Department of
Computer Science and Engineering
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CS215 DISCRETE MATH

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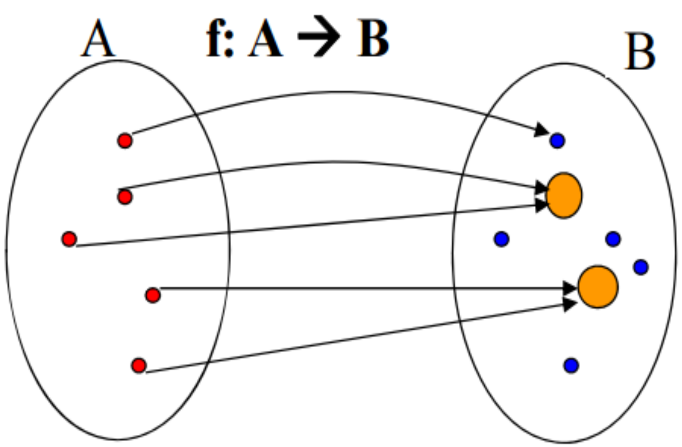
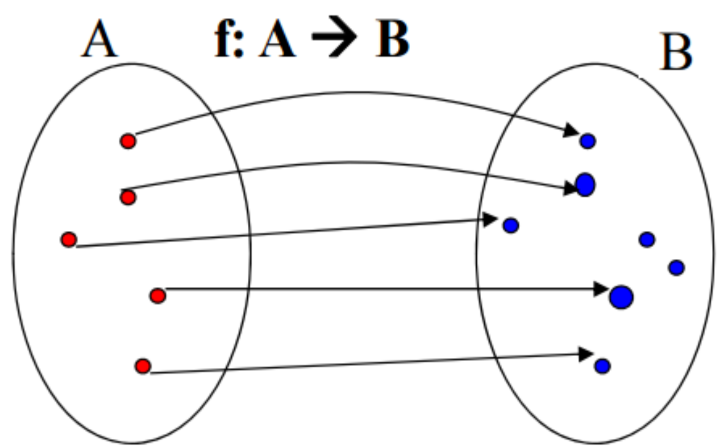
Injective (One-to-One) Function

- A function f is called *one-to-one* or *injective*, if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f. In this case, f is called an *injection*.

  Alternatively: A function is *one-to-one* if and only if $f(x) \neq f(y)$ whenever $x \neq y$. (contrapositive!)



