CS215 DISCRETE MATH

Dr. QI WANG

Department of Computer Science and Engineering
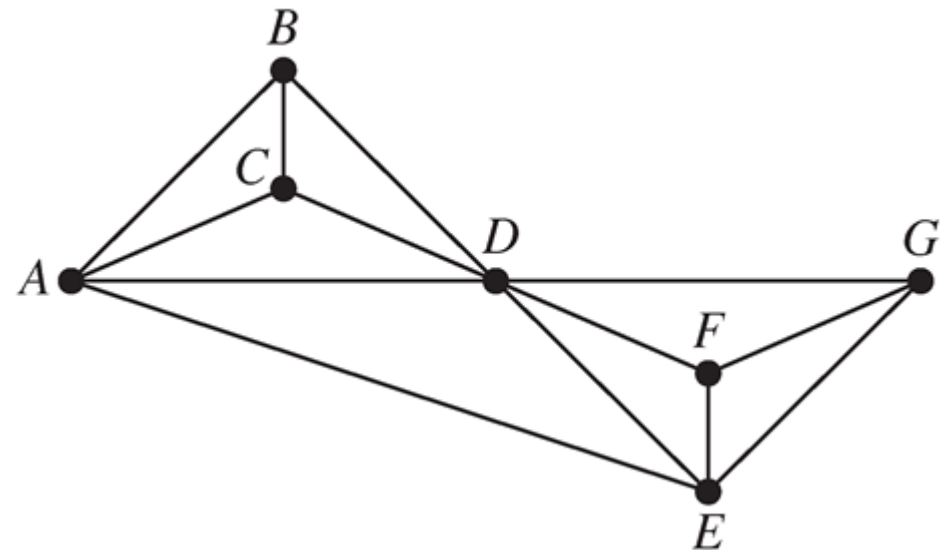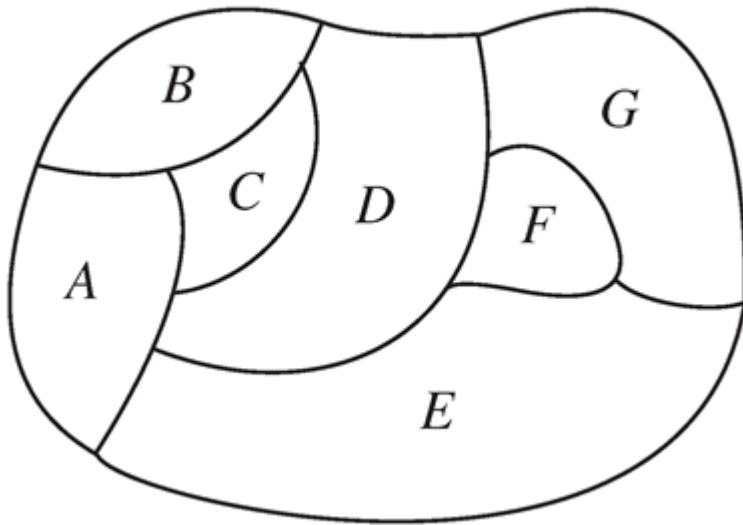Office: Room413, CoE South Tower
Email: wangqi@sustech.edu.cn

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

  The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph, denoted by $\chi(G)$.
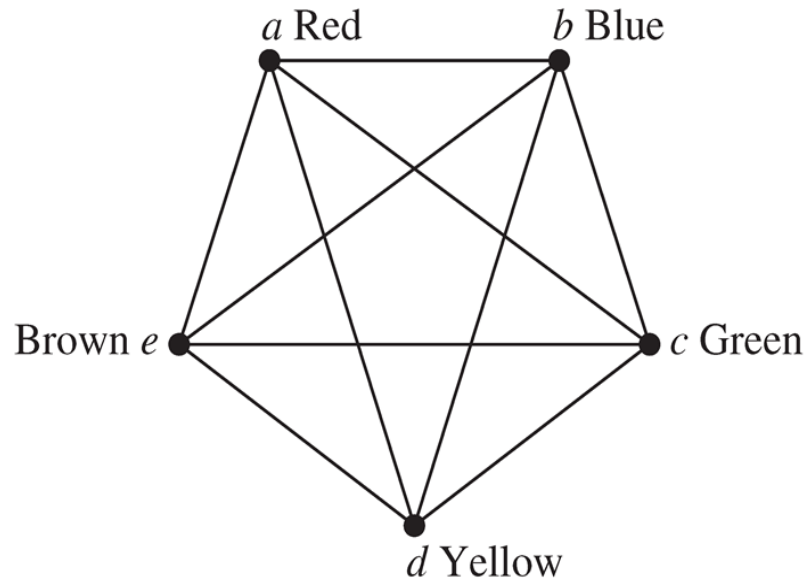
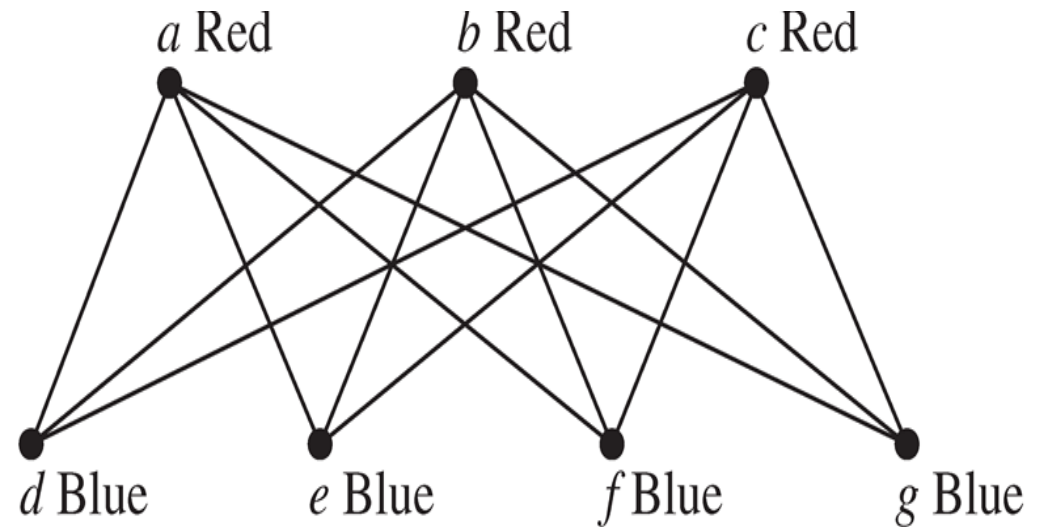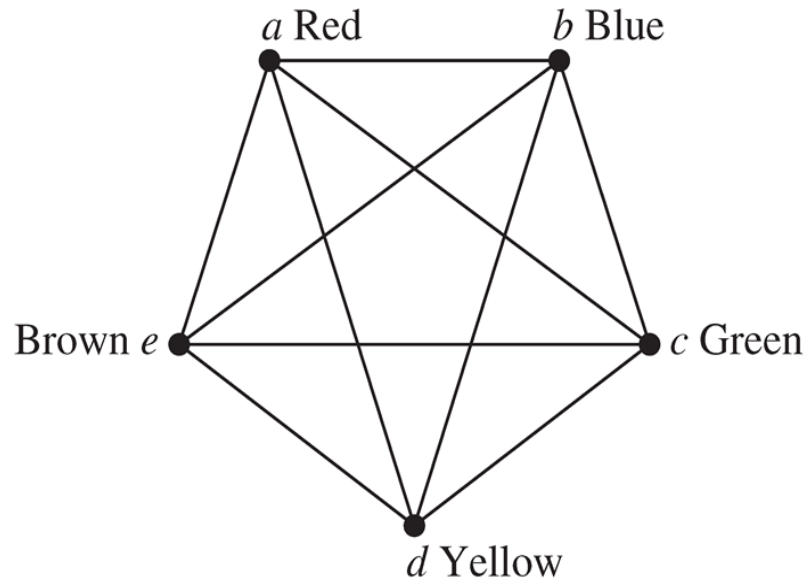- What is the chromatic number of K_n, $K_{m,n}$, C_n?

  完成图染色需要的最少的染色数目

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二分图染色数一定是2

- What is the chromatic number of K_n, $K_{m,n}$, C_n?

a Red   *b* Blue

Brown *e*   *c* Green
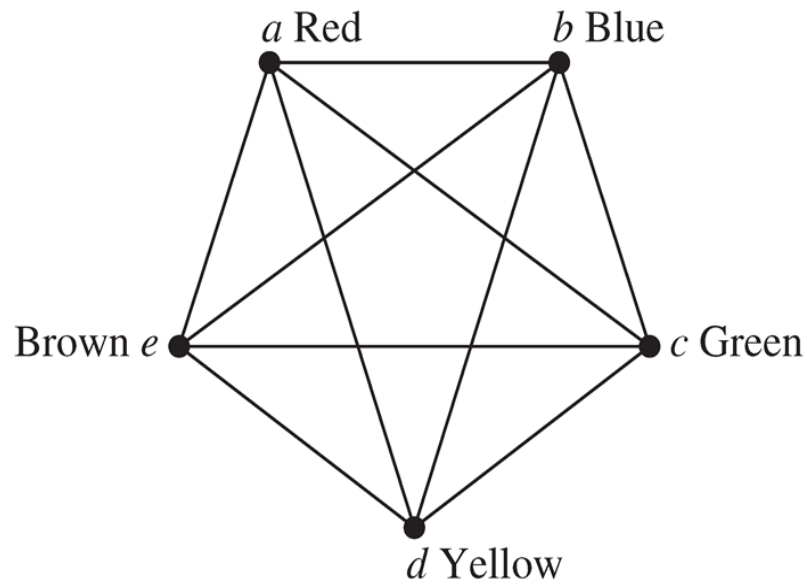
d Yellow

a Red   *b* Red   *c* Red

d Blue   *e* Blue   *f* Blue   *g* Blue

轮图 取决于是奇数还是偶数，2或3

a   *b*

Red   Blue

f Blue   Red *c*

Red   Blue

e   *d*

a   *b*

Red   Blue

e Yellow   Red *c*

Blue

d
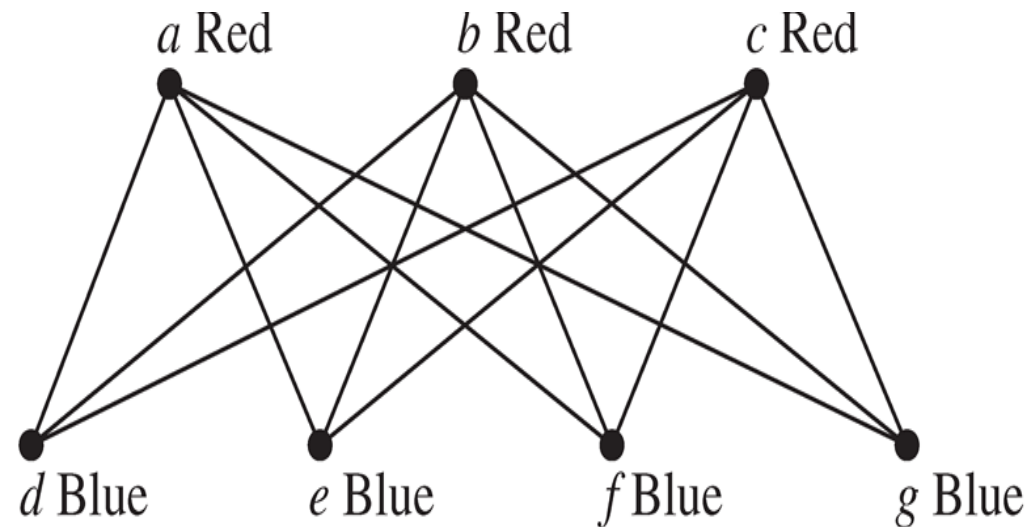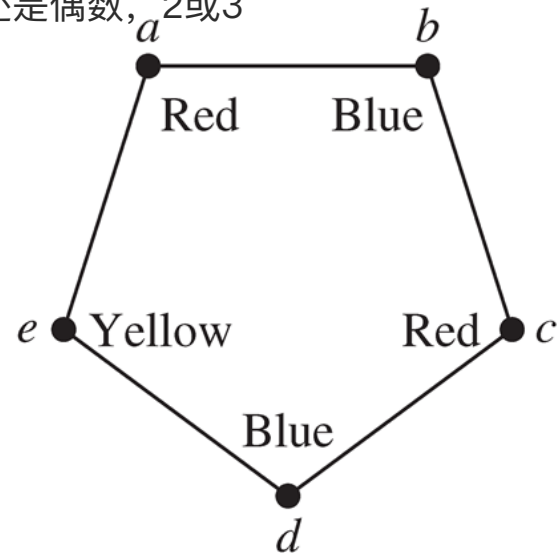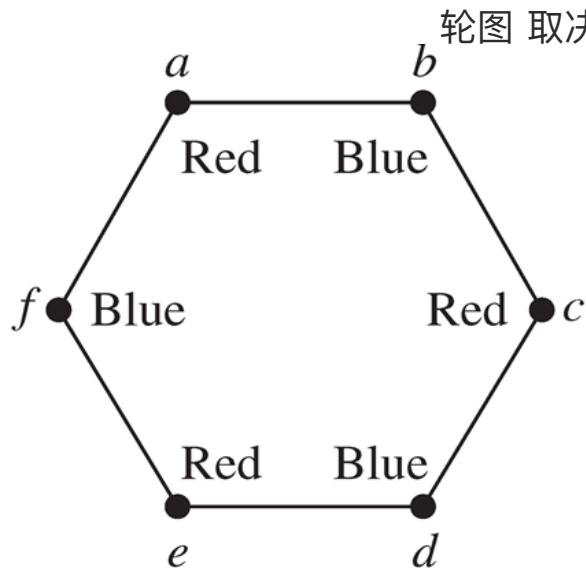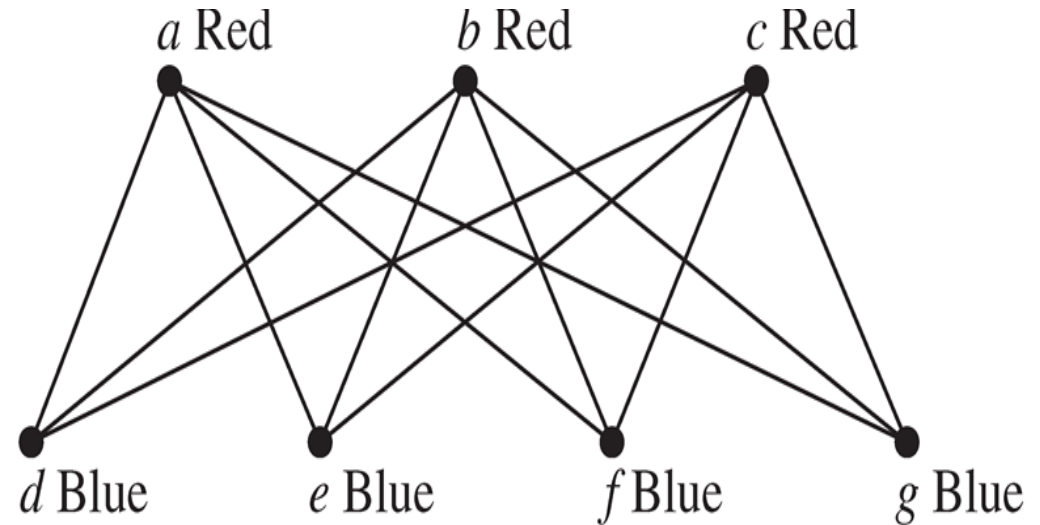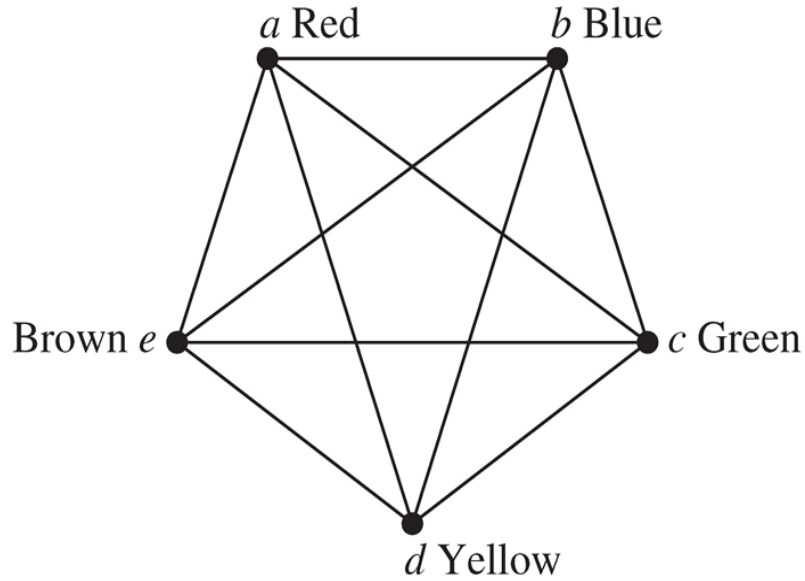
■ **Scheduling Final Exams**

Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.
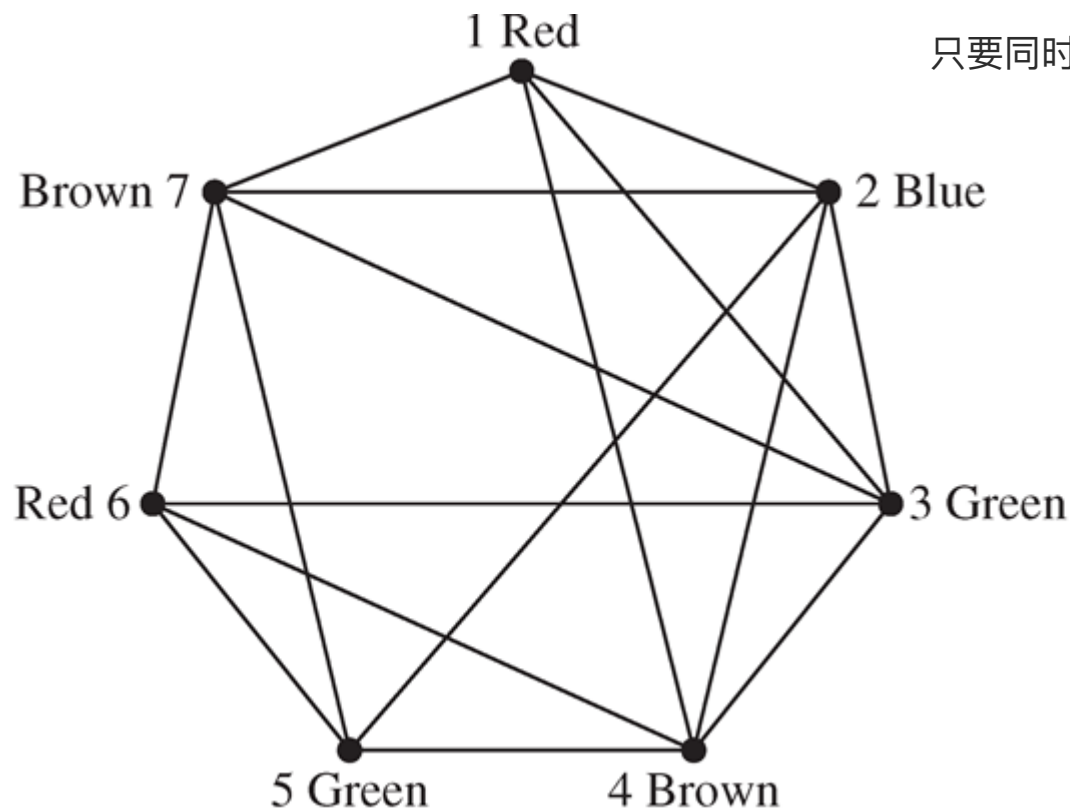
只要同时选择这两门课程，就连一条边

Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

4

Applications of Graph Coloring

- **Channel Assignments**

   Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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    k>=3

    Graph Coloring $\in$ NPC

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连通，无回路
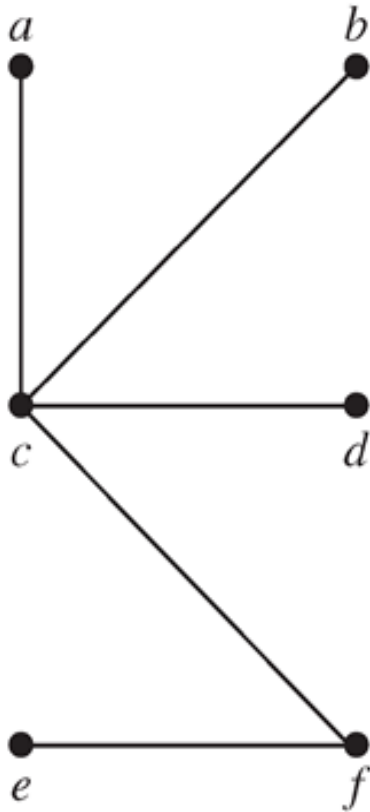
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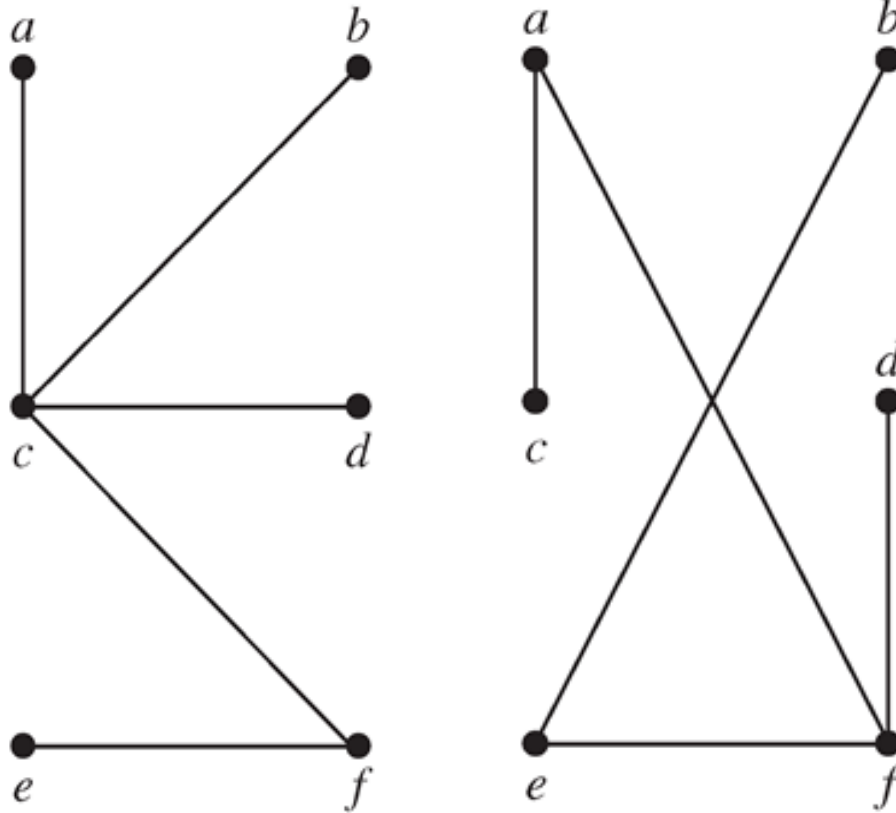
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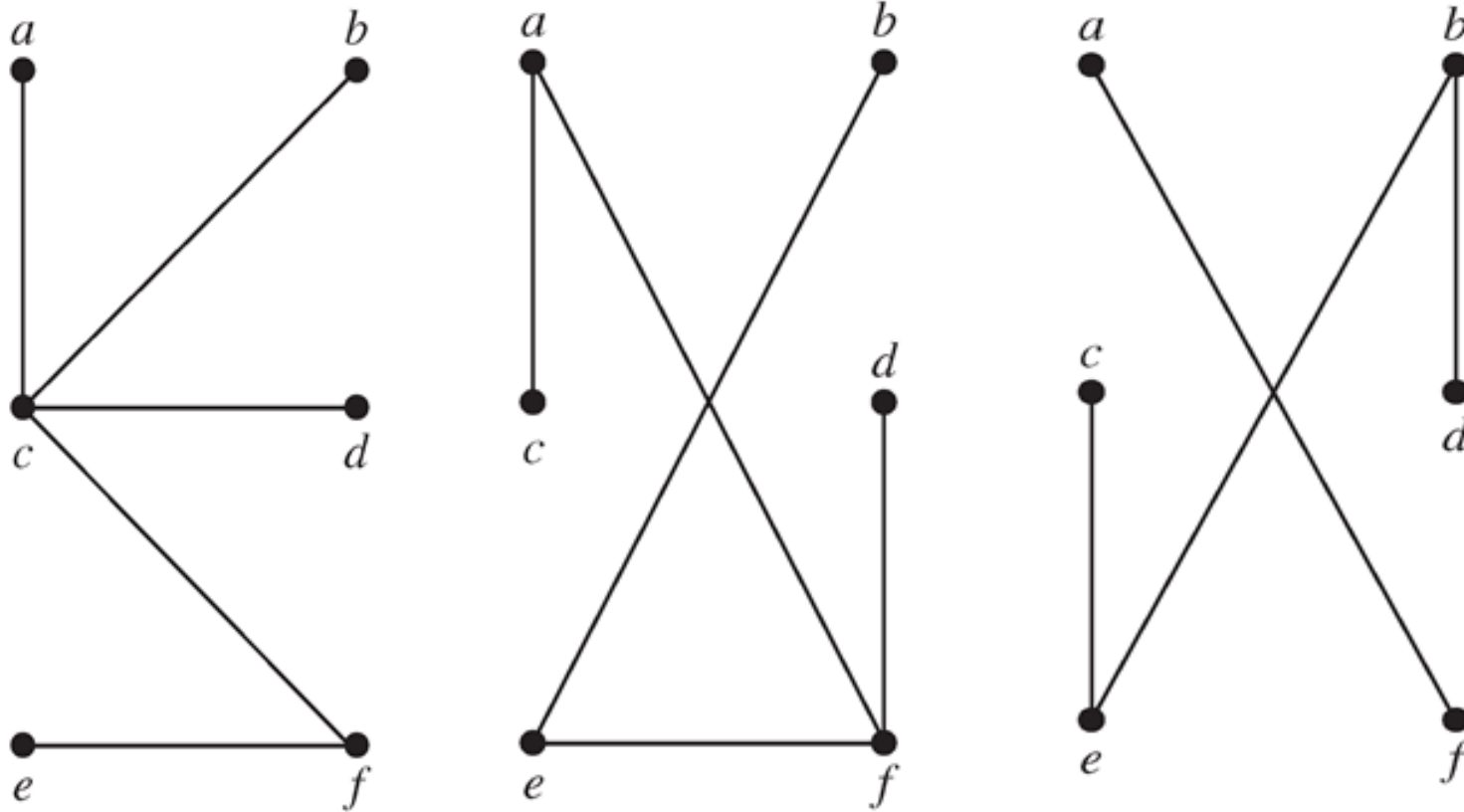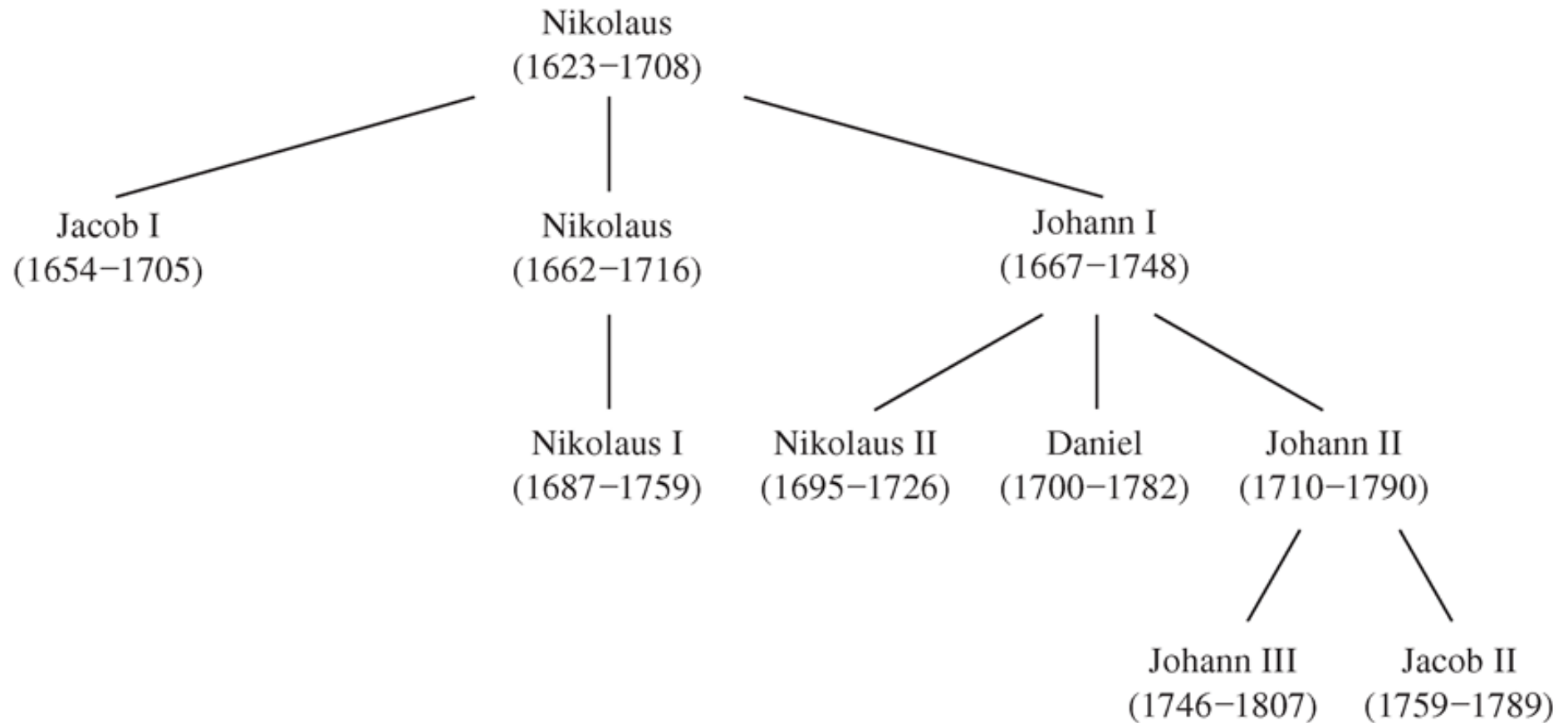
- **Definition** A *tree* is a connected undirected graph with no simple circuits.

