Introduction to Mathematical Logic

For CS Students

CS104/CS108

Yida TAO (陶伊达)

2024 年 3 月 19 日

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Recall that logic is the science of reasoning.
One important goal of logic is to infer that a conclusion is true based on a set of premises.

A logical argument:

Premise 1

Premise 2

......

Premise 3

Conclusion

A common problem is to prove that an argument is valid, that is the set of premises semantically entails (可推导) the conclusion.

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2 Semantic Entailment

Definition

Let Σ be a set of formulas ($\Sigma \subseteq Form(\mathscr{L}^p)$), A be a formula ($A \in Form(\mathscr{L}^p)$). We say:

- A is a logical consequence (逻辑推论) of Σ, or
- Σ (semantically) entails (逻辑蕴含) A, or
- $\Sigma \models A$

if and only if

$$\text{For all truth valuation } v \text{, if } \Sigma^v = 1 \text{ then } A^v = 1.$$

We use $\Sigma \not\models A$ to denote "not $\Sigma \models A$", which is:

There exists a truth valuation v such that $\Sigma^v = 1$ and $A^v = 0$.

How do we prove $\Sigma \models A$?

- **Direct proof**: For every truth valuation under which all of the premises are true, show that the conclusion is also true under this valuation.
- **Using a truth table**: Consider all rows of the truth table in which all of the formulas in Σ are true. Verify that A is true in all of these rows.

Example: Prove $\neg p \models p \land \neg q \to p \land q$

Proving entailment

How do we prove $\Sigma \models A$?

- **Proof by contradiction**: Assume that the entailment does not hold, which means that there is a truth valuation under which all of the premises are true and the conclusion is false. Derive a contradiction.

Example: Prove $A \to B, B \to C \models A \to C$

Disproving entailment

2 Semantic Entailment

How do we prove $\Sigma \not\models A$?

- Find one truth valuation v under which all of the premises in Σ are true and the conclusion A is false.

Example: $\neg(\overset{0}{A} \leftrightarrow \overset{1}{B}) \vee \overset{0}{C}, \overset{1}{B} \wedge \neg\overset{0}{C} \not\models \neg A \wedge (\overset{1}{B} \to \overset{0}{C})$

0 1 0

Socrate says: p q

$\{p \to q, p\} \models q$) 相同

$(p \to q) \wedge p \models q$

"If I'm guilty, I must be punished; I'm guilty. Thus I must be punished."

"If I'm guilty, I must be punished; I'm not guilty. Thus I must not be punished."

Which argument(s) is logically correct?

第二句话: $\{p \to q, \neg p\} \models \neg q$

0 1 违反 ×

Properties

2 Semantic Entailment

Semantic Entailment 语义蕴含

$A \vDash B$ if and only if $A \to B$ is tautology.

Logical Equivalence

$A \equiv B$ if and only if both $A \vDash B$ and $B \vDash A$.

Empty set

$\varnothing \vDash A$ means that A is tautology.

空集永远可推出A

Theorem

$A_1, \ldots, A_n \vDash A$ if and only if $\varnothing \vDash A_1 \wedge \ldots \wedge A_n \to A$.

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3 Logical Connectives

How many n-ary logical connectives?

So far, we have discussed:

- One unary connective symbol: $\neg$
- Four binary connective symbols: $\wedge, \vee, \rightarrow, \leftrightarrow$

In fact, there are more unary and binary connective symbols, as well as n-ary connectives for $n > 2$.

How many n-ary connectives are there?

Unary Connectives

Number of n-ary Connectives

For any $n \geq 1$, there are 2^{2^n} different n-ary connective symbols.

We temporarily use f, g (with subscripts) to represent any connective. $fA_1 \ldots A_n$ denotes a formula formed by connecting formulas $A_1, \ldots, A_n$ using an n-ary connective f.

When $n = 1$, there are $2^{2^1} = 4$ different unary connectives, f_1, f_2, f_3, f_4.

A	f_1A	f_2A	f_3A	f_4A
1	1	1	0	0
0	1	0	1	0

Binary Connectives

3 Logical Connectives

When $n = 2$, there are $2^{2^2} = 16$ binary connectives, $g_1, g_2, \ldots, g_{16}$。

A	B	g_1AB	g_2AB	g_3AB	g_4AB	g_5AB	g_6AB	g_7AB	g_8AB
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1
A	B	g_9AB	$g_{10}AB$	$g_{11}AB$	$g_{12}AB$	$g_{13}AB$	$g_{14}AB$	$g_{15}AB$	$g_{16}AB$
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Binary Connective

- g_9: Joint denial (或非词, neither...nor...), denoted by $\downarrow$, $p \downarrow q \equiv \neg(p \vee q)$
- g_{15}： Alternative denial (与非词, not both), denoted by $\uparrow$, $p \uparrow q \equiv \neg(p \wedge q)$
- g_7： Exclusive disjunction (异或词, either...or...), denoted by $\otimes$, $p \otimes q \equiv \neg(p \leftrightarrow q)$

Redundancy

Formulas $(p \to q)$ and $((\neg p) \vee q)$ are logically equivalent.

可定义的

Hence, $\to$ is said to be definable in terms of $\neg$ and $\vee$.

We never need to use $\to$; we can always write an equivalent formula without it.

Adequate Set

A set of connectives is said to be adequate (完备的) iff every well-formed formula is logically equivalent to a well-formed formula using only connectives from the set. Or, every n-ary connectives is definable in terms of only the connectives from the adequate set.

n元的只要用

1,2元定义即可

Theorem 5.1

$\{\neg, \wedge, \vee\}$ is an adequate set of connectives.

Proof: By structural induction. Let $P(\gamma)$ be the statement that γ is logically equivalent to a well-formed formula that uses only connectives from the set $S = \{\neg, \wedge, \vee\}$. We prove this by structural induction.

Base case: If γ is a propositional formula, the statement holds since there are no connectives.
Inductive Hypothesis: Assume $P(\alpha)$ and $P(\beta)$ holds for some α and β.

Theorem 5.1

$\{\neg, \wedge, \vee\}$ is an adequate set of connectives.

Proof (Continued):

Induction step:

If $\gamma = (\neg\alpha)$, then since α is logically equivalent to a formula using only connectives from S, we have that γ must also be since it is the negation of this formula.

If $\gamma = (\alpha \star \beta)$:

- If $\star \in S$, we are done by the logic above.
- If $\star = \rightarrow$, then $\gamma \equiv ((\neg\alpha) \vee \beta)$ by the implication rule, and we're done.
- If $\star = \leftrightarrow$, then $\gamma \equiv ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$ and using the implication rule, we're done.

对 $\{\neg, \wedge\}$: $\alpha \vee \beta \equiv \neg(\neg(\alpha \vee \beta))$

$\equiv \neg(\neg\alpha \wedge \neg\beta)$

$\alpha \to \beta \equiv \neg\alpha \vee \beta$

$\alpha \leftrightarrow \beta \equiv (\alpha \to \beta) \wedge (\beta \to \alpha)$

Lemma 5.2

Each of the sets $\{\neg, \wedge\}$, $\{\neg, \vee\}$ and $\{\neg, \to\}$ is adequate.

Proof Sketch: The first two follow from DeMorgan's Laws. The third can follow from the first two. They can all be proven by structural induction or reduced to another known adequate set of connectives.

Assignment

Coursework

- Assignment 3 on PL semantics.

Readings

Optional

- TextI: 1.3
- Text1: 2.5, 2.8

Introduction to Mathematical Logic

Thank you for listening!
Any questions?