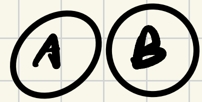
Discrete Math

12310520

芮煜涵

Q1 (a) False



When $A \cap B = \emptyset$, $A - B = A$ is true,

which means $\forall b (b \in B \wedge b \notin A)$;

then let $B \neq \emptyset$, then $\exists b (b \in B \wedge b \notin A)$, then $B \nsubseteq A$

(b) True

$(A \cap B \cap C) \subseteq A \subseteq (A \cup B)$

(c) False

when $\exists a (a \in A \wedge a \in B)$, then $B - A \subsetneq B$

then $\overline{(A-B)} \cap (B-A) \subseteq B - A \nsubseteq B$

$\overline{(A-B)} \cap (B-A) \neq B$

Q2 (1) False

A	B	C	$A \wedge B$	C-A	C-B	$C-(A \wedge B)$	$(C-A) \wedge (C-B)$
0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	1
0	1	0	0	0	0	0	0
0	1	1	0	1	0	1	0
1	0	0	0	0	0	0	0
1	0	1	0	0	1	1	0
1	1	0	1	0	0	0	0
1	1	1	1	0	0	0	0

(2) True

Prove $P(A) \cap P(B) \subseteq P(A \cap B)$: prove by case

if $x \notin A \wedge x \notin B$ then $\forall S\big((x \in S) \to S \notin P(A) \cap P(B)\big)$

if $x \in A \wedge x \notin B$ then $\forall S\big((x \in S) \to S \notin P(B)\big)$

then $\forall S\big((x \in S) \to S \notin P(A) \cap P(B)\big)$

if $x \notin A \wedge x \in B$, the same as above

Thus. $\forall x\big[\neg(x \in A \wedge x \in B) \to \forall S\,(x \in S) \to S \notin P(A) \cap P(B)\big]\big)$

then $\forall x \forall S\,\big(S \in P(A) \cap P(B) \wedge x \in S\big) \to \big(x \in P(A) \cap P(B)\big)$

$P(A) \cap P(B) \subseteq \{S \mid x \in A \wedge x \in B \wedge x \in S\}$

$= P(A \cap B)$

Prove $P(A \cap B) \subseteq P(A) \cap P(B)$: direct prove

$P(A \cap B) = \{S \mid x \in A \wedge x \in B \wedge x \in S\}$

$= \{S \mid x \in A \wedge x \in S\} \cap \{S \mid x \in B \wedge x \in S\}$

$= P(A) \cap P(B)$

Therefore : $P(A) \cap P(B) = P(A \cap B)$

(3) False

let $A = \{0\}$ $B = \{1\}$ $P(A) = \{\{0\}, \phi\}$ $P(B) = \{\{1\}, \phi\}$

$P(A \cup B) = \{\{0,1\}, \{0\}, \{1\}, \phi\} \neq P(A) \cup P(B)$

(4) False $f: x \to x^2$

$A = \{1\}$ $B = \{-1\}$

$f(A \cap B) = \emptyset$, $f(A) \cap f(B) = \{1\}$

$f(A \cap B) \neq f(A) \cap f(B)$

Q3. (a) Symmetric difference is associative.

A	B	C	$B \oplus C$	$A \oplus B$	$A \oplus (B \oplus C)$	$(A \oplus B) \oplus C$
0	0	0	0	0	0	0
0	0	1	1	0	1	1
0	1	0	1	1	1	1
0	1	1	0	1	0	0
1	0	0	0	1	1	1
1	0	1	1	1	0	0
1	1	0	1	0	0	0
1	1	1	0	0	1	1

(b) Yes

if $x \in C$, $A \oplus C = B \oplus C$ then $(x \in A \wedge x \in B) \vee (x \notin A \wedge x \notin B)$

if $x \notin C$, $A \oplus C = B \oplus C$ then $(x \in A \wedge x \in B) \vee (x \notin A \wedge x \notin B)$

Thus, any element should be in both A and B, or neither A nor B. $\Rightarrow A = B$

Q4. (a) countable

assume that there are n students in CS201

n is a finite integer.

then the cardinality is 2^n, finite.

(b) countable

Constructing the list: first list (a,b) with $a+b=0$, next list (a,b) with $a+b=1$, and so on.

$(0,0)$ ¦ $(1,0)$ $(0,1)$ ¦ $(2,0)$ $(1,1)$ $(0,1)$ ¦ ...

