Tutorial 01 for Calculus I (Sect. 2.1 - 2.4)

Sun Lulu

sunll@mail.sustech.edu.cn

数学系1楼03
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1. Section 2.1: Definitions of average rates of change, instantaneous rates of change, slope of a curve.
2. Section 2.2: Common used the limit laws (Thm 1-3), the Sandwich Theorem, a property of limits (Thm 5).
3. Section 2.3: The precise definition of a limit, and the steps to find a δ for given f, L, c, and $\epsilon > 0$.(不考)
4. Section 2.4: Definitions of one-sided limits, the relation between one-sided limits and limit, the limit of ratio $(\sin x)/x$ as $x \to 0$.

Review of Sect. 2.2

Theorem (The Limit Laws)

If $\lim\limits_{x\to x_0} f(x) = L$, $\lim\limits_{x\to x_0} g(x) = M$*,then*

(1) $\lim\limits_{x\to x_0} (f(x) \pm g(x)) = M \pm L$*;*

(2) $\lim\limits_{x\to x_0} (k \cdot f(x)) = k \cdot L$*;*

(3) $\lim\limits_{x\to x_0} (f(x) \cdot g(x)) = L \cdot M$*;*

(4) $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, M \neq 0$*;*

(5) $\lim\limits_{x\to x_0} f(x) = 0$*, and* $g(x)$ *is bounded for all x in some open interval containing* x_0*, except possibly at* $x = x_0$ *itself, then* $\lim\limits_{x\to x_0} (f(x) \cdot g(x)) = 0$*.*

Remark: $\lim\limits_{x\to c} x \sin \frac{1}{x} \neq 0 \cdot \sin \frac{1}{0}$.

Review of Sect. 2.2

Theorem (The Sandwich Theorem)

Suppose that $g(x) \leq f(x) \leq h(x)$ *for all* x *in some open interval containing* c*, except possibly at* $x = c$ *itself. Suppose also that*

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$*.*

Remark: This theorem states two results: the limit of $f(x)$ exists, and the value is exactly L. For example, by letting $g(x) = -|x|$, $h(x) = |x|$, and $f(x) = x \sin \frac{1}{x}$, we know that $\lim_{x \to 0} x \sin \frac{1}{x}$ exists, and it equals to 0.

Theorem (Thm 5 in P69)

If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and provide that both limits exist, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Remark: If the inequality is strict, i.e., $f(x) < g(x)$, the limits can still be equal. To see this, $\lim_{x \to 0^+} [\frac{1}{x}] \sin x$.

Review of Sect. 2.4

Theorem (Thm 7)

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Remark: Also, $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$.

And if $\lim\limits_{x \to 0} f(x) = 0$, then $\lim\limits_{x \to 0} \frac{\sin f(x)}{f(x)} = 1$. Hence, $\lim\limits_{x \to 0} \frac{\sin \alpha x}{x} = \alpha$.

But note that $\lim\limits_{x \to 0} x \sin \frac{1}{x} = 0$, $\lim\limits_{x \to \infty} \frac{\sin x}{x} = 0$. (Sect. 2.6)

Homework of Section 2.2

专题一: 分子分母有理化, 特别地, $\frac{0}{0}, \infty - \infty$.

例1: $\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$. (书本22)

延伸: (1) $\lim_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

(2) $\lim_{x\to 2} \dfrac{x^2-x-2}{\sqrt[3]{x^2+23}-3}$.

(3) $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1-\dfrac{x}{2}}{x^2}$.

(4) $\lim_{x\to 1} \dfrac{x^4-1}{x^3-1}$.

(5) $\lim_{x\to 1} \dfrac{x^{2020}-1}{x^{2019}-1}$.

例2: $\lim_{x\to 1} (\dfrac{3}{1-x^3} - \dfrac{4}{1-x^4})$.

Homework of Section 2.2

专题二：分式极限存在,分母极限不是0，则乘法法则；若分母极限是0,则分母极限也是0.

