Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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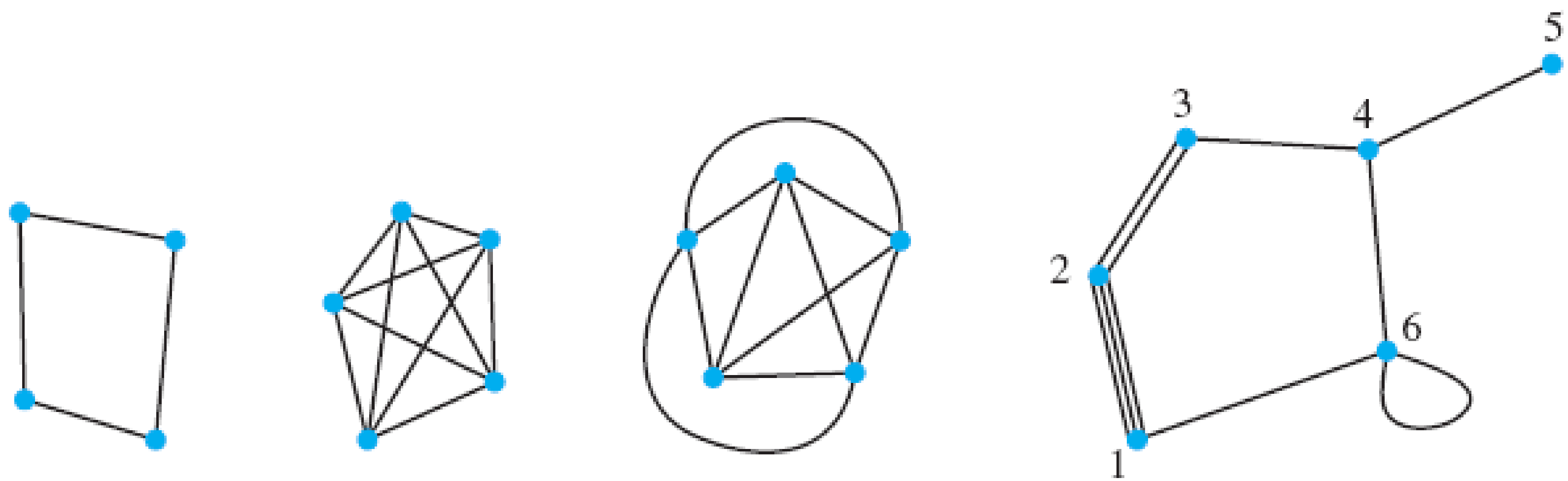
Definition of a Graph

- **Definition**. A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to be *incident to* (or *connect* its endpoints.



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Complete Graphs

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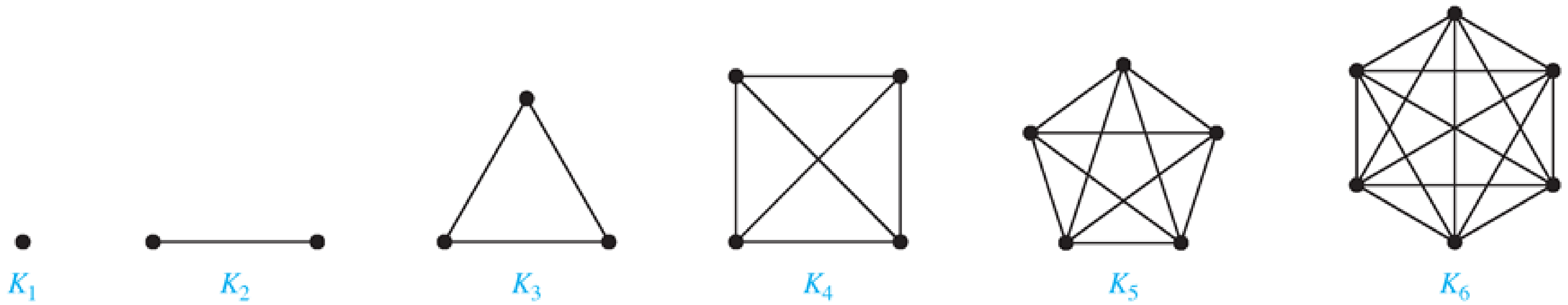

Cycles

- A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

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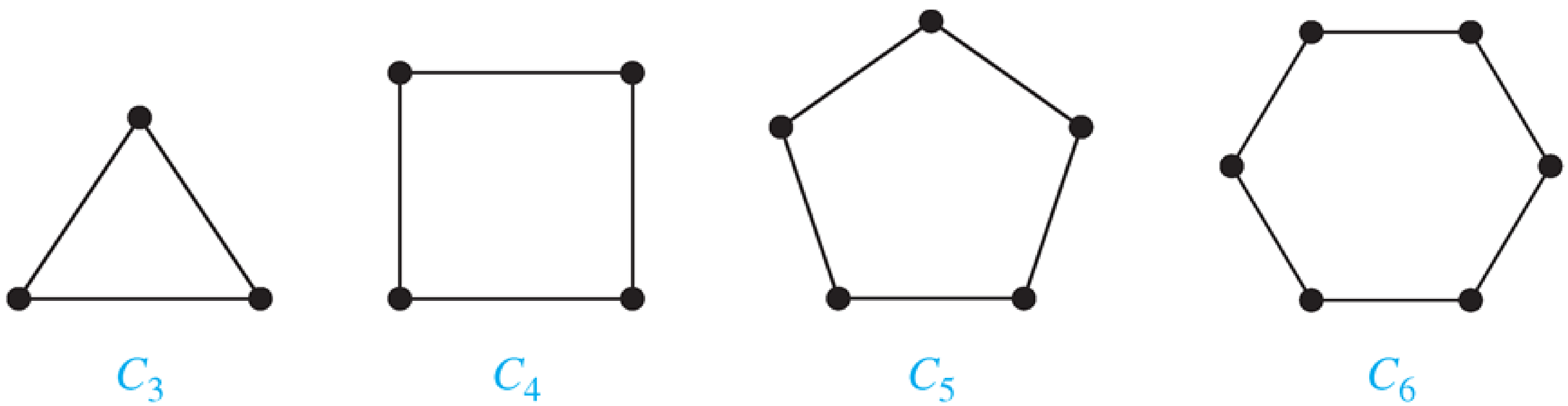

Wheels

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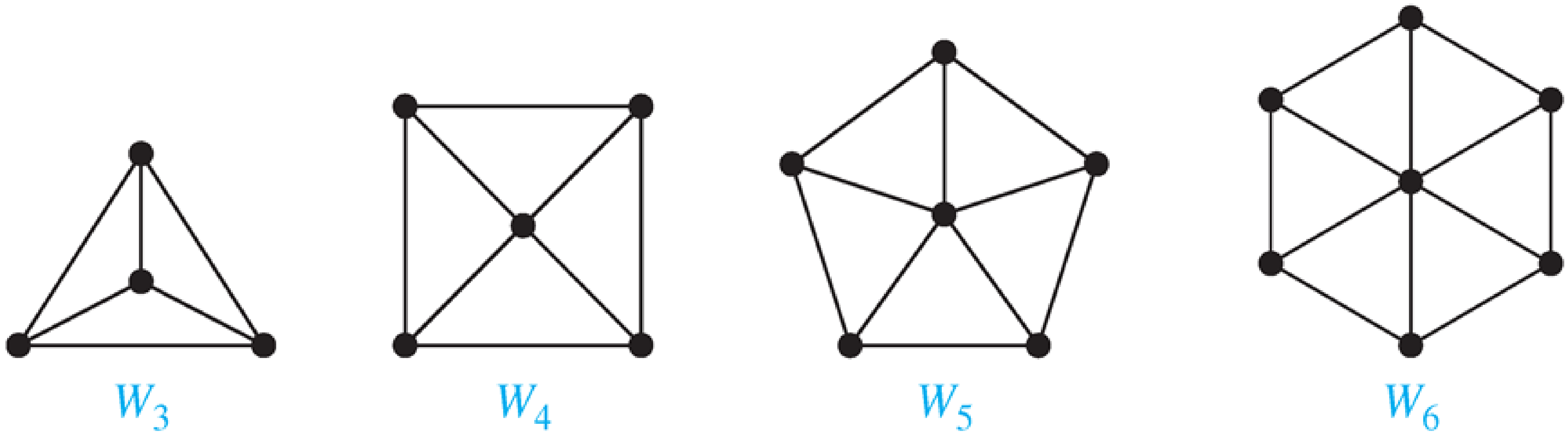

N-dimensional Hypercube

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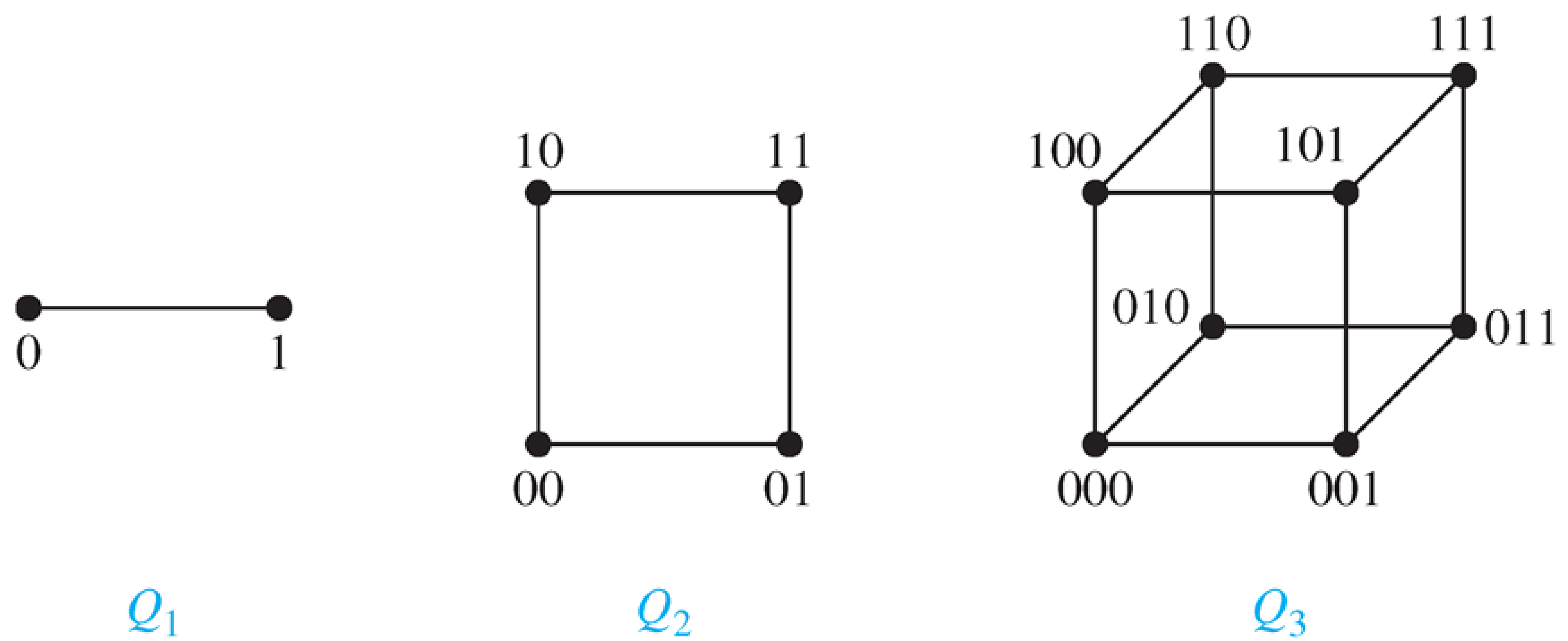

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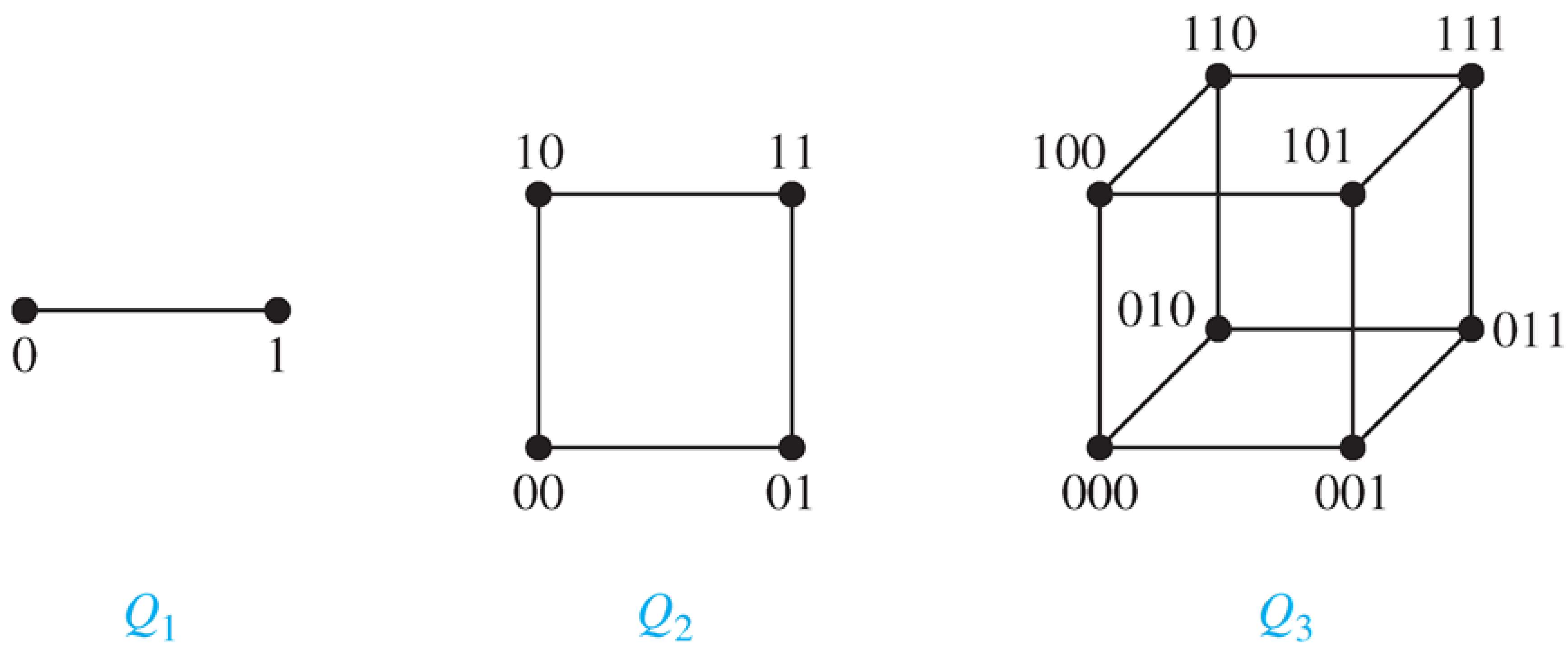

How many vertices? How many edges?

