Introduction to Mathematical Logic

For CS Students

CS104/CS108

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1 Warm Up

What’s the meaning of well-formed formulas

The meaning of a natural language sentence depends on the meaning of words and connectives.

- ”The sky is blue and the grass is green.”
- ”Nothing right in my left brain, and nothing left in my right brain.”

Similarly, the meaning of well-formed formulas in a formal language (e.g., $p \wedge q$) depends on the truth values of the *atoms* and how the *logical connectives* manipulate these truth values.

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2 Semantics

Truth Table

- Truth values of atomic propositions: True(1) or False(0)
- Truth values of compound propositions: depends on its atomic propositions and logical connectives
- Truth table for $\neg p$:

p	$\neg p$
1	0
0	1

Truth Table

- Truth values of atomic propositions: True(1) or False(0)
- Truth values of compound propositions: depends on its atomic propositions and logical connectives
- Truth table for $p \wedge q$:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Logical Connectives in NL & PL

Logical connectives in formal languages are not completely equivalent to words in natural language.

She became violently sick and she went to the doctor.
She went to the doctor and she became violently sick.

In natural language, ”and” indicates time progression; whereas logical connectives in formal language like PL are solely concerned with truth values.

Truth Table

- Truth values of atomic propositions: True(1) or False(0)
- Truth values of compound propositions: depends on its atomic propositions and logical connectives
- Truth table for $p \vee q$:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

"He will come today or tomorrow"

"From $(a-1)(a-2)=0$, we get $a=1$ or $a=2$."

In natural language, sometimes "A or B" means "A is true or B is true but not both". But in PL, "A or B" means "A is true or B is true or both are true."

Truth Table

Truth table for $p \to q$ (logical implication): if p (is true), then q (is true).

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Also known as: It's not the case that p is true and q is false.

Truth Table

Why $p \to q$ is true when p is false?

- If $x > 7$, then $x > 5$.
- ”If you stick a fork in an electrical outlet, you will get hurt.”

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Truth Table

2 Semantics

$p \leftrightarrow q$ (iff) is true when both p and q carry the same truth value, and is false otherwise.

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Truth Valuation 赋值函数（真值指派）

To interpret a formula, we have to give meanings to the propositional variables and the connectives.

A propositional variable has no intrinsic meaning; it gets a meaning via a valuation.

Definition

A truth valuation is a function $v : Atom(\mathscr{L}^P) \to \{0, 1\}$.

For $p \in Atom(\mathscr{L}^P)$, we use $v(p)$ or p^v to denote the truth value of p under the truth valuation v, $p^v \in \{0, 1\}$

p^v 表示真值

Truth value of a formula

Fix a truth valuation v. Every formula A has a value under v, denoted as A^v, can be recursively defined as:

Definition

1. $p^v \in \{0, 1\}$
2. $(\neg A)^v = \begin{cases} 1 & \text{if } A^v = 0 \\ 0 & \text{else} \end{cases}$
3. $(A \wedge B)^v = \begin{cases} 1 & \text{if } A^v = B^v = 1 \\ 0 & \text{else} \end{cases}$
4. $(A \vee B)^v = \begin{cases} 1 & \text{if } A^v = 1 \text{ or } B^v = 1 \\ 0 & \text{else} \end{cases}$
5. $(A \to B)^v = \begin{cases} 1 & \text{if } A^v = 0 \text{ or } B^v = 1 \\ 0 & \text{else} \end{cases}$
6. $(A \leftrightarrow B)^v = \begin{cases} 1 & \text{if } A^v = B^v \\ 0 & \text{else} \end{cases}$

Theorem

Fix a truth valuation v. Every formula $\alpha \in Form(\mathscr{L}^p)$ has a value α^v in $\{0, 1\}$.

Proof: By structural induction. Let $P(\alpha)$ be "α has a value α^v in $\{0, 1\}$".

