Assignment 4

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Q1. ① Base step: $n=1$ $n^3+2n=3$ then $3 \mid n^3+2n$

② induction: let $P(x)$ be $3 \mid x^3+2x$, where x is a positive integer

Assume $P(n-1)$, then $3 \mid (n-1)^3+2(n-1)$

$n^3+2n = (n-1)^3+2(n-1)+3n^2-3n+3 = (n-1)^3+2(n-1)+3(n^2-n+1)$

$3 \mid (n-1)^3+2(n-1)$, $3 \mid 3(n^2-n+1)$

then $P(n)$ $\Rightarrow P(n-1) \to P(n)$

③ conclusion: we have $P(1)$ and $P(n-1) \to P(n)$ for $n \geq 2$

then the prosition holds for every positive integer

Q2. First we prove when $t \in (0,1)$, for every positive integer n, we have $\frac{1-t^n}{1-t} \leq n$

let $A_i = \frac{1-t^i}{1-t}$ $\Rightarrow A_i = 1+t+\cdots+t^{i-1}$

① Base step: when $i=1$ $A_i = 1 \leq i$

② induction let $P(k)$ be $A_k \leq k$

Assume $P(n-1)$, then $A_{n-1} \leq n-1$

then $A_n = A_{n-1} + t^{n-1} \leq n-1+1 = n$ $(t \in (0,1))$

$\Rightarrow P(n)$ we have $P(n-1) \to P(n)$

③ conclusion: for $P(1)$ and $P(n-1) \to P(n)$ $n \geq 2$

we have $P(n)$ for every positive integer n, $\frac{1-t^n}{1-t} \leq n$

let $t = \frac{b}{a}$, for $0<b<a$ then $0<t<1$

$\Rightarrow 1-(\frac{b}{a})^n \le n(1-\frac{b}{a})$ for $\forall n \in N^*$

$\Rightarrow a^n - b^n \le n a^{n-1}(a-b)$ the proposition holds

Q3. We can make \$10 and \$(5k+20) where $k \in N$

① Base case: let $P(x)$ be \$x can be made by \$10 and \$25

$P(10)$: use one \$10, $P(20)$: use two \$10.

$P(25)$: use one \$25

② induction: Assume that we have

$P(10) \wedge P(20) \wedge P(25) \cdots \wedge P(20+5k)$ $k \in N^*$

for $20+5(k+1)$, we can use

$(20+5(k-1))$ and 10 to make it

for $k \in N^*$, $(k-1) \in N$ then $20+5(k-1)$ is in $10, 20, 25, \cdots 20+5k$

$\Rightarrow P(20+5(k-1))$ and $P(10)$

$\Rightarrow P(20+5(k+1))$

then we have $P(10) \wedge P(20) \wedge P(25) \cdots \wedge P(20+5k) \rightarrow P(20+5(k+1))$

③ conclusion: for strong induction, we can make

\$10, and \$(20+5k) where $k \in N$

Q4. $\Rightarrow$: assume the prinple of mathematical induction is valid,

we have $P(b)$ and $P(n-1) \rightarrow P(n)$ for all $n > b$

then we can induct that $P(b+1)$, $P(b+2) \cdots P(n)$ one by one

then we have $P(b) \wedge P(b+1) \cdots \wedge P(n-1) \to P(n)$
which shows strong induction is valid.

$\Leftarrow$: assume the strong induction is valid.
we have $P(b)$ and $P(b) \wedge P(b+1) \cdots \wedge P(n-1) \to P(n)$
for all $n > b$
then we have $P(b) \to P(b+1)$ and $P(b) \Rightarrow P(b+1)$
$P(b) \wedge P(b+1) \to P(b+2)$ and $P(b)$ and $P(b+1) \Rightarrow P(b+2)$
$\cdots$
$P(b) \wedge P(b+1) \cdots \wedge P(n-1) \to P(n)$ and $P(b) \cdots P(n-1) \Rightarrow P(n)$

by each iteration we have every $P(n-1) \to P(n)$ for $n > b$
which shows weak induction is valid.