Therefore, $\{(a,b) \mid a,b \in N\}$ is countable

(c) uncountable

The set of real numbers $\mathbb{R}$ is uncountable

let $S = \{(a,b) \mid a=0, b \in R\}$

$f: x \to (0,x)$ $\quad |\mathbb{R}| \leq |S|$

$g: (0,x) \to x$ $\quad |S| \leq |\mathbb{R}|$

then $|\mathbb{R}| = |S|$ $\quad S$ is uncountable

$S \subseteq \{(a,b) \mid a \in N, b \in R\}$ Thus, uncountable.

Q5.

(a) $A = \mathbb{R}$ $\quad B = \{x \mid x \in R, x \neq 0\}$

(b) $A = \mathbb{R}$ $\quad B = \{x \mid x \in R, x \notin N\}$

(c) $A = \mathbb{R}$ $\quad B = \mathbb{Q}$

Q6. prove $A \subseteq B \rightarrow P(A) \subseteq P(B)$:

if $A \subseteq B$, then $\forall x \, (x \in A \rightarrow x \in B)$

$P(A) = \{S \mid \forall x \, (x \in S) \rightarrow (x \in A)\}$

$P(B) = \{S \mid \forall x \, (x \in S) \rightarrow (x \in B)\}$

$\{S \mid \forall x \, (x \in S) \rightarrow (x \in A)\} \subseteq \{S \mid \forall x (x \in S) \rightarrow (x \in B)\}$

$\Rightarrow P(A) \subseteq P(B)$

prove $P(A) \subseteq P(B) \rightarrow A \subseteq B$: prove by contradiction

Assume $P(A) \subseteq P(B) \wedge (A \nsubseteq B)$

for $A \nsubseteq B$ then $\exists x \, (x \in A \wedge x \notin B)$

let $S = \{x\}$ $S \in P(A) \wedge S \notin P(B)$

thus $P(A) \nsubseteq P(B)$, leading to a contradiction

Therefore $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$

Q7. $g \circ f : x \rightarrow x$

prove f is one-to-one: prove by contradiction

if f is not one-to-one, then

$\exists x, y \in A$ $x \neq y$ and $f(x) = f(y)$

then $(g \circ f)(x) = (g \circ f)(y)$, however $g \circ f = 1_A$

$g \circ f(x) = x$ $g \circ f(y) = y$ $g \circ f(x) \neq g \circ f(y)$

leading to a contradiction, thus f is one-to-one prove g is onto: prove by contradiction

if g is not onto, then

$\exists a, b \in f(A), a \neq b \land g(a) = g(b)$

for we have proved f is one-to-one,

thus, if $x, y \in A$ $f(x) = a, f(y) = b$ $a \neq b$

then $x \neq y$

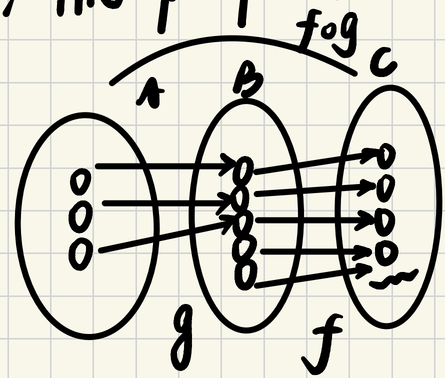
however $g \circ f(x) = g \circ f(y) = g(a) = g(b)$,

which leads to a contradiction, thus g is onto.

Therefore, the proposition holds.

Q8.

(a) No



x not one-to-one

(b) Yes, the proof is the same as (c)

(c) Yes, prove by contradiction

Assume that, $f \circ g$ is one-to-one and

g is not one-to-one, then

$\exists x, y \in A$ $x \neq y \land g(x) = g(y)$

and $f(g(x)) = f(g(y)) \Leftrightarrow f \circ g(x) = f \circ g(y)$

which leads to a contradiction.

Therefore, if $f \circ g$ is one to one, g must be one-to-one

(d) Yes, prove by contradiction

Assume that $f \circ g$ is onto and f is not onto

then $\forall c\big((c \in C) \rightarrow \exists a\,(f \circ g(a) = c)\big)$ (*)

$\exists c\big((c \in C) \wedge \forall b\,(b \in B \rightarrow f(b) \neq c)\big)$

let t, $t \in C \wedge \forall b\,(b \in B \rightarrow f(b) \neq t)$

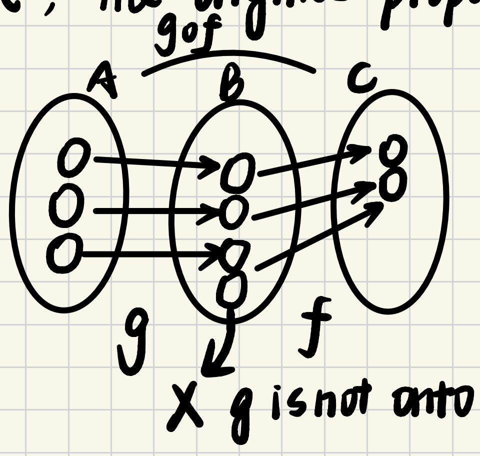
and for $f(A) \subseteq B$

then $t \in C \wedge \forall a\,(a \in B \rightarrow g \circ f(a) \neq t)$

which leads to contradiction with (*)

Therefore, the original proposition holds.

