Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Binary Relation

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- **Theorem** The number of binary relations on a set A, where $|A| = n$ is 2^{n^2}.

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  Yes. If $a|b$ and $b|c$, then $a|c$.

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- **Example**: Assume that $R_{\neq} = \{(a, b) : a \neq b\}$ on $A = \{1, 2, 3, 4\}$.

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  No. $(1, 2), (2, 1) \in R_{\neq}$ but $(1, 1) \notin R_{\neq}$.

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  Set operations: union, intersection, difference, etc.

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- **Example**: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and $R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$, $R_2 = \{(1, v), (3, u), (3, v)\}$

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  We may also combine relations by matrix operations.

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

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$R^k = ?$ for $k > 3$

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"only if" part: by induction.

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  How many subsets on $n(n-1)$ elements are there?

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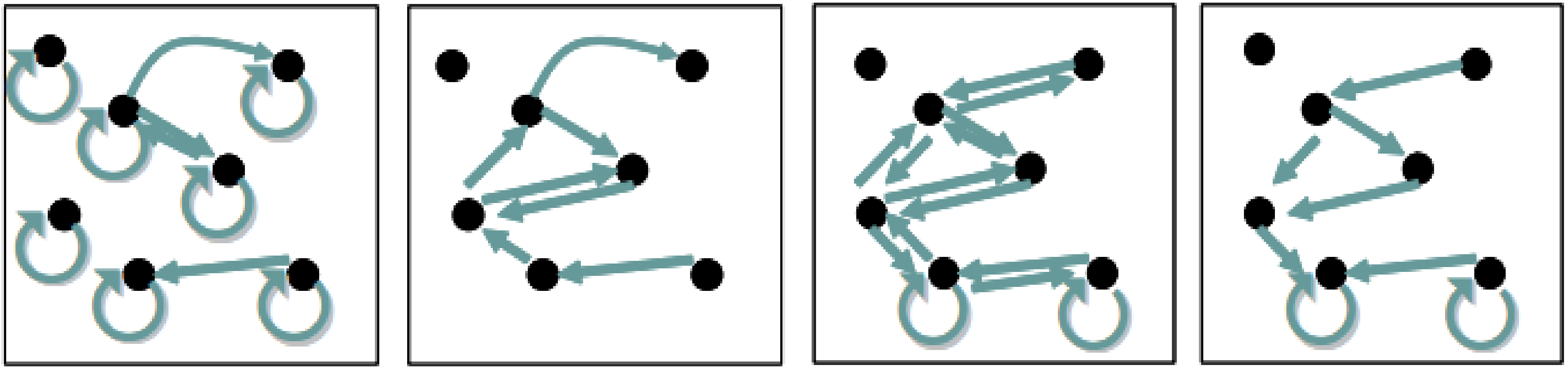

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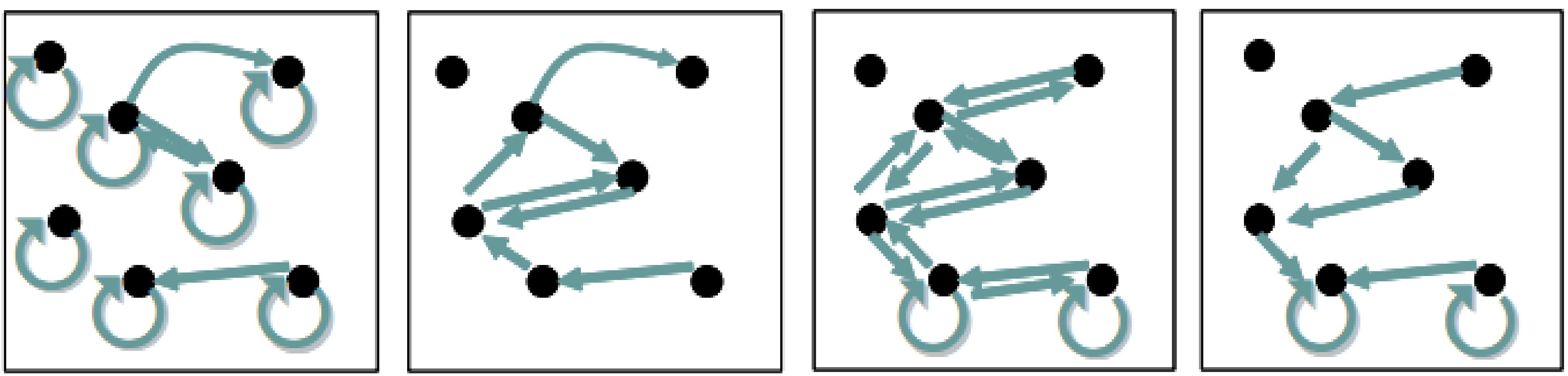

