Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Hamilton Paths and Circuits

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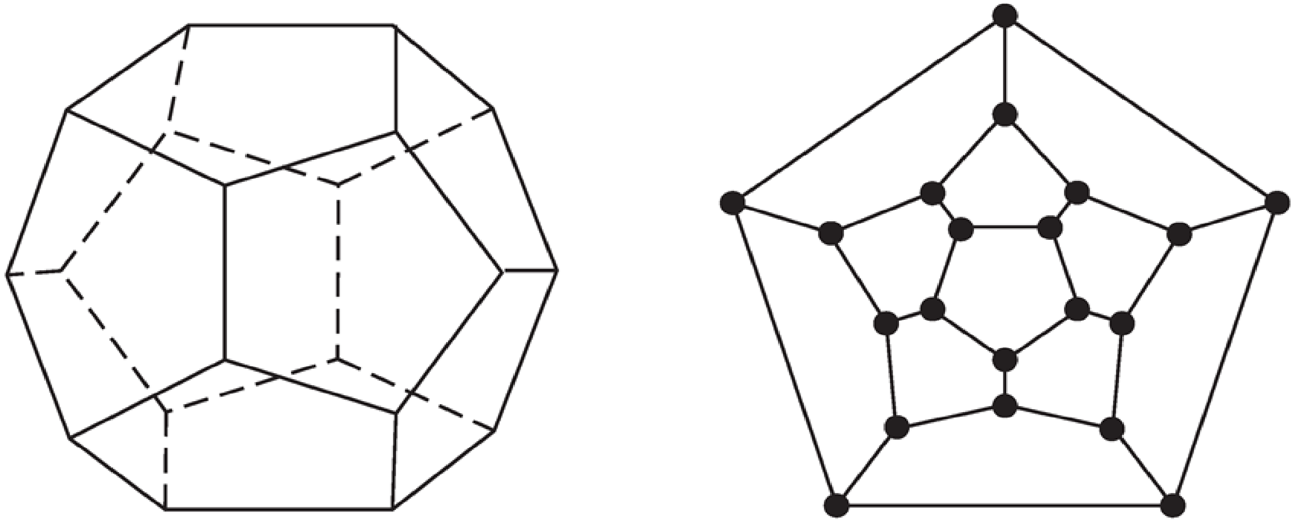

(a) (b)

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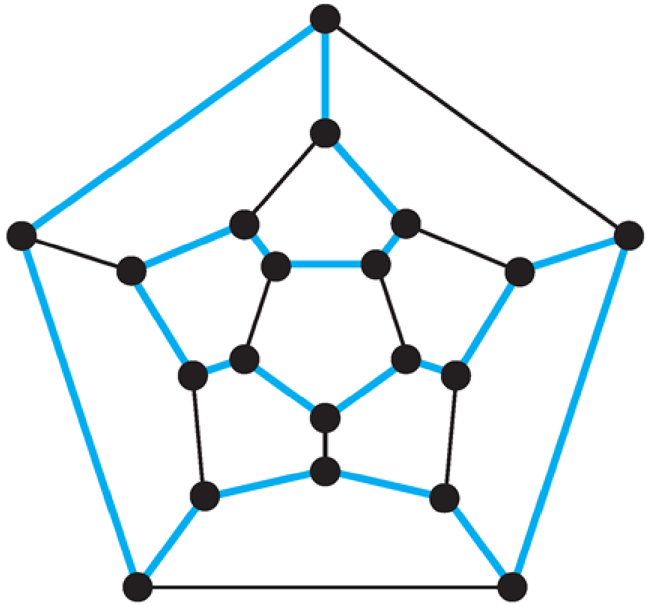

Hamilton Paths and Circuits

- **Definition**: A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*.

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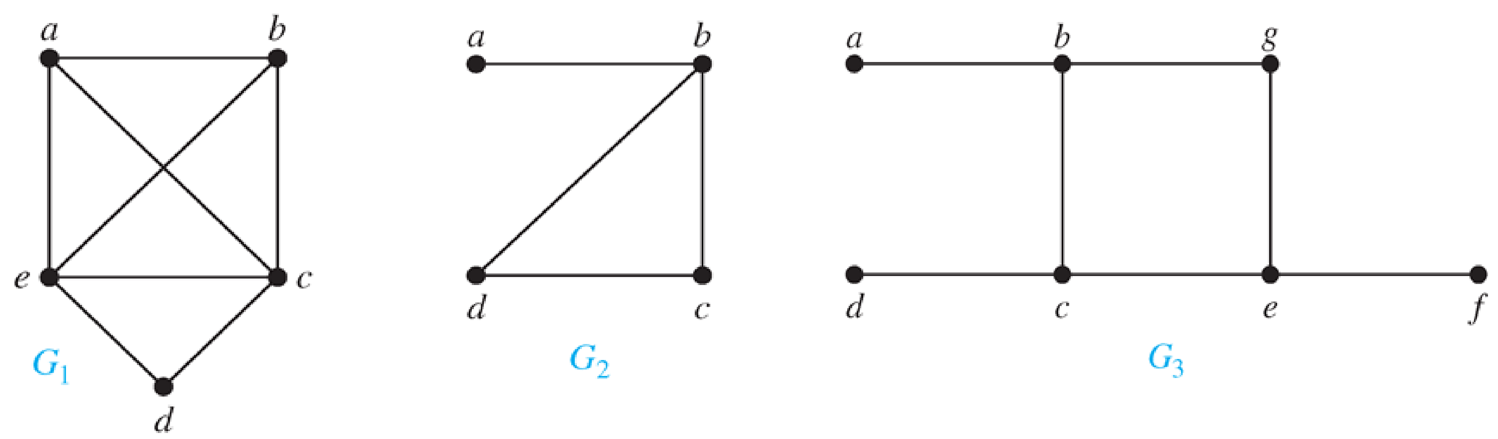
Example Which of these simple graphs has a Hamilton circuit or, if not, a Hamilton path?



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每一对不相邻的顶点的degree和>=n

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  Hamilton path problem $\in$ NPC

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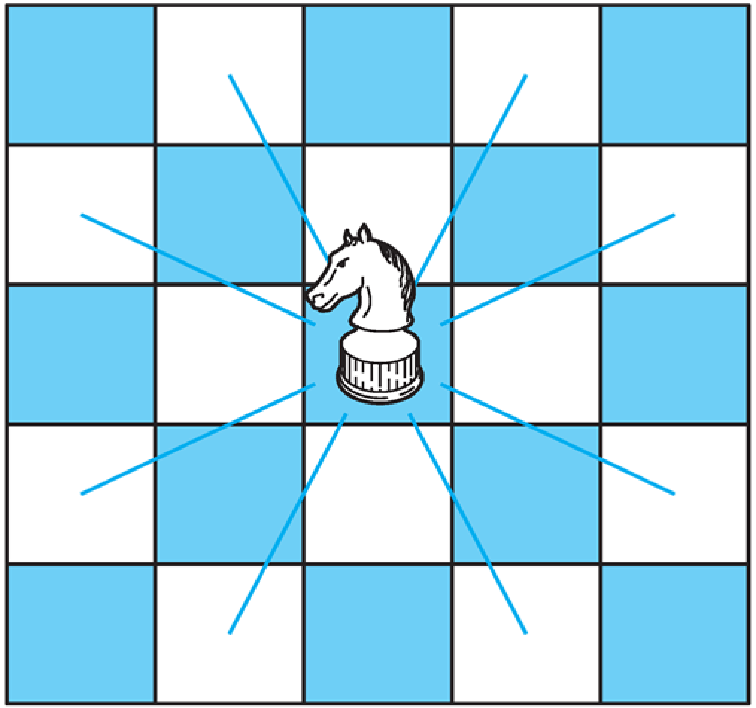
  the decision version of the TSP $\in$ NPC

Applications of Hamilton Paths and Circuits

- Can we traverse every space (and come back) in the 5×5 chessboard?



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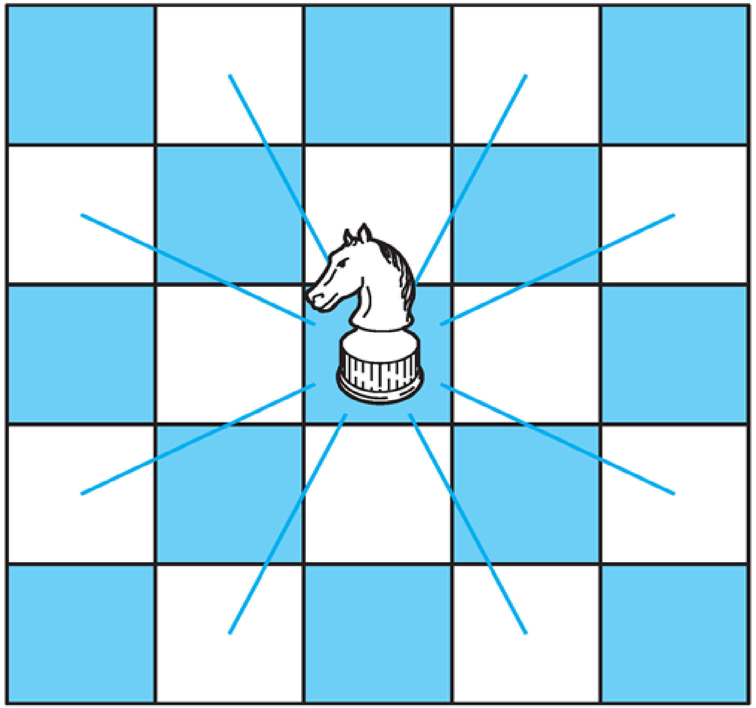

What about in 6×6 chessboard?

Shortest Path Problems

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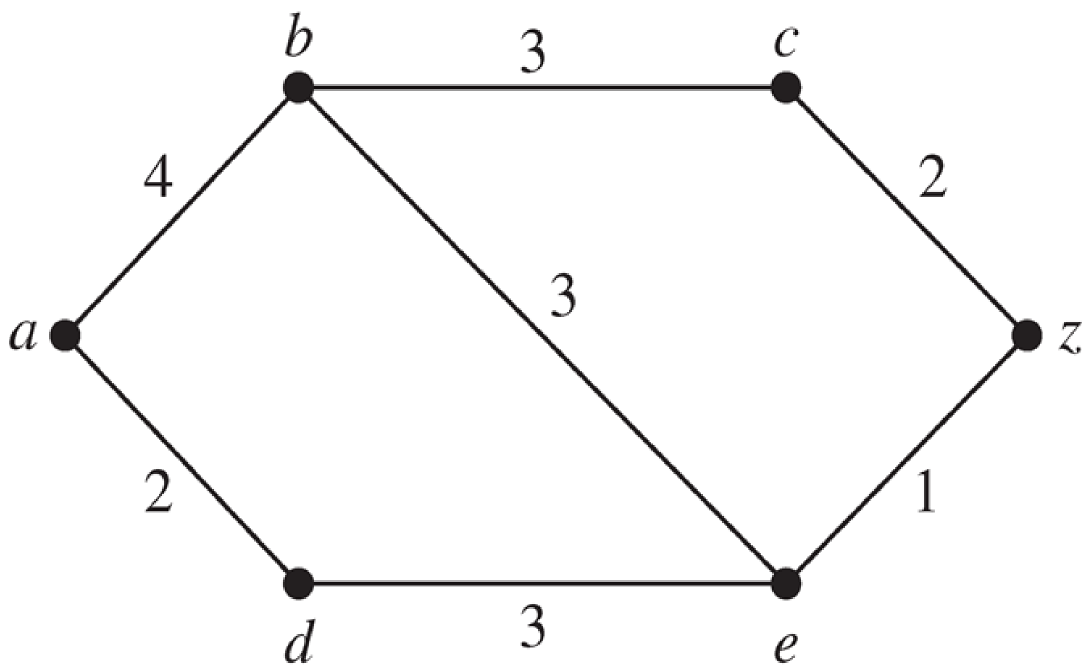

- **Definition** Let G^α be an weighted graph, with a weight function $\alpha : E \to \mathbf{R}^+$ on its edges. If $P = e_1 e_2 \cdots e_k$ is a path, then its weight is $\alpha(P) = \sum_{i=1}^{k} \alpha(e_i)$. The *minimum weighted distance* between two vertices is

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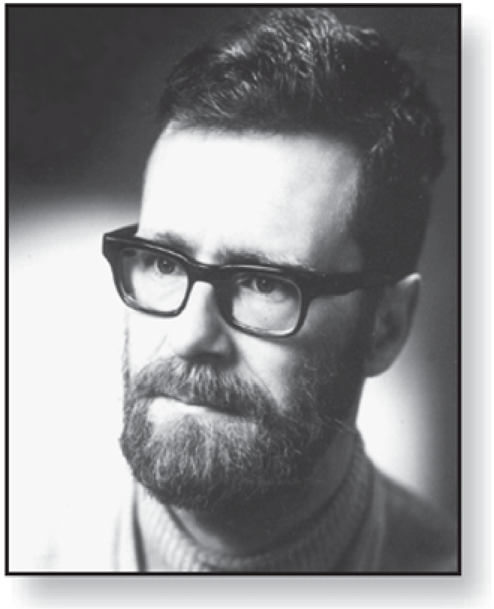

