专题二 导数.

知识点一:

1. 选(B)

$$f(x)=\begin{cases} x^4\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases} \Rightarrow f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}h^3\sin\frac{1}{h}=0$$

$$\Rightarrow f'(x)=\begin{cases} 4x^3\sin\frac{1}{x}-x^2\cos\frac{1}{x}, & x\neq 0 \\ 0. & x=0 \end{cases} \Rightarrow f''(0)=\lim_{h\to 0}\frac{f'(h)-f'(0)}{h}=\lim_{h\to 0}\left(4x^2\sin\frac{1}{h}-h\cos\frac{1}{h}\right)$$

$$=0$$

$$\Rightarrow f''(x)=\begin{cases} 12x^2\sin\frac{1}{x}-4x\cos\frac{1}{x}-2x\cos\frac{1}{x}-\sin\frac{1}{x}, & x\neq 0. \\ 0 & , x=0. \end{cases}$$

$$\Rightarrow f^{(3)}(0)=\lim_{h\to 0}\frac{f''(h)-f''(0)}{h}=\lim_{h\to 0}\Big(\underbrace{12h\sin\frac{1}{h}}_{\downarrow 0}\underbrace{-6\cos\frac{1}{h}-\frac{1}{h}\sin\frac{1}{h}}_{\text{不存在}}\Big)\text{ 不存在}$$

$\Rightarrow n=2.$

2. $$\lim_{x\to a}\left(\frac{1}{f'(a)(x-a)}-\frac{1}{f(x)-f(a)}\right)=\lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{f'(a)(x-a)[f(x)-f(a)]}$$

$$=\frac{1}{f'(a)}\lim_{x\to a}\frac{f'(x)-f'(a)}{[f(x)-f(a)]+(x-a)f'(x)}=\frac{1}{f'(a)}\lim_{x\to a}\frac{\frac{f'(x)-f'(a)}{x-a}}{\frac{f(x)-f(a)}{x-a}+f'(x)}=\frac{1}{f'(a)}\cdot\frac{f''(a)}{2f'(a)}$$

$$=\frac{f''(a)}{2[f'(a)]^2}$$

3. $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\frac{g(x)+g(h)}{1-g(x)g(h)} - g(x)}{h} = \lim_{h\to 0} \frac{g(h)+[g(x)]^2 g(h)}{h[1-g(x)g(h)]}$

$= \lim_{h\to 0} \frac{g(h)}{h} \cdot \frac{1+g^2(x)}{1-g(x)g(h)} = [1+g^2(x)] \lim_{h\to 0} \frac{g(h)}{h} = 1+g^2(x)$

$\Rightarrow \frac{dy}{dx} = 1+y^2$, let $y=g(x)$

$\Rightarrow \int \frac{dy}{1+y^2} = \int dx$

$\Rightarrow \tan^{-1} y = x + C$

since $g(x)$ is differentiable $\Rightarrow \lim_{h\to 0} g(h) = g(0) = 0$

$\Rightarrow C = 0$

$\Rightarrow g(x) = \tan x$.

4. $f(x) = \begin{cases} a\cdot\sin x, & x \le \frac{\pi}{4} \\ 1+b\cdot\tan x, & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$ $f(x)$ is differentiable at $x = \frac{\pi}{4}$

$\Rightarrow \lim_{x\to \frac{\pi}{4}^-} f(x) = \lim_{x\to (\frac{\pi}{4})^+} f(x) \Rightarrow \frac{\sqrt{2}}{2}a = 1+b$

$f'_+(\frac{\pi}{4}) = f'_-(\frac{\pi}{4}) \Rightarrow \frac{\sqrt{2}}{2}a = 2b$

$\Rightarrow \begin{cases} a = 2\sqrt{2} \\ b = 1 \end{cases}$.