Bipartite Graphs

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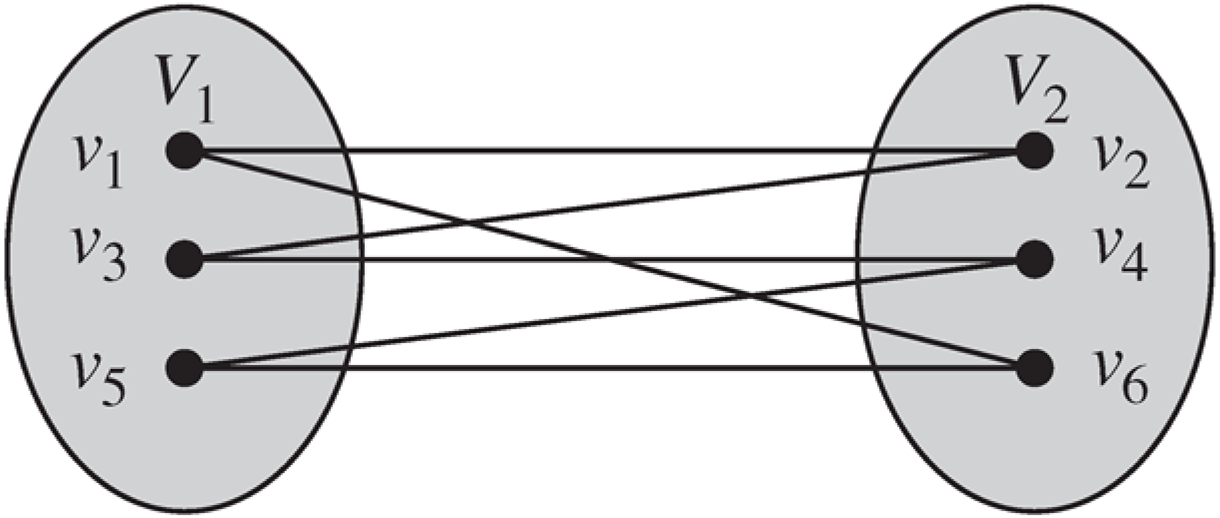

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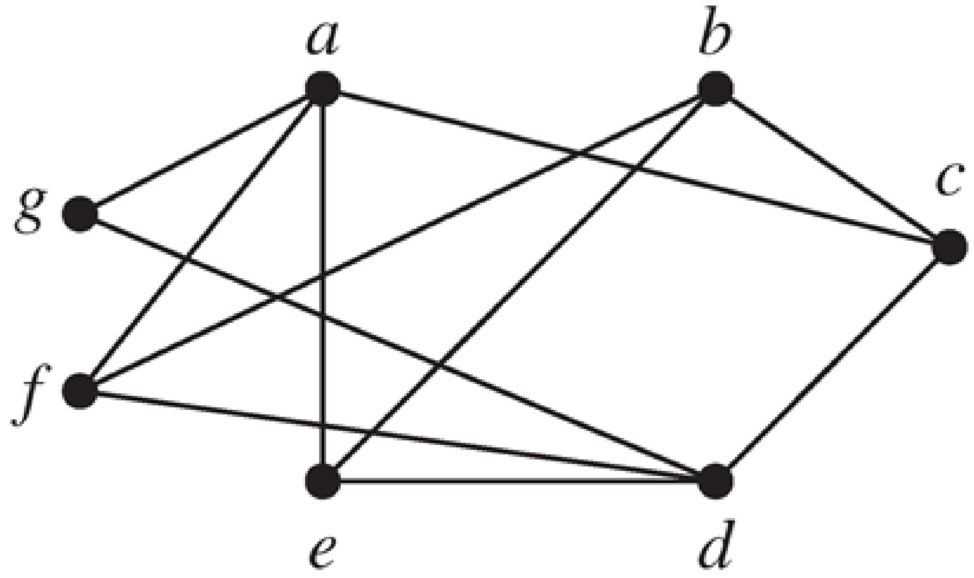

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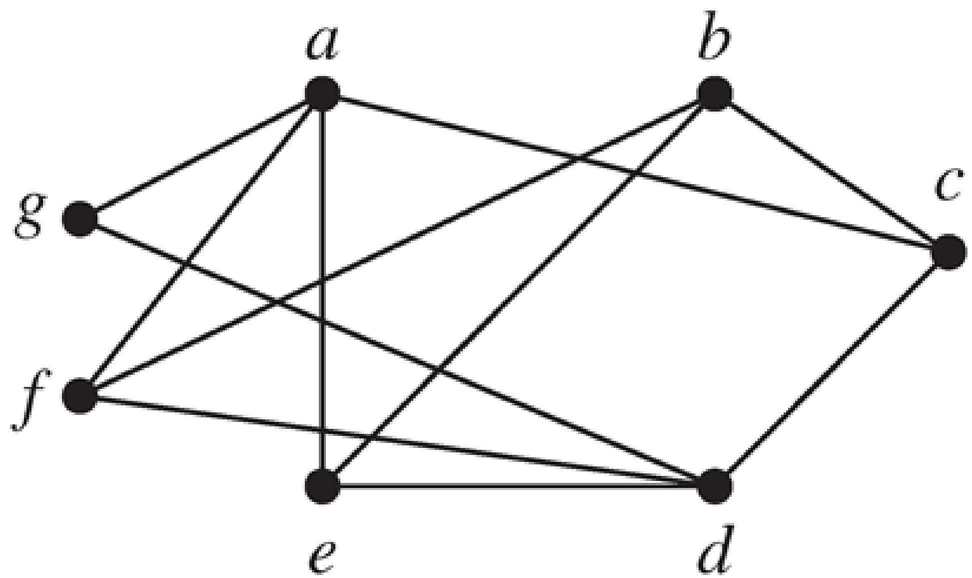
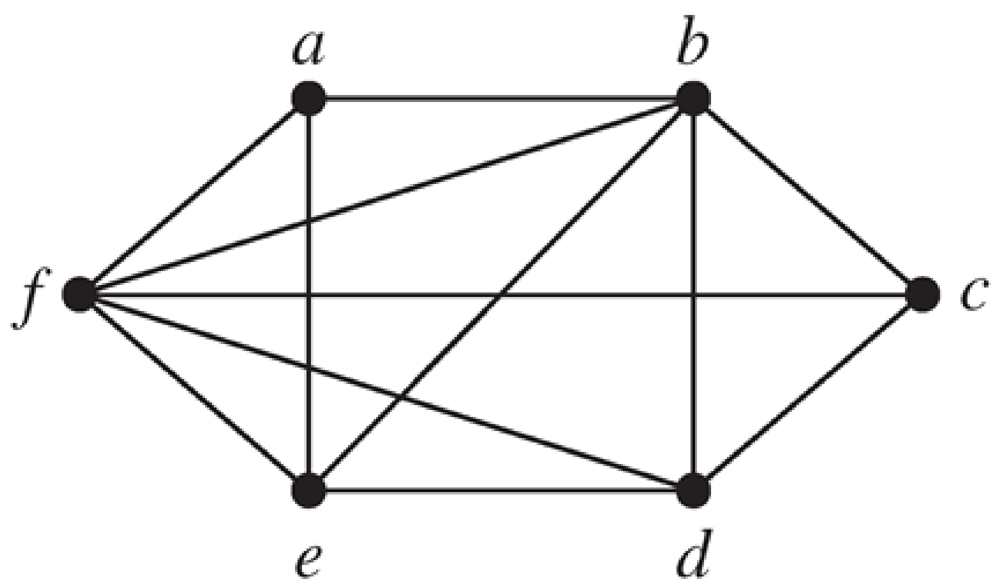
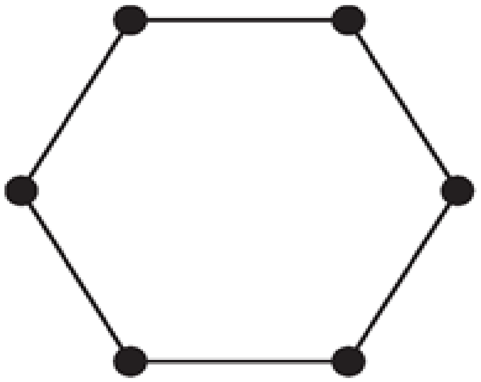




Bipartite Graphs

- **Example** Show that C_6 is bipartite.



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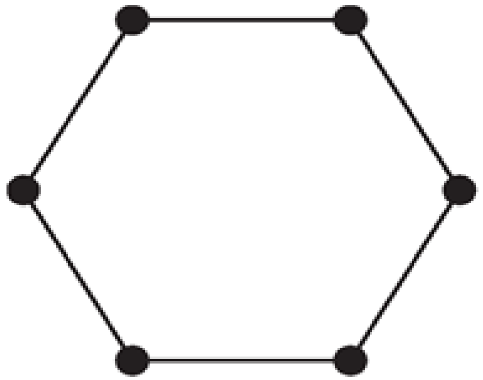

C_6

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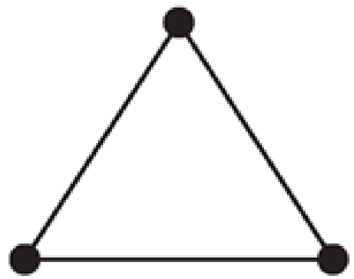

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Complete Bipartite Graphs

- **Definition** A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

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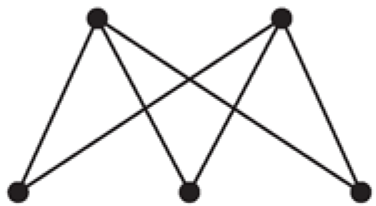

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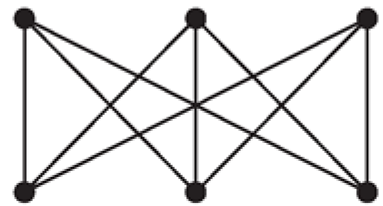

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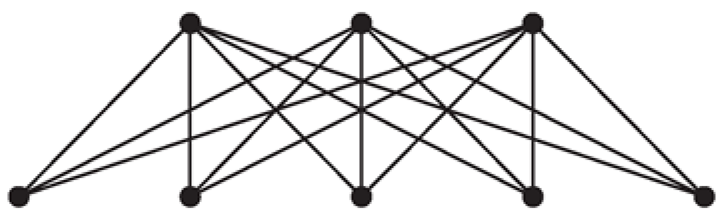
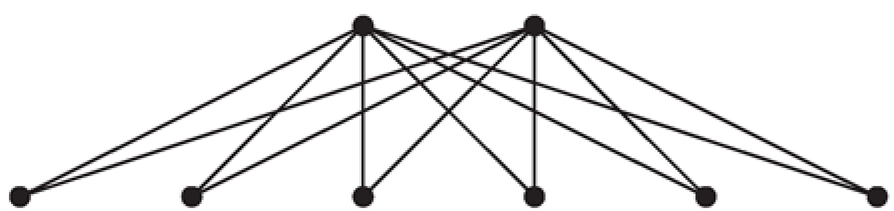


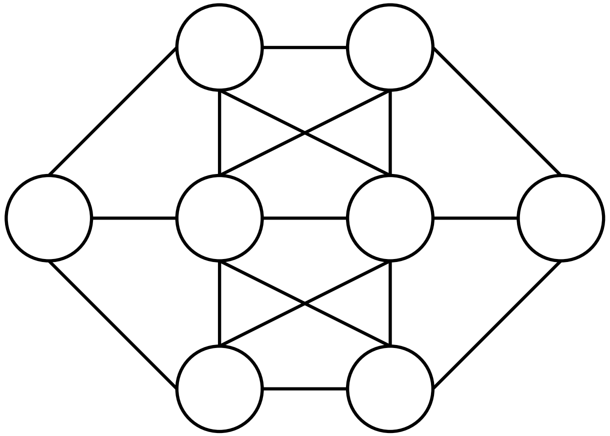
$K_{3,5}$



$K_{2,6}$

Puzzles using Graphs

- **The eight-circles problem** Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure, in such a way that no letter is adjacent to a letter that is next to it in the alphabet.



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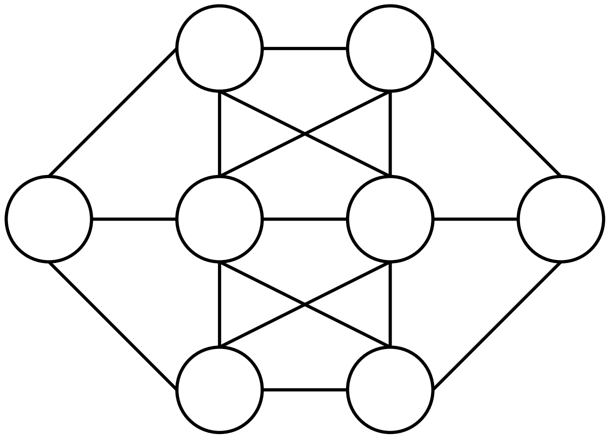

- **Six people at a party** Show that, in any gathering of six people, there are either three people who all know each other, or three people none of which knows either of the other two.

Bipartite Graphs and Matchings

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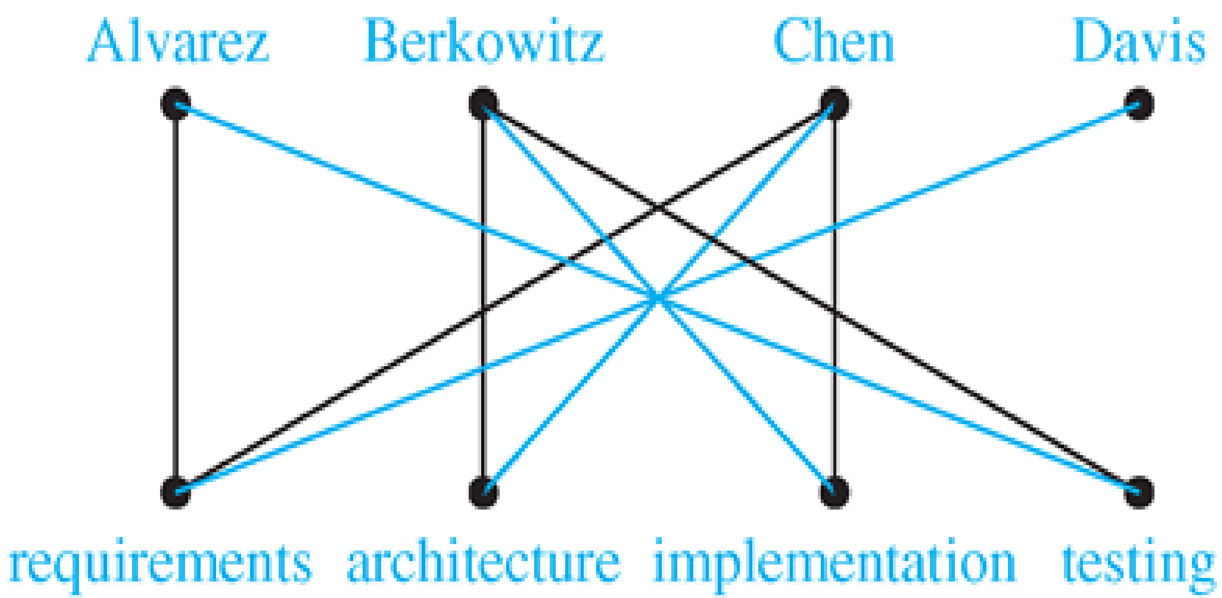

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Case (i): For all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2

Case (ii): For some integer j with $1 \leq j \leq k$, there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2

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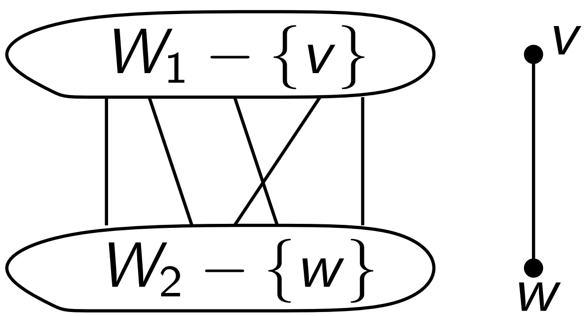

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If not, there is a subset B of t vertices with $1 \leq t \leq k + 1 - j$ s.t. $|N(B)| < t$. 这个是因为前面inductive