Edsger Wybe Dijkstra

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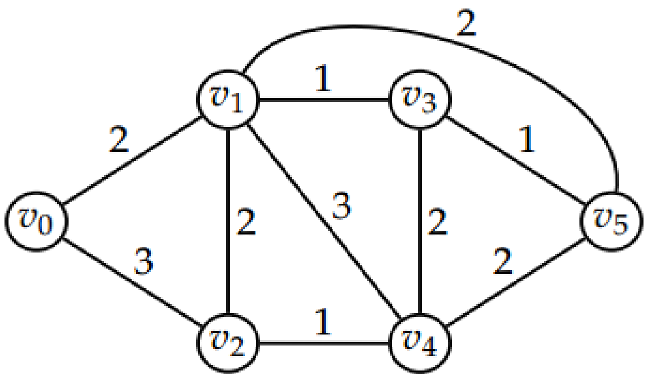

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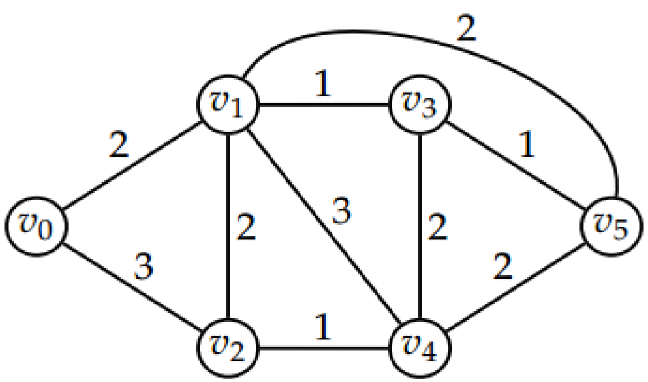

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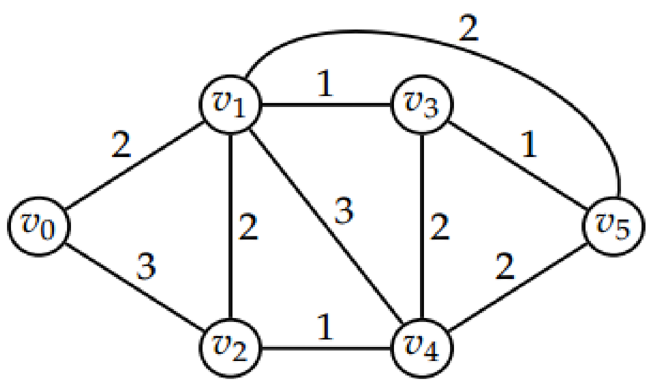

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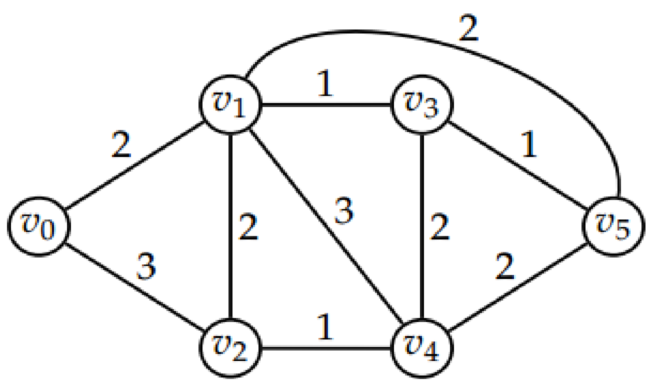

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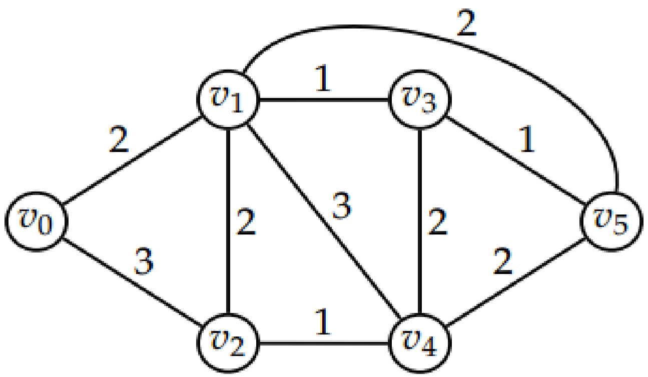

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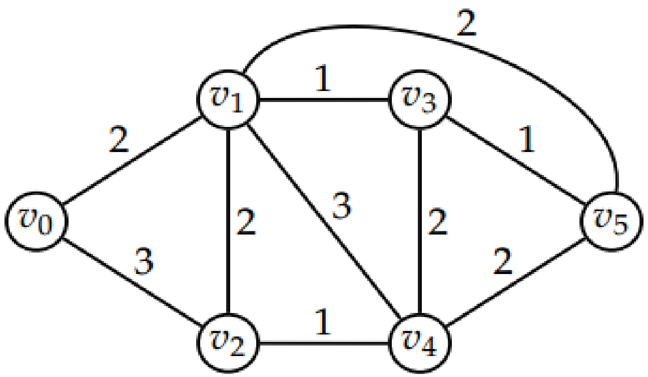

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$d(v_3) = 2 + 1 = 3$, $d(v_4) = 2 + 3 = 5$,

$d(v_5) = 2 + 2 = 4$

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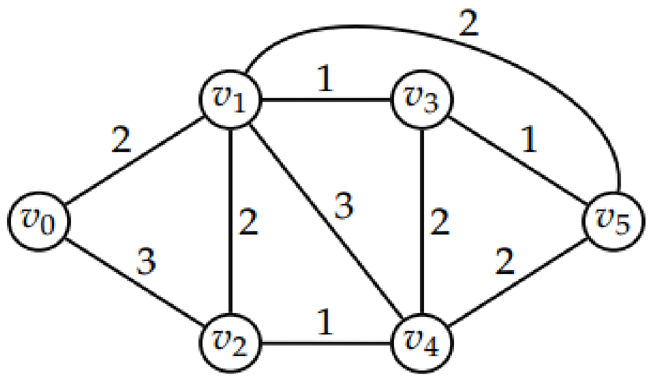

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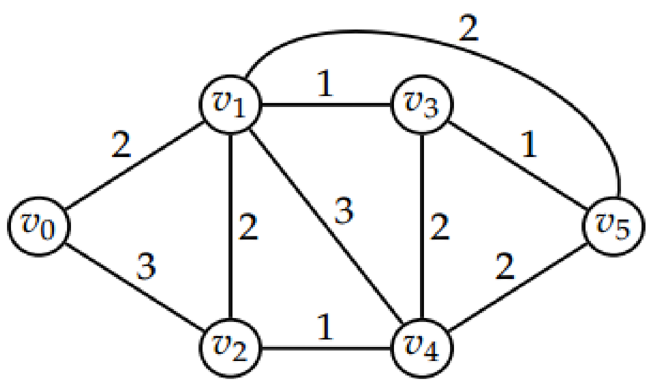

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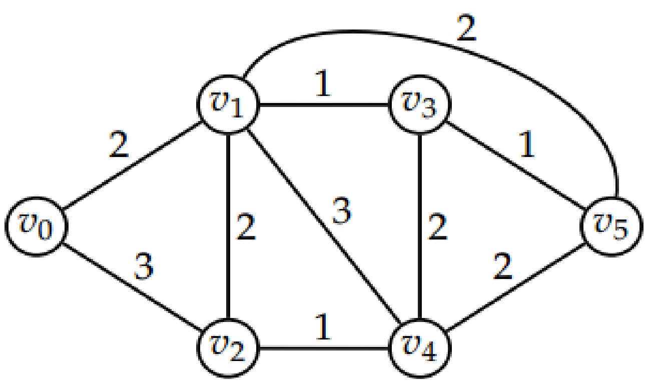

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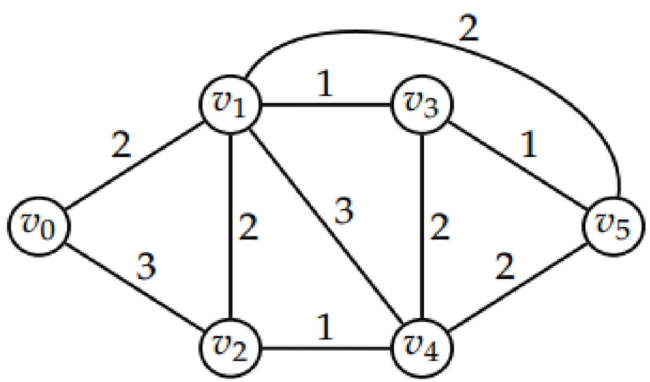

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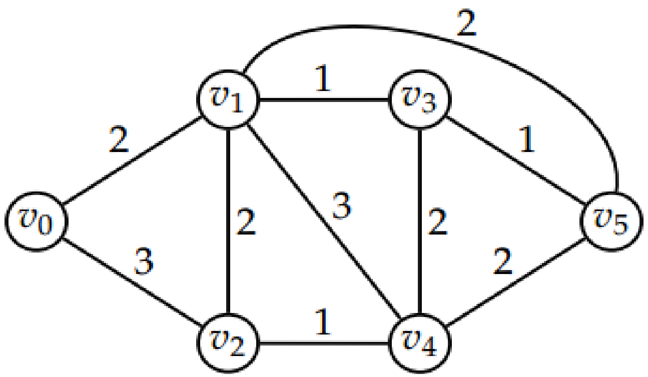

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  read the Textbook p.712 – p.714

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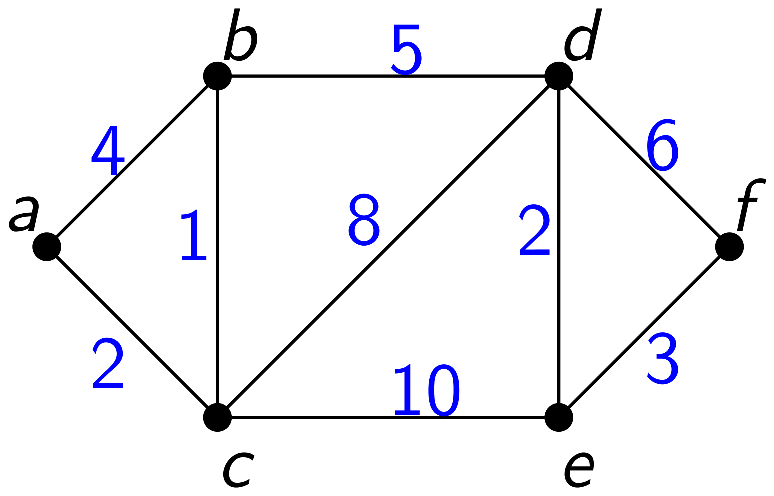
Another Example



The Single-Source Shortest Paths (SSSP) Problem

- Dijkstra's algorithm

  $O(v^2)$ using an *array* [Dijkstra 1956]

  $O(e + v \log v)$ using a *Fibonacci heap min-priority queue* [Fredman & Tarjan 1984]

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- New result

Negative-Weight Single-Source Shortest Paths in Near-linear Time

Aaron Bernstein* Danupon Nanongkai† Christian Wulff-Nilsen‡

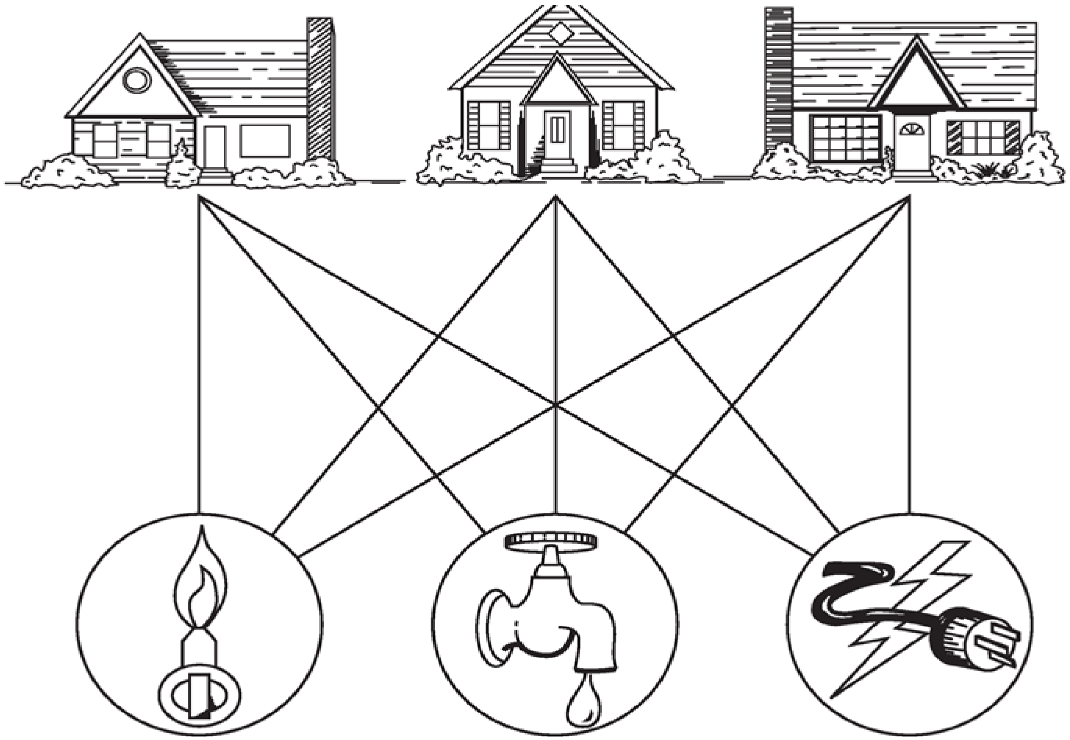
Abstract

We present a randomized algorithm that computes single-source shortest paths (SSSP) in $O(m\log^8(n)\log W)$ time when edge weights are integral and can be negative.[1] This essentially resolves the classic negative-weight SSSP problem. The previous bounds are $\tilde{O}((m+n^{1.5})\log W)$ [BLNPSSSW FOCS'20] and $m^{4/3+o(1)}\log W$ [AMV FOCS'20]. Near-linear time algorithms were known previously only for the special case of planar directed graphs [Fakcharoenphol and Rao FOCS'01].