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知识点二：

5. $\begin{cases} x=\log_a x=\frac{\ln x}{\ln a} \Rightarrow \ln a=\frac{\ln x}{x} \\ 1=\frac{1}{x\ln a} \Rightarrow x\ln a=1 \end{cases} \Rightarrow \ln x=1 \Rightarrow x=e \Rightarrow \begin{cases} x=e \\ a=e^{\frac{1}{e}} \end{cases}$

$\ln a=\frac{1}{e}$

6. $\begin{array}{l} f(x)=x^2 \\ y=kx-1 \end{array} \Rightarrow \begin{cases} x^2=kx-1 \\ 2x=k \end{cases} \Rightarrow \frac{k^2}{4}=\frac{k^2}{2}-1 \Rightarrow \frac{k^2}{4}=1 \Rightarrow k=\pm 2$

7. $\begin{cases} (b-a)^3=b+a \\ 3(y-x)^2\cdot\left(\frac{dy}{dx}-1\right)=\frac{dy}{dx}+1 \end{cases}$

let $\frac{dy}{dx}=3$

$\Rightarrow 3(b-a)^2\cdot(3-1)=3+1=4$

$\Rightarrow (b-a)^2=\frac{1}{6}$

$\Rightarrow b-a=\pm\frac{1}{\sqrt{6}}$

$\Rightarrow \begin{cases} b-a=\frac{1}{\sqrt{6}} \\ b+a=\frac{1}{6\sqrt{6}} \end{cases}$ 或 $\begin{cases} b-a=-\frac{1}{\sqrt{6}} \\ b+a=-\frac{1}{6\sqrt{6}} \end{cases}$

$\Rightarrow \begin{cases} a=\frac{-5}{12\sqrt{6}} \\ b=\frac{7}{12\sqrt{6}} \end{cases}$ 或 $\begin{cases} a=\frac{5}{12\sqrt{6}} \\ b=-\frac{7}{12\sqrt{6}} \end{cases}$

8. $f(x) = (1+x)(1+2x)\cdots(1+10x)$.

$\Rightarrow f'(x) = (1+2x)\cdots(1+10x) + 2(1+x)\cdots(1+10x) + \cdots \quad + 10(1+x)(1+2x)\cdots(1+9x)$

$\Rightarrow f'(0) = 1+2+\cdots+10 = \frac{1}{2}\times 10\times 11 = \underline{55}$

9. $f(x) = (x^2+1)(x^2+2)(x^2+3)(x^2+4) = (x^4+3x^2+2)(x^4+7x^2+12)$

$f^{(6)}(0)$ 即 x^6 的系数 $10x^6$ $\Rightarrow f^{(6)}(0) = 10$.

10. $f(x) = \arctan\frac{1+x}{1-x} \Rightarrow f'(x) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \cdot \frac{1-x+(1+x)}{(1-x)^2} = \frac{\cancel{(1-x)^2}}{\cancel{2}(1+x^2)} \cdot \frac{\cancel{2}}{\cancel{(1-x)^2}}$

$= \frac{1}{1+x^2}$

$\Rightarrow f'(0) = 1$

11. $g(x) = f(\ln x) \Rightarrow g'(x) = f'(\ln x)\cdot\frac{1}{x} \Rightarrow g''(x) = f''(\ln x)\cdot\frac{1}{x^2} - \frac{1}{x^2}f'(\ln x)$

$f(0) = 3$, $f''(0) = 5$, $\Rightarrow g''(1) = f''(0) - f'(0) = \underline{2}$

$f'(1) = -4$, $f''(1) = -7$

12. $g(1) = 3 \Rightarrow g^{-1}(3) = 1$

选(A)

$\Rightarrow (g^{-1})'(3) = \frac{1}{g'(1)} = \frac{1}{4}$

13. $y = (\cos x)^x \Rightarrow \ln y = x\ln\cos x \Rightarrow \frac{y'}{y} = \ln\cos x + x\cdot\frac{-\sin x}{\cos x}$

