



CS215 DISCRETE MATH

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Department of Computer Science and Engineering

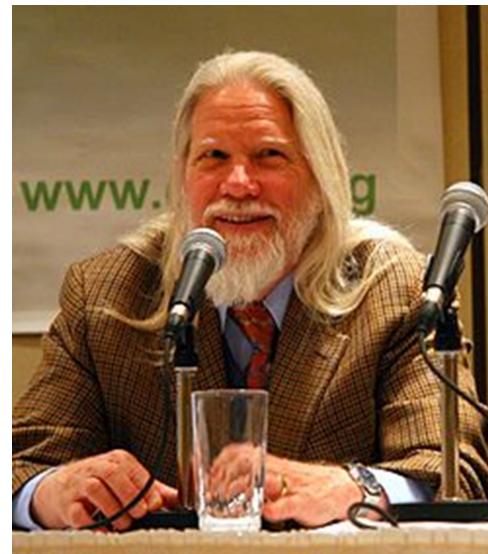
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Cryptography History

- History (from 1976)
 - ◊ W. Diffie, M. Hellman, “New direction in cryptography”, *IEEE Transactions on Information Theory*, vol. 22, pp. 644-654, 1976.

“We stand today on the brink of a revolution in cryptography.”



Bailey W. Diffie

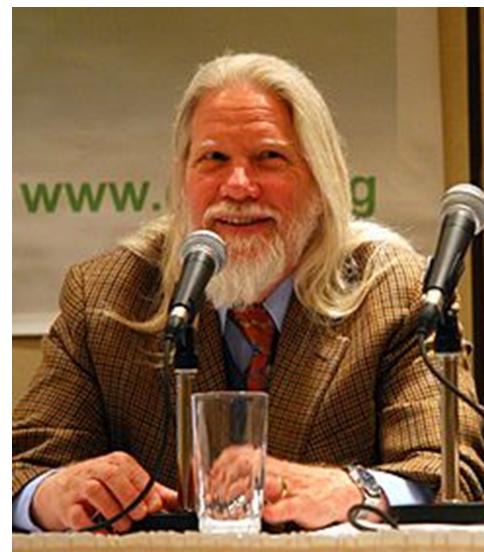
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2015 Turing Award

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2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to modern cryptography . Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography," ^[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today. ^[40]
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Public Key Cryptography

- Alice wants to send a message to Bob



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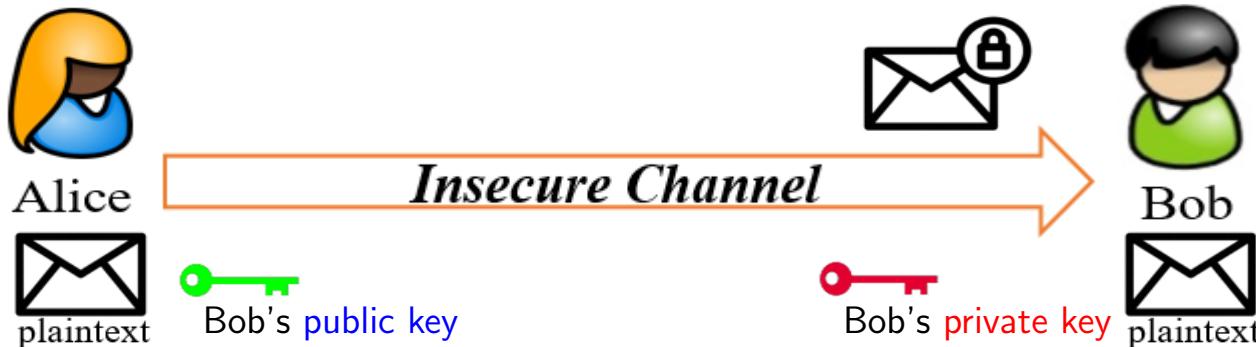
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Ronald L. Rivest



Adi Shamir



Leonard M. Adleman

R. Rivest, A. Shamir, L. Adleman, “A method for obtaining digital signatures and public-key cryptosystems”,
Communications of the ACM, vol. 21-2, pages 120-126, 1978.