In contrast to all recent developments that rely on sophisticated continuous optimization methods and dynamic algorithms, our algorithm is simple: it requires only a simple graph decomposition and elementary combinatorial tools. In fact, ours is the first combinatorial algorithm for negative-weight SSSP to break through the classic $\tilde{O}(m\sqrt{n}\log W)$ bound from over three decades ago [Gabow and Tarjan SICOMP'89].

Planar Graphs

- Join three houses to each of three seperate utilities.



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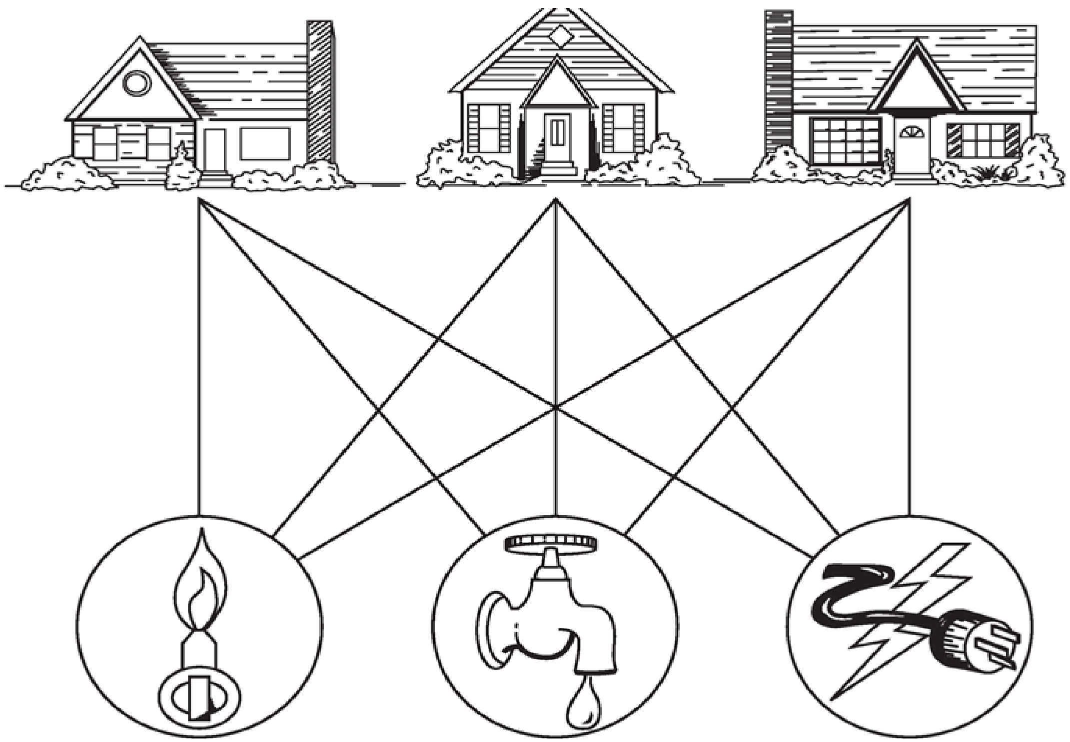

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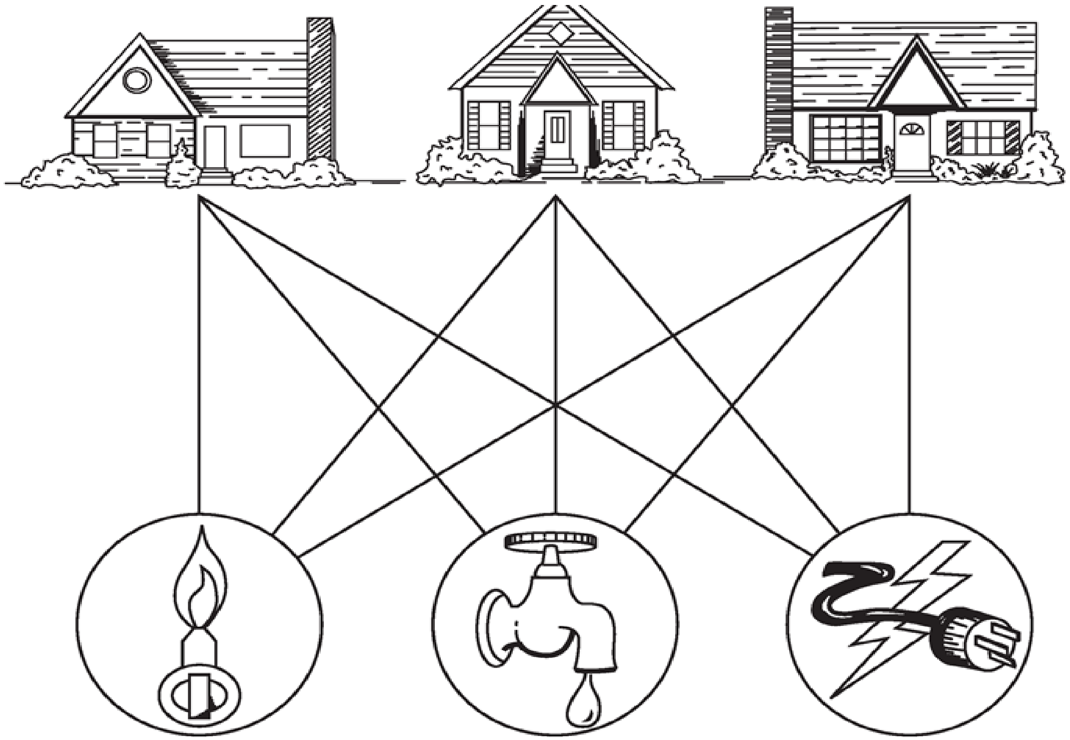

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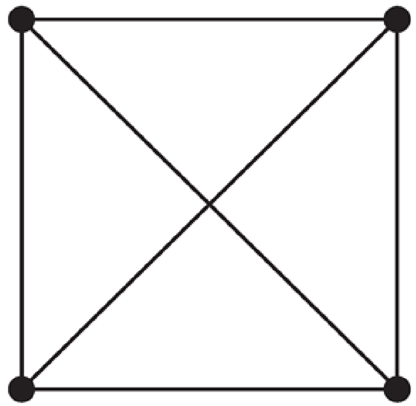

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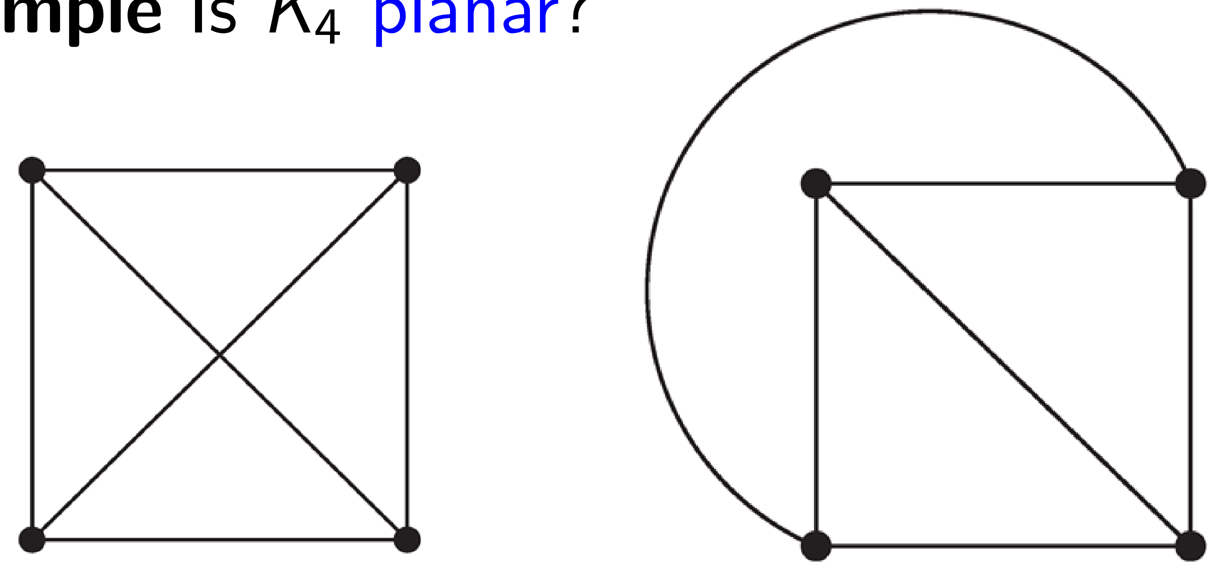
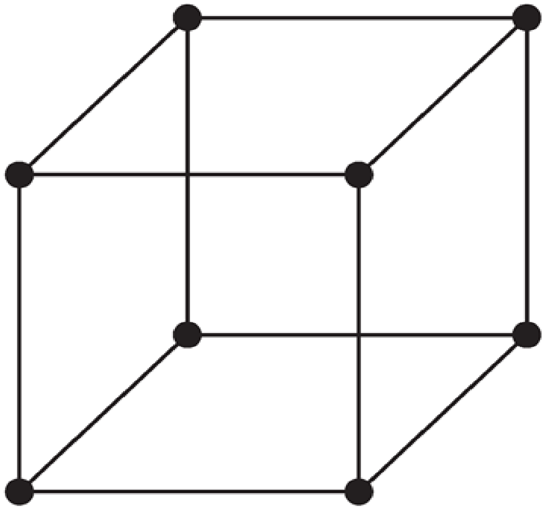
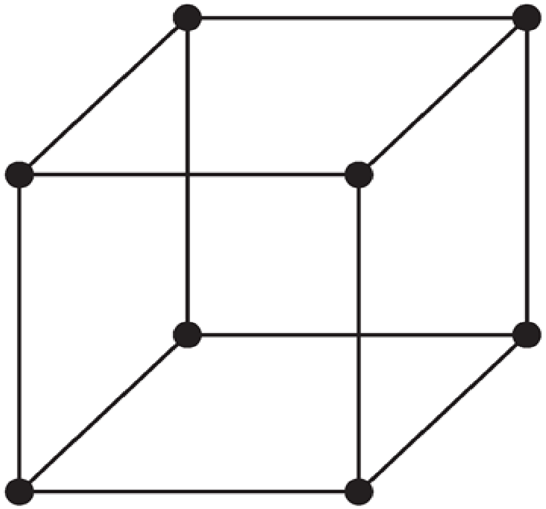


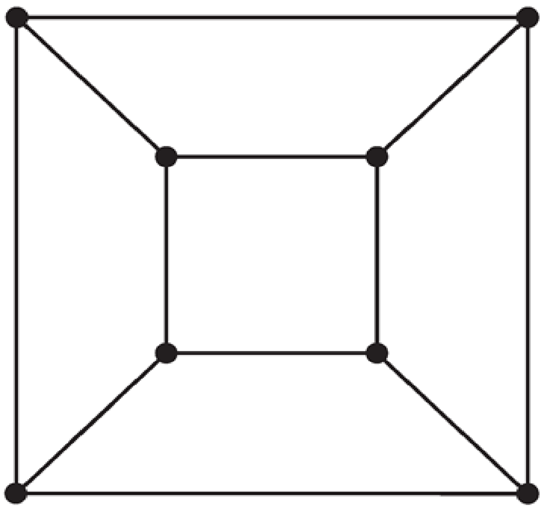
- **Example**



Planar Graphs

- **Example**





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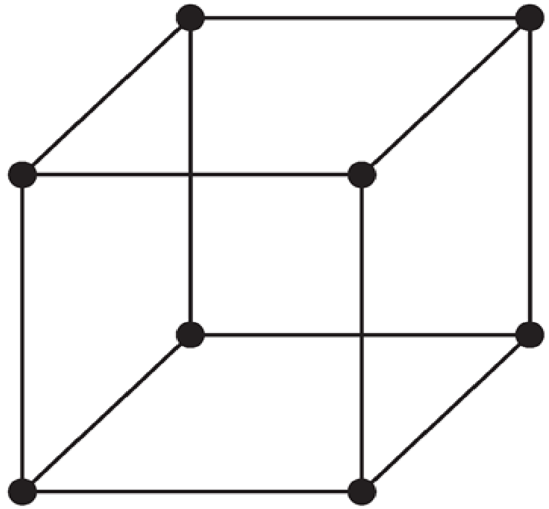

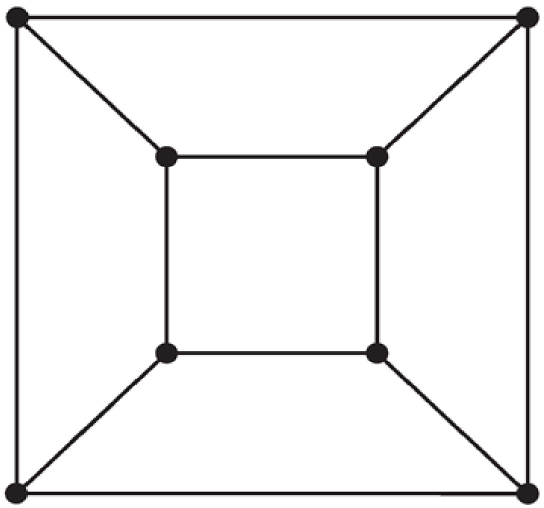

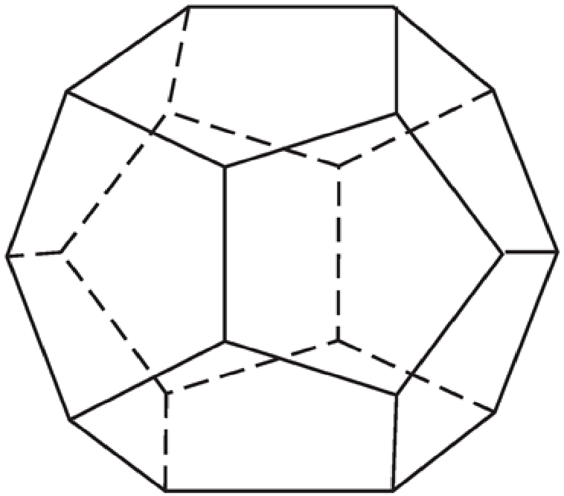

(a)