- **Theorem** An undirected graph is a tree if and only if there is a <mark>unique simple path between any two of its vertices.</mark>

  任意两个顶点间有唯一路径

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  Proof

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Two properties of tree: connected, no circuit

Rooted Trees

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- *parent, child, sibling*

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  ancestor, descendant
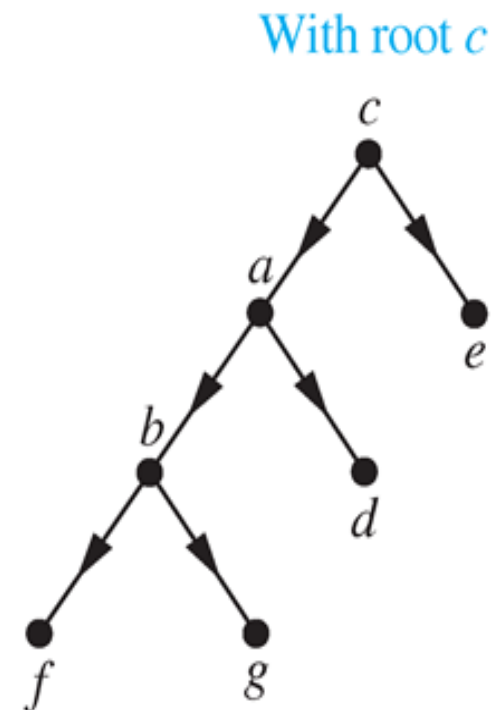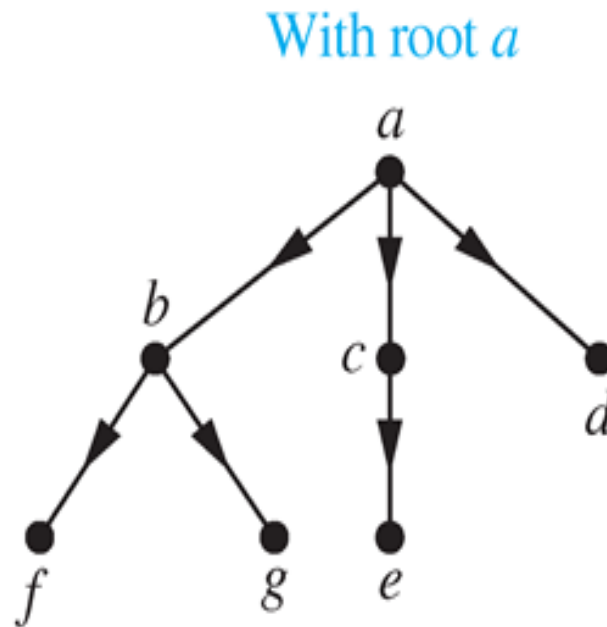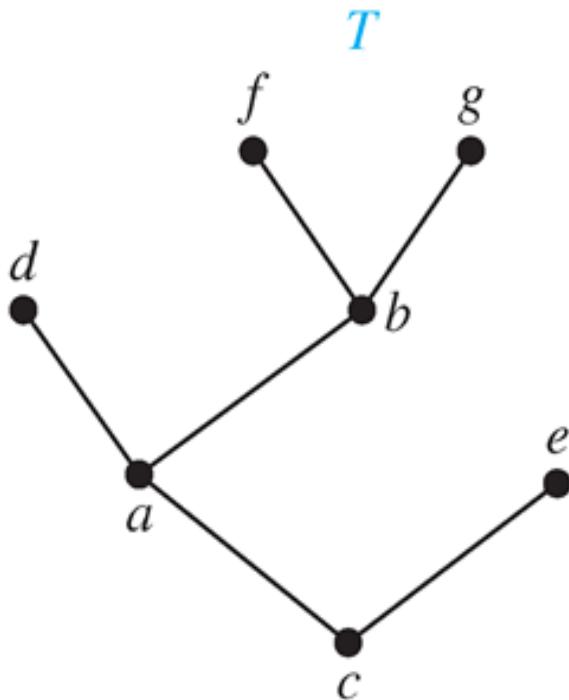
Rooted Trees

- *parent, child, sibling*

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  subtree with a as its root: consists of a and its descendants and all edges incident to these descendants

最多m个子节点

- **Definition** A rooted tree is called an *m-ary tree* if every internal vertex has <mark>no more than *m* children</mark>. The tree is called a *full m-ary tree* if every internal vertex has exactly *m* children. In particular, an *m*-ary tree with $m = 2$ is called a *binary tree*.

刚好每个都有m个子节点

full：有子节点的话必须满
complete：填充的顺序从上到下，从左到右
balanced：

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  left subtree, *right subtree*

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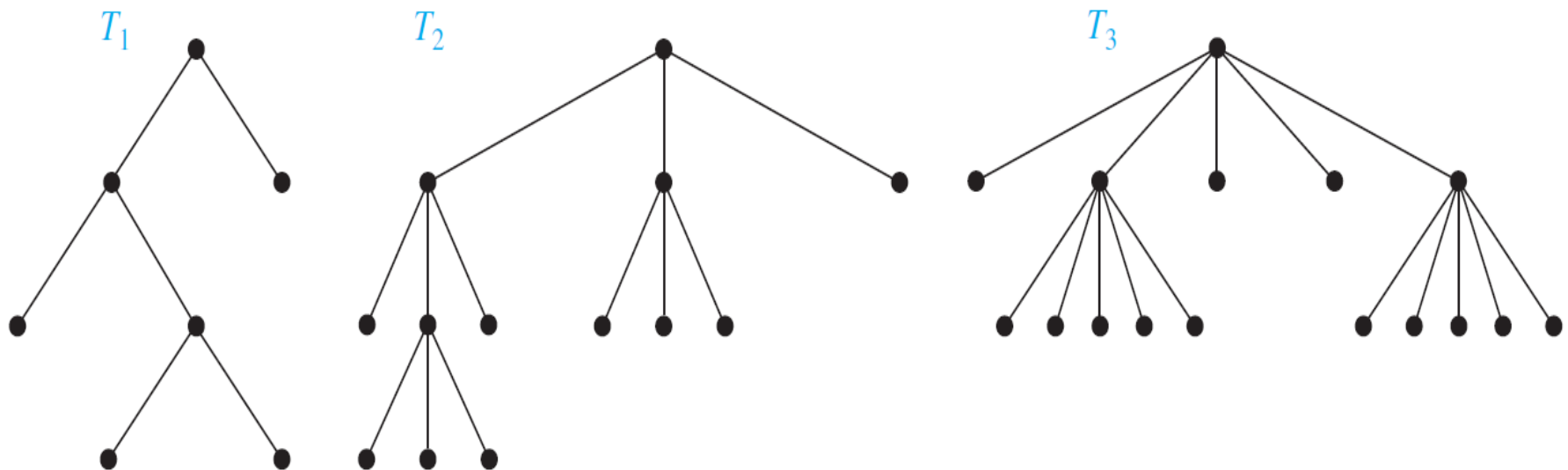
- **Theorem** A full *m*-ary tree with *i* internal vertices has $n = mi + 1$ vertices.

  多一个root

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  (i) n vertices, $i = (n-1)/m$, $\ell = [(m-1)n+1]/m$
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  (iii) ℓ leaves, $n = (m\ell - 1)/(m-1)$, $i = (\ell-1)/(m-1)$

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  using $n = mi + 1$ and $n = i + \ell$

Level and Height

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- The *level* of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.

  The *height* of a rooted tree is the maximum of the levels of the vertices.

  Definition A rooted m-ary tree of height h is *balanced* if all leaves are at levels h or $h - 1$. (differ no greater than 1)

- **Theorem** There are at most m^h leaves in an m-ary tree of height h.

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- **Definition** A *binary tree* is an ordered rooted tree where each internal tree has two children, the first is called the *left child* and the second is the *right child*. The tree rooted at the left child of a vertex is called the *left subtree* of this vertex, and the tree rooted at the right child of a vertex is called the *right subtree* of this vertex.

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  The three most commonly used traversals are *preorder traversal*, *inorder traversal*, *postorder traversal*.