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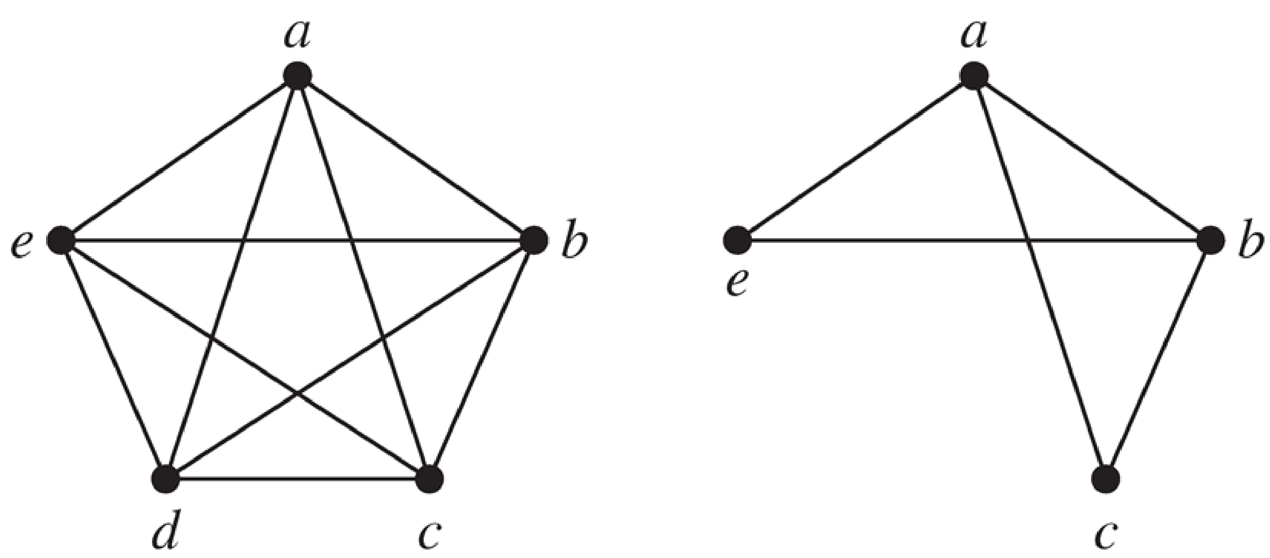

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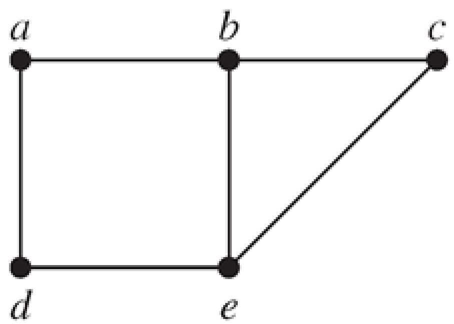
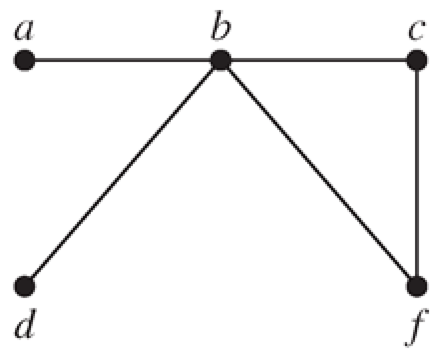


G_1



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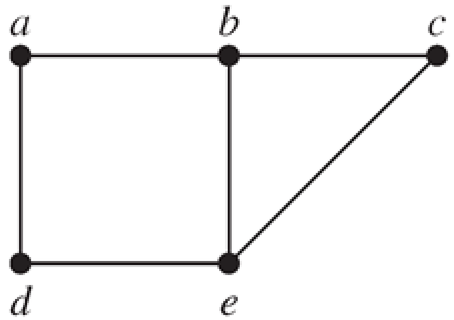

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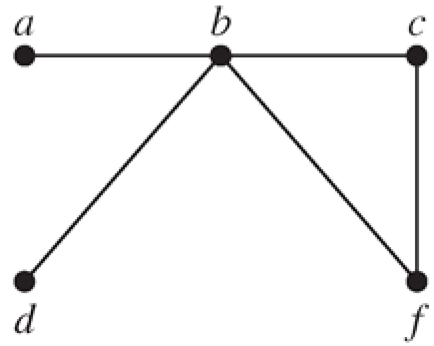
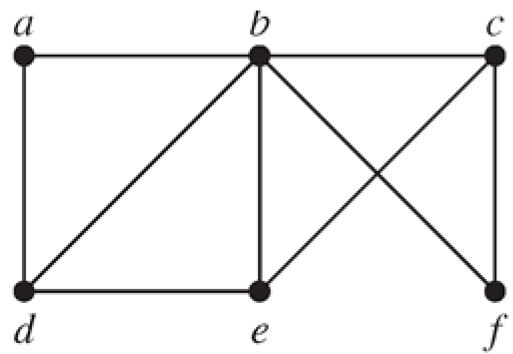


G_2



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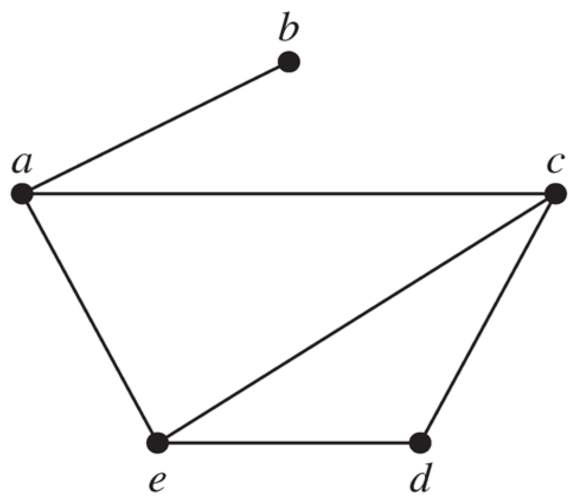

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  Definition An *adjacency list* can be used to represent a graph with no multiple edges by specifying the vertices that are adjacent to each vertex of the graph.

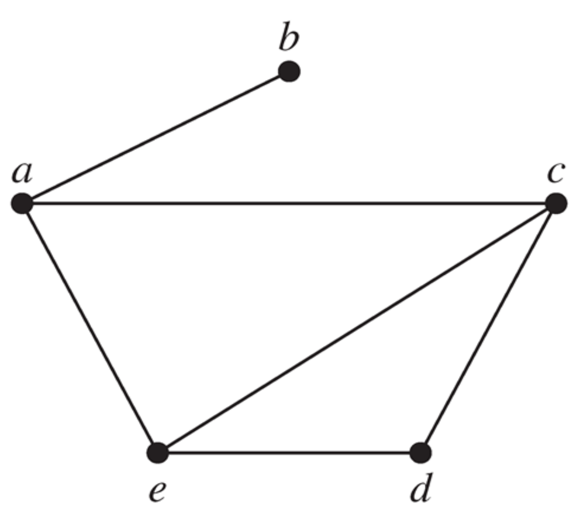

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

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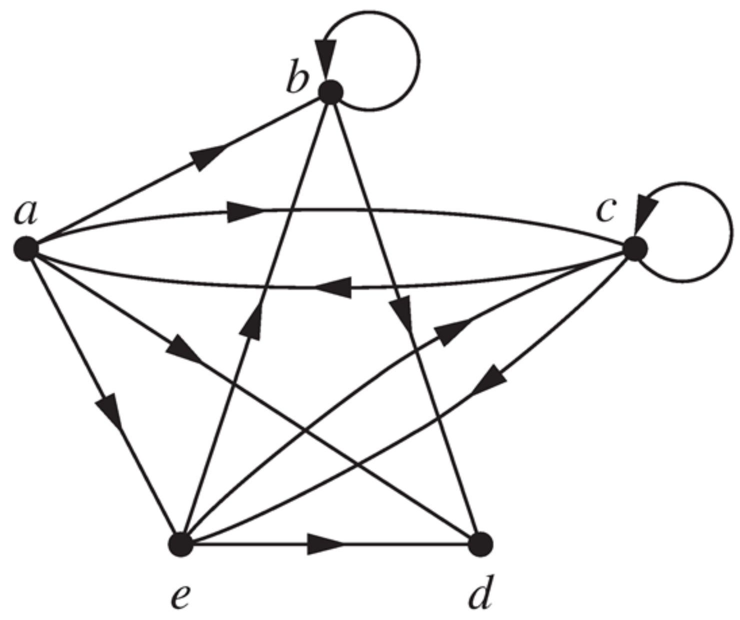

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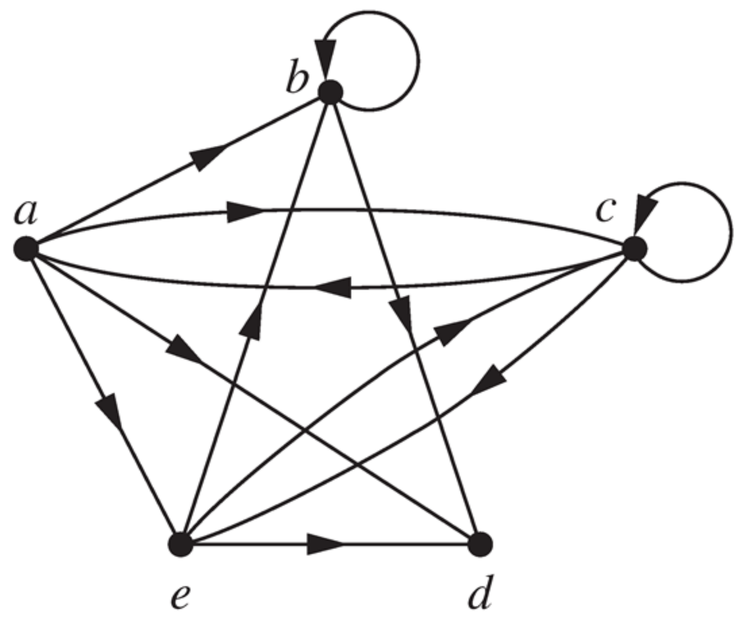

TABLE 2 **An Adjacency List for a Directed Graph.**

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrices

- **Definition** Suppose that $G = (V, E)$ is a simple graph with $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $\mathbf{A}_G$ of G, is the $n \times n$ zero-one matrix with 1 as its (i, j)-th entry when v_i and v_j are adjacent, and 0 as its (i, j)-th entry when they are not adjacent.

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$$\mathbf{A}_G = [a_{ij}]_{n\times n}, \text{ where}$$

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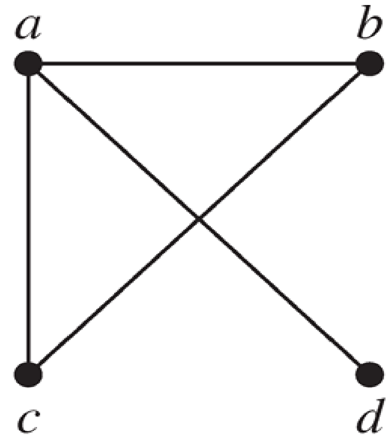

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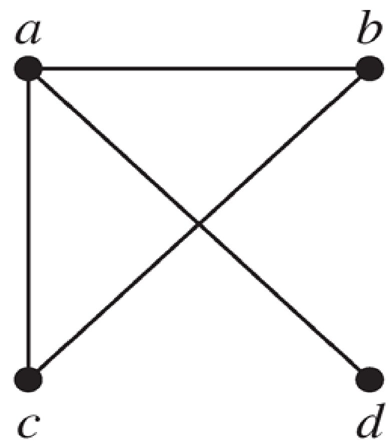

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrices

- Adjacency matrices can also be used to represent graphs with loops and multiple edges. The matrix is no longer a zero-one matrix.

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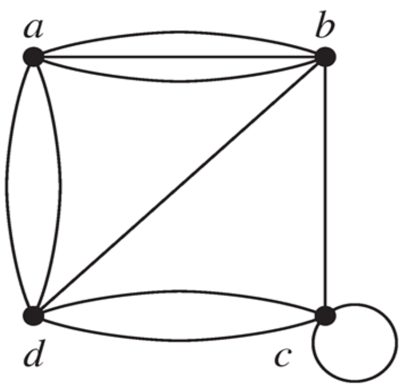

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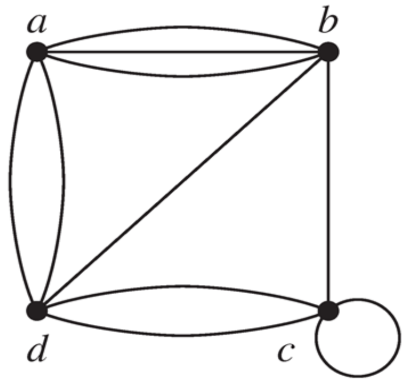

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

- **Definition** Let $G = (V, E)$ be an undirected graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_m$. The *incidence matrix* with respect to the ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]_{n \times m}$, where

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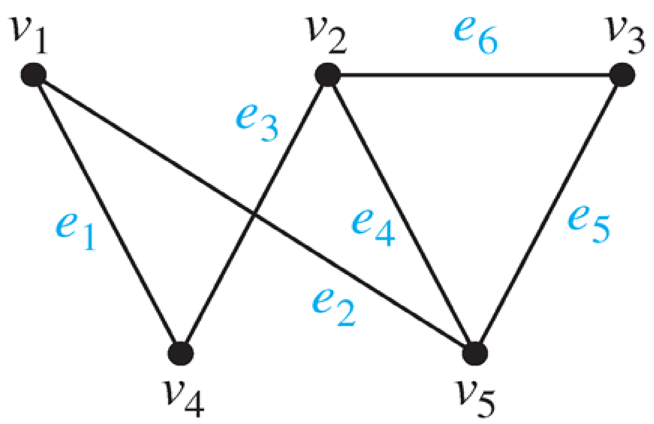

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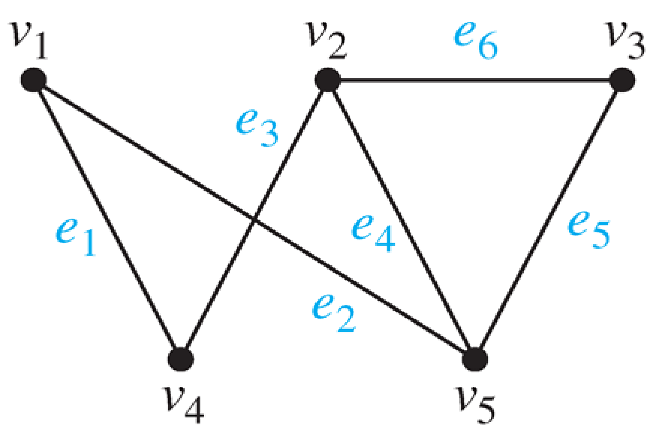

刻画边和顶点的连接关系

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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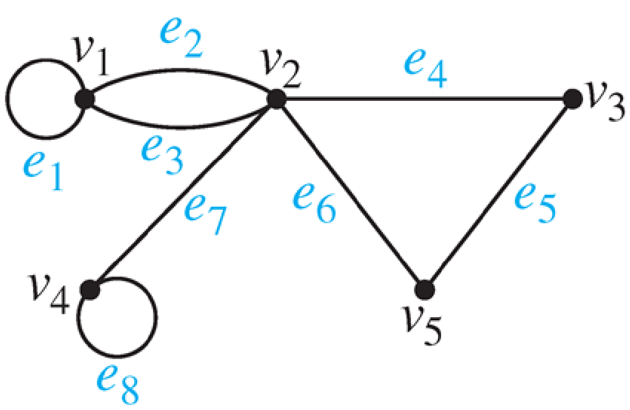

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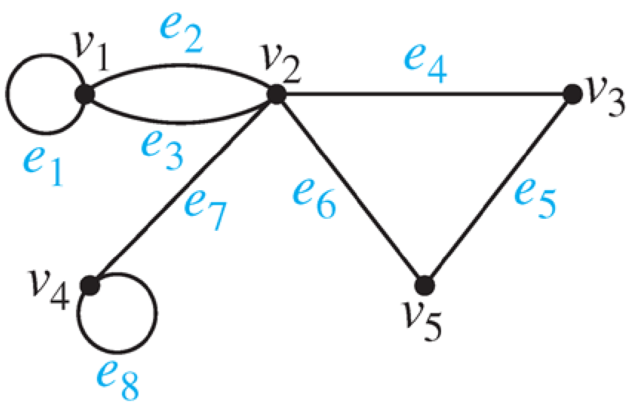

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Isomorphism of Graphs

- **Definition** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is called an *isomorphism*.