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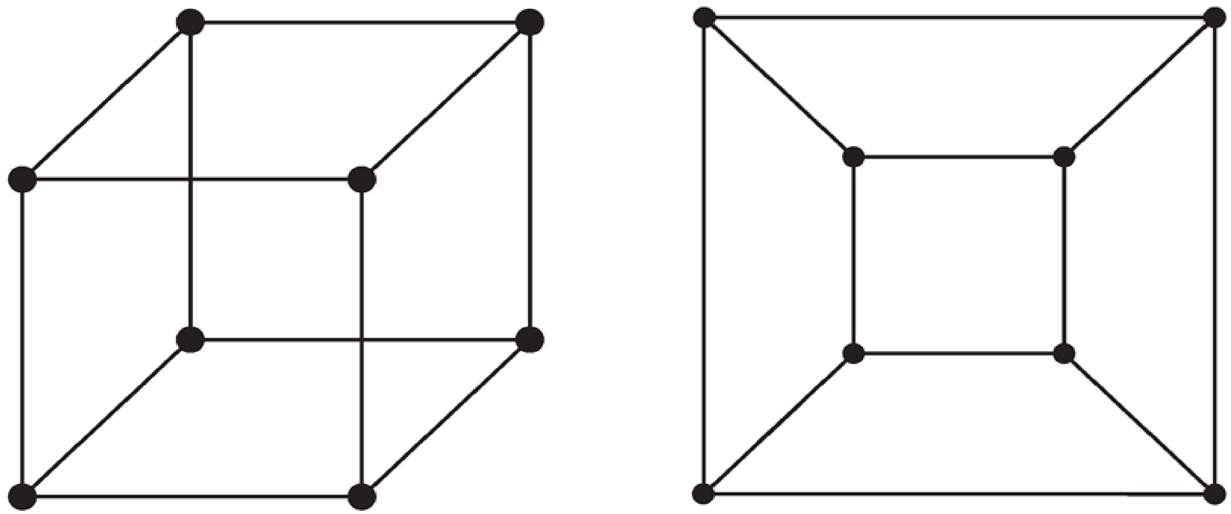

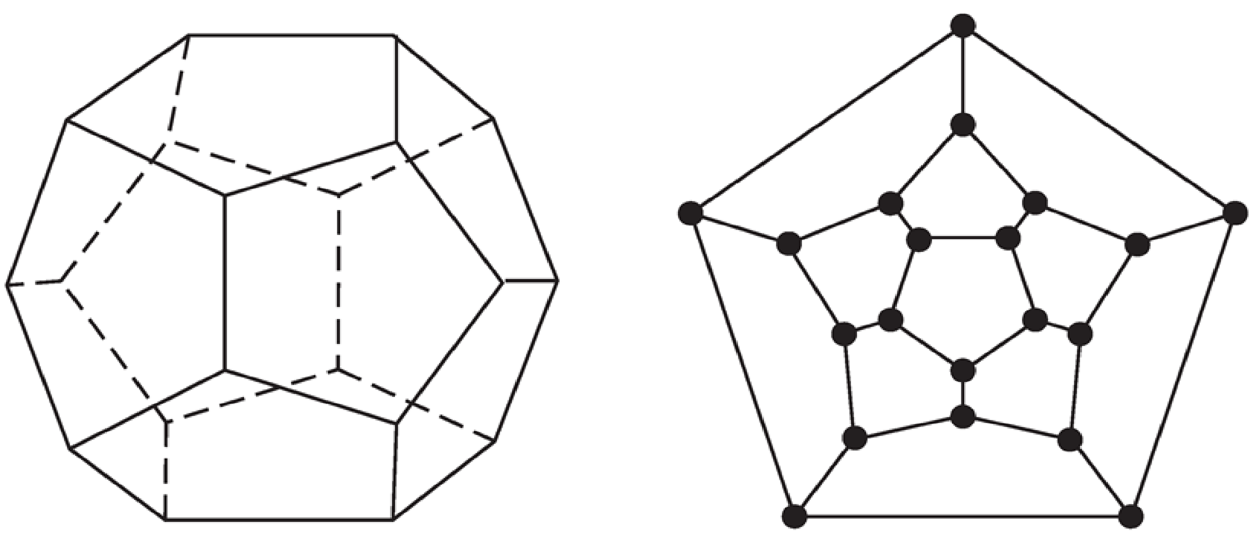

(a) (b)

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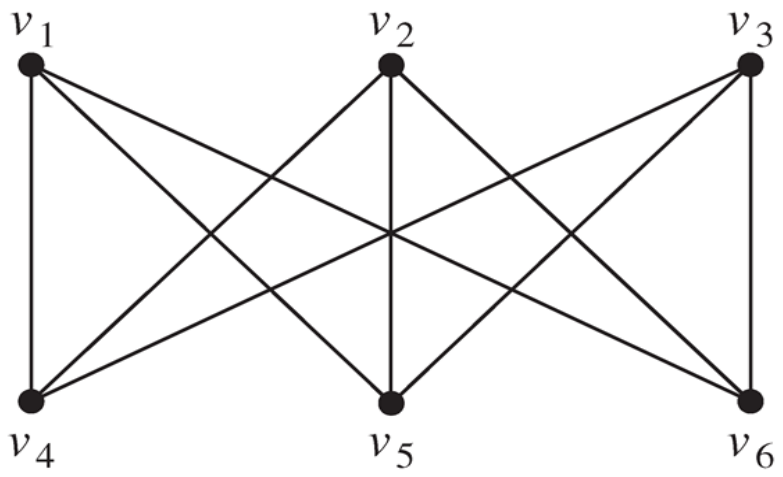

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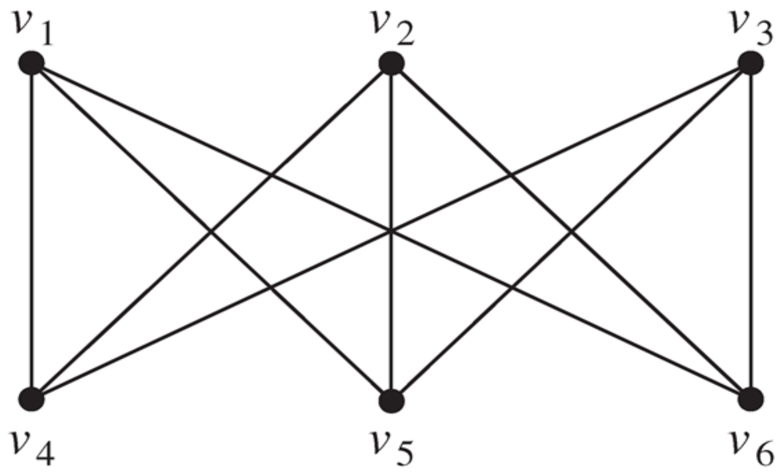

Applications

- ◇ IC design
- ◇ design of road networks

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

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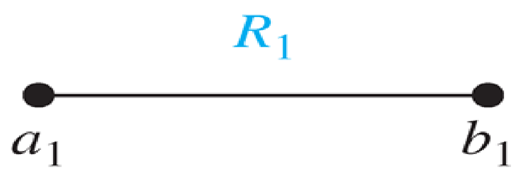

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  连通平面图

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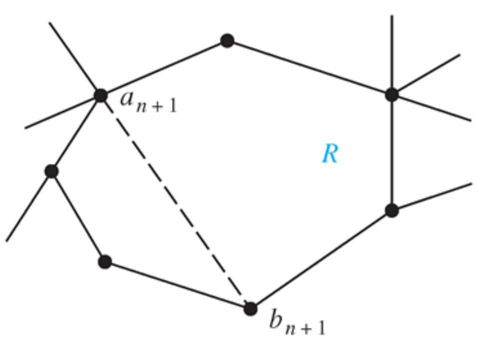

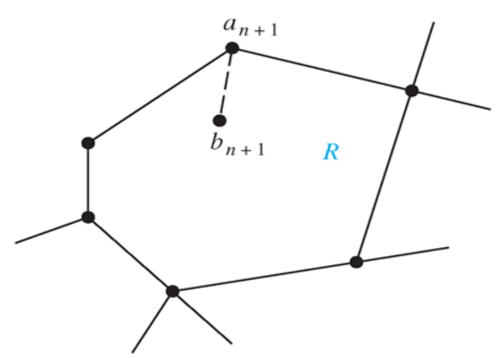

The Degree of Regions

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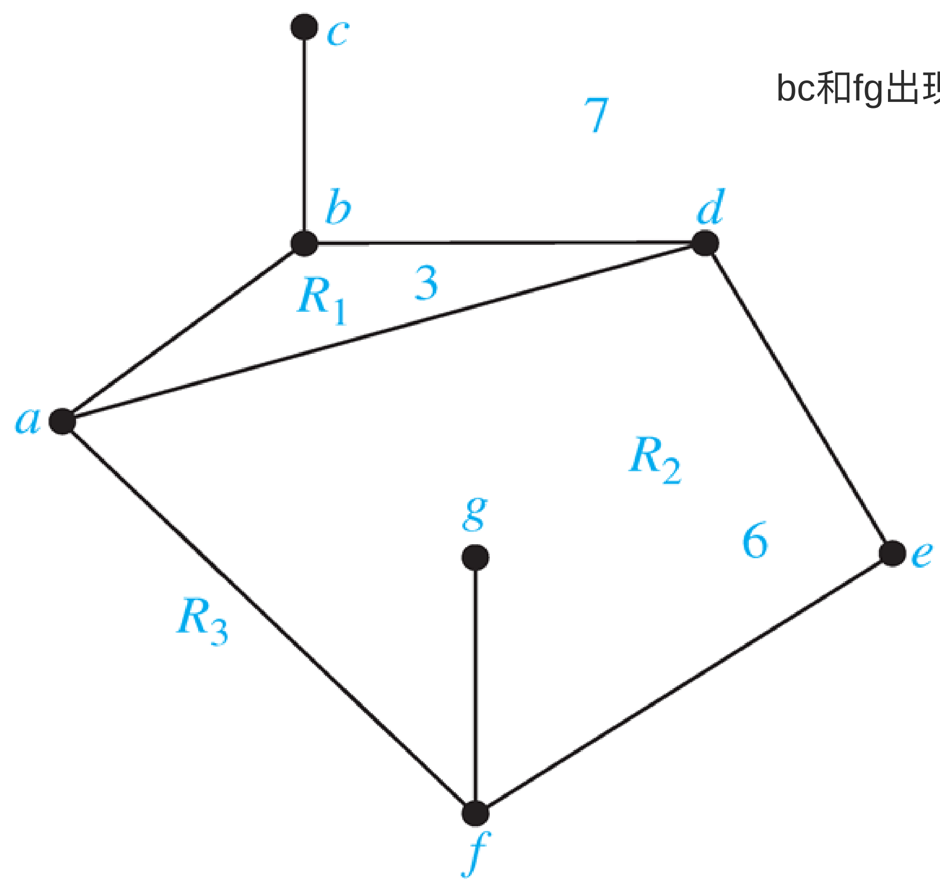

bc和fg出现了两次

Corollaries

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每条边会给region一共贡献两个degree

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  By Euler's formula, the proof is completed.

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By Corollary 1 and the Handshaking Theorem.

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  图中所有顶点度数之和等于边数的两倍
  （region 的度数也类似）

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  Proof similar to that of Corollary 1.

Examples

- Show that K_5 is nonplanar.

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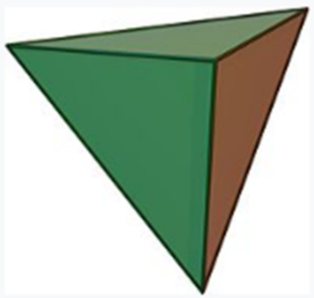
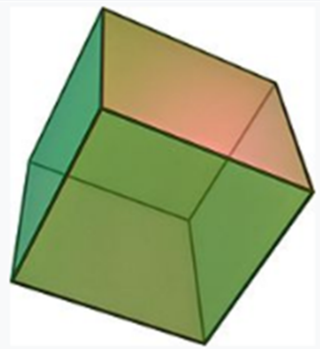
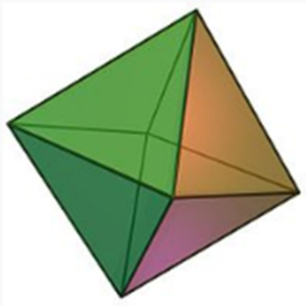
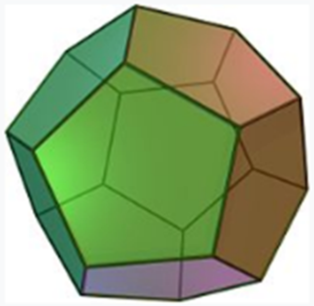
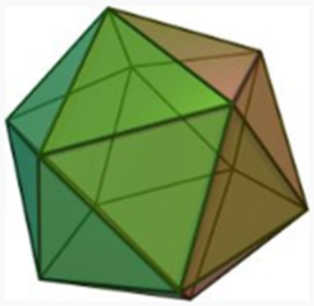
  Corollary 2 is used in the proof of Five Color Theorem.

