


Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Application of Number Theory

- G. H. Hardy (1877 - 1947)

  In his 1940 autobiography *A Mathematician's Apology*, Hardy wrote

  "The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present, **almost as 'useless' as the theory of numbers**."

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  "The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present, **almost as 'useless' as the theory of numbers**."

If he could see the world now, Hardy would be spinning in his grave.

Number Theory

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  RSA 公钥加密机制

- At one point, the largest employer of mathematicians in the United States, and probably the world, was the National Security Agency (NSA). The NSA is the largest spy agency in the US (bigger than CIA, Central Intelligence Agency), and has the responsibility for code design and breaking.

Division

- If a and b are integers with $a \neq 0$, we say that *a divides b* if there is an integer k such that $b = ak$, or equivalently b/a is an integer. In this case, we say that a is a *factor* or *divisor* of b, and b is a *multiple* of a. (We use the notations $a|b$, $a \nmid b$)

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Example

- $4 \mid 24$
- $3 \nmid 7$

Divisibility

- **All integers divisible by** $d > 0$ can be enumerated as:

$$\ldots, -kd, \ldots, -2d, -d, 0, d, 2d, \ldots, kd, \ldots$$

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- **Question:** Let n and d be two positive integers. How many positive integers not exceeding n are divisible by d?

Answer: Count the number of integers such that $0 < kd \leq n$. Therefore, there are $\lfloor n/d \rfloor$ such positive integers.

Divisibility

- **Properties**

  Let a, b, c be integers. Then the following hold:

  (i) if $a|b$ and $a|c$, then $a|(b+c)$
  (ii) if $a|b$ then $a|bc$ for all integers c
  iii) if $a|b$ and $b|c$, then $a|c$

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Proof. (1) $a|b$ $a|c$
$b=ak_1$ $c=ak_2$
$b+c=a(k_1+k_2)$

(2) $a|b$ $b=ak$
$bc=a(kc)$

(3) $b=ak_1$
$c=bk_2$
$c=a(k_1k_2)$

Divisibility

- **Corollary** If a, b, c are integers, where $a \neq 0$, such that $a|b$ and $a|c$, then $a|(mb + nc)$ whenever m and n are integers.

线性组合

Divisibility

- **Corollary** If a, b, c are integers, where $a \neq 0$, such that $a|b$ and $a|c$, then $a|(mb + nc)$ whenever m and n are integers.

Proof. By part (ii) and part (i) of Properties.

The Division Algorithm

- If a is an integer and d a positive integer, then there are **unique** integers q and r, with $0 \leq r < d$, such that $a = dq + r$. In this case, d is called the *divisor*, a is called the *dividend*, q is called the *quotient*, and r is called the *remainder*.

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  In this case, we use the notations $q = a \ div \ d$ and $r = a \bmod d$.

Congruence Relation

- If a and b are integers and m is a positive integer, then a is *congruent to b modulo m if m divides $a - b$*, denoted by $a \equiv b \pmod{m}$. This is called *congruence* and m is its *modulus*.

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Example

- $15 \equiv 3 \pmod{6}$
- $-1 \equiv 11 \pmod{6}$

More on Congruences

- Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

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Proof.

"only if" part

"if" part

$(\mathrm{mod}\ m)$ and $\mathrm{mod}\, m$ Notations

- $a \equiv b \pmod{m}$ and $a \bmod m = b$ are different.
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Proof. 标准除法 找范围

$a = q_1 m + r_1$

$b = q_2 m + r_2$

$0 \le r_1, r_2 < m$

$a \equiv b \bmod m \Leftrightarrow m \mid a-b$

$\Rightarrow m \mid (q_1 - q_2)m + r_1 - r_2$

$m \mid (q_1 - q_2)m$

$\Rightarrow m \mid r_1 - r_2$ and $-m < r_1 - r_2 < m \Rightarrow r_1 = r_2$

then $a \bmod m = r_1$

$b \bmod m = r_2$

$a \bmod m = b \bmod m$

Congruences of Sums and Products

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Proof. $m \mid a-b$, $m \mid c-d$ $\Rightarrow$ $m \mid (a-b)+(c-d)$ $\Rightarrow a+c \equiv b+d$

$m \mid (a-b)c$

$m \mid (c-d)b$

相加 $m \mid ac-bc+bc-db$

凑出倍数相消 $\Rightarrow m \mid ac-bd$

Algebraic Manipulation of Congruences

- If $a \equiv b \bmod m$, then
  - $c \cdot a \equiv c \cdot b \pmod{m}$?
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$\gcd(c, m) = 1$ 才可以除

$14 \equiv 8 \pmod 6$ but $7 \not\equiv 4 \pmod 6$

Computing the mod Function

- **Corollary** Let m be a positive integer and let a and b be integers. Then

$$(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$

$$ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$$

$2^{1000} \bmod 23$ 每次 >23 就 mod 23

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- Let $\mathbf{Z}_m$ be the set of nonnegative integers less than m: $\{0, 1, \ldots, m-1\}$.

