

#### **Introduction to Mathematical Logic**

For CS Students

CS104/CS108

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#### **Types of Proof Systems**

1 Warm up

We will introduce 3 formal proof systems.

- Hilbert-style system ( $\Sigma \vdash_H A$ ): many axioms and only one rule. The deduction is linear.
- Natural Deduction System ( $\Sigma \vdash_{ND} A$ ): Few axioms (even none) and many rules. The deductions are tree-like.<sup>1</sup>
- Resolution ( $\Sigma \vdash_{Res} A$ ): used to prove contradictions.

<sup>&</sup>lt;sup>1</sup>Part of this slide is based on the course notes of UWaterloo CS245.



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Warm up

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#### **Alphabet of ND**

$$\Sigma = \{(,),\neg,\wedge,\vee,\rightarrow,\leftrightarrow,p,q,r,\ldots\}$$

#### Formulas of ND

- 1. Atoms p, q, r, ... are formulas.
- 2. If A, B are formulas, then  $(\neg A), (A \land B), (A \lor B), (A \to B), (A \leftrightarrow B)$  are also formulas.
- 3. Only expressions of  $\Sigma$  that are generated by 1 and 2 are formulas.



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#### **Reflexivity (Premise)**

$$\Sigma \cup \{\alpha\} \vdash \alpha \quad \text{( or } \Sigma, \alpha \vdash \alpha \text{)}$$

If you want to vrite down a previous formula in the proof a ain, you can do it by reflexivity (自反).



#### An example of using reflexivity

2 The ND Proof System

A proof of  $\{p, q\} \vdash_{ND} p$ 

Premise

*q* Premise

p Reflexivity: 1 A 5 Z RI A 3 7

Alternatively, we could simply write:

**Premise** 



For the following inference rules:

For each logical symbol, the rules come in pairs.

- An "introduction rule" adds the symbol to the formula.
- An "elimination rule" removes the symbol from the formula.





### **Inference Rules for Conjunction**

2 The ND Proof System

Name	⊢-notation	inference notation
$\land$ -introduction $(\land i)$	$\begin{array}{c} \text{If } \Sigma \vdash_{\textit{ND}} \alpha \text{ and } \Sigma \vdash_{\textit{ND}} \beta, \\ \text{then } \Sigma \vdash_{\textit{ND}} (\alpha \wedge \beta) \end{array}$	$\frac{\alpha  \beta}{(\alpha \land \beta)}$

Name	⊢-notation	inference notation
$\land$ -elimination $(\land e)$	$\begin{array}{c} \text{If } \Sigma \vdash_{\textit{ND}} (\alpha \land \beta), \\ \text{then } \Sigma \vdash_{\textit{ND}} \alpha \text{ and } \Sigma \vdash_{\textit{ND}} \beta \end{array}$	$\frac{(\alpha \wedge \beta)}{\alpha}  \frac{(\alpha \wedge \beta)}{\beta}$

Intuition from tautology:  $\alpha \to (\beta \to \alpha \land \beta), \alpha \land \beta \to \alpha, \alpha \land \beta \to \beta$ 



#### **Inference Rules for Conjunction**

2 The ND Proof System

Example: Show that  $\{(p \land q)\} \vdash_{ND} (q \land p)$ 

Proof:

1. 
$$(p \land q)$$
 Premise  
2.  $q$   $\land$ e: 1  
3.  $p$   $\land$ e: 1  
4.  $(q \land p)$   $\land$ i: 2,3



2 The ND Proof System

Name	⊢-notation	inference notation
$ ightarrow$ -elimination $( ightarrow e)$ $(modus\ ponens)$	$\begin{array}{c} \text{If } \Sigma \vdash_{\textit{ND}} (\alpha \rightarrow \beta) \text{ and } \Sigma \vdash_{\textit{ND}} \alpha, \\ \text{then } \Sigma \vdash_{\textit{ND}} \beta \end{array}$	$\frac{(\alpha \to \beta)  \alpha}{\beta}$

Intuition: If you assume  $\alpha$  is true and  $\alpha$  implies  $\beta$ , then you may conclude  $\beta$ .



