Discrete Math Assign01

Q.1

(a)

$r \land \neg q$

(b)

$r \to p$

(c)

$p \land \neg q \land r$

(d)

$p \land q \to r$

(e)

$r \leftrightarrow p \lor q$

Q.2

(a)

p	q	$p \oplus q$	$p \land q$	$(p \oplus q) \to (p \land q)$
F	F	F	F	T
F	T	T	F	F
T	F	T	F	F
T	T	F	T	T

p	q	$p \oplus q$	$\neg q$	$p \oplus \neg q$	$(p \oplus q) \to (p \oplus \neg q)$
F	F	F	T	T	T
F	T	T	F	F	F
T	F	T	T	F	F
T	T	F	F	T	T

Two propositions are equivalent.

(b)

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
F	F	F	T	T	T	T
F	T	F	T	F	F	F
T	F	F	F	T	F	F
T	T	T	F	F	F	T

Two propositions are equivalent.

(c)

p	q	$\neg q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$(\neg q \wedge \neg(p \rightarrow q))$
F	F	T	T	F	F
F	T	F	T	F	F
T	F	T	F	T	T
T	T	F	T	F	F

p	q	$\neg p$
F	F	T
F	T	T
T	F	F
T	T	F

Two propositions are NOT equivalent.

(d)

p	q	r	$p \to \neg q$	$p \lor \neg q$	$r \to (p \lor \neg q)$	$(p \to \neg q) \leftrightarrow (r \to (p \lor \neg q))$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	T	T
F	T	T	T	F	F	F
T	F	F	T	T	T	T
T	F	T	T	T	T	T
T	T	F	F	T	T	F
T	T	T	F	T	T	F

p	q	r	$\neg p \land \neg r$	$q \lor (\neg p \land \neg r)$
F	F	F	T	T
F	F	T	F	F
F	T	F	T	T
F	T	T	F	T
T	F	F	F	F
T	F	T	F	F
T	T	F	F	T
T	T	T	F	T

Two propositions are NOT equivalent.

Q.3

(a)

$$\begin{aligned}
& (p \land \neg q) \to r \\
\equiv\ & \neg(p \land \neg q) \lor r \quad \text{Useful law} \\
\equiv\ & (\neg p \lor \neg(\neg q)) \lor r \quad \text{De Morgan's law} \\
\equiv\ & (\neg p \lor q) \lor r \quad \text{Double negation law} \\
\equiv\ & \neg p \lor (q \lor r) \quad \text{Associative law} \\
\equiv\ & p \to (q \lor r) \quad \text{Useful law}
\end{aligned}$$

(b)

$$
\begin{aligned}
&((p \to q) \land (q \to r)) \to (p \to r) \\
&\equiv ((\neg p \lor q) \land (\neg q \lor r)) \to (\neg p \lor r) \quad \text{Useful law} \\
&\equiv \neg((\neg p \lor q) \land (\neg q \lor r)) \lor (\neg p \lor r) \quad \text{Useful law} \\
&\equiv (\neg(\neg p \lor q) \lor \neg(\neg q \lor r)) \lor (\neg p \lor r) \quad \text{De Morgan's law} \\
&\equiv (\neg\neg p \land \neg q) \lor (\neg\neg q \land \neg r) \lor (\neg p \lor r) \quad \text{De Morgan's law} \\
&\equiv (p \land \neg q) \lor (q \land \neg r) \lor (\neg p \lor r) \quad \text{Double negation law} \\
&\equiv \neg p \lor (p \land \neg q) \lor (q \land \neg r) \lor r \quad \text{Associative law \& Commutative law} \\
&\equiv (\neg p \lor p) \land (\neg p \lor \neg q) \lor (q \lor r) \land (\neg r \lor r) \quad \text{Distributive law} \\
&\equiv T \land (\neg p \lor \neg q) \lor (q \lor r) \land T \quad \text{Negation law} \\
&\equiv (\neg p \lor \neg q) \lor (q \lor r) \quad \text{Identity law} \\
&\equiv (q \lor \neg q) \lor (\neg p \lor r) \quad \text{Associative law \& Commutative law} \\
&\equiv T \lor (\neg p \lor r) \quad \text{Negation law} \\
&\equiv T \quad \text{Domination law}
\end{aligned}
$$

Q.4

We can find a counterexample to show that the two expression are not logically equivalent.