Only 5 Platonic Solids

				
Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}

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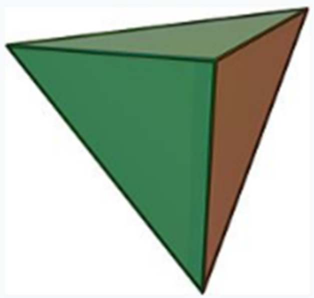	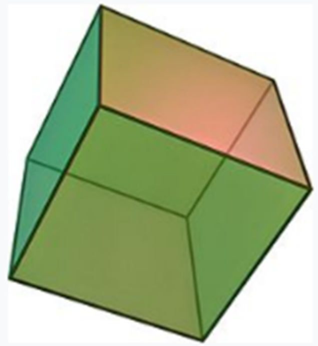	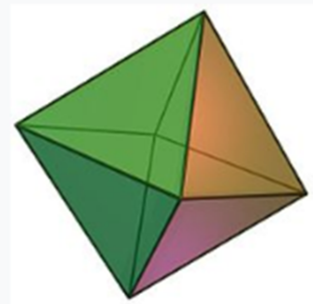	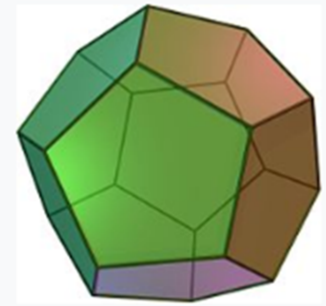	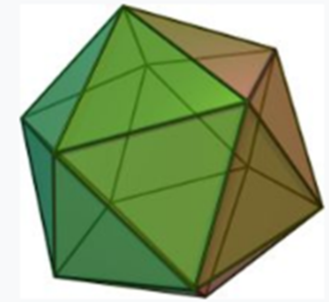
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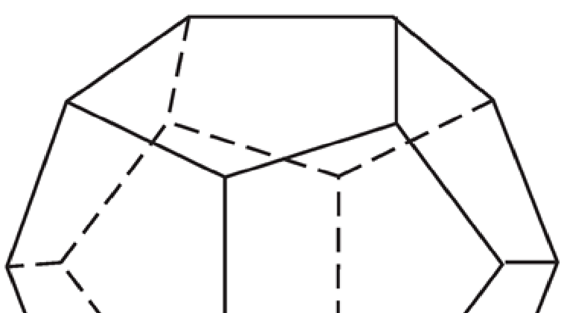

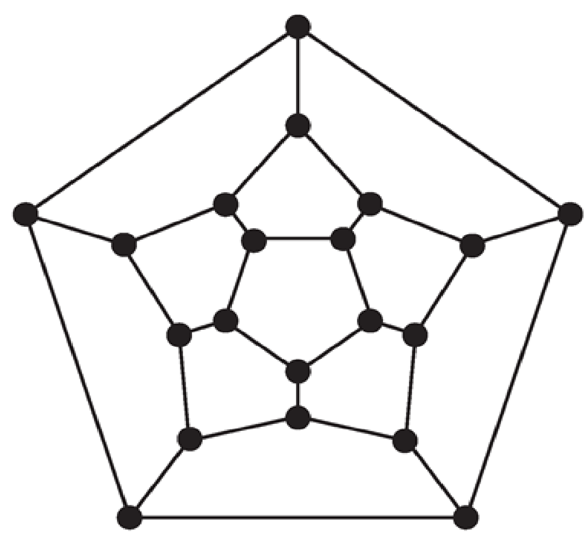

(b)

柏拉图立体的定义特性

1. 所有的面都是全等的正多边形。
2. 每个顶点的邻接面数相同（即顶点是规则的）。
3. 内部角度规则。
4. 三维空间中的唯一性：只有五种这样的正多面体。

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

1. **初等细分 (Elementary Subdivision)**：
   - 如果一个图是平面图（planar graph），那么通过以下操作所得的图仍然是平面图：
     - **移除一条边** $\{u, v\}$。
     - **在这条边上添加一个新顶点** w，并连接两条新边 $\{u, w\}$ 和 $\{w, v\}$。
   - 这种操作被称为**初等细分**（Elementary Subdivision）。
2. **同胚图 (Homomorphic Graphs)**：
   - 如果两个图 $G_1 = (V_1, E_1)$ 和 $G_2 = (V_2, E_2)$ 可以通过一系列初等细分从同一个图生成，那么它们被称为**同胚图**。

Kuratowski's Theorem

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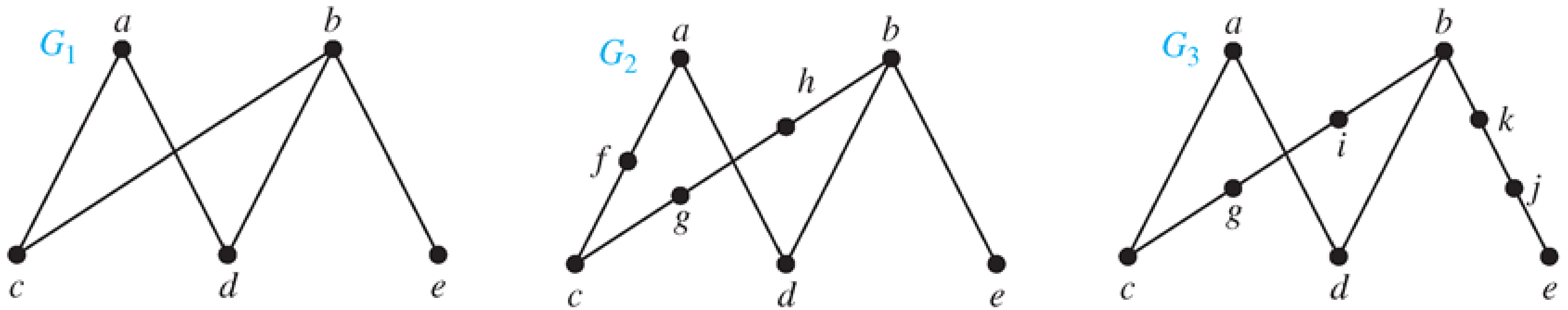

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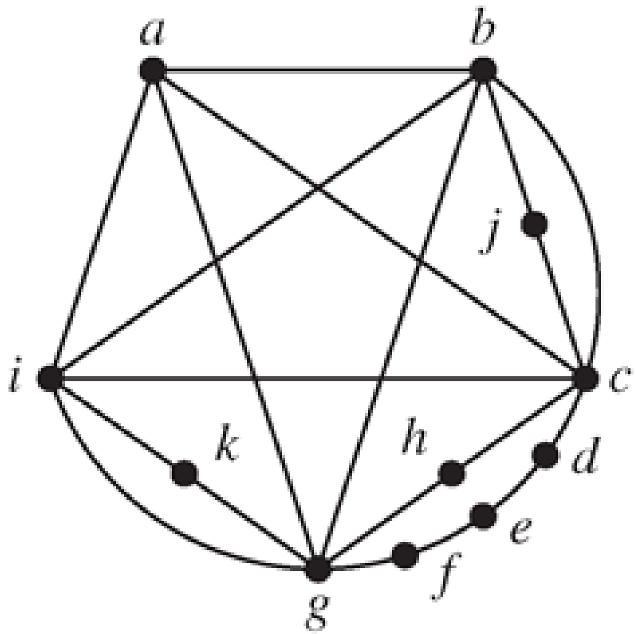
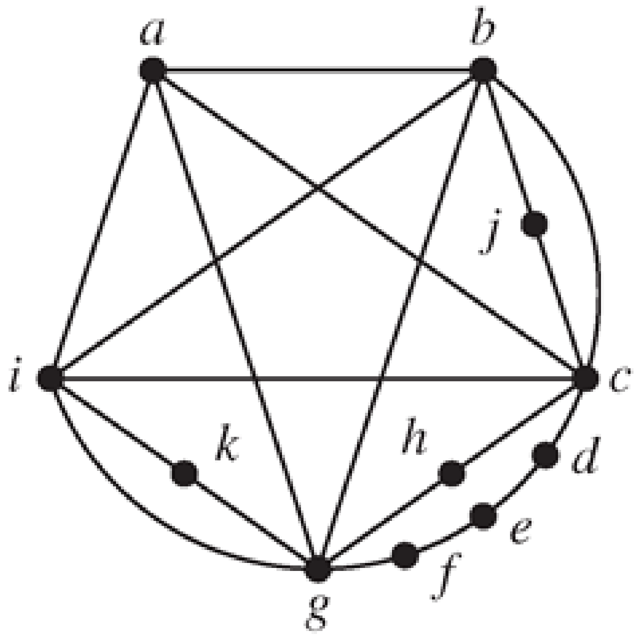
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples



G

Examples



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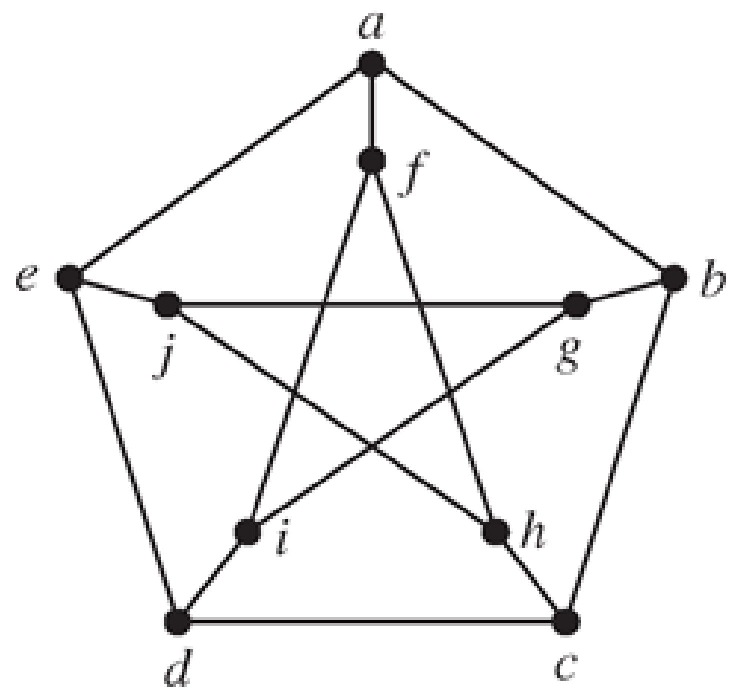

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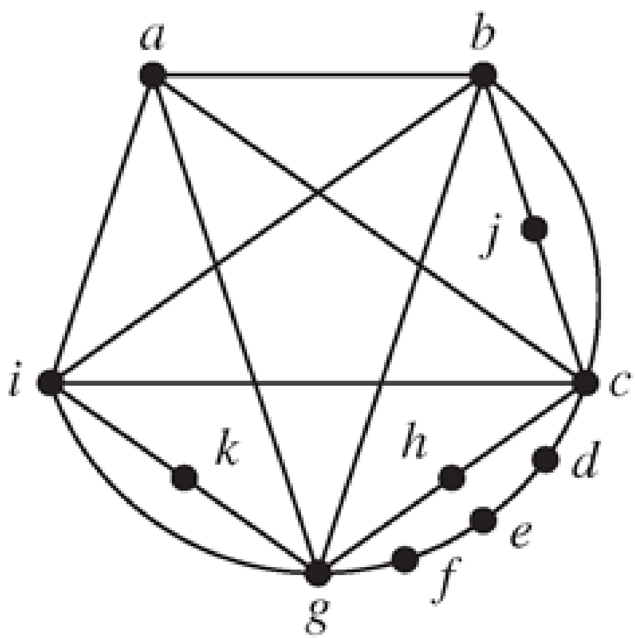

