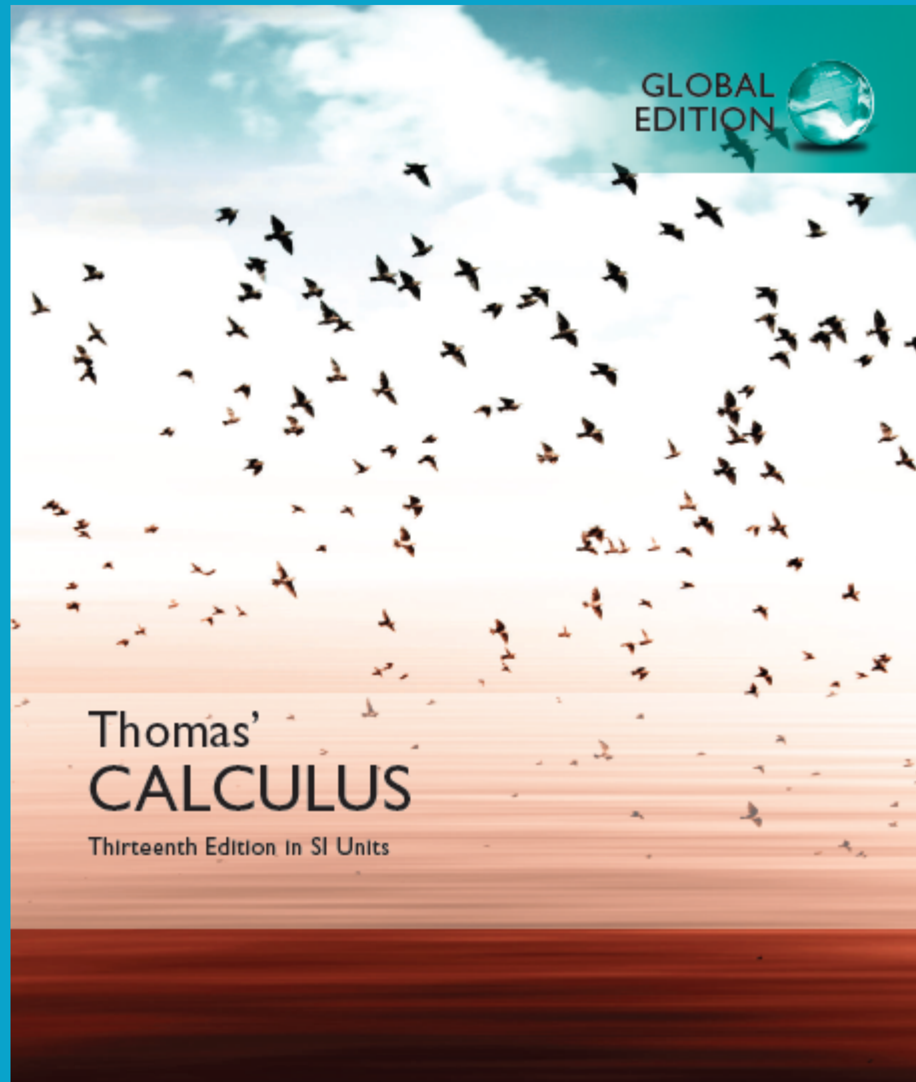

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 6

Applications of Definite Integrals

6.1

Volumes Using Cross-Sections

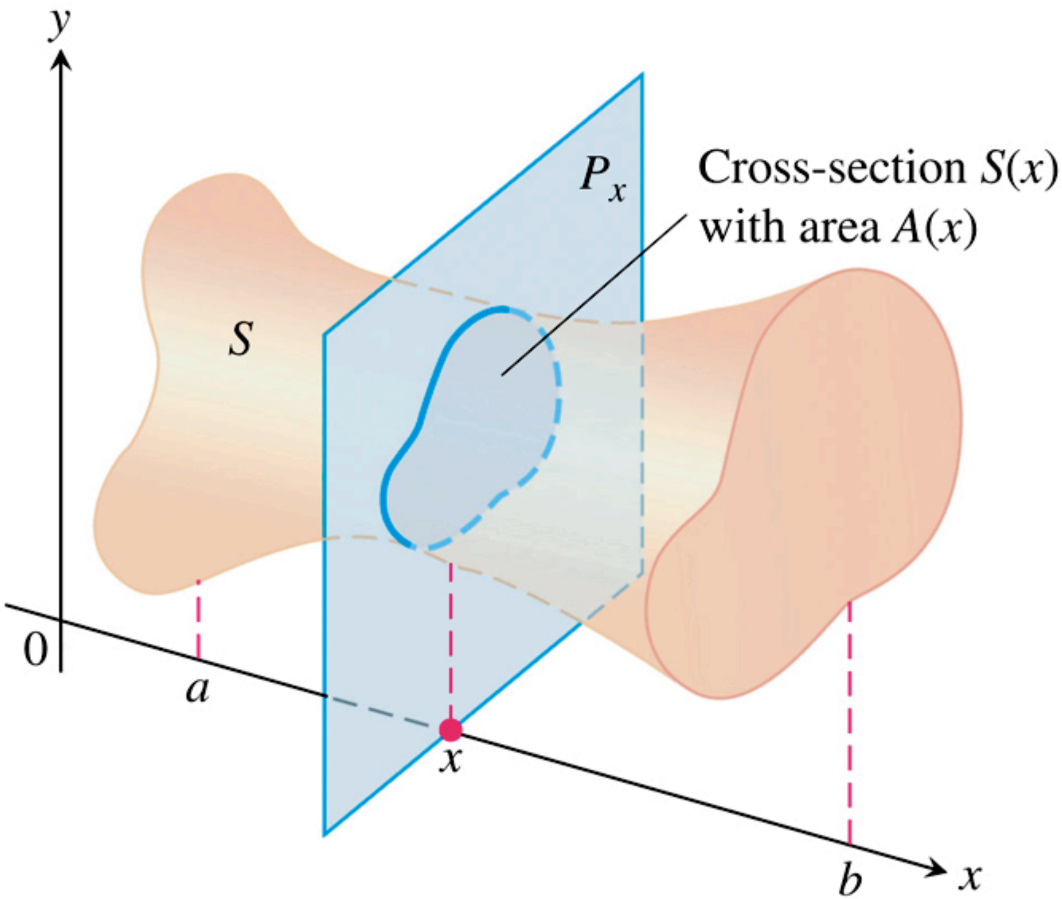


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

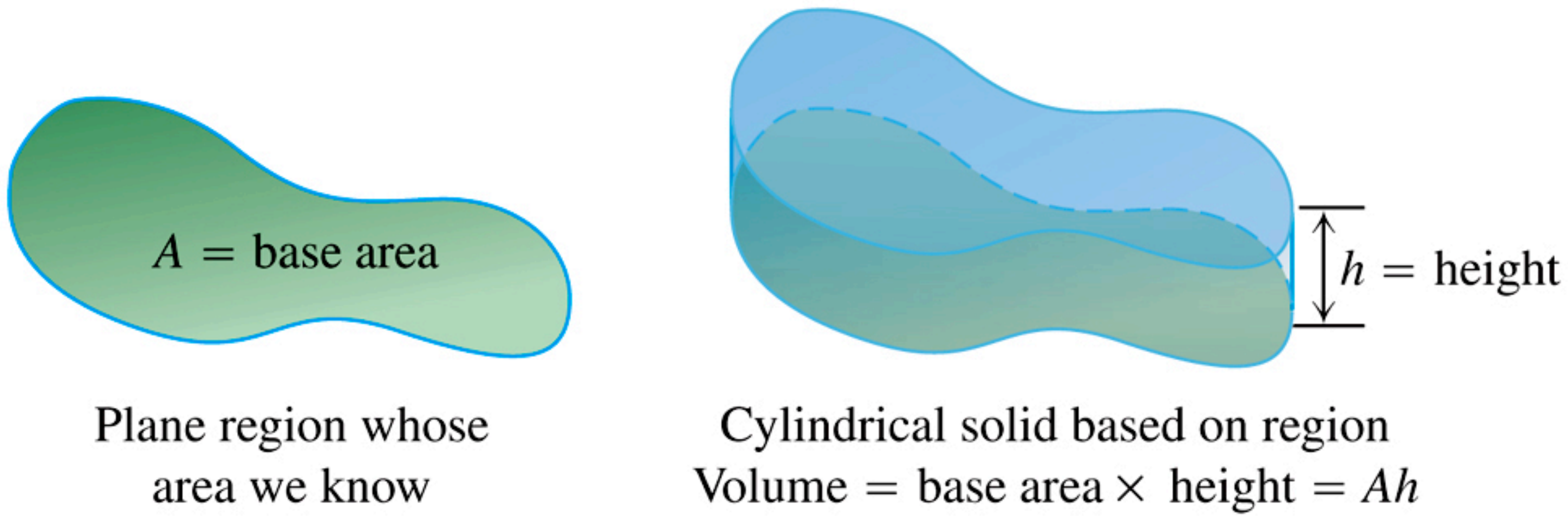
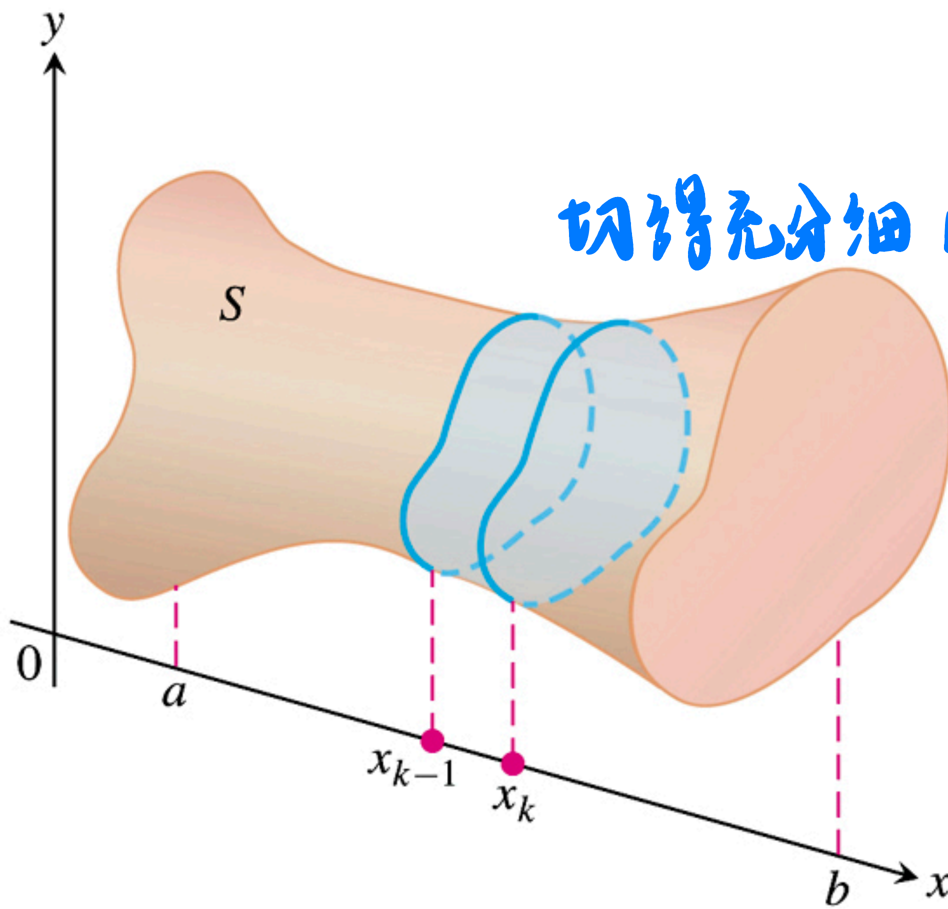
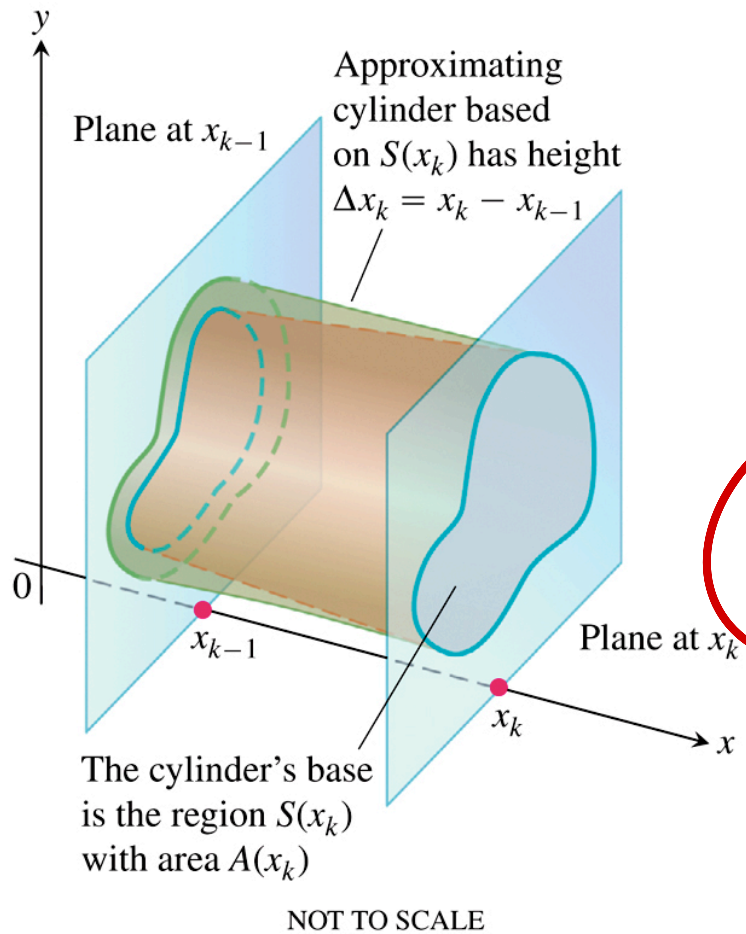