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  Add $(2,2)$ and $(3,3)$

  Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ are not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

  How to make R reflexive by minimum number of additions?

  Add $(2,2)$ and $(3,3)$

  Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$

  The minimal set $S \supseteq R$ is called *the reflexive closure of R*.

Reflexive Closure

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Reflexive Closure

- The set S is called *the reflexive closure of R* if it:
  - ◇ contains R
  - ◇ is reflexive
  - ◇ is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

分解条件

- **包含 R:**
  S 中必须包括 R 中的所有原始关系。例如，如果 $R = \{(a,b),(b,c)\}$，那么 S 必须包含这些对。
- **是自反的:**
  S 必须保证集合中所有元素与自身相关联。例如，如果集合是 $\{a,b,c\}$，那么 S 必须包括 $(a,a),(b,b),(c,c)$，即使这些对不在 R 中。
- **是最小的:**
  S 不能多加多余的对。例如，如果集合中只有 $R = \{(1,2),(2,3)\}$，则 S 不能包含 $(1,3)$ 或其他与自反无关的对。

Closures on Relations

- Relations can have different properties:
  - reflexive
  - symmetric
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  We define:
  - reflexive closures
  - symmetric closures
  - transitive closures

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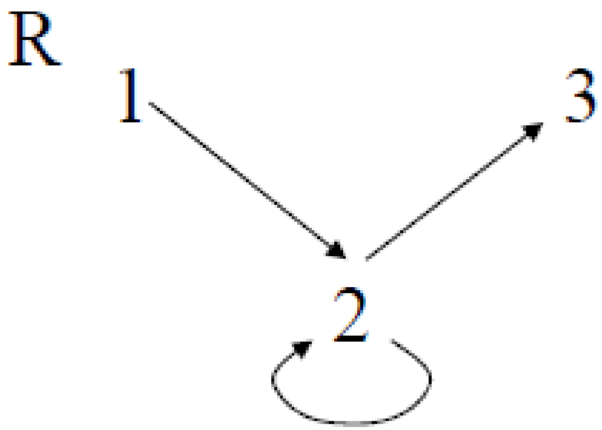

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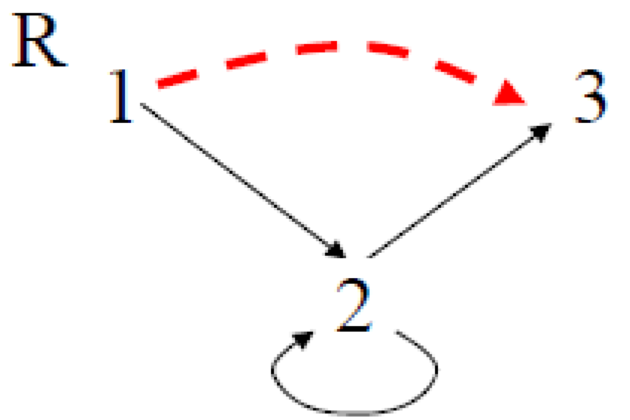

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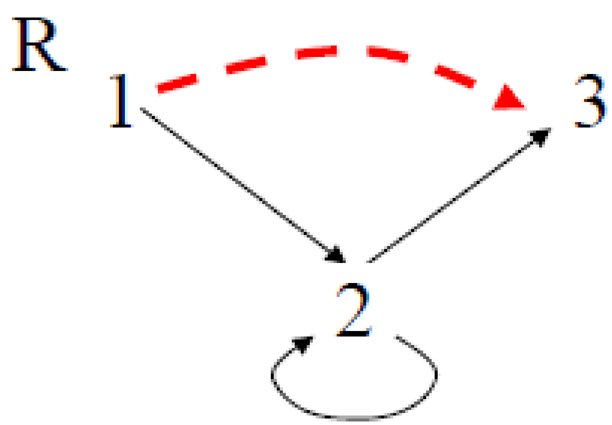