Let p is **false**, q is **true**, r is **false**; then $(p \to q) \to r$ is **false**, while $p \to (q \to r)$ is **true**. If the two expression are logically equivalent, they should have the same truth value all the time. Therefore, the two expression are not logically equivalent.

Q.5

We manage to prove by contradiction.

Assume that r is true, then for the given statement, we have:

$$
\begin{aligned}
&(q \to (T \lor p)) \to ((\neg T \lor s) \land \neg s) \\
&\equiv (q \to T) \to ((F \lor s) \land \neg s) \quad \text{Domination law} \\
&\equiv (q \to T) \to (s \land \neg s) \quad \text{Identity law} \\
&\equiv (q \to T) \to F \quad \text{Negation law} \\
&\equiv \neg(\neg q \lor T) \lor F \quad \text{De Morgan's law} \\
&\equiv \neg T \lor F \quad \text{Domination law} \\
&\equiv F \lor F \\
&\equiv F \quad \text{Idempotent law}
\end{aligned}
$$

This is a contradiction to the assumption that r is **true**,. Therefore, this implies $\neg r$.

Q.6

(a)

$\forall x \exists y L(x, y)$

(b)

$\exists y \forall x L(x, y)$

(c)

$\neg \exists x \forall y L(x, y)$

(d)

$\exists y(\neg \exists x L(x, y))$

(e)

$\exists x \forall y(L(y, x) \wedge \forall z(L(y, z) \rightarrow(z=x)))$

(f)

$\exists x \exists y(L(Lynn, x) \wedge(x \neq y) \wedge L(Lynn, y) \wedge \forall z(L(Lynn, z) \rightarrow(x=z \vee y=z)))$

(g)

$\exists x(L(x, x) \wedge \forall y((x \neq y) \rightarrow \neg L(x, y)))$

Q.7

(1)

$\forall x(I(x) \rightarrow E(x))$

(2)

$L(x, y) \rightarrow \neg Q(x, y)$

(3)

$\forall x \exists y L(x, y)$

Q.8

(1)

Assume that the universe is all real number, then $P(x, y)$ predicates "$x>y$" with the universe of the real numbers.

$\exists x \forall y P(x, y)$ is **false**, while $\forall y \exists x P(x, y)$ is **true**.

(2)

No

Assume that the universe is all real number, then $P(x,y)$ predicates "$x+y$ is even" with the universe of the real numbers.

$\forall y \exists x P(x,y)$ is **true**, while $\exists x \forall y P(x,y)$ is **false**.

Q.9

(1)

Not equivalent

Assume that the universe is all the people in the world, then $P(x)$ predicates "x is male" with the universe of all the people, $Q(x)$ predicates "x is female" with the universe of all the people.

A person can be either male or female.(Ignore special cases) People in the world are not all male or female. Therefore, $(\forall x \in \mathbb{R}\ P(x)) \vee (\forall x \in \mathbb{R}\ Q(x))$ is **false**, while $\forall x \in \mathbb{R}\ (P(x) \vee Q(x))$ is **true**. Therefore, they are **not equivalent**.

(2)

Equivalent

(3)

Equivalent

(4)

Equivalent

Q.10

(a)

$\exists n \in \mathbb{N}\ (n^3 + 6n + 5\ is\ odd \rightarrow n\ is\ odd)$

(b)

The original statement is true.

We can prove by contradiction. Assume that the original statement is **false**, then we have $\exists n \in \mathbb{N}\ (n^3 + 6n + 5\ is\ odd \rightarrow n\ is\ odd)$.

Assume $a \in \mathbb{N}\ (a^3 + 6a + 5\ is\ odd \rightarrow a\ is\ odd)$, and $a\ is\ odd$.

Then we prove by cases.

If a is odd, let $a = 2k+1, k \in \mathbb{N}$. $a^3 + 6a + 5 = (2k+1)^3 + 6(2k+1) + 5 = 8k^3 + 12k^2 + 18k + 12$, is even, which contradicts to the premise.

If a is even, let $a = 2k, k \in \mathbb{N}$. $a^3 + 6a + 5 = (2k)^3 + 6(2k) + 5 = 8k^3 + 12k + 5$ is odd.

Therefore, when "$a^3 + 6a + 5\ is\ odd$" holds, "$a\ is\ odd$" is always **false**.