(e) No



Q9. $(k+1)^3 - k^3 = 3k^2 + 3k + 1$

$\Rightarrow \sum_{k=1}^{n}\big((k+1)^3 - k^3\big) = \sum_{k=1}^{n}(3k^2 + 3k + 1)$

$$= 3\sum_{k=1}^{n} k^2 + 3\frac{(1+n)n}{2} + n$$

And notice that $\sum_{k=1}^{n} (k+1)^3 - k^3$

$$= (n+1)^3 - n^3 + n^3 - (n-1)^3 \cdots + 2^3 - 1^3$$

$$= (n+1)^3 - 1$$

thus, $n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n} k^2 + \frac{5}{2}n + \frac{3}{2}n^2$

$$\Rightarrow \sum_{k=1}^{n} k^2 = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Q10. $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$

$$\sum_{k=1}^{n}\left((k+1)^4 - k^4\right) = \sum_{k=1}^{n} (4k^3 + 6k^2 + 4k + 1)$$

then $(n+1)^4 - 1 = 4\sum_{k=1}^{n} k^3 + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n$

$$\Rightarrow \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Q11. $\underbrace{\sqrt{0}}_{1}, \underbrace{\sqrt{1}, \sqrt{2}, \sqrt{3}}_{3}, \underbrace{\sqrt{4}, \sqrt{5} \cdots \sqrt{8}}_{5}, \sqrt{9}$

from $\sqrt{b^2}$ to $\sqrt{(b+1)^2-1}$ where $b \in \mathbb{N}^*$

$\lfloor \sqrt{k} \rfloor$ holds the same, $\lfloor \sqrt{b^2} \rfloor = \lfloor \sqrt{(b+1)^2 - 1} \rfloor = b$

$(b+1)^2 - 1 - b^2 + 1 = 2b + 1$, $2b+1$ numbers

$$\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \sum_{k=0}^{\lfloor \sqrt{m} \rfloor^2 - 1} \lfloor \sqrt{k} \rfloor + \sum_{k=\lfloor \sqrt{m} \rfloor^2}^{m} \lfloor \sqrt{k} \rfloor$$

$$= \sum_{k=0}^{\lfloor \sqrt{m} \rfloor - 1} k(2k+1) + \sum_{k=\lfloor \sqrt{m} \rfloor^2}^{m} \lfloor \sqrt{k} \rfloor$$

$$= \frac{2}{3}(\lfloor \sqrt{m} \rfloor - 1)^3 + (\lfloor \sqrt{m} \rfloor - 1)^2 + \frac{1}{3}(\lfloor \sqrt{m} \rfloor - 1) + \frac{(\lfloor \sqrt{m} \rfloor - 1)\lfloor \sqrt{m} \rfloor}{2} + \lfloor \sqrt{m} \rfloor (m - \lfloor \sqrt{m} \rfloor + 1)$$

$$= m\lfloor \sqrt{m} \rfloor - \frac{1}{3}\lfloor \sqrt{m} \rfloor^3 - \frac{1}{2}\lfloor \sqrt{m} \rfloor^2 + \frac{5}{6}\lfloor \sqrt{m} \rfloor$$

Q12. $|A| = |B|$, $|C| = |D|$ then

one-to-one correspondence between A and B, C and D

assume that $f: A \to B$ $g: C \to D$

$A \times C = \{(a, c) \mid a \in A, c \in C\}$ $B \times D = \{(b, d) \mid b \in B, d \in D\}$

$f: a_i \to b_i$ $f^{-1}: b_i \to a_i$ $g: c_i \to d_i$ $g^{-1}: d_i \to c_i$

then we can define $h: (a_i, c_i) \to (b_i, d_i)$

and $k: (b_i, d_i) \to (a_i, c_i)$ $(k = h^{-1})$

Thus, $|A \times C| = |B \times D|$

Q13.

We can prove by contradiction

There is at least one in $|A| \geq |B|$ and $|A| \leq |B|$ to be true. We assume $|A| \geq |B|$ and not $|A| \leq |B|$

(The same proof for the opposite case.)

then $|A| \neq |B|$, otherwise leads to contradiction to "Not $|A| \leq |B|$", thus $|A| > |B|$.