2 The ND Proof System

Name	⊢-notation	inference notation
$ ightarrow$ -introduction $\left( ightarrow$ i $ ight)$	$\begin{array}{c} \text{If } \Sigma, \alpha \vdash_{\mathit{ND}} \beta, \\ \text{then } \Sigma \vdash_{\mathit{ND}} (\alpha \rightarrow \beta) \end{array}$	$\frac{\begin{bmatrix} \alpha \\ \vdots \\ \beta \end{bmatrix}}{(\alpha \to \beta)}$

Intuition: If by assuming  $\alpha$  is true we can get  $\beta$ , then  $\alpha$  implies  $\beta$ .

The "box" denotes a sub-proof. Nothing inside the sub-proof may come out. Outside of the sub-proof, we could only use the sub-proof as a whole.



2 The ND Proof System

Example: Give a proof of  $\{(p \to q), (q \to r)\} \vdash_{ND} (p \to r)$ 

#### Proof:

1. 
$$(p \rightarrow q)$$
 Premise

2. 
$$(q \rightarrow r)$$
 Premise

$$q \rightarrow e: 1, 3$$

5. 
$$\mid r \rightarrow e: 2, 4$$

6. 
$$(p \rightarrow r) \rightarrow i: 3-5$$

$$p$$
 Assumption  $q$   $\rightarrow$ e: 1, 3  $r$   $\rightarrow$ e: 2, 4  $r$   $\rightarrow$ i: 3-5  $r$ 





#### **Inference Rules for Disjunction**

2 The ND Proof System

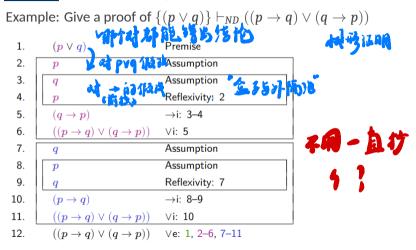
Name	⊢-notation	inference notation
∨-introduction (∨i)	$\begin{array}{c} \text{If } \Sigma \vdash_{\textit{ND}} \alpha, \\ \text{then } \Sigma \vdash_{\textit{ND}} (\alpha \vee \beta) \end{array}$	$\frac{\alpha}{(\alpha \vee \beta)} \qquad \frac{\alpha}{(\beta \vee \alpha)}$
( ∨ 1)	and $\Sigma \vdash_{\mathit{ND}} (\beta \lor \alpha)$	
	If $\Sigma$ , $\alpha_1 \vdash_{ND} \beta$	$\alpha_1$ $\alpha_2$
∨-elimination	and $\Sigma$ , $\alpha_2 \vdash_{ND} \beta$ ,	
(∨e)	then	$  (\alpha_1 \lor \alpha_2)   \dot{\beta}   \dot{\beta}  $
		$\beta$

 $\vee e$  is also known as "proof by cases".

Intuition: from the tautology  $(\alpha \lor \beta) \land (\alpha \to \gamma) \land (\beta \to \gamma) \to \gamma$ 



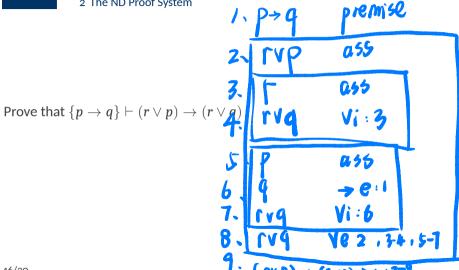
2 The ND Proof System





#### **Examples**

2 The ND Proof System





2 The ND Proof System

If an assumption  $\alpha$  leads to a contradiction, then we have  $(\neg \alpha)$ .

Name	⊢-notation	inference notation
$\neg$ -introduction $(\neg i)$	$\begin{array}{ c c c } \text{If } \Sigma, \alpha \vdash_{\mathit{ND}} \bot, \\ \text{then } \Sigma \vdash_{\mathit{ND}} (\neg \alpha) \end{array}$	$\frac{\alpha}{\vdots}$ $\frac{\Box}{(\neg \alpha)}$

We shall use the notation  $\perp$  to represent any contradiction. It may appear in proofs as if it were a formula.



2 The ND Proof System

If we have both  $\alpha$  and  $(\neg \alpha)$ , then we have a contradiction, also known as  $\neg e$  $(\neg$ -elimination).



