পারবে না

n আর m সমান হতেও পারে নাও হতে পারে

$$= \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)dx+\cos(m-n)]dx$$

n = m হলে π

$$= \frac{1}{2}\int_{-\pi}^{\pi}\cos(m+n)x+\frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x\,dx$$

$$= \frac{1}{2}\times 0+\frac{1}{2}\begin{cases}0 & \text{if } n\neq m\\ 2\pi & \text{if } n=m\end{cases} = \begin{cases}0 & \text{if } n\neq m\\ \pi & \text{if } n=m\end{cases}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_m\pi$$

$$\Rightarrow a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

③ To determine b_n, we multiply both sides of ① with $\sin mx$ and integrate with respect to x from $-\pi$ to π, we have,

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]\sin mx\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin mx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx.\sin mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx.\sin mx\,dx\right]$$

Here $\int_{-\pi}^{\pi}\sin mx\,dx = 0$ $\quad \int_{-\pi}^{\pi}\cos nx.\sin mx.dx = 0$

∴ Now $\int_{-\pi}^{\pi}\sin nx.\sin mx\,dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(n-m)x - \cos(n+m)x\right]dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x\,dx$$

$$= \frac{1}{2}\begin{cases} 0 \text{ if } n\neq m \\ 2\pi \text{ if } n=m \end{cases} \qquad = \int_{-\pi}^{\pi} f(x)\sin mx\,dx = b_m\pi$$

$$= \begin{cases} 0 \text{ if } n\neq m \\ \pi \text{ if } n=m \end{cases} \qquad \Rightarrow b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

* Dirichlet condition for the convergence of fourier series:

series যদি চলতে থাকে আর যদি তার কোনো fixed value পর্যন্ত যায় হয় তা convergence

1. If $f(x)$ is defined on $(-\pi, \pi)$
2. If $f(x)$ is periodic
3. If $f(x)$ can be extend to other values by the periodically condition,

   $f(x) = f(x+2\pi)$; $k = 1, 2 \ldots\ldots$
4. $f(x)$ is sectionally continuous in $(-\pi, \pi)$ with a finite number of discontinuities

   discontinuity $-\pi, \pi \to$ ✓

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· Then the fourier series converges to $f(x)$ at all points where $f(x)$ is continuous or $\frac{f(x-0)+f(x+0)}{2}$ if x is point of discontinuity.

discont. $= \frac{f(-\pi-0)+f(\pi+0)}{2}$

Math → 3.75

Lec-2 : Zakaria Sir

* Find the fourier series of $f(x)=x+x^2$ for $-\pi<x<\pi$, Hence deduce $\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\dots$

Solⁿ: By the definition of fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

Solⁿ: Now,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]$$

$$=\frac{1}{\pi}\times\frac{2\pi^3}{3}$$

$$\Rightarrow \frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n}-(1+2x)\frac{(-\cos nx)}{n^2}+(0+2)\frac{(-\sin nx)}{n^3}\right]_{-\pi}^{\pi}$$

P.T.O.

$$= \frac{1}{\pi}\left[-\frac{(1+2x)\cos nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(1+\cancel{2}\pi)\cos n\pi}{n^2} - \frac{(1-\cancel{2}\pi)\cos(n)(\pi)}{n^2}\right]$$

$$= \frac{1}{\pi n^2}\left[\cancel{(1+2\pi)}\ 4\pi\cos n\pi\right]$$

$$\Rightarrow \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{-\cos nx}{n} - (1+2x)\frac{\cancel{\cos nx}\sin nx}{n^2} + (0+2)\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\cancel{x}+\cancel{x}^2)\frac{-\cos n\cancel{x}}{n} - 2\frac{\cos n\cancel{x}}{n^3}\right]_{-\pi}$$

$$= \frac{1}{\pi}\left[\frac{2(\cos n\cancel{\pi} - \cos(-n)\cancel{\pi})}{n^3} - (\pi+\pi^2)\frac{\cos n\pi}{n} + (-\pi+\pi^2)\frac{\cos(-n\pi)}{n}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\frac{\cos n\pi}{n}\right]' = -\frac{2}{n}(-1)^n$$

Lec-3 Zakaria Sir

Now substituing the values of a_0, a_n, b_n in eq^n ①, we have

$$f(x) = \frac{2\pi^2}{2.3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx + \sum_{n=1}^{\infty} -(-1)^n \frac{2}{n} \sin nx$$

$$= \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$-2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

2nd part :

$-\pi < x < \pi$

নির্দিষ্ট point ~~dis~~continuity থাকে fourier ও func এর value same

$$f(\pi) = \frac{\pi^2}{3} \quad \frac{f(-\pi+0)+f(\pi-0)}{2}$$

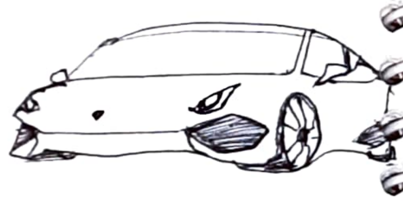
discontinuity হলে ঐ point এ (discontinuous) point এ relation use করতে হবে

Now at extremum π and $-\pi$, the sum of the series

$$f(\pi) = \frac{1}{2}\{f(-\pi+0) + f(\pi-0)\}$$

$$= \frac{-\pi+\pi^2+\pi+\pi^2}{2}$$

$$= \pi^2$$



Putting $x = \pi$ in the eqn (i), we obtain

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

If the function $f(x) = \begin{cases} -\pi & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi \end{cases}$ then show that $f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$

$+\left[3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right]$

and finally show that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ (120)

Sol^n: Given that $f(x) = \begin{cases} -\pi & \text{when } -\pi < x < 0 \\ x & \text{when } 0 < x < \pi \end{cases}$ → composit function

By the definition of fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{—①}$$

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\pi\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= -\left[x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= -\pi + \frac{\pi}{2} = -\pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{-1}{\pi}\left[\pi\,\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\,\frac{\sin nx}{n} - 1\cdot\frac{(-\cos nx)}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{n^2\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\pi\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[x\,\frac{-\cos nx}{n} - 1\cdot\frac{-\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{n}\left[1-(-1)^n\right] + \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right]$$

$$= \frac{1}{n}\left[1-(-1)^n\right] - \frac{1}{n}(-1)^n$$

$$= \frac{1}{n}\left[1-2(-1)^n\right]$$

Now putting the values of a_0, a_n, b_n in ① we have

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left((-1)^n - 1\right)\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\left[1-2(-1)^n\right]\sin nx$$

$n=1$ বসিয়ে

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right] + \left(3\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right)$$

(২)

Here $f(x)$ is discontinuous at $x=0$ $\therefore f(0) = \dfrac{f(0+0)+f(0-0)}{2}$

$$= \frac{0+(-\pi)}{2} = -\pi/2$$

Putting this values in ②, we have

$$f(0) = -\pi/2 = \frac{-\pi}{4} - \frac{2}{A}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\Rightarrow \frac{-\pi}{2}+\frac{\pi}{4} = -\frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

