Question 6.1

1、Loop invarient: At the end of each inner loop, $A[j-1]$ is not larger than any elements in $A[j \cdots A.length]$

Initialization: when $j = A.length$, $j-1 = A.length$, if $A[j] > A[j-1]$ then the loop invarient holds; otherwise, exchange $A[j]$ with $A[j-1]$ then the loop invarient holds.

Maintanence: When $j=k$, $A[k-1]$ is not larger than any elements in $A[k \cdots A.length]$; In the next loop, $j=k-1$, if $A[k-1] \geq A[k-2]$, loop invarient holds; otherwise exchange $A[k-1]$ with $A[k-2]$. After exchanging, $A[k-2] < A[k-1] \leq A[k \cdots A.length]$ it holds.

Termination: When $j = i+1$, after the inner loop $A[i] \leq A[i+1] \leq A[i+2 \cdots A.length]$, and $A[i]$ is the smallest element in $A[i \cdots A.length]$, loop invarient holds.

2、loop invarient: At the end of each outer loop, $A[1 \cdots i]$ is sorted and have the smallest elements.

Initialization: when $i=1$, after the inner loop, $A[1]$ is the smallest element in $[1 \cdots A.length]$, loop invarient holds.

Maintanence: When $i=k$, $A[1 \cdots k]$ is sorted and have the smallest elements; In the next loop, after the termination inner loop, $A[k+1]$ is the smallest element in $A[k+1 \cdots A.length]$, and $A[k+1] \geq A[1 \cdots k]$. The loop invarient holds.

Termination: When $i = A.length - 1$, there is no exchange in inner loop, then $A[A.length]$ is the largest element and

A[1 ⋯ A.length−1] has been sorted. The loop invarient holds.

3. The best case:

1: n times

2: $(n-i+1)$ times

3: $n-i$ times

4: 0 times

$$n+\sum_{i=1}^{n-1}\left((n-i+1)+(n-i)\right)$$

$$=n+\frac{n-1}{2}(2n-1+3)$$

$$=n^2+n-1=\Theta(n^2)$$

The worst case:

1: n times

2: $(n-i+1)$ times

3: $n-i$ times

4: $n-i$ times

$$n+\sum_{i=1}^{n-1}\left((n-i+1)+2(n-i)\right)$$

$$=n+\frac{n-1}{2}(n+n-1+n-1+2+1+1)$$

$$=\frac{3n^2+n-2}{2}=\Theta(n^2)$$

$\Rightarrow$ runtime: $\Theta(n^2)$

Question 6.2

1. $Pr(Z_7 \text{ is compared to } Z_2)=\frac{2}{7-2+1}=\frac{1}{3}$
2. $Pr(Z_8 \text{ is compared to } Z_9)=\frac{2}{9-8+1}=1$
3. $Pr(Z_1 \text{ is compared to } Z_9)=\frac{2}{9-1+1}=\frac{2}{9}$
4. $Pr(Z_7 \text{ is compared to } Z_3)=\frac{2}{7-3+1}=\frac{2}{5}$

Question 6.3

We prove $T(n)\geq cn\log n$ by induction $c>0$

① base case: when $n=1$ $T(n)\geq c\cdot 1\log 1=0$ it holds

② induction step: for every k, $k\leq n$ that $T(k)\geq ck\log k$

$T(n+1)=T(x)+T(n-x)+\Theta(n+1)$ it means that there are x elements smaller than pivot

$\geq cx\log x+c(n-x)\log(n-x)+\Theta(n+1)$

then let $f(x) = cx\log x + c(n-x)\log(n-x)$

$$\frac{df(x)}{dx} = c\left(\log x + 1 - 1 - \log(n-x)\right)$$
$$= c\,\log\frac{x}{n-x}$$

let $\frac{df(x)}{dx} = 1 \Rightarrow x = \frac{n}{2}$ where we have $f(x)_{min}$

then $T(x) + T(n-x) \geqslant 2T(\frac{n}{2}) \geqslant 2c\frac{n}{2}\log\frac{n}{2} = cn\log\frac{n}{2}$

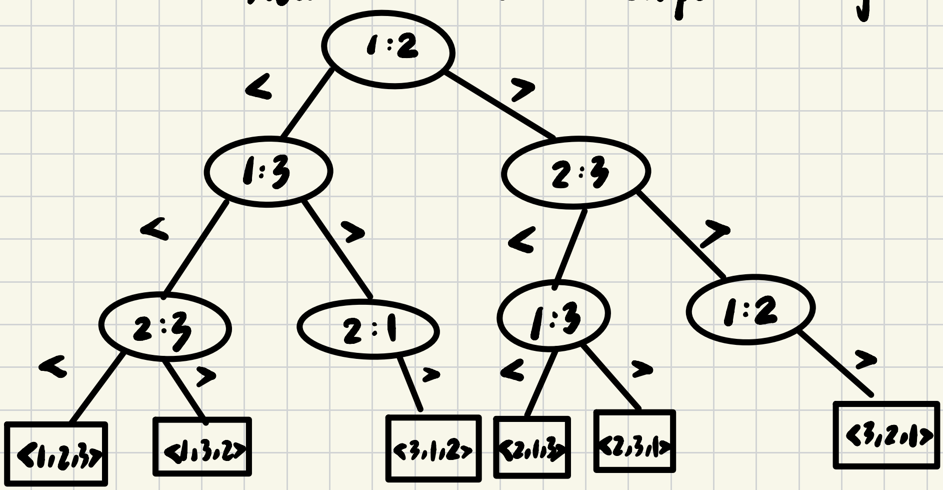
then $T(n+1) \geqslant cn\log n - cn\log 2 + \Theta(n+1)$

$\Rightarrow T(n+1) \geqslant c_0(n+1)\log(n+1)$ when $n \geqslant 1$, $0 < c_0 \leq \frac{c}{4}$

thus $T(n) = \Omega(n\log n)$

Question 6.4

A[smallest]: A[i] and compare the original indices



Question 6.5

n-1 , for the best case. we have a array that has been sorted. Then by comparing adjacent elements in sequence, we can finish the sort.

It takes n-1 times of comparison, and the height of decision tree is (n-1).