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  Example

  - $7 +_{11} 9 = ?$ 5
  - $7 \cdot_{11} 9 = ?$ 8

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- **Distributivity**: if $a, b, c \in \mathbf{Z}_m$, then
  $a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$ and
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单位元(么元)
$0 +_m a = a$
$1 \cdot_m a = a$
$a +_m \bar{a} = 0$

加法逆

乘法逆不一定存在

mod m 的整数环
ring 环

Group

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  Example:
  $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{M}_{n\times n}, +)$?
  $(\mathbb{Z}^*, \times)$, $(\mathbb{Q}^*, \times)$, $(\mathbb{R}^*, \times)$, $(\mathbb{M}^*_{n\times n}, \cdot)$?

Permutation Group

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  For example, $s_3 =< 1, 2, 3 >$
  $P_3 = \{< 1, 2, 3 >, < 1, 3, 2 >, < 2, 1, 3 >, < 2, 3, 1 >, < 3, 1, 2 >, < 3, 2, 1 >\}$

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- Define a binary operation $\circ$ on the elements of P_n: for $\rho, \pi \in P_n$, $\pi \circ \rho$ denotes a *re-permutation* of the elements of ρ according to the elements of π.

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- Consider $s_3 =< 1, 2, 3 >$, and
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- $\pi =< 3, 2, 1 >$, $\rho =< 1, 3, 2 >$, what is $\pi \circ \rho$? <2,3,1>

  根据元中顺序重排 ρ

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- We can verify the other three properties.

  $\rho_1 \circ (\rho_2 \circ \rho_3) = (\rho_1 \circ \rho_2) \circ \rho_3$

  $< 1, 2, 3 > \circ \rho = \rho \circ < 1, 2, 3 > = \rho$

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  $< 1, 2, 3 > \circ \rho = \rho \circ < 1, 2, 3 > = \rho$

  For each $\rho \in P_3$, there exists another unique $\pi \in P_3$ such that

  $\rho \circ \pi = \pi \circ \rho =< 1, 2, 3 >$

$(P_n, \circ)$ is called a *permutation group*.

$G = (V, E)$ 看图置换后是否仍具某些性质

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- If the group operation is referred to as addition (*multiplication*), then the group also allows for *subtraction* (*division*).

  $a - b = a + (-b)$ 在加法下逆元

  $a/b = a \cdot b^{-1}$ 在乘法下逆元

Ring

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  Associativity: $(a \times b) \times c = a \times (b \times c)$

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Group 群 $(G, +)$

closure
associativity
identity
inverse

+ abelian group

Ring 环 $(R, +, \times)$

closure
associativity
distributivity

+ "×" commutativity

+ commutative ring

"×" commutative

integral domain (整环)

+ "×" identity
nonzero product
if $ab=0$ then a or b must be 0

field 域

+ "×" inverse

finite field

characteristic p $\quad \mathbb{F}_{p^m} \cong \mathbb{F}_p^m$

Commutative Ring, Integral Domain

- A *ring* is *commutative* if the multiplication operation is *commutative* for all elements in the ring. ($ab = ba$)

交换环　　$(M, +, \cdot)$ 不是交换环

Commutative Ring, Integral Domain

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- An *integral domain* $(R, +, \times)$ is a *commutative ring* that satisfies the following two additional properties.

  Identity element for multiplication: $a1 = 1a = a$
  Nonzero product for any two nonzero elements:
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$(Z_m, +_m, \times_m)$ 不是整环

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  $(\mathbb{Z}_m, +, \times)$, $(\mathbb{M}_{n\times n}, +, \cdot)$?

Field

- A *field*, denoted by $(F, +, \times)$, is an *integral domain* whose elements satisfy the following additional property.

  Inverse for multiplication: For every $a \in F$, there exists an element b, denoted by a^{-1}, such that $ab = ba = 1$.

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  $(\mathbb{Z}_p, +, \times)$?

Representations of Integers

- We may use *decimal* (*base* 10) or *binary* or *octal* or *hexadecimal* or other notations to represent integers.

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- Let $b > 1$ be an integer. Then if n is a positive integer, it can be expressed uniquely in the form $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$, where k is nonnegative, a_i's are nonnegative integers less than b. The representation of n is called *the base-b expansion of n* and is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$. $0 \le a_i < b$

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Example

- $(101011111)_2 = 2^8 + 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 351$
- $(7016)_8 = 7 \cdot 8^3 + 1 \cdot 8 + 6 = 3598$

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  Example

  - $(101011111)_2 = 2^8 + 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 351$
  - $(7016)_8 = 7 \cdot 8^3 + 1 \cdot 8 + 6 = 3598$

- Conversions between binary, octal, hexadecimal expansions are easy.