Not injective

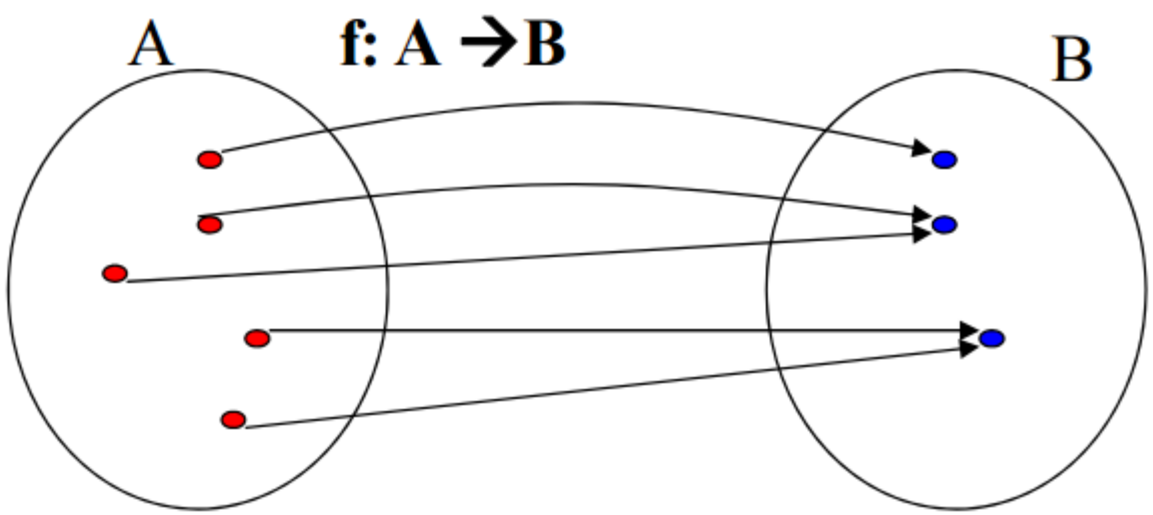


Injective function

Surjective (Onto) Function

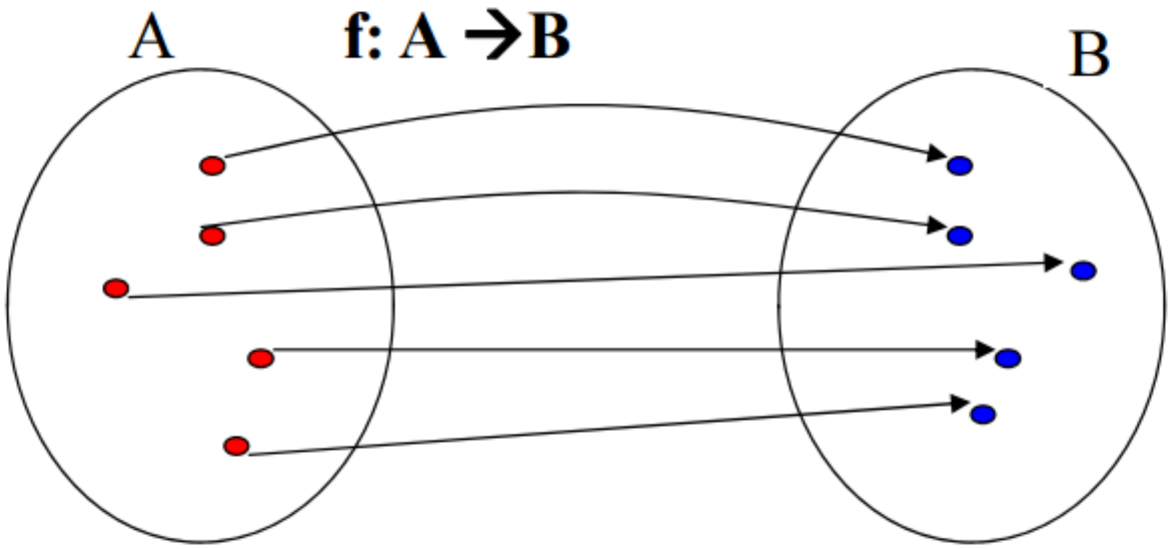
- A function f is called *onto* or *surjective*, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$. In this case, f is called a *surjection*.

  Alternatively: A function is *onto* if and only if all codomain elements are covered ($f(A) = B$).