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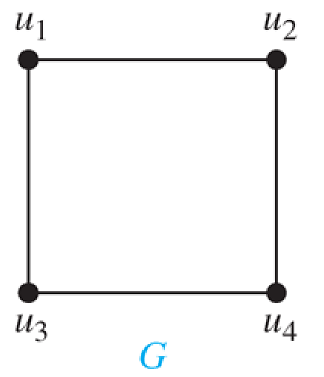

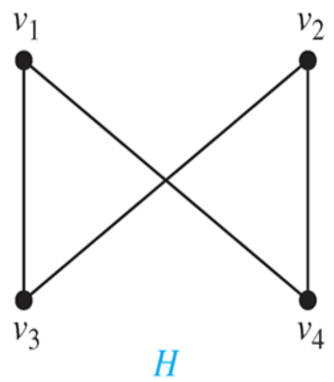

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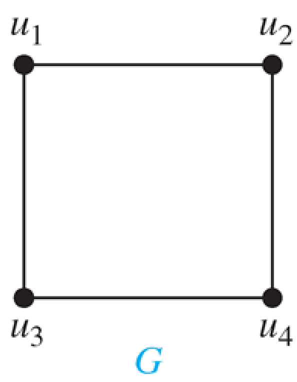

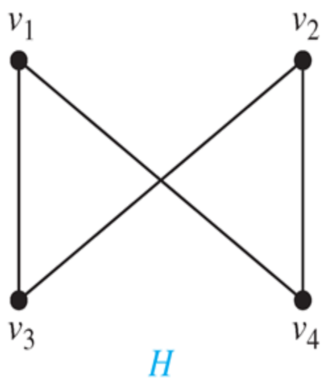

Are the two graphs isomorphic?

Define a one-to-one correspondence:
$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$

同构关系

Isomorphism of Graphs

- It is usually difficult to determine whether two simple graphs are isomorphic using brute force since there are $n!$ possible one-to-one correspondences.

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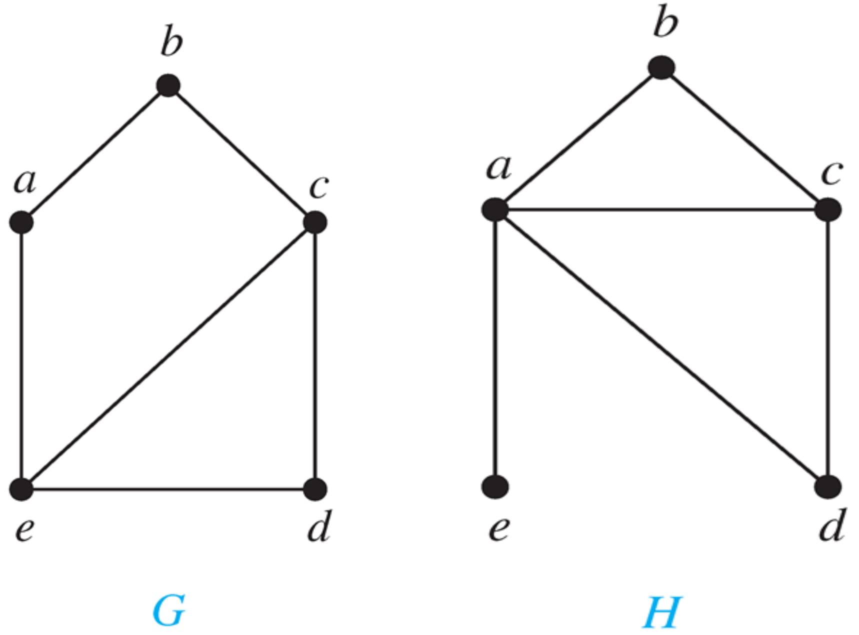
- Useful graph invariants include the number of vertices, number of edges, degree sequence, etc.

Isomorphism of Graphs

- **Example** Determine whether these two graphs are isomorphic.



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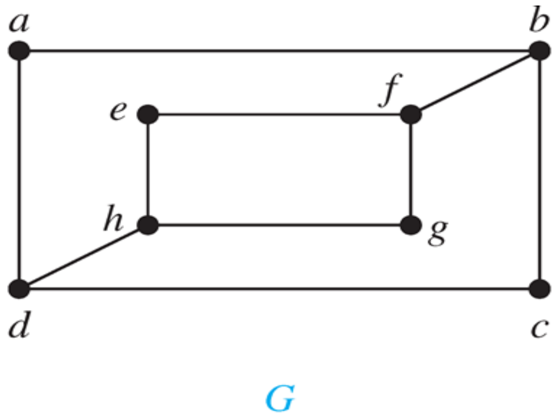

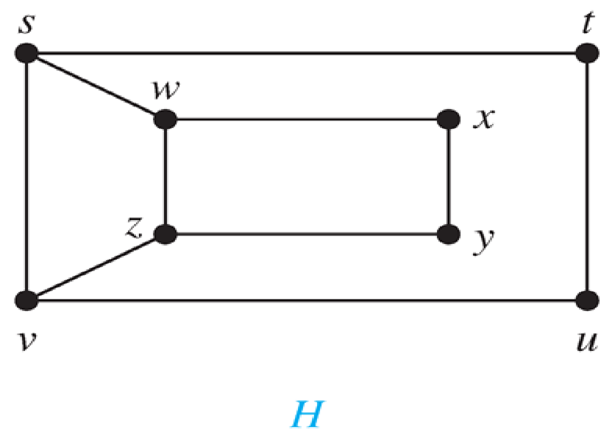

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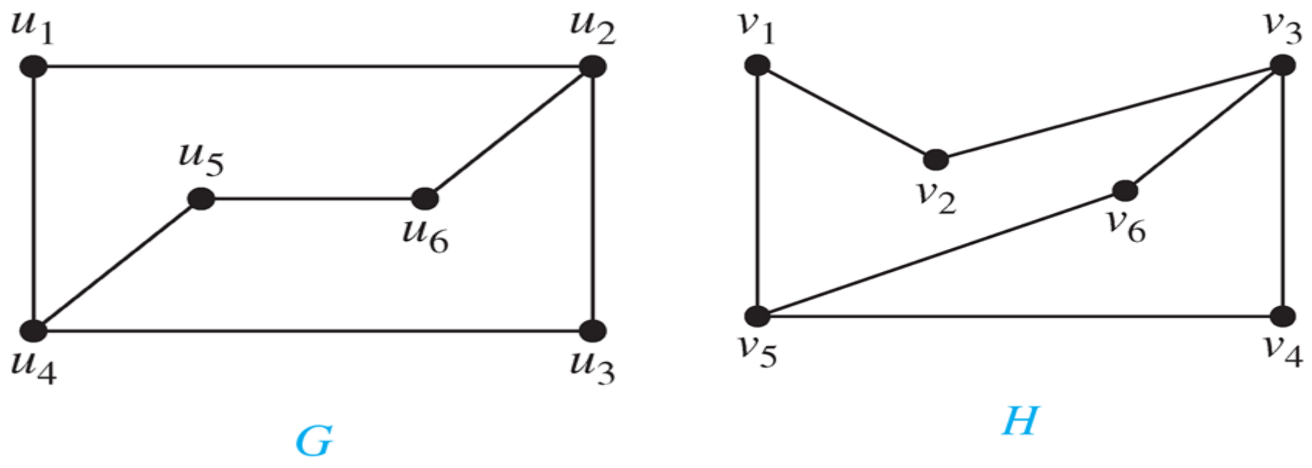

Path

- **Definition** Let n be a nonnegative integer and G an undirected graph. A *path of length n* from u to v in G is a sequence of n edges $e_1, e_2, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has the endpoints x_{i-1} and x_i for $i = 1, \ldots, n$. The path is a *circuit* if it begins and ends at the same vertex, i.e., if $u = v$ and has length greater than zero. A path or circuit is *simple* if it does not contain *repeating vertices*.

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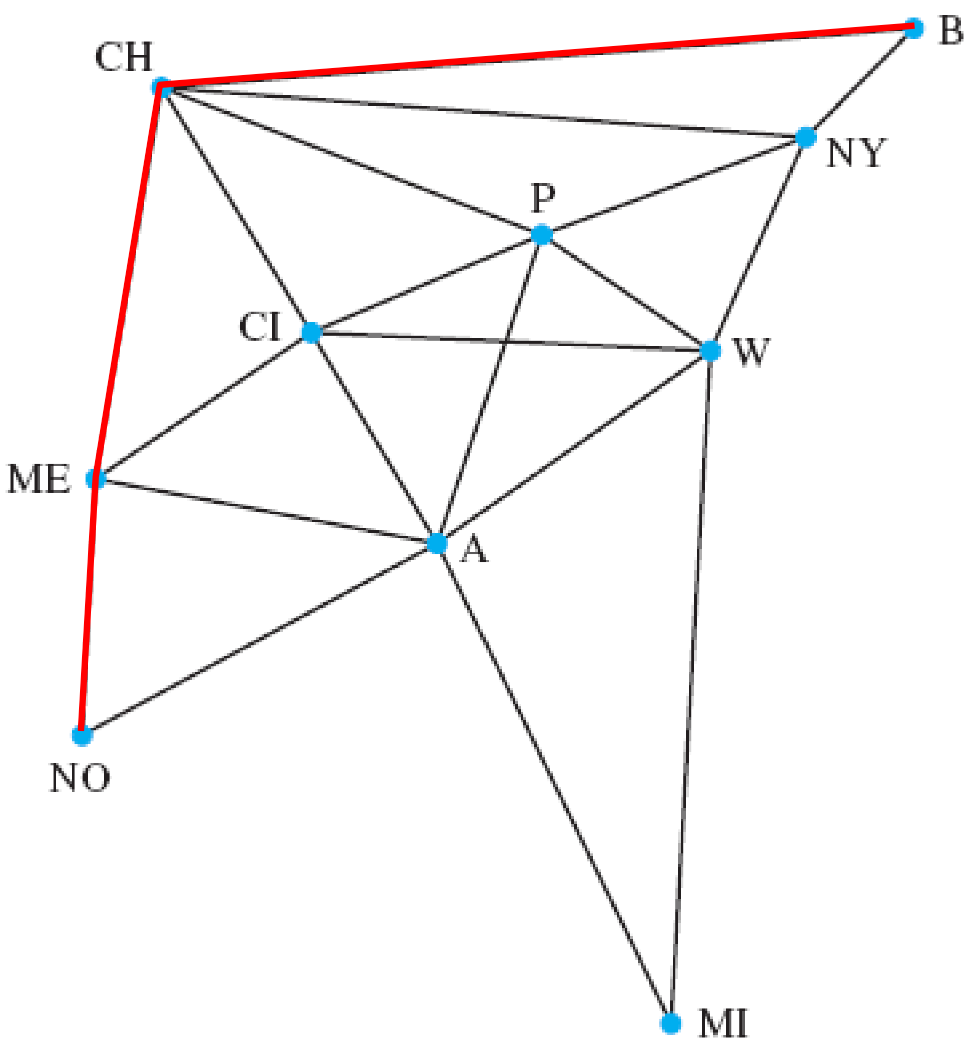
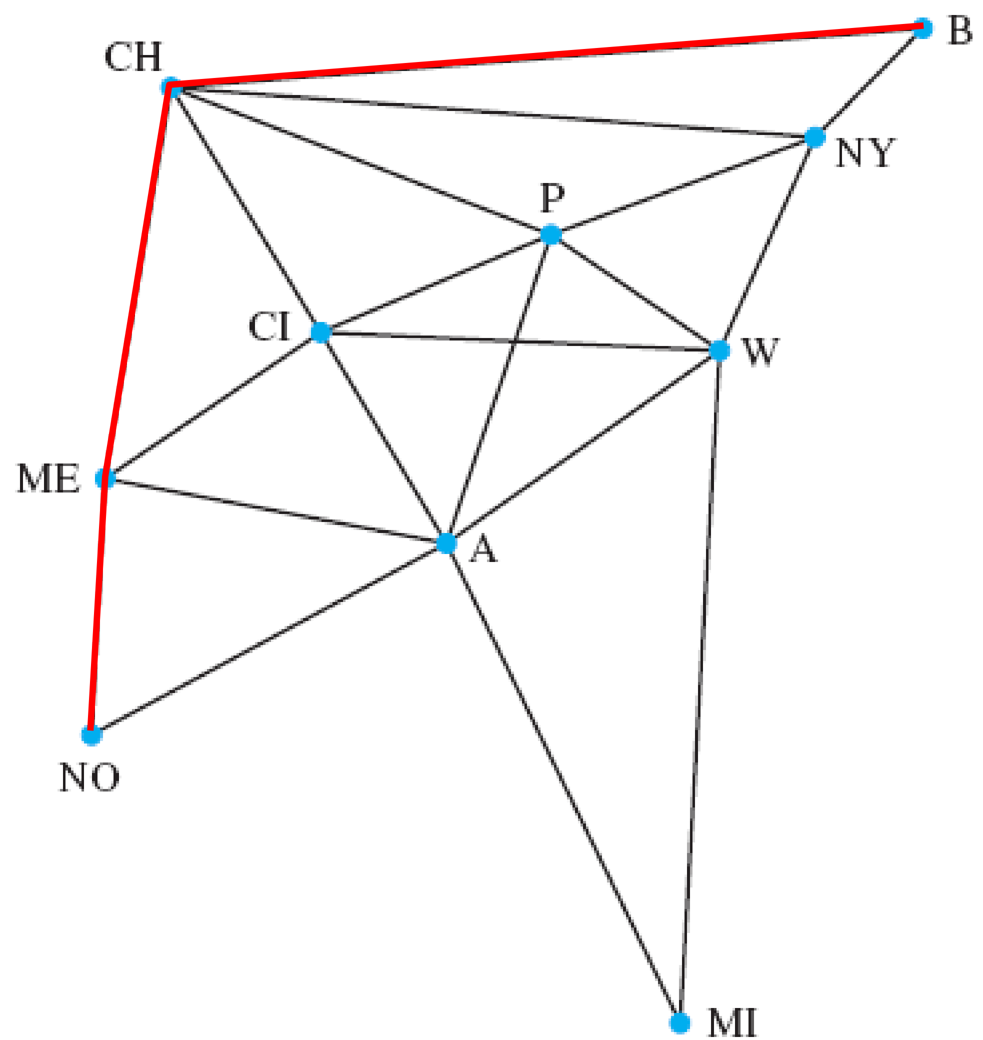
Length of a path = # of edges on path