It means that the cardinality of A is larger than the cardinality of B, leading to contradiction to assumption that "A and B have the same cardinality".

Therefore, $|A| \leq |B|$ and $|B| \leq |A|$

Q14. $f: A \to B$ is onto, then $\forall b (b \in B \to \exists a (a \in A \land f(a) = b))$

then $|B| \leq |A|$. And A is countable, then B is countable

Q15. prove one-to-one: prove by contradiction

Assume $\exists (a,b) \neq (c,d) \land f(a,b) = f(c,d)$ $a,b,c,d \in \mathbb{Z}^+$

then $\frac{(a+b-2)(a+b-1)}{2} + a = \frac{(c+d-2)(c+d-1)}{2} + c$

$\Rightarrow$ $(a+b)^2 - 3(a+b) \cancel{+2} + 2a = (c+d)^2 - 3(c+d) \cancel{+2} + 2c$

if $a+b = c+d$, then $a \neq c \Rightarrow f(a,b) \neq f(c,d)$

if $a+b \neq c+d$ then $(a+b+c+d)(a+b-c-d)$

assume $a+b = c+d+t$ $\quad = 3(a+b-c-d) + 2(c-a)$

$t \in \mathbb{Z}^+$ (The opposite is the same)

$\Rightarrow (a+b+c+d)t = 3t + 2(c-a)$ for $a,b,c,d \in \mathbb{Z}^+$

$$(a+b+c+d) = 3 + \frac{2(c-a)}{t} \Rightarrow t \mid 2(c-a)$$

if $c-a=0$ then $b \neq d \Rightarrow 2a+b+d=3$ but $2a+b+d \geq 4$, thus $c-a \neq 0$

then $a+b-c-d \leq 2(c-a)$

$$\Rightarrow \quad \begin{matrix} 3a+b \leq 3c+d \\ a+b > c+d \end{matrix} \Rightarrow \begin{matrix} a<c \\ b>d \end{matrix}$$

let $c=a+p \quad p \geq 1 \quad b=d+q \quad q \geq 2$ and $q>p$

then $\quad 2(a+d)+(p+q) = 3 + \dfrac{2p}{q-p}$

$$3+\frac{2p}{q-p} \leq 3+2p \, , \quad 2(a+d)+(p+q) \geq 2\times 2+2p$$

then $\quad 3+\dfrac{2p}{q-p} < 2(a+d)+(p+q)$

leads to contradiction $\Rightarrow f(a,b) \neq f(c,d)$

Therefore $\forall (a,b),(c,d) \; (a,b) \neq (c,d) \rightarrow f(a,b) \neq f(c,d)$

$\Rightarrow$ one - to - one

prove onto: to prove $\forall x(x \in \mathbb{Z}^+) \rightarrow \exists (a,b)\big((a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \wedge f(a,b) = x\big)$

to find a constructive proof

$$x = \frac{(a+b-2)(a+b-1)}{2} + a$$

$$\Rightarrow 2x = (a+b)^2 - 3(a+b) + 2 + 2a$$

$x=1 \Rightarrow (a,b)=(1,1) \quad x=2 \Rightarrow (a,b)=(1,2)$

$x=3 \Rightarrow (a,b)=(2,1) \quad x=4 \Rightarrow (a,b)=(3,1)$

① when $a+b=s$ a from 1 to $s-1$

x increase one for each time

② if $a+b=c+d+1=s+1$

$f(a,b)_{min}=\frac{1}{2}(s-1)s+1$

$f(c,d)_{max}=\frac{1}{2}(s-2)(s-1)+(s-1)=\frac{1}{2}(s-1)s$

we find that $f(a,b)$ is always larger than $f(c,d)$

for (a,b) $a+b=s$ we have $(s-1)$ pairs in total

$S=2$ $f(a,b)=1$

$S=3$ $f(a,b)=2,3$

$S=4$ $f(a,b)=4,5,6$

... In this way, we can cover $\mathbb{Z}^+$

with $\{f(a,b) \mid a,b\in\mathbb{Z}^+\}$ $\Rightarrow$ onto

Therefore, we finish the proof.