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.



切得充分细 面积相差不远

FIGURE 6.3 A typical thin slab in the solid S.



$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k$$

黎曼和

$$\int_a^b A(x)\,dx$$

FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

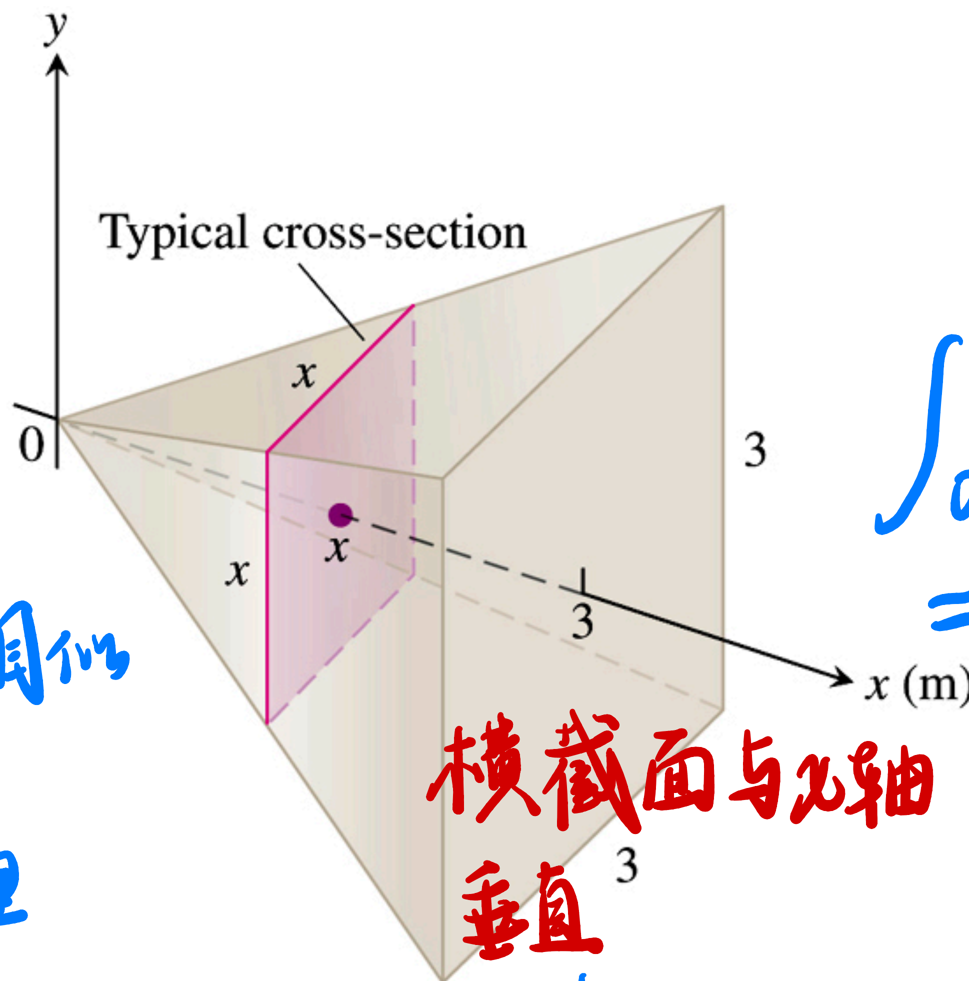
$$V = \int_a^b A(x)\,dx.$$

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

金字塔

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



$$\int_0^3 x^2\,dx = \frac{1}{3}x^3$$

相似

横截面与x轴垂直

FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

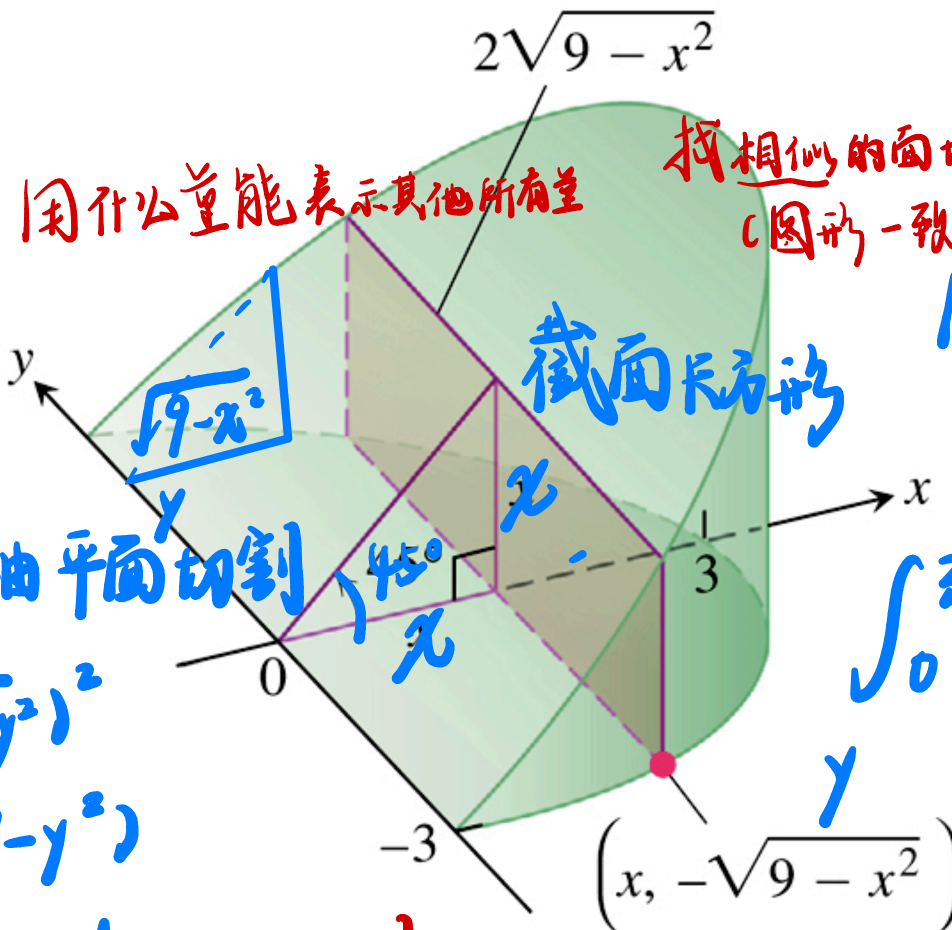
圆锥同理

$$\frac{\tilde r}{r} = \frac{x}{h} \rightarrow \tilde r = \frac{r}{h}x$$

$$A(x) = \pi \frac{r^2}{h^2}x^2$$

$$V = \int_0^h \pi \frac{r^2}{h^2}x^2\,dx = \pi \frac{r^2}{h^2}\frac{1}{3}x^3\Big]_0^h = \frac{1}{3}\pi r^2 h$$

EXAMPLE 2 A curved wedge 楔子 is cut from a circular cylinder of radius 3 by two planes. 平面 One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.