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *preorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *preorder traversal* begins by visiting r, and continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.
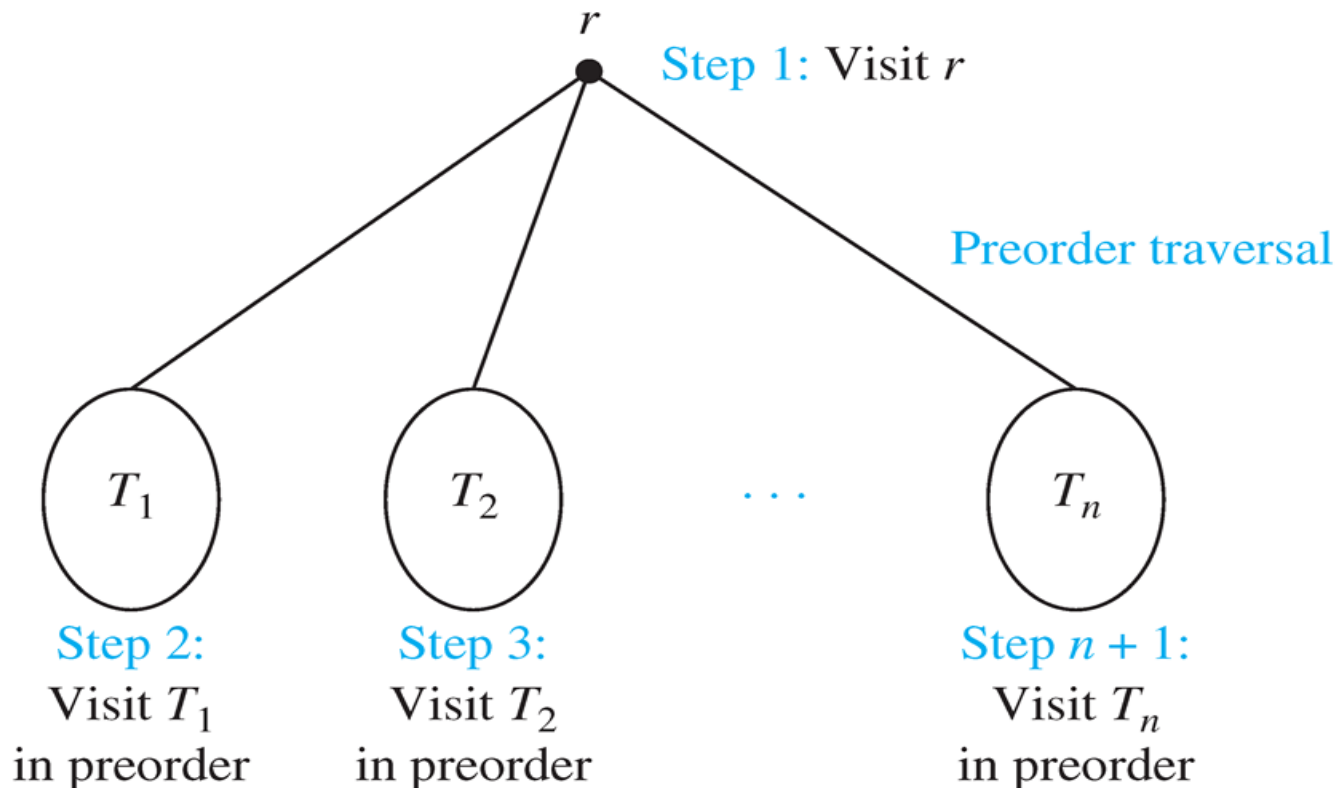
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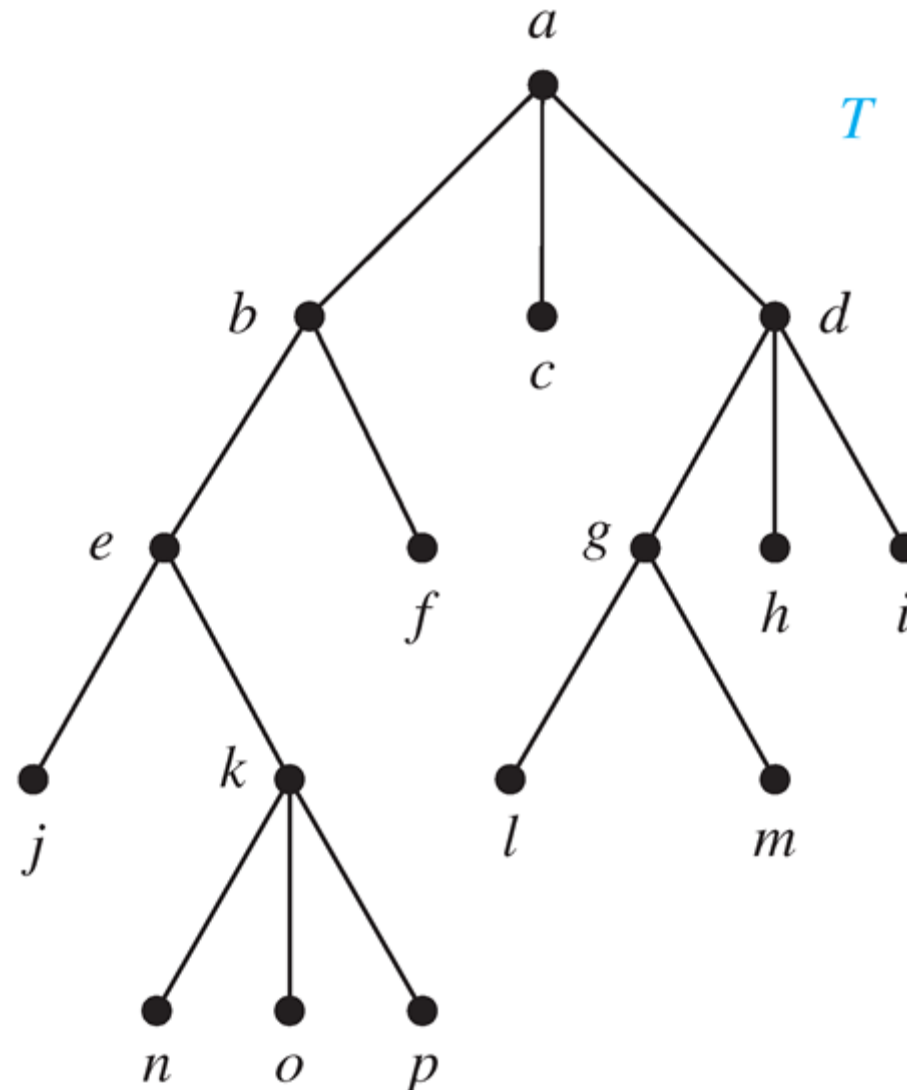

Step 1: Visit r

Preorder traversal

T_1 T_2 $\ldots$ T_n

Step 2: Visit T_1 in preorder   Step 3: Visit T_2 in preorder   Step $n+1$: Visit T_n in preorder

- **Example**

procedure *preorder* (T: ordered rooted tree)
$r :=$ root of T
list r
for each child c of r from left to right
    $T(c) :=$ subtree with c as root
    preorder($T(c)$)

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *inorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *inorder traversal* begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.
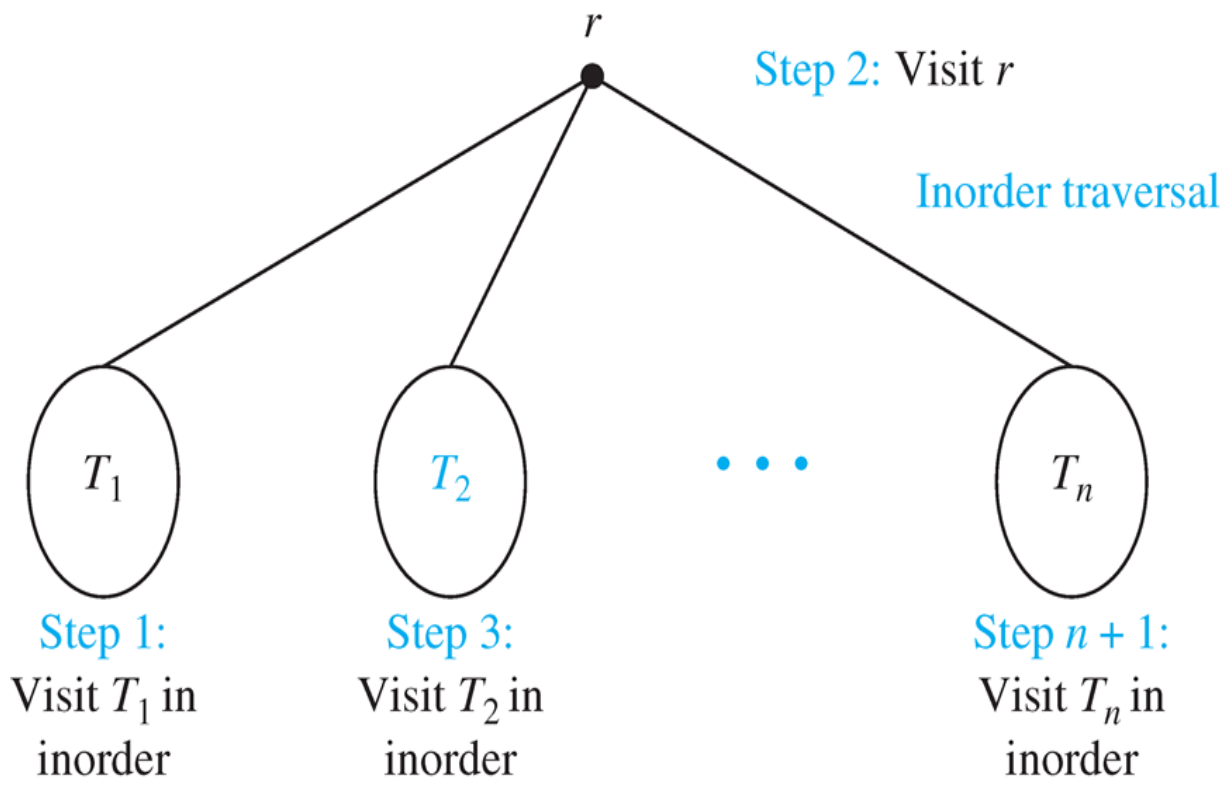
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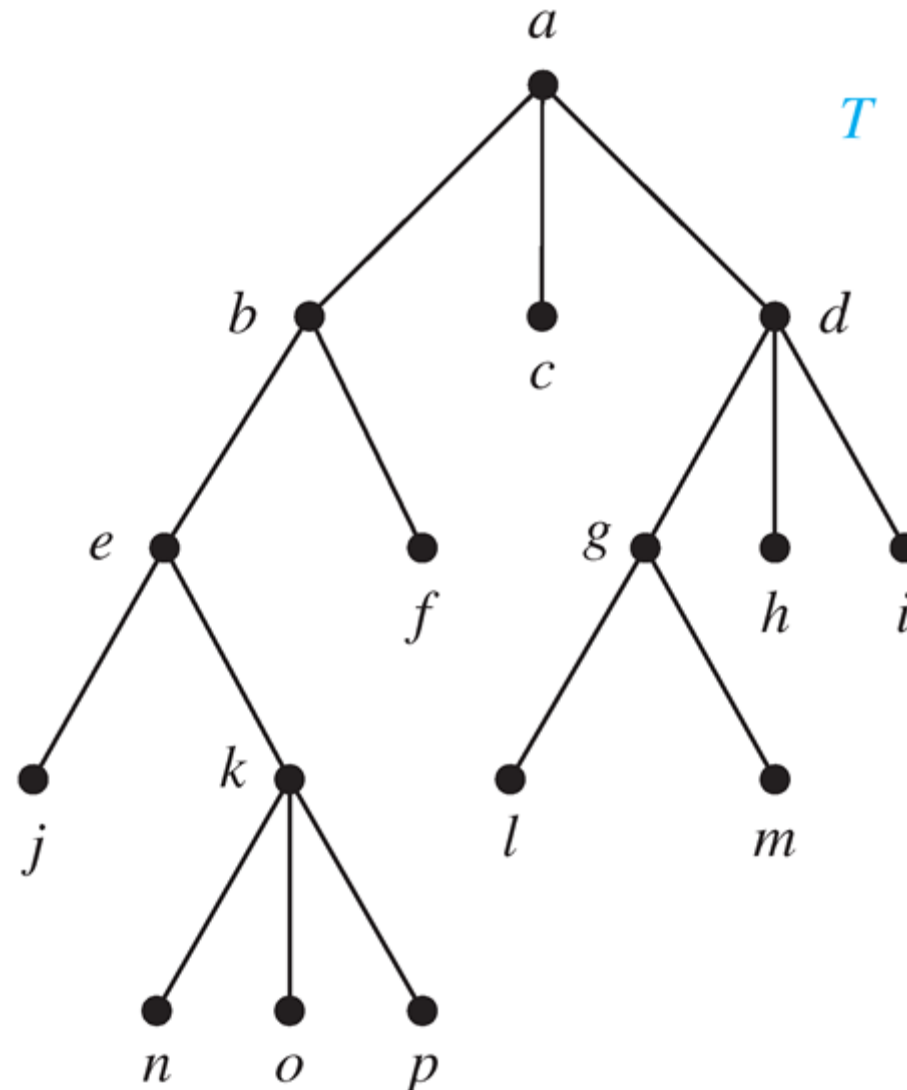Step 2: Visit r

Inorder traversal

T_1	T_2	T_n
Step 1: Visit T_1 in inorder	Step 3: Visit T_2 in inorder	Step $n+1$: Visit T_n in inorder

- **Example**

procedure *inorder* (*T*: ordered rooted tree)
r := root of *T*
if *r* is a leaf **then** list *r*
else
    l := first child of *r* from left to right
    T(*l*) := subtree with *l* as its root
    inorder(*T*(*l*))
    list(*r*)
    for each child *c* of *r* from left to right
        T(*c*) := subtree with *c* as root
        inorder(*T*(*c*))