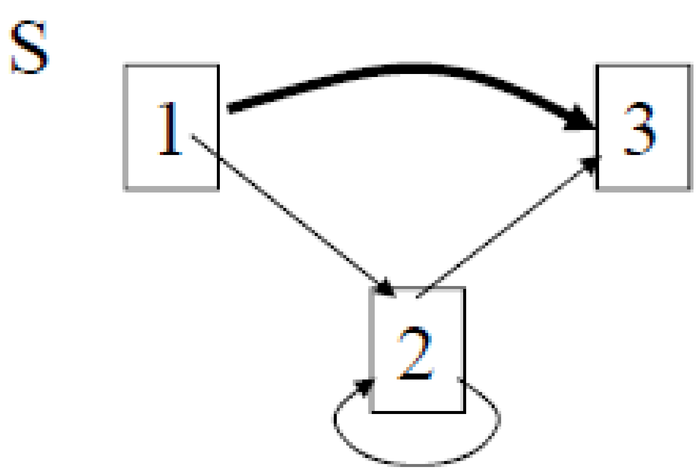

Paths in Directed Graphs

- **Definition** A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where n is nonnegative and $x_0 = a$ and $x_n = b$. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

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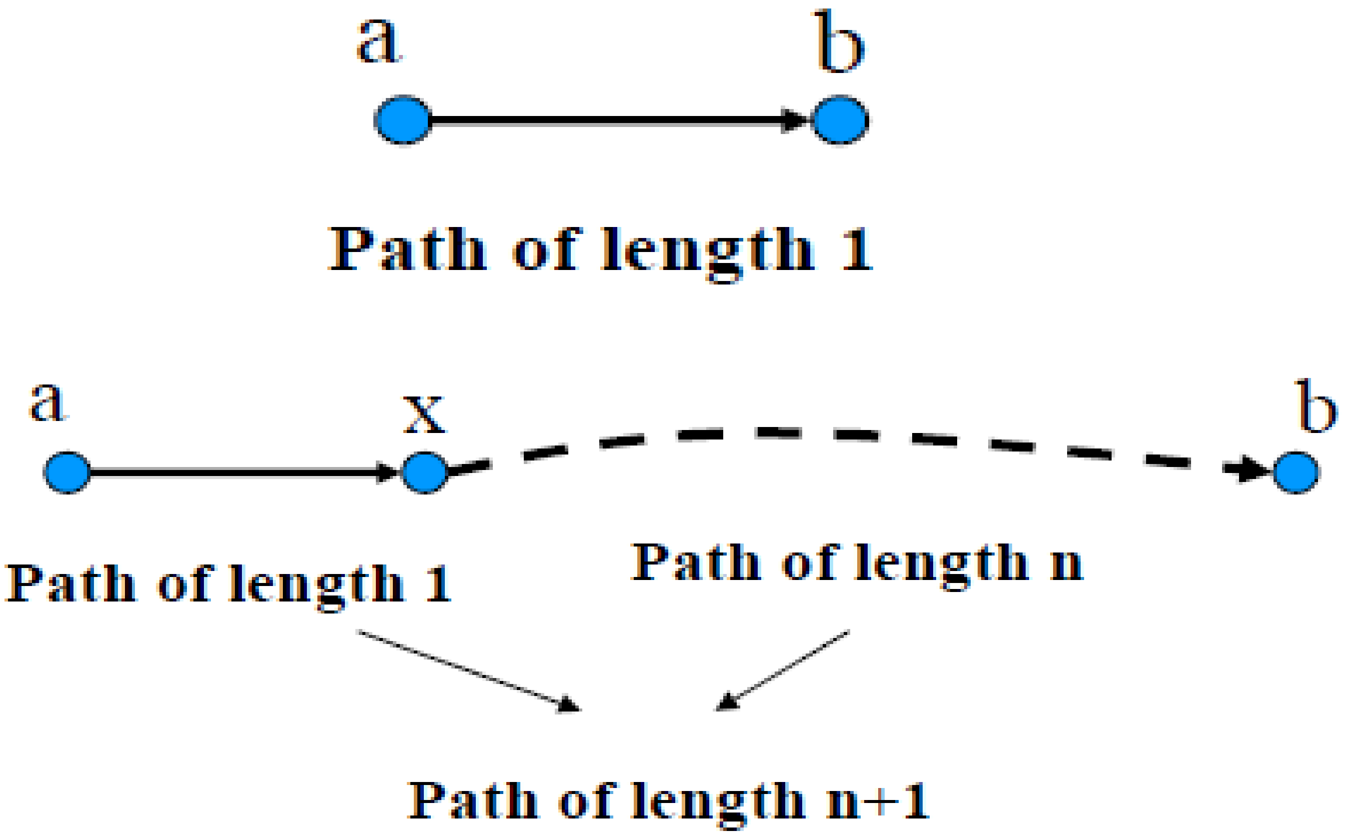

Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

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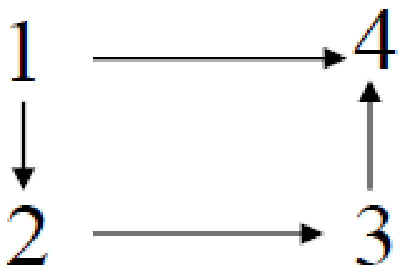

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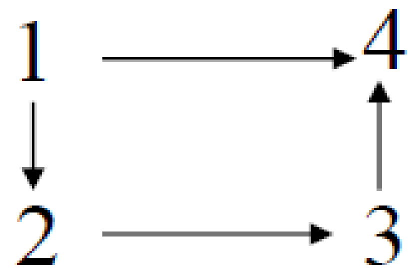

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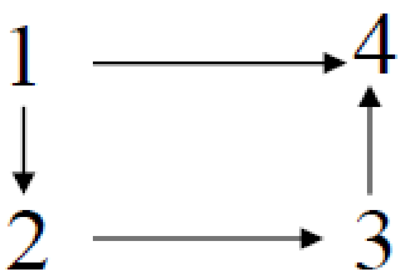

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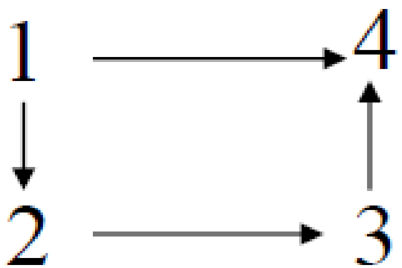

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1. **路径的概念：**
   - 如果在 R 中有一条从 a 到 b 的路径，无论路径多长，(a,b) 都属于 R^*。
2. **包含性：**
   - $R \subseteq R^*$：连通关系包含原始关系 R，因为 R^* 至少需要保留原始的点对。
3. **闭包的过程：**
   - 通过反复组合 R，加入所有间接连接的点对，直到不能再加入新的点对为止。

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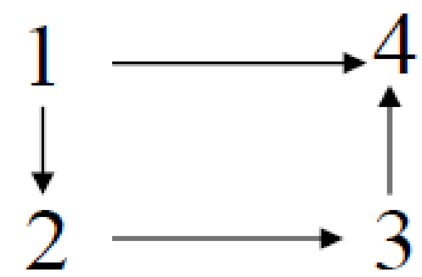

$$\begin{aligned}
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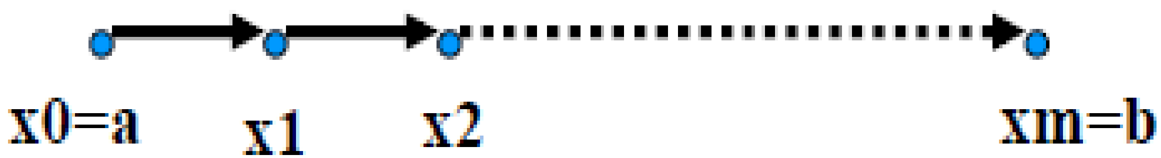

