Introduction to Mathematical Logic

For CS Students

CS104/CS108

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Table of Contents

Natural Deduction for FOL

FOL 中只介绍 ND Proof system
可直接用命题逻辑中 rules
但要加上 $\forall x$, $\exists x$

Natural Deduction for FOL extends Natural Deduction for propositional logic by including rules for introduction and elimination of quantifiers.

Other proof techniques and tricks remain the same as Natural Deduction for propositional logic. In fact, all the rules from Natural Deduction extend to our FOL setting, however we need new rules to handle quantifiers.

Table of Contents

2 Substitution

Substitution

1. $\forall x\,(Man(x) \to Mortal(x))$ Premise
2. $Man(So\cdots)$ Premise
3. $Man(So) \to Mortal(x)$
4. $Mortal(So\cdots)$ $\to e$ 2, 3

When writing natural deduction proofs in predicate logic, it is often useful to replace a variable in a formula with a term.
Suppose that the following sentences are true:

$$\forall x (Man(x) \to Mortal(x))$$
$$Man(Socrates)$$

To conclude that "Socrates is mortal", we need to replace every occurrence of the variable x in the implication by the term *Socrates*.

$$Man(Socrates) \to Mortal(Socrates)$$

By the $\to$e rule, we conclude that $Mortal(Socrates)$.
Formally, we use substitution to refer to this process of replace x by *Socrates* in the formula.

可用domain中任意一个元素替换

① α $\exists x P(x)$ ↓ $\exists x P(y)$ 含义不同 remain semantic unchanged

只能换 free variable

② $\exists x (x<y)$

$\alpha[w/y]$ $\exists x(x<w)$ 含义不变

$\alpha[Q(w)/y]$ $\exists x(x<Q(w))$ 只能换 term

For a variable x, a term t, and a formula α, $\alpha[t/x]$ denotes the resulting formula by replacing each free occurrence of x in α with t.
In other words, substitution does NOT affect bound occurrences of the variable.

Intuitively, $\alpha[t/x]$ answers the question:

"What happens to α if x has the value specified by term t?"

只能将 free variable 换成不相关的 term

③ $\exists x (x<y)$ y 原来是 free，换成 x 后变 bounded

$\alpha[x/y]$ $\exists x(x<x)$ 含义改变

Examples

2 Substitution

Let α be $E(f(x))$:

- $\alpha[(y+y)/x]$ is $E(f((y+y)))$
- $\alpha[f(x)/x]$ is $E(f(f(x)))$.

x换成f(x)

Let β be $P(x) \wedge (\exists x Q(x))$:

- $\beta[y/x]$ is $P(y) \wedge (\exists x Q(x))$ (only the free x gets substituted).

Let γ be $\forall x(E(f(x)) \wedge S(x,y))$:

- $\gamma[g(x,y)/x]$ is still γ, since γ has no free occurrence of x.

闭别的 顶换

$\alpha[(u+1)/z]$

If α is $\forall x(\exists y((x+y) = z))$, what is $\alpha[(y-1)/z]$?

There is a problem if we have:

$$\forall x(\exists y((x+y) = (y-1)))$$

Because the free variable y in the term $(y-1)$ got "captured" by the quantifier $\exists y$.

We want to avoid this capture.

Avoid Capture

If α is $\forall x(\exists y((x+y)=z))$, what is $\alpha[(y-1)/z]$?

We can prevent capture by renaming the quantified variable to something harmless, that is, a variable that occurs in neither α nor t.
For example, α can be rewritten by renaming y by a new variable w, without changing its meaning.

$$\forall x(\exists w((x+w)=z))$$

换前面量词

Now, $\alpha[(y-1)/z]$ is:

$$\forall x(\exists w((x+w)=(y-1)))$$

Table of Contents

∀-elimination

	e	i
$\forall$		
$\exists$		

Name	$\vdash$-notation	inference notation
∀-elimination (∀e)	If $\Sigma \vdash_{ND} (\forall x\ \alpha)$ then $\Sigma \vdash_{ND} \alpha[t/x]$	$\dfrac{(\forall x\ \alpha)}{\alpha[t/x]}$

Intuition: Given that a formula is true for every value of x, we can conclude it is true for any particular value, such as that of t.

∃-introduction (∃i)	If $\Sigma \vdash_{ND} \alpha[t/x]$, then $\Sigma \vdash_{ND} (\exists x\ \alpha)$	$\dfrac{\alpha[t/x]}{(\exists x\ \alpha)}$

Intuition: Given that a formula is true for a particular value (t), we can conclude it is true for some value.

Prove that

$$\forall x P(x) \vdash_{ND} \exists x P(x)$$

Proof:

1.	$\forall x P(x)$	Premise
2.	$P(u)$	$\forall e : 1$
3.	$\exists x P(x)$	$\exists i : 2$

Note: u represents an individual object in the domain.