Path



Path

Path from Boston to New Orleans is B, CH, ME, NO



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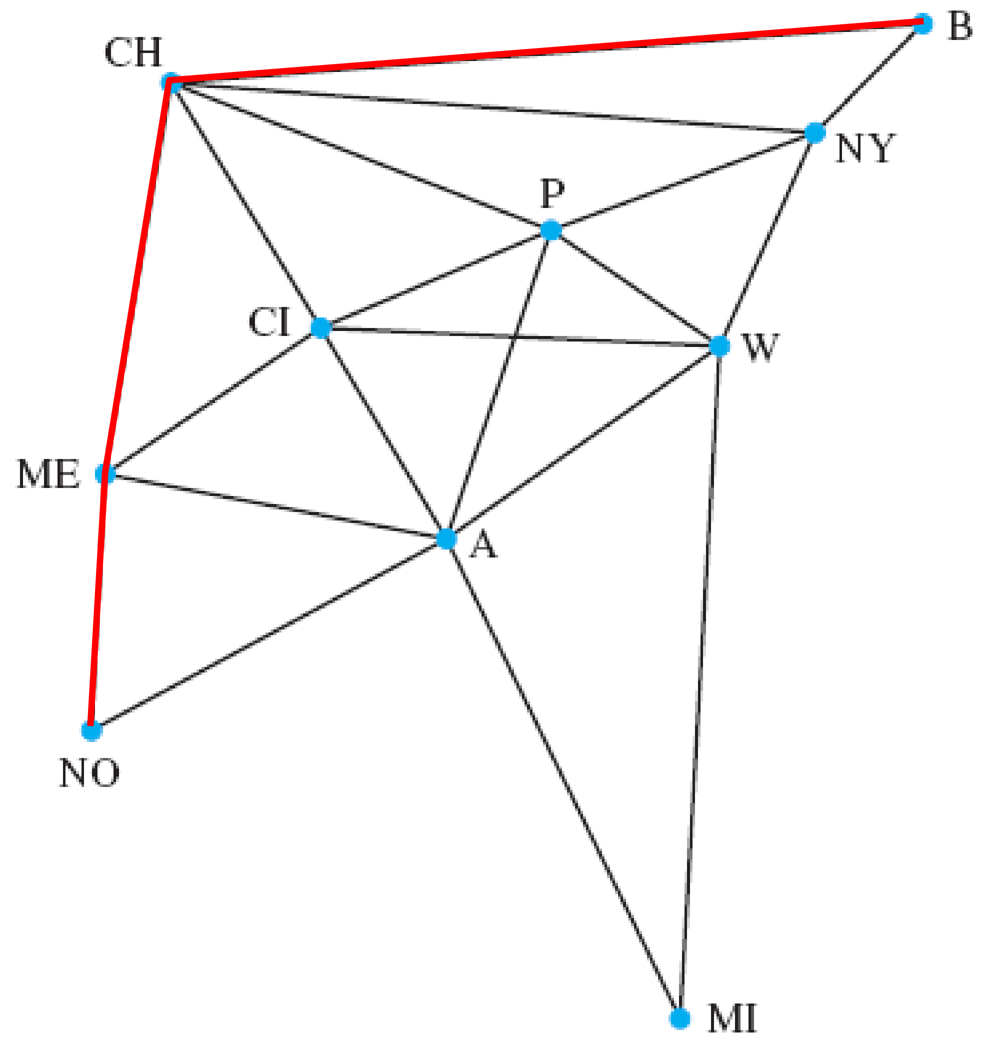
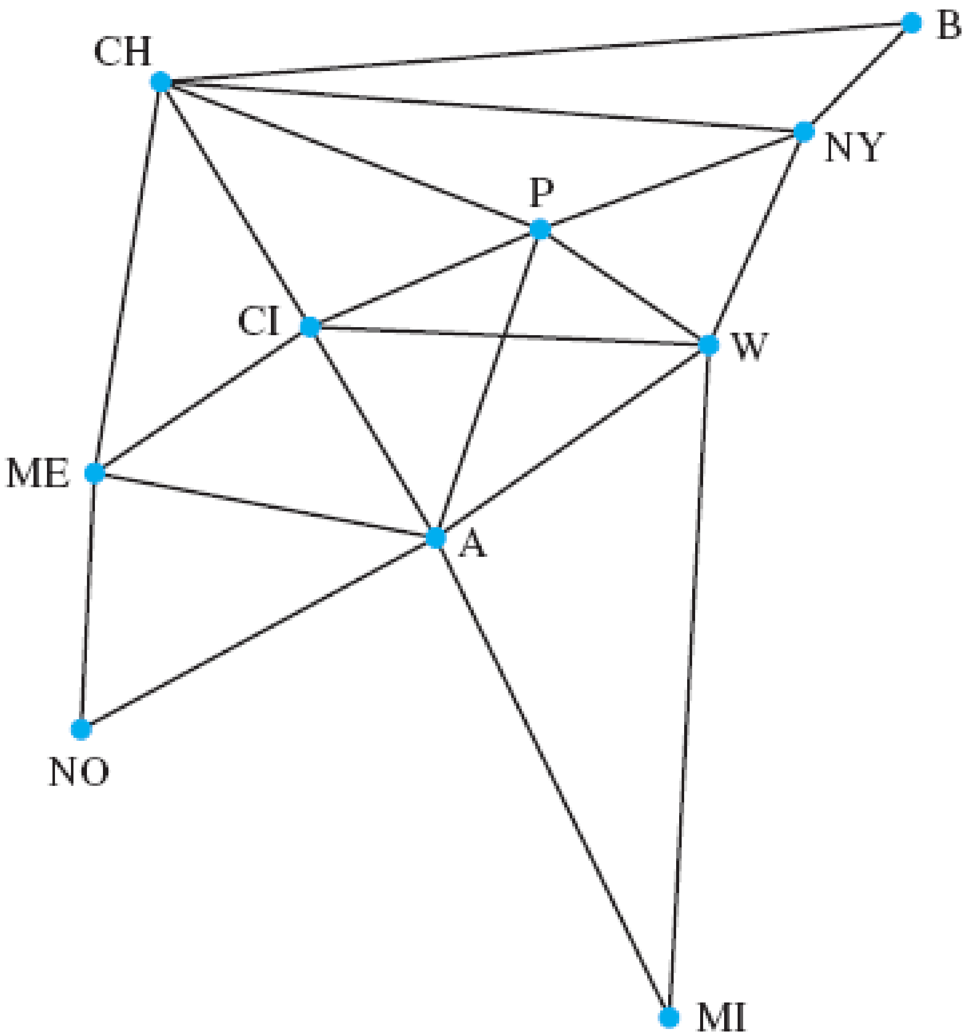
This path has length 3.



Connectivity



Company decides to lease only minimum number of communication lines it needs to be able to send a message from any city to any other city by using any number of intermediate cities.

What is the minimum number of lines it needs to lease?

Connectivity

- Choosing 10 edges?

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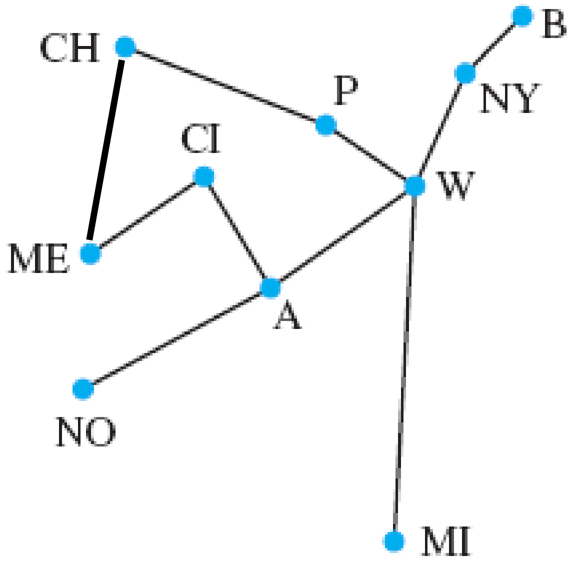

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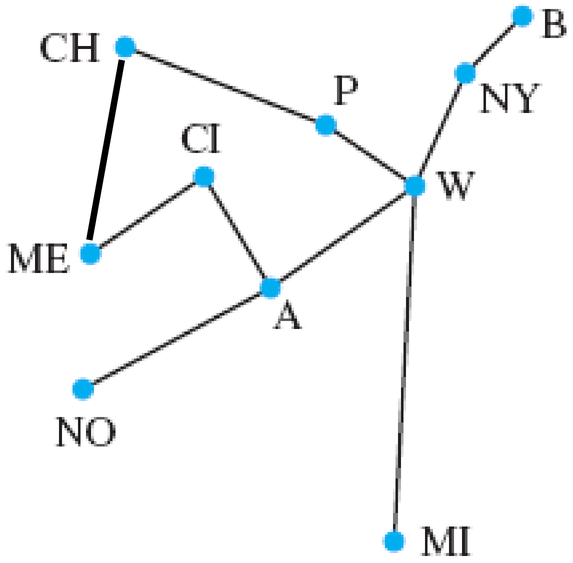

Too many.

Could throw away edge CI, A, and still have a solution.

Connectivity

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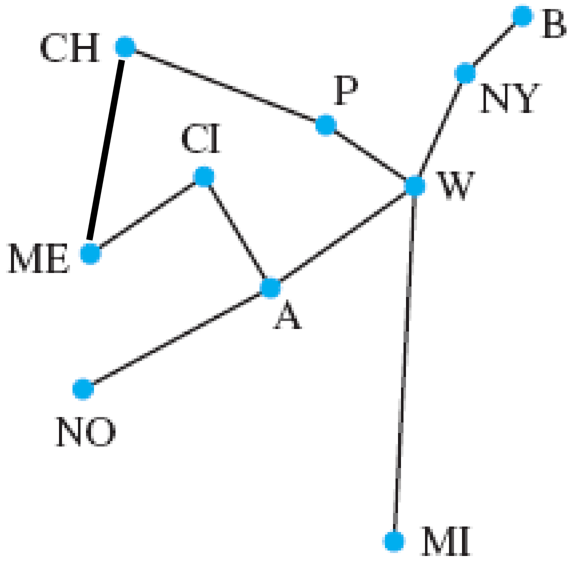

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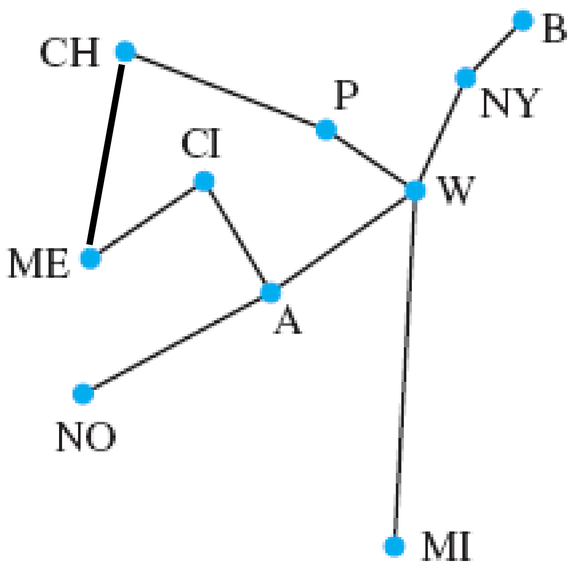

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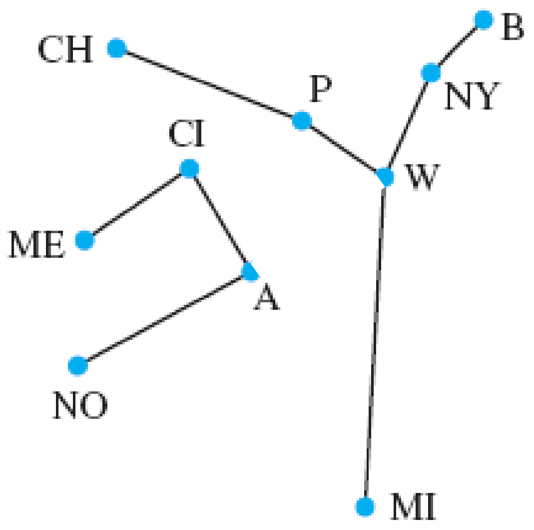

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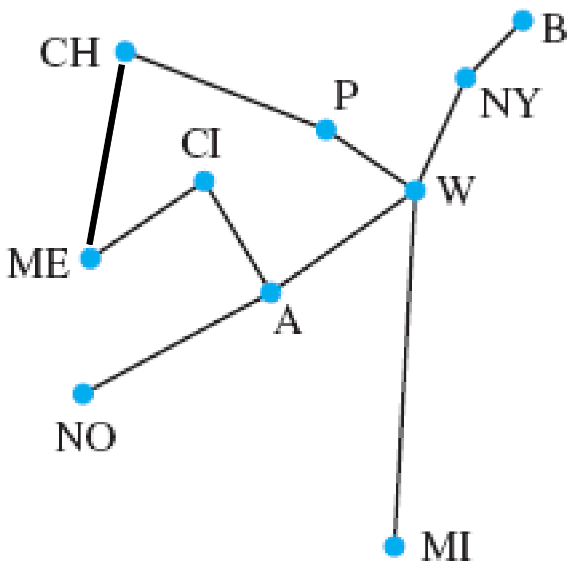

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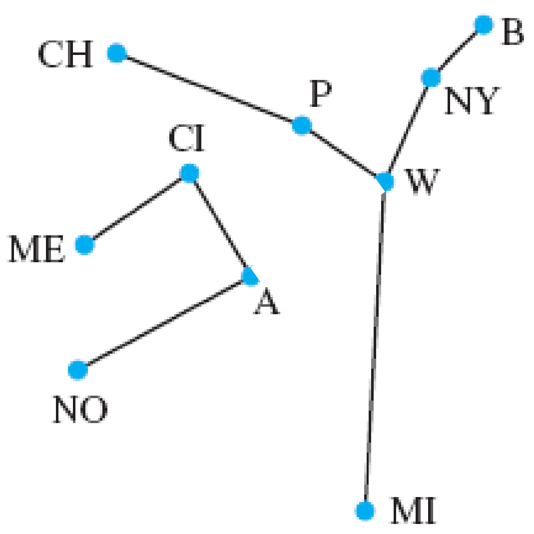

Not enough.

There is no path from, e.g., NO to B.

Connectivity

- Choosing 9 edges:

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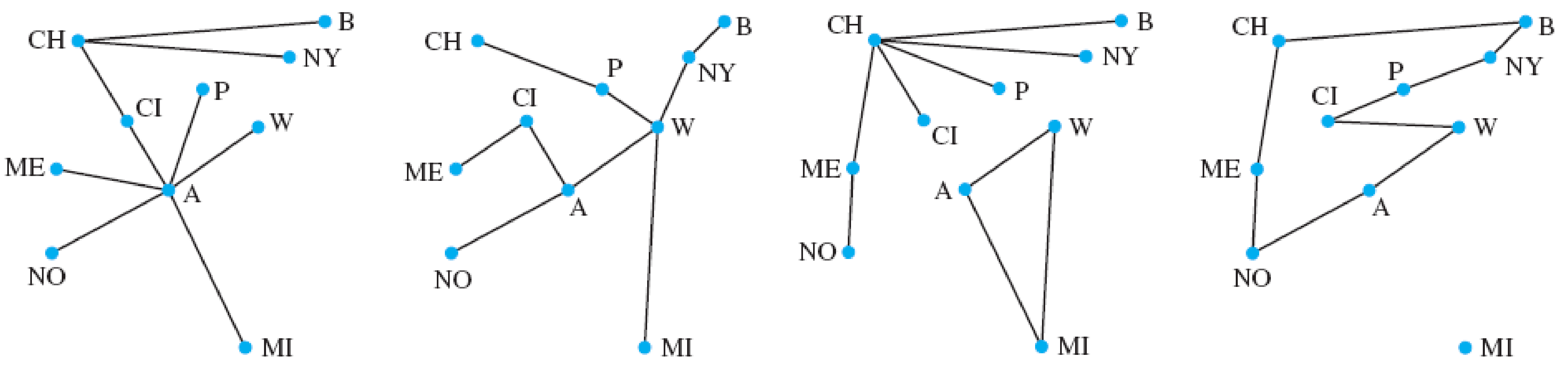

a b c d

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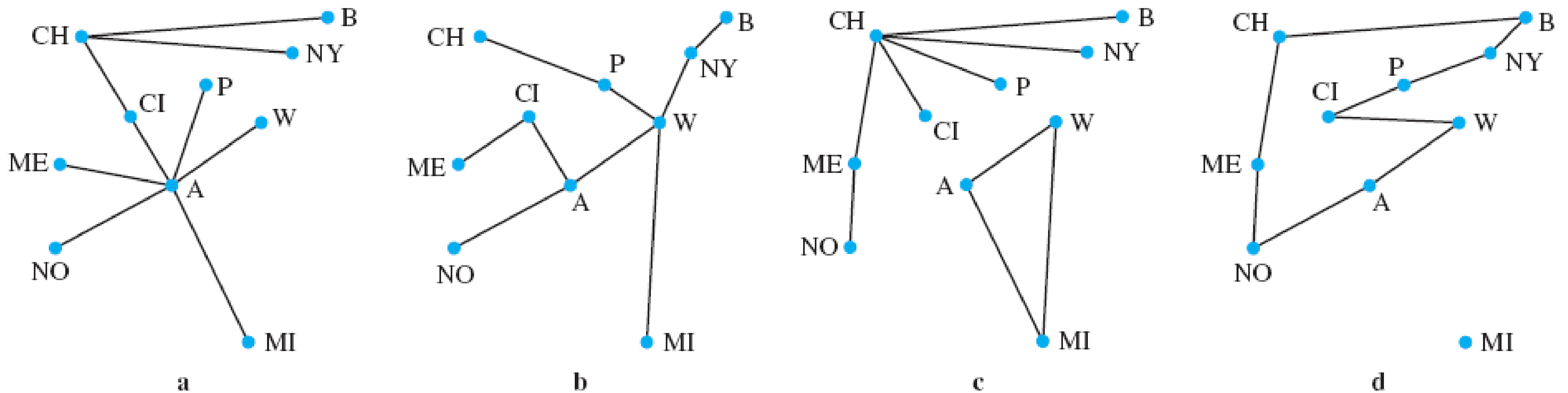