用什么量能表示其他所有量

找相似的面切
（图形一致→算法一致）

圆柱

截面长方形

$A = 2\sqrt{9-x^2}\,x$

$\int_0^3 2\sqrt{9-x^2}\,x\,dx$

$u = 9-x^2$

$du = -2x\,dx$

$\int_9^0 -du\sqrt{u}$

$=18$

若垂直于y轴平面切割

$A = \frac{1}{2}(\sqrt{9-y^2})^2$

$= \frac{1}{2}(9-y^2)$

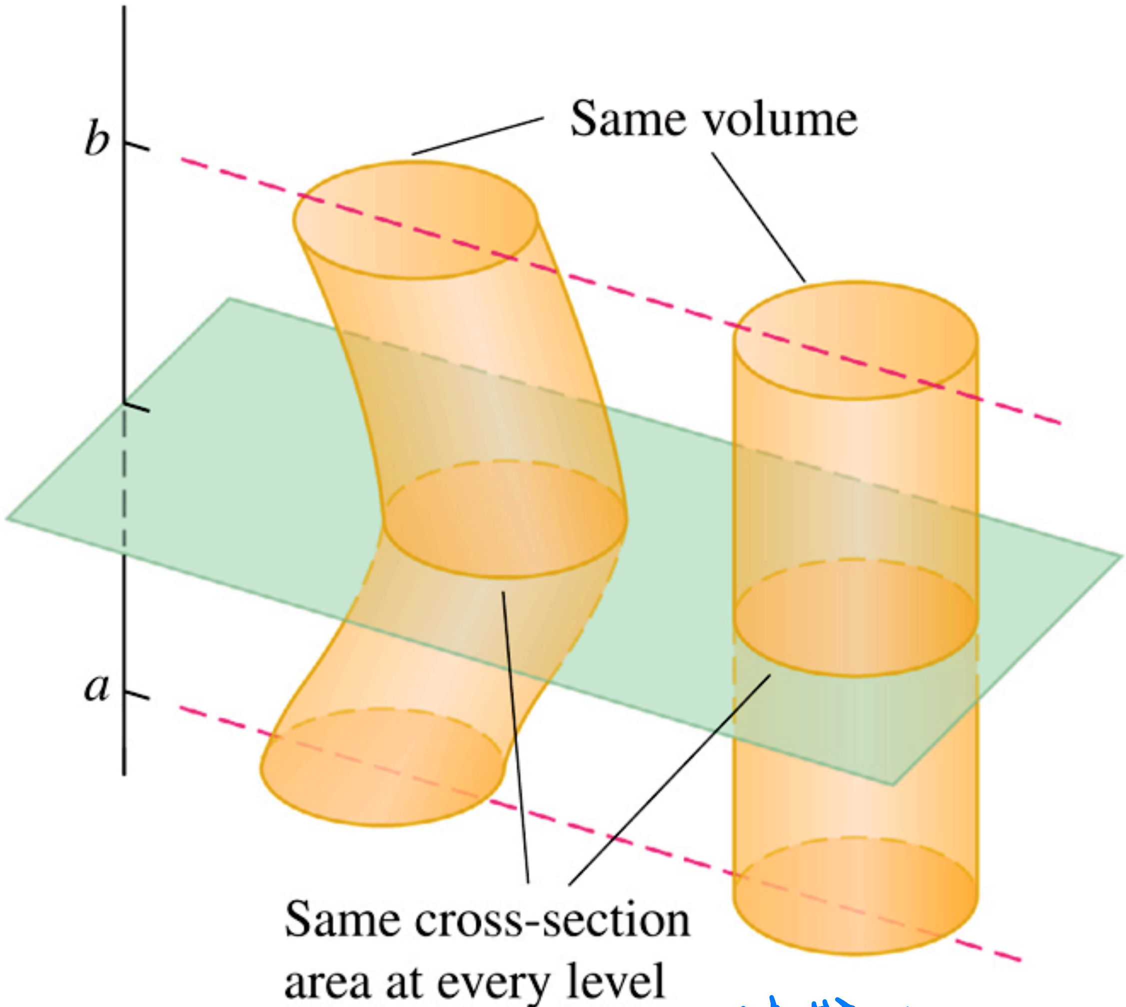
$\int_{-3}^{3} \frac{1}{2}(9-y^2)\,dy$

$= \frac{1}{2}\left(9y - \frac{1}{3}y^3\right)\Big]_{-3}^{3}$

$=18$

<0时S>0
积时也是正的
（几何含义）

FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x-axis. The cross sections are rectangles.



横截面积 + 上下限一样

FIGURE 6.7 *Cavalieri's principle:* These solids have the same volume, which can be illustrated with stacks of coins.

旋转体

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume in this case gives

Volume by Disks for Rotation About the x-axis

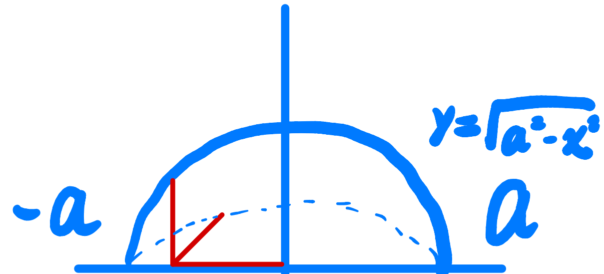
$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

EXAMPLE 5 The circle

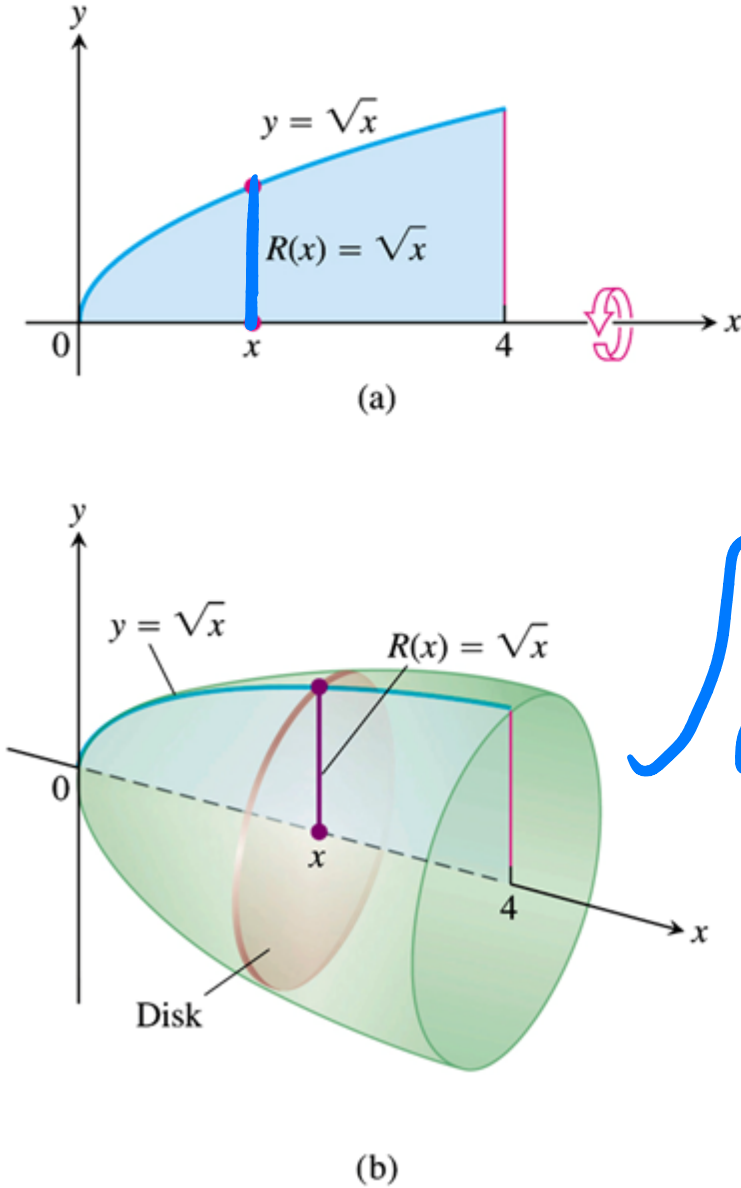
$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.



$$\int_{-a}^{a} \pi (a^2-x^2)\,dx$$
$$= \pi\left(a^2x-\frac{1}{3}x^3\right)\Big]_{-a}^{a}$$
$$= \frac{4}{3}\pi a^3$$

EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.



绕x轴转

$$\int_0^b \pi (f(x))^2 dx$$

FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

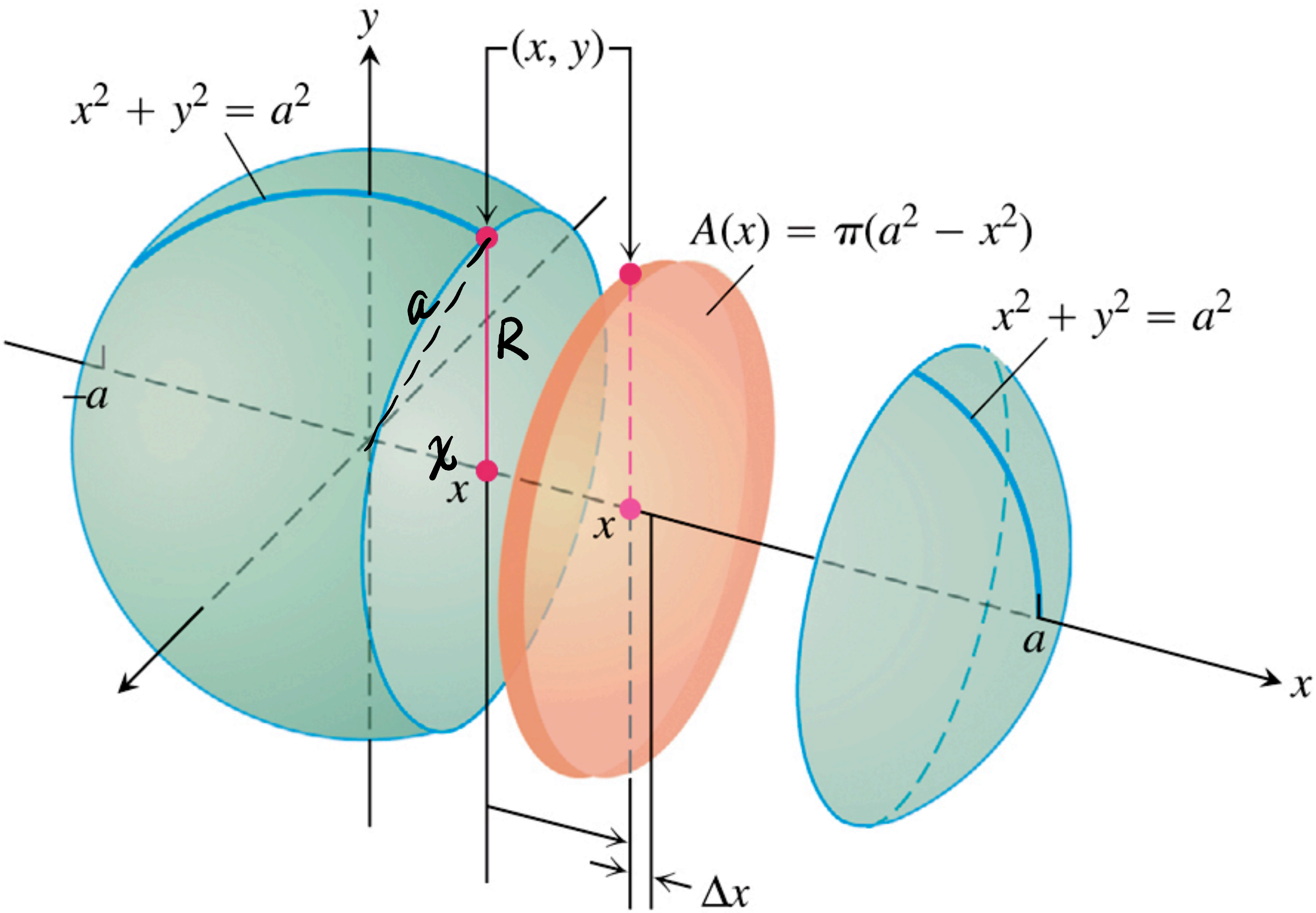
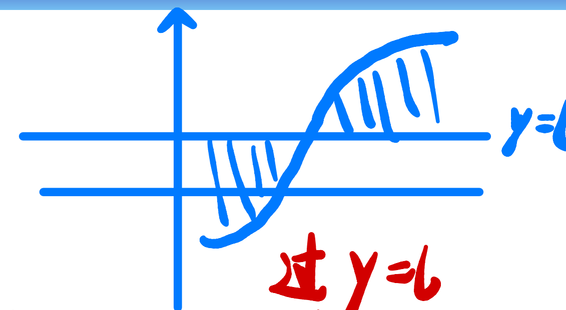