- **Definition** Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *postorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The *postorder traversal* begins by traversing T_1 in postorder, then T_2 in postorder, and so on, after T_n is traversed in postorder, r is visited.
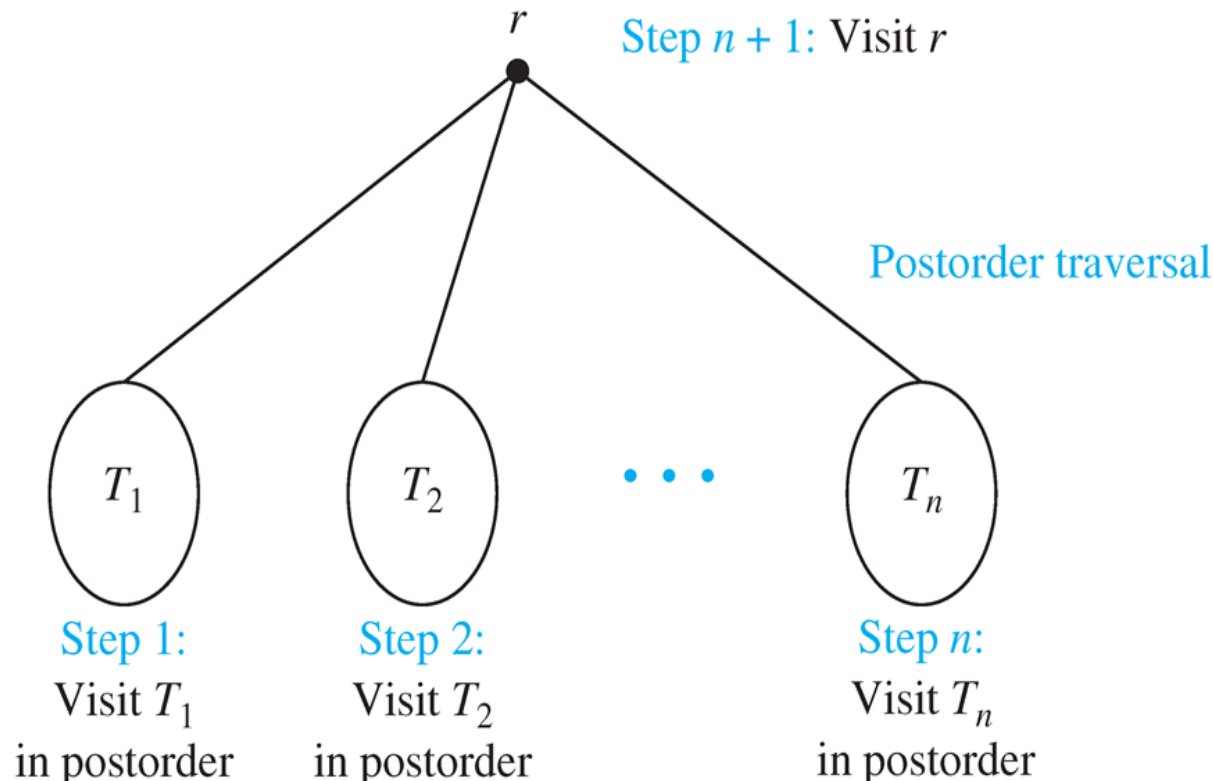
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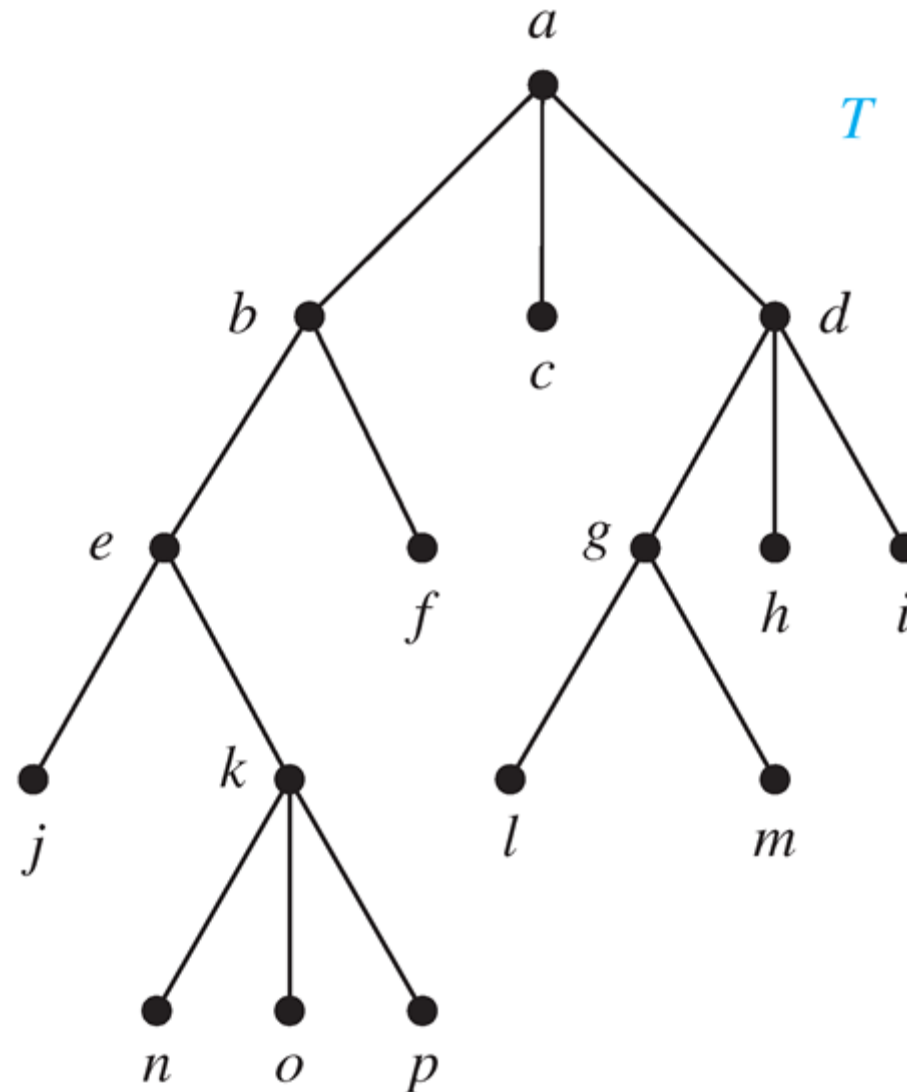

23 - 2

- **Example**

procedure *postordered* (T: ordered rooted tree)

$r :=$ root of T

for each child c of r from left to right

    $T(c) :=$ subtree with c as root

    postorder($T(c)$)

list r

- Complex expressions can be represented using ordered rooted trees
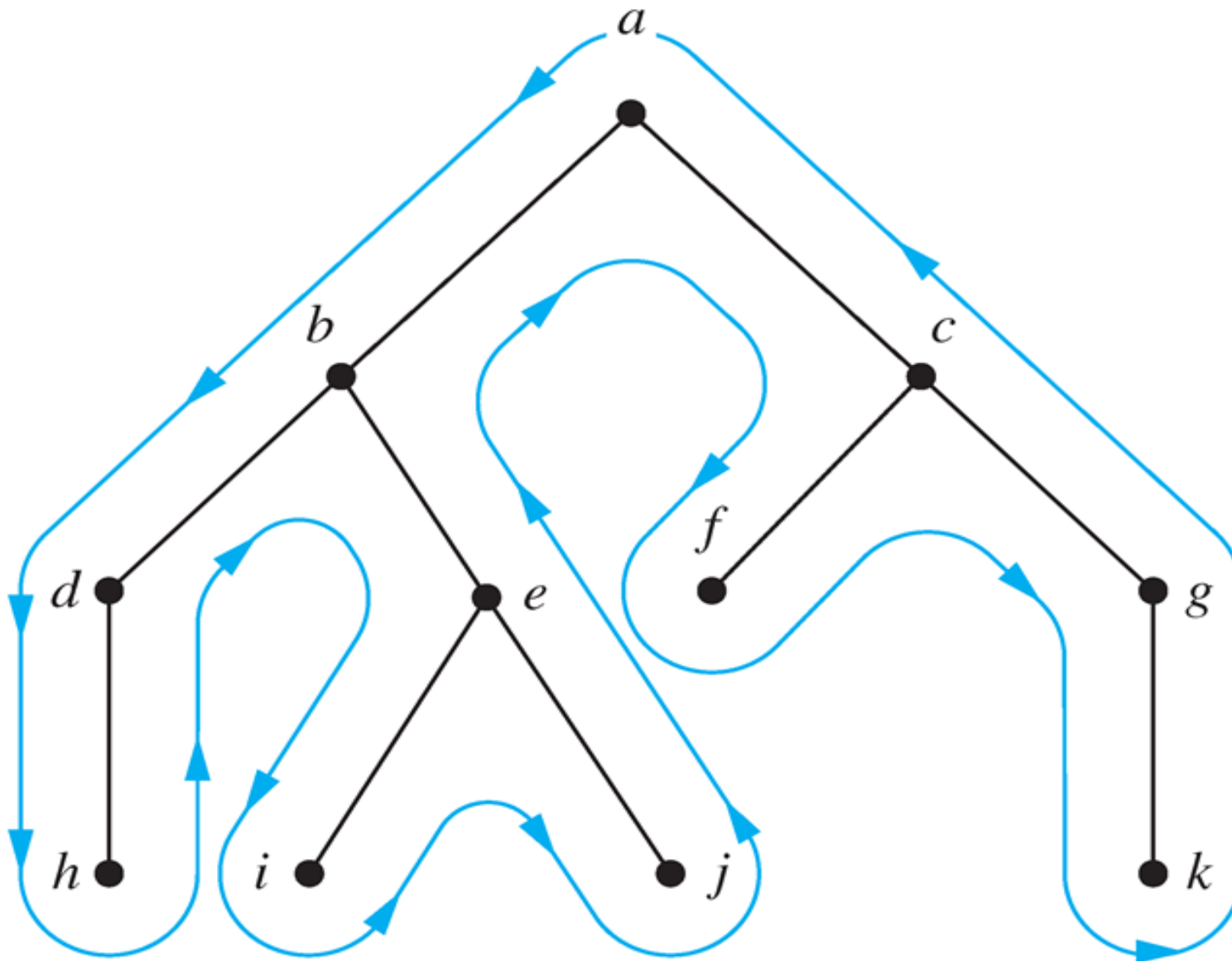
Expression Trees

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Example
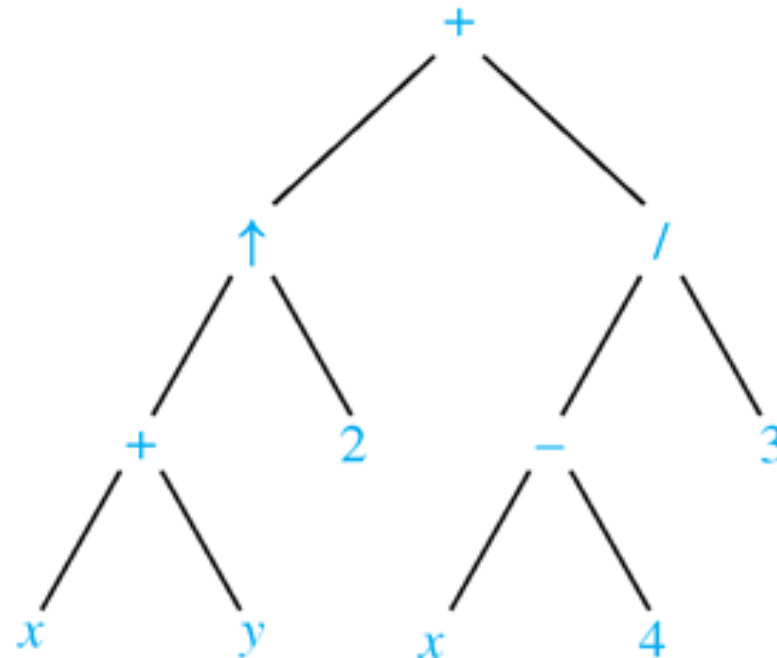
consider the expression $((x + y) \uparrow 2) + ((x - 4)/3)$

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Example

consider the expression $((x + y) \uparrow 2) + ((x - 4)/3)$

- An inorder traversal of the tree representing an expression produces the original expression when parentheses are included except for unary operation.

Infix Notation

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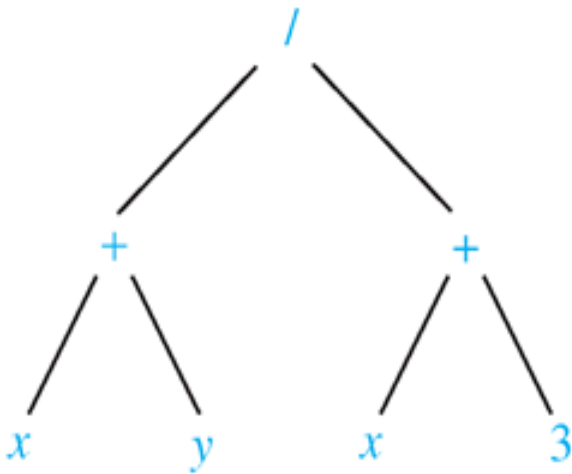
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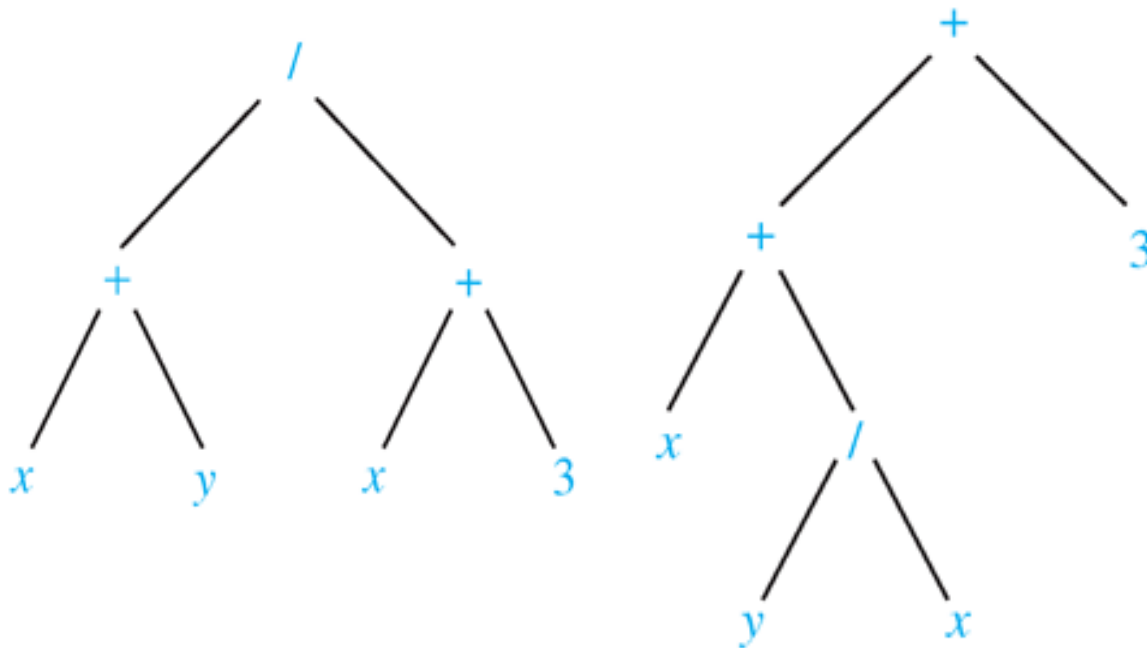
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  Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