Q5.

$$a^{2^{n+1}} = \left(a^{2^n}\right)^2 \quad (*)$$

Let $f(n) = a^{2^n}$ $\quad f(1) = a^2$

for the equality $(*)$, $f(n+1) = (f(n))^2$

$$\Rightarrow a^{2^n} = \begin{cases} a^2 & n = 1 \\ \left(a^{2^{n-1}}\right)^2 & n \geq 2, n \in N^* \end{cases}$$

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int calPower(a, n):
    if n==1 then
        return a*a
    else
        last = calPower(a, n-1)
        return last * last
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Q6. (a) $f(16) = 2f(4) + \log 16$

$= 2(2f(2) + \log 4) + \log 16$

$= 2 \times (2 \times 1 + 2) + 4$

$= 12$

(b) let $m = \log n \Rightarrow n = 2^m \quad n > 2$

$$f(n) = 2f(\sqrt{2^m}) + \log 2^m$$
$$= 2f(2^{\frac{m}{2}}) + m$$
$$= 2(2f(2^{\frac{m}{4}}) + \log 2^{\frac{m}{2}}) + m$$
$$= 4f(2^{\frac{m}{4}}) + m + m$$
$$\cdots = 2^{\log m} f(2^{\frac{m}{2^{\log m}}}) + \log m \cdot m$$
$$= m + m\log m = O(m\log m)$$

$$\Rightarrow f(n) = O(\log n \cdot \log(\log n))$$

$n = 2^k \quad k \in N^*$

Q7. (a) $S(n) = 9S(\frac{n}{2}) + n^2 = 9(9S(\frac{n}{4}) + (\frac{n}{2})^2) + n^2$

$$\cdots = 9^{\log n} b + n^2 + 9(\frac{n}{2})^2 + \cdots + 9^{\log n - 1}(\frac{n}{2^{\log n - 1}})^2$$
$$= 9^{\log n} b + n^2 (1 + \frac{9}{4} + (\frac{9}{4})^2 + \cdots + (\frac{9}{4})^{\log n - 1})$$
$$= n^{\log 9} b + n^2 \frac{(\frac{9}{4})^{\log n} - 1}{\frac{9}{4} - 1}$$
$$= bn^{2\cdot\log 3} + \frac{4}{5} n^2[(\frac{9}{4})^{\log n} - 1] = bn^{2\log 3} + \frac{4}{5}n^2 \cdot n^{\log\frac{9}{4}} - \frac{4}{5}n^2$$
$$= (b + \frac{4}{5})n^{2\log 3} - \frac{4}{5}n^2$$

(b) $n = 2^k, k \in N^*$

$$T(n) = aT(\frac{n}{4}) + n^2 = a(aT(\frac{n}{16}) + (\frac{n}{4})^2) + n^2$$
$$\cdots = a^{\log_4 n} T(1) + n^2 + a(\frac{n}{4})^2 + \cdots + a^{\log_4 n - 1}(\frac{n}{4^{\log_4 n - 1}})^2$$
$$= cn^{\log_4 a} + n^2(1 + \frac{a}{16} + \frac{a^2}{16^2} + \cdots + \frac{a^{\log_4 n - 1}}{16^{\log_4 n - 1}})$$

$$= cn^{\log_4 a} + n^2 \frac{\frac{a^{\log_4 n}}{16^{\log_4 n}} - 1}{\frac{a}{16} - 1} = cn^{\log_4 a} + \frac{16}{a-16} n^2 (\frac{a^{\log_4 n}}{n^2} - 1)$$

$$= cn^{\log_4 a} + \frac{16}{a-16} n^{\log_4 a} - \frac{16}{a-16} n^2$$

$$= (c + \frac{16}{a-16}) n^{\log_4 a} - \frac{16}{a-16} n^2$$

(c) $c > 0$, $a > 16$, $b > 0$

$$S_n = \Theta(cn^{2\log 3}) = \Theta(cn^{\log_4 81})$$

for $a > 16$, $c + \frac{16}{a-16} > 0$, $\log_4 a > 2$

$\Rightarrow T(n) = \Theta(cn^{\log_4 a})$

when $16 < a \leq 81$ $T(n) = O(S(n))$

Q8. let A: start with 010, B: start with 11, C: end with 00

$A \cap B = \emptyset$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= |A| + |B| + |C| - |A \cap C| - |B \cap C|$$
$$= 2^3 + 2^4 + 2^4 - 2^1 - 2^2$$
$$= 34$$

Q9. (a) product rule : 3^n

(b) if $n=1$ number is 3

if $n=2$ number is $3 \times 2 = 6$

if $n=3$ number is $3 \times 2 \times 1 = 6$

if $n > 3$, the cardinality of domain is larger than codomain

$\Rightarrow$ number is 0

(c) ① if $n \geq 3$ to count how many functions that are not onto

let A_i be the set of function that i is not in the range

$|A_1| = |A_2| = |A_3| = 2^n$

$|A_{12}| = |A_{13}| = |A_{23}| = 1^n$

$\Rightarrow$ not onto : $|A_1| + |A_2| + |A_3| - |A_{12}| - |A_{13}| - |A_{23}|$