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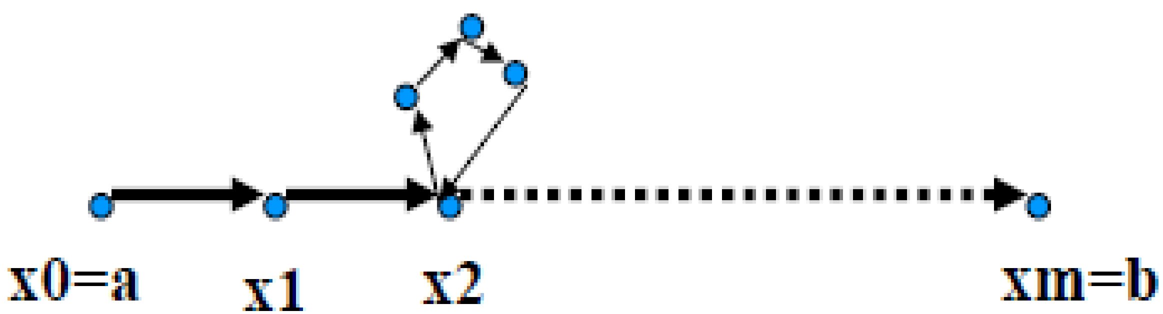

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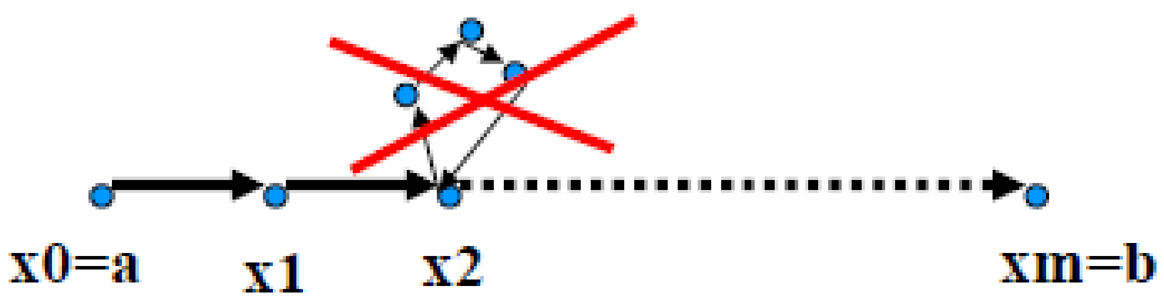

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  1. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.

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We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

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Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathbf{M}_{R^*} = ?$

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
for $i := 2$ to n
  $A := A \odot \mathbf{M}_R$
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Roy-Warshall Algorithm

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for $k := 1$ to n
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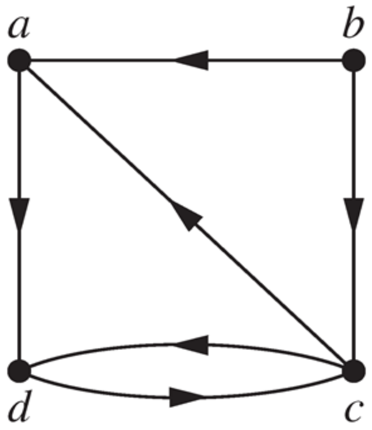
Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



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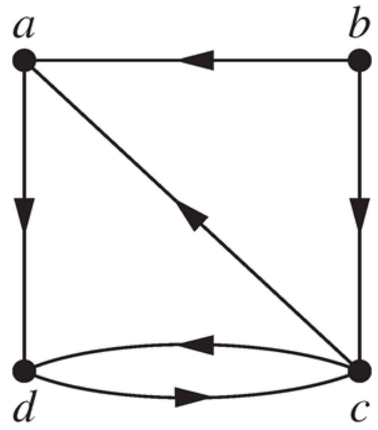

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$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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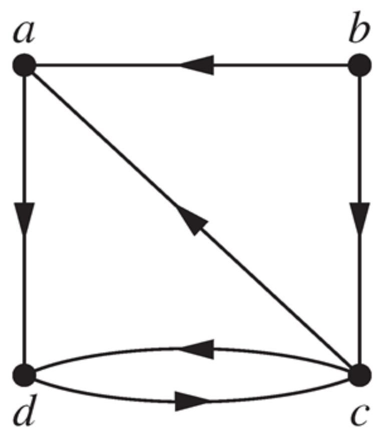