- **Example**

$$+ - * \ 2 \ 3 \ 5 \ / \ \uparrow \ 2 \ 3 \ 4$$

- **Example**

$$+ - * 2\ 3\ 5\ / \uparrow 2\ 3\ 4$$

$+$	$-$	$*$	2	3	5	/	$\uparrow$	2	3	4

$2 \uparrow 3 = 8$

	$+$	$-$	$*$	2	3	5	/	8	4

$8 / 4 = 2$

$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad 2$

$2 * 3 = 6$

$+ \quad - \quad 6 \quad 5 \quad 2$

$6 - 5 = 1$

$+ \quad 1 \quad 2$

$1 + 2 = 3$

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- **Example**

  $7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$

- **Example**

$$7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$$

7   2   3   *   −   4   ↑   9   3   /   +

$$2 * 3 = 6$$

每次遇到符号，就弹出两个数字进行运算

7   6   −   4   ↑   9   3   /   +

$$7 - 6 = 1$$

1   4   ↑   9   3   /   +

$$1^4 = 1$$

1   9   3   /   +

$$9 / 3 = 3$$

1   3   +

$$1 + 3 = 4$$

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$$C_0 = C_1 = 1$$

$$C_2 = 2$$

$$C_3 = 5$$
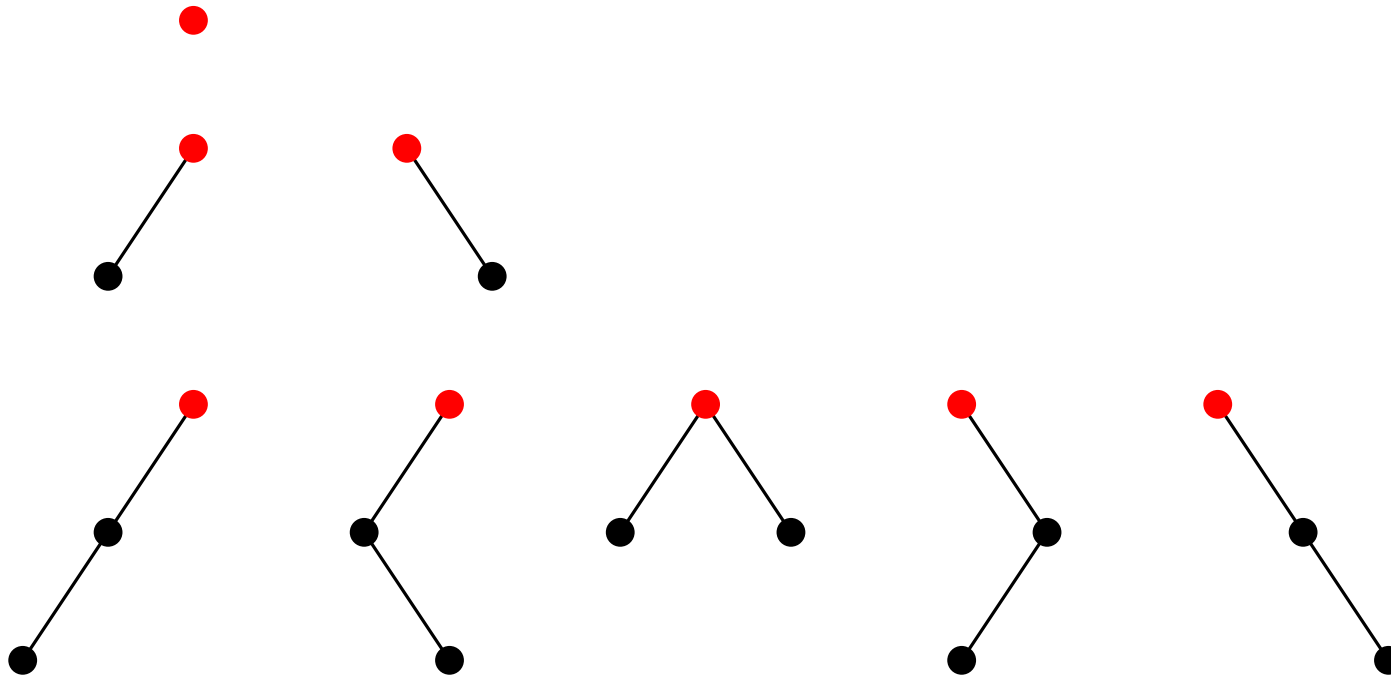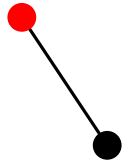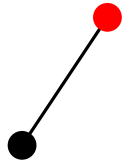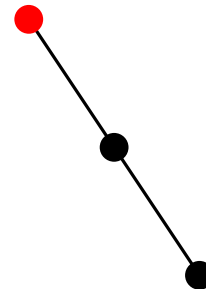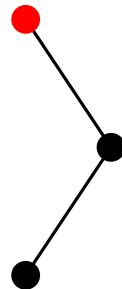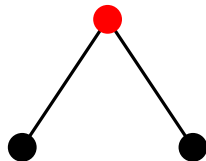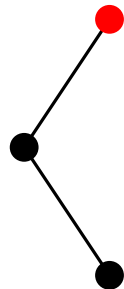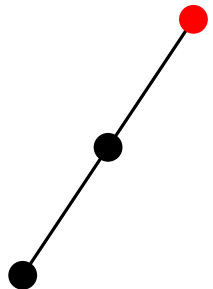
- How many different binary tress are there with n vertices? We denote this number as C_n.
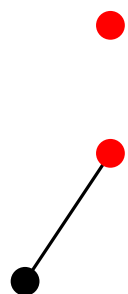
$$C_0 = C_1 = 1$$

$$C_2 = 2$$
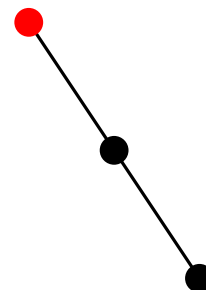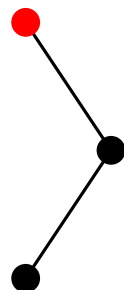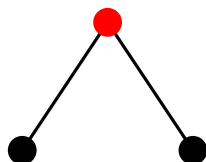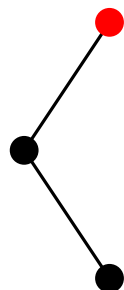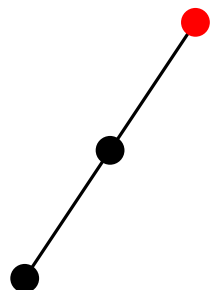
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How to find a formula for C_n?

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Catalan Numbers

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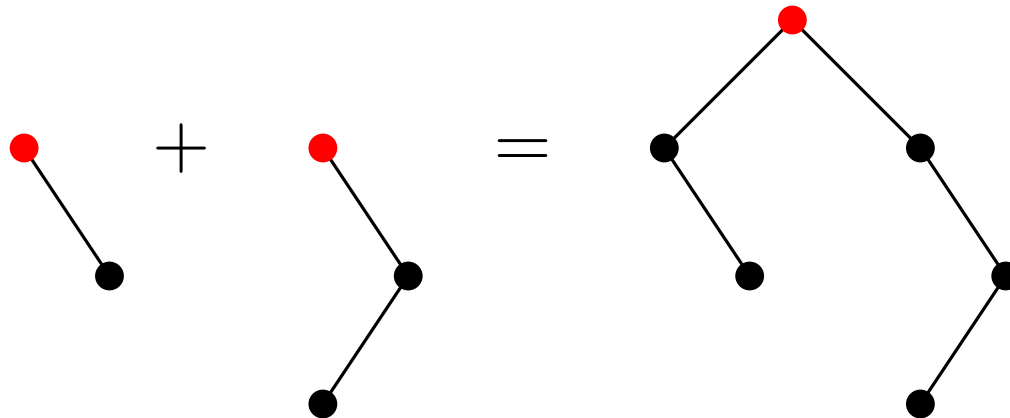
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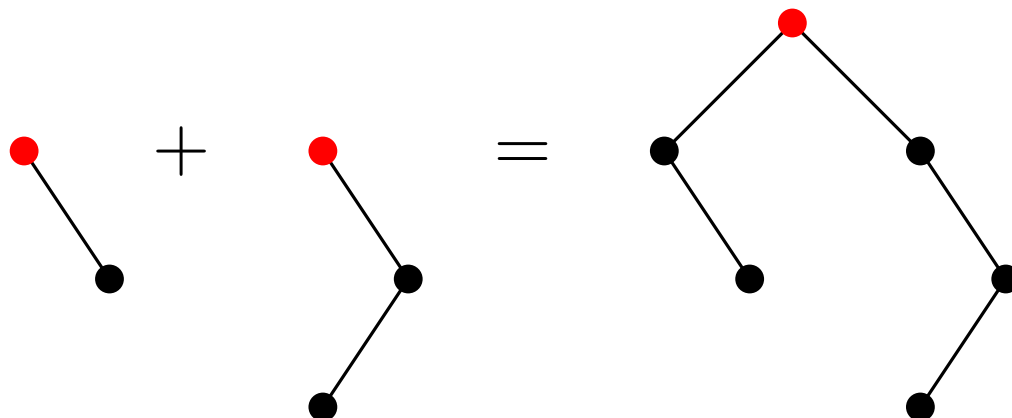
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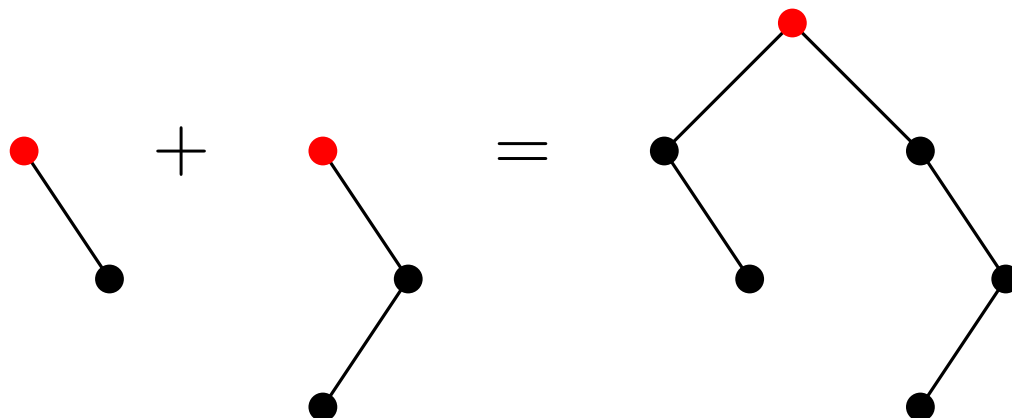
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For example, $C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 1 * 2 + 1 * 1 + 2 * 1 = 5.$

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  C_n – the coefficient of x^n in the expansion of f.

- $f = \frac{1-\sqrt{1-4x}}{2x}$, by the extended Binomial Theorem,

$$\sqrt{1-4x} = (1 + (-4x))^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n}(-4x)^n.$$

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Then we have $C_n = \frac{1}{n+1}\binom{2n}{n}$.

This is called the n-th *Catalan number*.

- **Theorem** The number of sequences $a_1, \ldots, a_{2n}$ of $2n$ terms that can be formed using exactly n $+1$'s and exactly n -1's whose partial sums are always nonnegative, i.e., $a_1 + a_2 + \cdots + a_k \geq 0$ for any $1 \leq k \leq 2n$, equals the n-th Catalan number C_n.

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  R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015. Includes 214 combinatorial interpretations of C_n, and 68 additional problems!

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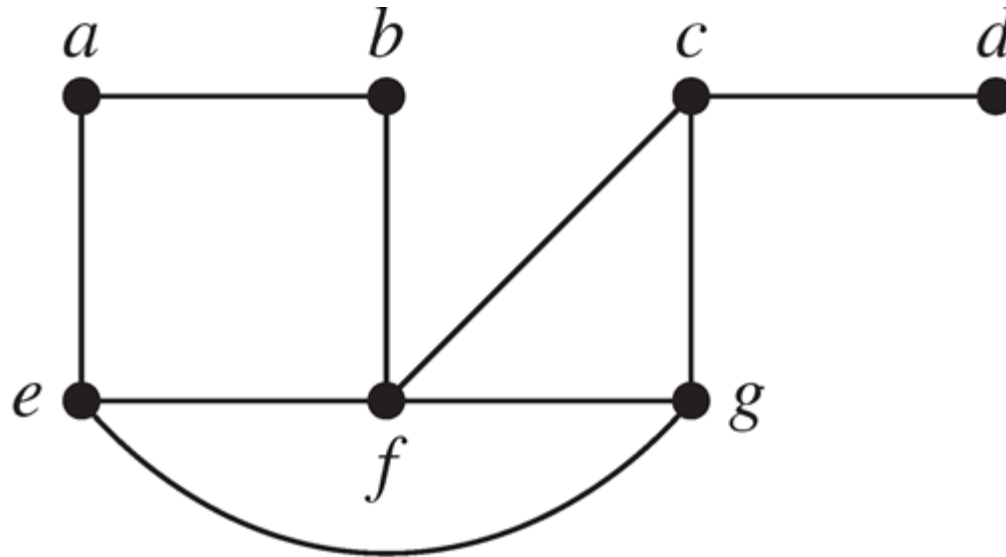
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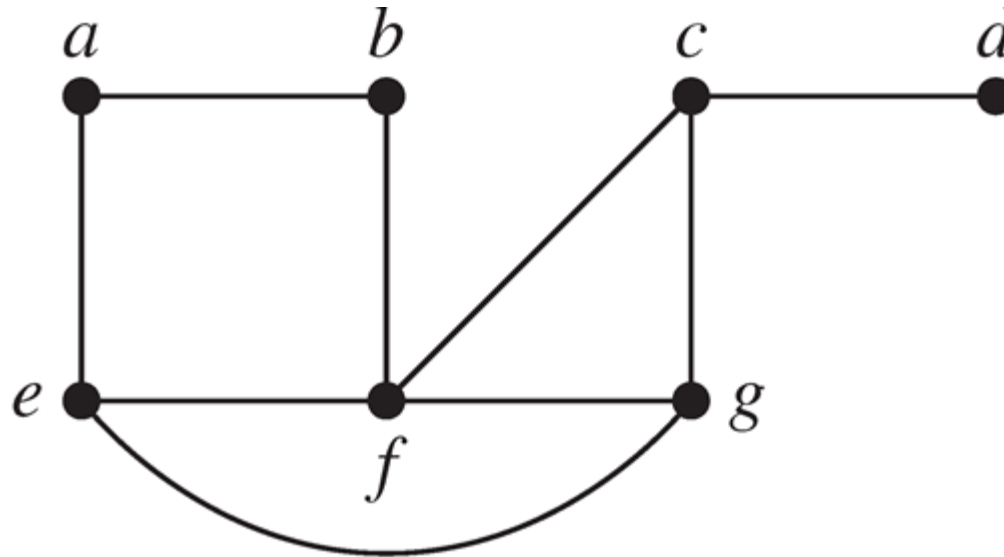
remove edges to avoid circuits