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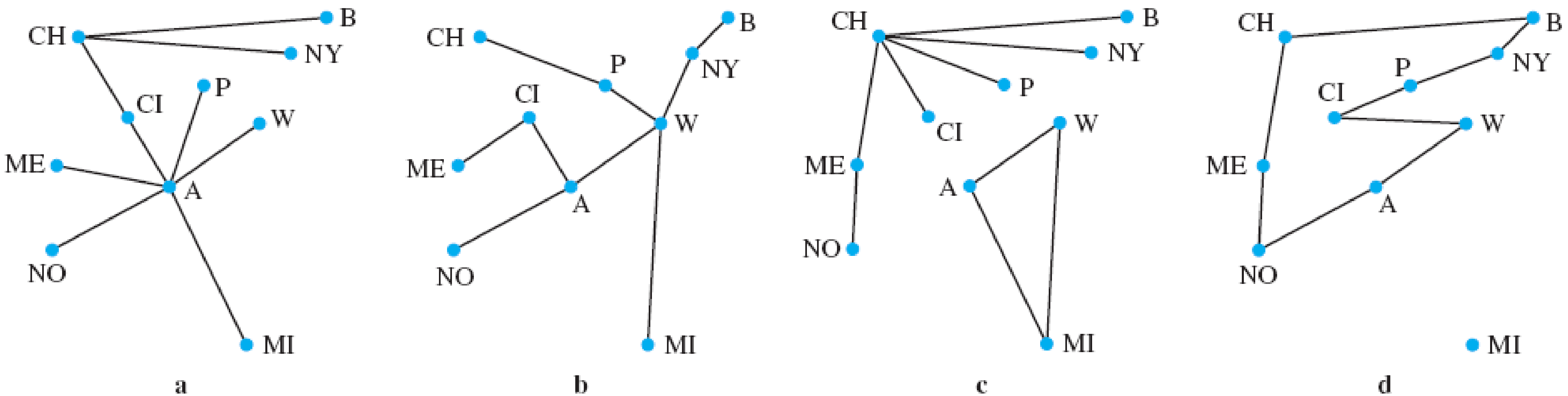

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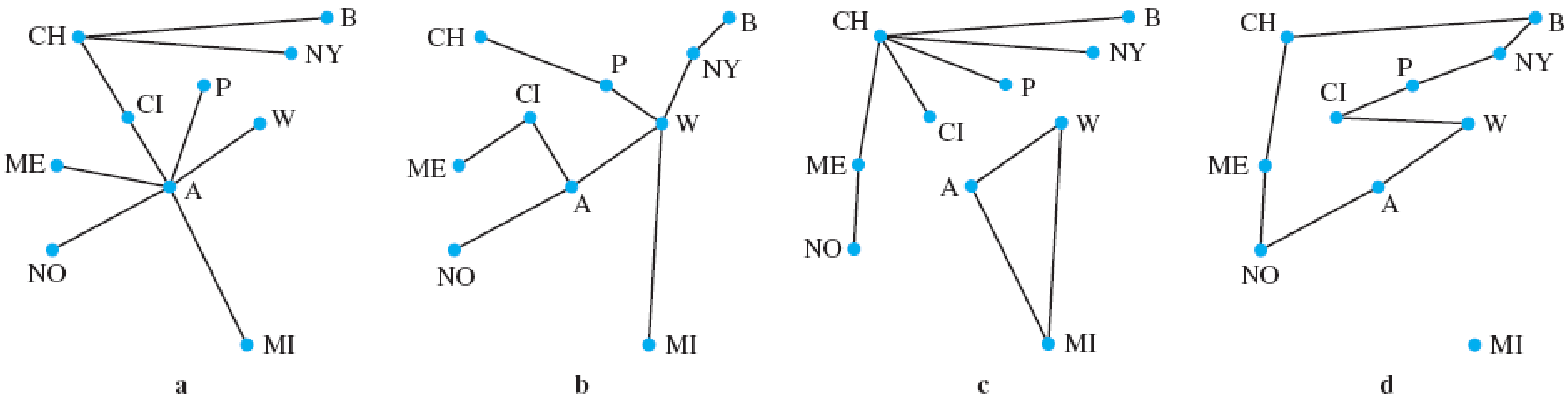

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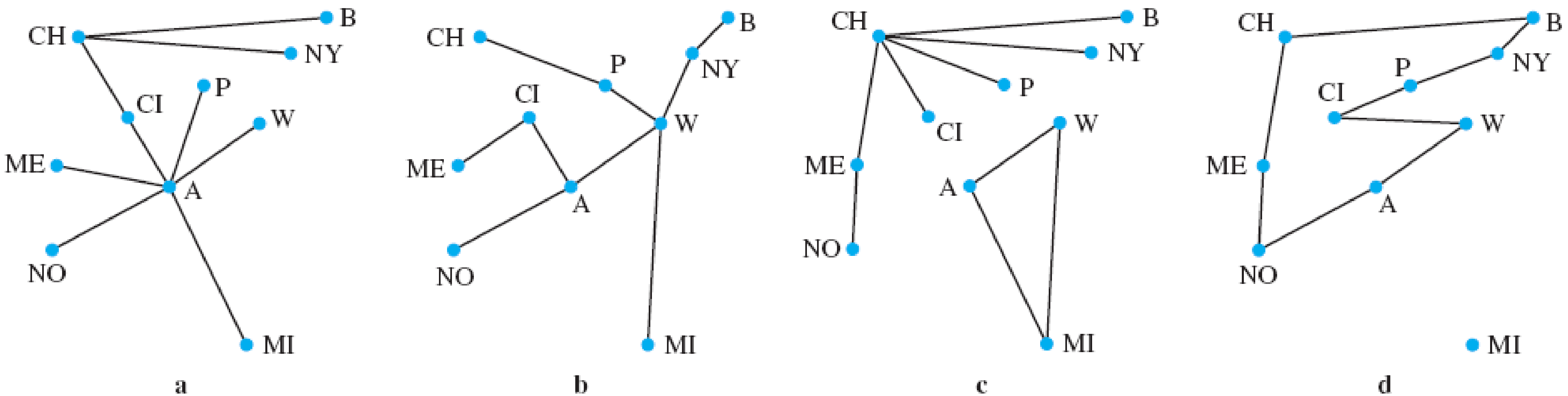

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Example: (a) and (b) are connected, (c) and (d) are disconnected.

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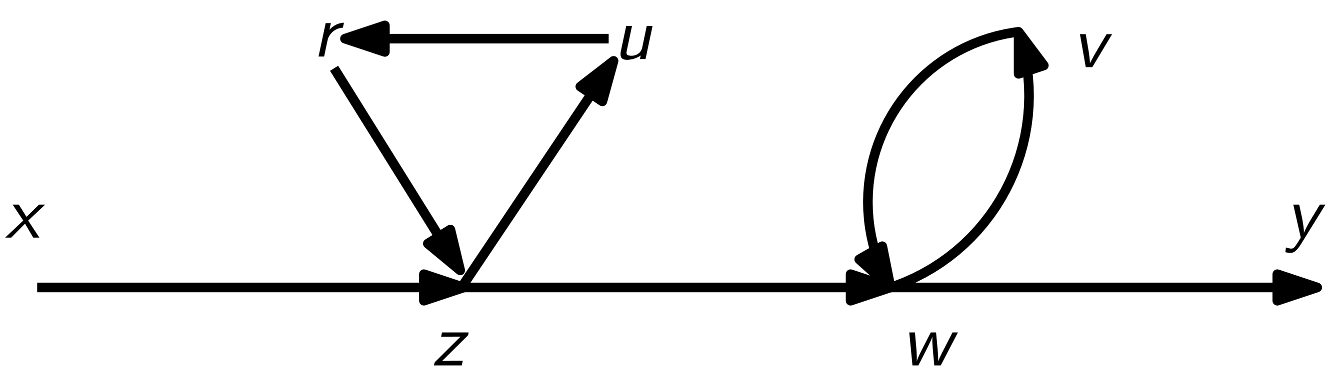

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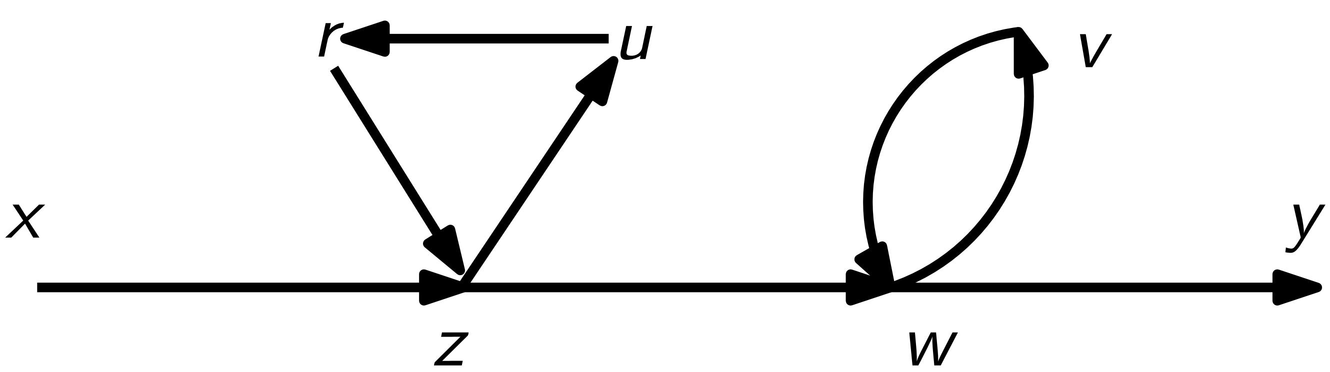

Path from x to y

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Path

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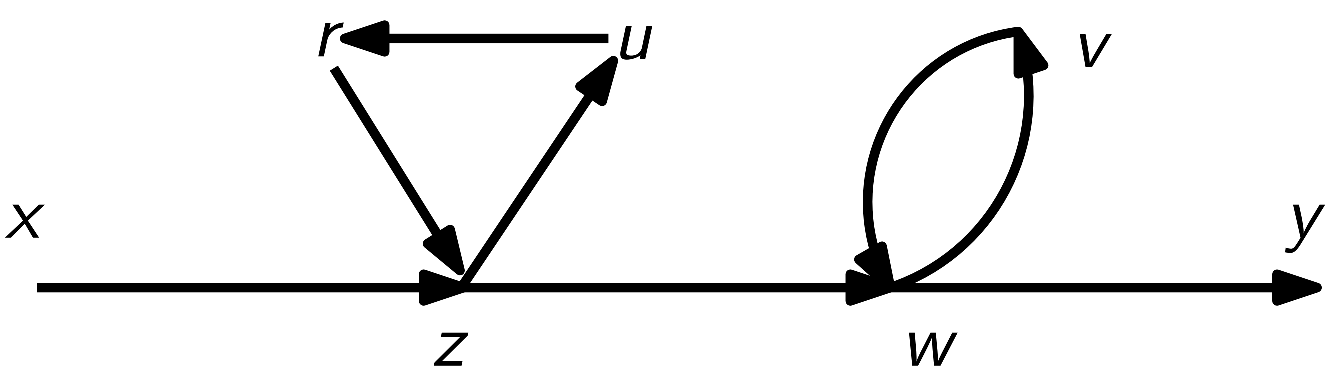

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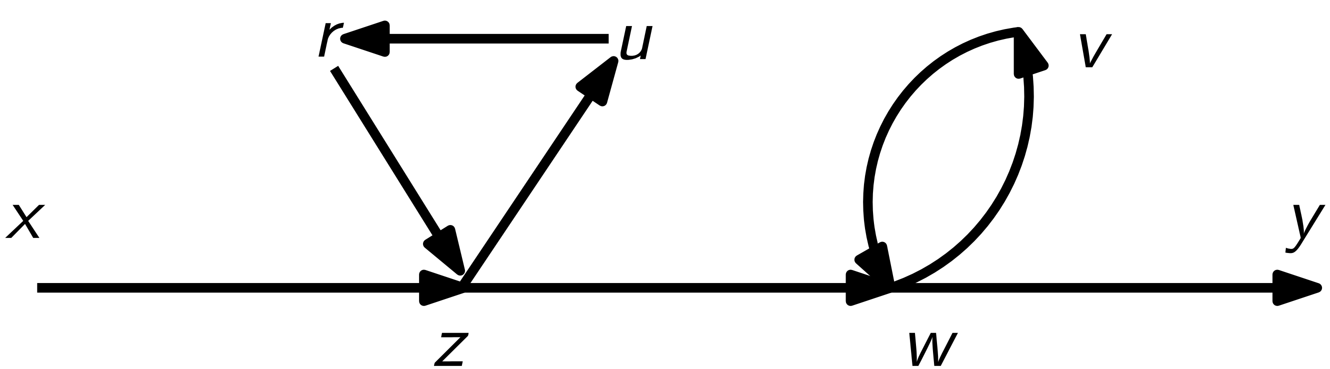

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Theorem There is a simple path between every pair of distinct vertices of a connected undirected graph.