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$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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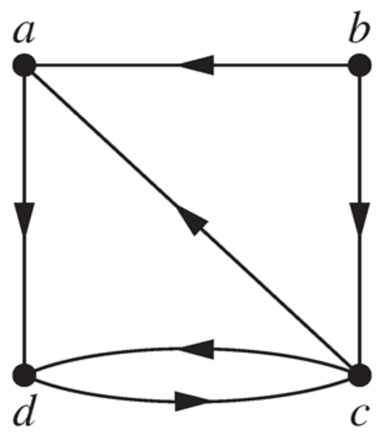

Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 0 & 1 \\ \color{red}{1} & 0 & 1 & \color{red}{1} \end{bmatrix}$$

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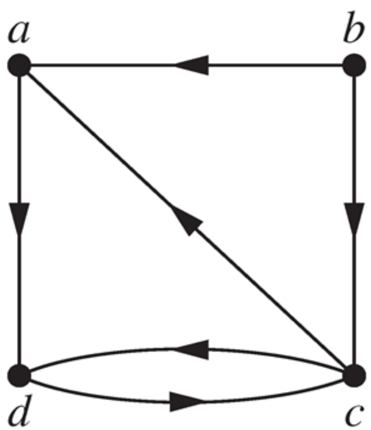

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n-ary Relations

- **Definition** An n-ary relation R on sets $A_1, \ldots, A_n$, written as $R : A_1, \ldots, A_n$, is a subset $R \subseteq A_1 \times \cdots \times A_n$.

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  – R is *functional* in domain A_i if it contains at most one n-tuple $(\cdots, a_i, \cdots)$ for any value a_i within domain A_i.

Relational Databases

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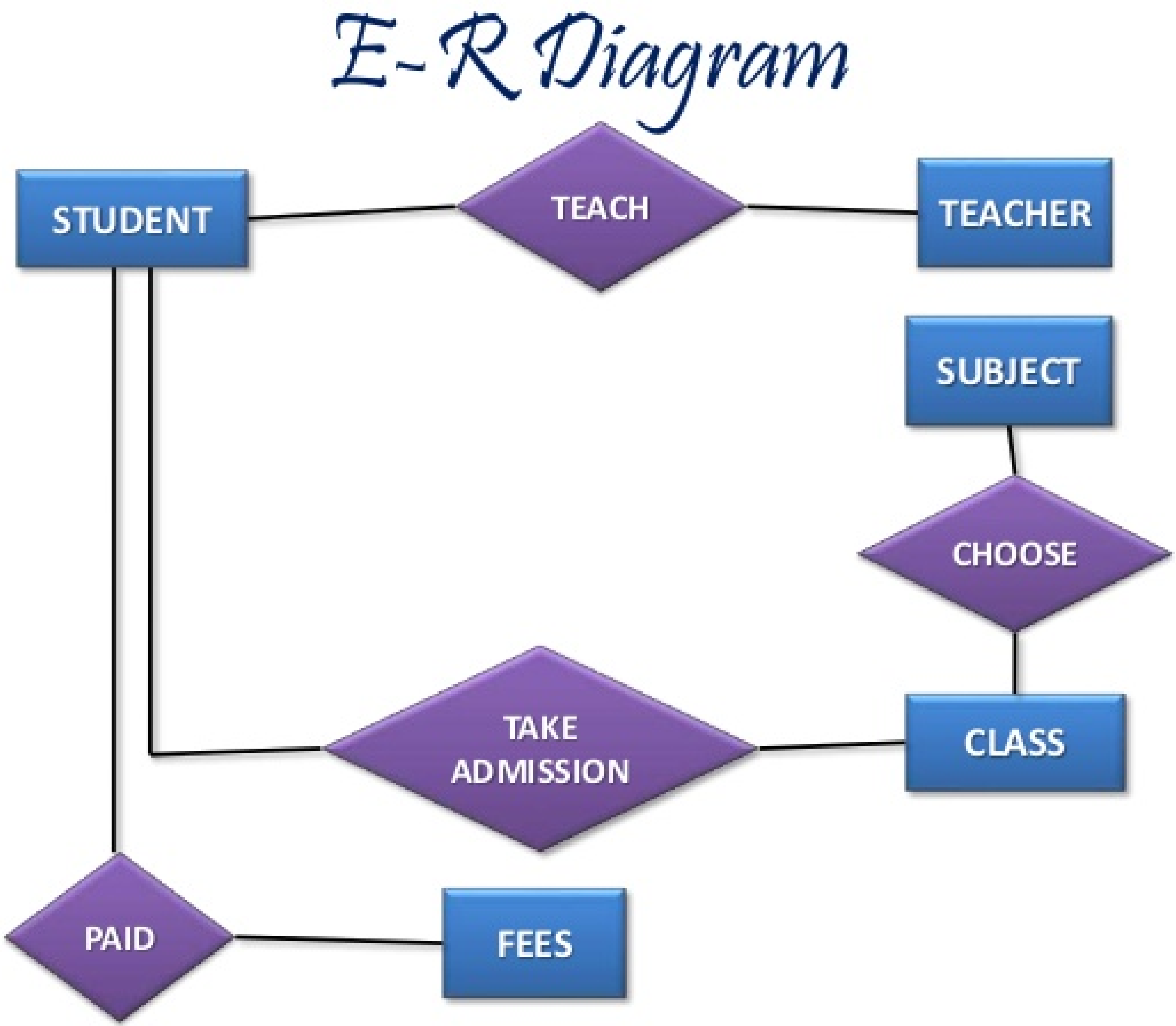
- A *composite key* for the database is a set of domains $\{A_i, A_j, \cdots\}$ such that R contains at most 1 n-tuple $(\cdots, a_i, \cdots, a_j, \cdots)$ for each composite value $(a_i, a_j, \cdots) \in A_i \times A_j \times \cdots$.

