Introduction to Mathematical Logic

For CS Students

CS104/CS108

Yida TAO (陶伊达)

2024 年 3 月 26 日

Table of Contents

What's reasoning?

- Everyday reasoning: Natural Language and intuition.
- Semantic entailment: $\Sigma \models A$, use semantic truth table (tableaux) to determine if A follows from Σ (i.e., to reason semantically).

Problems with a purely semantical approach?

What’s reasoning?

1 Warm up

Problems with a purely semantical approach:

- Entries in the semantic truth table could be large or even infinite (think about common sense in real life and axioms in math).
- Semantic approach directly produces a ’yes/no’ answer, so it is difficult to recognize intermediate results.

The *deductive* approach overcomes the above problems, by reasoning syntactically or symbolically (don’t care about the semantics).

What’s a proof?

A proof is:

- a formal demonstration that a statement is true.
- A proof is generally syntactic, rather than semantic.
- Generically, a proof consists of a sequence of formulas.
- Inference rules justify subsequent lines of the proof which are inferred by the previous lines.

Remember Leibniz’s vision: Let’s calculate!

Definition

2 Overview of Proof Systems

A formal **proof system** (deductive system, 形式推演系统) consists of the language part and the inference part.

- Language part
  - Symbols, alphabet
  - Set of formulas
- Inference part
  - Set of axioms 公理
  - Inference rules

Proof: Starting from a set of formulas and a set of axioms, obtaining new formulas through finitely many times of mechanically applying inference rules.

Notation

2 Overview of Proof Systems

We notate ”there is a proof with premises Σ and conclusion A” by

$$\Sigma \vdash A$$

形式推演（语法、机械化）

只用公理与推理

$\Sigma \models A$ 用真值表（语义）逻辑推论

Types of Proof Systems

2 Overview of Proof Systems

We will introduce 3 formal proof systems.

- Hilbert-style system ($\Sigma \vdash_H A$): many axioms and only one rule. The deduction is linear.
- Natural Deduction System ($\Sigma \vdash_{ND} A$): Few axioms (even none) and many rules. The deductions are tree-like.
- Resolution ($\Sigma \vdash_{Res} A$): used to prove contradictions.

Table of Contents

3 The Hilbert-style Proof System

Language

Alphabet of $\mathscr{H}$

$\Sigma = \{(,), \neg, \rightarrow, p, q, r, \ldots\}$

Formulas of $\mathscr{H}$

1. Atoms $p, q, r, \ldots$ are formulas.
2. If A, B are formulas, then $(\neg A), (A \rightarrow B)$ are also formulas.
3. Only expressions of Σ that are generated by 1 and 2 are formulas.

Axioms & Inference Rules

Axioms

A, B, C are arbitrary well-formed formulas. 永真式 → 公理

- $A_1 : A \to (B \to A)$
- $A_2 : (A \to (B \to C)) \to ((A \to B) \to (A \to C))$
- $A_3 : (\neg A \to \neg B) \to (B \to A)$

Inference rule

Modus ponens (MP, 分离规则): A and $A \to B$ imply B。

$$r_{mp} : \frac{A, A \to B}{B}$$

Formal Proof

3 The Hilbert-style Proof System

A formal proof is a “logical chain” from assumptions to conclusions.

- First, the “chain” must be finite.
- Second, each “link” in the ”chain” may be:
  - Axioms (common sense)
  - Assumptions (premises)
  - Intermediate conclusions derived by using inference rules.

Formal Proof

Definition 6.1

A **proof** of formula A in $\mathscr{H}$ is a finite sequence of formulas

$$A_1, A_2, \dots, A_n$$

such that $A_n = A$, and for any $i \leq n$, A_i

- is either an axiom in $\mathscr{H}$ 公理
- or is A_j $(j < i)$ 定理
- or is derived from A_j, A_k $(j, k < i)$ by the MP rule (e.g., $A_k = A_j \to A_i$)

前面有Aj、Ak

$A_k = A_j \to A_i$

MP规则

Definition 6.2

If a formula A has a proof with premises Σ, then we say "A can be proved from Σ", denoted as $\Sigma \vdash_H A$, or simply $\Sigma \vdash A$.

