Introduction to Mathematical Logic

For CS Students

CS104/CS108

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- ▶ Satisfiability and Validity
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An interpretation $\mathcal{I}$ and environment E satisfy a formula α, denoted $\mathcal{I} \models_E \alpha$, iff $\alpha^{(\mathcal{I},E)} = 1$. They do not satisfy α, denoted $\mathcal{I} \not\models_E \alpha$, if $\alpha^{(\mathcal{I},E)} = 0$.

Form of α	Condition for $\mathcal{I} \models_E \alpha$
$P(t_1, \ldots, t_k)$	$\langle t_1^{(\mathcal{I},E)}, \ldots, t_k^{(\mathcal{I},E)} \rangle \in P^{\mathcal{I}}$
$(\neg\beta)$	$\mathcal{I} \not\models_E \beta$
$(\beta \wedge \gamma)$	both $\mathcal{I} \models_E \beta$ and $\mathcal{I} \models_E \gamma$
$(\beta \vee \gamma)$	either $\mathcal{I} \models_E \beta$ or $\mathcal{I} \models_E \gamma$ (or both)
$(\beta \rightarrow \gamma)$	either $\mathcal{I} \not\models_E \beta$ or $\mathcal{I} \models_E \gamma$ (or both)
$(\forall x\ \beta)$	for every $a \in dom(\mathcal{I})$, $\mathcal{I} \models_{E[x \mapsto a]} \beta$
$(\exists x\ \beta)$	there is some $a \in dom(\mathcal{I})$ such that $\mathcal{I} \models_{E[x \mapsto a]} \beta$

If $\mathcal{I} \models_E \alpha$ for every E, then $\mathcal{I}$ satisfies α, denoted $\mathcal{I} \models \alpha$.

Validity and Satisfiability

Validity and satisfiability of formulas have definitions analogous to the ones for propositional logic (the term "tautology" is not used in FOL).

A formula α is

- valid: if every interpretation and environment satisfy α; that is, if $\mathcal{I} \models_E \alpha$ for every $\mathcal{I}$ and E.
- satisfiable: if some interpretation and environment satisfy α; that is, if $\mathcal{I} \models_E \alpha$ for some $\mathcal{I}$ and E.
- unsatisfiable: if no interpretation and environment satisfy α; that is, if $\mathcal{I} \not\models_E \alpha$ for every $\mathcal{I}$ and E.

Example 1

Let $f^{(1)}$ and $h^{(2)}$ be function symbols, $P^{(1)}$ and $Q^{(2)}$ be predicate symbols, a, b, c be constant symbols.

Define an interpretation $\mathcal{I}$ by

- Domain: $D = \{1, 2, 3\}$
- Constant: $a^{\mathcal{I}} = 1$, $b^{\mathcal{I}} = 2$, $c^{\mathcal{I}} = 3$
- Functions: $f^{\mathcal{I}}(1) = 2, f^{\mathcal{I}}(2) = 3, f^{\mathcal{I}}(3) = 1, h^{\mathcal{I}} : (x, y) \mapsto min\{x, y\}$
- Predicates: $P^{\mathcal{I}} = \{1, 3\}$, $Q^{\mathcal{I}} = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$

Define an environment E by

$$E(x) = 3, E(y) = 3, E(z) = 1$$

Example 1

1 Satisfiability and Validity

(continued from the previous slide)

We have:

- $\mathcal{I} \vDash_E P(h(f(a), z))$
- $\mathcal{I} \vDash_E Q(y, h(a, b))$

Give a new interpretation $\mathcal{J}$ and environment G such that

- $\mathcal{J} \nvDash_G P(h(f(a), z))$
- $\mathcal{J} \nvDash_G Q(y, h(a, b))$

Example 2

Let L be a language consisting of variables x, y, z, function symbols $f^{(2)}$, $g^{(1)}$ and predicate symbol $P^{(2)}$.

Define an interpretation $\mathcal{I}$ by

- $dom(\mathcal{I})$: $\mathbb{N}$
- $f^{\mathcal{I}}$: sum
- $g^{\mathcal{I}}$: square

Consider the formula $\alpha \stackrel{\text{def}}{=} f(g(x), g(y)) = g(z)$

- Give an environment such that α is satisfiable
- Whether α is valid?

Example 3

Whether the following formulas are valid?

- $\exists y \forall x R(x, y) \rightarrow \forall x \exists y R(x, y)$ valid
- $\forall x \exists y R(x, y) \rightarrow \exists y \forall x R(x, y)$ not valid
- $\exists x (P(x) \rightarrow \forall x P(x))$ valid

case 1: all x has P

case 2: there exists x doesn't have P

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We write $\mathcal{I} \models_E \Sigma$ iff for every formula $\varphi \in \Sigma$, we have $\mathcal{I} \models_E \varphi$.

Let Σ be a set of FOL formulas and α be a FOL formula. We say $\Sigma \models \alpha$ if and only if

For any interpretation $\mathcal{I}$ and environment E, if $\mathcal{I} \models_E \Sigma$ then $\mathcal{I} \models_E \alpha$ (or $\alpha^{(\mathcal{I},E)} = 1$).

Interpretation: every pair of interpretation and environment that makes Σ true must also make α true.

$\emptyset \models \alpha$ means that α is valid.