Lec-4 : Zakaria Sir

1. Express $f(x) = x - x^2$, $-\pi < x < \pi$ in a fourier series and hence prove that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\cdots$$

2) Expand the function $f(x) = e^x$ in the interval $-\pi < x < \pi$ in fourier series

3) Expand the function $f(x)$ in fourier series where $f(x) = \begin{cases} x - \pi, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$

Hence deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

4) Expand the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \frac{\pi x}{4}, & 0 < x < \pi \end{cases}$$

in fourier series, Hence show that

(i) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ (ii) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots\cdots$

* Half range fourier series:

If a function $f(x)$ is defined $(0, \pi)$. To find a fourier series of $f(x)$, we extend the function $f(x)$ on $(-\pi, \pi)$ either as an even function or as odd function, then obtain either a cosine series or a sine series. This series is called half range cosine series or " " sine series

Even function:

$f(x) = f(x) \rightarrow \cos x$

Odd function

$f(-x) = -f(x) = \rightarrow \sin x$

* If we extend the half range $(0, \pi)$ to $(-\pi, \pi)$ as $f(x)$ on even function then

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

and $b_n = 0$

an odd-function

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$a_0=0,\ a_n=0$

and $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

* Expand the fourier series in the function $f(x)$ in the interval $-\pi<x<\pi$, Hence deduce that $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots$ $\qquad f(x)=x\sin x$

Soln: By definition of fourier series,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

The given function, $f(x)=x\sin x$

$f(-x)=-x\sin(-x)=x\sin x=f(x)$

So, $f(x)=x\sin x$ is an even function

Now $a_0=\frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$

$$=\frac{2}{\pi}\left[-x\cos x-1(-\sin x)\right]_0^{\pi}$$

$$=\frac{2}{\pi}[\pi]=2$$

Also,

$$a_n=\frac{2}{\pi}\int_0^{\pi} x\sin x.\cos nx=\frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x-\frac{1}{\pi}\int x\sin(n-1)x\,dx$$

$\boxed{\begin{array}{l}\cos(n+1)\pi=-\cos n\pi\\ \cos(n-1)\pi=-\cos n\pi\end{array}}$

$$=\frac{1}{\pi}\left[\frac{-x\cos(n+1)x}{n+1}+\frac{\sin(n+1)x}{(n+1)^2}+\frac{x\cos(n-1)x}{n-1}-\frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}$$

$\cos n\pi\cos\pi$ $+\sin n\pi\sin\pi$

$-\cos n\pi$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n+1}\cos(n+1)\pi+\frac{\pi}{n-1}\cos(n-1)\pi\right]$$

$$=-\frac{1}{n+1}(-1)^n-\frac{1}{n-1}(-1)^n$$

$$=\frac{2}{n^2-1}(-1)^{n+1}$$

Lec-5 Zakariadin

$$a_n = \frac{2(-1)^{n+1}}{n^2-1} \qquad a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x = \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos 2\pi}{2}\right] = -\frac{1}{2}$$

Now substituing the values of a_0, a_1, a_n in (I) we have

$$f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} (-1)^{n+1}\frac{2}{n^2-1}\cos nx$$

$$= 1 - \frac{1}{2}\cos x - \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x - \frac{4}{4^2-1}\cos 4x + \cdots$$

$$\cancel{= 1 - \frac{1}{2}\cos x - 2(}$$

$$= 1 - 2\left[\frac{1}{4}\cos x + \frac{1}{3}\cos 2x - \frac{1}{8}\cos 3x + \frac{1}{15}\cos 4x - \cdots\right]$$

Now putting $x = \frac{\pi}{2}$ in (II), we have

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$

$$= 1 - 2\left[\frac{1}{4}\times 0 + \frac{1}{3}(-1) - \frac{1}{8}\times 0 + \frac{1}{15}\times 1 + \cdots\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{3.5} + \cdots$$

$$\Rightarrow \frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{15}$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{3.5}$$

5| $f(x) = x\cos x \quad , -\pi < x < \pi$

$a_0 = a_n = 0$

$b_n = \frac{1}{n^2-1}$

* If $f(x)$ is defined on (a,b) with period $(b-a)$, then the fourier series of $f(x)$ is defined by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi}{b-a} x + b_n \sin \frac{2n\pi}{b-a} x \right)$$

where, $a_0 = \frac{2}{b-a} \int_a^b f(x)dx$, $\quad a_n = \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a}$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a}$$

If $f(x)$ is defined on $(-c, c)$ with period $2c$, then the fourier series of $f(x)$ is defined by,

$$\cancel{f(x) = \frac{a_0}{2} + \int_{-c}^{c} f(x)dx}, \quad a_n =$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{c} x + b_n \sin \frac{n\pi}{c} x \right)$$

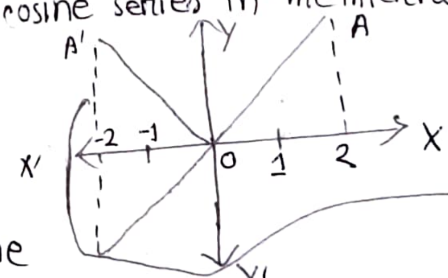
$$a_0 = \frac{1}{c} \int_{-c}^{c} f(x)dx, \quad a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos \frac{n\pi}{c} x$$

$$b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin \frac{n\pi}{c} x$$

Lec-6 Zakaria Sir

Expand $f(x) = x$ as a half range cosine series in the interval $0 < x < 2$

Solⁿ: $\quad -2 < x < 2 \quad -2 < x < 0$



even function হয়ে behave করবে

The graph of the function $f(x) = x$ in the interval $0 < x < 2$ in the line OA. Let us extend the function $f(x)$ in the interval $-2 < x < 0$ (shown by the line OA') So that the new function is symmetrical about the Y axis and therefore represents an even function in the interval $-2 < x < 2$

P.T.O.