If $\Sigma = \varnothing$, then $\Sigma \vdash A$ is simply $\varnothing \vdash A$, meaning that A is a **theorem** in $\mathscr{H}$, denoted as $\vdash A$.

If $A \in \Sigma$, then $\Sigma \vdash A$. (Called *Assumption* which will be used a lot in later proofs)

假定

Theorem (in $\mathscr{H}$) 6.3

$\vdash A \to A$

请用三个永真式证

Proof:

1.	$\vdash (A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$	Axiom 2
2.	$\vdash A \to ((A \to A) \to A)$	Axiom 1
3.	$\vdash (A \to (A \to A)) \to (A \to A)$	MP 1, 2
4.	$\vdash A \to (A \to A)$	Axiom 1
5.	$\vdash A \to A$	MP 3, 4

■

The proof here is purely syntactic (mechanic), no semantic involved.

Theorem

3 The Hilbert-style Proof System

Axiom ×3	Inference rule MP
Theorem 定理	Derived rule (可直接用)

指在 Hilbert-style Proof System 中的定理

Theorem 6.3

$\vdash A \to A$

(只是简化了证明)

The proof is rather complicated for such a trivial formula.
In order to formalize the powerful methods of inference used in mathematics, we introduce new rules of inference called derived rules.

Derived rules in $\mathscr{H}$

Rule 6.5 (Contrapositive rule)

$$\frac{\vdash \neg B \to \neg A}{\vdash A \to B}$$

Read as: (In $\mathscr{H}$) If $\neg B \to \neg A$ has a proof , then $A \to B$ has a proof.

This rule immediately follows from Axiom 3 and MP.

Deduction Rule

The most important derived rule is the deduction rule.

Rule 6.4 Deduction rule

For any set of formulas Γ and formulas A, B in $\mathscr{H}$, $\Gamma \cup \{A\} \vdash B$ if and only if $\Gamma \vdash A \to B$.

Specifically, $\{A\} \vdash B$ if and only if $\vdash A \to B$。

Derived rules in $\mathscr{H}$

3 The Hilbert-style Proof System

Theorem 6.6

$\vdash (A \to B) \to ((B \to C) \to (A \to C))$ 用5;中的

Proof

1.	$\{A \to B, B \to C, A\} \vdash A$	Assumption
2.	$\{A \to B, B \to C, A\} \vdash A \to B$	Assumption
3.	$\{A \to B, B \to C, A\} \vdash B$	MP 1, 2
4.	$\{A \to B, B \to C, A\} \vdash B \to C$	Assumption
5.	$\{A \to B, B \to C, A\} \vdash C$	MP 3, 4
6.	$\{A \to B, B \to C\} \vdash A \to C$	Deduction 5
7.	$\{A \to B\} \vdash [(B \to C) \to (A \to C)]$	Deduction 6
8.	$\vdash (A \to B) \to [(B \to C) \to (A \to C)]$	Deduction 7

Derived rules in $\mathscr{H}$

Rule 6.7 (Transitivity rule)

$$\frac{\vdash A \to B \quad \vdash B \to C}{\vdash A \to C}$$

Followed from step 6 in the previous proof.

Derived rules in $\mathscr{H}$

assumption ⇒ 利用前提

Theorem 6.8

$\vdash (A \to (B \to C)) \to (B \to (A \to C))$

*取前面作前提，去构造后面的结构

Proof

1.	$\{A \to (B \to C), B, A\} \vdash A$	Assumption
2.	$\{A \to (B \to C), B, A\} \vdash A \to (B \to C)$	Assumption
3.	$\{A \to (B \to C), B, A\} \vdash B \to C$	MP 1, 2
4.	$\{A \to (B \to C), B, A\} \vdash B$	Assumption
5.	$\{A \to (B \to C), B, A\} \vdash C$	MP 3, 4
6.	$\{A \to (B \to C), B\} \vdash A \to C$	Deduction 5
7.	$\{A \to (B \to C)\} \vdash B \to (A \to C)$	Deduction 6
8.	$\vdash [A \to (B \to C)] \to [B \to (A \to C)]$	Deduction 7

Derived rules in $\mathscr{H}$

前件交换

Rule 6.9 (Exchange of antecedent rule)

$$\frac{\vdash A \to (B \to C)}{\vdash B \to (A \to C)}$$

Followed from step 7 in the previous proof.