Base-b Expansions

- To get the decimal expansion is easy.

  Example

  - $(101011111)_2 = 2^8 + 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 351$
  - $(7016)_8 = 7 \cdot 8^3 + 1 \cdot 8 + 6 = 3598$

- Conversions between binary, octal, hexadecimal expansions are easy.

  Example

  - $(101011111)_2 = (\underline{101}\overline{011}\underline{111}) = (537)_8$
  - $(7016)_8 = (\underline{111}\overline{000}\underline{001}\overline{110})_2$
    $= (\underline{1110}\overline{0000}\underline{1110})_2 = (E0E)_{16}$

Base-b Expansions

$$
\begin{aligned}
n &= a_k b^k + a_{k-1} b^{k-1} + a_{k-2} b^{k-2} + \cdots + a_2 b^2 + a_1 b + a_0 \\
&= b(a_k b^{k-1} + a_{k-1} b^{k-2} + a_{k-2} b^{k-3} + \cdots + a_2 b + a_1) + a_0 \\
&= b(b(a_k b^{k-2} + a_{k-1} b^{k-3} + a_{k-2} b^{k-4} + \cdots + a_2) + a_1) + a_0 \\
&= \cdots
\end{aligned}
$$

Base-b Expansions

$$
\begin{aligned}
n &= a_k b^k + a_{k-1} b^{k-1} + a_{k-2} b^{k-2} + \cdots + a_2 b^2 + a_1 b + a_0 \\
&= b(a_k b^{k-1} + a_{k-1} b^{k-2} + a_{k-2} b^{k-3} + \cdots + a_2 b + a_1) + a_0 \\
&= b(b(a_k b^{k-2} + a_{k-1} b^{k-3} + a_{k-2} b^{k-4} + \cdots + a_2) + a_1) + a_0 \\
&= \cdots
\end{aligned}
$$

To construct the base-b expansion of an integer n,

- Divide n by b to obtain $n = bq_0 + a_0$, with $0 \leq a_0 < b$
- The remainder a_0 is the rightmost digit in the base-b expansion of n. Then divide q_0 by b to get $q_0 = bq_1 + a_1$ with $0 \leq a_1 < b$
- a_1 is the second digit from the right. Continue by successively dividing the quotients by b until **the quotient is** 0

Algorithm: Constructing Base-b Expansions

procedure *base b expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while ($q \neq 0$)
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return(a_{k-1} ,..., a_1,a_0){(a_{k-1} ... a_1a_0)$_b$ is base b expansion of n}

$k \times O(q \div b)$

bit operations $O(k)$ $O(\log n)$

Example

- $(12345)_{10} = (30071)_{8}$

Example

- $(12345)_{10} = (30071)_8$

$$12345 = 8 \cdot 1543 + 1$$

$$1543 = 8 \cdot 192 + 7$$

$$192 = 8 \cdot 24 + 0$$

$$24 = 8 \cdot 3 + 0$$

$$3 = 8 \cdot 0 + 3$$

Binary Addition of Integers

$a = (a_{n-1}a_{n-2}\ldots a_1a_0)$, $b = (b_{n-1}b_{n-2}\ldots b_1b_0)$

procedure *add*(a, b: positive integers)

{the binary expansions of a and b are $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and $(b_{n-1},b_{n-2},\ldots,b_0)_2$, respectively}

$c := 0$

for $j := 0$ to $n-1$

    $d := \lfloor (a_j + b_j + c)/2 \rfloor$

    $s_j := a_j + b_j + c - 2d$

    $c := d$

$s_n := c$

return$(s_0,s_1,\ldots, s_n)$ {the binary expansion of the sum is $(s_n,s_{n-1},\ldots,s_0)_2$}

Binary Addition of Integers

$a = (a_{n-1}a_{n-2}\ldots a_1a_0)$, $b = (b_{n-1}b_{n-2}\ldots b_1b_0)$

procedure *add*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and $(b_{n-1},b_{n-2},\ldots,b_0)_2$, respectively}
$c := 0$
for $j := 0$ to $n-1$
    $d := \lfloor (a_j + b_j + c)/2 \rfloor$
    $s_j := a_j + b_j + c - 2d$
    $c := d$
$s_n := c$
return $(s_0,s_1,\ldots,s_n)$ {the binary expansion of the sum is $(s_n,s_{n-1},\ldots,s_0)_2$}