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$ where $a_0 = \frac{2}{c}\int_0^c f(x)dx$

$a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$ $\qquad = \frac{2}{2}\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2$

$= 2$

$= \left[\frac{x\sin n\pi x}{n\pi/2} + 1.\frac{\cos n\pi x}{(n\pi/2)^2}\right]_0^2$

$= \frac{4}{n^2\pi^2}\left[0 + \cos n\pi - \cos 0\right] = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$

$\therefore f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\right]$

* Complex form of fourier series

Let the function $f(x)$ defined in the interval $(-c, c)$ with period $2c$, then the fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta,\quad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},\quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$1 = (-1)(-1)$
$= -i^2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\frac{1}{2}\left(e^{in\pi x/c} + e^{-in\pi x/c}\right) + b_n\frac{1}{2i}\left[e^{in\pi x/c} - e^{-in\pi x/c}\right]\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\frac{1}{2}\left(e^{in\pi x/c} + e^{-in\pi x/c}\right) - b_n\frac{i^2}{2i}\left(e^{in\pi x/c} - e^{-in\pi x/c}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{in\pi x/c} + \frac{1}{2}(a_n + ib_n)e^{-in\pi x/c}\right]$$

$$= c_0 + \sum_{n=1}^{\infty}\left[c_n e^{in\pi x/c} + c_{-n}e^{-in\pi x/c}\right] \quad \text{--- (11)}$$

where, $c_0 = \frac{a_0}{2} = \frac{1}{2c}\int_{-c}^{c} f(x)dx$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2c}\int_{-c}^{c} f(x)e^{-in\pi x/c}dx$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2c}\int_{-c}^{c} f(x)e^{in\pi x/c}dx$$

Lec-7 Zakariadir

* Find the complex form of the fourier series of the periodic function whose definition in one period is

$$f(x) = e^{-x}, -1<x<1$$

Soln: By the definition of fourier series in complex form we have

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{in\pi x/c} + \sum_{n=1}^{\infty} C_{-n} e^{-in\pi x/c}$$

$$C_0 = \frac{1}{2c}\int_{-c}^{c} f(x)dx, \quad C_n = \frac{1}{2c}\int_{-c}^{c} f(x) e^{-in\pi x/c} dx \quad C_{-n} = \frac{1}{2c}\int_{-c}^{c} f(x) e^{in\pi x/c} dx$$

Now,

$$C_0 = \frac{1}{2}\int_{-1}^{1} e^{-x} dx$$

$$= \frac{1}{2}\left[-e^{-x}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[-e^{-1} + e^{+1}\right]$$

$$= \frac{1}{2}\left[e^{1} - e^{-1}\right]$$

$$= \sin 1 . h$$

$\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ $\quad \cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$

$e^{in\pi} = \cos n\pi + i \sin n\pi = (-1)^n$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x} dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x} dx$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-(1+n\pi i)} - e^{(1+in\pi)}\right]$$

$$= \frac{1}{2(1+n\pi i)}\left[e^{1}.e^{in\pi} - e^{-1}.e^{-in\pi}\right]$$

$$= \frac{1}{2(1+n\pi i)}\left[e^{1} - e^{-1}\right](-1)^n$$

$$= \frac{(-1)^n}{1+in\pi} \sinh 1$$

$$C_{-n} = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{in\pi x} dx = \frac{-1}{2(1-in\pi)}\left[e^{-(1-in\pi)x}\right]_{-1}^{1}$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1-in\pi)x} dx \quad = \frac{-1}{2(1-in\pi)}\left[e^{-(1-in\pi)} - e^{(1-in\pi)}\right]$$

$$= \frac{1}{2(1-in\pi)}\left[e^{1}.e^{-in\pi} - e^{-1}e^{in\pi}\right]$$

$$= \frac{1}{2(1-in\pi)}\left[e^{1} - e^{-1}\right](-1)^n$$

$$= \frac{(-1)^n}{1-in\pi} \sinh 1$$

from ①

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$$f(x) = \sinh 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+in\pi} \sinh 1\, e^{in\pi x} + \sum_{n=1}^{\infty} \frac{(-1)}{1-n\pi i} \sinh 1\, e^{-in\pi x}$$

$c = 1$

H.W. $f(x) = e^x, -1 < x < 1$

The function x^2 is periodic with period $2l$ on the interval $(-l, l)$

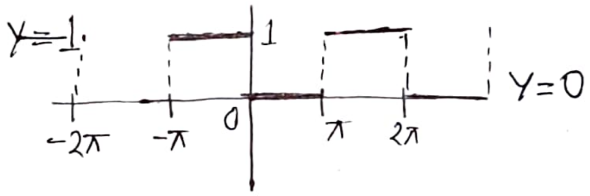
find its fourier series

$\partial x \quad \partial y / \quad \partial y = \partial x$

Lec-8 Zakaria Sir:

Y axis এর সাপেক্ষে symmetric হলে even function | graph দেখে চেনার উপায়
X " " " " " odd "

* $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$ $y = 1$



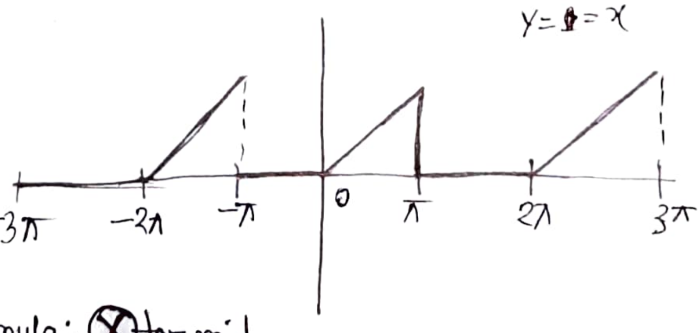
sketching → 2 marks

$\boxed{\text{extrimumpoint}/0}$ দিয়ে math করতে হবে

$-\pi, \pi, 0 \to \pi/2\pi$

* $f(x) = \begin{cases} 0, & -\pi, x < 0 \\ x, & 0 < x, < \pi \end{cases}$

$y = 0$
$y = x$



Parsevals Formula: ⊗ for mid

If the function $f(x)$ defined and convergence on $(-c, c)$ then $\int_{-c}^{c} [f(x)]^2 dx$

$$= c \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\}$$

P.T.O.

By the definition fourier series

Date: ..../.........../.......

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \quad \text{— (I)}$$

Multiplying (I) by $f(x)$, we have

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n f(x) \sin\frac{n\pi x}{c} \quad \text{— (II)}$$

Integrating term by term from $-c$ to c, we have

$$\int_{-c}^{c} [f(x)]^2 dx = \frac{a_0}{2}\int_{-c}^{c} f(x)\, dx + \sum_{n=1}^{\infty} a_n \int_{-c}^{c} f(x) \cos\frac{n\pi x}{c}\, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-c}^{c} f(x) \sin\frac{n\pi x}{c}\, dx \quad \text{— (III)}$$

Now, $\frac{1}{c}\int_{-c}^{c} f(x)\, dx = a_0$

$\Rightarrow \int_{-c}^{c} f(x)\, dx = c a_0$

$\int_{-c}^{c} f(x) \cos\frac{n\pi x}{c}\, dx = c a_n$

$\int_{-c}^{c} f(x) \sin\frac{n\pi x}{c}\, dx = c b_n$

Substituting this values in,

(III), we have,

$$\int_{-c}^{c} [f(x)]^2 dx = c\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$$

Mid: 1 derivation, 3 maths → included graph

Lec-9 Zakaria Sir

→ fourier series, periodic func → 3 marks

definition, proof 3 টা থেকে 1 টা

2 maths { fourier series, complex form, half range, CN + Assignment.