Thus, $\neg\exists n \in \mathbb{N}\ (n^3 + 6n + 5\ is\ odd \rightarrow n\ is\ odd)$, and the original statement is true.

Q.11

(a)

We can use truth table to prove they are equivalent.

p	q	$q \vee \neg p$	$(p \leftrightarrow (q \vee \neg p))$	$\neg(p \leftrightarrow (q \vee \neg p))$	$\neg p \vee \neg q$
F	F	T	F	T	T
F	T	T	F	T	T
T	F	F	F	T	T
T	T	T	T	F	F

The two propositions always have the same truth. Therefore, they are equivalent.

(b)

First, we can prove that the combination of logical connectives and p ,q can create at most $2^4 = 16$ different logical symbols, for we can build a form of truth table:

p	q	P
F	F	2 different results
F	T	2 different results
T	F	2 different results
T	T	2 different results

Each row of P's truth value can be independent of other rows.

Then, all we need is to find out all the corresponding $A \square B$. We can find all 16 combinations to prove by case. (Though it must be stupid, but I have no other ideas and need to learn further.)

p	q	$p \wedge \neg p$
F	F	F
F	T	F
T	F	F
T	T	F

p	q	$\neg p \land \neg q$
F	F	T
F	T	F
T	F	F
T	T	F

p	q	$\neg p \land q$
F	F	F
F	T	T
T	F	F
T	T	F

p	q	$\neg p \land \neg p$
F	F	T
F	T	T
T	F	F
T	T	F

p	q	$p \land \neg q$
F	F	F
F	T	F
T	F	T
T	T	F

p	q	$\neg q \land \neg q$
F	F	T
F	T	F
T	F	T
T	T	F

p	q	[illegible]
F	F	F

p	q	$\neg p \leftrightarrow q$
F	T	T
T	F	T
T	T	F

p	q	$\neg p \vee \neg q$
F	F	T
F	T	T
T	F	T
T	T	F

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

p	q	$q \wedge q$
F	F	F
F	T	T
T	F	F
T	T	T

p	q	[illegible]
F	F	T
F	T	T

p	q	$\neg p \vee q$
T	F	F
T	T	T

p	q	$p \wedge p$
F	F	F
F	T	F
T	F	T
T	T	T

p	q	$p \vee \neg q$
F	F	T
F	T	F
T	F	T
T	T	T

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

p	q	$p \vee \neg p$
F	F	T
F	T	T
T	F	T
T	T	T

Therefore, we can prove that every combination can be represented in the form of $A \square B$.

Q.12

Let $J(x)$: The movie x is produced by John Sayles, with the universe of all movies.$

$W(x)$: The movie x is wonderful, with the universe of all movies.

$C(x)$: The movie is about coal miners, with the universe of all movies.

Step		Reason
1.	$\forall x(J(x) \to W(x))$	*Premise*
2.	$\exists x(J(x) \wedge C(x))$	*Premise*
3.	$J(a) \wedge C(a)$	*EI from* (2)
4.	$J(a) \to W(a)$	*UI from* (1)
5.	$J(a)$	*Simplification from* (3)
6.	$C(a)$	*Simplification from* (3)
7.	$W(a)$	*MP from* (4) *and* (5)
8.	$W(a) \wedge C(a)$	*Conj from* (6) *and* (7)
9.	$\exists x(W(x) \wedge C(x))$	*EG from* (8)

Q.13

(1)

We can disprove by contradiction.

Assume the proposition is **true**, let $\text{irrational numbers } a = b = \sqrt{2}$, then $\sqrt{2}^{\sqrt{2}}$ $\text{is also irrational}$; then we have $\frac{1}{\sqrt{2}^{\sqrt{2}}}$ is irrational, and $\frac{2}{\sqrt{2}^{\sqrt{2}}}$ is irrational. Use the proposition again, we can get $\sqrt{2}^{\frac{2}{\sqrt{2}^{\sqrt{2}}}}$ $\text{is also irrational}$

Now we use the proposition again. Let $\text{irrational numbers } a = \sqrt{2}^{\frac{2}{\sqrt{2}^{\sqrt{2}}}}, b = \sqrt{2}^{\sqrt{2}}$, then $a^b = (\sqrt{2}^{\frac{2}{\sqrt{2}^{\sqrt{2}}}})^{(\sqrt{2}^{\sqrt{2}})} = \sqrt{2}^2 = 2.\ 2$ is rational. This leads to a contradiction to the proposition.