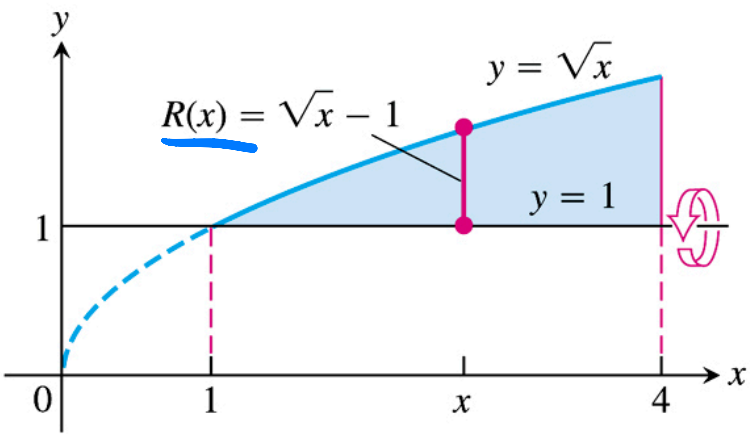
FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

由x轴 → 平行于x轴

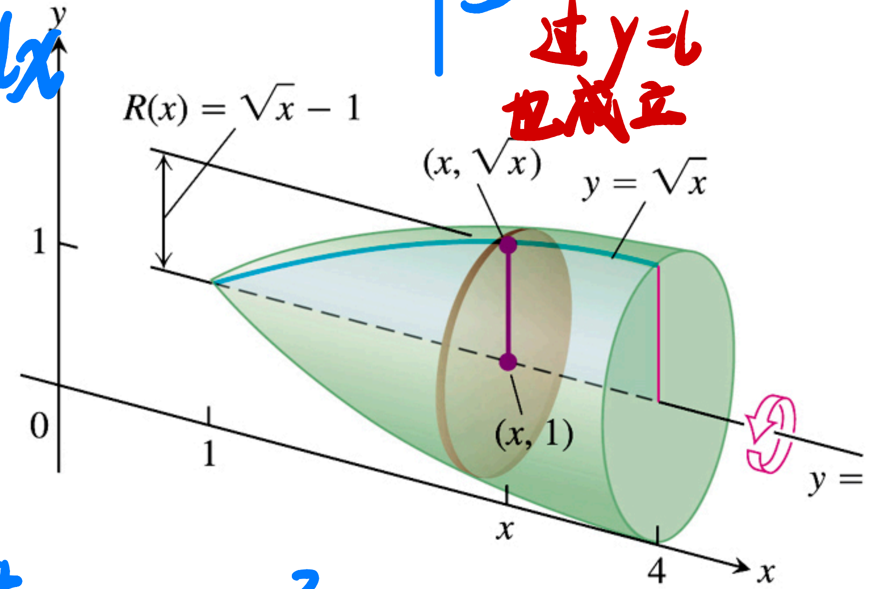
$$\int_a^b \pi(f(x)-L)^2\,dx$$



过 y=L 也成立



(a)



(b)

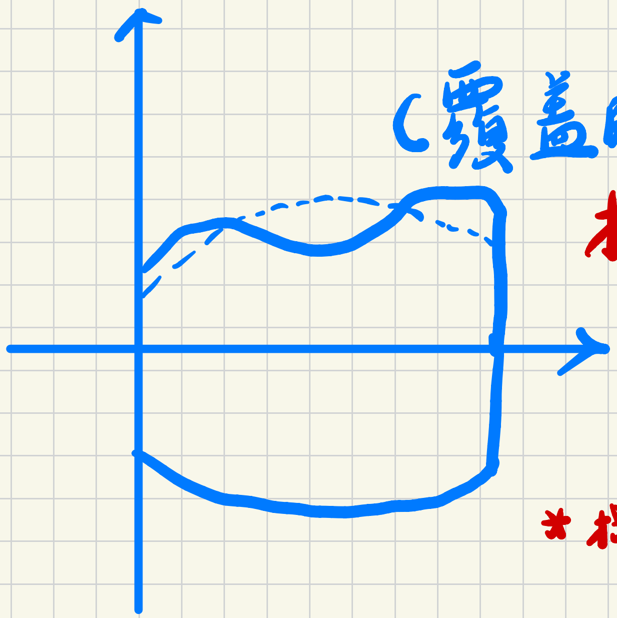
$$\int_1^4 \pi(\sqrt{x}-1)^2\,dx$$

FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6.

$$=\int_1^4 \pi(x-2\sqrt{x}+1)\,dx$$

$$=\pi\left(\frac{1}{2}x^2-\frac{4}{3}x^{\frac{3}{2}}+x\right)\Big]_1^4$$

* 若



(覆盖的情况)

(在对称轴两侧同样)

找离对称轴最远的部分

* 找到产生最大体积的曲线

Volume by Disks for Rotation About the y-axis

横截面与旋转轴垂直

$$V = \int_c^d A(y)\,dy = \int_c^d \pi\left[R(y)\right]^2 dy.$$

EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about the y-axis.

