Assignment 5

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Q1、let a relation R on a set A be antisymmetric. then if $(a,b) \in R$ and $(b,a) \in R$ implies $a=b$ for $\forall a,b \in R$

prove by contradiction:

let S be a subset of the antisymmetric relation R

$\Rightarrow S \subseteq R$, assume that S is not antisymmetric,

that is $\exists (x,y) \in S \wedge (y,x) \in S \wedge x \neq y$. $x,y \in A$

then for $S \subseteq R \Rightarrow \exists (x,y) \in R \wedge (y,x) \in R \wedge x \neq y$, $x,y \in A$

leading to contradiction that A is antisymmetric

Therefore, the proposition holds.

Q2. (1) Yes, for every $x \in \mathbb{R}$, $x-x=0$ is rational

$\Rightarrow (x,x) \in R$ for $\forall x \in \mathbb{R}$ $\Rightarrow$ reflexive

(2) Yes, if $(x,y) \in R$ for $x,y \in \mathbb{R}$, then $x-y$ is rational

$y-x$ is also rational $\Rightarrow (y,x) \in R$ $\Rightarrow$ symmetric

(3) No, let $x=2$, $y=1$ $(x,y) \in R$, $(y,x) \in R$

but $x \neq y$

(4) Yes, if $(x,y) \in R$ and $(y,z) \in R$ for $x,y,z \in \mathbb{R}$

then let $x-y=a$, $y-z=b$ a,b rational

then $x-z=a+b$ rational $\Rightarrow (x,z) \in R$ $\Rightarrow$ transitive

Q3. (a) $2^{\frac{n(n+1)}{2}}$ (b) $2^{n} \cdot 3^{\frac{n(n-1)}{2}}$ (c) $2^{n(n-1)}$

(d) $2^{\frac{n(n-1)}{2}}$ (e) $2^{n^2} - 2^{n^2-n+1}$ (f) $3^{\frac{n(n-1)}{2}}$

(g) 2^n

Q4. if R is symmetric, if $(x,y) \in R$ then $(y,x) \in R$ for $\forall x.y$ in set.

The transitive closure R^* is formed by adding pairs to R so that transitivity is satisfied. If $(a,b) \in R^*$, $(b,c) \in R^*$, then $(a,c) \in R^*$.

① Base case: $R \subseteq R^*$, R is symmetric, all pairs $(a,b) \in R$ satisfy $(b,a) \in R$. Then, it's symmetric over pairs in R.

② induction: Assume $R' = R \cup R^2 \cdots \cup R^k$, which means R' contains all pairs that there is a path from two elements and the path length $\leq k$. And assume R' is symmetric.

Then we try to prove R^{k+1} is symmetric.

if $R^{k+1} = \emptyset$ we're done.

if $R^{k+1} \neq \emptyset$ all pairs in R^{k+1} can be constructed by pairs in R'

then if $(a,b) \in R^{k+1}$, we can find $(a,c) \in R'$ and $(c,b) \in R'$

for R' symmetric $\Rightarrow$ $(c,a) \in R'$ and $(b,c) \in R'$
and total path length keeps $(k+1)$

$\Rightarrow (b,a) \in R^{k+1}$

So R^{k+1} symmetric and thus $R' \cup R^{k+1}$ symmetric

③ conclusion: for base case and induction, we have R^* symmetric

Q5. Not true, counter example

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$

but R is not transitive, for $(1,2) \in R$, $(2,3) \in R$

but $(1,3) \notin R$

Q6. R is reflexive ⇒ for $\forall a \in A$, $(a,a) \in R$

R^2 contain all pairs of (a,c) that $(a,b) \in R$ and $(b,c) \in R$

let (x,y) be an arbitary pair on set A that $(x,y) \in R$
then we prove that $(x,y) \in R^2$.

① if $x=y$ ⇒ $(x,x) \in R$ and $(x,x) \in R$, then $(x,x) \in R^2$
② if $x \neq y$ ⇒ $(x,y) \in R$ and $(y,y) \in R$, then $(x,y) \in R^2$

⇒ $(x,y) \in R^2$ for $\forall (x,y) \in R$
the proposition holds.

Q7. (1) transitive
prove by contradition, assume that $R \cap S$ is not transitive
then we have $(a,b) \in R \cap S$, $(b,c) \in R \cap S$, and $(a,c) \notin R \cap S$
for some $a, b, c \in A$.

Then we have $(a,b) \in R$, $(a,b) \in S$; $(b,c) \in R$, $(b,c) \in S$
for R、S transitive ⇒ $(a,c) \in R$, $(a,c) \in S$ ⇒ $(a,c) \in R \cap S$
which leads to contradition. Thus $R \cap S$ transitive.