$\Rightarrow y' = (\cos x)^x[\ln\cos x - x\tan x]$

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14. $\cancel{\lim_{n\to\infty}(f(1+\frac{1}{n}))^n}$ let $y=(f(1+\frac{1}{n}))^n \Rightarrow \ln y = n\ln f(1+\frac{1}{n})$

方法错

$$\Rightarrow \lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{\ln f(1+\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{f(1+\frac{1}{n})}\cdot f'(1+\frac{1}{n})\cdot(\cancel{-\frac{1}{n^2}})}{\cancel{-\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty}\frac{f'(1+\frac{1}{n})}{f(1+\frac{1}{n})} = 2. \Rightarrow \lim_{n\to\infty}(f(1+\frac{1}{n}))^n = e^2.$$

→ 导函数连续的条件没有

方法二: $\lim_{n\to\infty}(f(1+\frac{1}{n}))^n = \lim_{n\to\infty}\left(\underbrace{[1+(f(1+\frac{1}{n})-1)]^{\frac{1}{f(1+\frac{1}{n})-1}}}_{e}\right)^{\underbrace{\frac{f(1+\frac{1}{n})-f(1)}{\frac{1}{n}}}_{f'(1)=2}}$

$$= e^2$$

知识点三:

15. (1) $\lim_{x\to\infty}\frac{\ln^{100}x}{x^{0.0001}} = 0 \Rightarrow \ln^{100}x < x^{0.0001}$

$\lim_{x\to\infty}\frac{x^{0.0001}}{2^{kx}} = \lim_{x\to\infty}\frac{1}{0.0001\cdot x^{1-0.0001}\cdot k\cdot 2^{kx}\cdot\ln 2} = 0 \Rightarrow x^{0.0001} < 2^{kx}$

$\Rightarrow \ln^{100}x < x^{0.0001} < 2^{kx}$ √

(2) √ 方法一: U

方法二: (⇒) 成立

(⇐) $f''(c) = \lim_{x\to c}\frac{f'(x)-f'(c)}{x-c} = \lim_{x\to c}\frac{f'(x)}{x-c} > 0 \Rightarrow$ $x<c$, $f'(x)<0$ ↓ ; $x>c$, $f'(x)>0$ ↑ local min.

16. 选 (D).

$f'(x)=\frac{a(x^2-1)-2x(ax+b)}{(x^2-1)^2}$

$\begin{cases} f(3)=1 \\ f'(3)=0 \end{cases} \Rightarrow \begin{cases} \frac{3a+b}{8}=1 \\ 8a-18a-6b=0 \end{cases} \Rightarrow \begin{cases} 3a+b=8 \\ 5a+3b=0 \end{cases} \Rightarrow \begin{cases} a=6 \\ b=-10 \end{cases}$

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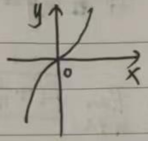
17. 选 (C)

方法一: let $f(x) = x^3$



$\Rightarrow$ inflection point at $x=0$

no local extreme

方法二: $f^{(3)}(x_0) = \lim_{x \to x_0} \frac{f''(x)}{x-x_0} > 0 \Rightarrow x > x_0,\ f''(x) > 0$ up $\Rightarrow f'(x) > f'(x_0) = 0 \uparrow$

$x < x_0,\ f''(x) < 0$ down $\Rightarrow f'(x) > f'(x_0) = 0\ \uparrow$

18. 选 (D)

$g(0) = f(0),\quad g(1) = f(1)$

~~$h(x) = f(x) - g(x) = f(x) - (1-x)f(0) - x f(1)$~~

let $h(x) = f(x) - g(x) \Rightarrow h(0) = 0$

$h(1) = 0$

$\Rightarrow$ ~~$h'(x) = f'(x) + f(0) - f(1) = f'(x) - f'(c),\ 0<c<1$~~

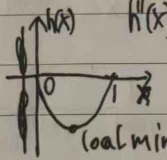
$\Rightarrow \exists c \in (0,1)$ s.t $h'(c) = 0$ · critical point. extreme.