EXAMPLE 8 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

$$\int_1^4 \pi \left(\frac{2}{y}\right)^2 \underset{\sim}{dy}$$

对"轴"积分

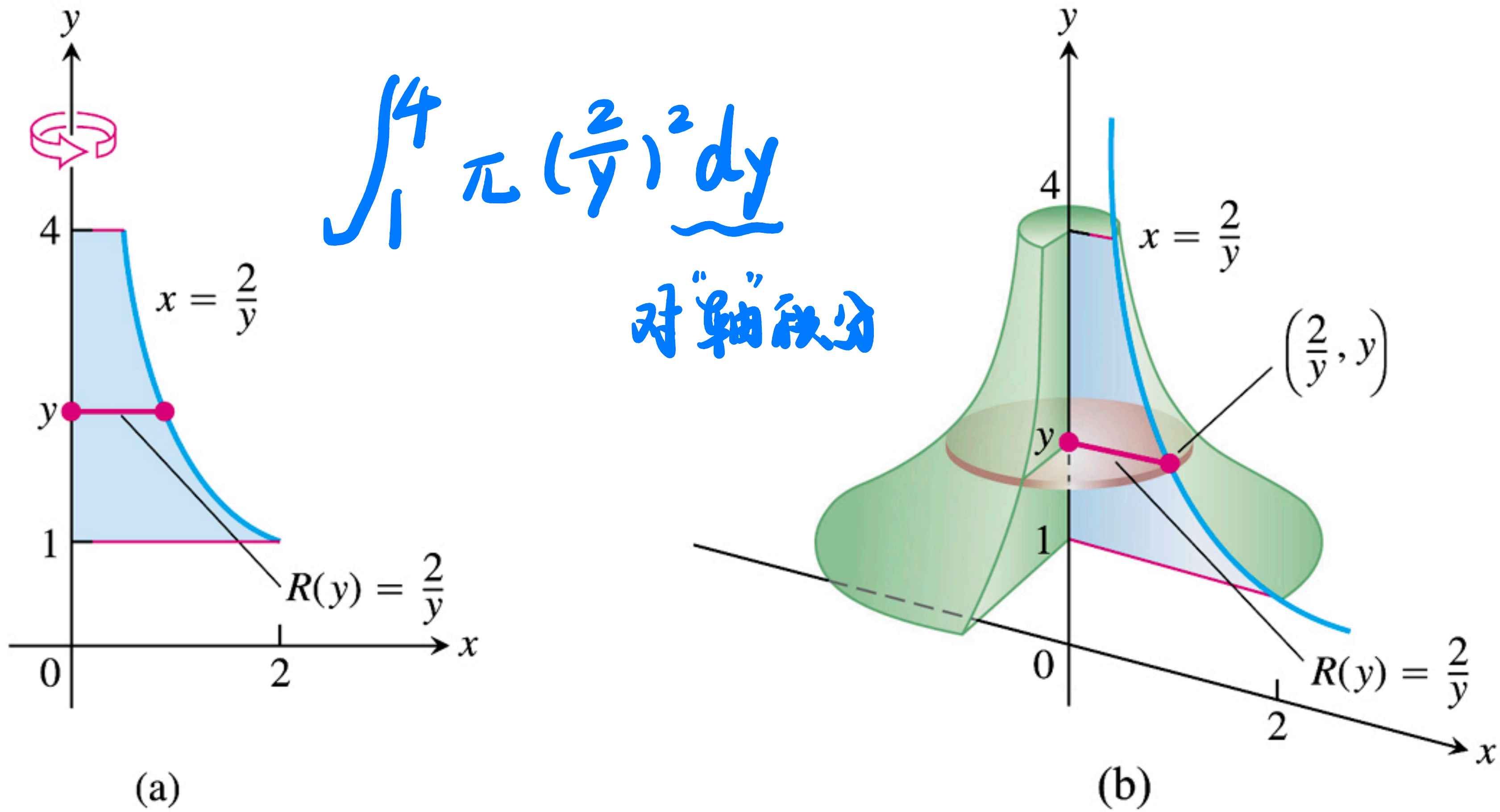


FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

先大致画图形

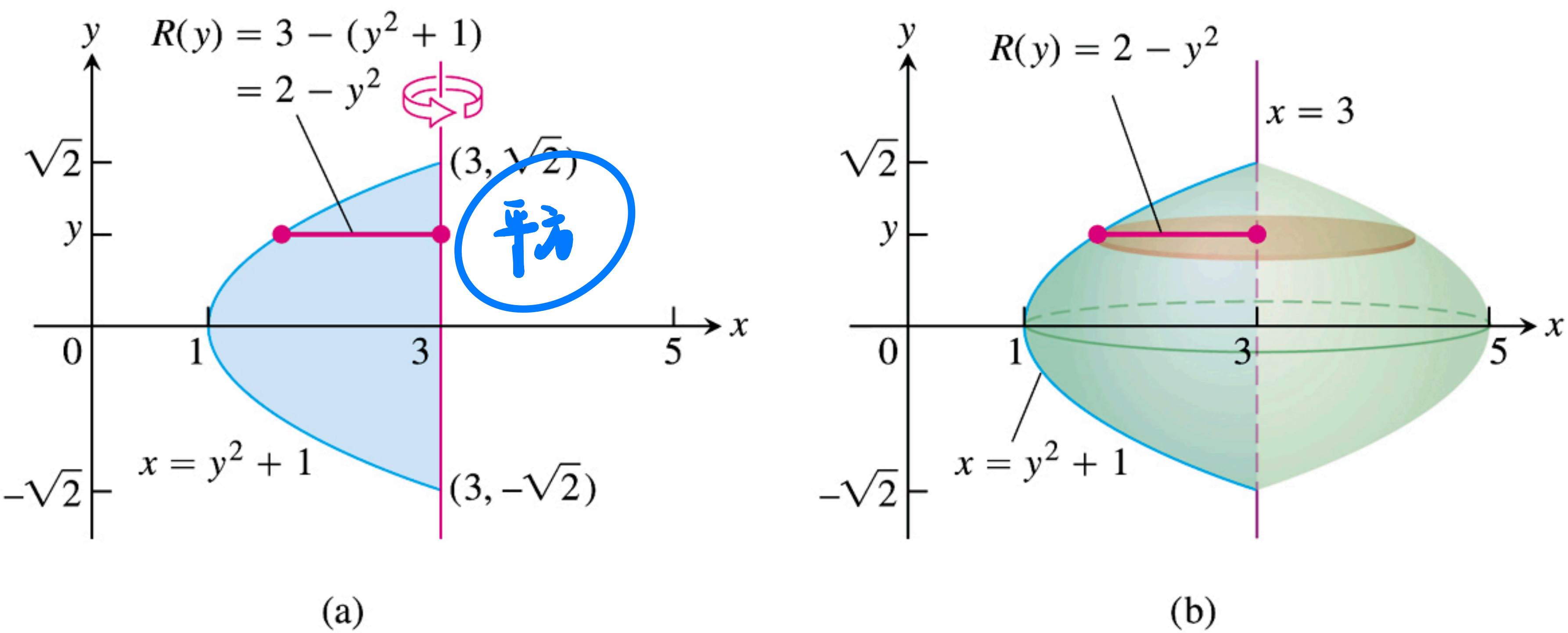
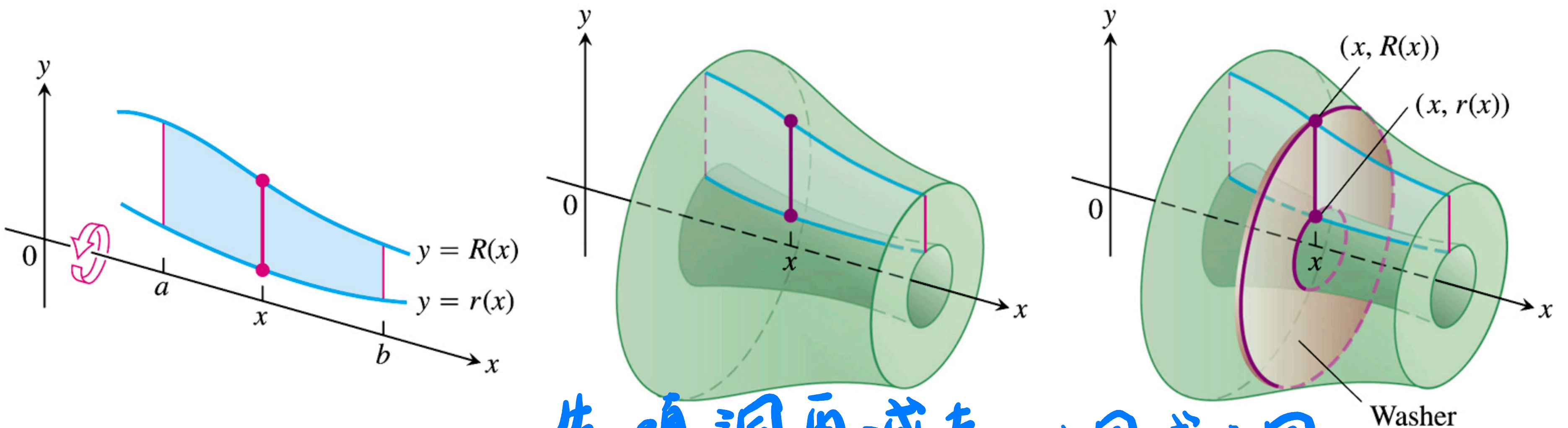


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.



先填洞再减去 (大圆减小圆

FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are *washers* (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

$$\text{Outer radius:} \quad R(x)$$

$$\text{Inner radius:} \quad r(x)$$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume in this case gives

洗衣机

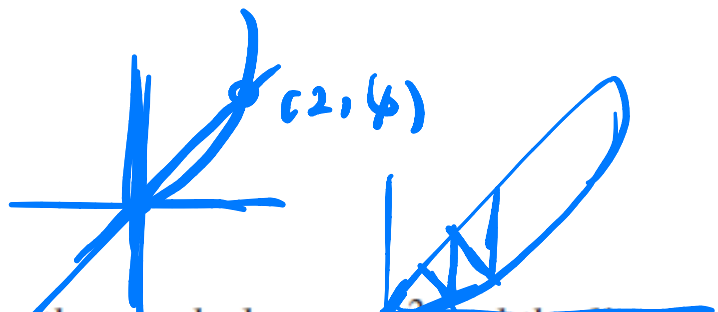
Volume by Washers for Rotation About the *x*-axis (若平行与x轴)

函数到旋转轴

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

注：平方差！

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

绕y轴 → 写成 $x=f(y)$

$x^2=2x$

$x=0/2$

$y=0/4$

$x=\sqrt{y}$

$x=\frac{1}{2}y$ 猜下大小

$$\int_0^4 \pi\left((\sqrt{y})^2-\left(\tfrac{1}{2}y\right)^2\right)dy$$

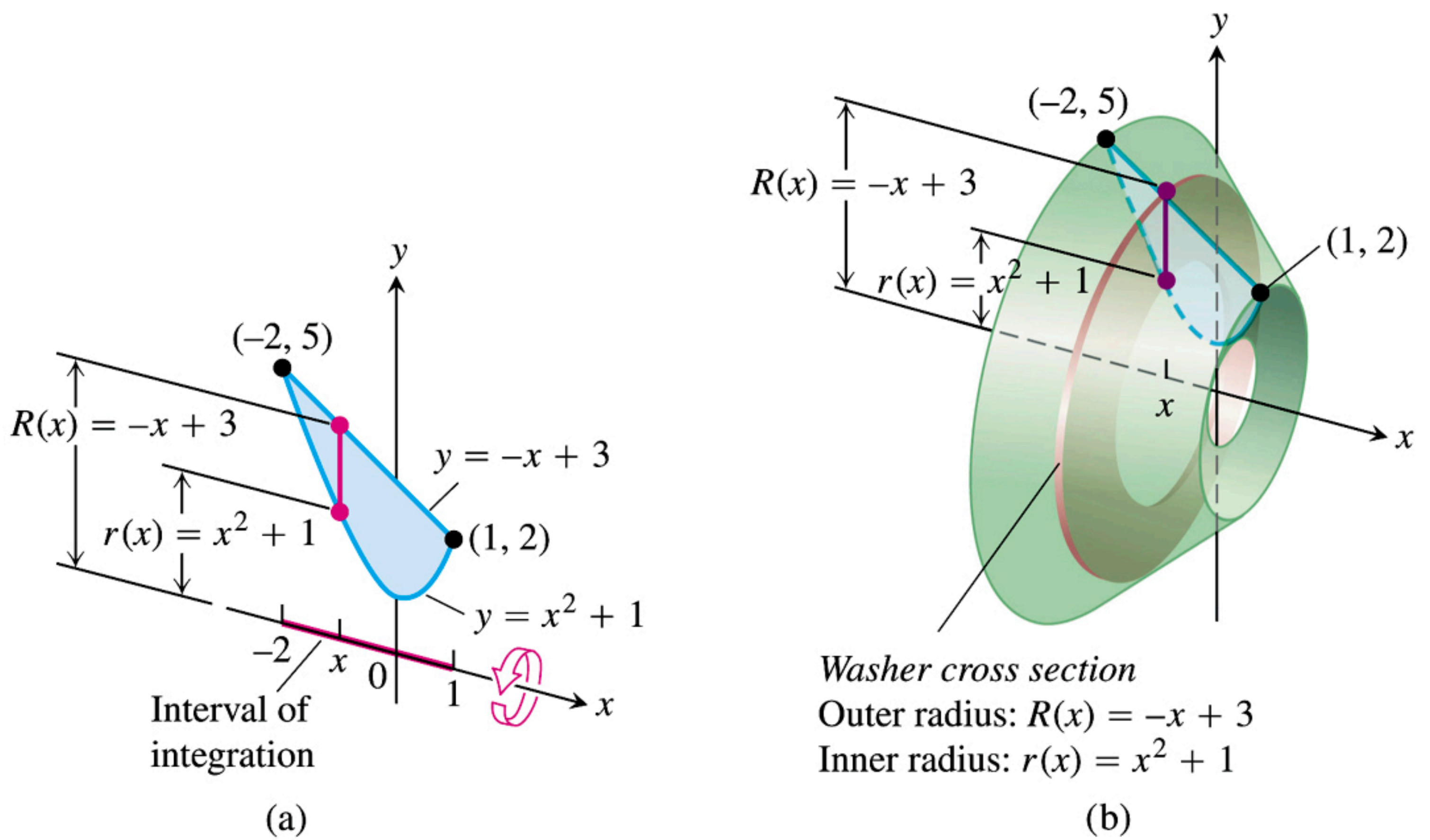


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

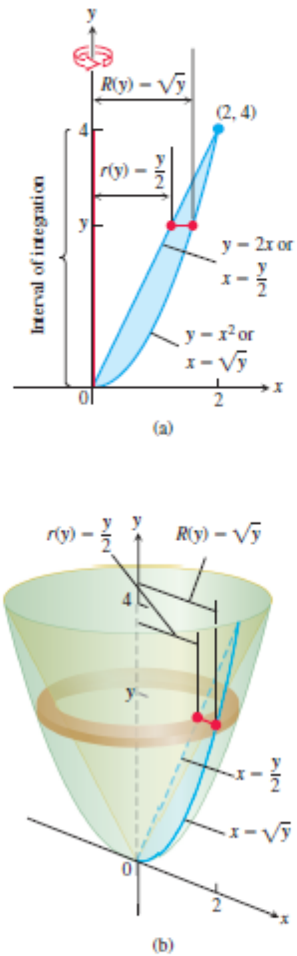


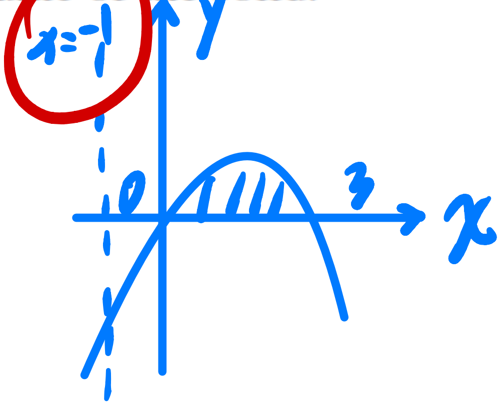
FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

6.2

Volumes Using Cylindrical Shells

只适用于旋转体

EXAMPLE 1 The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid (Figure 6.16). Find the volume of the solid.