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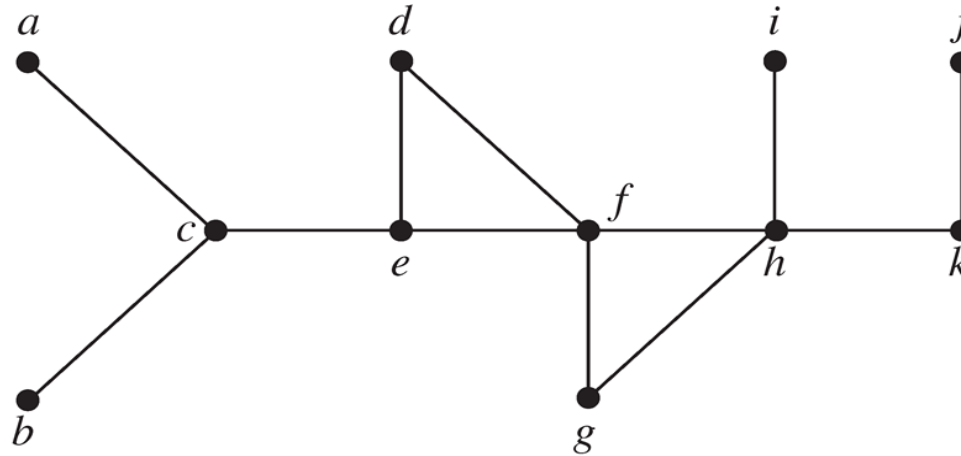
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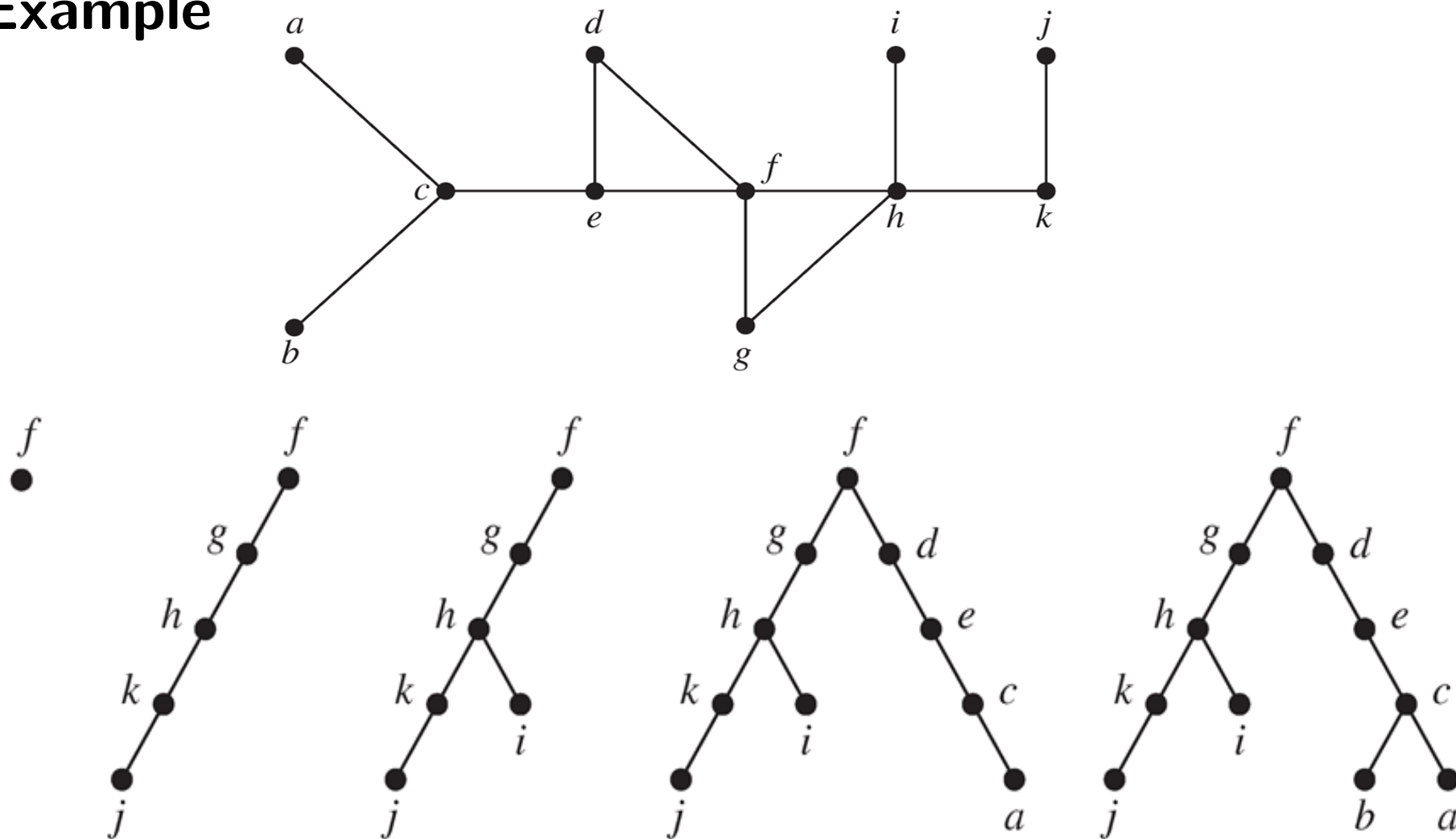
Depth-First Search Algorithm

procedure $DFS(G$: connected graph with vertices $v_1, v_2, ..., v_n)$
$T :=$ tree consisting only of the vertex v_1
$visit(v_1)$

procedure $visit(v$: vertex of $G)$
for each vertex w adjacent to v and not yet in T
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Breadth-First Search

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  ◇ First arbitrarily choose a vertex of the graph as the root.

  ◇ Form a path by adding all edges incident to this vertex and the other endpoint of each of these edges

  ◇ For each vertex added at the previous level, add edge incident to this vertex, as long as it does not produce a simple circuit.

  ◇ Continue in this manner until all vertices have been added.

- **Example**

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procedure $BFS(G$: connected graph with vertices $v_1, v_2, ..., v_n)$
T := tree consisting only of the vertex v_1
L := empty list $visit(v_1)$
put v_1 in the list L of unprocessed vertices
while L is not empty
    remove the first vertex, v, from L
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        if w is not in L and not in T **then**
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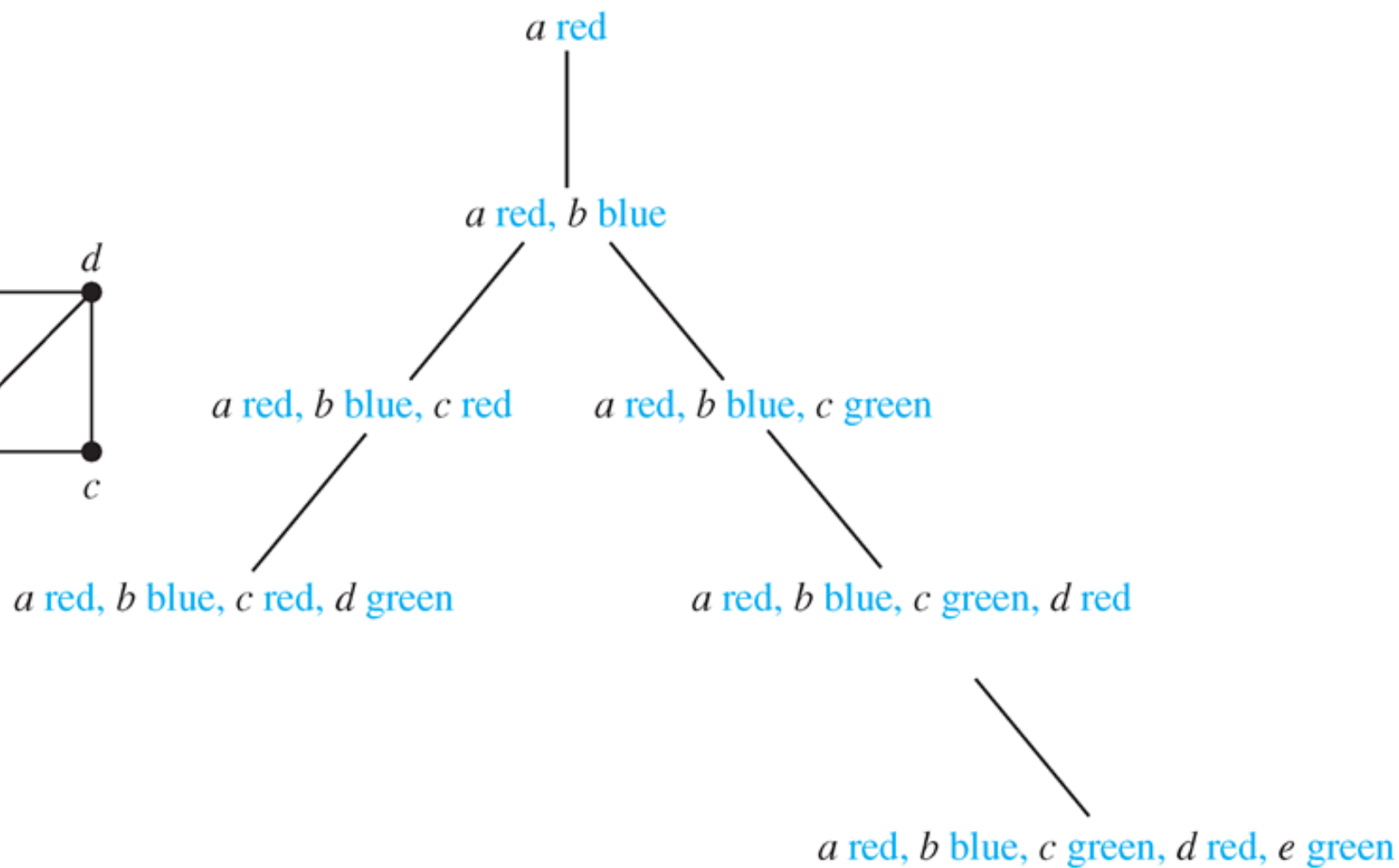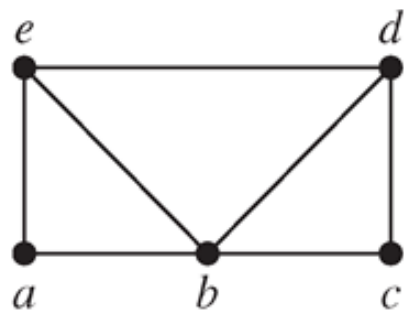
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find

grap

Ø
Sum = 0

{31}
Sum = 31

{27}
Sum = 27

{31, 7}
Sum = 38

{31, 5}
Sum = 36

{27, 11}
Sum = 38

{27, 7}
Sum = 34

{27, 7, 5}
Sum = 39
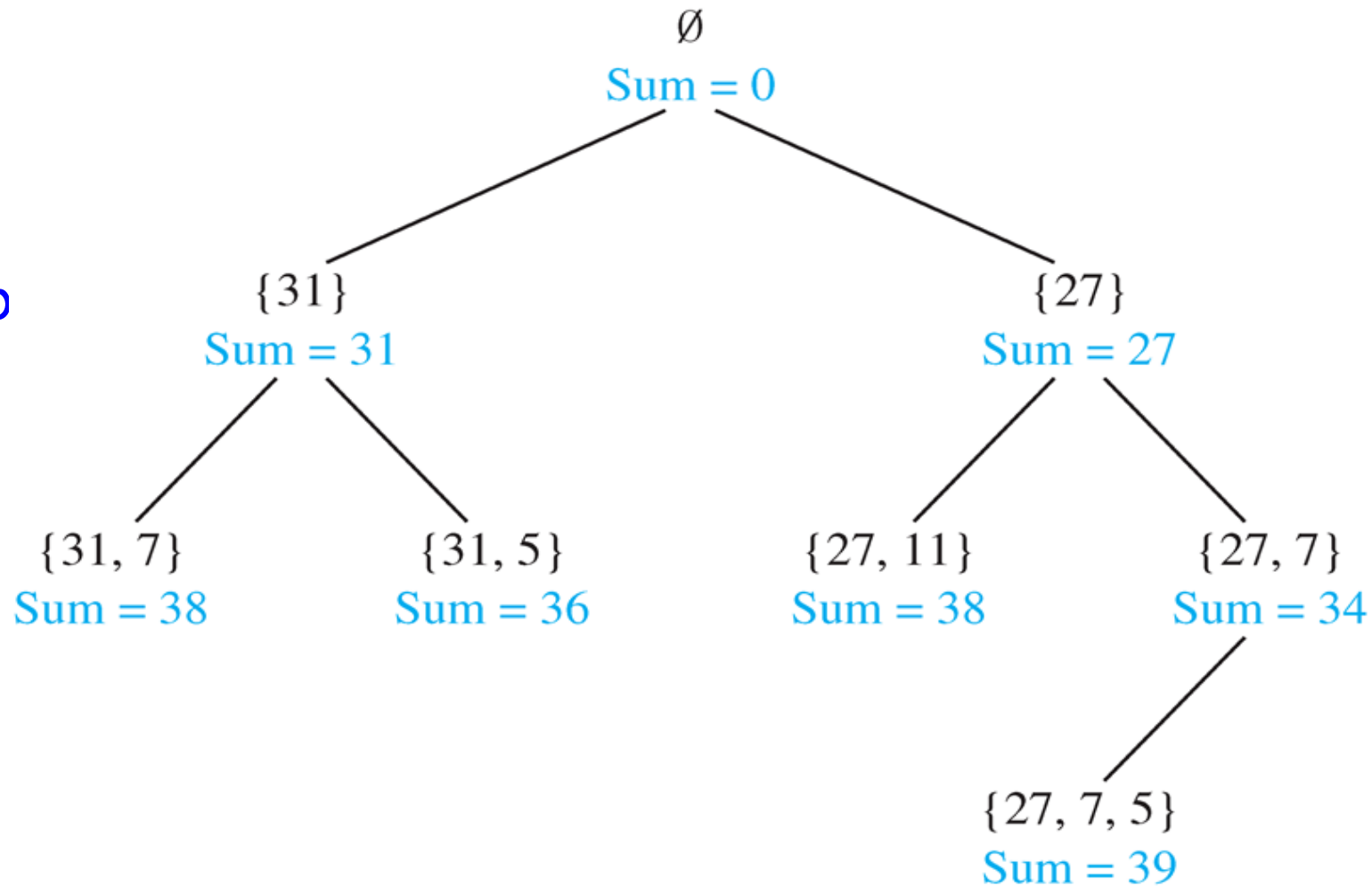
find a subset of $\{31, 27, 15, 11, 7, 5\}$ with the sum 39