$h(x) = f(x) - (1-x)f(0) - x f(1)$

$\Rightarrow h'(x) = f'(x) + f(0) - f(1)$

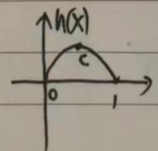
$h''(x) = f''(x)$.

if $f''(x) > 0 \Rightarrow$ concave up

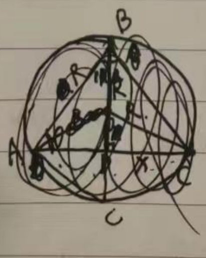


$\Rightarrow h(x) < 0 \Rightarrow f(x) < g(x)$ C错

if $f''(x) < 0 \Rightarrow$ concave down



$\Rightarrow h(x) > 0 \Rightarrow f(x) > g(x)$ D对



~~$AB = AC,\ OB = OC$~~

~~$\triangle ABO \cong \triangle ACO$~~

~~$\Rightarrow \angle 1 = \angle 2$~~

~~等腰三角形三线合一~~

~~$\Rightarrow$ D是中点~~

~~let $CD = x \Rightarrow CB = 2x$~~

~~$OD = \sqrt{R^2 - x^2}$~~

~~$\Rightarrow AC^2 = CD^2 + AD^2$~~

~~$= x^2 + (R + \sqrt{R^2 - x^2})^2$~~

19. $AB=BC$, let $\angle BAC=\theta$. $\Rightarrow AB=BC=2R\sin\theta$, $AC=4R\sin\theta\cos\theta=2R\sin2\theta$

$\Rightarrow P(\theta)=AB+BC+AC=2R(2\sin\theta+\sin2\theta)$, $0<\theta<\frac{\pi}{2}$

$P'(\theta)=4R(\cos\theta+\cos2\theta)=4R(2\cos^2\theta+\cos\theta-1)=4R(2\cos\theta-1)(\cos\theta+1)$

$P''(\theta)=-4R(\sin\theta+2\sin2\theta)$

let $P'(\theta)=0 \Rightarrow \theta_0=\frac{\pi}{3}$, $P''(\frac{\pi}{3})<0$

$\Rightarrow$ The perimeter has a maximum $P(\frac{\pi}{3})=3\sqrt{3}R$.

20. $y=\frac{x^2+1}{x+1}=x-1+\frac{2}{x+1}$ $\Rightarrow$ 斜渐近线 $y=x-1$

竖直渐近线 $x=-1$

$\Rightarrow$ (1) $y'=1-\frac{2}{(x+1)^2}$, $y''=\frac{4}{(x+1)^3}$

let $y'=0 \Rightarrow (x+1)^2=2 \Rightarrow x=\sqrt{2}-1$ or $x=-\sqrt{2}-1$

$x<-\sqrt{2}-1 \Rightarrow y'>0$ $\uparrow$ local max $f(-\sqrt{2}-1)=-2\sqrt{2}-2$

$-\sqrt{2}-1<x<\sqrt{2}-1 \Rightarrow y'<0$ $\downarrow$

local min $f(\sqrt{2}-1)=2\sqrt{2}-2$

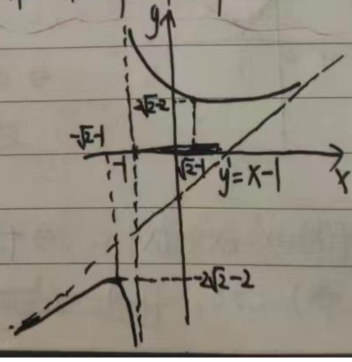
$x>\sqrt{2}-1 \Rightarrow y'>0$ $\uparrow$

$\Rightarrow$ has no absolute max and min.