RSA Public Key Cryptosystem

■ Rivest-Shamir-Adleman

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- $\gcd(e, \phi(n)) = 1$
- $ed \equiv 1 \pmod{\phi(n)}$

$$\text{Enc : } C = M^e \pmod{n}$$

$$\text{Dec : } M' = C^d \pmod{n}$$

complexity : polynomial time but slow

correctness : $M' = M$

$$M^d \equiv M \pmod{n}$$

Case I:
 $\gcd(m, n) = 1$ Euler Thm $m^{\phi(n)} \equiv 1 \pmod{n}$
 $M^{ed} \equiv M^{k\phi(n)+1} \equiv (M^{\phi(n)})^k \cdot M \equiv M \pmod{n}$

Case II:

$\gcd(m, n) = p$ $M = tp$ $\gcd(M, q) = 1$

$M^{ed} - M \equiv 0 \pmod{n}$ $M(M^{ed-1} - 1) \equiv 0 \pmod{pq}$

要在 $M^{ed} - 1 \equiv 0 \pmod{q}$ (M 有 p 的因子)

$\Leftrightarrow M^{ed} \equiv 1 \pmod{q}$

$M^{ed-1} = M^{k\phi(n)} \equiv M^{k(p-1)(q-1)} \pmod{q}$

$\equiv (M^{q-1})^{kp-1} \pmod{q}$ $M^{q-1} \equiv 1 \pmod{q}$

$M^{ed} \equiv 1 \pmod{q}$ 成立 美妙之理

Case III. $\gcd(m, n) = q$ 同 II

Case IV $\gcd(m, n) = pq$

$\Rightarrow M = 0$ 成立

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$M = C^d \pmod{n}$ (RSA **decryption**)

RSA Public Key Cryptosystem

- $C = M^e \bmod n$ (RSA encryption)
- $M = C^d \bmod n$ (RSA decryption)

Theorem (Correctness) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 \leq x < n$,

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Q : How to prove this?

RSA Public Key Cryptosystem: Example

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	5	11	55	40	7	23

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	5	11	55	40	7	23

Public key: (7, 55)

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Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

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Parameters: p q n $\phi(n)$ e d

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RSA Public Key Cryptosystem: Parameters

Parameters: p q n $\phi(n)$ e d

Public key: (e, n)

对 RSA 的安全机制：
做 $n = pq$ 分解

Private key: d

$p, q, \phi(n)$ must be kept secret!

Q : Why?

已知 $\phi(n)$ 与 n $\phi(n) = pq - p - q + 1$
可求出 pq 与 $p + q = pq - \phi(n) + 1$
 $pq = n$

Comment: It is believed that determining $\phi(n)$ is equivalent to factoring n . Meanwhile, determining d given e and n , appears to be at least as time-consuming as the integer factoring problem.

CS 208 – Algorithm Design and Analysis

The Security of the RSA

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Q : Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \text{lcm}(p - 1, q - 1)$$

$p-1$ 与 $q-1$ 至少有2的因子

and compute $d' = e^{-1} \pmod{\lambda(n)}$. Will decryption using d' instead of d still work?

