5 Assignment 5 Solutions (100 points)

5.1 Soundness and Completeness of ND (20 points)

Solution 1: If $\{\alpha, \beta\} \vdash_{ND} \gamma$ holds, then there is a natural deduction proof which starts with α and β as the premises and ends with γ.

1.	α	Premise
2.	β	Premise
3.	...	...
4.	γ	...

Hence, we can construct a ND proof sequence for $\varnothing \vdash (\alpha \wedge \beta) \to \gamma$:

1.	$\alpha \wedge \beta$	Assumption
2.	α	$\wedge e$,1
3.	β	$\wedge e$,1
4.	...	...
5.	γ	...
6.	$(\alpha \wedge \beta) \to \gamma$	$\to i$ 1-5

By the soundness of ND, the entailment $\varnothing \models (\alpha \wedge \beta) \to \gamma$ holds.

Solution 2: Assume $\{\alpha, \beta\} \vdash_{ND} \gamma$ holds. By the soundness of natural deduction, the entailment $\{\alpha, \beta\} \models \gamma$ holds. By definition, $\{\alpha, \beta\} \models \gamma$ means that for all truth valuation v under which $\alpha^v = T$ and $\beta^v = T$, $\gamma^v = T$. Hence, for all truth valuation v under which $\alpha^v = T$ and $\beta^v = T$, $(\alpha \wedge \beta) \to \gamma$ is true.

If $\alpha^v = F$ or $\beta^v = F$, $(\alpha \wedge \beta) \to \gamma$ is also true. Hence, $\varnothing \models (\alpha \wedge \beta) \to \gamma$, i.e., it is a tautology.

5.2 CNF and Resolution (20 points)

1. (10 points)

Formalization (3 points)

$$\begin{aligned}\text{Let:}\quad & A = \text{Meeting A is scheduled}\\ & B = \text{Meeting B is scheduled}\\ & C = \text{Meeting C is scheduled}\end{aligned}$$

$$\begin{aligned}\text{Constraint 1:}\quad & A \to (\neg B \wedge C)\\ \text{Constraint 2 :}\quad & \neg C \to ((A \wedge \neg B) \vee (\neg A \wedge B))\\ \text{Constraint 3 :}\quad & \neg A \to \neg C\end{aligned}$$

Then, conjunct all three formulas.

Convert to CNF (3 points)

$$\begin{aligned}\text{Constraint 1:}\quad & A \to (\neg B \wedge C)\\ & \equiv \neg A \vee (\neg B \wedge C)\\ & \equiv (\neg A \vee \neg B) \wedge (\neg A \vee C)\end{aligned}$$

$$\begin{aligned}\text{Constraint 2 :}\quad & \neg C \to ((A \wedge \neg B) \vee (\neg A \wedge B))\\ & \equiv \ldots\ldots\\ & \equiv C \vee ((A \vee B) \wedge (\neg A \vee \neg B))\\ & \equiv (C \vee A \vee B) \wedge (C \vee \neg A \vee \neg B)\end{aligned}$$

$$\begin{aligned}\text{Constraint 3 :}\quad & \neg A \to \neg C\\ & \equiv A \vee \neg C\end{aligned}$$

Final CNF Clauses (4 points):

$$(\neg A \vee \neg B) \wedge (\neg A \vee C) \wedge (C \vee A \vee B) \wedge (C \vee \neg A \vee \neg B) \wedge (A \vee \neg C)$$

2. (10 points) Yes. Schedule Meeting B only, or schedule Meeting A and C.

5.3 Syntax (30 points)

2,3,6,8,10,13,14

5.4 Formalization (30 points)

1. All Students are smart.

$$\forall x(Student(x) \rightarrow Smart(x))$$

2. Every course has at least one prerequisite course.

$$\forall x\,(\text{Course}(x) \rightarrow \exists y\,(\text{Course}(y) \land \text{Prerequisite}(y,x)))$$

3. Some students registered for all courses.

$$\exists x\,(\text{Student}(x) \land \forall y\,(\text{Course}(y) \rightarrow \text{Registered}(x,y)))$$

4. No student is both a TA and a professor.

$$\forall x\,(\text{Student}(x) \rightarrow \neg(\text{TA}(x) \land \text{Professor}(x)))$$

$$\neg\exists x\,(\text{Student}(x) \land \text{TA}(x) \land \text{Professor}(x))$$

5. Only professors can access the restricted section of the library.

$$\forall x\,(\text{CanAccess}(x,s) \land \text{Restricted}(s) \rightarrow \text{Professor}(x))$$

6. There is a professor who has never taught any course.

$$\exists x\,(\text{Professor}(x) \land \forall y\,(\text{Course}(y) \rightarrow \neg\text{Teaches}(x,y)))$$

7. Every student loves some student.

$$\forall x(Student(x) \rightarrow \exists y(Student(y) \land Loves(x,y)))$$

8. Every student loves some other student.

$$\forall x(Student(x) \rightarrow \exists y(Student(y) \land \neg(x=y) \land Loves(x,y)))$$

9. There is a student who is loved by every other student.

$$\exists x(Student(x) \wedge \forall y(Student(y) \wedge \neg(x = y) \rightarrow Loves(y, x)))$$

10. Some students love only themselves.

$$\exists x\,(\text{Student}(x) \wedge \text{Loves}(x, x) \wedge \forall y\,(\text{Loves}(x, y) \rightarrow x = y))$$

11. There is at least one student.

$$\exists x\ \text{Student}(x)$$

12. There is only one student.

$$\exists x\,(\text{Student}(x) \wedge \forall y\,(\text{Student}(y) \rightarrow y = x))$$

13. There are at least two students.

$$\exists x \exists y\ \ (\text{Student}(x) \wedge \text{Student}(y) \wedge \neg(x = y))$$

14. There are more than two students.

$$\exists x \exists y \exists z\,(\text{Student}(x) \wedge \text{Student}(y) \wedge \text{Student}(z) \wedge \neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z))$$

15. Exactly two students failed Geometry.

$$\begin{aligned}
&\exists x\,\exists y\,(\text{Student}(x) \wedge \text{Student}(y) \wedge \neg(x = y) \wedge \\
&\qquad \text{Failed}(x, \text{geometry}) \wedge \text{Failed}(y, \text{geometry}) \wedge \\
&\qquad \forall z\,(\text{Student}(z) \wedge \text{Failed}(z, \text{geometry}) \rightarrow (z = x \vee z = y)))
\end{aligned}$$