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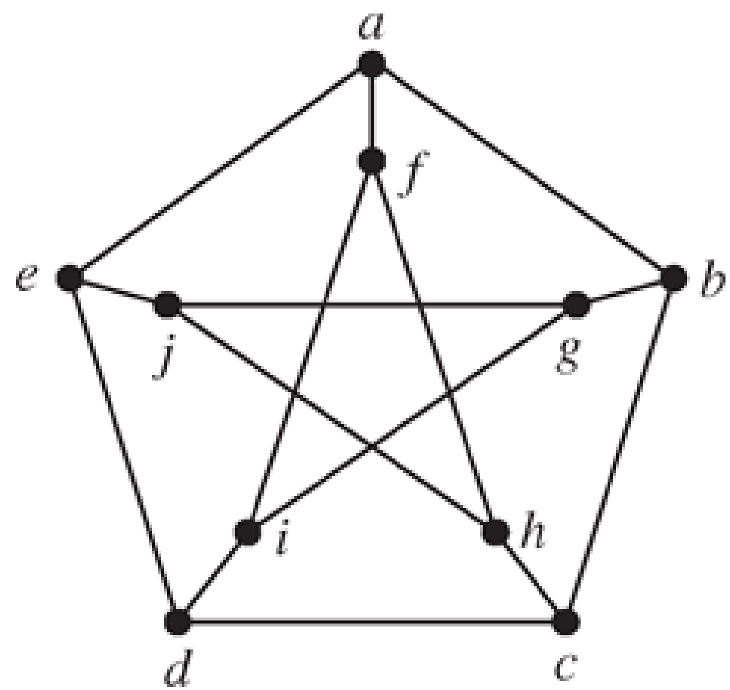
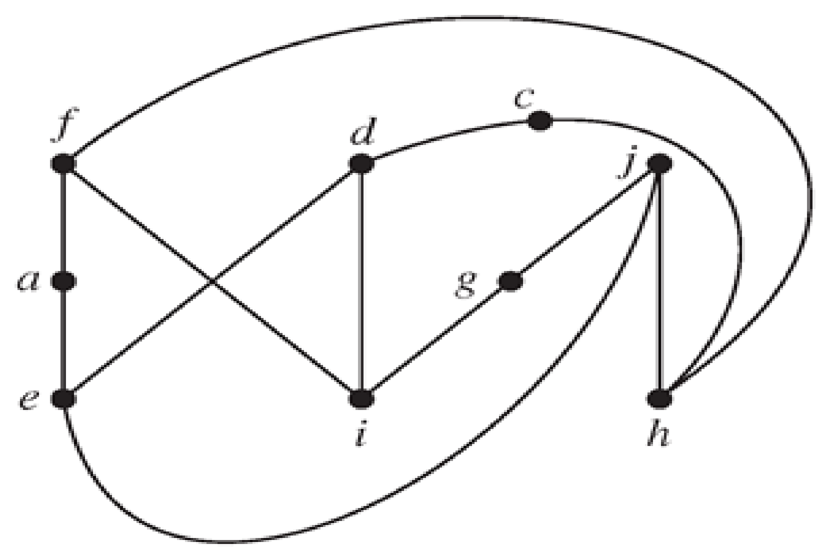
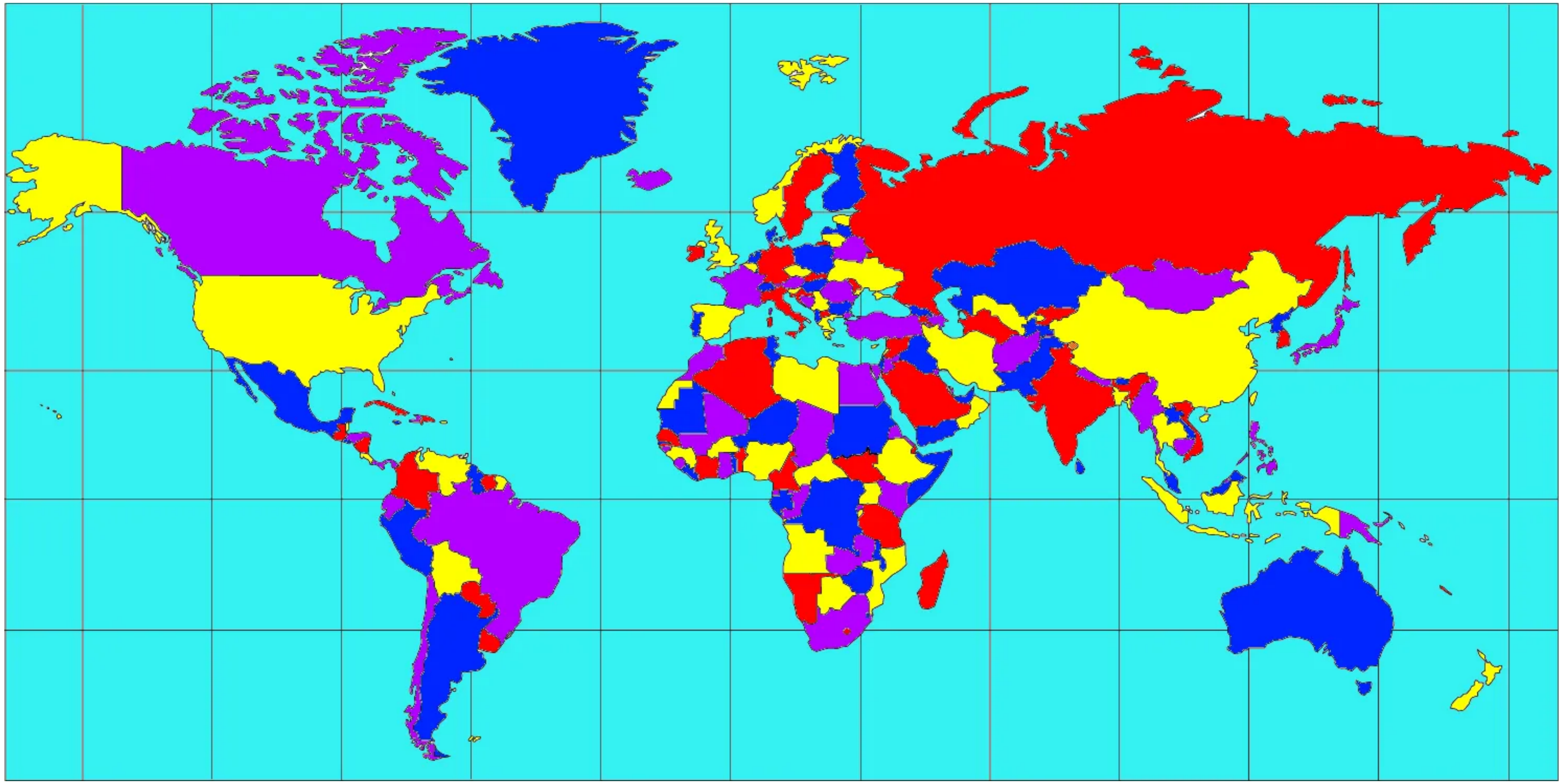




Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



- **Four-color theorem**
  - ◇ first proposed by Francis Guthrie in 1852
  - ◇ his brother Frederick Guthrie told Augustus De Morgan
  - ◇ De Morgan wrote to William Hamilton
  - ◇ Alfred Kempe proved it incorrectly in 1879
  - ◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*
  - ◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)
  - ◇ Kempe's incorrect proof serves as a basis

Graph Coloring

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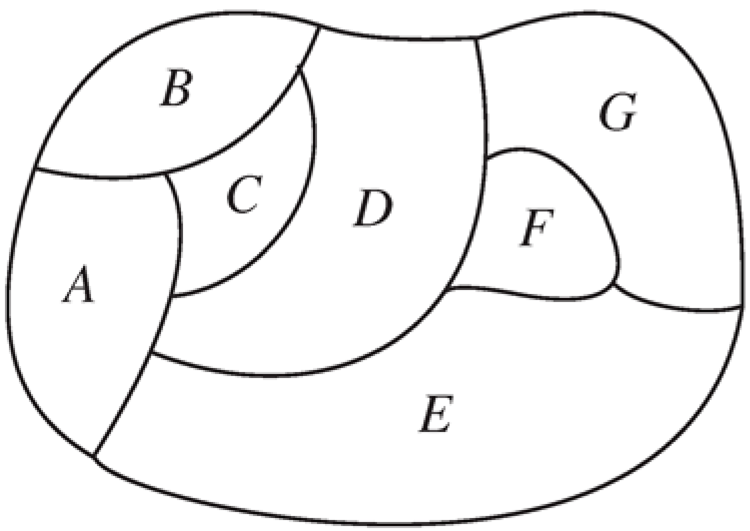

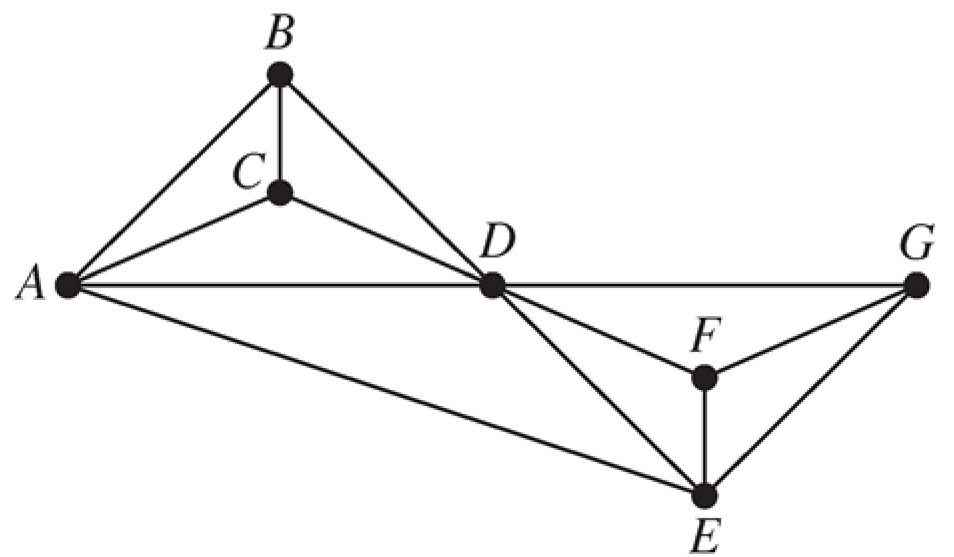

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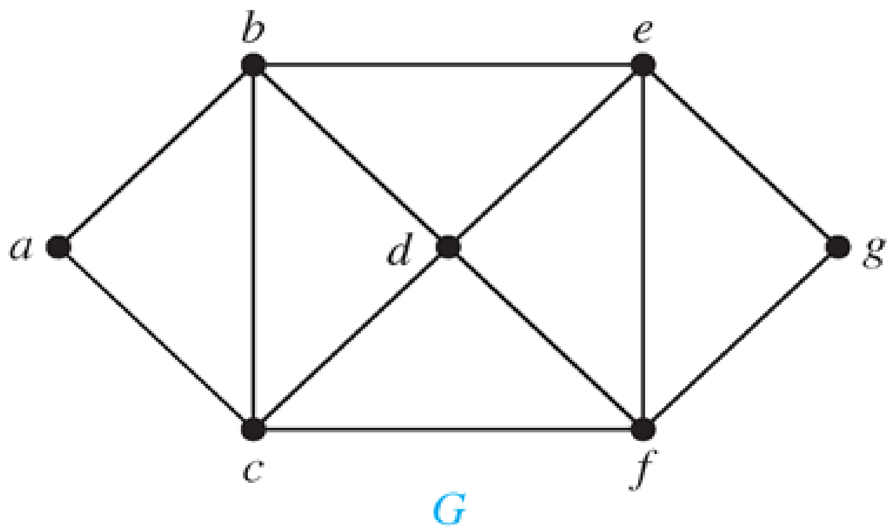

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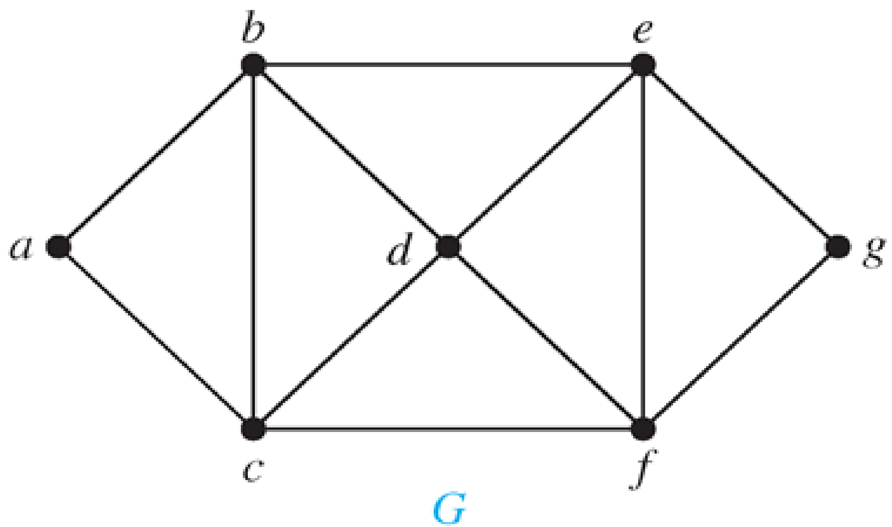

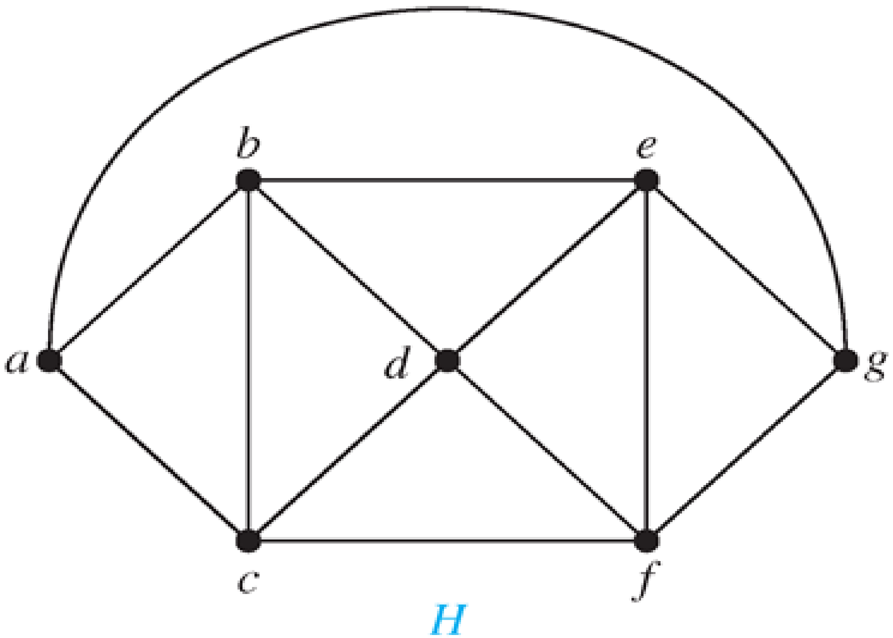

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  Inductive step: Consider a planar graph with $k + 1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in. Since there are at most 5 colors adjacent, so we have at least one color left.