Connected Components

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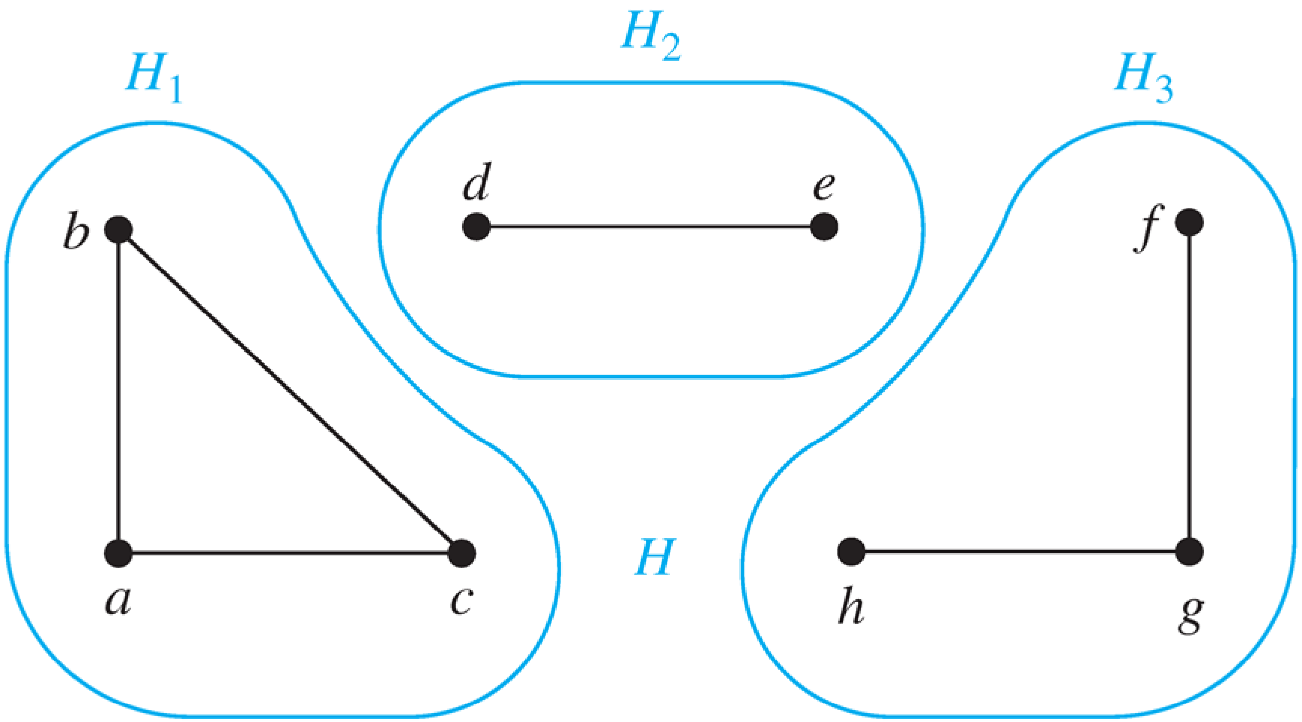
最大的连通分量



Connectedness in Directed Graphs

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强连通，双向连接

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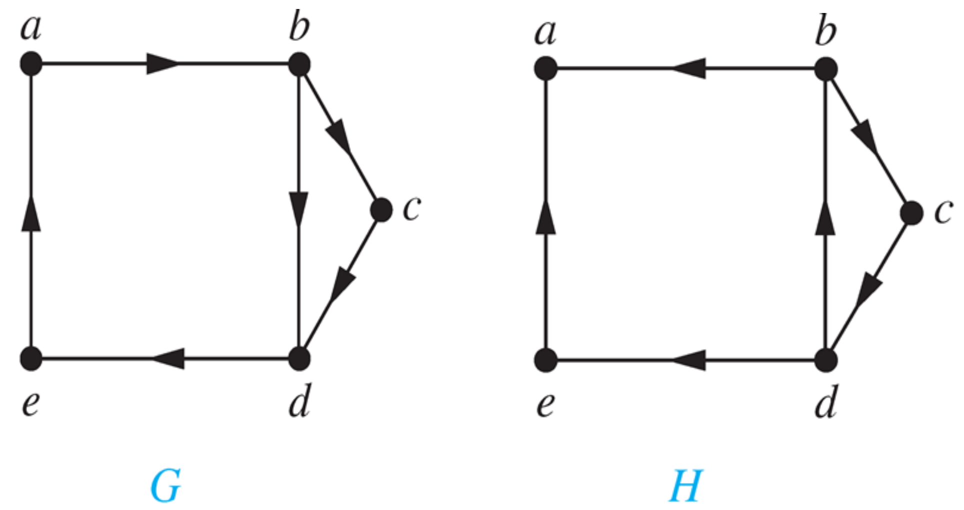

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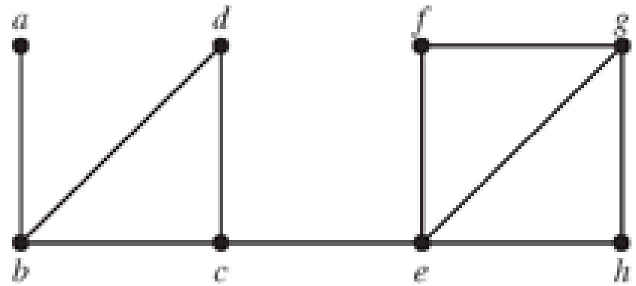

G_1

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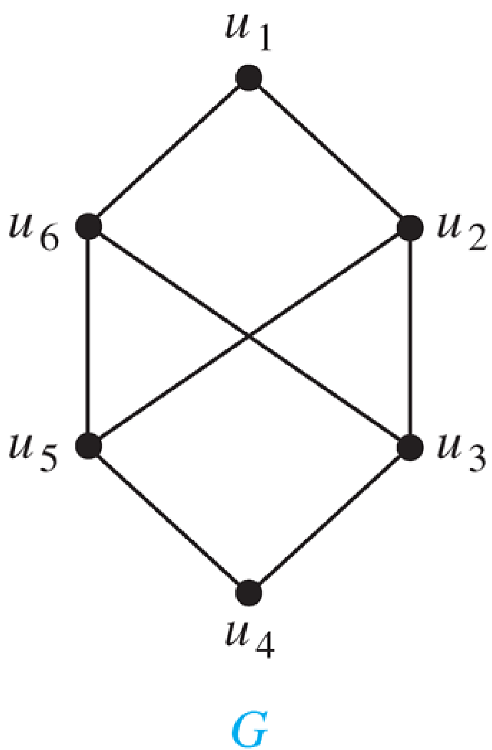

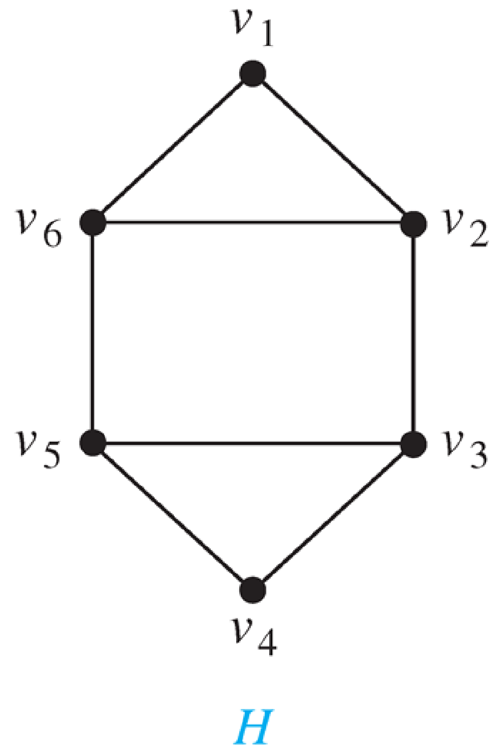

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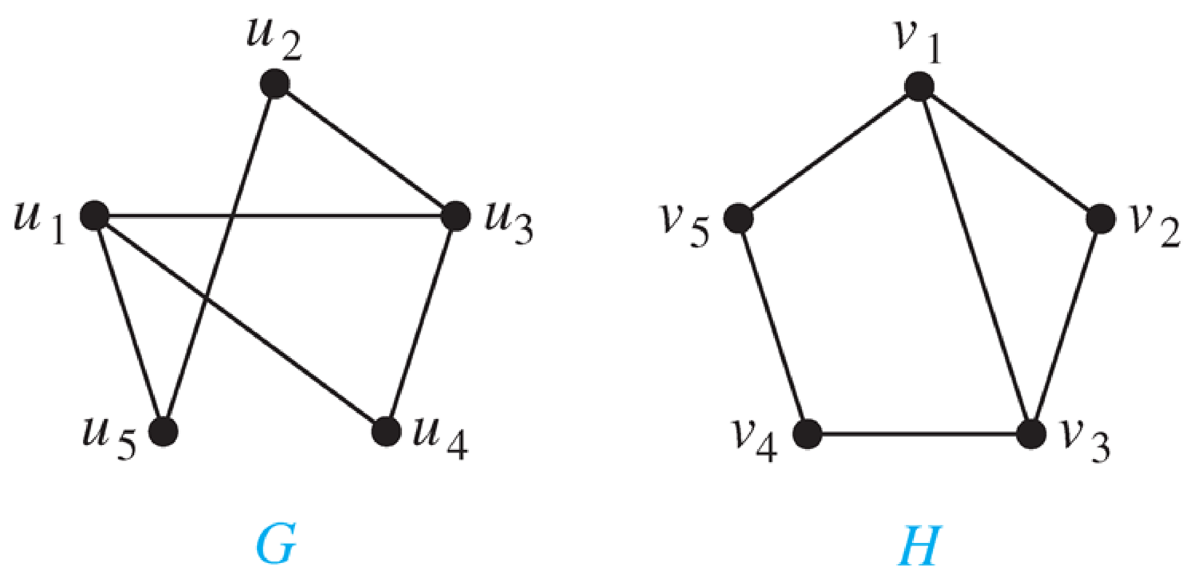

长度为k 的简单回路（simple circuit） 的存在是一种同构不变性（isomorphic invariant）。

Counting Paths between Vertices

- **Theorem** Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where $r > 0$ is positive, equals the (i, j)-th entry of $\mathbf{A}^r$.

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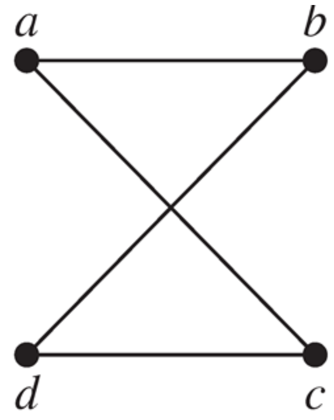
  $\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$, the (i, j)-th entry of $\mathbf{A}^{r+1}$ equals $b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj}$, where b_{ik} is the (i, k)-th entry of $\mathbf{A}^r$.

Counting Paths between Vertices

- **Example** How many paths of length 4 are there from a to d in the graph G?



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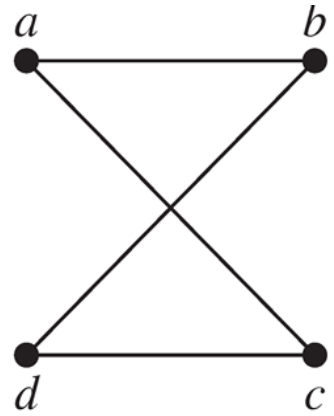

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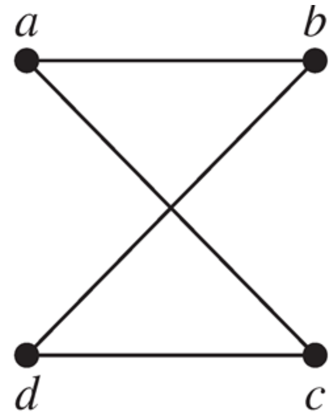

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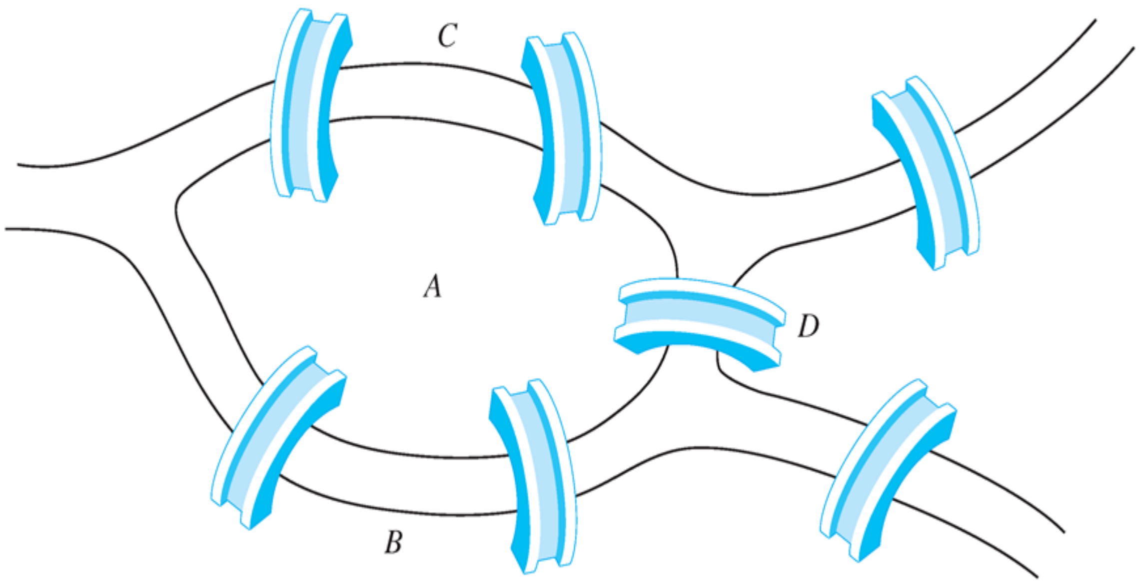
- **Königsberg seven-bridge problem**

  People wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.



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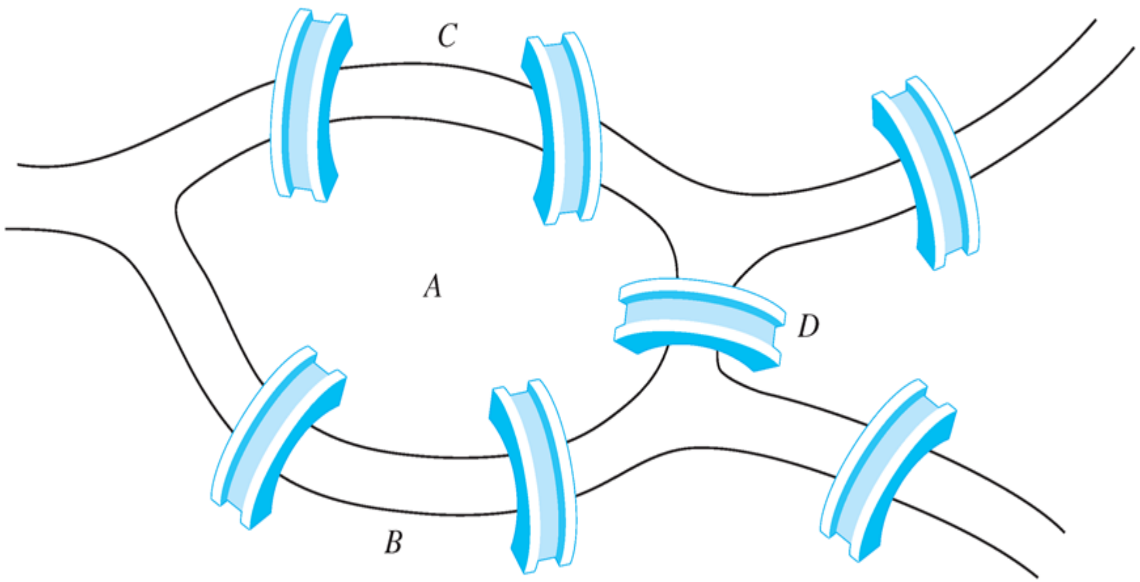

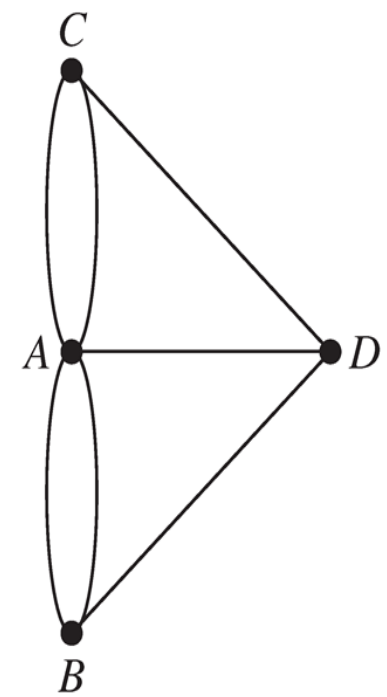

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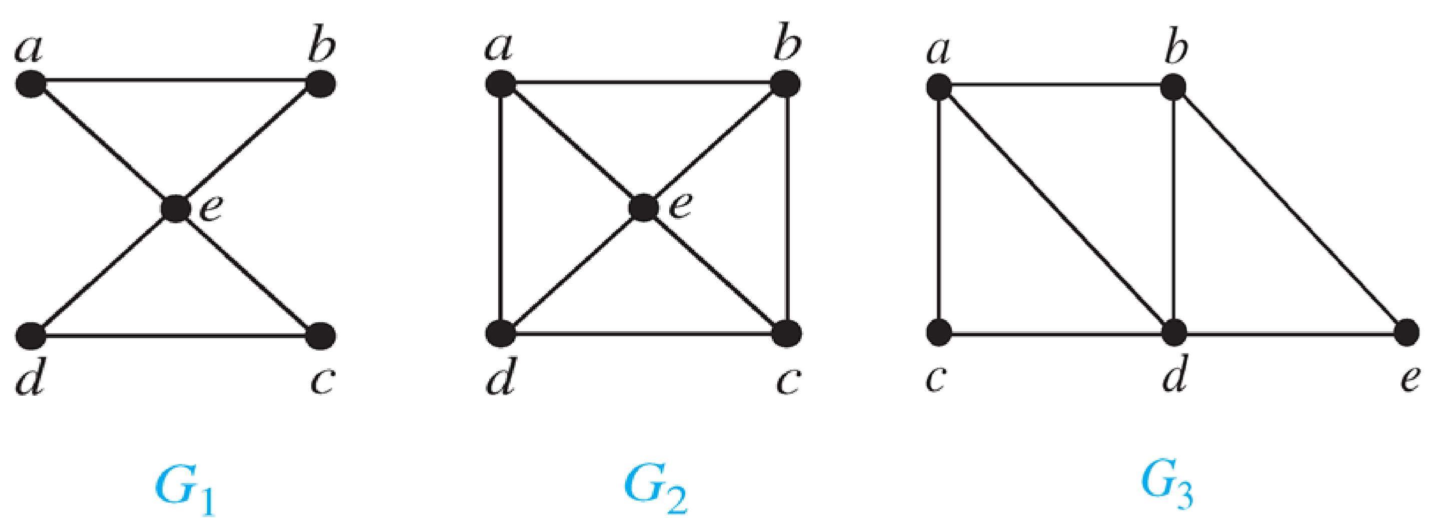
每个顶点只经过一次（环的话起点和终点重合）

Example Which of the undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?