$= 3 \times 2^n - 3$

$\Rightarrow$ onto : $3^n - (3 \times 2^n - 3) = 3^n - 3 \times 2^n + 3$

② if $1 \leq n < 3$ the cardinality of domain is smaller than codomain

number is 0

Q10. $\binom{13}{2}\binom{4}{2}\binom{4}{2} \cdot \binom{11}{2}\binom{4}{1}\binom{4}{1}$

$= \frac{13!}{2! \times 11!} \frac{4!}{2! \times 2!} \frac{4!}{2! \times 2!} \cdot \frac{11!}{2! \times 9!} \times 4 \times 4$

$= \frac{13!}{4 \times 9!} \times (4!)^2$

Q11. let A be the set of strings with 5 consecutive 0s.

max 5 consecutive 0s :

000001----	2^4
1000001---	2^3
-1000001--	2^3
--1000001-	2^3
---1000001	2^3
----100000	2^4

$2^3 \times 4 + 2^4 \times 2$

$= 2^6$

$= 64$

max 6 consecutive 0s :

0000001---	2^3
10000001--	2^2
-10000001-	2^2
--10000001	2^2
---1000000	2^3

$2^2 \times 3 + 2^3 \times 2$

$= 28$

max 7 consecutive 0s :

00000001--	2^2
100000001-	2^1
0100000001	2^1
--10000000	2^2

$2^2 \times 2 + 2^1 \times 2$

$= 12$

max 8 consecutive 0s : $2^1+1+2^1=5$

max 9 consecutive 0s : 2

max 10 consecutive 0s : 1

in total : $64+28+12+5+2+1=112$

for 1s , the similar classification : 112

minus the overlaping : 5 consewtive 0s and 5 consecutive 1s

$\Rightarrow 2$

$\Rightarrow$ ans is : $112\times2-2=222$

Q12 : $(x_1-3)+x_2+(x_3+2)+x_4+x_5=9$

(1)

let $x_1-3=x_1'$ $x_1'\geq0$, $x_3+2=x_3'$ $x_3'\geq0$

then it's equivalent to the problem of k-combinations from a set of n elements when repetition is allowed

$C(n+k-1,k)=C(9+5-1,9)=C(13,4)$

$=\dfrac{13\times12\times11\times10}{4\times3\times2\times1}=715$

(2) when $0\leq x_1\leq10$, there are $C(10+5-1,10)=C(14,4)$

$=\dfrac{14\times13\times12\times11}{4\times3\times2\times1}=1001$ solutions

when $x_1=6$, $x_2+x_3+x_4+x_5=4$, $C(4+3,3)=35$

when $x_1=7$, $x_2+x_3+x_4+x_5=3$. $C(3+3,3)=20$

when $x_1=8$, $x_2+x_3+x_4+x_5=2$, $C(2+3,3)=10$

when $x_1=9$, $x_2+x_3+x_4+x_5=1$, $C(1+3,3)=4$

when $x_1=10$, $x_2+x_3+x_4+x_5=0$, $C(0+3,3)=1$

therefore, $0\leq x_1\leq5$ $1001-35-20-10-4-1=931$ solutions

Q13. $a_1 \bmod 5 = a_2 \bmod 5$

$b_1 \bmod 5 = b_2 \bmod 5$

$a_i \bmod 5 = 0, 1, \cdots 4$

$b_i \bmod 5 = 0, 1, \cdots 4$

let $a_i \bmod 5 = m_i$, $b_i \bmod 5 = n_i$ $\quad m_i, n_i \in \{0,1,2,3,4\}$

then we need to find the maximum pairs needed to guaranteen that there are $(m_i, n_i) = (m_j, n_j)$ where $i \neq j$

for product rule, we have $5 \times 5 = 25$ different pairs

for pigeonhole principle, for 26 pairs, there must exist $(m_i, n_i) = (m_j, n_j)$ where $i \neq j$

$\Rightarrow$ 26 pairs

Q14. if p is prime, $1 \le k \le p-1$

for combinatorial argument, $\binom{p}{k}$ means the number of k-subsets of a p-element set $(1 \le k \le p-1)$, which must be a positive integer.