→ 7 marks

→ 1 math → 13 marks half range / complex form

→ 1 " → 7 "

Date: ..../............/.......

Theorem:

Fourier Integral:

If the function $f(x)$ is satisfied the Dirichlet's condition in the interval $(-l, l)$ and $\int_{-\infty}^{\infty} f(x)\,dx$ is absolute convergent, then for every continuous point

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,du\,dt.$$

***

Proof: By the definition of fourier series, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l} f(t)\,dt$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$$

Substituting the values of a_0, a_n and b_n in --- (1)

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}.\cos\frac{n\pi x}{l}\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}.\frac{\sin n\pi x}{l}\,dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\left\{\cos\frac{n\pi t}{l}.\cos\frac{n\pi x}{l} + \sin\frac{n\pi t}{l}.\sin\frac{n\pi x}{l}\right\}dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\cos\left\{\frac{n\pi x}{l} - \frac{n\pi t}{l}\right\}dt$$

P.T.O.

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\cos \frac{n\pi(x-t)}{l}\,dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\cancel{dt}\left[1 + 2\sum_{n=1}^{\infty}\int \cos \frac{n\pi(x-t)}{l}\cancel{dt}\right] dt$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\frac{\pi}{l} + \frac{2\pi}{l}\sum_{n=1}^{\infty}\cos \frac{n\pi(x-t)}{l}\right] dt$$

Lec-10 Zakaria sir : ***   Date: ..../.........../.......

Rest Part:

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\frac{\pi}{l} + \sum_{n=1}^{\infty}\frac{2\pi}{l}\cos\frac{n\pi(x-t)}{l}\right]dt$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\frac{\pi}{l}\cdot\cos\frac{0\cdot\pi(x-t)}{l} + \sum_{n=1}^{\infty}\frac{\pi}{l}\cos\frac{n\pi(x-t)}{l} + \sum_{n=1}^{\infty}\frac{\pi}{l}\cos\frac{-n\pi(x-t)}{l}\right]dt$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\sum_{n=0}^{\infty}\frac{\pi}{l}\cos\frac{n\pi(x-t)}{l} + \sum_{n=-\infty}^{-1}\frac{\pi}{l}\cos\frac{n\pi(x-t)}{l}\right]dt$$

$$\sum_{n=1}^{5}(-x) = \sum_{-5}^{-1} x$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\sum_{n=-\infty}^{\infty}\frac{\pi}{l}\cos\frac{n\pi(x-t)}{l}\right]dt$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\lim_{n\to\infty}\sum_{s=-n}^{n}\frac{1}{l/\pi}\cos\frac{s(x-t)}{l/\pi}\right]dt$$

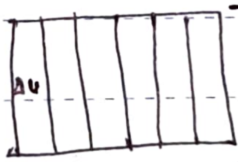
When $l \to \infty$ then $\frac{l}{\pi}\to\infty$. Therefore we have

↓ ধরে নেয়া

$$\lim_{l\to\infty}\sum_{s=-l}^{l}\frac{1}{l/\pi}\cos\frac{s(x-t)}{l/\pi}$$

Let, $\left[\frac{1}{l/\pi} = \Delta u \text{ then } l\to\infty,\ \Delta u \to 0\right]$

$$= \lim_{\Delta u\to 0}\sum_{s=-\infty}^{\infty}\Delta u \cos s\Delta u(x-t)$$



$s\Delta u = u$

$S \to$ number

$$= \int_{-\infty}^{\infty}\cos u(x-t)\,du \quad \left[\because s\Delta u = u \text{ and } \Delta u = du\right]$$

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,du\,dt$$

⊛ Different forms of fourier integral

1. General form :- $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,du\,dt$

P.T.O.

2. Fourier cosine integral: $f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos ux.\cos ut\,du\,dt$ $[\because f(x) = \text{even function}]$

3. Fourier sine integral: $f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin ux\sin ut\,du\,dt$ $[\because f(x) = \text{odd function}]$

* Complex form of fourier integral:

By the definition of fourier integral, we have $f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\cos u(x-t)\,du\,dt$ — ①

Now, $\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\sin u(x-t)\,du\,dt$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\left\{\int_{-\infty}^\infty \sin u(x-t)\,du\right\}dt$$

[$\because$ Here $\phi(-u) = -\phi(u)$

$\therefore \int_{-a}^{a}\phi(u)\,du = 0$]

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(t).\,0\,dt = 0$$

double integral

$$\int_a^b\int_c^d f(x,y)\,dx\,dy$$

$$= \int_c^d\int_a^b f(x,y)\,dy\,dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\cos u(x-t)\,du\,dt + i\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\sin u(x-t)\,du\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\left[\cos u(x-t) + i\sin u(x-t)\right]du\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)\,e^{iu(x-t)}\,du\,dt$$

Lec - 11 Zakaria Sir

Q. Evaluate the fourier integral of the function

$$f(x) = \begin{cases} 0 & \text{when } x<0 \\ 1/2 & \text{when } x=0 \\ e^{-x} & \text{when } x>0 \end{cases}$$

Solⁿ: Given that,

By the definition of fourier integral, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,du\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left\{\int_{-\infty}^{\infty}\cos u(x-t)\,du\right\}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left\{2\int_{0}^{\infty}\cos u(x-t)\,dt\right\}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left\{\int_{0}^{\infty}(\cos ut.\cos ux+\sin ut.\sin ux)\,du\right\}dt$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\,(\cos ut.\cos ux+\sin ut.\sin ux)\,dt\right\}du$$

$$= \frac{1}{\pi}\left[\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\cos ut.\,dt\right\}\cos ux\,du\right]$$

$$+ \frac{1}{\pi}\left[\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\sin ut\,dt\right\}\sin ux\,du\right] \text{ ——— (2)}$$

Now

$$\int_{-\infty}^{\infty} f(t)\cos ut\,dt$$

$$\left[\because \int e^{ax}\cos bx = \frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}\right.$$

$$= \int_{-\infty}^{0} f(t)\cos ut\,dt + \int_{0}^{\infty} f(t)\cos ut\,dt = 0 + \int_{0}^{\infty} e^{-t}\cos ut\,dt$$

P.T.O.

$$= \left[e^{-t} \frac{(-\cos ut + u \sin ut)}{1+u^2} \right]_0^\infty = \frac{1}{1+u^2}$$

$$\int_{-\infty}^{\infty} f(t) \sin ut \, dt = \int_{-\infty}^{0} f(t) \sin ut \, dt + \int_0^\infty f(t) \sin ut \, dt$$

$$= 0 + \int_0^\infty e^{-t} \sin ut \, dt = \left[e^{-t} \frac{(-\sin ut - u \cos ut)}{1+u^2} \right]_0^\infty = \frac{u}{1+u^2}$$

Substituting this values in (2)

$$f(x) = \frac{1}{\pi} \left[\int_0^\infty \frac{1}{1+u^2} \cos ux \, du + \int_0^\infty \frac{u}{1+u^2} \sin ux \, du \right]$$

$$= \frac{1}{\pi} \left[\int_0^\infty \frac{\cos ux + u \sin ux}{1+u^2} \, du \right] \quad \text{—— (iii)}$$

substituting $x = 0$ in (3), we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{1}{1+u^2} \, du = \frac{1}{\pi} \left[\tan^{-1} u \right]_0^\infty = \frac{1}{\pi} \times \frac{\pi}{2} = \frac{1}{2}$$

Eqn (3) will be the fourier integral of $f(x)$.

