6 Assignment 6 Solutions (100 points)

6.1 Free and Bound Variables (15 points)

- 1. No free variables
- 2. The last occurrence of y
- 3. The last occurrence of x
- 4. No free variables
- 5. u is free.

6.2 Semantics I (15 points)

- 0
- 0
- 1

6.3 Semantics II (20 points)

- $0 (\forall x(-x < 0))$
- $1 (\forall x(x-1 < x) \to \exists y(0 \ge y-2))$
- $0 (\forall x (x < 1 \rightarrow \forall y (y < 2 \rightarrow \forall z (x < z))))$
- $1 (\forall x \exists y (x < x 2y))$

6.4 Semantic Entailment (20 points)

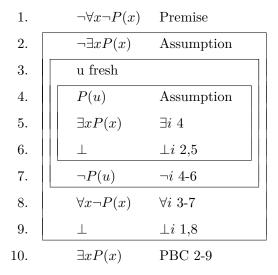
- No. We may construct a graph with a, b, c, d as nodes and b to a, b, c as edges. As a counter example for the formula, let's choose b for x and a for y. There is an edge from b to a, but a doesn't point to any node.
- Yes. Similar to the above case, we can construct a graph. For any choice of x and y, if there is an edge from x to y, then there exists a node z with an edge from y.

6.5 ND Proof (30 points)

Proof 1:

1.	$\exists x P(x) \vee \exists x Q(x)$	Premise
2.	$\exists x P(x)$	Assumption
3.	P(u) u fresh	Assumption
4.	$P(u) \vee Q(u)$	$\forall i \ 3$
5.	$\exists x (P(x) \lor Q(x))$	$\exists i \ 4$
6.	$\exists x (P(x) \lor Q(x))$	$\exists e \ 2, \ 3\text{-}5$
7.	$\exists x Q(x)$	Assumption
8.	Q(u) u fresh	Assumption
9.	$P(u) \vee Q(u)$	<i>∨i</i> 8
10.	$\exists x (P(x) \lor Q(x))$	$\exists i \ 9$
11.	$\exists x (P(x) \lor Q(x))$	$\exists e \ 7, \ 8-10$
12.	$\exists x (P(x) \lor Q(x))$	$\forall e \ 1, 2-6,7-11$

Proof 2:



Proof 3:

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(\forall x \; (Q(x) \to R(x)))
 1.
                                        premise
        (\exists x\ (P(x) \land Q(x)))
 2.
                                        premise
 3.
        (P(u) \wedge Q(u)), u \text{ fresh}
                                        assumption
 4.
        P(u)
                                        ∧e: 3
        Q(u)
 5.
                                        ∧e: 3
        (Q(u) \to R(u))
 6.
                                        ∀e: 1
 7.
        R(u)
                                        \rightarrowe: 5, 6
        (P(u) \wedge R(u))
                                       ∧i: 4, 7
 8.
       (\exists x\ (P(x) \land R(x)))
 9.
                                        ∃i: 8
        (\exists x\ (P(x) \land R(x)))
                                        ∃e: 2, 3-9
10.
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Proof 4:

1.	$\forall x P(a, x, x)$	Premise
2.	$\forall x \forall y \forall z (P(x,y,z) \to P(f(x),y,f(z)))$	Premise
3.	P(a, a, a)	$\forall e \ 1$
4.	$\forall y \forall z (P(a,y,z) \rightarrow P(f(a),y,f(z)))$	$\forall e \ 2$
5.	$\forall z (P(a,a,z) \rightarrow P(f(a),a,f(z)))$	$\forall e \ 4$
6.	$P(a,a,a) \to P(f(a),a,f(a))$	$\forall e \ 5$
7.	P(f(a), a, f(a))	$\rightarrow e \ 3,\!6$