用 d' 是不可以解密

Applications of RSA

- SSL/TLS protocol

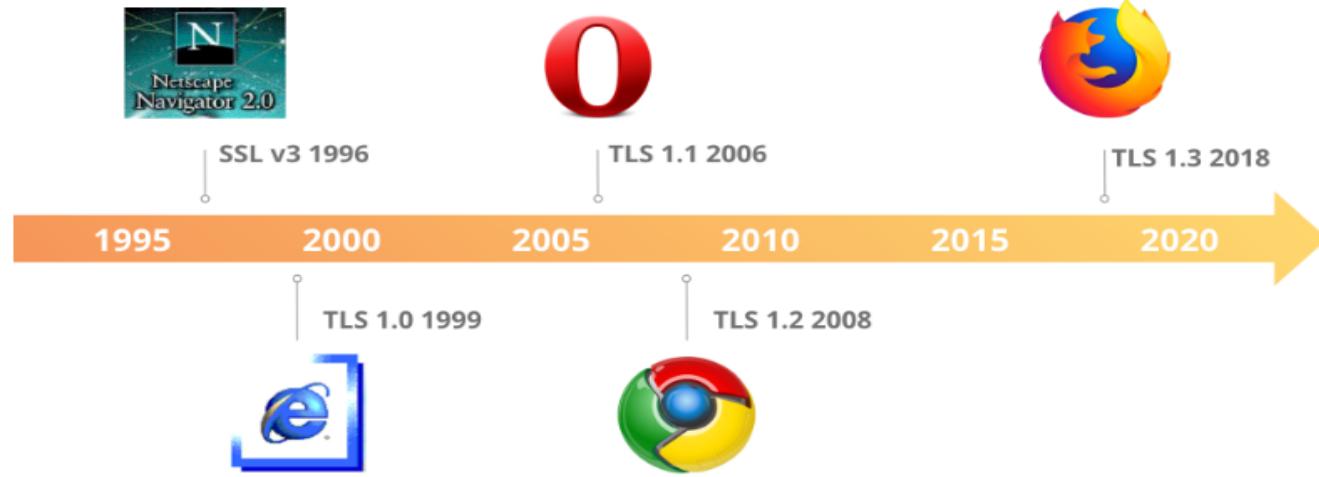
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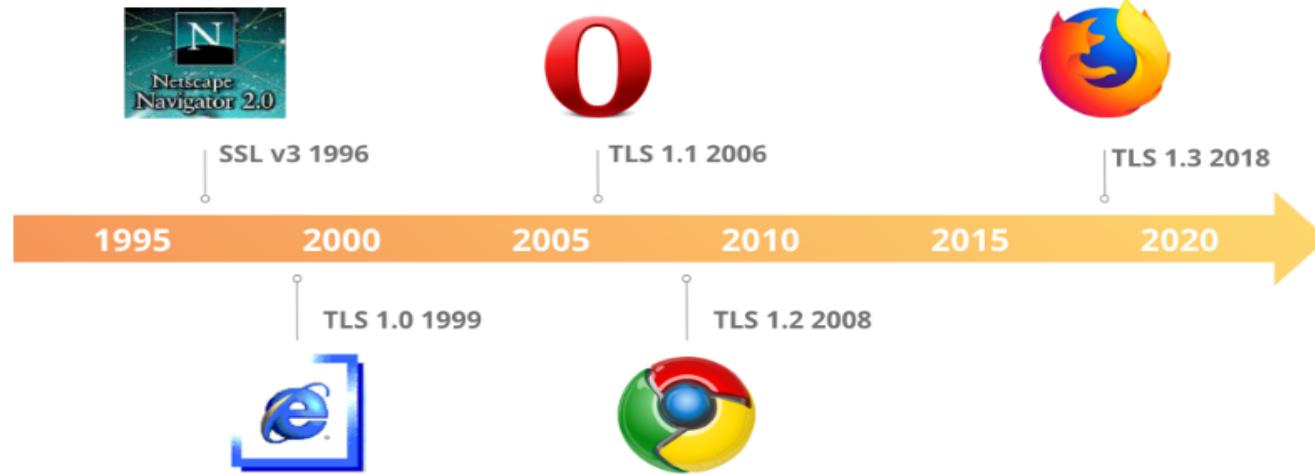
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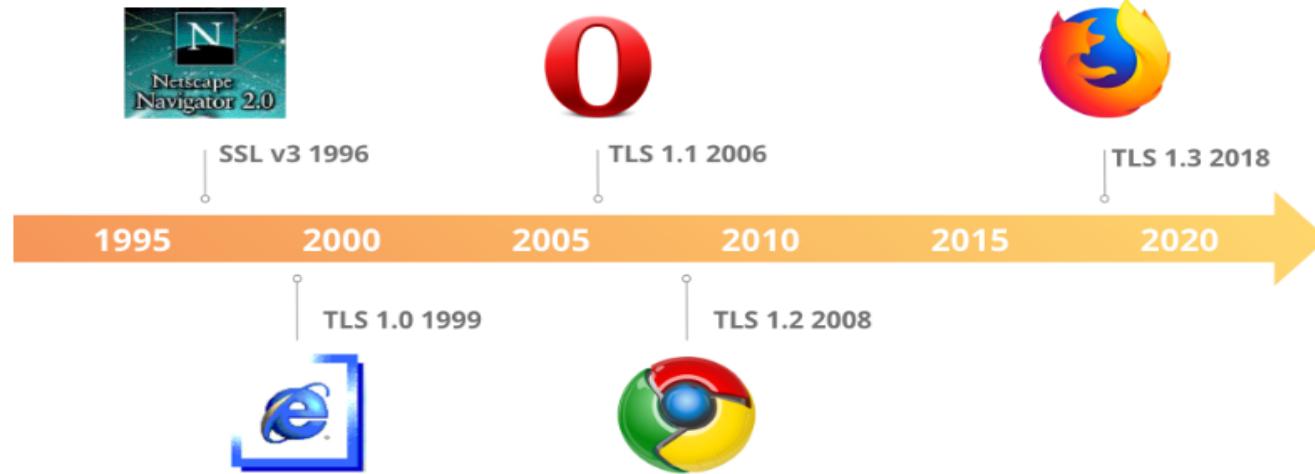