1. If α is a propositional variable, then v assigns it a value of 1 or 0 (by the definition of a truth valuation).
2. If α has a value in $\{0, 1\}$, then $(\neg\alpha)$ also does (by the truth table of $(\neg\alpha)$).
3. If α and β each has a value in $\{0, 1\}$, then $(\alpha \star \beta)$ also does for every binary connective $\star$, as shown by the corresponding truth tables.

By the principle of structural induction, every formula has a value.
By the unique readability of formulas, we have proved that a formula has only one truth value under any truth valuation v. QED

Truth value of a formula

What's the truth value of $A = p \wedge q \to (\neg q \vee r)$ given the following truth valuation?

- $p^v = q^v = r^v = 1, A^v = ?$
- $p^v = q^v = r^v = 0, A^v = ?$

Properties of a formula

Let $A \in Form(\mathscr{L}^p)$.

- If for every truth valuation v, $A^v = 1$, then A is tautology (永真式或重言式)
- If for every truth valuation v, $A^v = 0$, then A is contradiction (永假式或矛盾式)
- If there exists a truth value v such that $A^v = 1$, then A satisfiable (可满足的)

How to determine the properties of a formula?

- Reasoning
  - $(p \vee (\neg p))$ is a tautology (and satisfiable)
  - $(p \wedge (\neg p))$ is a contradiction (and not satisfiable)
- Truth table
- Valuation tree

Truth table

We could use **truth table** to determine the properties of a formula

p	q	r	$(p \lor q)$	$(q \land r)$	$((p \lor q) \to (q \land r))$
F	F	F	F	F	T
F	F	T	F	F	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	T	F	F
T	T	T	T	T	T

p	q	$q \to p$	$p \to (q \to p)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

Use truth tables to show that

- $p \to (q \to p)$ is tautology.
- $\neg(p \to p)$ is contradiction.
- $(p \to \neg p) \to p$ is satisfiable.

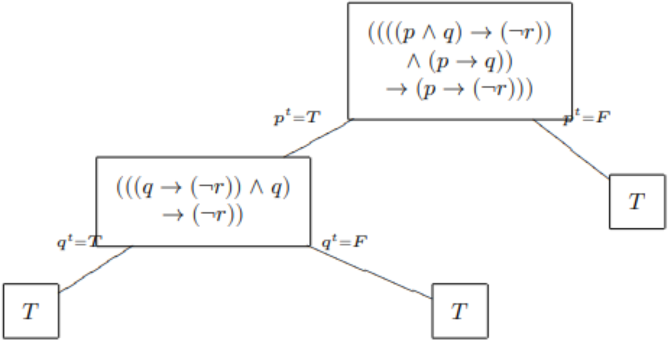
Valuation Tree

Rather than fill out an entire truth table, we can analyze what happens if we plug in a truth value for one variable.

$$
\begin{array}{l|l}
\neg \mathrm{T} & \mathrm{F} \\
\neg \mathrm{F} & \mathrm{T}
\end{array}
\qquad
\begin{array}{l|l}
(p \wedge \mathrm{T}) & p \\
(p \wedge \mathrm{F}) & \mathrm{F} \\
(p \wedge p) & p
\end{array}
\qquad
\begin{array}{l|l}
(p \vee \mathrm{T}) & \mathrm{T} \\
(p \vee \mathrm{F}) & p \\
(p \vee p) & p
\end{array}
\qquad
\begin{array}{l|c}
(p \rightarrow \mathrm{T}) & \mathrm{T} \\
(p \rightarrow \mathrm{F}) & (\neg p) \\
(\mathrm{T} \rightarrow p) & p \\
(\mathrm{F} \rightarrow p) & \mathrm{T} \\
(p \rightarrow p) & \mathrm{T}
\end{array}
$$

We can evaluate a formula by using these rules to construct a valuation tree.

Show that $((((p \wedge q) \to (\neg r)) \wedge (p \to q)) \to (p \to (\neg r)))$ is a tautology by using a valuation tree.