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  Inductive step: Consider a planar graph with $k + 1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 5 colors. Put the vertex back in.

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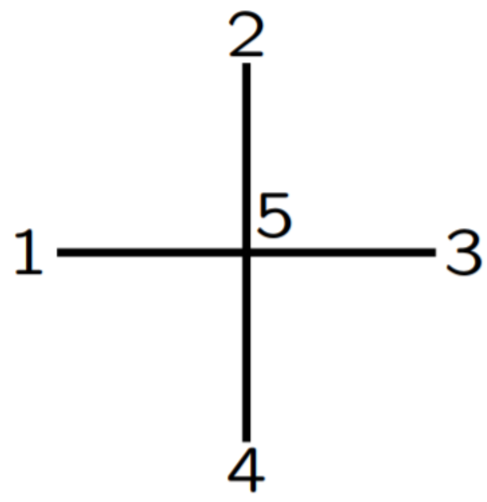
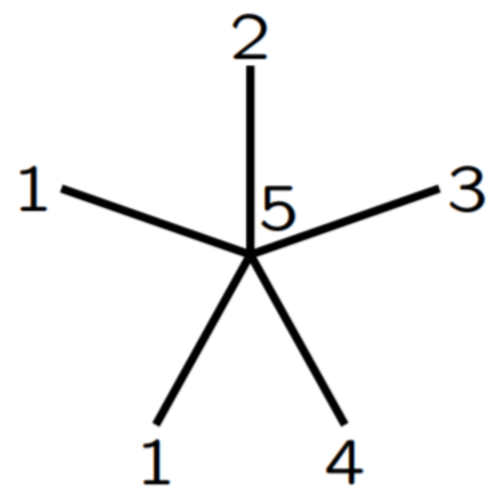
  If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we can pick an available color for it.





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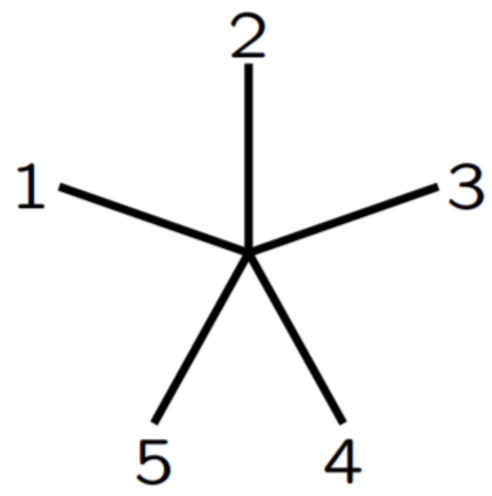
  If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the "special" vertex (degree 5) 1 to 5 (in order).



Graph Coloring

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  We make a subgraph out of all the vertices colored 1 or 3. If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in the subgraph.

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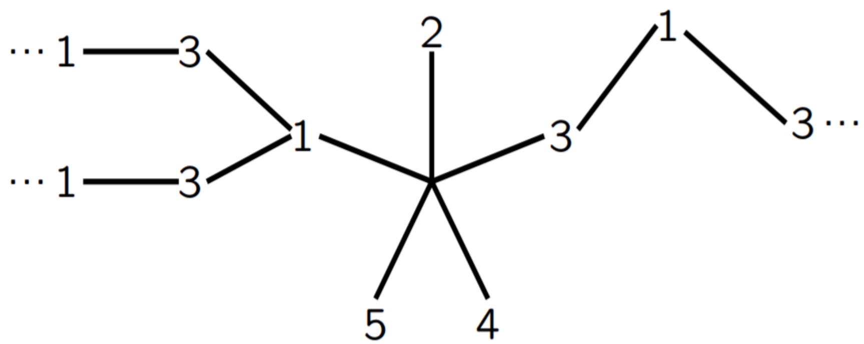

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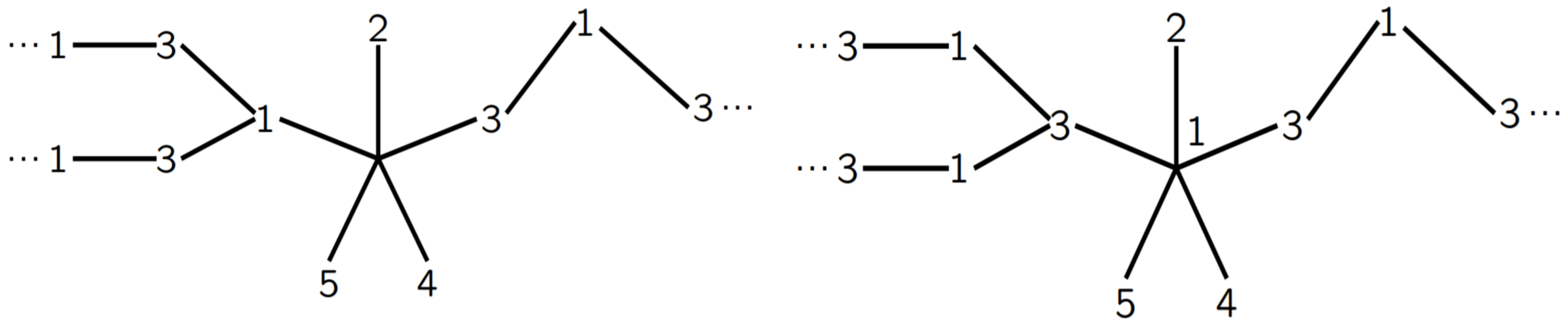

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  On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the the same for the vertices colored 2 and 4. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3 (Why?)

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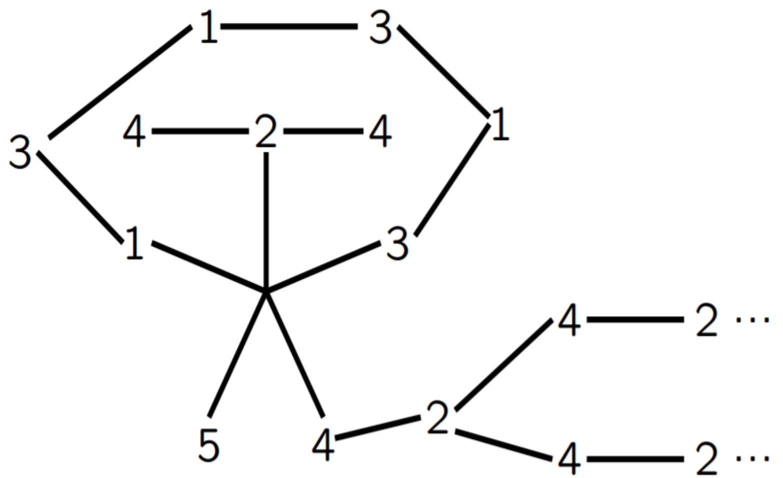

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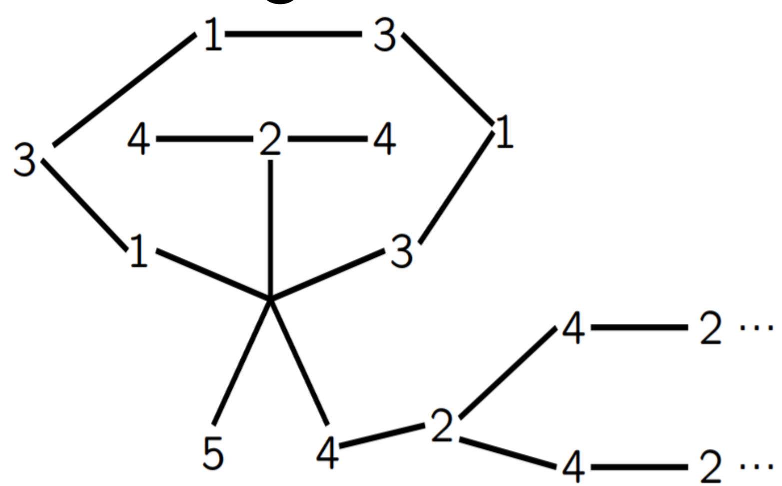

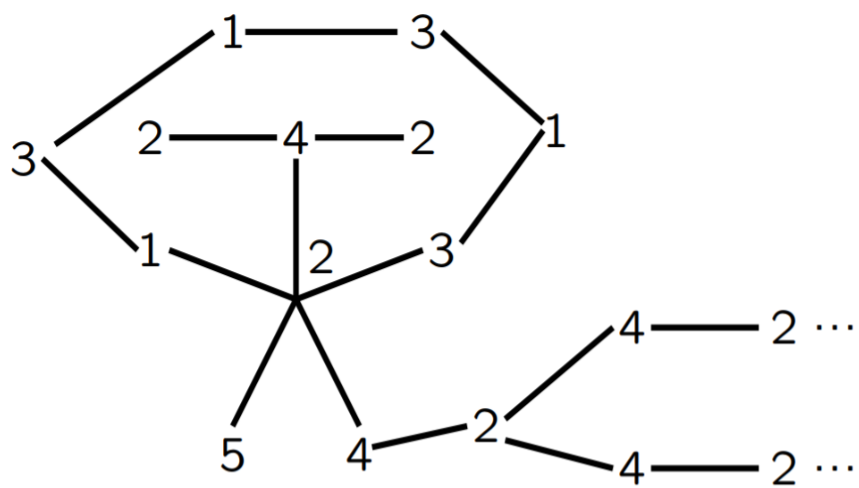

Examples

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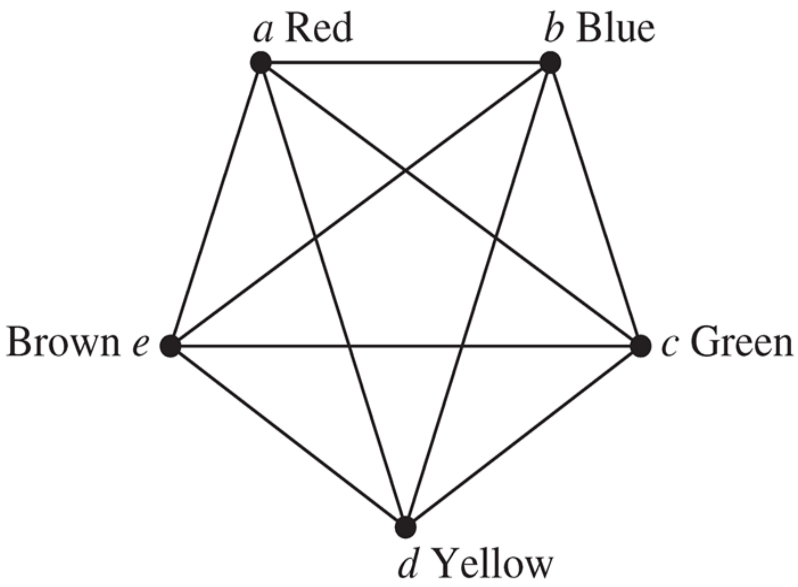

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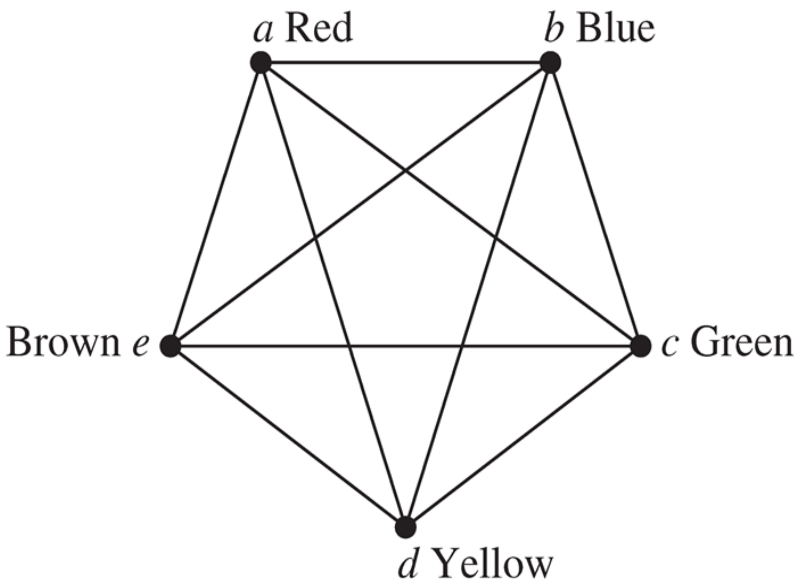