$x=-1$, $y''(-1)$ is undefined, but $x\neq-1$ $\Rightarrow$ has no inflection point

$x<-1$, $y''<0$ concave down

$x>-1$, $y''>0$ concave up



(2) $y=x-1+\frac{2}{x+1}$ $\Rightarrow$ 斜渐近线 $y=x-1$

竖直渐近线 $x=-1$

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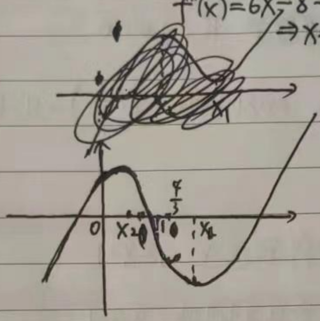
知识点四:

21. $f(x)=x^3-4x^2+x+1 \Rightarrow f'(x)=3x^2-8x+1=0 \Rightarrow x=\frac{8\pm\sqrt{64-12}}{2\times3}=\frac{8\pm2\sqrt{13}}{6}$

$f''(x)=6x-8=0 \Rightarrow x=\frac{4}{3}$　　$x_1=\frac{4}{3}+\frac{\sqrt{13}}{3}$, $x_2=\frac{4}{3}-\frac{\sqrt{13}}{3}$

$f(0)=1$

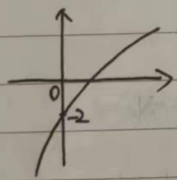
$f(1)=-1$



$\Rightarrow$ 三个根.

22. $f(x)=5x-2-\int_0^x\frac{dt}{1+t^8} \Rightarrow f'(x)=5-\frac{1}{1+x^8}=\frac{4+5x^8}{1+x^8}>0$. ↑

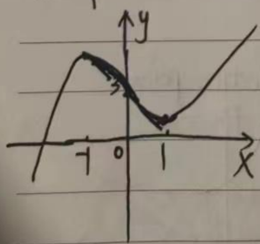
$f(0)=-2$



$\Rightarrow$ one root.

选(B)

23. $f(x)=x^3-3x+3 \Rightarrow f'(x)=3x^2-3=0 \Rightarrow x_1=1$, $x_2=-1$



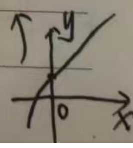
$f''(x)=6x \Rightarrow x=0$. $f(0)=3$

$f(1)=1$, $f(-1)=-1+3+3=5$

$\Rightarrow$ one zero root

选(B)

24. $f(x)=x^3-6x^2+9x+c \Rightarrow f'(x)=3x^2-6x+9=3(x^2-2x+3)=3[(x-1)^2+2]>0$ ↑



$f(0)=c>0$, $\Rightarrow$ one zero 选(B)

知识点五:

25. $y=\int_{x^2+1}^{2x^2+3} t\cdot\tan\sqrt{x+t}\,dt$

$\Rightarrow$ let $u=x+t \Rightarrow du=dt$

$$\Rightarrow y=\int_{x^2+x+1}^{2x^2+x+3}(u-x)\cdot\tan\sqrt{u}\,du=\int_{x^2+x+1}^{2x^2+x+3}u\tan\sqrt{u}\,du-x\int_{x^2+x+1}^{2x^2+x+3}\tan\sqrt{u}\,du$$

$$\Rightarrow \frac{dy}{dx}=(4x+1)(2x^2+x+3)\tan\sqrt{2x^2+x+3}-(2x+1)(x^2+x+1)\tan\sqrt{x^2+x+1}$$

$$-\int_{x^2+x+1}^{2x^2+x+3}\tan\sqrt{u}\,du-x\left[(4x+1)\tan\sqrt{2x^2+x+3}-(2x+1)\tan\sqrt{x^2+x+1}\right]$$