Key exchange/agreement and authentication

Algorithm	SSL 2.0	SSL 3.0	TLS 1.0	TLS 1.1	TLS 1.2	TLS 1.3
RSA	Yes	Yes	Yes	Yes	Yes	No
DH-RSA	No	Yes	Yes	Yes	Yes	No
DHE-RSA (forward secrecy)	No	Yes	Yes	Yes	Yes	Yes
ECDH-RSA	No	No	Yes	Yes	Yes	No
ECDHE-RSA (forward secrecy)	No	No	Yes	Yes	Yes	Yes

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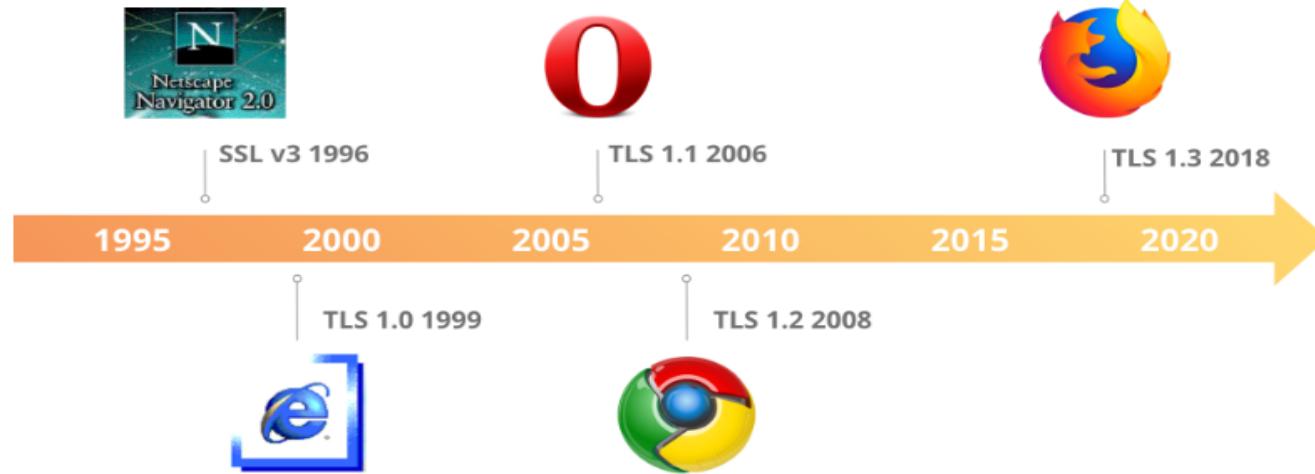
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CS 305 – Computer Networks

Applications of RSA

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CS 305 – Computer Networks

CS 403 – Cryptography and Network Security

Using RSA for Digital Signature

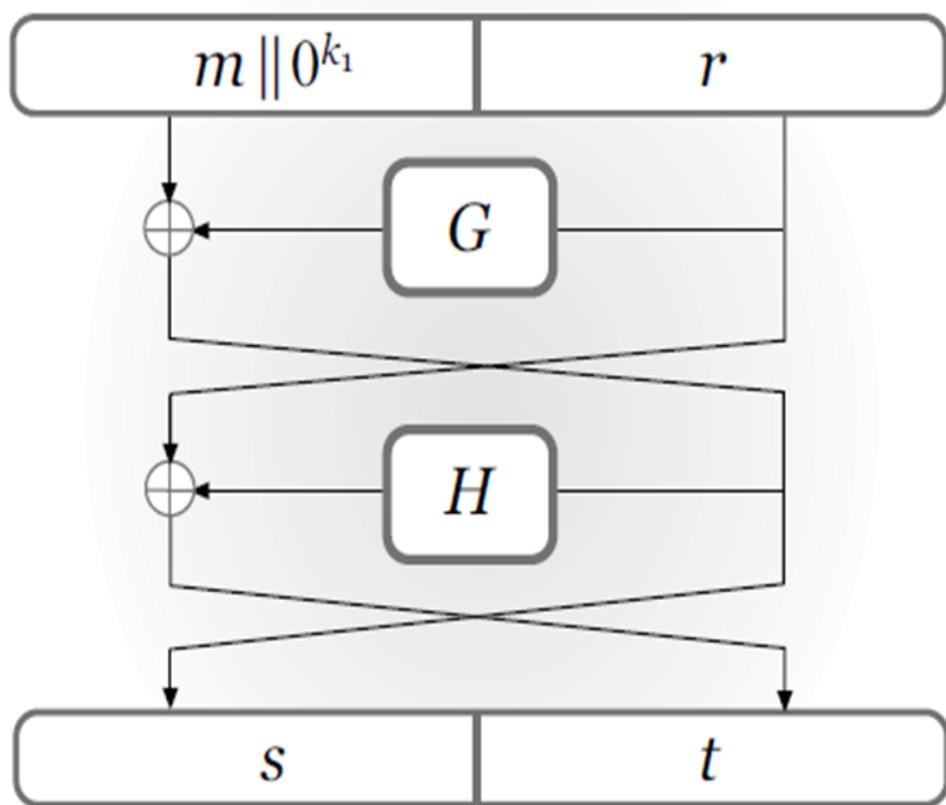
$$S = M^d \bmod n \text{ (RSA signature)}$$