NL vs. PL

2 Semantics

T 写出来判断

Here's a question about playing Monopoly:

> *If you get more doubles than any other player then you will lose or if you lose then you must have bought the most properties.*

True or false? We will answer this question, and won't need to know anything about Monopoly. Instead we will look at the logical form of the statement.

$$(p \to q) \lor (q \to r)$$

if $q^v = 0$ $\quad (p \to F) \lor T = T$

$q^v = 1$ $\quad T \lor (T \to r) = T$

Applications

2 Semantics

不会执行的

Finding live/dead code: can code block P_1, P_2, P_3, P_4 be executed?

```
if ( (input > 0) or (not output) ) {
    if ( not (output and (queuelength < 100) ) ) {
        P_1
    } else if ( output and (not (queuelength < 100)) ) {
        P_2
    } else { P_3 }
} else { P_4 }
```

Finding live/dead code: can code block P_1, P_2, P_3, P_4 be executed?

$i \quad \vee \quad \neg u$

```
if ( (input > 0) or (not output) ) {
    if ( not (output and (queuelength < 100) ) ) {
```

$\neg(u \wedge q)$

```
        P1
    } else if ( output and (not (queuelength < 100)) ) {
```

$u \wedge (\neg q)$

```
        P2
    } else { P3 }
} else { P4 }
```

Let's define i: input > 0, u: output, and q: queuelength < 100

① 化表达
② 真值
③ 计关系、程序终止?

Finding live/dead code: can code block P_1, P_2, P_3, P_4 be executed?

i	u	q	$(i \vee (\neg u))$	$(\neg(u \wedge q))$	$(u \wedge (\neg q))$	
T	T	T	T	F	F	
T	T	F	T	T	T	
T	F	T	T	T	F	
T	F	F	T	T	F	
F	T	T	F			P_4
F	T	F	F			P_4
F	F	T	T	T	F	
F	F	F	T	T	F	

不满足才判

都不执行 (前为T)

Finding live/dead code: can code block P_1, P_2, P_3, P_4 be executed?

i	u	q	$(i \vee (\neg u))$	$(\neg(u \wedge q))$	$(u \wedge (\neg q))$	
T	T	T	T	F	F	P_3
T	T	F	T	T		P_1
T	F	T	T	T		P_1
T	F	F	T	T		P_1
F	T	T	F			P_4
F	T	F	F			P_4
F	F	T	T	T		P_1
F	F	F	T	T		P_1

Satisfiability of sets of formulas

Let $\Sigma \subseteq Form(\mathscr{L}^p)$ (a set of well-formed formulas). v is a truth valuation. Define:

$$\Sigma^v = \begin{cases} 1 & \text{if for each } B \in \Sigma, B^v = 1 \\ 0 & \text{else} \end{cases}$$

有情况可全T

Σ is satisfiable iff there is some valuation v such that $\Sigma^v = 1$; we say v satisfies Σ.

Example: The set $\{(p \to q) \vee r, p \vee (q \vee s)\}$ is satisfiable.

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3 Logical Equivalence

Definition of Logical Equivalence

Two formulas A and B are logically equivalent if and only if they have the same value under any valuation.

- $A^v = B^v$ for every truth valuation v.
- A and B must have the same final column in their truth tables.
- $A \leftrightarrow B$ is a tautology.

Why do we care about logical equivalence?

- Will I lose marks if I provide a solution that is syntactically different but logically equivalent to the provided solution?
- Do these two circuits behave the same way?
- Do these two pieces of code fragments behave the same way?

Logical Equivalence

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws 恒等率
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws 支配率
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws 幂等率
$\neg(\neg p) \equiv p$	Double negation laws 双非率
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws 交换率

同符号

不同符号

Equivalence	Name
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws 结合率
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws 分配率
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws 德摩根率
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws 吸收率
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws 否定率

Equivalence	**Name**
$p \rightarrow q \equiv \neg p \vee q$	Implication
$p \rightarrow q \equiv \neg q \rightarrow \neg p$	Contrapositive (逆否)
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Equivalence

Example: Prove $((\neg p) \wedge q) \vee p \equiv p \vee q$

Exercise

"If it is sunny, I will play golf, provided that I am relaxed."

