3.6 Soundness and Completeness of $\mathscr{H}$

We now prove the soundness and completeness of the Hilbert system $\mathscr{H}$. As usual, soundness is easy to prove. Proving completeness will not be too difficult because we already know that the Gentzen system $\mathscr{G}$ is complete so it is sufficient to show how to transform any proof in $\mathscr{G}$ into a proof in $\mathscr{H}$.

Theorem 3.37 *The Hilbert system $\mathscr{H}$ is sound*: *If* $\vdash A$ *then* $\models A$.

Proof The proof is by structural induction. First we show that the axioms are valid, and then we show that *MP* preserves validity. Here are closed semantic tableaux for the *negations* of Axioms 1 and 3:

$$
\begin{array}{c}
\neg\,[A \rightarrow (B \rightarrow A)]\\
\downarrow\\
A, \neg\,(B \rightarrow A)\\
\downarrow\\
A, B, \neg\,A\\
\times
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & \neg\,[(\neg\,B \rightarrow \neg\,A) \rightarrow (A \rightarrow B)] & \\
 & \downarrow & \\
 & \neg\,B \rightarrow \neg\,A, \neg\,(A \rightarrow B) & \\
 & \downarrow & \\
 & \neg\,B \rightarrow \neg\,A, A, \neg\,B & \\
 \swarrow & & \searrow\\
\neg\,\neg\,B, A, \neg\,B & & \neg\,A, A, \neg\,B\\
\downarrow & & \times\\
B, A, \neg\,B & & \\
\times & &
\end{array}
$$

The construction of a tableau for the negation of Axiom 2 is left as an exercise.

Suppose that *MP* were not sound. There would be a set of formulas $\{A, A \rightarrow B, B\}$ such that A and $A \rightarrow B$ are valid, but B is not valid. Since B is not valid, there is an interpretation $\mathscr{I}$ such that $v_{\mathscr{I}}(B) = F$. Since A and $A \rightarrow B$ are valid, for *any* interpretation, in particular for $\mathscr{I}$, $v_{\mathscr{I}}(A) = v_{\mathscr{I}}(A \rightarrow B) = T$. By definition of $v_{\mathscr{I}}$ for implication, $v_{\mathscr{I}}(B) = T$, contradicting $v_{\mathscr{I}}(B) = F$. ■

There is no circularity in the final sentence of the proof: We are not using the syntactical proof rule *MP*, but, rather, the semantic definition of truth value in the presence of the implication operator.

Theorem 3.38 *The Hilbert system $\mathscr{H}$ is complete*: *If* $\models A$ *then* $\vdash A$.

By the completeness of the Gentzen system $\mathscr{G}$ (Theorem 3.8), if $\models A$, then $\vdash A$ in $\mathscr{G}$. The proof of the theorem showed how to construct the proof of A by first constructing a semantic tableau for $\neg\,A$; the tableau is guaranteed to close since A is valid. The completeness of $\mathscr{H}$ is proved by showing how to transform a proof in $\mathscr{G}$ into a proof in $\mathscr{H}$. Note that all three steps can be carried out algorithmically: Given an arbitrary valid formula in propositional logic, a computer can generate its proof.

We need a more general result because a proof in $\mathscr{G}$ is a sequence of *sets* of formulas, while a proof in $\mathscr{H}$ is a sequence of formulas.

Theorem 3.39 *If* $\vdash U$ *in* $\mathscr{G}$, *then* $\vdash \bigvee U$ *in* $\mathscr{H}$.

The difficulty arises from the clash of the data structures used: U is a set while $\bigvee U$ is a single formula. To see why this is a problem, consider the base case of the induction. The set $\{\neg p, p\}$ is an axiom in $\mathscr{G}$ and we immediately have $\vdash \neg p \vee p$ in $\mathscr{H}$ since this is simply $\vdash p \rightarrow p$. But if the axiom in $\mathscr{G}$ is $\{q, \neg p, r, p, s\}$, we can't immediately conclude that $\vdash q \vee \neg p \vee r \vee p \vee s$ in $\mathscr{H}$.

Lemma 3.40 *If* $U' \subseteq U$ *and* $\vdash \bigvee U'$ *in* $\mathscr{H}$ *then* $\vdash \bigvee U$ *in* $\mathscr{H}$.

Proof The proof is by induction using weakening, commutativity and associativity of disjunction (Theorems 3.34–3.35). We give the outline here and leave it as an exercise to fill in the details.