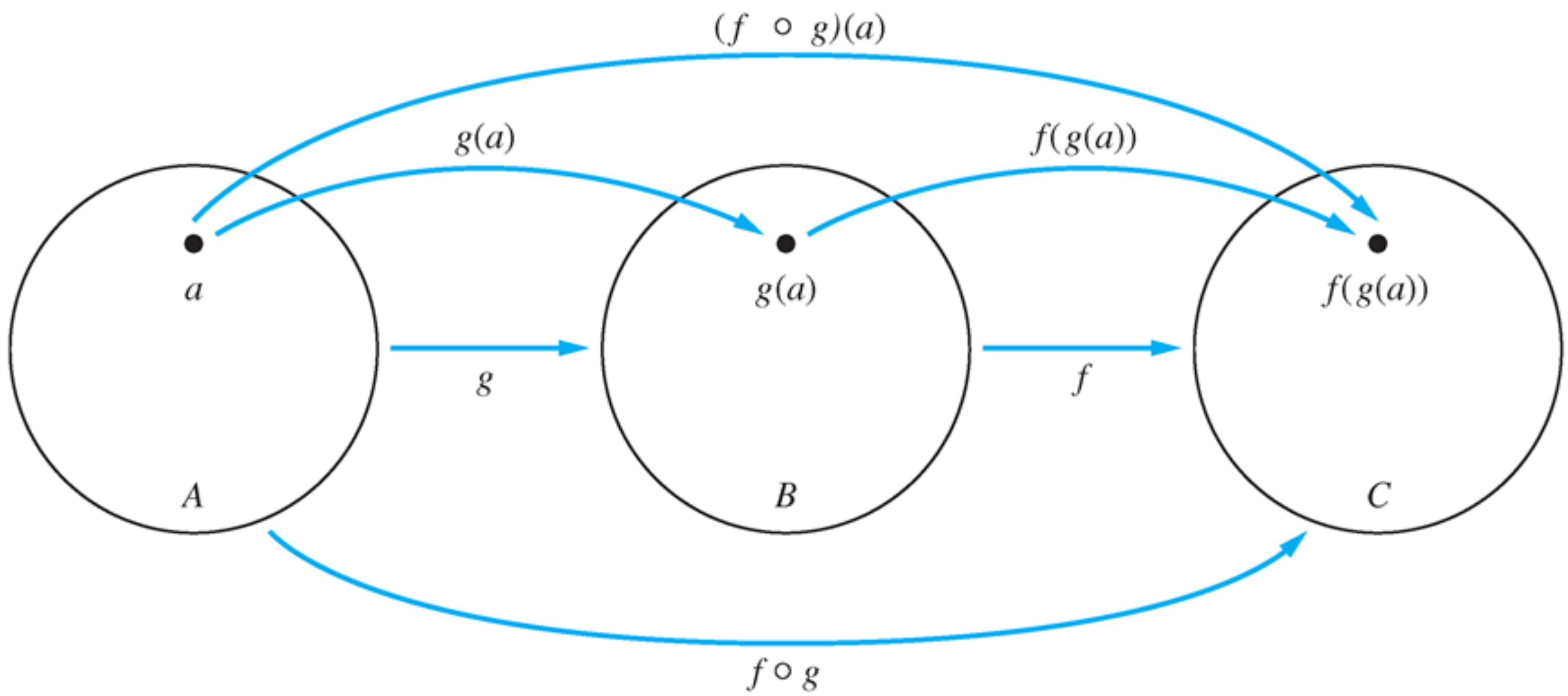
Bijective Function (One-to-One Correspondence)

- A function f is called *bijective*, if and only if it is both one-to-one and onto.



Composition of Functions

- Let f be a function from B to C and let g be a function from A to B. The *composition of the functions f and g*, denoted by $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$.



Composition of Functions

- Suppose that f is a bijection from A to B. Then $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$, Since

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$$
$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b,$$

where I_A, I_B denote the *identity functions* on the sets A and B, respectively.

Some Important Functions

- The *floor function* assigns a real number x the largest integer that is $\leq x$, denoted by $\lfloor x \rfloor$.
- The *ceiling function* assigns a real number x the smallest integer that is $\geq x$, denoted by $\lceil x \rceil$.

Some Important Functions

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TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x+1$
(2)	$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Some Important Functions

Ex. 1: Prove or disprove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Ex. 2: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

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Ex. 2: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

- The factorial function $f : \mathbf{N} \to \mathbf{Z}^+$ is the product of the first n positive integers when n is a nonnegative integer, denoted by $f(n) = n!$.

Sequences

- A *sequence* is a function from a subset of the set of integers (typically the set $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$ to a set S. We use the notation a_n to denote the image of the integer n. ($\{a_n\}$ represents the ordered list $a_1, a_2, a_3, \ldots$)

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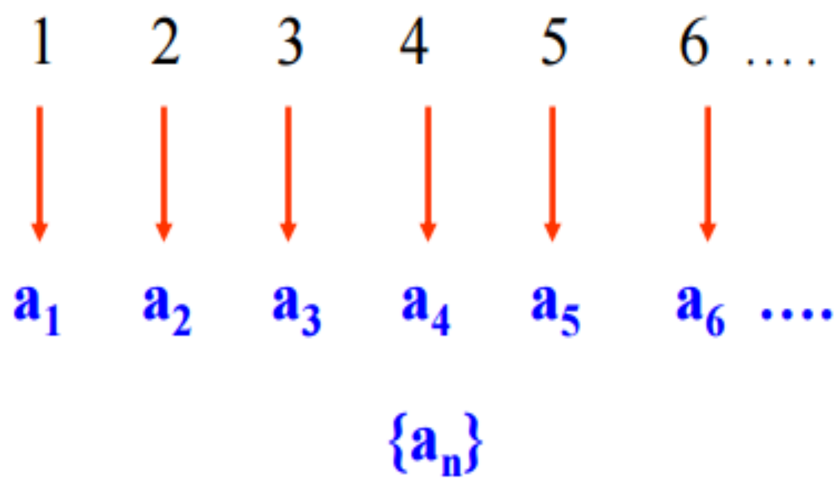

