6 Assignment 6 Solutions (100 points)

6.1 Free and Bound Variables (15 points)

1. No free variables
2. The last occurrence of y
3. The last occurrence of x
4. No free variables
5. u is free.

6.2 Semantics I (15 points)

- 0
- 0
- 1

6.3 Semantics II (20 points)

- 0 ($\forall x(-x < 0)$)
- 1 ($\forall x(x - 1 < x) \to \exists y(0 \geq y - 2)$)
- 0 ($\forall x(x < 1 \to \forall y(y < 2 \to \forall z(x < z)))$)
- 1 ($\forall x \exists y(x < x - 2y)$)

6.4 Semantic Entailment (20 points)

- No. We may construct a graph with a, b, c, d as nodes and b to a, b, c as edges. As a counter example for the formula, let's choose b for x and a for y. There is an edge from b to a, but a doesn't point to any node.
- Yes. Similar to the above case, we can construct a graph. For any choice of x and y, if there is an edge from x to y, then there exists a node z with an edge from y.

6.5 ND Proof (30 points)

Proof 1:

1.	$\exists x P(x) \lor \exists x Q(x)$	Premise
2.	$\exists x P(x)$	Assumption
3.	$P(u)$ u fresh	Assumption
4.	$P(u) \lor Q(u)$	$\lor i$ 3
5.	$\exists x(P(x) \lor Q(x))$	$\exists i$ 4
6.	$\exists x(P(x) \lor Q(x))$	$\exists e$ 2, 3-5
7.	$\exists x Q(x)$	Assumption
8.	$Q(u)$ u fresh	Assumption
9.	$P(u) \lor Q(u)$	$\lor i$ 8
10.	$\exists x(P(x) \lor Q(x))$	$\exists i$ 9
11.	$\exists x(P(x) \lor Q(x))$	$\exists e$ 7, 8-10
12.	$\exists x(P(x) \lor Q(x))$	$\lor e$ 1, 2-6,7-11

Proof 2:

1.	$\neg \forall x \neg P(x)$	Premise
2.	$\neg \exists x P(x)$	Assumption
3.	u fresh	
4.	$P(u)$	Assumption
5.	$\exists x P(x)$	$\exists i$ 4
6.	$\bot$	$\bot i$ 2,5
7.	$\neg P(u)$	$\neg i$ 4-6
8.	$\forall x \neg P(x)$	$\forall i$ 3-7
9.	$\bot$	$\bot i$ 1,8
10.	$\exists x P(x)$	PBC 2-9

Proof 3:

1.	$(\forall x\ (Q(x) \to R(x)))$	premise
2.	$(\exists x\ (P(x) \wedge Q(x)))$	premise
3.	$(P(u) \wedge Q(u))$, u fresh	assumption
4.	$P(u)$	$\wedge$e: 3
5.	$Q(u)$	$\wedge$e: 3
6.	$(Q(u) \to R(u))$	$\forall$e: 1
7.	$R(u)$	$\to$e: 5, 6
8.	$(P(u) \wedge R(u))$	$\wedge$i: 4, 7
9.	$(\exists x\ (P(x) \wedge R(x)))$	$\exists$i: 8
10.	$(\exists x\ (P(x) \wedge R(x)))$	$\exists$e: 2, 3-9

Proof 4:

1.	$\forall x P(a,x,x)$	Premise
2.	$\forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z)))$	Premise
3.	$P(a,a,a)$	$\forall e$ 1
4.	$\forall y \forall z (P(a,y,z) \to P(f(a),y,f(z)))$	$\forall e$ 2
5.	$\forall z (P(a,a,z) \to P(f(a),a,f(z)))$	$\forall e$ 4
6.	$P(a,a,a) \to P(f(a),a,f(a))$	$\forall e$ 5
7.	$P(f(a),a,f(a))$	$\to e$ 3,6