$$M = S^e \bmod n \text{ (RSA verification)}$$

Why?

RSA-OAEP Standard

- RSA-OAEP (Optimal Asymmetric Encryption Padding) is *IND-CCA2 secure*.
- PKCS#1 V2, RFC2437 Standard



The Discrete Logarithm

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Given n , b and y , find x .

This is very hard!

El Gamal Encryption

- **Setup** Let p be a prime, and g be a generator of \mathbb{Z}_p . The **private key** x is an integer with $1 < x < p - 2$. Let $y = g^x \bmod p$. The **public key** for *El Gamal encryption* is (p, g, y) .

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El Gamal Encryption: Pick a **random** integer k from \mathbb{Z}_{p-1} ,

$$\begin{aligned} a &= g^k \bmod p \\ b &= M y^k \bmod p \end{aligned}$$

The ciphertext C consists of the pair (a, b) .

El Gamal Decryption:

$$M = b(a^x)^{-1} \bmod p$$

Using El Gamal for Digital Signature

$$\begin{aligned} a &= g^k \pmod{p} \\ b &= k^{-1}(M - xa) \pmod{p-1} \end{aligned}$$

(El Gamal **signature**)

$$y^a a^b \equiv g^M \pmod{p}$$

(El Gamal **verification**)

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(El Gamal signature)

$$y^a a^b \equiv g^M \pmod{p}$$

(El Gamal verification)

Q : How to verify it?

An Example

Choose $p = 2579$, $g = 2$, and $x = 765$. Hence
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Encryption: Let $M = 1299$ and choose a random $k = 853$,

$$\begin{aligned}(a, b) &= (g^k \bmod p, My^k \bmod p) \\ &= (2^{853} \bmod 2579, 1299 \cdot 949^{853} \bmod 2579) \\ &= (435, 2396).\end{aligned}$$

Decryption:

$$M = b(a^x)^{-1} \bmod p = 2396 \times (435^{765})^{-1} \bmod 2579 = 1299.$$

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Question 2: Given a ciphertext (a, b) , is it feasible to derive the plaintext M ?

Attack 1: Use $M = by^{-k}$. However, k is **randomly** picked.

Attack 2: Use $M = b(a^x)^{-1} \bmod p$, but x is **secret**.