$\binom{p}{k} = \frac{p!}{k!(p-k)!}$ $\quad$ p is a prime, then $1, 2, \cdots p-1$ are pairwisely prime with p $\Rightarrow p \nmid k!$, $p \nmid (p-k)!$

And $\binom{p}{k} = \frac{p(p-1)!}{k!(p-k)!} \Rightarrow p \mid \frac{p(p-1)!}{k!(p-k)!}$

$\Rightarrow p \mid \binom{p}{k}$

Q15. (a) use mathematical induction

① Base case: when $r=0$, we need to prove $\binom{n+0}{0}=\binom{n+0+1}{0}$

LHS = number of 0-subset of n-element set

RHS = number of 0-subset of $(n+1)$-element set $\Rightarrow$ LHS=RHS

② induction: let $P(j)$ be $\sum_{k=0}^{j}\binom{n+k}{k}=\binom{n+j+1}{j}$

assume that $P(r-1)$, $\sum_{k=0}^{r-1}\binom{n+k}{k}=\binom{n+r}{r-1}$

then we try to prove $P(r)$: $\sum_{k=0}^{r}\binom{n+k}{k}=\binom{n+r+1}{r}$

for LHS $=\sum_{k=0}^{r-1}\binom{n+k}{k}+\binom{n+r}{r}=\binom{n+r}{r-1}+\binom{n+r}{r}$

it's equivalent to number of $(r-1)$-subset of $(n+r)$-element set (A)

plus number of r-subset of $(n+r)$-element set (B)

the sum is equivalent to choose r element from a $(n+r+1)$-element set: if a specific element x is chosen, we need to choose another $(r-1)$ elements from $(n+r)$ elements; otherwise, choose r elements from $(n+r)$ elements.

$\Rightarrow \binom{n+r}{r-1}+\binom{n+r}{r}=\binom{n+r+1}{r}$, which means $P(r)$

③ conclusion: we have $P(1)$ and $P(n-1)\Rightarrow P(n)$

then we have $P(1)$, $P(2)$ ... $P(n)$ for $\forall n \geq 2$

$\Rightarrow$ the identity holds

(b) $\sum_{k=0}^{r}\binom{n+k}{k}=\binom{n+0}{0}+\binom{n+1}{1}+\cdots+\binom{n+r}{r}$

$=\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+r}{r}$

for pascal's identity $\binom{n+k}{k}+\binom{n+k}{k-1}=\binom{n+k+1}{k}$

$$\Rightarrow \sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+2}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r}$$

$$= \binom{n+3}{2} + \binom{n+3}{3} + \cdots + \binom{n+r}{r}$$

$$\vdots = \binom{n+r}{r-1} + \binom{n+r}{r} = \binom{n+r+1}{r}$$

the identity holds

Q16. $\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!} \frac{r!}{k!(r-k)!} = \frac{n!}{(n-r)!k!(r-k)!}$

$$\sum_{r=k}^{n} \binom{n}{r}\binom{r}{k} = \frac{n!}{k!} \sum_{r=k}^{n} \frac{1}{(n-r)!(r-k)!}$$

while $\binom{n}{k} 2^{n-k} = \frac{n!}{k!(n-k)!} 2^{n-k}$

all we need to do is to prove $\frac{2^{n-k}}{(n-k)!} = \sum_{r=k}^{n} \frac{1}{(n-r)!(r-k)!}$

for: $\sum_{r=k}^{n} \frac{(n-k)!}{(n-r)!(r-k)!}$ let $n-k=m$

$$= \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m} = 2^m = 2^{n-k}$$

$\Rightarrow$ we prove that $\sum_{r=k}^{n} \frac{(n-k)!}{(n-r)!(r-k)!} = 2^{n-k}$

$$\Rightarrow \frac{2^{n-k}}{(n-k)!} = \sum_{r=k}^{n} \frac{1}{(n-r)!(r-k)!} \Rightarrow \frac{n!}{(n-k)!k!} 2^{n-k} = \sum_{r=k}^{n} \frac{n!}{(n-r)!(r-k)!k!}$$

$$\Rightarrow \sum_{r=k}^{n} \binom{n}{r}\binom{r}{k} = \binom{n}{k} 2^{n-k}$$

Q17. characteristic equation

$$r^4 = 2r^3 + r^2 - 2r$$

$$(r+1)(r-1)(r-2) = 0$$

$$r_1 = -1,\ r_2 = 1,\ r_3 = 2$$

$$\Rightarrow a_n = \alpha_1 (-1)^n + \alpha_2 1^n + \alpha_3 2^n$$

$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 3$

$a_1 = -\alpha_1 + \alpha_2 + 2\alpha_3 = 6 \Rightarrow \begin{cases} \alpha_1 = -2 \\ \alpha_2 = 6 \\ \alpha_3 = -1 \end{cases}$