Suppose we have a proof of $\bigvee U'$. By repeated application of Theorem 3.34, we can transform this into a proof of $\bigvee U''$, where U'' is a permutation of the elements of U. By repeated applications of commutativity and associativity, we can move the elements of U'' to their proper places. ■

Example 3.41 Let $U' = \{A, C\} \subset \{A, B, C\} = U$ and suppose we have a proof of $\vdash \bigvee U' = A \vee C$. This can be transformed into a proof of $\vdash \bigvee U = A \vee (B \vee C)$ as follows, where Theorems 3.34–3.35 are used as derived rules:

1.	$\vdash A \vee C$	Assumption
2.	$\vdash (A \vee C) \vee B$	Weakening, 1
3.	$\vdash A \vee (C \vee B)$	Associativity, 2
4.	$\vdash (C \vee B) \rightarrow (B \vee C)$	Commutativity
5.	$\vdash A \vee (C \vee B) \rightarrow A \vee (B \vee C)$	Weakening, 4
6.	$\vdash A \vee (B \vee C)$	MP 3, 5

■

Proof of Theorem 3.39 The proof is by induction on the structure of the proof in $\mathscr{G}$. If U is an axiom, it contains a pair of complementary literals and $\vdash \neg p \vee p$ can be proved in $\mathscr{H}$. By Lemma 3.40, this can be transformed into a proof of $\bigvee U$.

Otherwise, the last step in the proof of U in $\mathscr{G}$ is the application of a rule to an α- or β-formula. As usual, we will use disjunction and conjunction as representatives of α- and β-formulas.

Case 1: A rule in $\mathscr{G}$ was applied to obtain an α-formula $\vdash U_1 \cup \{A_1 \vee A_2\}$ from $\vdash U_1 \cup \{A_1, A_2\}$. By the inductive hypothesis, $\vdash ((\bigvee U_1) \vee A_1) \vee A_2$ in $\mathscr{H}$ from which we infer $\vdash \bigvee U_1 \vee (A_1 \vee A_2)$ by associativity.

Case 2: A rule in $\mathscr{G}$ was applied to obtain a β-formula $\vdash U_1 \cup U_2 \cup \{A_1 \wedge A_2\}$ from $\vdash U_1 \cup \{A_1\}$ and $\vdash U_2 \cup \{A_2\}$. By the inductive hypothesis, $\vdash (\bigvee U_1) \vee A_1$ and $\vdash (\bigvee U_2) \vee A_2$ in $\mathscr{H}$. We leave it to the reader to justify each step of the following deduction of $\vdash \bigvee U_1 \vee \bigvee U_2 \vee (A_1 \wedge A_2)$:

1. $\vdash \bigvee U_1 \vee A_1$
2. $\vdash \neg \bigvee U_1 \rightarrow A_1$
3. $\vdash A_1 \rightarrow (A_2 \rightarrow (A_1 \wedge A_2))$
4. $\vdash \neg \bigvee U_1 \rightarrow (A_2 \rightarrow (A_1 \wedge A_2))$
5. $\vdash A_2 \rightarrow (\neg \bigvee U_1 \rightarrow (A_1 \wedge A_2))$
6. $\vdash \bigvee U_2 \vee A_2$
7. $\vdash \neg \bigvee U_2 \rightarrow A_2$
8. $\vdash \neg \bigvee U_2 \rightarrow (\neg \bigvee U_1 \rightarrow (A_1 \wedge A_2))$
9. $\vdash \bigvee U_1 \vee \bigvee U_2 \vee (A_1 \wedge A_2)$

■

Proof of Theorem 3.38 If $\models A$ then $\vdash A$ in $\mathscr{G}$ by Theorem 3.8. By the remark at the end of Definition 3.2, $\vdash A$ is an abbreviation for $\vdash \{A\}$. By Theorem 3.39, $\vdash \bigvee\{A\}$ in $\mathscr{H}$. Since A is a single formula, $\vdash A$ in $\mathscr{H}$. ■

3.7 Consistency

What would mathematics be like if both $1+1=2$ and $\neg(1+1=2) \equiv 1+1 \neq 2$ could be proven? An inconsistent deductive system is useless, because *all* formulas are provable and the concept of proof becomes meaningless.

Definition 3.42 A set of formulas U is *inconsistent* iff for some formula A, both $U \vdash A$ and $U \vdash \neg A$. U is *consistent* iff it is not inconsistent. A deductive system is *inconsistent* iff it contains an inconsistent set of formulas. ■

Theorem 3.43 *U is inconsistent iff for all A, $U \vdash A$.*

Proof Let A be an arbitrary formula. If U is inconsistent, for some formula B, $U \vdash B$ and $U \vdash \neg B$. By Theorem 3.21, $\vdash B \rightarrow (\neg B \rightarrow A)$. Using *MP* twice, $U \vdash A$. The converse is trivial. ■

Corollary 3.44 *U is consistent if and only if for some A, $U \nvdash A$.*

If a deductive system is sound, then $\vdash A$ implies $\models A$, and, conversely, $\not\models A$ implies $\nvdash A$. Therefore, if there is even a single falsifiable formula A in a sound system, the system must be consistent! Since $\not\models$ *false* (where *false* is an abbreviation for $\neg(p \rightarrow p)$), by the soundness of $\mathscr{H}$, $\nvdash$ *false*. By Corollary 3.44, $\mathscr{H}$ is consistent.