Exchanging the antecedent simply means that it doesn't matter in which order we use the lemmas necessary in a proof.

Derived rules in $\mathscr{H}$

(前提中不能既有A又有¬A)

Theorem 6.10

$\vdash \neg A \to (A \to B)$

$\vdash A \to (\neg A \to B)$

↓ 用前提交换

Proof

1.	$\{\neg A\} \vdash \neg A \to (\neg B \to \neg A)$	Axiom 1
2.	$\{\neg A\} \vdash \neg A$	Assumption
3.	$\{\neg A\} \vdash \neg B \to \neg A$	MP 1, 2
4.	$\{\neg A\} \vdash (\neg B \to \neg A) \to (A \to B)$	Axiom 3
5.	$\{\neg A\} \vdash A \to B$	MP 3, 4
6.	$\vdash \neg A \to (A \to B)$	Deduction 5

If you can prove some formula A and its negation $\neg A$, then you can prove any formula B.

Derived rules in $\mathscr{H}$

1. $\{\neg\neg A\} \vdash \neg\neg A$ Assumption

Theorem 6.11

2. $\{\neg\neg A\} \vdash \neg\neg A \to (\neg\neg\neg\neg A \to \neg\neg A)$ Axiom 1

$\vdash \neg\neg A \to A$

3. $\{\neg\neg A\} \vdash \neg\neg\neg\neg A \to \neg\neg A$ MP 1, 2

$\vdash A \to \neg\neg A$

4. $\{\neg\neg A\} \vdash \neg A \to \neg\neg\neg A$ Contrapositive

Rule 6.12 (Double negation rule)

5. $\{\neg\neg A\} \vdash \neg\neg A \to A$ Contrapositive

6. $\{\neg\neg A\} \vdash A$ MP 1, 5

$$\frac{\vdash \neg\neg A}{\vdash A}, \quad \frac{\vdash A}{\vdash \neg\neg A}$$

7. $\vdash \neg\neg A \to A$ Deduction rule

Double negation is a very intuitive rule. We expect that "it is raining" and "it is not true that it is not raining" will have the same truth value.

1. $\{\neg A \to false\} \vdash \neg A$ ~~false~~ Assumption

2. $\{\neg A \to false\} \vdash \neg false \to \neg\neg A$ Contrapositive

Theorem 6.13

$\vdash$ true

$\vdash \neg$false

3. $\{\neg A \to false\} \vdash \neg false$ Th 6.13

4. $\{\neg A \to false\} \vdash \neg\neg A$ MP 2,3

Theorem 6.14

$\vdash (\neg A \to \text{false}) \to A$

5. $\{\neg A \to false\} \vdash A$ Double negation

6. $\vdash (\neg A \to false) \to A$ Deduction rule

Derived rules in $\mathscr{H}$

Rule 6.14 (Reductio ad absurdum)

$$\frac{\vdash \neg A \to \text{false}}{\vdash A}$$

Reductio ad absurdum is a very useful rule in mathematics: Assume the negation of what you wish to prove and show that it leads to a contradiction.

Derived rules in $\mathscr{H}$

1. $\{A \to \neg A, \neg\neg A\} \vdash \neg\neg A$ Assumption
2. $\{A \to \neg A, \neg\neg A\} \vdash A$ Double negation
3. $\{A \to \neg A, \neg\neg A\} \vdash A \to \neg A$ Assumption
4. $\{A \to \neg A, \neg\neg A\} \vdash \neg A$ MP 2, 3
5. $\{A \to \neg A, \neg\neg A\} \vdash A \to (\neg A \to false)$ Th 6.10
6. $\{A \to \neg A, \neg\neg A\} \vdash \neg A \to false$ MP 2, 5
7. $\{A \to \neg A, \neg\neg A\} \vdash false$ MP 4, 6
8. $\{A \to \neg A\} \vdash \neg\neg A \to false$ Deduction rule
9. $\{A \to \neg A\} \vdash \neg A$ Reductio..
10. 得证

Theorem 6.15

$\vdash (A \to \neg A) \to \neg A$

Proved in class.

Readings

Optional

- TextF: Section 3.1, 3.3, 3.4
- Text3: 第二章 2.6
- Video1: part 7, part 8

Introduction to Mathematical Logic

Thank you for listening!
Any questions?