Prove that:

$$\{P(t), \forall x(P(x) \to \neg Q(x))\} \vdash_{ND} \neg Q(t)$$

Proof:

1.	$P(t)$	Premise
2.	$\forall x(P(x) \to \neg Q(x))$	Premise
3.	$P(t) \to \neg Q(t)$	$\forall e : 2$
4.	$\neg Q(t)$	$\to e : 1, 3$

Prove that:

$$\neg P(y) \vdash_{ND} \exists x(P(x) \rightarrow Q(y))$$

Proof:

* 矛盾系统中任何命题都可被推断出来

Q:?

1.	$\neg P(y)$	Premise
2.	$P(y)$	Assumption
3.	$\perp$	$\neg e : 1, 2$ 或 $\perp i : 1, 2$
4.	$Q(y)$	$\perp e : 3$
5.	$P(y) \rightarrow Q(y)$	$\rightarrow i : 2 - 4$
6.	$\exists x(P(x) \rightarrow Q(y))$	$\exists i : 5$

$\alpha : P(x) \rightarrow Q(y)$

$\alpha[y/x]$

找到哪里用∃代替

Note: here we took $P(x) \rightarrow Q(y)$ for α in the $\exists i$ rule, i.e., $\alpha[y/x]$ is $P(y) \rightarrow Q(y)$.

$\forall$-introduction

类似数学证明

Name	$\vdash$-notation	inference notation
$\forall$-introduction ($\forall$i)	If $\Sigma \vdash_{ND} \alpha[y/x]$ and y not free in Σ or α, then $\Sigma \vdash_{ND} (\forall x\ \alpha)$	$\dfrac{\boxed{\begin{array}{c} y \text{ fresh} \\ \vdots \\ \alpha[y/x] \end{array}}}{(\forall x\ \alpha)}$

To prove $\forall x\ \alpha$, prove $\alpha[y/x]$ for arbitrary y. This rule follows ordinary mathematical usage:

- To prove a property holds for all integers, one often starts with “Let x be an integer”, then one proves that x has the property.
- Since we know nothing about the value x, except that it is an integer, this justifies that every integer has the property.

$\forall$-introduction

Name	$\vdash$-notation	inference notation
$\forall$-introduction ($\forall$i)	If $\Sigma \vdash_{ND} \alpha[y/x]$ and y not free in Σ or α, then $\Sigma \vdash_{ND} (\forall x\ \alpha)$	$\dfrac{\boxed{\begin{array}{c} y \text{ fresh} \\ \vdots \\ \alpha[y/x] \end{array}}}{(\forall x\ \alpha)}$

因不出现在 formula 中的变量

A variable is fresh in a subproof if it occurs nowhere outside the box of the subproof.

- It's safest to always use variables that aren't in any formula in Σ and not in α.
- Your fresh variable must be used only in the subproof. They cannot escape boxes.
- Use different fresh variables in different subproofs to avoid confusion.

Prove that:

$$\forall x P(x) \vdash \forall y P(y)$$

Proof:

1.	$\forall x P(x)$	Premise
2.	u fresh	
3.	$P(u)$	$\forall$ e:1
4.	$\forall y P(y)$	$\forall$ i: 2-3

Example: $\forall$i

3 Natural Deduction for FOL

Prove that:

$$\emptyset \vdash \forall x (P(x) \to P(x))$$

Proof:

1.	u fresh	
2.	$P(u)$	Assumption
3.	$P(u)$ 在假设下可推出	Reflexive: 2 闭自反 增加一项
4.	$P(u) \to P(u)$	$\to$i:2-3
5.	$\forall x(P(x) \to P(x))$	$\forall$i: 1-4

Example: $\forall$i

3 Natural Deduction for FOL

Prove that:

$$\{\forall x(P(x) \to Q(x)), \forall x P(x)\} \vdash \forall x Q(x)$$

1. $\forall x(P(x) \to Q(x))$ premise
2. $\forall x P(x)$ Premise
3. u fresh
4. $P(u) \to Q(u)$ $\forall e:1$
5. $P(u)$ $\forall e:2$
6. $Q(u)$ $\to e:4,5$
7. $\forall x Q(x)$ $\forall i:3\text{-}6$

∃-elimination

Name	$\vdash$-notation	inference notation
∃-elimination ($\exists$e)	If $\Sigma, \alpha[u/x] \vdash_{ND} \beta$, with u fresh, then $\Sigma, (\exists x\ \alpha) \vdash_{ND} \beta$	$\dfrac{(\exists x\ \alpha) \qquad \boxed{\begin{array}{c} \alpha[u/x],\ u \text{ fresh} \\ \vdots \\ \beta \end{array}}}{\beta}$

In $\exists e$, the variable u should not occur free in Σ, α, or β.