26. $f(x)=2^{g(x)}$, $g(x)=\int_2^{\frac{x^2}{2}}\frac{t}{1+t^4}\,dt$

$$\Rightarrow f'(x)=2^{g(x)}\ln 2\cdot g'(x),\quad g'(x)=x\cdot\frac{\frac{x^2}{2}}{1+\left(\frac{x^2}{2}\right)^4}$$

$$\Rightarrow f'(2)=2^{g(2)}\ln 2\cdot g'(2),\quad g(2)=0,\quad g'(2)=2\cdot\frac{2}{1+16}=\frac{4}{17}$$

$$\Rightarrow f'(2)=\frac{4\ln 2}{17}$$

27. $F(x)=\int_{\frac{1}{x}}^{1}xf(u)\,du+\int_1^{\frac{1}{x}}\frac{f(u)}{u^2}\,du=x\int_{\frac{1}{x}}^{1}f(u)\,du+\int_1^{\frac{1}{x}}\frac{f(u)}{u^2}\,du$

$$\Rightarrow F'(x)=\int_{\frac{1}{x}}^{1}f(u)\,du+x\cdot\left[-f\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)\right]+\left(-\frac{1}{x^2}\right)\cdot\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x^2}}=\int_{\frac{1}{x}}^{1}f(u)\,du-\left(1-\frac{1}{x}\right)f\left(\frac{1}{x}\right)$$

$=\left(1-\frac{1}{x}\right)f(c)-\left(1-\frac{1}{x}\right)f\left(\frac{1}{x}\right)=\left(f(c)-f\left(\frac{1}{x}\right)\right)\left(1-\frac{1}{x}\right)$, c介于$1, \frac{1}{x}$之间, $f(c)\neq f\left(\frac{1}{x}\right)$.

let $F'(x)=0 \Rightarrow x=1 \Rightarrow 0<x<1,\ \frac{1}{x}>1,\ 1<c<\frac{1}{x} \Rightarrow f(c)<f(\frac{1}{x})$, $1-\frac{1}{x}<0$ $\Rightarrow F'(x)>0 \uparrow$

$x>1,\ \frac{1}{x}<1,\ \frac{1}{x}<c<1,\ f(c)>f(\frac{1}{x})$, $1-\frac{1}{x}>0$ $\Rightarrow F'(x)>0 \uparrow$

$\Rightarrow$ increasing on $(0,1)$ and $(1,+\infty)$.

(2) $F'(x)=\int_{\frac{1}{x}}^{1} f(u)\,du-(1-\frac{1}{x})f(\frac{1}{x})=\int_{\frac{1}{x}}^{1} f(u)\,du-f(\frac{1}{x})+\frac{1}{x}f(\frac{1}{x})$

$\Rightarrow F''(x)=\cancel{-(-\frac{1}{x^2})f(\frac{1}{x})}+\frac{1}{x^2}f'(\frac{1}{x})-\cancel{\frac{1}{x^2}f(\frac{1}{x})}+\frac{1}{x}f'(\frac{1}{x})(-\frac{1}{x^2})$

$=(\frac{1}{x^2}-\frac{1}{x^3})f'(\frac{1}{x})=\frac{x-1}{x^3}f'(\frac{1}{x})$

$0<x<1,\ F''(x)<0$ concave down

$x>1,\ F''(x)>0$ concave up.

知识点，大：

28. (1) $g(x)=f(x)-x$, $g(1)=f(1)-1=0-1=-1<0$, $g(\frac{1}{2})=f(\frac{1}{2})-\frac{1}{2}=1-\frac{1}{2}>0$ $\Rightarrow \exists c\in(\frac{1}{2},1)$ s.t $g(c)=0$ $\Rightarrow f(c)=c$

(2) let $g(x)=e^{-kx}(f(x)-x)$, $f(0)=f(1)=0$,

$\Rightarrow g(0)=0,\ g(c)=0.$

$\Rightarrow$ By Rolle's theorem, $\exists \xi\in(0,c)$ s.t $g'(\xi)=0$

$\Rightarrow \cancel{-ke^{-k\xi}}(f(\xi)-\xi)+\cancel{e^{-k\xi}}(f'(\xi)-1)=0$

$\Rightarrow f'(\xi)-k[f(\xi)-\xi]=1$