4 Answers to Assignment 4: PL Proof Systems

4.1 The Hilbert-style proof system

Proof 1:

1.	$\{\neg A \to A\} \vdash \neg A \to A$	Assumption
2.	$\{\neg A \to A\} \vdash \neg A \to \neg\neg A$	Contrapositive
3.	$\{\neg A \to A\} \vdash (\neg A \to \neg\neg A) \to \neg\neg A$	Theorem H6
4.	$\{\neg A \to A\} \vdash \neg\neg A$	MP 2,3
5.	$\{\neg A \to A\} \vdash A$	Double negation
6.	$\vdash (\neg A \to A) \to A$	Deduction 5

Proof 2:

1.	$\{\neg A \to \text{false}\} \vdash \neg A \to \text{false}$	Assumption
2.	$\{\neg A \to \text{false}\} \vdash \neg false \to \neg\neg A$	Contrapositive
3.	$\{\neg A \to \text{false}\} \vdash \neg\text{false}$	Theorem H5
4.	$\{\neg A \to \text{false}\} \vdash \neg\neg A$	MP 2,3
5.	$\{\neg A \to \text{false}\} \vdash A$	Double negation 4
6.	$\vdash (\neg A \to \text{false}) \to A$	Deduction 5

Proof 3:

1.	$\{A \to B, \neg A \to B\} \vdash \neg A \to B$	Assumption
2.	$\{A \to B, \neg A \to B\} \vdash \neg B \to A$	Contrapositive
3.	$\{A \to B, \neg A \to B\} \vdash A \to B$	Assumption
4.	$\{A \to B, \neg A \to B\} \vdash \neg B \to B$	Transitivity 2,3
5.	$\{A \to B, \neg A \to B\} \vdash (\neg B \to B) \to B$	Question 1
6.	$\{A \to B, \neg A \to B\} \vdash B$	MP 4,5
7.	$\{A \to B\} \vdash (\neg A \to B) \to B$	Deduction
8.	$\vdash (A \to B) \to ((\neg A \to B) \to B)$	Deduction

Proof 4:

1.	$\{\neg A, \neg B \to A\} \vdash \neg A$	Assumption
2.	$\{\neg A, \neg B \to A\} \vdash \neg B \to A$	Assumption
3.	$\{\neg A, \neg B \to A\} \vdash \neg A \to \neg\neg B$	Contrapositive
4.	$\{\neg A, \neg B \to A\} \vdash \neg\neg B$	MP 1,3
5.	$\{\neg A, \neg B \to A\} \vdash B$	Double negation
6.	$\{\neg A\} \vdash (\neg B \to A) \to B$	Deduction 5

4.2 The ND proof system

Proof 1:

1	$\neg(\neg p \lor q)$	premise
2	$\neg p$	assumption
3	$\neg p \lor q$	$\lor\mathrm{i}_1$ 2
4	$\bot$	$\neg$e 3, 1
5	$\neg\neg p$	$\neg$i 2 $-$ 4
6	p	$\neg\neg$e 5

Proof 2:

1	$(p \vee q) \vee r$	premise
2	$(p \vee q)$	assumption
3	p	assumption
4	$p \vee (q \vee r)$	$\vee i_1$ 3
5	q	assumption
6	$q \vee r$	$\vee i_1$ 5
7	$p \vee (q \vee r)$	$\vee i_2$ 6
8	$p \vee (q \vee r)$	$\vee e$ $2, 3{-}4, 5{-}7$
9	r	assumption
10	$q \vee r$	$\vee i_2$ 9
11	$p \vee (q \vee r)$	$\vee i_2$ 10
12	$p \vee (q \vee r)$	$\vee e$ $1, 2{-}8, 9{-}11$

Proof 3:

1.	$p \to (q \to r)$	Premise
2.	$p \to q$	Assumption
3.	p	Assumption
4.	q	$\to$ e 2,3
5.	$q \to r$	$\to$ e 1,3
6.	r	$\to e$ 4,5
7.	$p \to r$	$\to$ i 3-6
8.	$(p \to q) \to (p \to r)$	$\to$ i 2-7

Proof 4:

1.	$\neg(p \lor q)$	Premise
2.	p	Assumption
3.	$p \lor q$	$\lor$ i 2
4.	$\bot$	$\bot$ i 1,3
5.	$\neg p$	$\neg$ i 2-4
6.	q	Assumption
7.	$p \lor q$	$\lor$ i 6
8.	$\bot$	$\bot$ i 1,7
9.	$\neg q$	$\neg$ i 6-8
10.	$\neg p \land \neg q$	$\land$ i 5,9

Proof 5:

1.	$\neg p \land \neg q$	Premise
2.	$p \lor q$	Assumption
3.	p	Assumption
4.	$\neg p$	$\land$ e 1
5.	$\bot$	$\bot$ i 3,4
6.	q	Assumption
7.	$\neg q$	$\land$ e 1
8.	$\bot$	$\bot$ i 6,7
9.	$\bot$	$\lor$ e 2-8
10.	$\neg(p \lor q)$	$\neg$ i 2-9

4.3 Formalization

p: the train arrives late

q: there are taxis at the station

r: John is late

Premises: $\{p \land \neg q \to r, \neg r, p\}$

Conclusion: q

So, we want to prove $\{p \land \neg q \to r, \neg r, p\} \vdash q$:

1	$p \wedge \neg q \to r$	premise
2	$\neg r$	premise
3	p	premise
4	$\neg q$	assumption
5	$p \wedge \neg q$	$\wedge$i $3, 4$
6	r	$\to$e $1, 5$
7	$\bot$	$\neg$e $6, 2$
8	$\neg\neg q$	$\neg$i $4-7$
9	q	$\neg\neg$e 8