例: If $\lim\limits_{x\to -2}\frac{f(x)}{x^2}=1$, find $\lim\limits_{x\to -2}\frac{f(x)}{x}$. (书本78)

If $\lim\limits_{x\to 2}\frac{f(x)-5}{x-2}=3$, find $\lim\limits_{x\to 2}f(x)$. (书本79)

延伸: (1) Find a and b such that $\lim\limits_{x\to 2}\dfrac{x-2}{x^2+ax+1}=b$.

(2) Find a and b such that $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{x^2+x+1}}{x}-\dfrac{a}{x}-b\right)=0$.

(3) Find a and b such that $\lim\limits_{x\to \pi/2}\dfrac{\sqrt{x}-a}{\cos x}=b$.

(4) Suppose that $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=1$, compute $\lim\limits_{x\to 0}\dfrac{f(x)}{1-\cos x}$.

Homework of Section 2.3

专题三: 夹逼定理.

例: Prove that $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$. (书本49)

Prove that $\lim\limits_{x\to 0} x^2\sin\frac{1}{x} = 0$. (书本50)

延伸: (1) $\lim\limits_{x\to 0^+} x[\frac{1}{x}]$ (2) $\lim\limits_{x\to 0^+} [\frac{1}{x}]\sin x$.

专题四: 极限的定义(不考).

$\lim_{x\to x_0} f(x) = L \Leftrightarrow \forall\epsilon > 0, \exists\delta > 0, 0 < |x - x_0| < \delta$, s.t $|f(x) - L| < \epsilon$.

例: Prove that $\lim_{x\to 1} f(x) = 1$, if $f(x) = \begin{cases} x^2, & \text{x} \neq 1 \\ 2, & \text{x=1} \end{cases}$. (书本41)

延伸: Prove that $\lim_{x\to 0} x \sin\frac{1}{x} = 0$. (书本49)

专题五: 单边极限 (如 $\lim\limits_{x\to 0^-} e^{\frac{1}{x}} = 0$, $\lim\limits_{x\to 0^+} e^{\frac{1}{x}} = +\infty$).

例: 1. Suppose that f is an odd function of x. Does knowing that $\lim\limits_{x\to 0^+} f(x) = 3$ tell you anything about $\lim\limits_{x\to 0^-} f(x)$? Give reasons for your answer. (书本45)

2. Suppose that f is an even function of x. Does knowing that $\lim\limits_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim\limits_{x\to -2^-} f(x)$ or $\lim\limits_{x\to -2^+} f(x)$? Give reasons for your answer. (书本46)

延伸: (1) suppose $\lim\limits_{x\to 0^+} f(x) = a$, $\lim\limits_{x\to 0^-} f(x) = b$, then $\lim\limits_{x\to 0^-} (f(x - \sin x) + 2f(x^2 + x)) = (\quad)$.

(A) $a + 2b$ (B) $b + 2a$ (C) $3a$ (D) $3b$

专题五: 单边极限 .

延伸: (2) Given the function $\lim\limits_{x\to 0^+} f(x) = l$, $\lim\limits_{x\to 0^-} f(x) = m$, determine the following limits exist or not. If so, find the limit.

$\lim\limits_{x\to 0} f(-x)$; $\quad \lim\limits_{x\to 0^+} f(x^2 - x)$; $\quad \lim\limits_{x\to 0^-} (2f(-x) + f(x^2))$.

(3) $\lim\limits_{x\to 0}\left(\dfrac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}} + \dfrac{\sin x}{|x|}\right)$.

(4) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} e^{\frac{1}{x-1}}$.

专题六：重要极限：$\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$，一定是 $\frac{0}{0}$.

常用：$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

例：(1) $\lim\limits_{x\to 0}\dfrac{\tan 2x}{x}$ (书本25) $\lim\limits_{x\to 1}\dfrac{\tan 2x}{x}$

(2) $\lim\limits_{x\to 0}\dfrac{\sin(\sin x)}{\sin x}$ (书本 34) (3) $\lim\limits_{x\to 0}\dfrac{\tan x}{x^2\cot 3x}$ (书本41)

延伸：(1) Compute the following limits.

a. $\lim\limits_{x\to 0}\dfrac{\sin(1-\cos x)}{(\tan x)^2}$ b. $\lim\limits_{x\to 0}\dfrac{\tan(2x)}{3x}$ c. $\lim\limits_{x\to 0}\dfrac{1-\cos(2x)}{x\sin(2x)}$.