Lec-11: Zakaria Sir

Find the fourier integral of the function, $f(x) = e^{-kx}$, $x>0$ for $k>0$ and $f(-x) = -f(x)$ and hence prove that

$$\int_0^\infty \frac{u \sin ux}{k^2+u^2}\, du = \frac{\pi}{2} e^{-kx},\ k>0$$

Soln: Here $f(-x) = -f(x)$, Hence $f(x)$ be an odd function.

By the definition of fourier integral for odd function, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t) \sin ut - \sin ux\, du\, dt$$

$$= \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(t) \sin ut\, dt\right] \sin ux\, dx$$

Here,

$$\int_0^\infty f(t) \sin ut\, dt$$

$$= \int_0^\infty e^{-tk} \sin ut\, dt$$

$$= \left[\frac{e^{-tk}(-k\sin ut - u\cos ut)}{k^2+u^2}\right]_0^\infty = \frac{u}{k^2+u^2}$$

from (1), we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{u}{k^2+u^2} \sin(ux)\, du$$

$$\Rightarrow \int_0^\infty \frac{u}{k^2+u^2} \sin ux\, du = \frac{\pi}{2} e^{-kx}$$

$$\Rightarrow e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{u}{k^2+u^2} \sin ux\, du$$

Q. Find the fourier integral of the function, $f(x) = e^{-kx}$

$x > 0$ for $k > 0$ and $f(-x) = f(x)$ and hence prove that

$$\int_0^\infty \frac{\cos ux}{k^2+u^2}\,du = \frac{\pi}{2k} e^{-kx}, \; k > 0$$

Soln: Here $f(-x) = f(x)$. Hence $f(x)$ be an even function by the definition of fourier integral for even function, we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos ut \cos ux \, du\, dt$$

$$= \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(t) \cos ut \, dt \right] \cos ux \, du$$

Here, $$\int_0^\infty f(t) \cos ut \, dt = \int_0^\infty e^{-tk} \cos ut \, dt$$

$$= \left[\frac{e^{-tk}(-k\cos ut + u \sin ut)}{k^2+u^2} \right]_0^\infty = \frac{k}{k^2+u^2}$$

from (1) we have, $$f(x) = \frac{2}{\pi} \int_0^\infty \frac{k}{k^2+u^2} \cos ux \, du$$

$$\Rightarrow e^{-kx} = \frac{2}{\pi} \int_0^\infty \frac{k}{k^2+u^2} \cos ux \, du$$

$$\Rightarrow \int_0^\infty \frac{\cos ux}{k^2+u^2} \, du = \frac{\pi}{2k} e^{-kx}$$

* Fourier Transformation,

Limiting Value + Functional value same

Continuous + Piecewise smooth and absolutely integrable function over $(-\infty, \infty)$ then the fourier transformation of $f(x)$ is denoted by $f(\alpha)$ and is defined by

$$\mathcal{F}[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$ The inverse of $F(\alpha)$ is denoted by

$\mathcal{F}^{-1}[F(\alpha)]$ and is given by $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha$$

piece wise → (1) Continuous → Isolated points which are not continuous

হতে পারে

(2) Limit exist

Lec- 12 Zakaria Sir :

* Properties of fourier Transformation :

1. Linear Property : $\mathcal{F}[af(x)+bg(x)] = a\,\mathcal{F}[f(x)] + b\,\mathcal{F}[g(x)]$

2. Shifting Property : $\mathcal{F}[f(t-c)] = e^{i\alpha c}\,\mathcal{F}[f(\alpha)]$

Proof: By the definition of fourier transformation, we have

$$\mathcal{F}[f(t-c)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-c)\, e^{i\alpha t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(c+u)}\, du$$

$$= e^{i\alpha c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du$$

$$= e^{i\alpha c}\,\mathcal{F}[f(u)]$$

Let, $t-c=u$
$\Rightarrow dt = du$
If $t\to\infty,\ u\to\infty$
$t\to-\infty,\ u\to-\infty$

3. Scaling property : $\mathcal{F}[f(t)] = \frac{1}{c}F\left(\frac{\alpha}{c}\right)$

Proof: By the definition of Fourier Transformation, we have

$$\mathcal{F}[f(ct)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ct)\, e^{i\alpha t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha\left(\frac{u}{c}\right)}\, du$$

$$= \frac{1}{c}\times\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{i\left(\frac{\alpha}{c}\right)u}\, du = \frac{1}{c}F\left(\frac{\alpha}{c}\right)$$

Let $ct=u$
$\Rightarrow dt = \frac{1}{c}du$

$t\to\infty,\ u\to\infty$
$t\to-\infty,\ u\to-\infty$

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* Differentiation Property: ***

$$\mathcal{F}\left[f^{n}(x)\right] = (-i\alpha)^n \mathcal{F}\left[f(x)\right]$$

Q: Find the fourier transformation of the function

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

also find its inversion. Hence deduce that $\int_{-\infty}^{\infty}\left(\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right)d\alpha = \frac{\pi}{2}$

Soln: Given that $f(x) = \begin{cases} 1-x^2, & -1<x<1 \\ 0, & x>1 \text{ or } x<-1 \end{cases}$

By the definition of fourier transformation we have.

$$\mathcal{F}\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-1}\overbrace{f(x)}^{\to 0}e^{i\alpha x}\,dx + \frac{1}{2\pi}\int_{-1}^{1} f(x)e^{i\alpha x}\,dx + \int_{1}^{\infty}\overbrace{f(x)}^{\to 0}e^{i\alpha x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{i\alpha x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{i\alpha x}}{i\alpha} - (0-2x)\frac{e^{i\alpha x}}{(i\alpha)^2} + (-2)\frac{e^{i\alpha x}}{(i\alpha)^3}\right]_{-1}^{1} \quad \left[\int uv\,dx = \dots\right]$$

$$= \left[\frac{2e^{i\alpha}}{(i\alpha)^2} - \frac{2e^{i\alpha}}{i^3\alpha^3} + 2\frac{e^{-i\alpha}}{i^2\alpha^2} + 2\frac{e^{-i\alpha}}{i^3\alpha^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2\times 2}{-\alpha^2}\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right) + \frac{2\times 2}{\alpha^3}\left(\frac{e^{i\alpha}-e^{-i\alpha}}{2i}\right)\right]$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}}\left[-\frac{1}{\alpha^2}\cos\alpha + \frac{1}{\alpha^3}\sin\alpha\right]$$ $\to$ Which is the fourier transformation of the given function.