- s: it is sunny
- g: I will play golf.
- r: I am relaxed.

Prove that all three translations are logically equivalent.

- $s \to (r \to g)$ $\equiv \neg s \vee (r \to g) \equiv \neg s \vee (\neg r \vee g)$
  $\equiv \neg r \vee (\neg s \vee g)$
  $\equiv \neg r \vee (s \to g)$
  $\equiv r \to (s \to g)$
- $r \to (s \to g)$
- $(s \wedge r) \to g$
  $\equiv \neg (s \wedge r) \vee g$
  $\equiv (s \wedge r) \to g$

Substitution

A substitution is a *syntactic transformation* on formal expressions.

Definition

$A, B \in Form(\mathscr{L}^p)$. B is a substitution instance of A if and only if B may be obtained from A by substituting formulas for propositional variables in A, replacing each occurrence of the same variable by an occurrence of the same formula.

替代同一公式

Example:

- $(r \to s) \land (t \to s)$ is a substitution instance of $p \land q$
- $(p \leftrightarrow p) \leftrightarrow (p \leftrightarrow p)$ is a substitution instance of $p \leftrightarrow p$

Theorem

$A, B \in Form(\mathscr{L}^p)$. If A is a tautology, and B is a substitution instance of A, then B is again a tautology.

Example: $(r \wedge s) \to (q \to (r \wedge s))$ is a tautology, since it is a substitution instance of $p \to (q \to p)$

Theorem

$A \in Form(\mathscr{L}^p)$. A contain a subformula C (i.e., C is a *segment* of A and is itself a well-formed formula). If $C \equiv D$, then replacing some occurrences (not necessarily all) of the subformula C in A with D to obtain the formula B, then $A \equiv B$.

用子公式替换

Example: $p \to q \equiv (\neg p \vee q)$. Then, $(p \to q) \wedge (r \to (p \to q)) \equiv ?$

$(\neg p \vee q) \wedge (r \to (\neg p \vee q))$

Logical Equivalence

关系验证

Logical equivalence is an equivalence relation on $Form(\mathscr{L}^p)$

- Reflexive: for any $A \in Form(\mathscr{L}^p)$, $A \equiv A$.
- Symmetric: for any $A, B \in Form(\mathscr{L}^p)$, $A \equiv B$, then $B \equiv A$.
- Transitive: for any $A, B, C \in Form(\mathscr{L}^p)$, if $A \equiv B$, $B \equiv C$, then $A \equiv C$.

All formulas in the same equivalence class have the same truth table.

Applications

The two code have different syntax, but equivalent semantics.

Listing 1: Your code

```
if (i || !u) {
  if (!(u && q)) {
    P1
  } else if (u && !q) {
    P2
  } else { P3 }
} else { P4 }
```

Listing 2: Your friend's code

```
if ((i && u) && q) {
  P3
} else if (!i && u) {
  P4
} else {
  P1
}
```

Applications

To prove that the two code are semantically equivalent, show that each code block is executed under logically equivalent conditions.

Block	Fragment 1	Fragment 2
P_1	$(i \vee (\neg u)) \wedge (\neg(u \wedge q))$	$(\neg(i \wedge u \wedge q)) \wedge (\neg((\neg i) \wedge u))$
P_2	$(i \vee (\neg u)) \wedge (\neg(\neg(u \wedge q)))$ $\wedge (u \wedge (\neg q))$	$\mathbf{F}$
P_3	$(i \vee (\neg u)) \wedge (\neg(\neg(u \wedge q)))$ $\wedge (\neg(u \wedge (\neg q)))$	$(i \wedge u \wedge q)$
P_4	$(\neg(i \vee (\neg u)))$	$(\neg(i \wedge u \wedge q)) \wedge ((\neg i) \wedge u)$

Readings

Optional

- TextB: 1.4.1
- TextI: 1.1, 1.2
- Text1: 第二章 2.4
- Text3: 第二章 2.3
- Reference: CS245 course notes, University of Waterloo

Introduction to Mathematical Logic

Thank you for listening!
Any questions?