$y = 3x - x^2$

$y = 0$

y 视为 Const

$x = \frac{3+\sqrt{9-4y}}{2} \;/\; \frac{3-\sqrt{9-4y}}{2}$

$$\int_0^{\frac{9}{4}} \pi \left[\left(\frac{3+\sqrt{9-4y}}{2} + 1 \right)^2 - \left(\frac{3-\sqrt{9-4y}}{2} + 1 \right)^2 \right] dy$$

转轴!

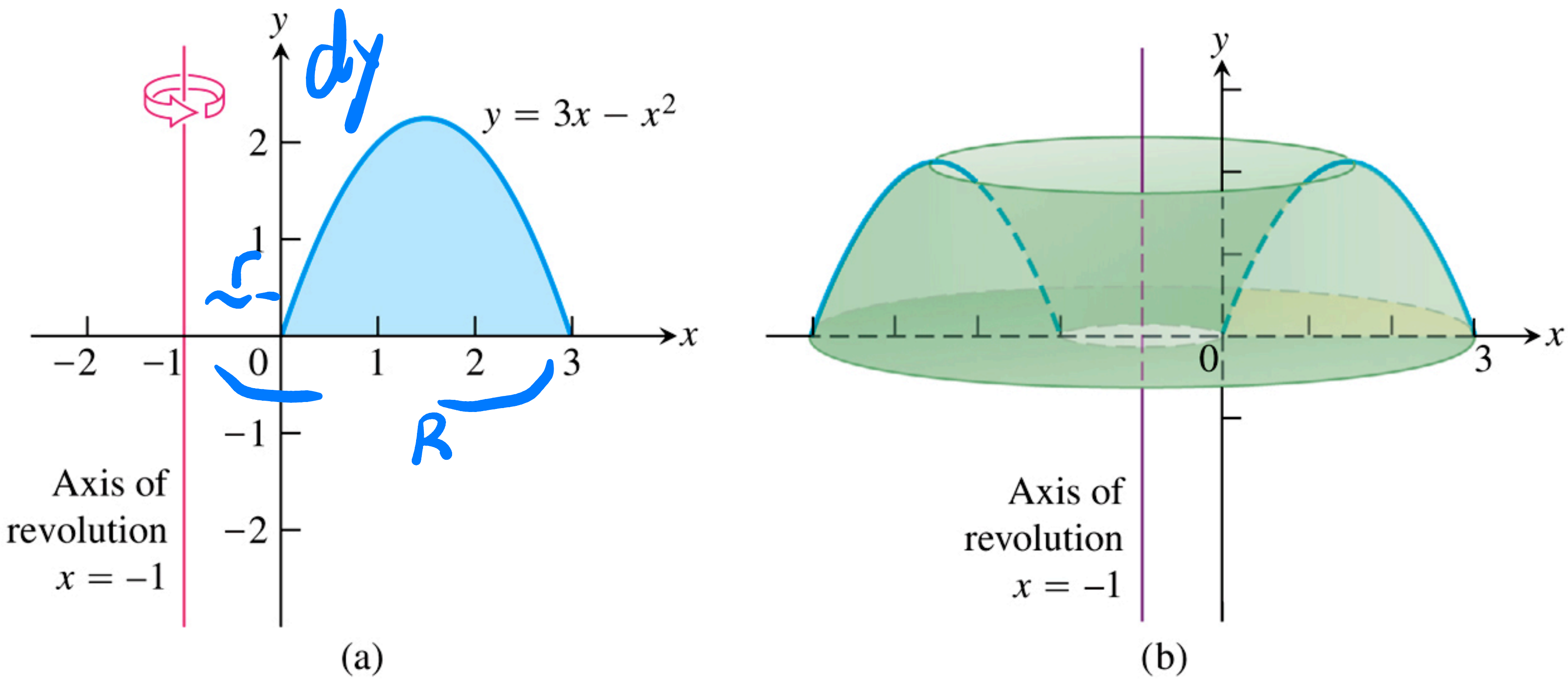
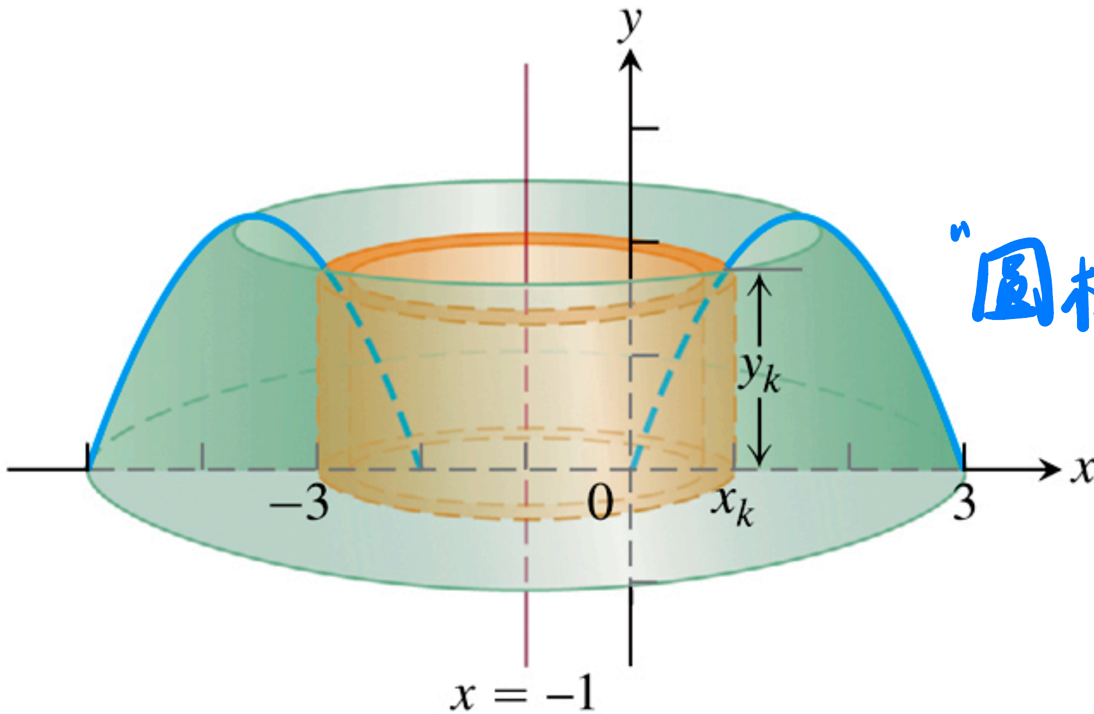
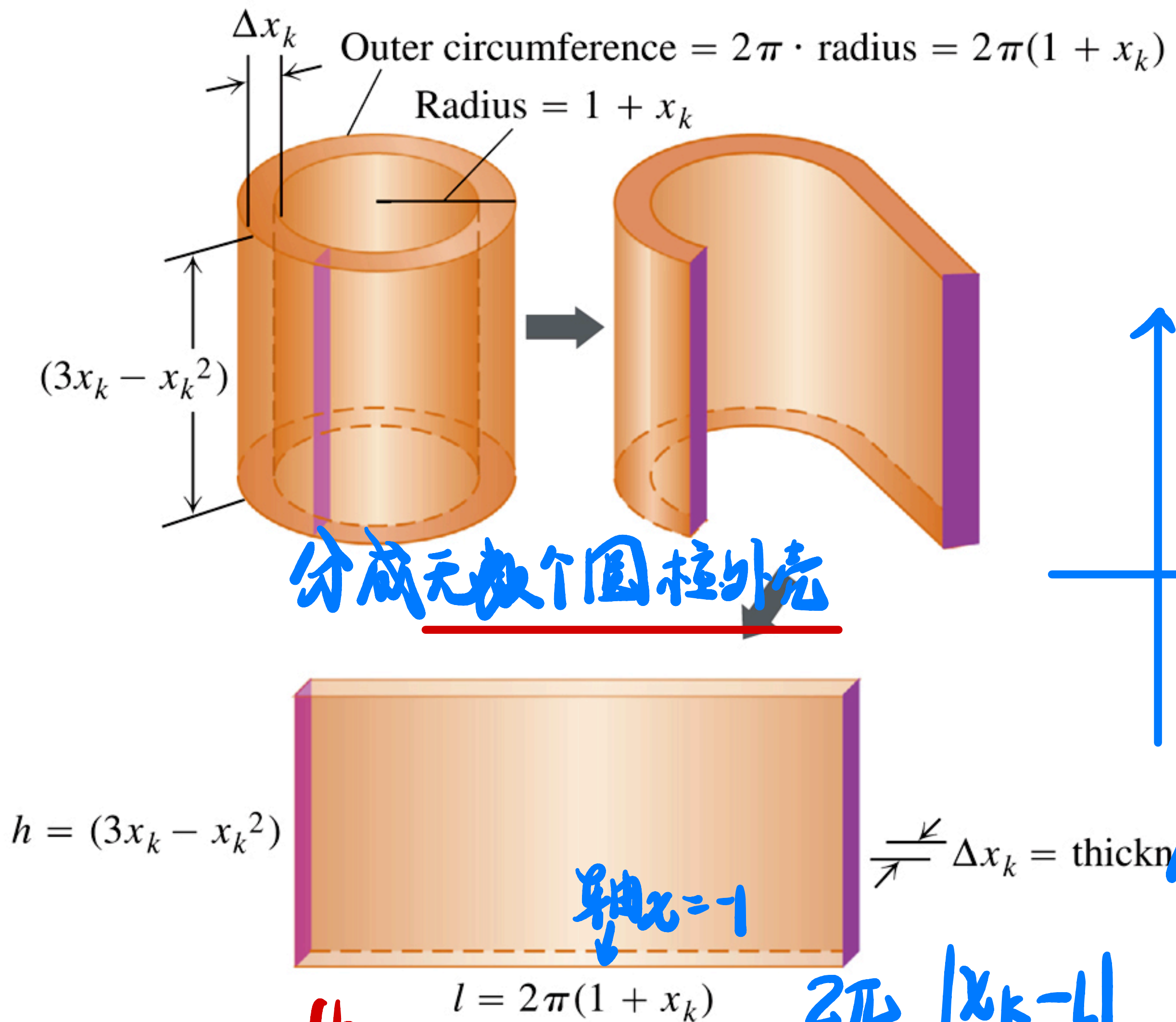


FIGURE 6.16 (a) The graph of the region in Example 1, before revolution. (b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.