1.1 Basic Concepts and Notation

In general, a *sequence* is an ordered list of elements from a set S. Formally, a *finite sequence* with elements over S is a function from the index set $\{0, 1, \ldots, N-1\}$ to S for some integer $N \geq 0$, and N is called the *length* of the sequence. An *infinite sequence* with elements over S is a function from the integer group $\mathbf{Z}$ to S, and a *semi-infinite* sequence with elements over S is a function from the semi-group $\{0, 1, \ldots\}$ to S. If the set S is a finite field $\mathbb{F}_q$ with q elements, we say that the sequence is a *q-ary sequence* over $\mathbb{F}_q$. In particular, if $S = \mathrm{GF}(2)$, the sequence is called a *binary* sequence.

For a sequence $\mathbf{s} = (s_i)_{i \geq 0}$, if there exist integers $r > 0$ and $u \geq 0$ such that

$$s_{i+r} = s_i \quad \text{for all } i \geq u, \tag{1.1}$$

the sequence is said to be *ultimately periodic* with parameters (r, u), and r is called a *period* of the sequence $\mathbf{s}$. The smallest number r satisfying (1.1) is called the *least period*

1

Sequences

- **Examples**:
  - ◇ $a_n = n^2$, where $n = 1, 2, 3, \ldots$
  - ◇ $a_n = (-1)^n$, where $n = 0, 1, 2, \ldots$
  - ◇ $a_n = 2^n$, where $n = 0, 1, 2, \ldots$

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- An *arithmetic progression* is a sequence of the form $a, a+d, a+2d, a+3d, \ldots, a+nd, \ldots$, where the *initial term* a and *common difference* d are real numbers.

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  Example:
  - ◇ $a_n = -1 + 4n$, where $n = 0, 1, 2, 3, \ldots$

Geometric Progression

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  Given a sequence, how to find a rule for generating the sequence?

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Question:

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$$8, 42, 226, 1232, 6646, 35362, 185868, \ldots$$

Recursively Defined Sequences

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Examples:

◇ $a_n = a_{n-1} + 2$ assuming $a_0 = 1$, for $n \geq 1$

◇ $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$ (*Fibonacci sequence*)

Summations

- The *summation of the terms of a sequence* is

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$$

  The variable j is referred to as *the index of summation* and the choice of the letter j is arbitrary.
  - m is the *lower limit*
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$$\sum_{j=1}^{n}(ax_j + by_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j = \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_j$$

Summations

- The sum of the first n terms of the arithmetic progression $a, a+d, a+2d, \ldots, a+nd$ is

$$S = \sum_{j=0}^{n}(a+jd) = (n+1)a + d\sum_{j=0}^{n} j = (n+1)a + d\frac{n(n+1)}{2}$$

- The sum of the first n terms of the geometric progression $a, ar, ar^2, \ldots, ar^k$ is

$$S = \sum_{j=0}^{n}(ar^j) = a\sum_{j=0}^{n} r^j = a\frac{r^{n+1}-1}{r-1}$$

Examples

- **Examples**:
  - $S = \sum_{j=1}^{5}(2+3j)$
  - $S = \sum_{j=3}^{5}(2+3j)$
  - $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)$
  - $S = \sum_{j=0}^{3} 2(5)^j$
  - $S = \sum_{i=1}^{4}\sum_{j=1}^{3} ij$

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  - ◇ $S = \sum_{j=1}^{5}(2+3j)$ 55
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  - ◇ $S = \sum_{i=1}^{4}\sum_{j=1}^{2}(2i-j)$ 28
  - ◇ $S = \sum_{j=0}^{3} 2(5)^j$ 312
  - ◇ $S = \sum_{i=1}^{4}\sum_{j=1}^{3} ij$ 60

Infinite Series

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$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Some Useful Summation Formulas

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

$1+3k+k^2 = (k+1)^3 - k^3$

telescoping
邻差

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} ij = \sum_{i=1}^{n}\sum_{j=1}^{i} ij \quad i \geq j$$

$$+ \sum_{j=1}^{n}\sum_{i=1}^{j-1} ij \quad i < j$$

$$= \sum_{k=1}^{n}\sum_{l=1}^{k} kl + \sum_{k=1}^{n}\sum_{l=1}^{k-1} kl$$

$$= \sum_{k=1}^{n} k\Big(\underbrace{\sum_{l=1}^{k} l + \sum_{l=1}^{k-1} k}_{=k^2}\Big) = \frac{(1+k)k}{2} + \frac{k(k-1)}{2} \;//$$

$$= \sum_{k=1}^{n} k^3$$

Cardinality of Sets

- Recall: the cardinality of a finite set is defined by the number of the elements in the set.