Name	⊢-notation	inference notation
$\perp$ -introduction	$\Sigma$ , $\alpha$ , $(\neg \alpha) \vdash_{ND} \bot$	$\frac{\alpha  (\neg \alpha)}{\perp}$



2 The ND Proof System

Example. Show that $\{\alpha \to (\neg \alpha)\} \vdash_{\mathit{ND}} (\neg \alpha)$		上:有みねる	
1.	$(\alpha \to (\neg \alpha))$	Premise	→ 1
2.	$\alpha$	Assumption	7 4
3.	$\begin{array}{c} \alpha \\ (\neg \alpha) \\ \bot \end{array}$	→e: 1, 2	71: 01 UN _
4.		⊥i: 2, 3	1 : d w 1
5.	$(\neg \alpha)$	¬i: 2–4	



2 The ND Proof System

The elimination rule for *double negations*:

Name	⊢-notation	inference notation
¬¬-elimination	$\begin{array}{c} \text{If } \Sigma \vdash_{\mathit{ND}} (\neg (\neg \alpha)), \\ \text{then } \Sigma \vdash_{\mathit{ND}} \alpha \end{array}$	$\frac{\left(\neg(\neg\alpha)\right)}{\alpha}$
ירן	3)	



2 The ND Proof System

#### Contradiction elimination:

Name	⊢-notation	inference notation
	If $\Sigma \vdash_{\mathit{ND}} \bot$ , then $\Sigma \vdash_{\mathit{ND}} \alpha$	$\frac{\perp}{\alpha}$

The rule of  $\perp$ -elimination is redundant. Why?



2 The ND Proof System

Any proof that uses  $\perp e$ 

27. 
$$\perp$$
  $\langle some rule \rangle$ 

28. 
$$\alpha$$
  $\perp$ e: 27.

...can be replaced by existing inference rules.

27. 
$$\perp$$
  $\langle some \ rule \rangle$   
28.  $(\neg \alpha)$  Assumption  
29.  $\perp$  Reflexivity: 27  
30.  $(\neg (\neg \alpha))$   $\neg i$ : 28–29  
31.  $\alpha$   $\neg \neg e$ : 30.



Whenever we have a proof of the form  $\Gamma \vdash_{ND} \alpha$ , we can consider it as a derived rule:

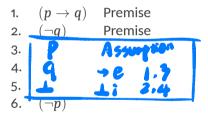
$$\frac{\Gamma}{\alpha}$$

If we use this in a proof, it can be replaced by the original proof of  $\Gamma \vdash_{ND} \alpha$ . The result is a proof using only the basic rules.

Using derived rules does not expand the things that can be proved. But they can make it easier to find a proof.



Modus tollens (MT, 否定后件):  $\{(p \to q), (\neg q)\} \vdash_{ND} (\neg p)$ 



Please finish the proof.





Modus tollens can be used as a derived rule:

$$\frac{\alpha \to \beta \quad \neg \beta}{\neg \alpha} \quad \mathbf{MT}$$



#### **Derived Rules**

2 The ND Proof System

Double-negation introduction:  $\Sigma, \alpha \vdash_{ND} (\neg(\neg\alpha))$ 

$$\frac{\alpha}{(\neg(\neg\alpha))}$$
  $\neg\neg i$ 

1. 
$$\alpha$$
 Premise

2. 
$$(\neg \alpha)$$
 Assumption

3. 
$$\downarrow \perp$$
  $\perp$  i: 1, 2

4. 
$$(\neg(\neg\alpha))$$
 ¬i: 2–3



#### **Derived Rules**

2 The ND Proof System

Proof by contradiction (reductio ad absurdum):



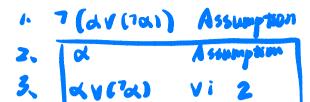
. 
$$((\neg \alpha) \rightarrow \bot)$$
 Premise

2. 
$$(\neg \alpha)$$
 Assumption

4. 
$$(\neg(\neg\alpha))$$
 ¬i: 2–3

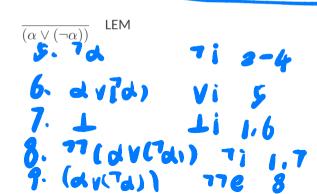


# **Derived Rules**2 The ND Proof System



Law of Excluded Middle (tertiam non datur, 其中律)

Proved in class.





• TextB: Section 1.2

• Text1: 第二章 2.6



## Introduction to Mathematical Logic

Thank you for listening!
Any questions?