Prove that:

$$\exists x P(x) \vdash \exists y P(y)$$

Proof:

1.	$\exists x P(x)$	Premise
2.	$P(u)$ u fresh	Assumption
3.	$\exists y P(y)$	∃ i: 2
4.	$\exists y P(y)$	∃ e:1, 2-3

Prove that:

$$\exists y(\forall x P(x,y)) \vdash \forall x(\exists y P(x,y))$$

Proof:

1.	$\exists y(\forall x P(x,y))$	Premise
2.	$\forall x P(x,w)$ w fresh	Assumption
3.	u fresh	
4.	$P(u,w)$	$\forall$e:2
5.	$\exists y P(u,y)$	$\exists$i:4
6.	$\forall x(\exists y P(x,y))$	$\forall$i:3-5
7.	$\forall x(\exists y P(x,y))$	$\exists$e:1,2-6

1. $\exists y(\forall x P(x,y))$ Premise
2. $\forall x P(x,w)$ w fresh Assumption
3. u fresh
4. $P(u,w)$ $\forall$e:2
5. $\exists y P(u,y)$ $\exists$i:4
6. $\forall x \exists y P(x,y)$ $\forall$i:3-5
7. $\forall x (\exists y P(x,y))$ $\exists$e:1,2-6

Example: ∃e

Prove that:

$$\{\exists x P(x), \forall x (P(x) \rightarrow Q(x))\} \vdash \exists x Q(x)$$

Proof:

1.	$\exists x P(x)$	Premise
2.	$\forall x (P(x) \rightarrow Q(x))$	Premise
3.	$P(u)$, u fresh	Assumption
4.	$P(u) \rightarrow Q(u)$	∀e:2
5.	$Q(u)$	→e:3,4
6.	$\exists x Q(x)$	∃i: 5
7.	$\exists x Q(x)$	∃e:1,3-6

Prove that:

$$\exists x(P(x) \lor Q(x)) \vdash \exists x P(x) \lor \exists x Q(x)$$

Proved in class.

1. $\exists x(P(x) \lor Q(x))$ Premise
2. $P(u) \lor Q(u)$ u fresh Assumption
3. $P(u)$ Assumption
4. $\exists x P(x)$ $\exists i$: 3
5. $\exists x P(x) \lor \exists x Q(x)$ $\lor i$: 4
6. $Q(u)$ Assumption
7. $\exists x Q(x)$ $\exists i$: 6
8. $\exists x P(x) \lor \exists x Q(x)$ $\lor i$: 7
9. $\exists x P(x) \lor \exists x Q(x)$ $\lor e$: 2, 3-5, 68
10. $\exists x P(x) \lor \exists x Q(x)$ $\exists e$: 1, 2-9

Exercises

3 Natural Deduction for FOL

尝试倒推

Prove the following:

- $\exists x(\neg P(x)) \vdash \neg(\forall x P(x))$
- $\neg(\forall x P(x)) \vdash \exists x(\neg P(x))$

(1)

1. $\exists x(\neg P(x))$ Premise
2. $\neg P(u)$ u fresh Assumption
3. $\forall x P(x)$ Assumption
4. $P(u)$ $\forall e: 3$
5. $\perp$ $\perp i: 2,4$
6. $\neg(\forall x P(x))$ $\neg i: 3\text{-}5$
7. $\neg(\forall x P(x))$ $\exists e: 1, 2\text{-}6$

另一种证明

1. $\exists x(\neg P(x))$ Premise
2. $\forall x P(x)$ Assumption
3. $\neg P(u)$ u fresh Assumption
4. $P(u)$ $\forall e: 2$
5. $\perp$ $\perp i: 3,4$
6. $\perp$ $\exists e$ 1,3-5
7. $\neg \forall x P(x)$ $\neg i: 2\text{-}6$

(2)

1. $\neg(\forall x P(x))$ Premise
2. $\neg \exists x(\neg P(x))$ Assumption
3. u fresh
4. $\neg P(u)$ Assumption
5. $\exists x(\neg P(x))$ $\exists i: 4$
6. $\perp$ $\perp i: 2,5$
7. $P(u)$ $\neg e: 4,6$
8. $\forall x P(x)$ $\forall i: 3\text{-}7$
9. $\perp$ $\perp i: 1,8$
10. $\exists x(\neg P(x))$ $\neg e: 2\text{-}9$

Coursework

- Assignment 6

Readings

- TextB: chapter 2.2.4, 2.3.1
- Reference: CS245 course notes at University of Waterloo

Introduction to Mathematical Logic

Thank you for listening!
Any questions?