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- The sets A and B have *the same cardinality* if there is a one-to-one correspondence between elements in A and B.

- If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B, denoted by $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinalities, we say that the cardinality of A is less than the cardinality of B, denoted by $|A| < |B|$.

Countable Sets

- A set that is **either finite** or **has the same cardinality as the set of positive integers $\mathbf{Z}^+$** is called *countable*. A set that is **not countable** is called *uncountable*.

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Why are these called **countable**?

◇ The elements of the set can be **enumerated and listed**.

prove countable

① find a bijection $f: S \to \mathbb{Z}^+$ / $f: \mathbb{Z}^+ \to S$

② find a way to enumerate all elements

Hilbert's Grand Hotel

- The Grand Hotel has **countably infinite number of rooms**, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

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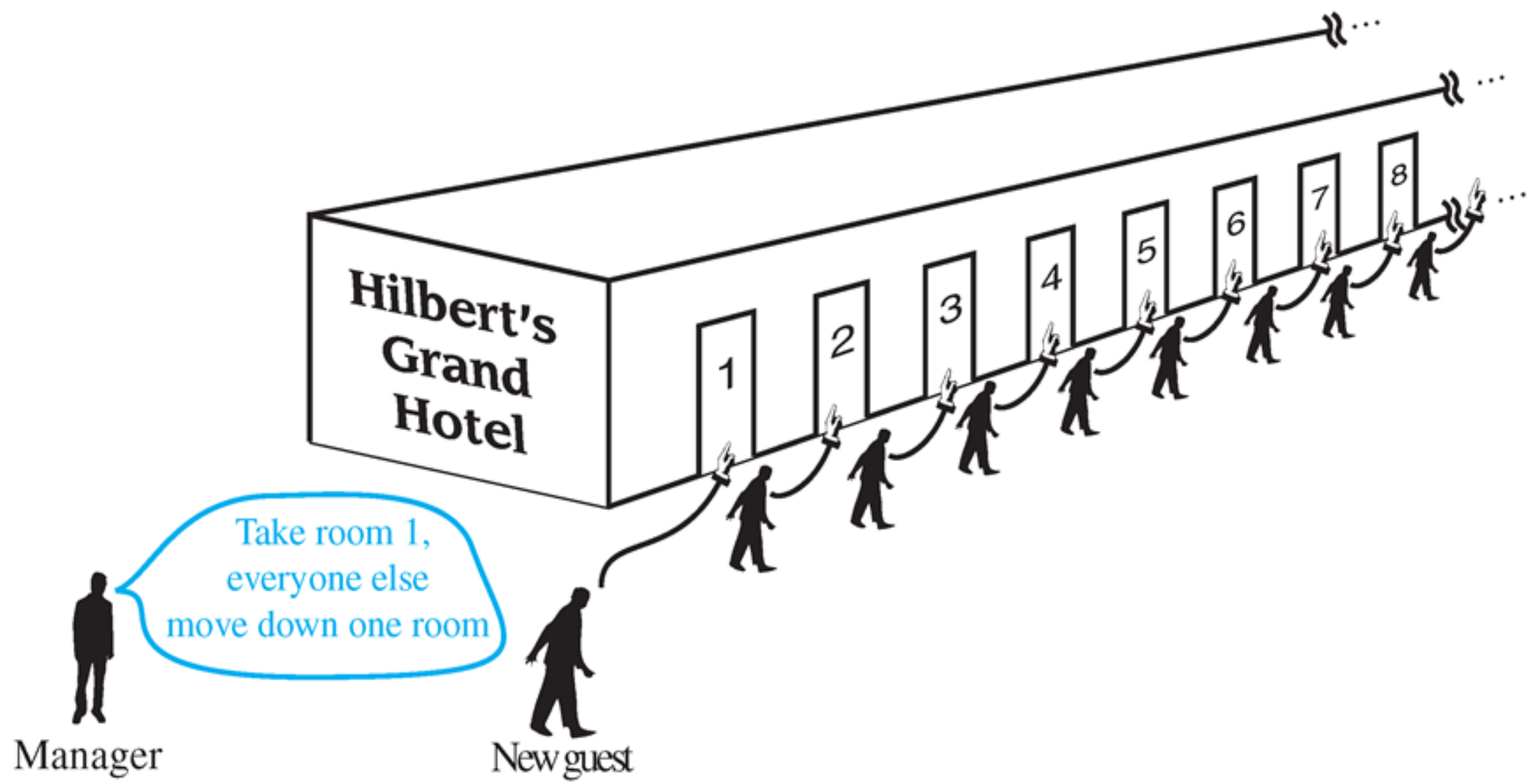