Relational Databases



Selection Operators

- Let A be any *n-ary domain* $A = A_1 \times \cdots \times A_n$, and let $C : A \to \{T, F\}$ be any *condition* (predicate) on elements (n-tuples) of A.

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- The *selection operator* s_C is the operator that maps any (n-ary) relation R on A to the n-ary relation of all n-tuples from R that satisfy C.

3. Selection operator (选择操作符):

- s_C 是选择操作符，它将关系 R（R 是 A 上的一个 n 元关系，即 R 包含 A 中的 n 元组）映射到一个新的 n 元关系，这个新关系只包含那些满足条件 C 的 n 元组。

4. 数学表达式:

- $\forall R \subseteq A$ 表示对于 A 的所有子集 R。
- $s_C(R) = R \cap \{a \in A \mid s_C(a) = T\}$ 表示选择操作符 s_C 应用于关系 R 的结果是 R 与所有满足 $s_C(a) = T$ 的 a 的集合的交集。
- $s_C(R) = \{a \in R \mid s_C(a) = T\}$ 是上述交集的另一种表达方式，它直接表示为 R 中所有满足 $s_C(a) = T$ 的元素 a 的集合。

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  – $\forall R \subseteq A$,

$$\begin{aligned} s_C(R) &= R \cap \{a \in A \mid s_C(a) = T\} \\ &= \{a \in R \mid s_C(a) = T\}. \end{aligned}$$

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$$UpperLevel(name, \ standing, \ ssn) \\ :\equiv \quad [(standing = junior) \vee (standing = senior)]$$

- Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of just the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \cdots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n.

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- Then the *projection operator* on n-tuples

  $$P_{\{i_k\}} : A \to A_{i_1} \times \cdots \times A_{i_m}$$

  is defined by

  $$P_{\{i_k\}}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m})$$

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  $\color{blue}{Cars = Model \times Year \times Color}$ $(n = 3)$

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- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of $model/color$ combinations available.

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- A, B, C can also be sequences of elements rather that single elements.

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- Suppose that R_1 is a teaching assignment table, relating *Professors* to *Courses*.

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- Suppose that R_2 is a room assignment table relating *Courses* to *Rooms* and *Times*.

- Then $J(R_1, R_2)$ is like your class schedule, listing $(professor, course, room, time)$.

Next Lecture

- relation III ...