(2) No
$R = \{(1,2), (2,3), (1,3)\}$
$S = \{(2,3), (3,4), (2,4)\}$
and $R \cup S = \{(1,2), (2,3), (1,3), (3,4), (2,4)\}$

$(1,3) \in R \cup S$, $(3,4) \in R \cup S$ but $(1,4) \notin R \cup S$
⇒ not transitive

(3) No $R = \{(a,b), (b,c)\}$ $S = \{(b,c), (c,a)\}$
$R \circ S = \{(a,c), (b,a)\}$ but $(b,c) \notin R \circ S$
⇒ not transitive

Q8 (1) $A=\{1,2,3\}$ $R=\{(1,2),(3,3)\}$

transitive closure of the symmetric closure of R:

$\{(1,2),(3,3),(2,1),(1,1)\}$

symmetric closure of the transitive closure of R:

$\{(1,2),(3,3),(2,1)\}$

Not equal

(2) let $(a,b)\in(R^T)^S$, then $(a,b)\in R^T$ or $(b,a)\in R^T$

① if $(a,b)\in R^T$, then $\{(a,x_1),(x_1,x_2),\cdots(x_n,b)\}\in R$, $R\subseteq R^S$

then $\{(a,x_1),(x_1,x_2),\cdots(x_n,b)\}\in R^S$ then $(a,b)\in(R^S)^T$

② if $(b,a)\in R^T$, then $\{(b,y_1),(y_1,y_2),\cdots(y_n,a)\}\in R$, $R\subseteq R^S$

then $\{(a,y_n),(y_n,y_{n-1}),\cdots(y_1,b)\}\in R^S$, then $(a,b)\in(R^S)^T$

$\Rightarrow$ Therefore, if $(a,b)\in(R^T)^S$, there must be $(a,b)\in(R^S)^T$

$\Rightarrow (R^T)^S\subseteq(R^S)^T$

Q9. (1) prove reflexive, symmetric, transitive

reflexive: $\forall m\in\mathbb{Z}$, $m^2-m^2=0$ $3\mid m^2-m^2\Rightarrow(m,m)\in R$

symmetric: if $\overset{m,n\in\mathbb{Z}}{(m,n)}\in R$ then $3\mid m^2-n^2\Rightarrow 3\mid(-1)(m^2-n^2)$

$\Rightarrow 3\mid n^2-m^2 \Rightarrow (n,m)\in R$

transitive: if $x,y,z\in\mathbb{Z}$, $(x,y)\in R$, $(y,z)\in R\Rightarrow 3\mid x^2-y^2$, $3\mid y^2-z^2$

$\Rightarrow 3\mid(x^2-y^2)+(y^2-z^2) \Rightarrow 3\mid x^2-z^2 \Rightarrow (x,z)\in R$

$\Rightarrow$ R is equivalence relationship

(2) $[m]_R=\{n:(m,n)\in R\}$

$3\mid m^2-n^2$ when $n=0$ $m^2\equiv 0 \pmod 3$ $m=3k$

when $n=1$ $m^2\equiv 1 \pmod 3$ $m=3k+1$ or $3k+2$

when $n=2$ $m^2\equiv 4\equiv 1 \pmod 3$ $m=3k+1$ or $3k+2$

$\Rightarrow [0]=\{n\mid n=3k, k\in\mathbb{Z}\}$ $[1]=\{n\mid n=3k+1$ or $n=3k+2, k\in\mathbb{Z}\}$

Q10. prove reflexive, symmetric, transitive.

reflexive: for $A \subseteq S$, $(A \cup A) \setminus (A \cap A) = \emptyset \subseteq T$

$\Rightarrow (A, A) \in R$

symmetric: for $A, B \subseteq S$ if $(A, B) \in R$ then $(A \cup B) \setminus (A \cap B) \subseteq T$

$B \cup A = A \cup B$, $B \cap A = A \cap B \Rightarrow (B \cup A) \setminus (B \cap A) = (A \cup B) \setminus (A \cap B) \subseteq T$

$\Rightarrow (B, A) \in R$

transitive: for $A, B, C \subseteq S$, if $(A, B) \in R$, $(B, C) \in R$

then $(A \cup B) \setminus (A \cap B) \subseteq T$, $(B \cup C) \setminus (B \cap C) \subseteq T$

And we need to prove that $(A \cup C) \setminus (A \cap C) \subseteq T$

prove by contradiction: assume that $\exists x \in S$, $x \in (A \cup C) \setminus (A \cap C)$

and $x \notin T$, then $x \in (A \cup C)$ and $x \notin (A \cap C)$

① if $x \in A$ and $x \notin C$

i. if $x \in B$ then $x \in (B \cup C)$ and $x \notin (B \cap C)$

$\Rightarrow x \in (B \cup C) \setminus (B \cap C)$, $(B \cup C) \setminus (B \cap C) \subseteq T$