FIGURE 2 **A New Guest Arrives at Hilbert's Grand Hotel.**

- **Example 1**

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  Define a function $f : x \mapsto 2x - 2$. This is a bijection!

  one-to-one Why?

  onto Why?

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  Using the definition: Is there a **bijection** $f : \mathbf{Z}^+ \to A$?

  Define a function $f : x \mapsto 2x - 2$. This is a bijection!

  one-to-one Why?

  if $2x - 2 = 2y - 2$, then $x = y$

  onto Why?

  $\forall x \in A$, $(x + 2)/2$ is the preimage in $\mathbf{Z}^+$

- **Example 2 (Theorem)**

  The set of integers $\mathbf{Z}$ is countable.

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  The set of integers **Z** is countable.

Solution:

We can list a sequence:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

or define a bijection from $\mathbf{Z}^+$ to $\mathbf{Z}$:
– when n is even: $f(n) = n/2$
– when n is odd: $f(n) = -(n-1)/2$

Countable Sets

- **Example 3 (Theorem)**

  The set of (positive) rational numbers is countable.

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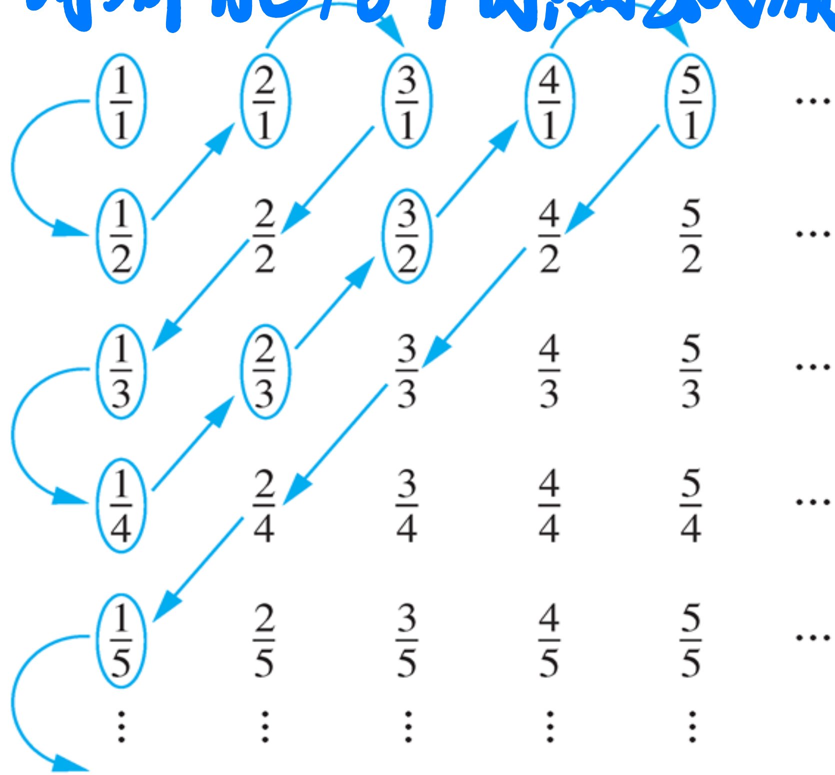
  The set of (positive) rational numbers is countable.

每一对都能给个自然数编号

Solution:

Constructing the list: first list p/q with $p+q=2$, next list p/q with $p+q=3$, and so on.