Using inversion form of F.T. in (II), we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha)\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2\sqrt{2}}{\sqrt{\pi}} \left[\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right] \left(\cos\alpha x - i\sin\alpha x\right) d\alpha$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right)\left(\cos\alpha x - i\sin\alpha x\right) d\alpha$$

Now equating real and imaginary part, we have

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right)\cos\alpha x\, d\alpha = f(x) \quad \text{—(III)}$$

$$\Rightarrow \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right)\sin\alpha x\, d\alpha = 0 \quad \text{—(IV)}$$

From (III)

$$\Rightarrow \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right)\cos\alpha x\, dx = \frac{\pi}{2} \begin{cases} (1-x^2), & |x|<1 \\ 0 & |x|>1 \end{cases} \quad \text{—(V)}$$

If $x=0$ then (V) becomes

$$\int_{-\infty}^{\infty} \left(\frac{1}{\alpha^3}\sin\alpha - \frac{1}{\alpha^2}\cos\alpha\right) d\alpha = \frac{\pi}{2}$$

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Find the fourier Transformation of $f(x) = \begin{cases} 1 & \text{when } |x| < a \\ 0 & \text{when } |x| > a \end{cases}$

Also find it's inversion form. Hence deduce that

$$\int_0^\infty \frac{\sin\alpha}{\alpha} d\alpha = \pi/2$$ H.W.

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* fourier ~~series~~ sine-transformation:

If $f(x)$ is defined on $(0,\infty)$ and it is piecewise smooth and absolutely integrable then $\mathcal{F}_s[f(x)] = F_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\alpha x\,dx$ with inversion fourier transformation

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\alpha)\sin\alpha x\,d\alpha$$

$$\left[\mathcal{F}_c(f(x)] = F_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\alpha x\,dx\right.$$

Ifcosine transform: $\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\alpha)\cos\alpha x\,d\alpha$

* find the fourier transform of $f(x) = e^{-x}, x \geq 0$

Soln: By the definition of fourier cosine transformation, we have $\mathcal{F}_c[f(x)] =$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\alpha x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos\alpha x\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}(-\cos\alpha x + \alpha\sin\alpha x)}{1+\alpha^2}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+\alpha^2}$$

* Find the fourier cosine transform of $f(x) = e^{-x^2}, x>0$

Soln: By the definition of fourier cosine transformation, we have

$$\mathcal{F}[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\alpha x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos x\,dx$$

$$I = \int_0^\infty e^{-x^2}\cos\alpha x\,dx = \frac{1}{2}\int_0^\infty \sin\alpha x(-2xe^{-x^2})dx$$

$$\Rightarrow \frac{dI}{d\alpha} = \int_0^\infty e^{-x^2}(-x)\sin\alpha x\,dx \quad = \frac{1}{2}\left[\sin\alpha x\, e^{-x^2} - \alpha\int \cos\alpha x . e^{-x^2}dx\right]_0^\infty$$

$$= -\frac{\alpha}{2}\int_0^\infty e^{-x^2}\cos\alpha x\,dx = -\alpha/_2\, I$$

$$\therefore \frac{dI}{d\alpha} = -\frac{dI}{2}$$

$$\int\frac{dI}{I} = -\int \alpha/_2\,d\alpha$$

$$\Rightarrow \ln I = -\frac{\alpha^2}{4} + \ln A$$

$$\Rightarrow I = Ae^{-\alpha^2/4}$$

From ① we have

$$\mathcal{F}_c[f(x)] = Ae^{-\alpha^2/2} \quad \text{— (ii)}$$

Here, $\int_0^\infty e^{-x^2}\cos\alpha x \, dx = Ae^{-\alpha^2/4}$

Substituting $\alpha = 0$ in (iii), we have

$$\int_0^\infty e^{-x^2} dx = A$$

$$\Rightarrow A = \int_0^\infty e^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-z} z^{-1/2} dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{1/2 - 1} dz$$

$$= \frac{1}{2} \times \Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{2}$$

Let, $x^2 = z$

$x = z^{1/2}$

$\Rightarrow dx = \frac{1}{2} z^{-1/2} dz$

If $x = 0$ then $z = 0$

$x = \infty$ then $z = \infty$

$$\left[\because \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \right.$$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} \cdot e^{-\alpha^2/4}$$

$$= \frac{1}{\sqrt{2}} e^{-\alpha^2/4}$$

$$\boxed{f(x) = e^{-ax} \quad \frac{a}{a^2+x^2}}$$ H.W.

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Finite fourier sine Transformation :

The finite fourier sine transformation of a function $f(x), (0<x<l)$ is derived as $F_s(\alpha)$ and defined by

$$\mathcal{F}_s[f(x)] = F_s(\alpha) = \int_0^l f(x) \sin \frac{\alpha \pi x}{l} \, dx$$

and the inversion of the finite fourier sine transformation of $F_s(\alpha)$ is ~~defined by~~ denoted $f(x)$ and defined by

$$f(x) = \frac{2}{l} \sum_{\alpha=1}^{\infty} F_s(\alpha) \sin \frac{\alpha \pi x}{l}$$

Finite Fourier cosine transformation:

The finite fourier sine transformation of a function $f(x)$, $(0<x<l)$ is denoted as $F_c(\alpha)$ and defined by

$$\mathcal{F}_c[f(x)] = F_c(\alpha) = \int_0^l f(x)\cos\frac{\alpha\pi x}{l}\,dx$$

$\forall$ and the inversion of the finite fourier cosine transformation of $F_c(\alpha)$

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{\alpha=1}^{\infty} F_c(\alpha)\cos\frac{\alpha\pi x}{l}$$

Boundary Value Problem:

A boundary value problem is a differential equation together with a set of additional constraints, called the boundary condition.

Example: $U = U(x,t)$; $\frac{\partial U}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$ → Heat eqⁿ

where $u(0,t) = u(1,t) = 0$ and $u(x,0) = x$

PDE, $\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$

→ Special domain
(x,t) ↳ time domain,
↳ x 2 types Boundary cond
↳ initial cond

Initial Condition I.C = $u(x,0) = x$

Boundary " B.C = $U(0,t) = u(1,t) = 0$

Condition: If boundary conditions are given as the form

a. $U(0,t)$ and $U(l,t)$

then we have used finite fourier sine transformation

b. $U_x(0,t)$ and $U_x(l,t)$ then we have used finite fourier cosine transform.

