6 Assignment 6 (100 points)

6.1 Free and Bound Variables (15 points)

Is there any free variable in the following formulas? If so, please also point out which variable(s) or which occurrence(s) of a variable is free.

1. $\forall x(P(x) \rightarrow \exists y \neg Q(f(x), y, f(y)))$
2. $\forall x(\exists y R(x, f(y)) \rightarrow R(x, y))$
3. $\forall z(P(z) \rightarrow \exists y(\exists x Q(x, y, z) \vee Q(z, y, x)))$
4. $\forall z \exists u \exists y(Q(z, y, g(u, y)) \vee R(u, g(z, u)))$
5. $\forall z \exists x \exists y(Q(z, u, g(u, y)) \vee R(u, g(z, u)))$

6.2 Semantics I (15 points)

Define an interpretation $\mathcal{I}$ by:

- Domain $dom(\mathcal{I}) : \{-2, 3, 6\}$
- Constant: $a^{\mathcal{I}} = 3$
- Unary predicates: $F(x)^{\mathcal{I}} : x \leq 3; G(x)^{\mathcal{I}} : x > 5; R(x)^{\mathcal{I}} : x \leq 7$

What are the values of the following formulas? In addition to the answers, please also show your steps or reasoning.

- $\forall x(F(x) \wedge G(x))$
- $\forall x(R(x) \rightarrow F(x)) \wedge G(a)$
- $\exists x(F(x) \vee G(x))$

6.3 Semantics II (20 points)

Define an interpretation $\mathcal{I}$ by:

- Domain:$\mathbb{R}$
- Constant: $a^{\mathcal{I}} = 0$
- Function: $f(x, y) = x - y$
- Predicate: $P(x, y) : x < y$

Define an environment E: $E(x) = 0, E(y) = 1, E(z) = 2$. Given $\mathcal{I}$ and E, what are the values of the following formulas? In addition to the answers, please also show your steps or reasoning.

- $\forall x P(f(a, x), a)$
- $\forall x P(f(x, y), x) \to \exists y \neg P(x, f(y, z))$
- $\forall x (P(x, y) \to \forall y (P(y, z) \to \forall z P(x, z)))$
- $\forall x \exists y P(x, f(f(x, y), y))$

6.4 Semantic entailment (20 points)

Let ϕ be the sentence $\forall x \forall y \exists z (R(x, y) \to R(y, z))$, where R is a binary predicate.

1. Let $dom(\mathcal{I}) \stackrel{\text{def}}{=} \{a, b, c, d\}$, and $R^{\mathcal{I}} \stackrel{\text{def}}{=} \{(b, c), (b, b), (b, a)\}$. Do we have $\mathcal{I} \vDash \phi$? Answer "yes" or "no" and also justify your answer.
2. Let $dom(\mathcal{I}) \stackrel{\text{def}}{=} \{a, b, c\}$, and $R^{\mathcal{I}} \stackrel{\text{def}}{=} \{(b, c), (a, b), (c, b)\}$. Do we have $\mathcal{I} \vDash \phi$? Answer "yes" or "no" and also justify your answer.

6.5 Formal proof (30 points)

Use the ND proof system of FOL to prove the following:

1. $\exists x P(x) \lor \exists x Q(x) \vdash \exists x (P(x) \lor Q(x))$
2. $\neg \forall x \neg P(x) \vdash \exists x P(x)$
3. $\{\forall x (Q(x) \to R(x)), \exists x (P(x) \land Q(x))\} \vdash \exists x (P(x) \land R(x))$
4. $\{\forall x P(a, x, x), \forall x \forall y \forall z (P(x, y, z) \to P(f(x), y, f(z)))\} \vdash P(f(a), a, f(a))$