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  two greedy algorithms:
  Prim's Algorithm, Kruscal's Algorithm

ALGORITHM 1 Prim's Algorithm.

procedure *Prim*(G: weighted connected undirected graph with n vertices)
T := a minimum-weight edge
for $i := 1$ **to** $n - 2$
    e := an edge of minimum weight incident to a vertex in T and not forming a
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We can maintain a *heap* of all the edges with at least one endpoint in T, and in each iteration, we do Extract-Mins until we see an edge that has one endpoint in T and one endpoint not in T.

time complexity: $e \log v$
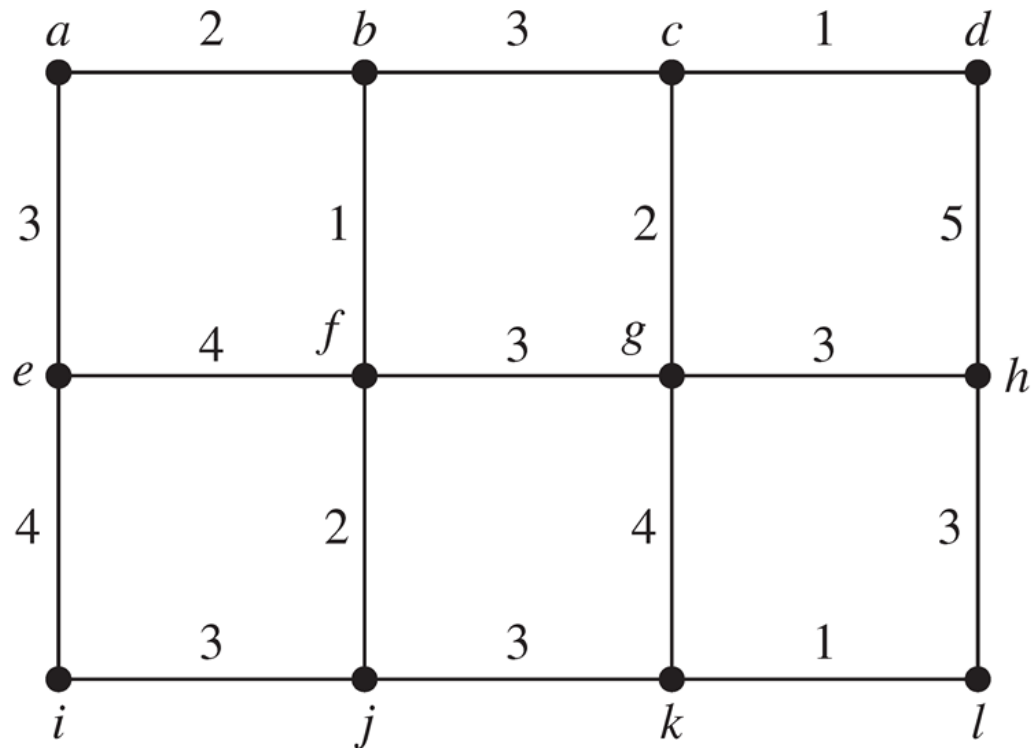
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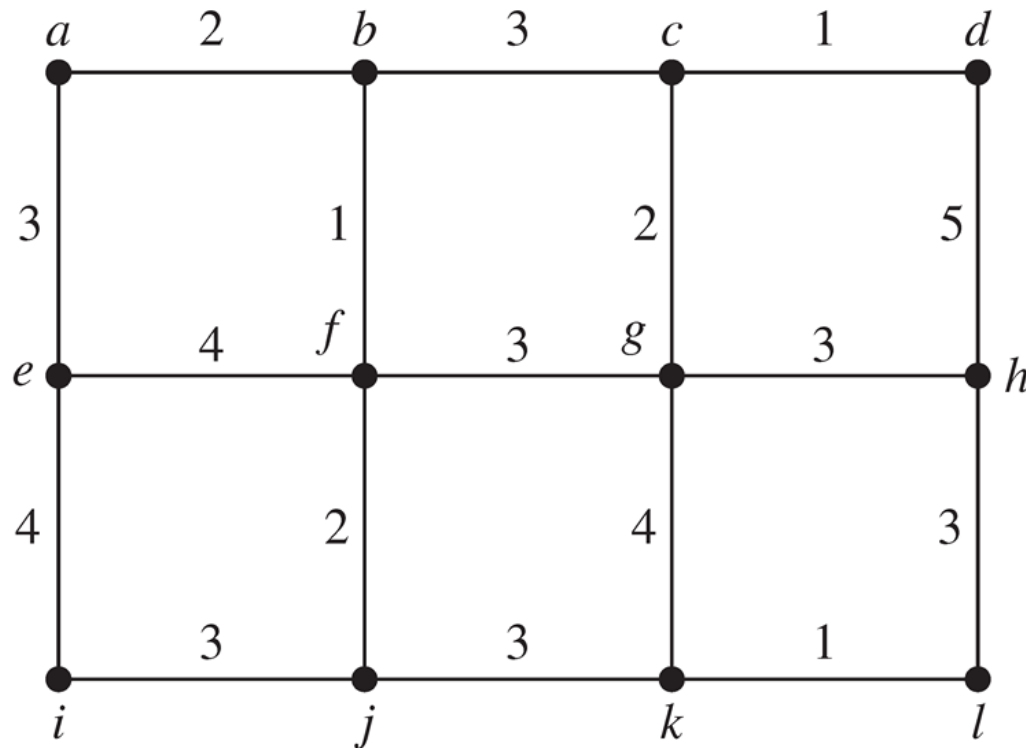
Choice	Edge	Weight
1	$\{b, f\}$	1
2	$\{a, b\}$	2
3	$\{f, j\}$	2
4	$\{a, e\}$	3
5	$\{i, j\}$	3
6	$\{f, g\}$	3
7	$\{c, g\}$	2
8	$\{c, d\}$	1
9	$\{g, h\}$	3
10	$\{h, l\}$	3
11	$\{k, l\}$	1
	Total:	24

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Prim's Algorithm: Correctness

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  Since Prim's algorithm has chosen to add e, we have $w(e) \leq w(e')$. So if we add e to M and remove e' from M, we will have a new tree M' whose total weight $\leq$ that of M, and $T \cup \{e\} \subset M'$.

Kruscal's Algorithm

ALGORITHM 2 Kruskal's Algorithm.

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
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    $e :=$ any edge in G with smallest weight that does not form a simple circuit
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time complexity: $e \log e$ *Union-Find*

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    $T := T$ with e added
return T {T is a minimum spanning tree of G}
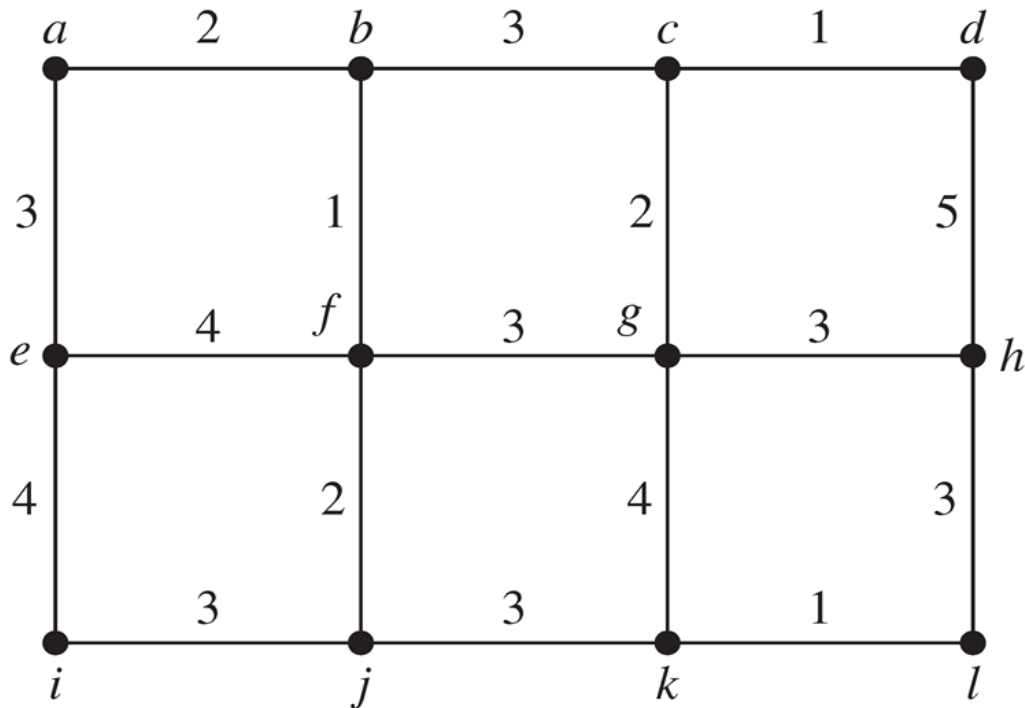
time complexity: $e \log e$ Union-Find
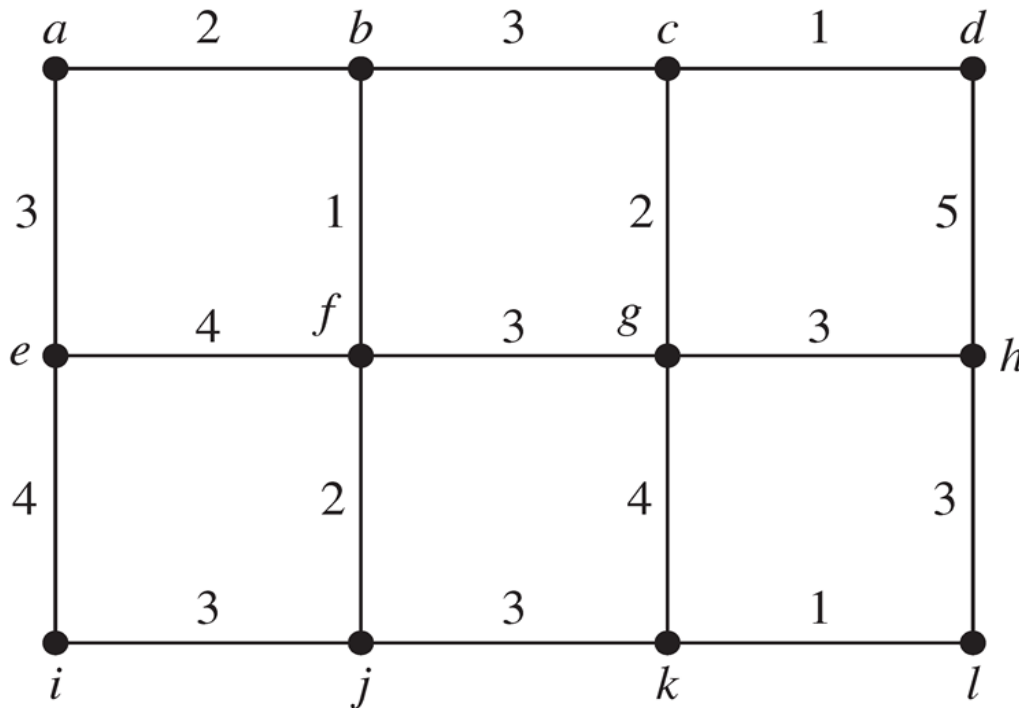see *CLRS / Algorithm Design*, J. Kleinberg, E. Tardos

■ **Example**

- **Example**

Choice	Edge	Weight
1	$\{c, d\}$	1
2	$\{k, l\}$	1
3	$\{b, f\}$	1
4	$\{c, g\}$	2
5	$\{a, b\}$	2
6	$\{f, j\}$	2
7	$\{b, c\}$	3
8	$\{j, k\}$	3
9	$\{g, h\}$	3
10	$\{i, j\}$	3
11	$\{a, e\}$	3
	Total:	24

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- A unifying structure: Given a graph $G = (V, E)$, every subset $S \subseteq V$ defines a *cut* $(S, \bar{S})$. We consider the edge set between S and $\bar{S}$.

  Theorem Let $(S, \bar{S})$ be an arbitrary cut, and let e be an edge across the cut (one endpoint in S, the other in $\bar{S}$) that has the smallest weight of all edges cross the cut. Then there must be an MST T containing e.

  Theorem Let $(S, \bar{S})$ be an arbitrary cut, and let E' be the set of edges across the cut of minimum weight ($w(e) = w(e')$ for any two edges $e, e' \in E'$ and $w(e) < w(e')$ for any $e \in E'$ and $e' \notin E'$). Let T be an arbitrary MST. Then T must contain some edge in E'.

- reduction, review ...