Use the suitable fourier transform to solve

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$$

where $u(0,t) = u(2,t) = 0, t>0$ and $U(x,0) = x\ (0<x<2)$

$v \quad u$

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Solⁿ: Given partial differential eqⁿ

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \quad \text{— (I)}$$

$$I.C \rightarrow u(x,0) = x \quad \text{— (II)}$$

$$B.C \rightarrow u(0,t) = u(2,t) = 0 \quad \text{— (III)}$$

According to the given boundary condition finite fourier sine transform is more useful.

Taking the finite fourier sine transform $(0<x<2)$ of both sides of (I) we have

$$\Rightarrow \int_0^2 \frac{\partial u}{\partial t} \sin\frac{\alpha\pi x}{2}\,dx = 3\int_0^2 \frac{\partial^2 u}{\partial x^2}\sin\frac{\alpha\pi x}{2}\,dx$$

$$\Rightarrow \frac{d}{dt}\int_0^2 u(x,t)\sin\frac{\alpha\pi x}{2}\,dx = 3\int_0^2 \frac{\partial^2 v}{\partial x^2}\sin\frac{\alpha x\pi}{2}\,dx \quad \text{— (IV)}$$

$$\text{Let } V(\alpha,t) = \int_0^2 u(x,t) \quad \text{— (V)}$$

$V \quad V$

from (IV) we have -

$$\Rightarrow \frac{dV}{dt} = 3\int_0^2 \frac{\partial^2 u}{\partial x^2}\sin\frac{\alpha\pi x}{2}\,dx$$

$$\Rightarrow \frac{dV}{dt} = 3\left[\sin\frac{\alpha\pi x}{2}\cdot\frac{\partial u}{\partial x} - \frac{\alpha\pi}{2}\int\cos\frac{\alpha\pi x}{2}\cdot\frac{\partial u}{\partial x}\,dx\right]_0^2$$

$$= -\frac{3\alpha\pi}{2}\left[\cos\frac{\alpha\pi x}{2}\,u(x,t) + \frac{\alpha\pi}{2}\int\frac{\sin\alpha\pi x}{2}\,u(x,t)\,dx\right]_0^2$$

$$= -\frac{3\alpha\pi}{2}\cdot\frac{\alpha\pi}{2}\int_0^2 u(x,t)\sin\frac{\alpha\pi x}{2}\,dx$$

$$\frac{dV}{dt} = -\frac{3\alpha^2\pi^2}{4}V$$

$$\Rightarrow \int\frac{dV}{V} = \int -\frac{3\alpha^2\pi^2}{4}\,dt$$

P.T.O.

$$\Rightarrow \ln V = -\frac{3}{4}\alpha^2\pi^2 t + \ln A$$

$$\Rightarrow V(\alpha,t) = Ae^{-3/4\,\alpha^2\pi^2 t} \quad \text{— (vi)}$$

$\ln a = b$

$a = e^b$

Substituting $t = 0$ in (vi), we have

$$V(\alpha,0) = Ae^{-0}$$

$$\Rightarrow A = V(\alpha,0)$$

$$= \int_0^2 u(x,0)\sin\frac{\alpha\pi x}{2}$$

$$= \int_0^2 x\sin\frac{\alpha\pi x}{2}\,dx$$

$$= \left[\frac{-x\cos\frac{\alpha\pi x}{2}}{\alpha\pi/2} - \frac{-\sin\frac{\alpha\pi x}{2}}{\left(\frac{\alpha\pi}{2}\right)^2}\right]_0^2$$

$$= \frac{-4}{\alpha\pi}\cos\alpha\pi$$

$$= \frac{-4}{\alpha\pi}(-1)^\alpha$$

$$= \frac{4}{\alpha\pi}(-1)^{\alpha+1} \text{ — (8)}$$

$$V(\alpha,t) = (-1)^{\alpha+1}\frac{4}{\alpha\pi}e^{-\frac{3}{4}\alpha^2\pi^2 t} \quad \text{(vii)}$$

Applying inverse finite fourier sine transform in both sides of (vii)

$$u(x,t) = \frac{2}{2}\sum_{\alpha=1}^{\infty}(-1)^{\alpha+1}\frac{4}{\alpha\pi}e^{-\frac{3}{4}\alpha^2\pi^2 t}\cdot\sin\frac{\alpha\pi x}{2}$$

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ZAKARIA SIR LEC-15

Q: Solve: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0<x<6$, $t>0$

Solⁿ: Given Partial Differential Equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ --- (i)}$$

$$I.C = u(x,0) = 2x \text{ --- (ii)}$$

$$B.C = u_x(0,t) = u_x(6,t) = 0 \text{ --- (iii)}$$

According to given boundary ~~equation,~~ condition, Here the finite fourier series transformation is more useful. Taking the finite fourier cosine transform $(0<x<6)$ of both sides of (i), we have

$$\int_0^6 \frac{\partial u}{\partial t} \cos\frac{\alpha\pi x}{t}\,dx = \int_0^6 \frac{\partial^2 u}{\partial x^2} \cos\frac{\alpha\pi x}{6}\,dx$$

$$\Rightarrow \frac{d}{dt}\int_0^6 u(x,t)\cos\frac{\alpha\pi x}{6}\,dx = \int_0^6 \frac{\partial^2 u}{\partial x^2}\cos\frac{\alpha\pi x}{6}\,dx \text{ --- (iv)}$$

Let $\int_0^6 u(x,t)\cos\frac{\alpha\pi x}{6}\,dx = v(\alpha,t) = v$

From (iv) we have $\frac{dv}{dt} = \int_0^6 \frac{\partial^2 u}{\partial x^2}\cos\frac{\alpha\pi x}{6}\,dx$

$$= \left[\underbrace{\cos\frac{\alpha\pi x}{6}}_{0}\,\underbrace{\frac{\partial u}{\partial x}}_{u_x(x,t)} + \frac{\alpha\pi}{6}\int \sin\frac{2\pi x}{6}\,\frac{\partial u}{\partial x}\,dx\right]_0^6$$

$$= \frac{\alpha\pi}{6}\left[\sin\frac{\alpha\pi x}{6}\,u(x,t) - \frac{\alpha\pi}{6}\int\cos\frac{\alpha\pi x}{6}\,u(x,t)\,dx\right]_0^6$$

$$= -\frac{\alpha^2\pi^2}{36}\int_0^6 \cos\frac{\alpha\pi x}{6}\cdot u(x,t)\,dx$$

P.T.O.