Diffie-Hellman Key Exchange Protocol

User A

Generate random
 $X_A < p$
calculate
 $Y_A = \alpha^{X_A} \text{ mod } p$

Calculate
 $k = (Y_B)^{X_A} \text{ mod } p$

Y_A
→
←
 Y_B

User B

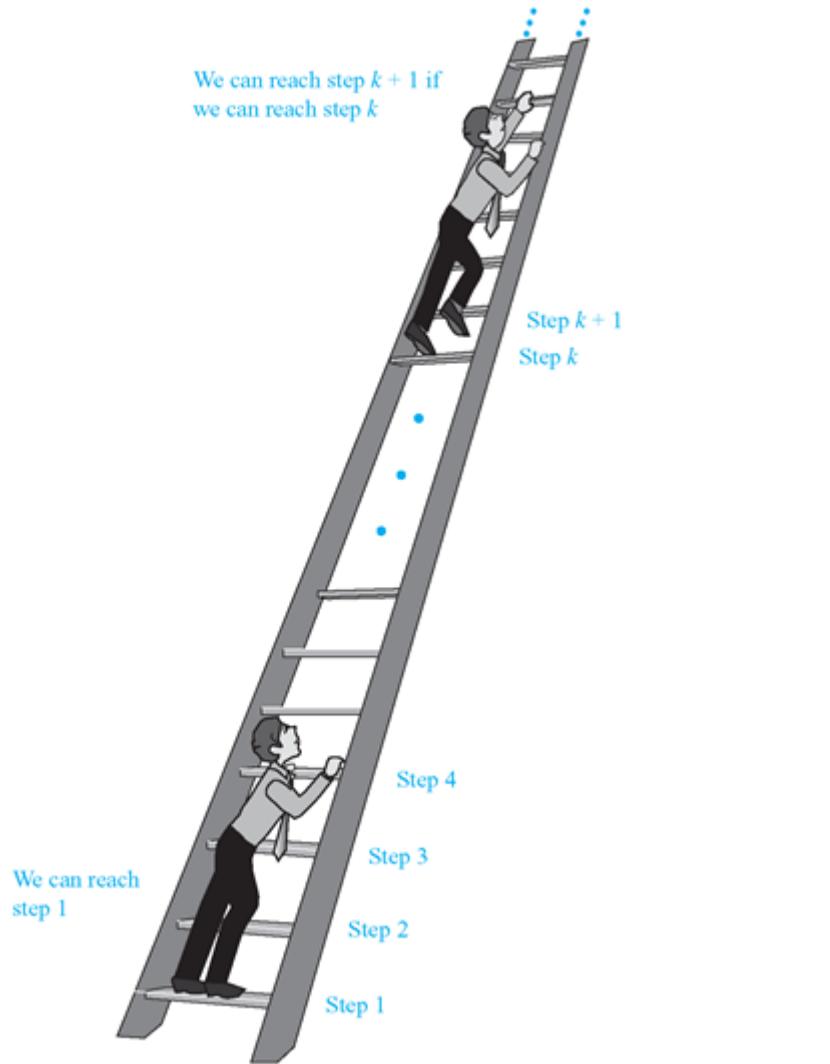
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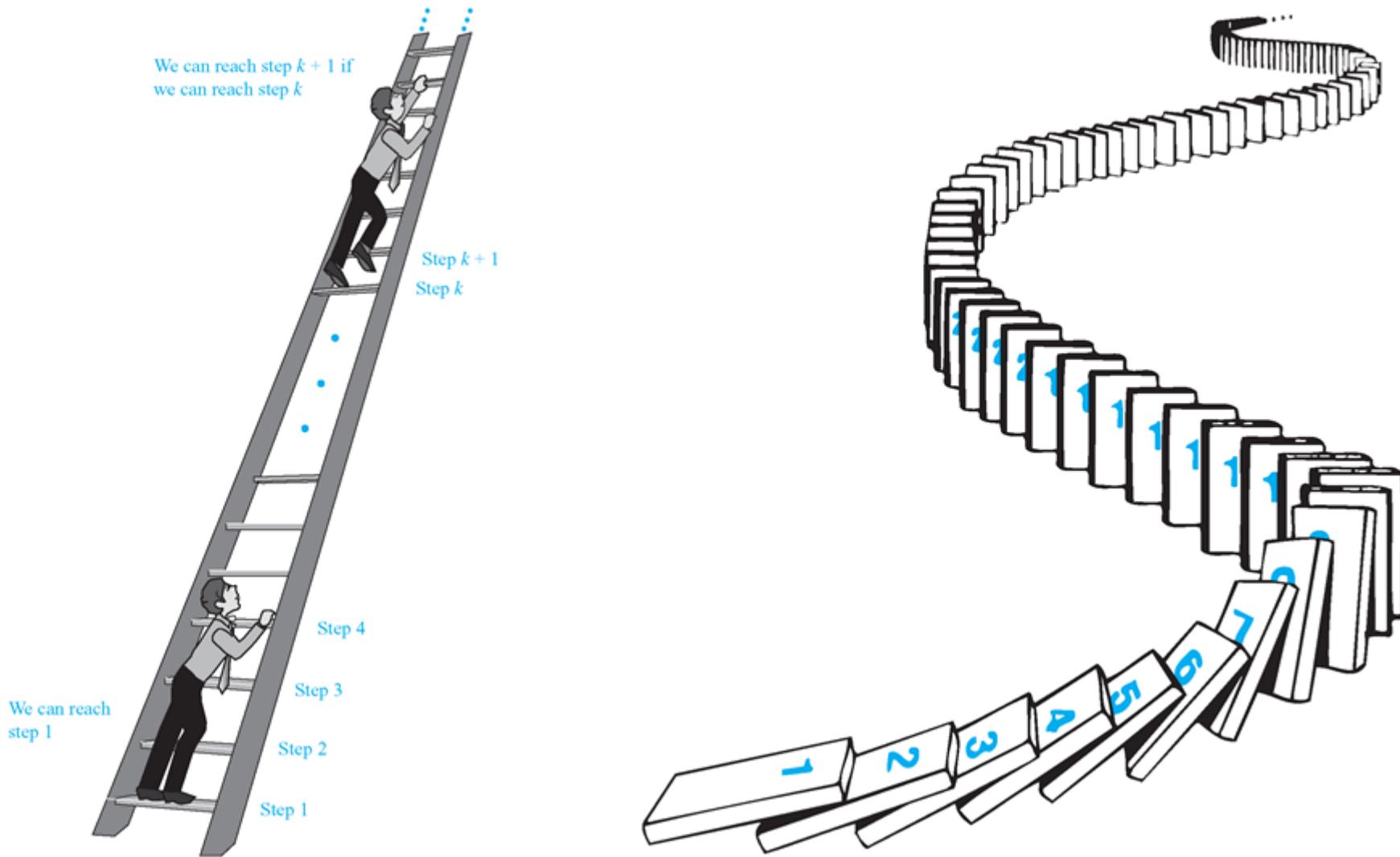
Cryptography Wonders