$O(n)$ bit additions

Algorithm: Binary Multiplication of Integers

$a = (a_{n-1}a_{n-2}\ldots a_1a_0)_2$, $b = (b_{n-1}b_{n-2}\ldots b_1b_0)_2$

$$\begin{aligned} ab &= a(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1}2^{n-1}) \\ &= a(b_0 2^0) + a(b_1 2^1) + \cdots + a(b_{n-1}2^{n-1}) \end{aligned}$$

procedure *multiply*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and $(b_{n-1},b_{n-2},\ldots,b_0)_2$, respectively}
for $j := 0$ to $n-1$
    if $b_j = 1$ **then** $c_j = a$ shifted j places
    else $c_j := 0$
{$c_0,c_1,\ldots, c_{n-1}$ are the partial products}
$p := 0$
for $j := 0$ to $n-1$
  $p := p + c_j$
return p {p is the value of ab}

Algorithm: Binary Multiplication of Integers

$a = (a_{n-1}a_{n-2}\ldots a_1a_0)_2$, $b = (b_{n-1}b_{n-2}\ldots b_1b_0)_2$

$$\begin{aligned} ab &= a(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1}2^{n-1}) \\ &= a(b_0 2^0) + a(b_1 2^1) + \cdots + a(b_{n-1}2^{n-1}) \end{aligned}$$

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return p {p is the value of ab}

$O(n^2)$ shifts and $O(n^2)$ bit additions

Algorithm: Computing div and mod

procedure *division algorithm* (a: integer, d: positive integer)

$q := 0$

$r := |a|$

while $r \geq d$

$r := r - d$

$q := q + 1$

if $a < 0$ and $r > 0$ **then**

$r := d - r$

$q := -(q+1)$

return (q, r) {$q = a$ ***div*** d *is the quotient,* $r = a$ **mod** d *is the remainder* }

$a<0$

$|a| = dq + r$

$-a = dq + r$

$a = -dq - r$

$= -d(q+1) + q - r$

Algorithm: Computing div and mod

procedure *division algorithm* (a: integer, d: positive integer)
$q := 0$
$r := |a|$
while $r \geq d$
    $r := r - d$
    $q := q + 1$
if $a < 0$ and $r > o$ **then**
    $r := d - r$
    $q := -(q+1)$
return (q, r) {$q = a$ ***div*** d *is the quotient,* $r = a$ **mod** d *is the remainder* }

$O(q \log a)$ bit operations. But there exist more efficient algorithms with complextiy $O(n^2)$, where $n = \max(\log a, \log d)$

Algorithm: Computing div and mod (cont)

- **procedure** *division2* $(a, d \in \mathbb{N},\ d \geq 1)$
  if $a < d$
  return $(q, r) = (0, a)$
  $(q, r) = \mathit{division2}(\lfloor a/2 \rfloor, d)$
  $q = 2q,\ r = 2r$
  if a is odd
  $r = r + 1$
  if $r \geq d$
  $r = r - d$
  $q = q + 1$
  return (q, r)

Algorithm: Computing div and mod (cont)

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  if a is odd
  $r = r + 1$
  if $r \geq d$
  $r = r - d$
  $q = q + 1$
  return (q, r)

  $O(\log q \log a)$ bit operations.

Algorithm: Binary Modular Exponentiation

$$b^n = b^{a_{k-1}\cdot 2^{k-1}+\cdots+a_1\cdot 2+a_0} = b^{a_{k-1}\cdot 2^{k-1}} \cdots b^{a_1\cdot 2} \cdot b^{a_0}$$

Successively finds $b \bmod m$, $b^2 \bmod m$, $b^4 \bmod m$, $\ldots$, $b^{2^{k-1}} \bmod m$, and multiplies together the terms b^{2^j} where $a_j = 1$.

procedure *modular exponentiation*(b: integer, $n = (a_{k-1}a_{k-2}\ldots a_1a_0)_2$, m: positive integers)
$x := 1$
$power := b$ **mod** m
for $i := 0$ to $k-1$
    if $a_i = 1$ **then** $x := (x \cdot power)$ **mod** m
    $power := (power \cdot power)$ **mod** m
return x {x equals b^n **mod** m}

Algorithm: Binary Modular Exponentiation

$$b^n = b^{a_{k-1}\cdot 2^{k-1}+\cdots+a_1\cdot 2+a_0} = b^{a_{k-1}\cdot 2^{k-1}} \cdots b^{a_1\cdot 2} \cdot b^{a_0}$$

Successively finds $b \bmod m$, $b^2 \bmod m$, $b^4 \bmod m$, $\ldots$, $b^{2^{k-1}} \bmod m$, and multiplies together the terms b^{2^j} where $a_j = 1$.

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    $power := (power \cdot power)$ **mod** m
return x {x equals b^n **mod** m }

$O((\log m)^2 \log n)$ bit operations

Next Lecture

- number theory, cryptography ...