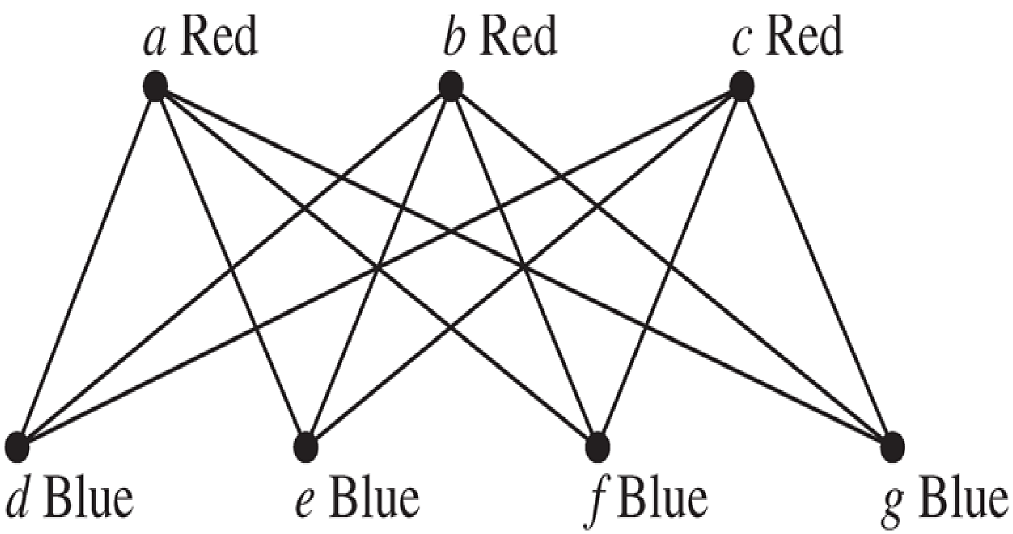

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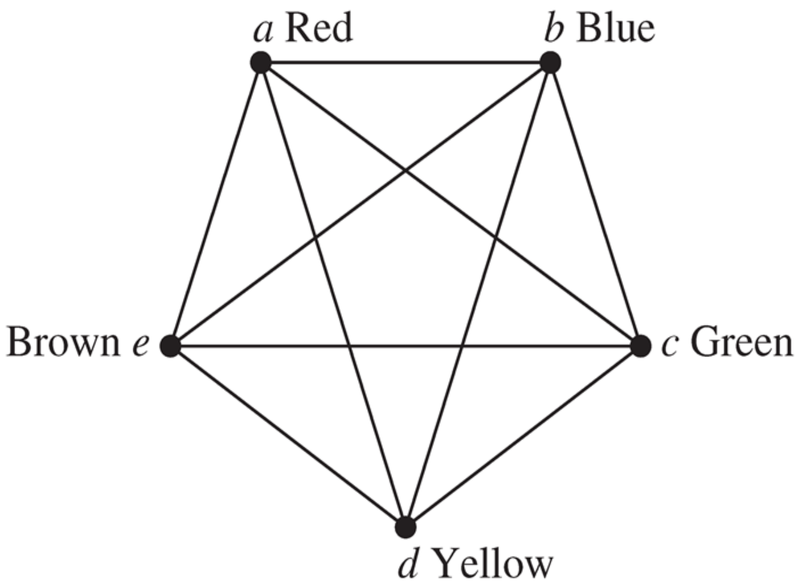

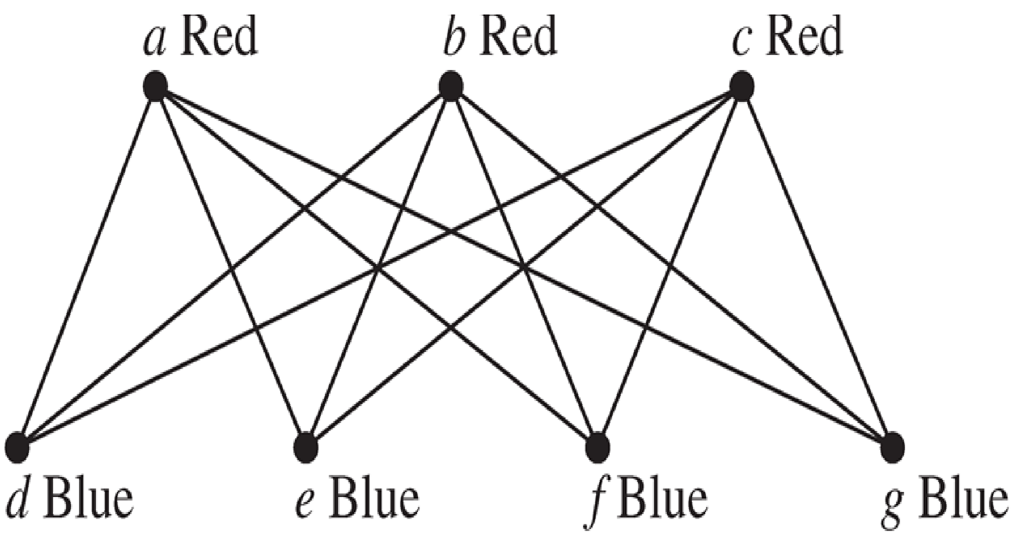

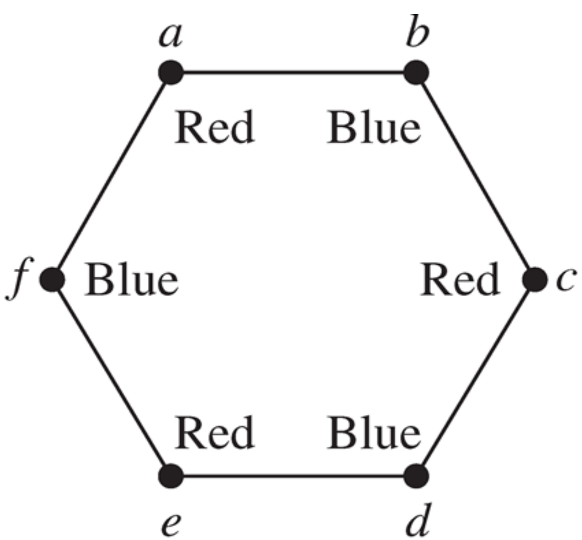

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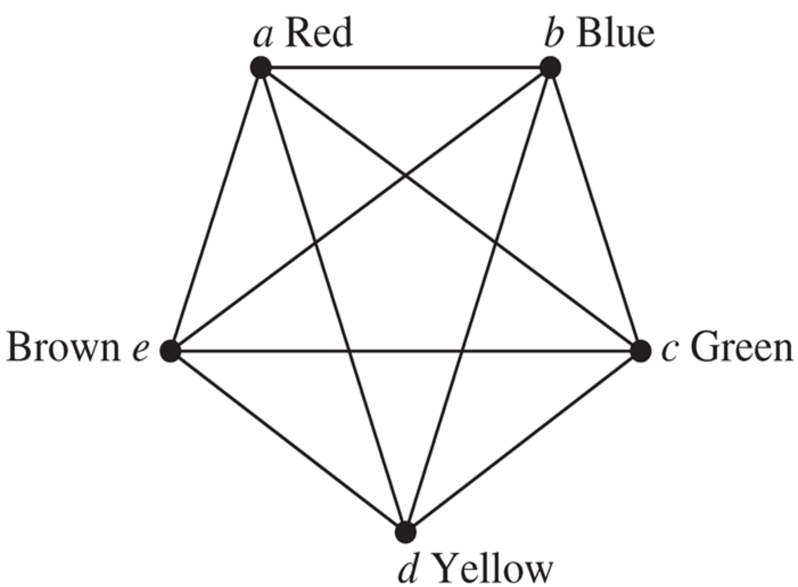
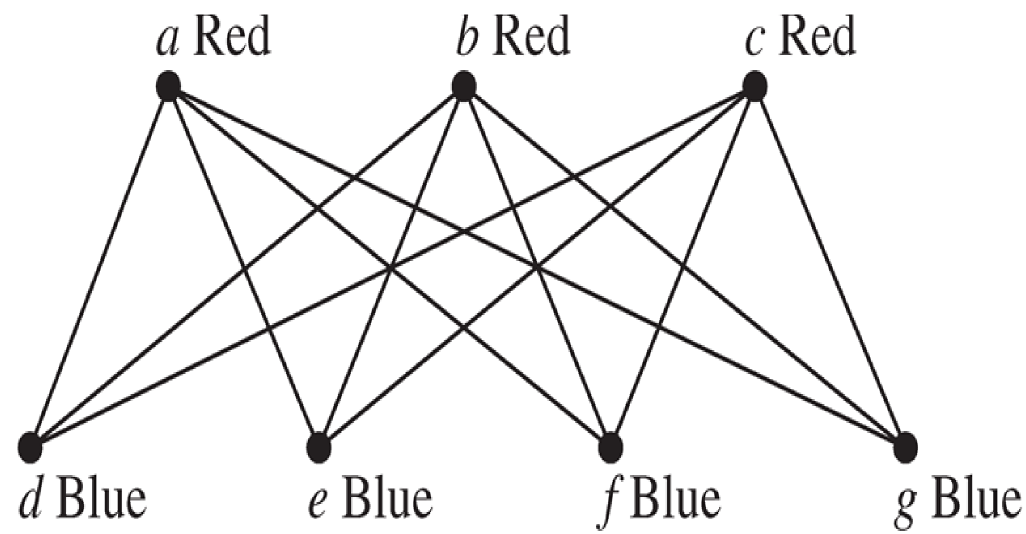
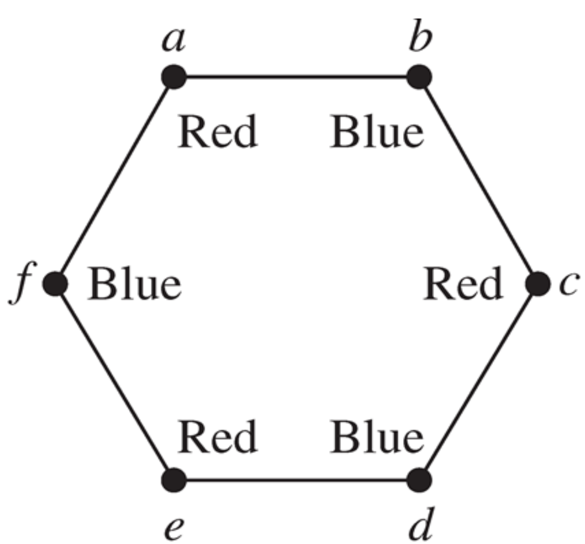
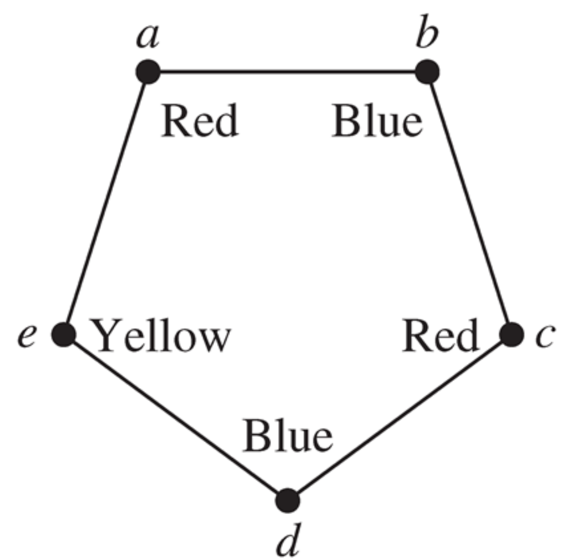




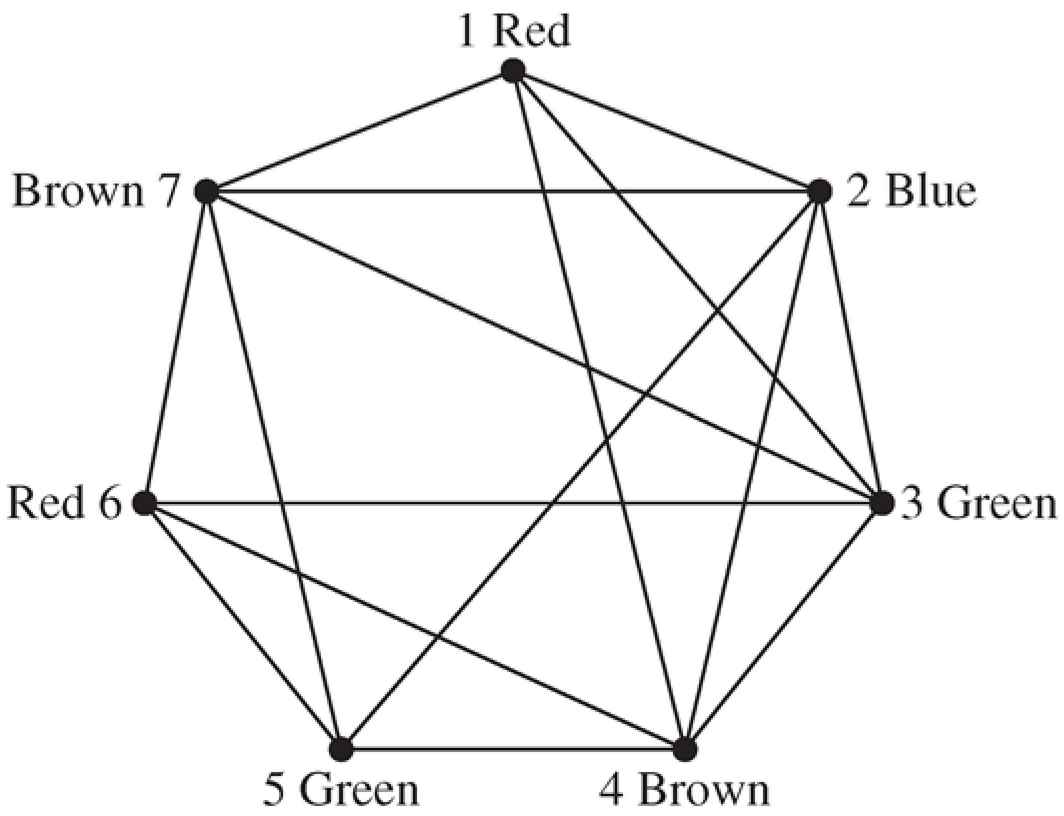




Applications of Graph Coloring

- Scheduling Final Exams

  Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Applications of Graph Coloring

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  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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Graph Coloring $\in$ NPC

Next Lecture

- tree ...