- *Digital Signatures.* Electronically sign documents
用交互代价换取零知识
- Zero-knowledge Proofs.* Alice proves to Bob that she earns < \$50k without Bob learning her income.
- Privacy-preserving data mining.* Bob holds DB. Alice gets answer to one query, without Bob knowing what she asked.
- Playing poker over the net.* Alice, Bob, Carol and David can play Poker over the net without trusting each other or any central server. (*E-Voting*)
- Electronic Auctions.* Can run auctions s.t. no one (even not seller) learns anything other than winning party and bid.
- Fully Homomorphic Encryption.* Encrypt $E(m)$ in a way that allows to compute $E(f(m))$.

Mathematical Induction



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The *strong principle* can actually be derived from the *weak principle*.

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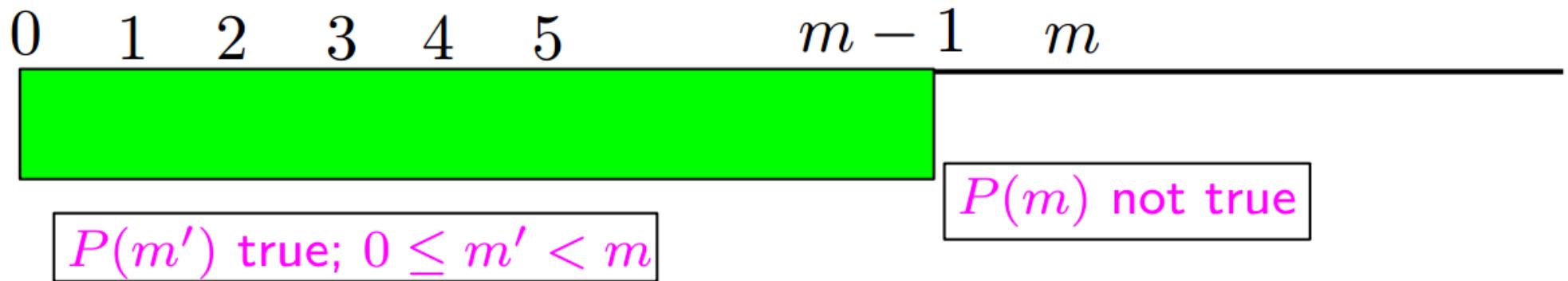
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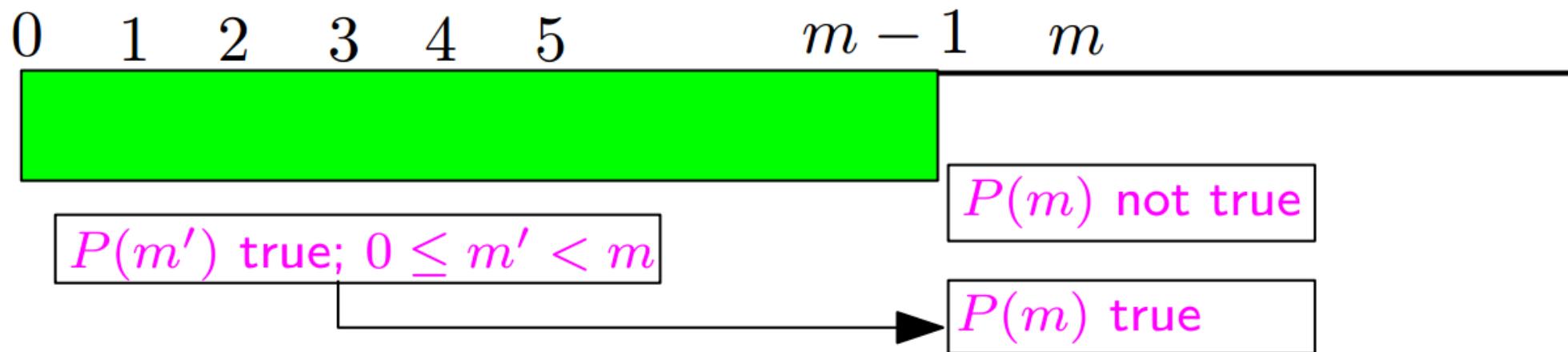