$\Rightarrow x \in T$, leading to contradiction

ii. if $x \notin B$ then $x \in (A \cup B)$ and $x \notin (A \cap B)$

$\Rightarrow x \in (A \cup B) \setminus (A \cap B)$, $(A \cup B) \setminus (A \cap B) \subseteq T$

$\Rightarrow x \in T$, leading to contradiction

② if $x \in C$ and $x \notin A$

i. if $x \in B$

ii. if $x \notin B$

it's similar to ①

in conclusion, $\forall x \in S$, $x \in (A \cup C) \setminus (A \cap C)$ then $x \in T$

$\Rightarrow (A \cup C) \setminus (A \cap C) \subseteq T \Rightarrow (A, C) \in R$

$\Rightarrow$ in conclusion, R is equivalence relation.

Q11. prove reflexive, symmetric, transitive

reflexive: for $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, then $a+b=a+b \Rightarrow (a,b) R (a,b)$

symmetric: for $(a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}$, if $(a,b) R (c,d)$ then $a+d=b+c \Rightarrow c+b=d+a \Rightarrow (c,d) R (a,b)$

transitive: for $(a,b),(c,d),(e,f) \in \mathbb{Z} \times \mathbb{Z}$, if $(a,b) R (c,d)$, $(c,d) R (e,f)$ then $a+d=b+c$, $c+f=d+e$

then $a+d+c+f = b+c+d+e \Rightarrow a+f=b+e \Rightarrow (a,b) R (e,f)$

Q12. prove reflexive, symmetric, transitive

reflexive: for $x \in N$ $x=2^0 x \Rightarrow x \sim x$

symmetric: for $x,y \in N$ if $x \sim y$ then $x=2^k y$ $k \in N$ or $y=2^k x$ $k \in N$

if $x=2^k y$ $y \sim x$; if $y=2^k x$ $y \sim x$

$\Rightarrow y \sim x$

transitive: for $x,y,z \in N$, $x \sim y$, $y \sim z$

prove by case

① if $x=2^{k_1} y$, $k_1 \in N$

i. $y=2^{k_2} z$, $k_2 \in N$ then $x=2^{k_1+k_2} z$, $k_1+k_2 \in N \Rightarrow x \sim z$

ii. $z=2^{k_2} y$, $k_2 \in N \Rightarrow \frac{x}{z}=2^{k_1-k_2} \Rightarrow x=2^{k_1-k_2} z$

if $k_1 \geq k_2$ then $k_1-k_2 \in N \Rightarrow x \sim z$

if $k_1 < k_2$ then $z=2^{k_2-k_1} x$ $k_2-k_1 \in N \Rightarrow x \sim z$

② if $y=2^{k_1} x$, $k_1 \in N$

i. $y=2^{k_2} z$, $k_2 \in N \Rightarrow 1=2^{k_1-k_2} \frac{x}{z} \Rightarrow z=2^{k_1-k_2} x$

if $k_1 \geq k_2$ then $k_1-k_2 \in N \Rightarrow x \sim z$

if $k_1 < k_2$ then $x=2^{k_2-k_1} z$ $k_2-k_1 \in N \Rightarrow x \sim z$

ii. $z=2^{k_2} y$, $k_2 \in N \Rightarrow z=2^{k_1+k_2} x$ $k_1+k_2 \in N$

$\Rightarrow x \sim z$

$\Rightarrow$ in any case, $x \sim z$

in conclusion $\sim$ is equivalence relation.

Q13. (a) yes (b) no (c) yes (d) no

Q14. (a) No, let $f(n)=n$ $g(n)=n^2$

then $f=O(g)$ and $g \neq O(f) \Rightarrow f \propto g$ but g not $\propto f$

(not symmetric)

(b) No let $f(n)=n$ $g(n)=2n$

then $f=O(g)$ and $g=O(f) \Rightarrow f \propto g$, $g \propto f$

but $f \neq g$ (not antisymmetric)

(c) No, $\propto$ is not partial ordering, then it can't be total ordering.

Q15. (1) prove reflexive, antisymmetric, transitive

reflexive: for $(a,b,c) \in X$, $2^a 3^b 5^c \leq 2^a 3^b 5^c \Rightarrow (a,b,c) R (a,b,c)$

antisymmetric: for $(a_1,b_1,c_1), (a_2,b_2,c_2) \in X$, if $(a_1,b_1,c_1) R (a_2,b_2,c_2)$ and $(a_2,b_2,c_2) R (a_1,b_1,c_1)$ then $2^{a_1}3^{b_1}5^{c_1} \leq 2^{a_2}3^{b_2}5^{c_2}$ and $2^{a_2}3^{b_2}5^{c_2} \leq 2^{a_1}3^{b_1}5^{c_1}$

