CS215: Discrete Math (H)
2024 Fall Semester Written Assignment # 2
Due: Oct. 28th, 2024, please submit at the beginning of class

Q.1 Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) $(A - B = A) \rightarrow (B \subseteq A)$

(b) $(A \cap B \cap C) \subseteq (A \cup B)$

(c) $\overline{(A - B)} \cap (B - A) = B$

Q.2 Prove or disprove the following.

(1) For any three sets A, B, C, $C - (A \cap B) = (C - A) \cap (C - B)$.

(2) For any two sets A, B, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$, where $\mathcal{P}(A)$ denotes the power set of the set A.

(3) For any two sets A, B, $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$, where $\mathcal{P}(A)$ denotes the power set of the set A.

(4) For a function $f : X \rightarrow Y$, $f(A \cap B) = f(A) \cap f(B)$, for any two sets $A, B \subseteq X$.

Q.3 The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

(a) Determine whether the symmetric difference is associative; that is, if A, B and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

(b) Suppose that A, B and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

Q.4 For each set defined below, determine whether the set is *countable* or *uncountable*. Explain your answers. Recall that $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers.

(a) The set of all subsets of students in CS201

(b) $\{(a, b) | a,\ b \in \mathbb{N}\}$

(c) $\{(a, b) | a \in \mathbb{N},\ b \in \mathbb{R}\}$

Q.5 Give an example of two uncountable sets A and B such that the difference $A - B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q.6 Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Q.7 For each set A, the *identity function* $1_A : A \to A$ is defined by $1_A(x) = x$ for all x in A. Let $f : A \to B$ and $g : B \to A$ be the functions such that $g \circ f = 1_A$. Show that f is one-to-one and g is onto.

Q.8 Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the *composition* function.

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one? Explain your answer.

(c) If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

(d) If $f \circ g$ is onto, must f be onto? Explain your answer.

(e) If $f \circ g$ is onto, must g be onto? Explain your answer.

Q.9 Derive the formula for $\sum_{k=1}^{n} k^2$.

Q.10 Derive the formula for $\sum_{k=1}^{n} k^3$.

Q.11 Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when m is a positive integer.

Q.12 Show that if A, B, C and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

Q.13 Show that if A and B are sets with the same cardinality, then $|A| \leq |B|$ and $|B| \leq |A|$.

Q.14 Suppose that A is a countable set. Show that the set B is also countable if there is an onto function from A to B.

Q.15 Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m, n) = (m + n - 2)(m + n - 1)/2 + m$ is one-to-one and onto.

Q.16 By the Schröder-Bernstein theorem, prove that $(0, 1)$ and $[0, 1]$ have the same cardinality.

Q.17 Suppose that $f(x), g(x)$ and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$.

Q.18 If $f_1(x)$ and $f_1(x)$ are functions from the set of positive integers to the set of positive real numbers and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, is $(f_1 - f_2)(x)$ also $\Theta(g(x))$? Either prove that it is or give a counter example.

Q.19 Show that if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$, and a_n are real numbers and $a_n \neq 0$, then $f(x)$ is $\Theta(x^n)$.

Q.20 Prove that for any $a > 1$, $\Theta(\log_a n) = \Theta(\log_2 n)$.

Q.21 The conventional algorithm for evaluating a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$ at $x = c$ can be expressed in pseudocode by where the final value

Algorithm 1 polynomial ($c, a_0, a_1, \ldots, a_n$: real numbers)

$power := 1$
$y := a_0$
for $i := 1$ to n **do**
  $power := power * c$
  $y := y + a_i * power$
end for
return y $\{y = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0\}$

of y is the value of the polynomial at $x = c$. Exactly how many multiplications and additions are used to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable).

Q.22 There is a more efficient algorithm (in terms of the number of multiplications and additions used) for evaluating polynomials than the conventional algorithm described in the previous exercise. It is called **Horner's method**. This pseudocode shows how to use this method to find the value of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ at $x = c$.

Algorithm 2 Horner ($c, a_0, a_1, \ldots, a_n$: real numbers)

$y := a_n$
for $i := 1$ to n **do**
  $y := y * c + a_{n-i}$
end for
return y $\{y = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0\}$

Exactly how many multiplications and additions are used by this algorithm to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable.)

Q.23

(1) Show that $(\log n)^{\log\log n} = O(\log(n^n))$, where the base of the logarithm is 2.

(2) Order the following function by asymptotic growth rates. That is, list them as $f_1(n), f_2(n), \ldots, f_9(n)$, such that $f_i(n) = O(f_{i+1}(n))$ for all i. You don't have to explain your answer.

6 9 7 3 1 4 5 8 2

n^2, $\log n$, $n \log n$, $n^{\log n}$, $(\log n)^n$, $(\log n)^{\log n}$, $(\log\log n)^{\log n}$, $(\log n)^{\log\log n}$, $3^{n/2}$.

Q.24 Aliens from another world come to the Earth and tell us that the *3SAT* problem is *solvable* in $O(n^8)$ time. Which of the following statements follow as a consequence?
A. All NP-Complete problems are solvable in polynomial time.
B. All NP-Complete problems are solvable in $O(n^8)$ time.
C. All problems in NP, even those that are not NP-Complete, are solvable in polynomial time.

D. No NP-Complete problem can be solved *faster* than in $O(n^8)$ in the worst case.
E. P = NP.