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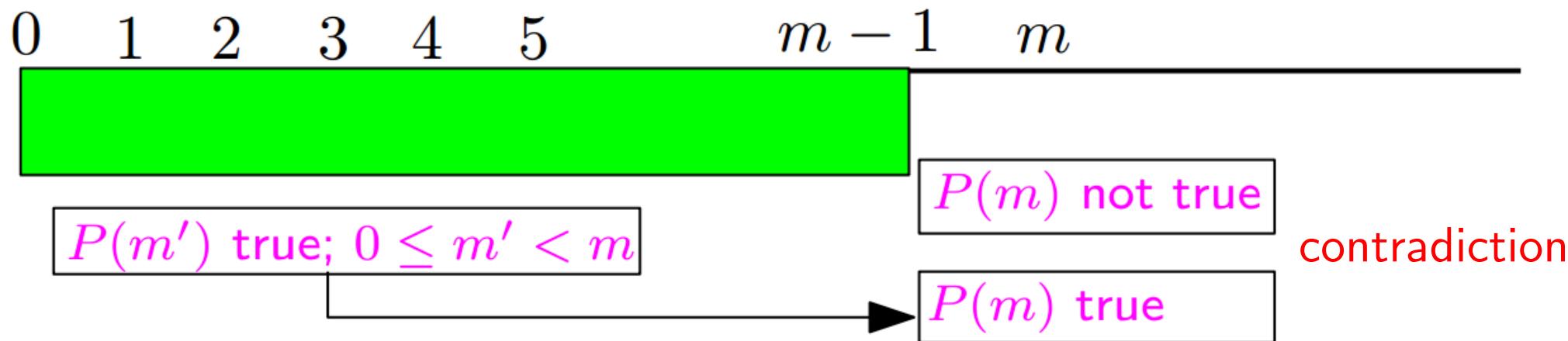


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- ◊ The smallest counterexample n is larger than 0

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- ◊ Therefore, (*) holds for all positive integers n .

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The **key step** was proving that

$$P(n - 1) \rightarrow P(n)$$

where $P(n)$ is the statement

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}$$

Example 2

- Use proof by smallest counterexample to show that, $\forall n \in N$,

$$2^{n+1} \geq n^2 + 2.$$

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When a **for all** quantifier is false, there must be some n for which it is false. Let n be the smallest nonnegative integer for which $2^{n+1} \not\geq n^2 + 2$.

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Then setting $i = n - 1$ gives

$$2^{(n-1)+1} \geq (n-1)^2 + 2.$$

or

$$(*) \quad 2^n \geq n^2 - 2n + 1 + 2 = n^2 - 2n + 3$$

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Thus, we write

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Since $P(n - 1) \rightarrow P(n)$, we see that

$P(0)$ implies $P(1)$, $P(1)$ implies $P(2)$, ...

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(a) – *Basic Step Inductive Hypothesis*

31 - 4 (b) – *Inductive Step Inductive Conclusion*

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Inductive Step

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Inductive Conclusion

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 - ◊ Iterating gives us a proof of $P(n)$ for all n

Strong Induction

■ Principle (*The Strong Principle of Mathematical Induction*)

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(b) for all $n > b$, the statement

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then $P(n)$ is true for all integers $n \geq b$.

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 - ◊ Then, if n is not a prime power, it is a product of two smaller numbers, each of which is, by the **inductive hypothesis**, a power of a prime or a product of powers of primes.
 - ◊ Thus, by the **strong principle of mathematical induction**, every positive integer is a power of a prime or a product of powers of primes.

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3. We conclude on the basis of **the principle of mathematical induction** that $P(n)$ is true for all $n \geq b$.

Next Lecture

- recurrence ...

