CS215: Discrete Math (H)
2024 Fall Semester Written Assignment # 5
Due: Dec. 23rd, 2024, please submit at the beginning of class

Q.1 Show that a subset of an *antisymmetric* relation is also *antisymmetric*.

Q.2 Define a relation R on $\mathbb{R}$, the set of real numbers, as follows: For all x and y in $\mathbb{R}$, $(x, y) \in R$ if and only if $x - y$ is rational. Answer the followings, and explain your answers.

(1) Is R reflexive?

(2) Is R symmetric?

(3) Is R antisymmetric?

(4) Is R transitive?

Q.3 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Q.4 Suppose that the relation R is symmetric. Show that R^* is symmetric.

Q.5 Prove or give a counterexample to the following: For a set A and a binary relation R on A, if R is reflexive and symmetric, then R must be transitive as well.

Q.6 Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.

Q.7 Let R and S both be *transitive* relations on a set A. For each of the relations below, either prove that it is transitive, or give a counterexample, showing that it may not be transitive.

(1) $R \cap S$

(2) $R \cup S$

(3) $R \circ S$

Q.8

(1) Give an example to show that the transitive closure of the symmetric closure of a relation is not necessarily the same as the symmetric closure of the transitive closure of this relation.

(2) Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Q.9 Let R be the relation on $\mathbb{Z}$, the set of integers, as follows: For all m and n in $\mathbb{Z}$, $(m, n) \in R$ if and only if 3 divides $(m^2 - n^2)$.

(1) Prove that R is an equivalence relation.

(2) Describe the equivalence classes of R.

Q.10 Let S be a finite set and T be a subset of S. We define a binary relation R on the power set $\mathcal{P}(S)$ of set S: for subsets A and B of S, $(A, B) \in R$ if and only if $(A \cup B) \setminus (A \cap B) \subseteq T$. Prove that the relation R is an equivalence relation.

Q.11 Show that the relation R on $\mathbb{Z} \times \mathbb{Z}$ defined on $(a, b)\mathbb{R}(c, d)$ if and only if $a + d = b + c$ is an *equivalence* relation.

Q.12 Let $\sim$ be a relation defined on $\mathbb{N}$ by the rule that $x \sim y$ if $x = 2^k y$ or $y = 2^k x$ for some $k \in \mathbb{N}$. Show that $\sim$ is an equivalence relation.

Q.13 Which of these are posets?

(a) $(\mathbf{Z}, =)$

(b) $(\mathbf{Z}, \neq)$

(c) $(\mathbf{Z}, \geq)$

(d) $(\mathbf{Z}, \nmid)$

Q.14 Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Q.15 The relation R on the set $X = \{(a, b, c) : a, b, c \in \mathbb{N}\}$ with $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ if and only if $2^{a_1}3^{b_1}5^{c_1} \leq 2^{a_2}3^{b_2}5^{c_2}$.

(1) Prove that R is a partial ordering.

(2) Write two comparable and two incomparable elements if they exist.

(3) Find the least upper bound and the greatest lower bound of the two elements $(5, 0, 1)$ and $(1, 1, 2)$.

(4) List a minimal and a maximal element if they exist.

Q.16 Define the relation $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ according to

$$(a, b) \preceq (c, d) \Leftrightarrow (a, b) = (c, d) \text{ or } a^2 + b^2 < c^2 + d^2.$$

Show that $(\mathbb{Z}\times\mathbb{Z}, \preceq)$ is a poset; Construct the Hasse diagram for the subposet $(B, \preceq)$, where $B = \{0, 1, 2\} \times \{0, 1, 2\}$.

Q.17 We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0, 1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0, 1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.

Q.18 Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

(1) Find the maximal elements.

(2) Find the minimal elements.

(3) Is there a greatest element?

(4) Is there a least element?

(5) Find all upper bounds of $\{3, 5\}$.

(6) Find the least upper bound of $\{3, 5\}$, if it exists.

(7) Find all lower bounds of $\{15, 45\}$.

(8) Find the greatest lower bound of $\{15, 45\}$, if it exists.