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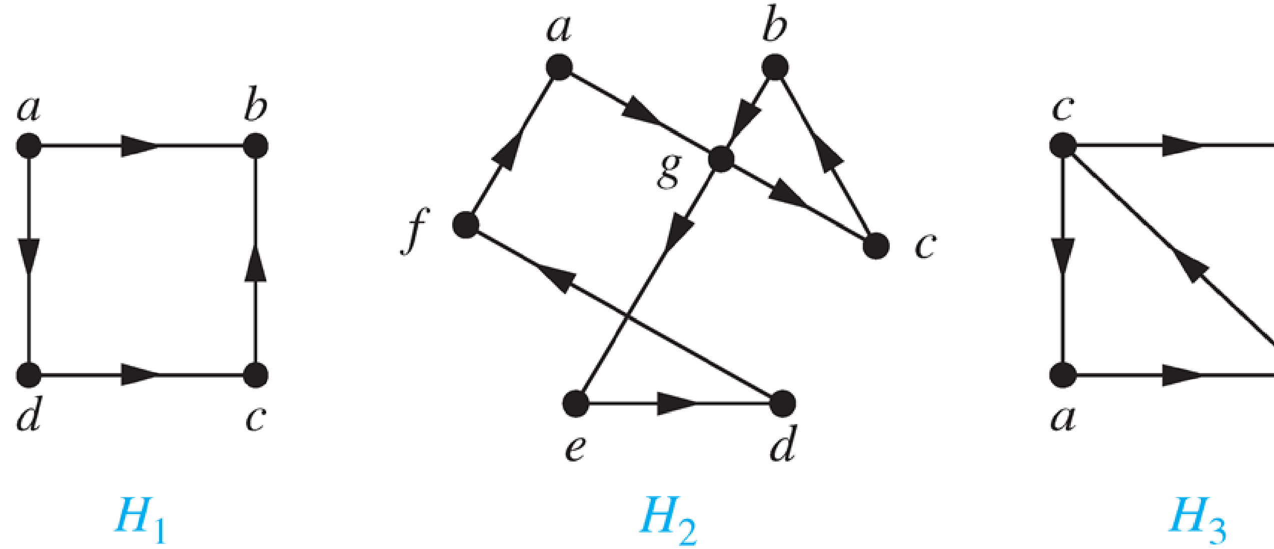

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- ◇ The initial vertex and the final vertex of an Euler path have odd degree.

Sufficient Conditions for Euler Circuits and Paths

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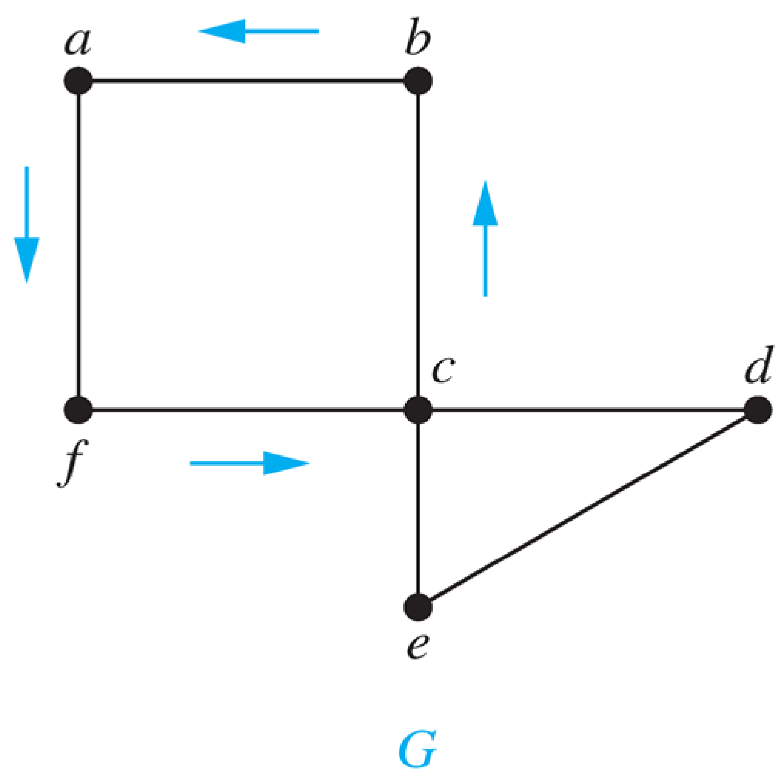

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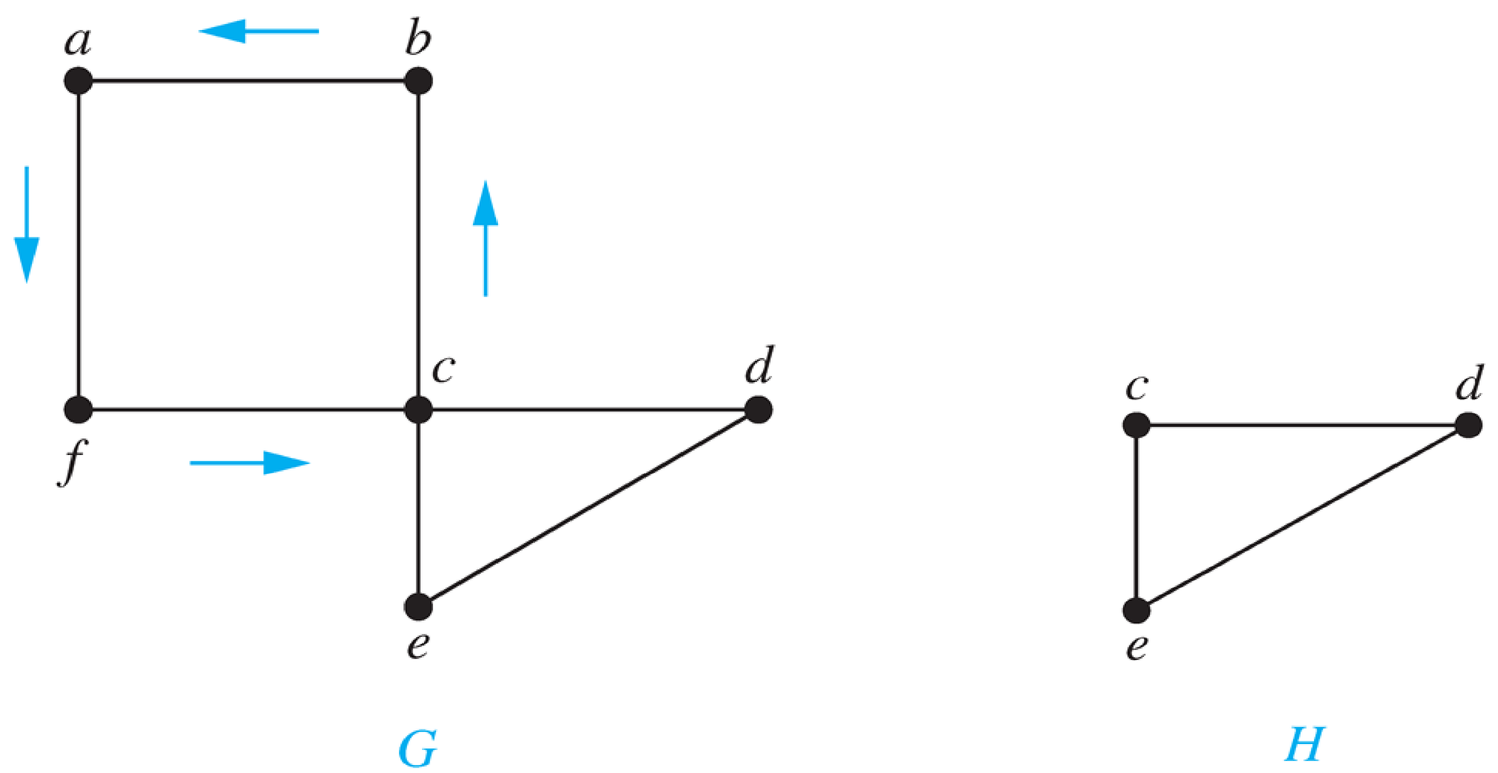

Algorithm for Constructing an Euler Circuit

procedure *Euler*(*G*: connected multigraph with all vertices of even degree)
circuit := a circuit in *G* beginning at an arbitrarily chosen vertex with edges
successively added to form a path that returns to this vertex.
H := *G* with the edges of this circuit removed
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共享定点连接不同circuit

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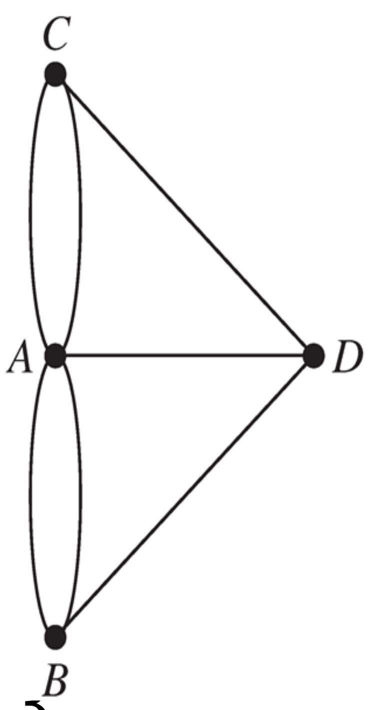

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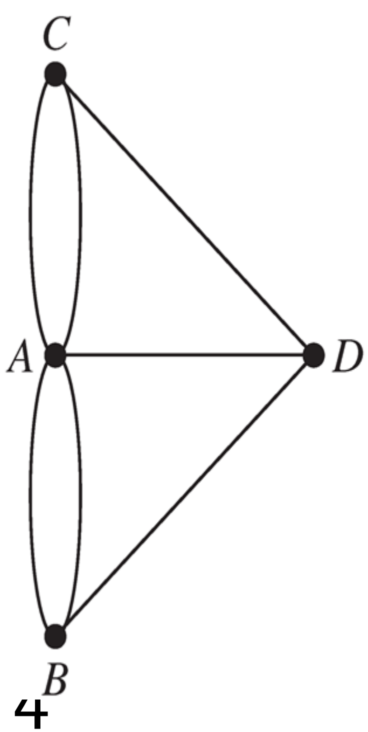


No Euler circuit

对于无向图，degree就是连接到这个顶点的边数

- **Example**

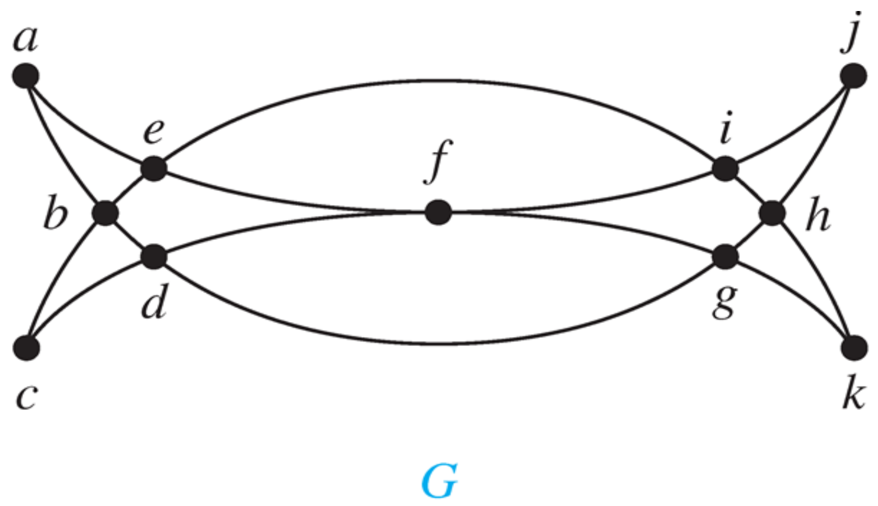
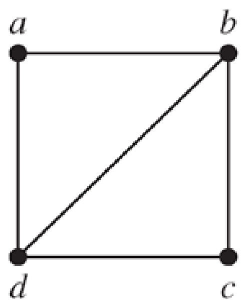
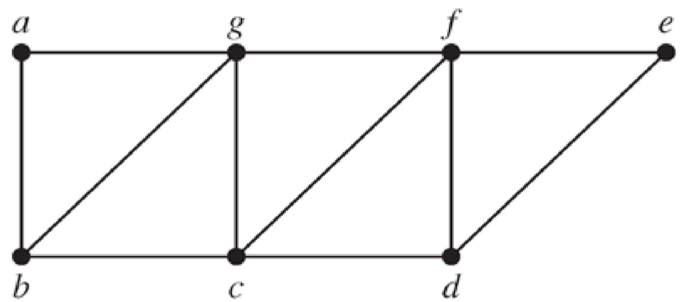


FIGURE 6 **Mohammed's Scimitars.**

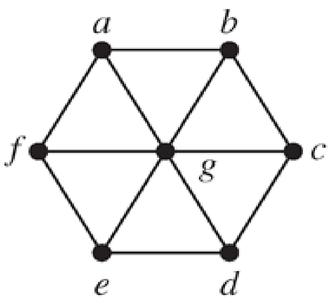
- **Example**



G_1



G_2



G_3

Applications of Euler Paths and Circuits

- Finding a path or circuit that traverses each
  - ◇ street in a neighborhood
  - ◇ road in a transportation network
  - ◇ link in a communication network
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Chinese Postman Problem Meigu Guan [60']

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k-Postman Chinese Postman Problem (k-PCPP)

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$\in$ NPC

Next Lecture

- Graph theory II ...