$$\Rightarrow \frac{dV}{dt} = \frac{-\alpha^2\pi^2}{6}V$$

$$\Rightarrow \int \frac{dV}{V} = \frac{-\alpha^2\pi^2}{36}\int dt$$

$$\Rightarrow \ln V = -\frac{\alpha^2\pi^2}{36}t + \ln A$$

$$V = Ae^{-\alpha^2\pi^2/36\,t} \quad \text{— (VI)}$$

Substituting $t=0$ in (VI) we have

$$V(\alpha,0) = Ae^{-0}$$

$$\Rightarrow A = V(\alpha, 0)$$

$$= \int_0^6 u(x,t)\cos\frac{\alpha\pi x}{6}dx$$

$$= \int_0^6 u(x,0)\cos\frac{\alpha\pi x}{6}dx$$

$$= \int_0^6 2x\cos\frac{\alpha\pi x}{6}dx$$

$$\left[2x\cdot\frac{\sin\frac{\alpha\pi x}{6}}{\frac{\pi\alpha}{6}} - 2\frac{\left(-\cos\frac{\alpha\pi x}{6}\right)}{\left(\frac{\pi\alpha}{6}\right)^2}\right]_0^6$$

$$= \frac{2\times6^2}{\alpha^2\pi^2}\left[(-1)^\alpha - 1\right]$$

$$= \frac{72}{\alpha^2\pi^2}\left[(-1)^\alpha - 1\right]$$

From (VI) we have, $V = V(\alpha,t) = \frac{72}{\pi^2\alpha^2}\left[(-1)^\alpha - 1\right]e^{-\frac{\alpha^2\pi^2}{36}t}$ — (VII)

Taking Inverse Fourier Cosine Transform we have,

$$u(x,t) = \frac{1}{6}V(0) + \frac{2}{6}\sum_{\alpha=1}^{\infty}\frac{72}{\alpha^2\pi^2}\left[(-1)^\alpha - 1\right]e^{-\frac{\alpha^2\pi^2}{36}t}\cdot\cos\frac{\alpha\pi x}{6} \quad \text{— (VIII)}$$

Here $V(\alpha) = \int_0^6 u(x,t)\cos\frac{\alpha\pi x}{6}dx$

$$V(0) = \int_0^6 u(x,0)\,1\cdot dx$$

$$V(0) = \int_0^6 2x\,dx = \left[x^2\right]_0^6 = 36$$

$$u(x,t) = \frac{36}{6} + \frac{24}{\pi^2}\sum_{\alpha=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{\alpha^2}e^{-\frac{\alpha^2\pi^2 t}{36}}\cdot\cos\frac{2\pi x}{6}$$

Zakariadin- Lec- 16 Date: ..../.........../.......

Fourier Transformation:

General form $\rightarrow \mathcal{F}[f(x)] = F(\alpha) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$

Complex " $\rightarrow \mathcal{F}[f(x)] = F(\alpha) = \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$

Inverse:

General form: $f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha$

Complex form: $f(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha$

Use the method of fourier transform to determine the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$

Here we have to solve the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ; -\infty < x < \infty, t > 0 \quad — (1)$$

Subject to the following $y(x,0) = f(x)$ — (ii)

t এর সাপেক্ষে differentiation $y_t(x,0) = 0$ — (iii)

Taking the complex fourier transform of both sides of (1), we have

$$\cancel{\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} = c^2 \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial x^2}} \quad \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} e^{-i\alpha x} = c^2 \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-i\alpha x} dx$$

$$\Rightarrow \frac{d^2}{dt^2} \int_{-\infty}^{\infty} y(x,t) e^{-i\alpha x} dx \qquad \left[\because \mathcal{F}[f^n(x)] = (-i\alpha)^n \mathcal{F}[f(x)]\right]$$

$$= c^2 (-i\alpha)^2 \int_{-\infty}^{\infty} y(x,t) e^{-i\alpha x} dx \quad — (iv)$$

Let $\displaystyle\int_{-\infty}^{\infty} y(x,t) e^{-i\alpha x} dx = Y(\alpha,t)$ — (v)

Hence (iv) becomes $\dfrac{d^2 Y}{dt^2} = -c^2\alpha^2 Y$

$$\Rightarrow \frac{d^2 Y}{dt^2} + c^2\alpha^2 = 0 \quad — (vi)$$

$\left[\because \cancel{\mathcal{F}[f^n(x)] = (-i\alpha)^n F}\right]$

which is a 2nd order ode where solution is $Y = Y(\alpha, t) = A\cos c\alpha t + B\sin c\alpha t$ — (vii)

~~A~~ Differentiating (vi) with respect to t, we have

$$Y_t = -Ac\alpha\sin c\alpha t + Bc\alpha\cos c\alpha t$$

Also from initial conditions, we have $y(x,0) = f(x)$

$\cancel{Y(\alpha,0)}$

Taking the fourier transform $Y(\alpha,0) = \int_{-\infty}^{\infty} y(x,0)e^{-i\alpha x}dx = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx$

$$= F(\alpha) \text{ (say)}$$

$$Y_t(\alpha,0) = \int_{-\infty}^{\infty} y_t(x,0)e^{-i\alpha x}dx$$

$$= 0$$

Substituting $t=0$ in ~~vii~~ (vii) and (viii) we have

$\Rightarrow$ ~~$Y(\alpha,0) = A \quad Y_t(\alpha,t)$~~

Y ~~$_t(\alpha,t)$~~

$Y(\alpha,0) = A$ | $Y_t(\alpha,0) = Bc\alpha$

$A = F(\alpha)$ | $\Rightarrow Bc\alpha = 0$

$\Rightarrow B = 0 \quad [\because c\alpha \neq 0]$

Hence (vii) becomes

$$Y(\alpha,t) = F(\alpha)\cos c\alpha t \quad \text{— (ix)}$$

Taking the inverse fourier Transform of (IX) we have

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\cos c\alpha t\, e^{i\alpha x}d\alpha$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} F(\alpha)\left(\frac{e^{ic\alpha t} + e^{-ic\alpha t}}{2}\right)e^{i\alpha x}d\alpha\right]$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha(x+ct)}d\alpha + \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha(x-ct)}d\alpha\right]$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

Lec 17 Zakaria Sir ①

↑ যেগুলো আসবেই compulsory, 2nd question fourier series, integral

First Question: Heat eqⁿ, wave eqⁿ এর application form

Q1 Use the method of fourier transform to determine the temperature at any point x at an instant of time in a solid bounded by the planes $x=0$ and $x=2$. At the boundary ends, the temperature of solid is zero while the initial temperature is x.

Solid এর temp নির্ণয়ের জন্যে heat equation নিতে হবে

Solⁿ:

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$$

B.C → $U(0,t) = u(2,t) = 0$ U_x থাকলে cosine transform

I.C → $u(x,0) = x$

2nd Set: ① Parseval's Identity → 15 marks

② fourier transform → 15 marks

3rd Set: Fourier Integral Theorem / math ①

② Fourier transformation এর math

4th set: Fourier series mid এর syllabus এর উপর