Q16. $A=(0,1)$ $B=[0,1]$

$A\to B: f(x)=x$

$B\to A: f(x)=\frac{1}{2}x+\frac{1}{3}$

therefore $|A|=|B|$

Q17. $f(x)=\Theta(g(x))$ $g(x)=\Theta(h(x))$

$\exists c_1, x_1$ $|f(n)|\leq c_1|g(n)|$ whenever $n>x_1$

$\exists c_2, x_2$ $|f(n)|\geq c_2|g(n)|$ whenever $n>x_2$

$\exists c_3, x_3 \quad |g(x)| \le c_3 |h(x)|$ whenever $n > x_3$

$\exists c_4, x_4 \quad |g(x)| \ge c_3 |h(x)|$ whenever $n > x_4$

$\Rightarrow \quad |f(n)| \le c_1 c_3 |h(x)|$ whenever $n > \max\{x_1, x_3\}$

$|f(n)| \ge c_2 c_4 |h(x)|$ whenever $n > \max\{x_2, x_4\}$

Q18. False

$f_1(x) = x$, $f_2(x) = x$ then $g(x) = x$

$(f_1 - f_2)(x) = 0 \qquad 0 \ne \Theta(g(x))$

Q19.

① prove $|f(n)| \le c_1 |x^n|$

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

$= x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)$

$\le x^n \left(a_n + k\left(\frac{1}{x} + \cdots + \frac{1}{x^{n-1}} + \frac{1}{x^n}\right)\right) \quad k = \max\{a_{n-1}, \cdots a_1, a_0\}$

$= x^n \left(a_n + k \frac{1 - \frac{1}{x^n}}{x-1}\right) \le x^n \left(a_n + \frac{k}{x-1}\right)$

let $c_1 = a_n + k \quad x_0 = 2$

$|f(n)| \le c_1 |x^n|$ whenever $n > a_n + k$

(or $n > a_0 + a_1 + \cdots + a_n$)

② prove $|f(n)| \ge c_2 |x^n|$

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \ge a_n x^n$

let $c_2 = a_n \quad x_0 = 2$

$|f(n)| \ge c_2 |x^n|$ whenever $n > a_n$

Q20. $\log_a n = \log_2 n \cdot \log_a 2$, $\log_a 2 > 0$

We define $f(n) = \Theta(\log_a n)$, then prove $f(n) = \Theta(\log_2 n)$

$|f(n)| \geq c_1 |\log_a n|$ whenever $x > x_1$

$|f(n)| \leq c_2 |\log_a n|$ whenever $x > x_2$

$\Rightarrow$ $|f(n)| \geq c_1 |\log_2 n \cdot \log_a 2| = c_1 \log_a 2 \, |\log_2 n|$

whenever $x > x_1$ let $c_1 \log_a 2 = c_3$

$|f(n)| \leq c_2 |\log_2 n \cdot \log_a 2| = c_2 \log_a 2 \, |\log_2 n|$

whenever $x > x_2$ let $c_2 \log_a 2 = c_4$

thus, $f(n) = \Theta(\log_2 n) \Rightarrow \Theta(\log_a n) = \Theta(\log_2 n)$

Q21. multiplication:

for power: = power $*c$ n times

for $y := y + a_i *$ power n times

$2n$ times of multiplication

addition:

for $y := y + a_i *$ power n times

n times of addition

Q22. multiplication:

for $y := y * c + a_{n-i}$ n times

n times of multiplication

addition :

for $y := y^* c + a_{n-i}$ n times

n times of addition

Q23. (1) let $h(t) = 2^t - t^2$

$h'(t) = \ln 2 \cdot 2^t - 2t$

when $t \geq 4$ $h'(t) > 0$ $h(t)$ increase

$h(4) = 2^4 - 4^2 = 0 \Rightarrow 2^t - t^2 > 0$ whenever $t \geq 5$

$2^t + t \geq t^2$ whenever $t \geq 5$

let $t = \log(\log n)$ then $2^t = \log n$; $\log n \geq 2^5$; $n \geq 2^{32}$

$\log n + \log(\log n) \geq \log(\log n) \cdot \log(\log n)$ whenever $n \geq 2^{32}$

$\Rightarrow \log(n(\log n)) \geq (\log(\log n))^2$

$\log(\log n^n) \geq (\log(\log n))^2$

$\Rightarrow \log n^n \geq (\log n)^{\log\log n}$

thus $c = 1$, $n_0 = 2^{32}$

$|(\log n)^{\log\log n}| \leq c\, |\log n^n|$ whenever $n \geq n_0$

then $(\log n)^{\log\log n} = O(\log(n^n))$

(2) $f_1(n) = \log n$ $\qquad f_2(n) = (\log n)^{\log\log n}$

$f_3(n) = n\log n$ $\qquad f_4(n) = n^2$

$f_5(n) = (\log\log n)^{\log n}$ $f_6(n) = (\log n)^{\log n}$

$f_7(n) = n^{\log n}$ $f_8(n) = 3^{n/2}$

$f_9(n) = (\log n)^n$

Q24. ACE