$\Rightarrow 2^{a_1}3^{b_1}5^{c_1} = 2^{a_2}3^{b_2}5^{c_2} \Rightarrow 2^{a_1-a_2}3^{b_1-b_2}5^{c_1-c_2}=1$ and $a_1,a_2,b_1,b_2,c_1,c_2 \in \mathbb{N}$

$\Rightarrow a_1=a_2$, $b_1=b_2$, $c_1=c_2$ then $(a_1,b_1,c_1)=(a_2,b_2,c_2)$

transitive: for $(a_1,b_1,c_1), (a_2,b_2,c_2), (a_3,b_3,c_3) \in X$

$(a_1,b_1,c_1) R (a_2,b_2,c_2)$, $(a_2,b_2,c_2) R (a_3,b_3,c_3)$

then $2^{a_1}3^{b_1}5^{c_1} \leq 2^{a_2}3^{b_2}5^{c_2} \leq 2^{a_3}3^{b_3}5^{c_3} \Rightarrow (a_1,b_1,c_1) R (a_3,b_3,c_3)$

in conclusion, R is partial ordering

(2) all elements are comparable, for example $(0,0,0) R (1,1,1)$

no incomparable elements.

(3) $2^5 \cdot 3^0 \cdot 5^1 = 160$, $2^1 \cdot 3^1 \cdot 5^2 = 150$

least upper bound $(5,0,1)$

greatest lower bound $(1,1,2)$

(4) minimal element $(0,0,0)$

no maximal element

Q16. (1) prove reflexive, antisymmetric, transitive

reflexive: for $(a,b) \in \mathbb{Z}\times\mathbb{Z}$. then $(a,b) = (a,b) \Rightarrow (a,b) \preceq (a,b)$

antisymmetric: for $(a,b), (c,d) \in \mathbb{Z}\times\mathbb{Z}$, if $(a,b) \preceq (c,d)$ and $(c,d) \preceq (a,b)$

① $a^2+b^2 < c^2+d^2$ and $c^2+d^2 < a^2+b^2$ $\Rightarrow$ contradicted

② $(a,b) = (c,d)$ it holds

$\Rightarrow$ then there must exist $(a,b) = (c,d)$

transitive: for $(a,b), (c,d), (e,f) \in \mathbb{Z}\times\mathbb{Z}$, if $(a,b) \preceq (c,d)$ and $(c,d) \preceq (e,f)$

$\Rightarrow \underbrace{(a,b) = (c,d)}_{①}$ or $\underbrace{a^2+b^2 < c^2+d^2}_{②}$ and $\underbrace{(c,d) = (e,f)}_{③}$ or $\underbrace{c^2+d^2 < e^2+f^2}_{④}$

if satisfy ①③: $\Rightarrow (a,b) = (c,d) = (e,f) \Rightarrow (a,b) \preceq (e,f)$

if satisfy ①④: $(a,b) = (c,d)$ $a^2+b^2 = c^2+d^2 < e^2+f^2 \Rightarrow (a,b) \preceq (e,f)$

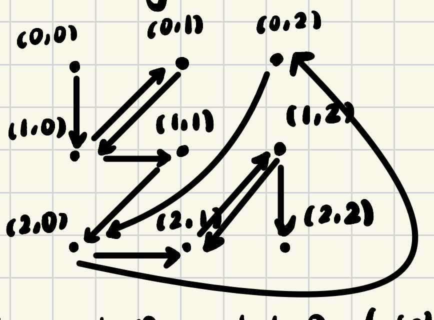
if satisfy ②③: $(c,d) = (e,f)$ $a^2+b^2 < c^2+d^2 = e^2+f^2 \Rightarrow (a,b) \preceq (e,f)$

if satisfy ②④: $a^2+b^2 < c^2+d^2 < e^2+f^2 \Rightarrow (a,b) \preceq (e,f)$

$\Rightarrow (a,b) \preceq (e,f)$

in conclusion: $(\mathbb{Z}\times\mathbb{Z}, \preceq)$ is poset

Hasse diagram:



Q17. (a) true, let $R = \{\{0\}, \{0,1\}, \{0,1,2\} \cdots\}$, for any set $\{0,1,2,\cdots k\}$ there exists $\{0,1,2,\cdots k, k+1\} \subsetneq R$, that $\{0,1,2,\cdots k\} \subseteq \{0,1,2,\cdots k+1\}$

$\Rightarrow$ no maximal element

(b) false, choose the subset B with minimal elements, then there doesn't exisit any set $S \subseteq R$, that $S \subsetneq B$.

(c) false, for minimal element must exist

Q18. (1) 24, 45 (2) 3, 5 (3) no, $24 \nmid 45$ (4) no, $3 \nmid 5$

(5) 15, 45 (6) 15 (7) 3, 5, 15 (8) 15