1 Answers for Assignment 1: Preliminaries

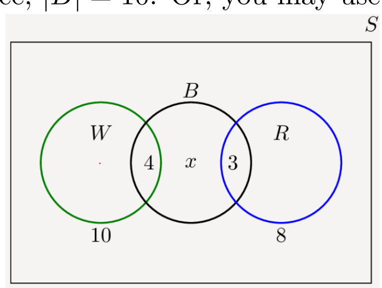
1.1 Set operations (10 points)

Let W, R and B be the number of people with white shirts, red shirts, and black shoes, respectively. Then we have:

$$|W| = 10$$
$$|R| = 8$$
$$|W \cap B| = 4$$
$$|R \cap B| = 3$$
$$|W \cup R \cup B| = 21$$

$$|W \cup R \cup B| = |W| + |R| + |B| - |W \cap R| - |W \cap B| - |R \cap B| + |W \cap R \cap B| = 10 + 8 + |B| - 0 - 4 - 3 + 0 = 21$$

Hence, $|B| = 10$. Or, you may use the following Venn diagram to calculate $|B|$:



1.2 Equivalence relation (20 points)

We need to prove that $\sim$ is reflexive, symmetric, and transitive.

1. (3 points) Reflexive: let p be any integer. Then we have $p^2 - p^2 = 0 = 4 \cdot 0$, thus $p \sim p$.

2. (3 points) Symmetric: suppose $p \sim q$. Then $p^2 - q^2 = 4k$ for some $k \in Z$. This implies $q^2 - p^2 = 4(-k)$, so we can conclude $q \sim p$.

3. (4 points) Transitivity: suppose $a \sim b$ and $b \sim c$. Then by definition of $\sim$, we have $a^2 - b^2 = 4k$ and $b^2 - c^2 = 4m$ for integers k and m. Combining these two equations yields:

$$4k = a^2 - b^2 = a^2 - (c^2 + 4m) = a^2 - c^2 - 4m$$
$$a^2 - c^2 = 4k + 4m = 4(k + m)$$

Hence, $a \sim c$.

There are two equivalence classes, the set of odd integers and the set of even integers.

(5 points) Let us first look at the equivalence class of [0]. By definition this set is $[0] = \{p \in Z | p \sim 0\} = \{p \in Z | p^2 = 4k, k \in Z\}$. Yet, if $p^2 = 4k$, dividing by 4 we see $(p/2)^2 = k$. That is, an

integer p is in [0] if $p/2$ squares to another integer. This is true for all even integers. Thus [0] is the set of all even integers.

(5 points) Let us now look at the equivalence class [1]. By definition this is the set of all integers p satisfying $p^2 - 1 = 4k$ for some integer k. Factoring we see $(p+1)(p-1) = 4k$. From here we can conclude [1] is the set of odd integers, since for any odd number both $p-1$ and $p+1$ will be even, and thus $(p-1)(p+1)$ will be a multiple of 4.

1.3 Equivalence classes (10 points)

$$[1]_R = \{1, 22\}$$
$$[5]_R = \{5, 12\}$$
$$[35]_R = \{35\}$$
$$[41]_R = \{41, 55\}$$

1.4 Partial order and total order relation (30 points)

a) Partial order, not total order

(4 points) 1. Reflexivity:

For any $x \in \mathbb{Z}^+$, take $z = 1$. Then,

$$x = x^1.$$

Thus, $(x, x) \in R$ for all $x \in \mathbb{Z}^+$, so R is reflexive.

(4 points) 2. Antisymmetry:

Suppose that for some $x, y \in \mathbb{Z}^+$, we have

$$x = y^z \quad \text{and} \quad y = x^w,$$

for some positive integers z and w. Then, we get

$$x = (x^w)^z = x^{wz}.$$

If $x > 1$, then the equation $x^{wz} = x$ implies that

$$wz = 1.$$

Since w and z are positive integers, it follows that $w = z = 1$, and hence $x = y$.

If $x = 1$, then $y = 1$. Thus, in all cases, $x = y$, establishing antisymmetry.

(4 points) 3. Transitivity:

Assume $x, y, z \in \mathbb{Z}^+$ and that

$$x = y^a \quad \text{and} \quad y = z^b,$$

for some positive integers a and b. Then,

$$x = y^a = (z^b)^a = z^{ab},$$

and since ab is a positive integer, it follows that $(x, z) \in R$. Therefore, R is transitive.

(3 points) Not a total order: since not every pair of positive integers is comparable under R (e.g., 2 and 3 are not comparable because neither $2 = 3^z$ nor $3 = 2^z$ holds for any $z \in \mathbb{Z}^+$).

b) Partial order, not total order

Proof of partial order

- (4 points) Reflexive: $\frac{a}{a} = 1 = 2^0 \Rightarrow aRa$.
- (4 points) Antisymmetric: $aRb \Rightarrow \frac{b}{a} = 2^{k_1}$, $bRa \Rightarrow \frac{a}{b} = 2^{k_2}$, hence $\frac{b}{a} \cdot \frac{a}{b} = 1 = 2^0 = 2^{k_1+k2}$, so we have $k1 = 0, k2 = 0$, hence $a = b$.
- (4 points) Transitive: $aRb \Rightarrow \frac{b}{a} = 2^{k_1}$, $bRc \Rightarrow \frac{c}{b} = 2^{k_2}$, hence $\frac{b}{a} \cdot \frac{c}{b} = \frac{c}{a} = 2^{k_1+k2} = 2^k$, hence aRc.

(3 points) Not total order: for example, $(3, 4)$ are incomparable (might have other examples).

1.5 One-to-one and onto functions (20 points)

a) Not one-to-one. Not onto.

- (5 points) Since $f(-2) = f(1) = 2$, the function f is not one-to-one.
- (5 points) Since $f(x) \geq 0$ for any $x \in \mathbb{Z}$, the function f is not onto.

b) one-to-one. Not onto.

- (5 points) One-to-one: let $x_1, x_2 \in \mathbb{Z}$ such that $f(x_1) = f(x_2)$. So,

$$\begin{aligned} &x_1^3 + x_1 = x_2^3 + x_2 \\ &x_1^3 - x_2^3 + x_1 - x_2 = 0 \\ &(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 1) = 0 \end{aligned}$$

  So $x_1 - x_2 = 0$, $x_1 = x_2$.

- (5 points) Not onto: when x is even or odd, $x^3 + x$ is always even. Thus, when $f(x)$ is odd, there is no $x \in \mathbb{Z}$ such that $f(x) = x^3 + x$.

1.6 Proof by induction (10 points)

Proof. We proceed by induction on n.

Base case: If $n = 1$, then $1^2 = 1 = \frac{1(2)(3)}{6}$, as desired.

Induction hypothesis (IH): Fix $n \geq 1$ and assume that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Induction step: We want to show that

$$\begin{aligned} 1^2 + 2^2 + \cdots + (n+1)^2 &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}. \\ &= \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$

Note that

$$\begin{aligned} 1^2 + 2^2 + \cdots + (n+1)^2 &= (1^2 + 2^2 + \cdots + n^2) + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \text{ by our IH} \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\ &= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}, \end{aligned}$$

as desired. $\square$