$a_2 = \alpha_1 + \alpha_2 + 4\alpha_3 = 0$

$\Rightarrow a_n = -2 \times (-1)^n + 6 + (-1) \times 2^n$

Q18. Base case: when $n=1$ only "0" is valid

$V(1) = 1 \qquad 10^1 - V(1) = 9$

induction: for n-digit number, if we decide the $(n-1)$-digit number:

if $(n-1)$-digit is valid, the last digit is $1 \sim 9$

if $(n-1)$-digit is invalid, the last digit is 0

$\Rightarrow V(n) = 9V(n-1) + (10^{n-1} - V(n-1)) = 8V(n-1) + 10^{n-1}$

characteristic equation of associated linear homogeneous recurrence relation is: $r^2 - 8r = 0$

thus, $V(n) = \alpha 8^n + (an+b)10^n$

$V(1) = 1 = 8\alpha + 10(a+b)$

$V(2) = 8 + 10^1 = 18 = 64\alpha + 100(2a+b)$

$V(3) = 8 \times 18 + 10^2 = 244 = 512\alpha + 1000(3a+b)$

$\Rightarrow \begin{cases} a = 0 \\ b = \frac{1}{2} \\ \alpha = -\frac{1}{2} \end{cases} \Rightarrow V(n) = \frac{1}{2} \times 10^n - \frac{1}{2} \times 8^n \quad (n \in N^*)$

Q19. Base case: $d_1 = |2| = 2$ $\quad d_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ $\quad d_0 = 0$

induction: $d_n = \begin{vmatrix} 2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2 \end{vmatrix} = 2|A_{n-1}| - \begin{vmatrix} 1 & 1 & & 0 \\ & 2 & \ddots & \\ & 1 & \ddots & 1 \\ 0 & & 1 & 2 \end{vmatrix} = 2|A_{n-1}| - |A_{n-2}|$

$\Rightarrow d_n = 2d_{n-1} - d_{n-2}$

characteristic equation: $r^3 = 2r^2 - r$

$(r-1)^2 = 0 \qquad \Rightarrow d_n = \alpha_1 1^n + \alpha_2 n \cdot 1^n = \alpha_1 + n\alpha_2$

$r_1 = r_2 = 1$

$d_1 = 2 = \alpha_1 + \alpha_2$

$d_2 = 3 = \alpha_1 + 2\alpha_2$

$\Rightarrow \begin{cases} \alpha_1 = 1 \\ \alpha_2 = 1 \end{cases}$

$\Rightarrow d_n = \begin{cases} n+1 & n \in N^* \\ 0 & n = 0 \end{cases}$

Q20. $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$

let $G_1(x) = (1+x)^n = \sum_{k=0}^{n} C(n,k)\, x^k$

$G_2(x) = (1+x)^{n-1} = \sum_{k=0}^{n-1} C(n-1,k)\, x^k$

$G_3(x) = x(1+x)^{n-1} = x \sum_{k=0}^{n-1} C(n-1,k)\, x^k = \sum_{k=0}^{n-1} C(n-1,k)\, x^{k+1}$

let $k' = k+1$ $\qquad = \sum_{k'=1}^{n} C(n-1,k'-1)\, x^{k'}$

for $G_1(x) = G_2(x) + G_3(x)$, coefficients of terms of the same degree are equal.

for $0 < r < n$: x^r: $C(n,r) = C(n-1,r) + C(n-1,r-1)$

thus the proposition holds.

Q21. $(1+x)^m = \sum_{k=0}^{m} C(m,k)\, x^k \quad (1+x)^n = \sum_{k=0}^{n} C(n,k)\, x^k$

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} C(m+n,k)\, x^k$$

for $(1+x)^{m+n} = (1+x)^m (1+x)^n$

then the coefficient of x^r must be same

$$x^r : \quad C(m+n, r) = C(m,0)C(n,r) + C(m,1)C(n,r-1) + \cdots + C(m,r)C(n,0)$$

$$= \sum_{k=0}^{r} C(m, r-k)\, C(n,k)$$

thus,
the proposition holds