(2) Let m and n be positive integers. Compute $\lim\limits_{x\to 0}\dfrac{\sin(mx)}{\sin(nx)}$ and $\lim\limits_{x\to \pi}\dfrac{\sin(mx)}{\sin(nx)}$.

$x-\frac{\pi}{2}$

$$\frac{(\sqrt{x}-\sqrt{\frac{\pi}{2}})(\sqrt{x}+\sqrt{\frac{\pi}{2}})}{\cos x\cdot(\sqrt{x}+\sqrt{\frac{\pi}{2}})}$$

专题六: 重要极限: $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$, 一定是 $\frac{0}{0}$.

延伸: (3) Find a and b such that $\lim\limits_{x\to\pi/2}\frac{\sqrt{x}-a}{\cos x}=b$. $\sin(\frac{\pi}{2}-x)$

(4) Suppose that $\lim\limits_{x\to 0}\frac{f(x)}{x^2}=1$, compute $\lim\limits_{x\to 0}\frac{f(x)}{1-\cos x}$.

Homework of Section 2.2-2.4

专题七: 各种判断题.

延伸: Determine whether the following statements are true or false?

$x = c\ 0$ $x \neq c\ 1$ 外函数不连续 $g(1)$

(1) If $\lim_{x\to c} f(x) = A$ and $\lim_{y\to A} g(y) = B$, imply that $\lim_{x\to c} g(f(x)) = B$.

(2) $\lim_{x\to c} |f(x)| = |l|$, then $\lim_{x\to c} f(x) = l$.

(3) If $\lim_{x\to x_0} f(x)$ exists, but $\lim_{x\to x_0} g(x)$ does not exist, then $\lim_{x\to x_0} [f(x) + g(x)]$, $\lim_{x\to x_0} [f(x) \cdot g(x)]$ do not exist.

(4) If $\lim_{x\to x_0} f(x)$, $\lim_{x\to x_0} g(x)$ all do not exist, then $\lim_{x\to x_0} [f(x) + g(x)]$ does not exist.

(5) If $\lim_{x\to x_0} [f(x) + g(x)]$ exists, then $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ both exist.

(6) If $f(x) > 0$ and $\lim_{x\to c} f(x) = l$, then $l > 0$.

≥ 0

1. Find the limits.

(1) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$ (2) $\lim\limits_{x \to 0^+} \dfrac{2^{1/x}+1}{2^{1/x}-1}$ (3) $\lim\limits_{x \to 0^-} \dfrac{2^{1/x}+1}{2^{1/x}-1}$

(4) $\lim\limits_{x \to 0^+} [\dfrac{1}{x}] \sin x$ (5) $\lim\limits_{x \to 0} \dfrac{\sin(1-\cos x)}{(\tan x)^2}$

$\cos x = 1 - 2\sin^2\frac{x}{2}$ $\sin(2\sin^2\frac{x}{2})$ $(\frac{x}{2})^2$ $\cos^2 x$ $2\sin^2\frac{x}{2}$ $\sin^2 x$ $2\sin^2\frac{x}{2}$ $(\frac{x}{2})^2$

2. Given the function $\lim\limits_{x \to 0^+} f(x) = l$, $\lim\limits_{x \to 0^-} f(x) = m$, determine the following limits exist or not. If so, find the limit.

(1) $\lim\limits_{x \to 0^+} f(x^2 - x)$ (2) $\lim\limits_{x \to 0^-} (f(x - \sin x) + 2f(x^2 + x))$

3. Find a and b such that $\lim\limits_{x \to \pi/2} \dfrac{\sqrt{x} - a}{\cos x} = b$.

4. Let m and n be positive integers. Compute $\lim\limits_{x \to 0} \dfrac{\sin(mx)}{\sin(nx)}$ and $\lim\limits_{x \to \pi} \dfrac{\sin(mx)}{\sin(nx)}$.