CS215: Discrete Math (H)
2024 Fall Semester Written Assignment # 4
Due: Dec. 9th, 2024, please submit at the beginning of class

Q.1 Use induction to prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Q.2 Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a - b)$.

Q.3 A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

Q.4 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Q.5 Devise a recursive algorithm to find a^{2^n}, where a is a real number and n is a positive integer. (use the equality $2^{2^{n+1}} = (a^{2^n})^2$)

Q.6 Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

(a) Find $f(16)$

(b) Find a big-O estimate for $f(n)$. [Hint: make the substitution $m = \log n$.]

Q.7 The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in *closed form* (in terms of b if necessary) and should *not* use summation.

(b) Find a solution for $T(n)$. Your solution should be in *closed form* (in terms of a and c if necessary) and should *not* use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Q.8 How many bit strings of length 6 have at least one of the following properties:

- start with 010
- start with 11
- end with 00

State clearly how you count and get your answer.

Q.9 Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Q.10 How many 6-card poker hands consist of exactly 2 pairs? That is two of one rank of card, two of another rank of card, one of a third rank, and one of a fourth rank of card? Recall that a deck of cards consists of 4 suits each with one card of each of the 13 ranks. You should leave your answer as an equation.

Q.11 How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Q.12 Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10.$$

with five variables.

(1) Count the number of integer solutions, with $x_1 \geq 3$, $x_2 \geq 0$, $x_3 \geq -2$, $x_4 \geq 0$, and $x_5 \geq 0$.

(2) Count the number of integer solutions, with $0 \leq x_1 \leq 5$ and $x_2, x_3, x_4, x_5 \geq 0$.

Q.13 How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Q.14 Show that if p is a prime and k is an integer such that $1 \leq k \leq p - 1$, then p divides $\binom{p}{k}$.

Q.15 Prove the hockeystick identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

Q.16 For $0 \leq k \leq n$, show that

$$\sum_{r=k}^{n} \binom{n}{r} \binom{r}{k} = \binom{n}{k} 2^{n-k}.$$

Your proof may be either combinatorial or algebraic.

Q.17 Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

Q.18 A computer system considers a string of decimal digits $(0, 1, \ldots, 9)$ to be a **valid** code word if and only if it contains an **odd number of zero digits**. For example, 12030 and 11111 are **not** valid, but 29046 is. Let $V(n)$ denote the number of valid n-digit code words. Find a recurrence relation for $V(n)$ with initial cases, and give a closed-form solution to this recurrence relation. Please explain how you find the recurrence relation. (Hint: notice that the number of non-valid code words is equal to $10^n - V(n)$.)

Q.19 Let $\mathbf{A}_n$ be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of $\mathbf{A}_n$. Solve this recurrence relation to find a formula for d_n.

Q.20 Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

Q.21 Use generating functions to prove Vandermonde's identity:

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k),$$

whenever m, n, and r are nonnegative integers with r not exceeding either m or n. [Hint: Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]

Q.22 Generating functions are very useful, for example, provide an approach to solving linear recurrence relations. Read pp. 537-548 of the textbook. [You do not need to write anything for this problem on your submitted assignment paper.]