“圆柱的外壳”

FIGURE 6.17 A cylindrical shell of height y_k obtained by rotating a vertical strip of thickness Δx_k about the line $x = -1$. The outer radius of the cylinder occurs at x_k, where the height of the parabola is $y_k = 3x_k - x_k^2$ (Example 1).

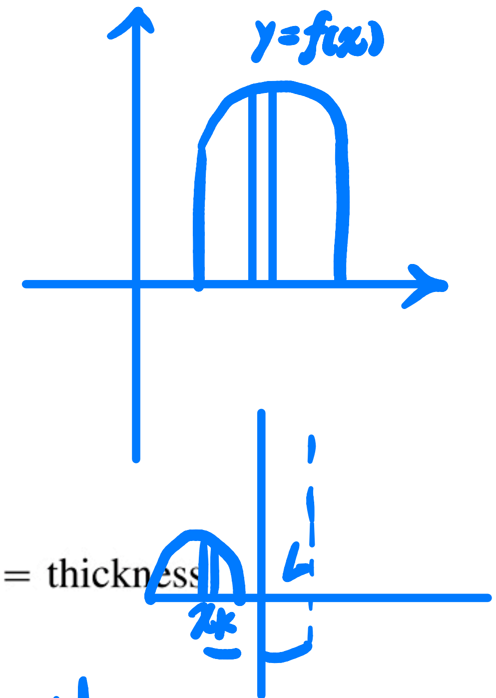


分成无数个圆柱外壳

轴 $x=-1$

$2\pi\,|x_k-1|$

正负？



$2\pi\int_a^b f(x)\,|x_k-1|$

先判

$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,2\pi\,|x_k-1|\,\Delta x_k$

FIGURE 6.18 Cutting and unrolling a cylindrical shell gives a nearly rectangular solid (Example 1).

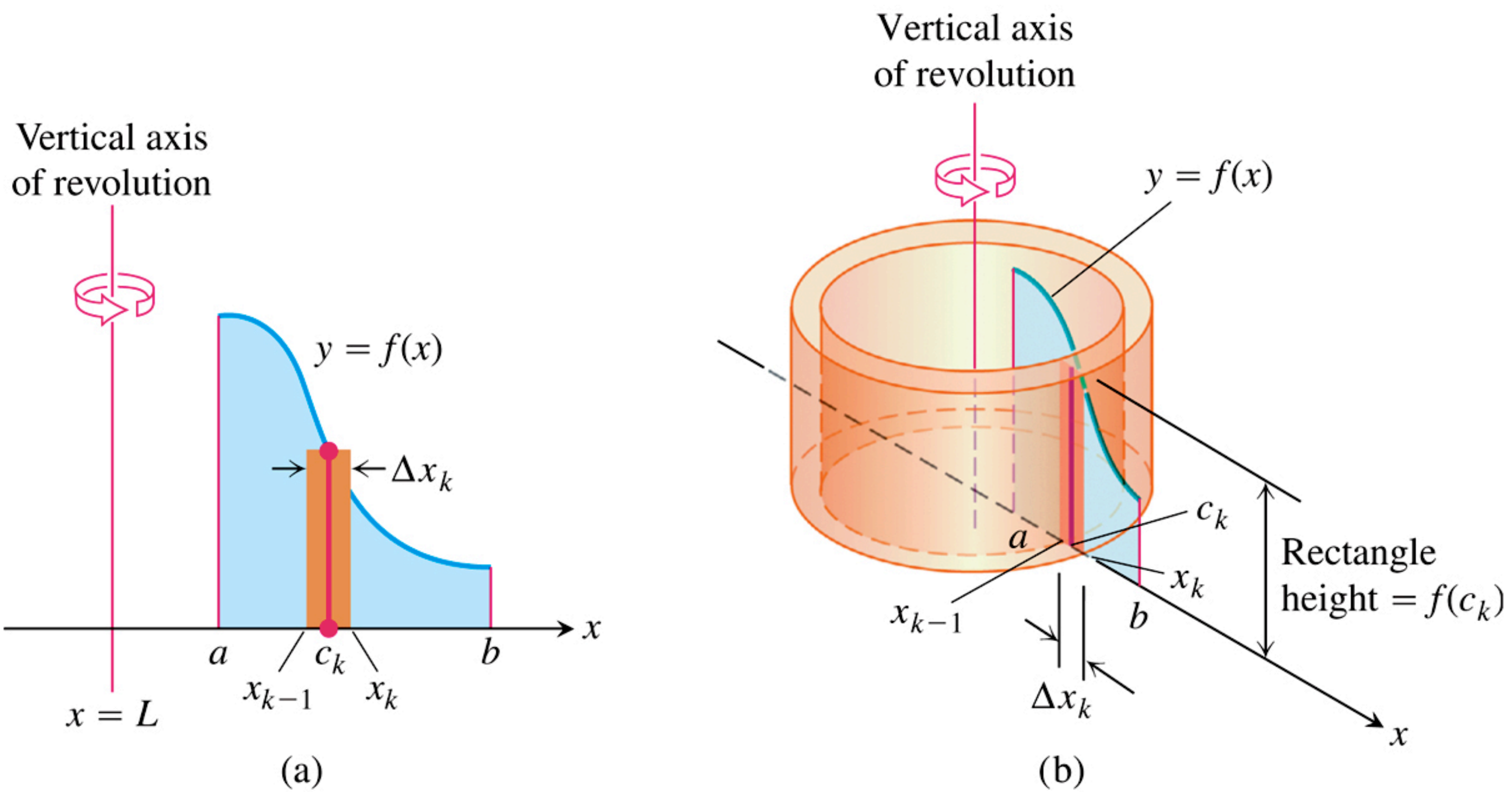


FIGURE 6.19 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

考虑正负

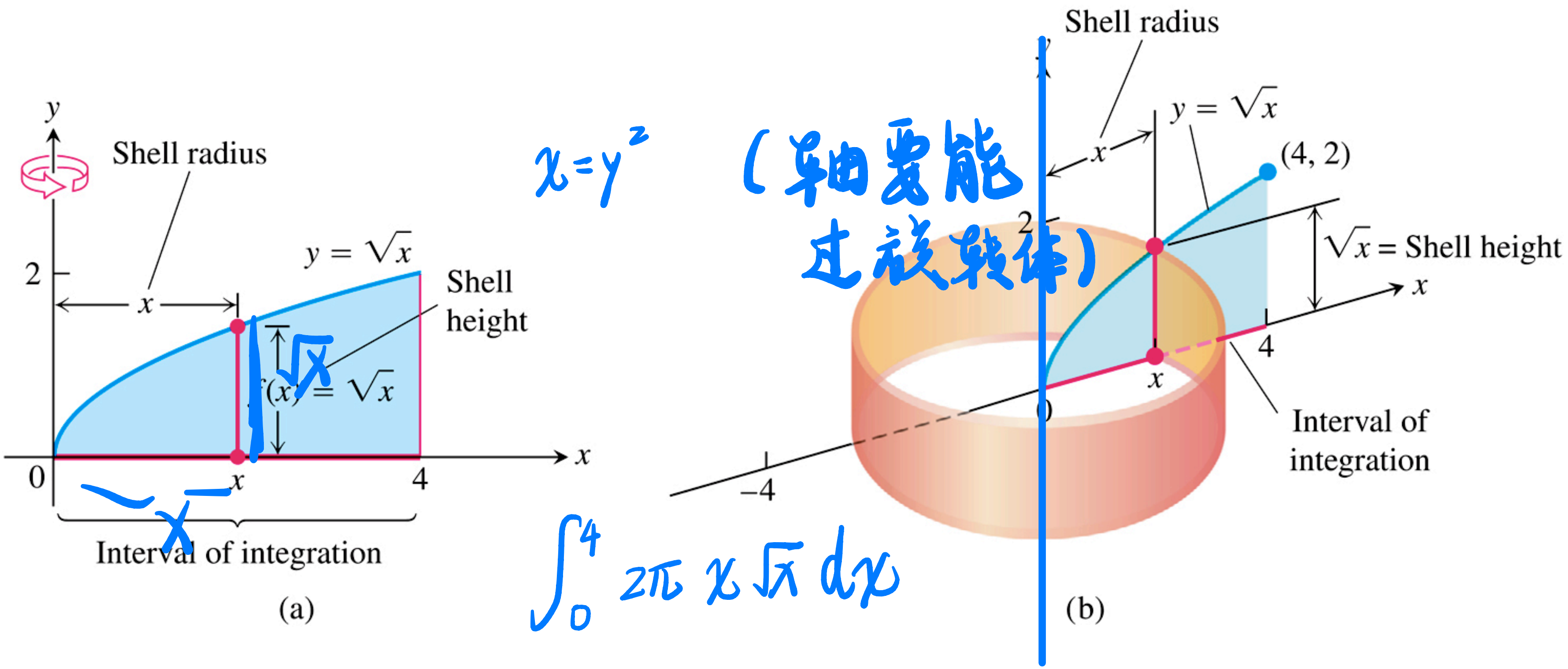
$$V = \int_a^b 2\pi \underbrace{\begin{pmatrix}\text{shell} \\ \text{radius}\end{pmatrix}}_{\text{离旋转轴距离}} \begin{pmatrix}\text{shell} \\ \text{height}\end{pmatrix} dx.$$

严格

切与轴平行！

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

EXAMPLE 3 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method.