Therefore, the proposition is **false**.

(2)

We can prove by contradiction. $a\ is\ irrational$, assume that $\sqrt{a}\ is\ rational$, we have $\sqrt{a} = \frac{m}{n}\ m, n \in \mathbb{N}$, then we have $a = \frac{m^2}{n^2},\ \text{which means a is rational}$. This leads to a contradiction to the premise "$a\ is\ irrational$".

Therefore, the original proposition is **true**.

(3)

We can prove by case.

Let $a = 2, b = \sqrt{2},\ then\ a^b\ is\ irrational.$

Therefore, the proposition is **true**.

Q.14

We can prove by contradiction. Assume that $\sqrt{2} + \sqrt{3}$ is rational, and for the theorem mentioned, we have the equivalent lemma, " $\sqrt{n}\ is\ rational, then$ $\text{n is a positive integer that is a perfect square}$".(By contrapositive)

$(\sqrt{2} + \sqrt{3})^2 = 2 + 3 + 2\sqrt{6} = 5 + \sqrt{6}$ $\text{is not a positive integer that a perfect square.}$ Therefore,this leads to a contradiction to the assumption "$\sqrt{n}\ is\ rational$", so $\sqrt{2} + \sqrt{3}\ is\ irrational.$

Q.15

We can give a constructive proof for all cases.

Let $x \ is \ a \ rational \ number, y \ is \ an \ irrational \ number$. Then $\frac{x+y}{2}$ must be between $x \ and \ y$.

Let's prove that $\frac{x+y}{2}$ is a irrational number. $x/2$ is a rational number, $y/2$ is an irrational number. Because of the theorem that "The sum of a rational number and an irrational number is an irrational number", $\frac{x+y}{2}$ is a irrational number.

Then we can get the irrational number between $x \ and \ y$ for all cases. Therefore, the proposition is **true**.

Q.16

We can find all cases that satisfy the premise.

If $a \ and \ b \ are \ both \ even$, then $a^2 + b^2 \ is \ even$;

If $a \ and \ b \ are \ both \ odd$, then $a^2 + b^2 \ is \ even$;

If one of the two integers is even, and the other is odd, then $a^2 + b^2 \ is \ odd$; only this case satisfy the premise.

Therefore, one of the two integers is even, and the other is odd, then $a + b$ is even.

Q.17

We can prove by contradiction.

Assume that there exist some roots of the equation that are neither integral nor irrational, then they must be fractions. Let t is a fraction root of the equation, t can be written as the form of $\frac{r}{s} \ \ r, s \ are \ integers, \ gcd(r, s) = 1$.

Then we have,

$$\begin{aligned} & a_0 + a_1 \cdot \frac{r}{s} + a_2 \cdot \frac{r^2}{s^2} + \cdots + a_{n-1} \cdot \frac{r^{n-1}}{s^{n-1}} + \frac{r^n}{s^n} = 0 \\ \iff & a_0 \cdot s^n + a_1 \cdot r \cdot s^{n-1} + a_2 \cdot r^2 \cdot s^{n-2} + \cdots + a_{n-1} \cdot r^{n-1} \cdot s + r^n = 0 \end{aligned}$$

We know LHS must be an integer.

If $s = 1$, then t is an integer, which satisfies the original proposition.

If $s \neq 1$,

$$a_0 \cdot s^n + a_1 \cdot r \cdot s^{n-1} + a_2 \cdot r^2 \cdot s^{n-2} + \cdots + a_{n-1} \cdot r^{n-1} \cdot s + r^n \equiv 0 \ (mod \ s)$$

then we find that, $a_0 \cdot s^n + a_1 \cdot r \cdot s^{n-1} + a_2 \cdot r^2 \cdot s^{n-2} + \cdots + a_{n-1} \cdot r^{n-1} \cdot s \equiv 0 \ (mod \ s)$.

Therefore, $r^n \equiv 0 \ (mod \ s)$, from the given **fact**, we can get $r \equiv 0 \ (mod \ s)$,which leads to a contradiction to the assumption that "$gcd(r, s) = 1$".

Thus, there exists no root of the equation that is neither integral nor irrational. The original proposition is **true**.