$1, 1/2, 2, 3, 1/3, 1/4, 2/3, \ldots$

A, B countable

$\Rightarrow$ A x B countable

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  Solution:

  We show that the strings can be listed in a sequence. First list

  (i) all the strings of length 0 in alphabetical order.

  (ii) then all the strings of length 1 in lexicographic order.

  (iii) and so on.

  This implies a bijection from $\mathbf{Z}^+$ to S.

Countable Sets

- **Example 5**

  The set of all Java programs is countable.

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Solution:

Let S be the set of strings constructed from the characters which may appear in a Java program. Use the ordering from the previous example. Take each string in turn

– feed the string into a Java compiler

– if the complier says YES, this is a syntactically correct Java program, we add this program to the list

– we move on to the next string

In this way, we construct a bijection from $\mathbf{Z}^+$ to the set of Java programs.

Uncountable Sets

- **Theorem**

  The set of real numbers $\mathbf{R}$ is uncountable.

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  The set of real numbers $\mathbf{R}$ is uncountable.

Proof by contradiction:

Assume that $\mathbf{R}$ is countable. Then every subset of $\mathbf{R}$ is countable (why?), in particular, the interval from 0 to 1 is countable. This implies that the elements of this set can be listed as $r_1, r_2, r_3, \ldots$, where

- $r_1 = 0.d_{11}d_{12}d_{13}d_{14}\cdots$
- $r_2 = 0.d_{21}d_{22}d_{23}d_{24}\cdots$
- $r_3 = 0.d_{31}d_{32}d_{33}d_{34}\cdots$

all $d_{ij} \in \{0, 1, 2, \ldots, 9\}$.

Uncountable Sets

- **Theorem**

  The set of real numbers **R** is uncountable.

Proof by contradiction:

We want to show that not all real numbers in the interval between 0 and 1 are in this list.

Form a new number called $r = 0.d_1d_2d_3d_4\cdots$, where $d_i = 2$ if $d_{ii} \neq 2$, and $d_i = 3$ if $d_{ii} = 2$.

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至少在对角线上与每个数不同

Example: suppose

r1 = 0.75243...	d1 = 2
r2 = 0.524310...	d2 = 3
r3 = 0.131257...	d3 = 2
r4 = 0.9363633...	d4 = 2
...	...
rt = 0.23222222...	dt = 3

- **Theorem**

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We claim that r is different from each number in the list.

Each expansion is unique, if we exclude an infinite string of 9's. r and r_i differ in the i-th decimal place for all i.

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This is called *Cantor diagonalization argument*.

Uncountable Sets

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Proof by contradiction:

Assume that $\mathcal{P}(\mathbb{N})$ is countable. This implies that the elements of this set can be listed as $S_0, S_1, S_2, \ldots$, where $S_i \subseteq \mathbb{N}$, and each S_i can be represented uniquely by the bit string $b_{i0}b_{i1}b_{i2}\ldots$, where $b_{ij} = 1$ if $j \in S_i$ and $b_{ij} = 0$ if $j \notin S_i$

- $S_0 = b_{00}b_{01}b_{02}b_{03}\cdots$
- $S_1 = b_{10}b_{11}b_{12}b_{13}\cdots$
- $S_2 = b_{20}b_{21}b_{22}b_{23}\cdots$

$\vdots$

用比特串表示 在Si中对应位置写1

all $b_{ij} \in \{0, 1\}$.

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We claim that R is different from each set in the list.

Each bit string is unique, and R and S_i differ in the i-th bit for all i.

一个集合与其幂集的大小可以不一样

$|S| < |P(S)|$ Cantor Theorem

Schröder-Bernstein Theorem

- **Theorem**

  不一定有限集

  If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

Schröder-Bernstein Theorem

- **Theorem**

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  Example

  Show that $|(0,1)| = |(0,1]|$.

  $f(x) = x$; $g(x) = x/2$

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Schröder-Bernstein Theorem

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  The set of functions from $\mathbf{Z}^+$ to the set $\{0, 1, 2, \ldots, 9\}$ is *uncountable*. Proof?

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Now f is bijective, and T is a subset of S, so there is an element $s_0 \in S$ s.t. $f(s_0) = T$.
$\mathcal{Q}$: Is $s_0 \in T$?

Next Lecture

- complexity ...