Tutorial 10 for Calculus I
Sect. 7.4-7.8

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专题一: 分离变量解微分方程 Sect. 7.4

分离不同微分

1. Separable Differentiable Equations: If the differential equation has the form: $\frac{dy}{dx} = g(x)H(y)$, then let $H(y) = \frac{1}{h(y)}$, $\Rightarrow \frac{dy}{dx} = \frac{g(x)}{h(y)}$
$\Rightarrow \underline{\int h(y)dy = \int g(x)dx}$. P408

2.Application:

(1)Unlimited Population Growth: $\frac{dy}{dt} = ky,\ y(0) = y_0 \Rightarrow y = y_0 e^{kt}$. P410

(2)Radioactivity: $\frac{dy}{dt} = -ky,\ k > 0,\ y(0) = y_0 \Rightarrow y = y_0 e^{-kt},\ k > 0$, and we know the Half-life$=\frac{\ln 2}{k}$. P411

$e^{-kt} = \frac{1}{2}$ $\quad -kt = \ln\frac{1}{2}$

(3)Heat Transfer: Newton's Law of cooling:

$\frac{dH}{dt} = -k(H - H_s),\ H(0) = H_0$, let $y = H - H_s \Rightarrow \frac{dy}{dt} = -ky$
$\Rightarrow H - H_s = (H_0 - H_s)e^{-kt}$. P413

Solve the differential equation:

22. $\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$

$\frac{dy}{dx} = (e^x+1)(e^{-y}+1)$

$H - H_s = (H_0 - H_s)e^{-kt}$

41. Cooling soup Suppose that a cup of soup cooled from $90°C$ to $60°C$ after 10 min in a room whose temperature was $20°C$. Use Newton's Law of Cooling to answer the following questions.

i. How much longer would it take the soup to cool to $35°C$?

ii. Instead of being left to stand in the room, the cup of $90°C$ soup is put in a freezer whose temperature is $-15°C$. How long will it take the soup to cool from $90°C$ to $35°C$?

L'Hôpital's Rule: Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$, assuming that the limit on the right side of this equation exists. P417

Remark: (1) As soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. P418

(2) Sometimes l'Hôpital's Rule can apply to the indeterminate form $\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty$. P419

Find the limits.

41. $\lim\limits_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$; $\frac{\ln x-(x-1)}{(x-1)\ln x}$

1^{∞}

61. $\lim\limits_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x$;

71. $\lim\limits_{x\to\infty}\frac{2^x-3^x}{3^x+4^x}$; $\lim\limits_{x\to\infty}\frac{3^x-5^x}{3^x+4^x}$;

73. $\lim\limits_{x\to\infty}\frac{e^{x^2}}{xe^x}$;

74. $\lim\limits_{x\to 0^+}\frac{x}{e^{-\frac{1}{x}}}$.

80. For what values of a and b is $\lim\limits_{x\to 0}\left(\frac{\tan 2x}{x^3}+\frac{a}{x^2}+\frac{\sin bx}{x}\right)=0$?

定义

88. Find $f'(0)$ for $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0 \\ 0, & x=0 \end{cases}$.

1. Definition of the inverse Trigonometric Functions. P426-428

(1) $y = \sin^{-1} x$ is the number in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for which $\sin y = x$.

(2) $y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

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(3) $y = \tan^{-1} x$ is the number in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for which $\tan y = x$.

(4) $y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

对 $|x| \geq 1$ 反常

(5) $y = \sec^{-1} x$ is the number in $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ for which $\sec y = x$.

(6) $y = \csc^{-1} x$ is the number in $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ for which $\csc y = x$.

2. Derivatives of the inverse trigonometric functions: P432

(1) $\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, |u| < 1$ (2) $\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, |u| < 1$

(3) $\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$ (4) $\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$

(5) $\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, |u| > 1$ (6) $\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, |u| > 1$

3. Integrals evaluated with inverse trigonometric functions. P433

(1) $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}(\frac{u}{a}) + C$ (Valid for $u^2 < a^2, a \neq 0$)

(2) $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}(\frac{u}{a}) + C$ (Valid for all u, $a \neq 0$)

(3) $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

Evaluate the integrals.

47. $\int \frac{dx}{x\sqrt{25x^2-2}}$

87. $\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1} x)\ dx}{x\sqrt{x^2-1}}$;

90. $\int \frac{e^x \sin^{-1} e^x}{\sqrt{1-e^{2x}}}\ dx$.

$v = \sec^{-1} x \in (0, \frac{\pi}{2})$

$\sec v = x \quad dx = \sec v \tan v\, dv$

$\frac{\sec^2 v}{\sec v \tan v} \quad \sec v \tan v\, dv$

Find the limits.

96. $\lim_{x\to\infty} \frac{e^x \tan^{-1} e^x}{e^{2x}+x}$

97. $\lim_{x\to 0^+} \frac{(\tan^{-1}\sqrt{x})^2}{x\sqrt{x+1}}$, $\lim_{x\to 0^+} \frac{(\cot^{-1}(\sqrt{4x}))^2}{x\sqrt{x+1}}$;

$\frac{\tan^{-1} u}{u + \frac{\ln u}{u}}$

$\frac{(\tan^{-1} v)^2}{v^2\sqrt{v^2+1}}$ $u = \tan^{-1} v$ $v = \tan u$ $\frac{v^2}{\tan^2 u \sec u}$

$8\frac{(\cot^{-1} u)^2}{u^2\sqrt{u^2+4}} =$

127. Find the domain and range of each composite function. Then graph

a. $y = \tan^{-1}(\tan x)$ b. $y = \tan(\tan^{-1} x)$

1. Definition of Rates of Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

(1) f grows faster than g as $x \to \infty$ if

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ or, equivalently, if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

we also say that g grows slower than f as $x \to \infty$.

(2) f and g grow at the same rate as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where L is finite and positive. $\Rightarrow f = O(g)$ and $g = O(f)$

1. A function f is of smaller order than g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $f = o(g)$. $\Rightarrow f = O(g)$

(2) Let f and g be positive for x sufficiently large. Then f is of at most the order of g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M$$

,

for x sufficiently large. We indicate this by writing $f = O(g)$.

第十周补充作业

1. Let $f(x) = \frac{1+e^{\frac{1}{x}}}{-1+e^{\frac{1}{x}}}$ for $x \neq 0$ and $f(0) = 1$. Then $x = 0$ is a

(A) jump discontinuity.

(B) removable discontinuity.

(C) continuous point.

(D) infinite discontinuity

4. Compute the following limits:

(1) $\lim_{x\to\infty} \left(\frac{x^2}{(x-1)(x+3)}\right)^x$

(4) $\lim_{x\to\infty} \left(\frac{\pi}{2} - \arctan x\right)^{\frac{1}{\ln x}}$

(8) $\lim_{x\to 1} \frac{x - x^x}{1 - x + \ln x}$

(14) $\lim_{x\to 0+} \left(e^x - x - 1\right)^{\frac{1}{\ln x}}$

8. Compute the following limit: $\lim_{x\to 0} \frac{\int_0^{x^2} \cos(t^2)\ dt}{x^2}$.

f'

16. Assume $f''(a)$ exists, compute the following limit $f''(a)(x-a)$

$\lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{\cancel{\sin(x-a)}\ x-a}$.