Prove that for any well-formed FOL formulas α and β:

$$\emptyset \models (\forall \boldsymbol{x}(\alpha \to \beta)) \to ((\forall \boldsymbol{x}\ \alpha) \to (\forall \boldsymbol{x}\ \beta))$$

Proof by contradiction: Suppose there are $\mathcal{I}$ and E such that:

$$\emptyset \not\models (\forall \boldsymbol{x}(\alpha \to \beta)) \to ((\forall \boldsymbol{x}\ \alpha) \to (\forall \boldsymbol{x}\ \beta))$$

Then we have $\mathcal{I} \models_E \forall \boldsymbol{x}(\alpha \to \beta)$, $\mathcal{I} \models_E \forall \boldsymbol{x}\ \alpha$, and $\mathcal{I} \not\models_E \forall \boldsymbol{x}\ \beta$.
By definition, for every $\boldsymbol{a} \in \boldsymbol{dom}(\mathcal{I})$, we have $\mathcal{I} \models_{E[\boldsymbol{x} \mapsto \boldsymbol{a}]} (\alpha \to \beta)$, $\mathcal{I} \models_{E[\boldsymbol{x} \mapsto \boldsymbol{a}]} \alpha$.
Hence, for every $\boldsymbol{a} \in \boldsymbol{dom}(\mathcal{I})$, we also have $\mathcal{I} \models_{E[\boldsymbol{x} \mapsto \boldsymbol{a}]} \beta$, which is $\mathcal{I} \models_E \forall \boldsymbol{x}\ \beta$.
Contradiction.

Prove that $\forall x(\neg\gamma) \vDash \neg(\exists x\ \gamma)$

Proof: Suppose $\mathcal{I} \vDash_E \forall x(\neg\gamma)$. By definition, we have:

$$\text{For every } a \in dom(\mathcal{I}), \mathcal{I} \vDash_{E[x\mapsto a]} (\neg\gamma)$$

which is equivalent to

$$\text{For every } a \in dom(\mathcal{I}), \mathcal{I} \nvDash_{E[x\mapsto a]} \gamma$$

which means

$$\text{There is no } a \in dom(\mathcal{I}) \text{ such that } \mathcal{I} \vDash_{E[x\mapsto a]} \gamma$$

If $\mathcal{I} \vDash_E (\exists x\ \gamma)$, then there is a $b \in dom(\mathcal{I})$ such that $\mathcal{I} \vDash_{E[x\mapsto b]} \gamma$, contradiction. Hence, $\mathcal{I} \vDash_E \neg(\exists x\ \gamma)$ holds as required.

Example 3

2 Semantic Entailment

注意 domain

Let's define a language $L = \langle R^{(2)} \rangle$, and the following interpretations:

- $\mathcal{I}_1 = \langle \mathbb{N}, \{(n, m) : n < m\} \rangle$
- $\mathcal{I}_2 = \langle \mathbb{N}, \{(n, m) : \cancel{m \% n = 0}\} \rangle$ n divides m : $m = xn$ 在自然数域中
- $\mathcal{I}_3 = \langle \mathcal{P}(\mathbb{N}), \{(A, B) : A \subseteq B\} \rangle$

$E(x)=2$ $E(y)=3$

Questions:

- Find an environment E such that $\mathcal{I}_1 \vDash_E R(x, y)$ and $\mathcal{I}_2 \nvDash_E R(x, y)$ ✓
- Find a sentence α such that $\mathcal{I}_1 \nvDash_E \alpha$ and $\mathcal{I}_2 \vDash_E \alpha$ $\exists x R(x, x)$
- Find a sentence α such that $\mathcal{I}_2 \vDash_E \alpha$ and $\mathcal{I}_3 \vDash_E \alpha$

$\forall x \exists y\, R(y, x)$

$\exists x \forall y\, R(x, y)$

Example 4

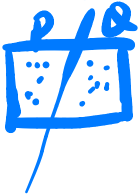
两者没有关联

1 0

Prove that:

$$(\forall x P(x)) \rightarrow (\forall x Q(x)) \nvDash \forall x (P(x) \rightarrow Q(x))$$

Idea: All we need to do is to find an $\mathcal{I}$ (and E) such that: $\mathcal{I} \vDash_E (\forall x P(x)) \rightarrow (\forall x Q(x))$ and $\mathcal{I} \nvDash_E \forall x (P(x) \rightarrow Q(x))$



$\forall x P(x)$ F

$\Rightarrow \forall x P(x) \rightarrow \forall x Q(x)$ T

$\forall x (P(x) \rightarrow Q(x))$ F

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3 Decidability

▶ Satisfiability and Validity

▶ Semantic Entailment

▶ Decidability

Undecidability of first-order logic

Given a formula φ in FOL, is φ valid, yes or no?

This problem is not solvable (i.e., we cannot write a correct C or Java program that works for all φ). In other words, first-order logic is not decidable in general.

Propositional logic is decidable, because the truth-table method can be used to determine whether an arbitrary propositional formula is logically valid.

在FOL中有$\forall x$ 量词 无法穷举完判断

Readings

- Text B: chapter 2.4.2
- Text F: chapter 7.3.2

Introduction to Mathematical Logic

Thank you for listening!
Any questions?