$x=y^2$

(轴要能过旋转体)

$\int_0^4 2\pi x\sqrt{x}\,dx$

FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

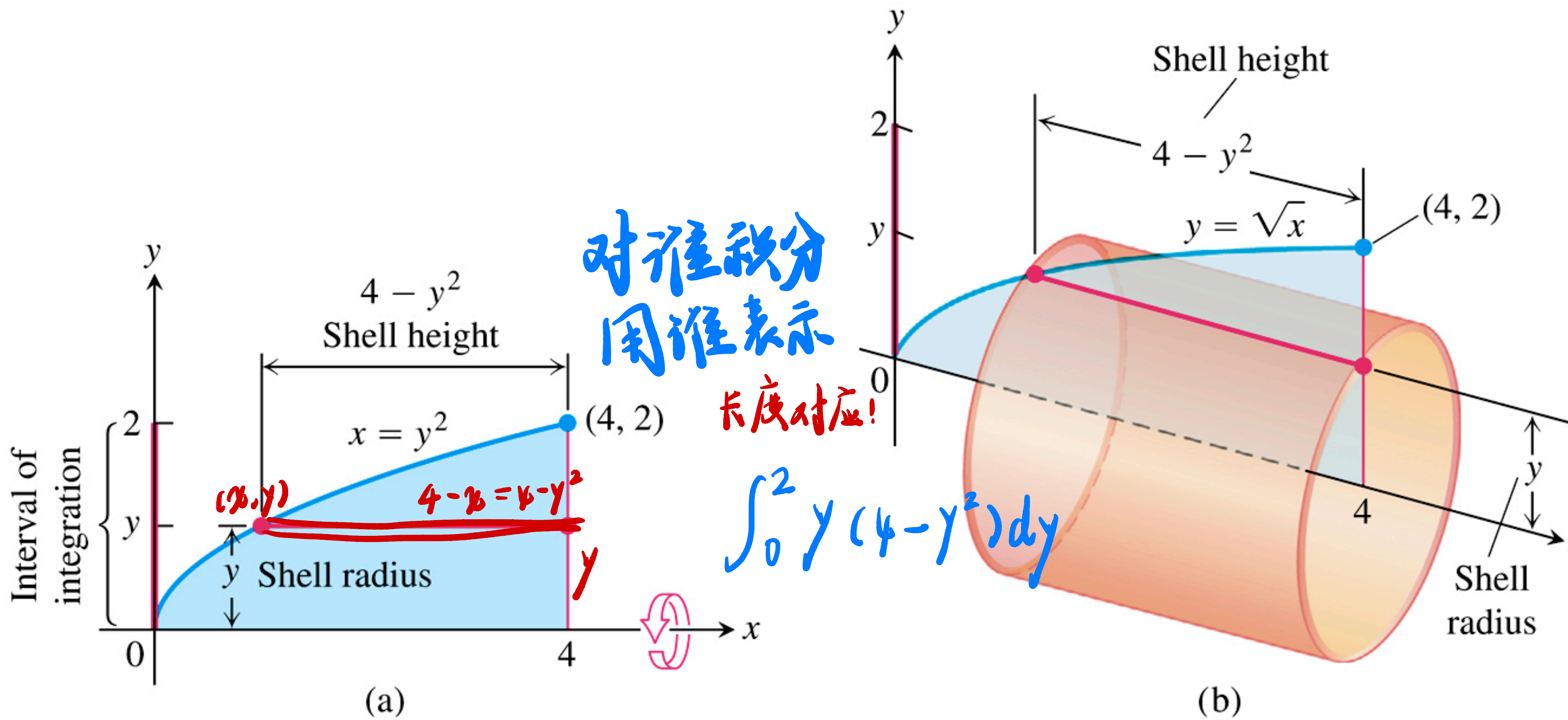


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy.

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).

   H , R

2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

6.3

Arc Length

弧长

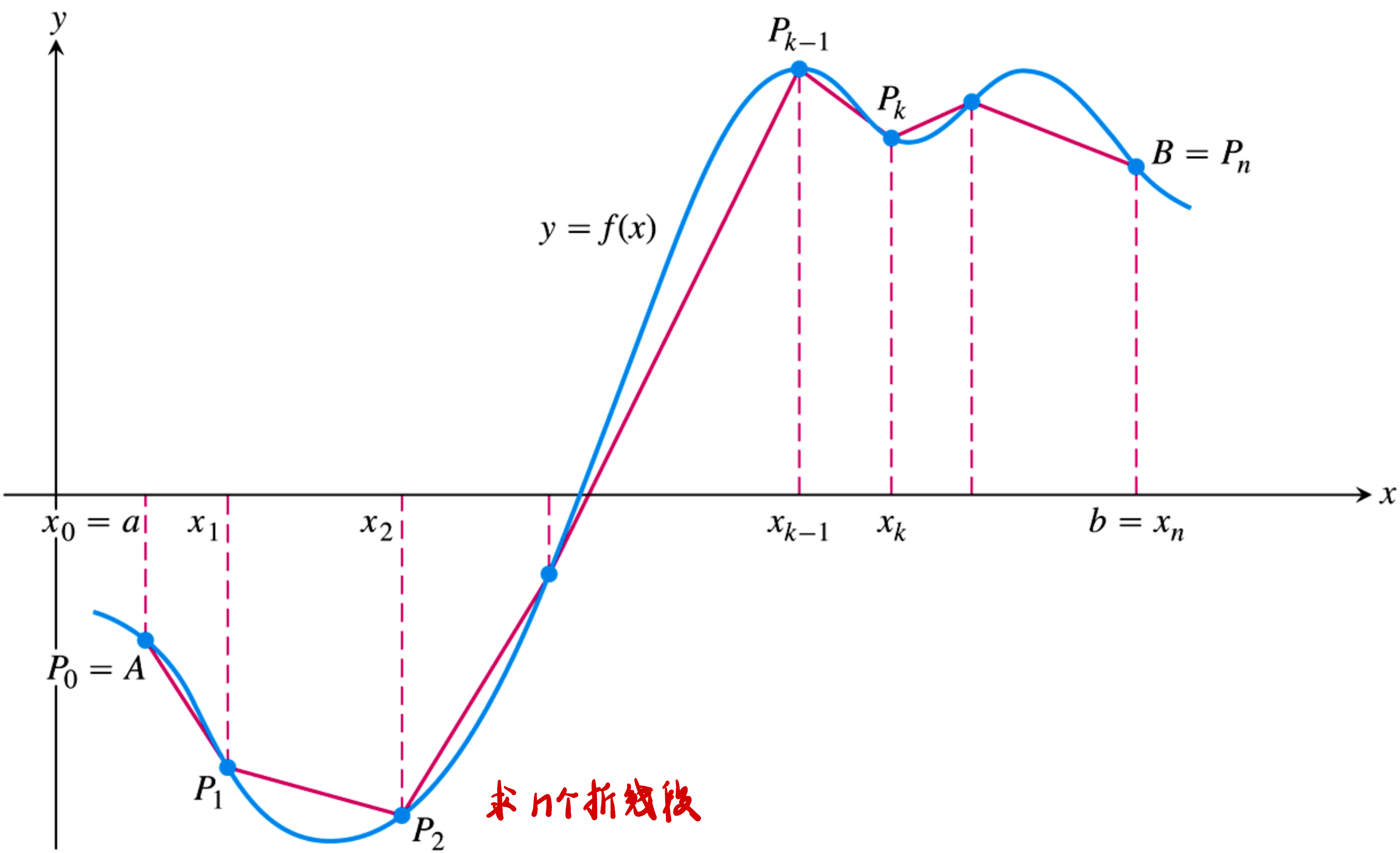
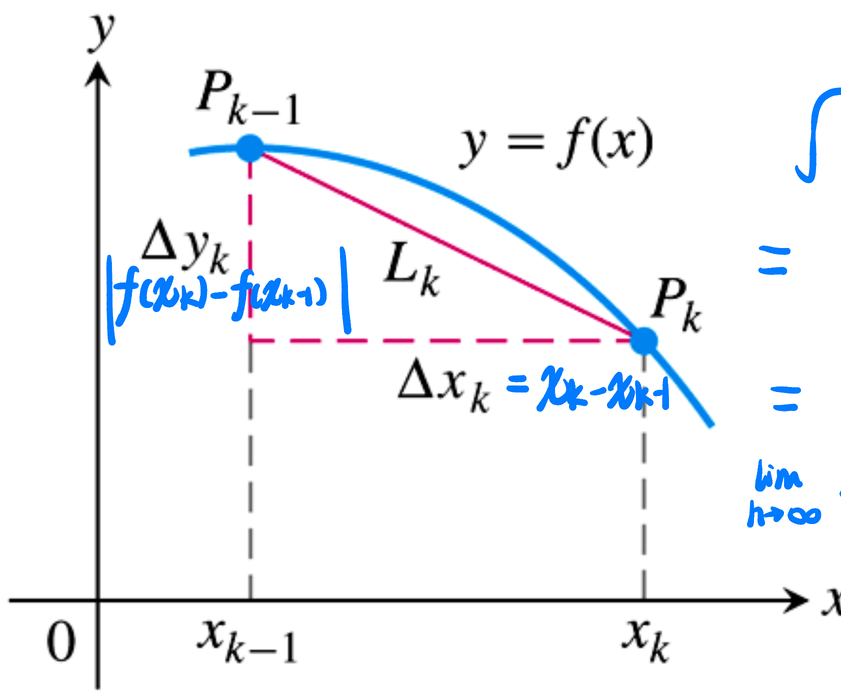


FIGURE 6.22 The length of the polygonal path $P_0P_1P_2 \cdots P_n$ approximates the length of the curve $y = f(x)$ from point A to point B.



$$\sqrt{\Delta x^2 + \Delta y^2}$$

$$= \sqrt{1+\left(\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}\right)^2}\,\Delta x$$

中值定理 $x_{k-1} < c < x_k$

$$= \sqrt{1+(f'(c_k))^2}\,\Delta x$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{1+(f'(c_k))^2}\,\Delta x_k$$

$$= \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

FIGURE 6.23 The arc $P_{k-1}P_k$ of the curve $y = f(x)$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

DEFINITION If f' is continuous on $[a, b]$, then the **length** (**arc length**) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{3}$$

EXAMPLE 1 Find the length of the curve (Figure 6.24)

在导数设点处

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

$$y' = \frac{4\sqrt{2}}{3} \times \frac{3}{2} x^{\frac{1}{2}} = 2\sqrt{2}x^{\frac{1}{2}}$$

$$\int_0^1 \sqrt{1+8x}\,dx = \int_{\cancel{0}\,1}^{9} \sqrt{u}\,du\,\frac{1}{8}$$

$u = 1+8x$ $du = 8dx$

$x=0$ $u=1$

$x=1$ $u=9$

不用重断[illegible]代入算

$$= \frac{1}{8} \times \frac{2}{3} u^{\frac{3}{2}} \Big]_{\cancel{0}\,1}^{9}$$

$$= \frac{13}{6}$$

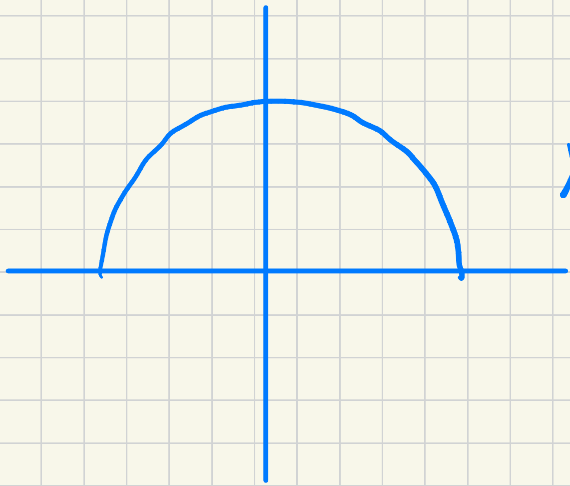
EXAMPLE 2 Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \le x \le 4.$$

$$f'(x) = \frac{1}{4}x^2 + \left(-\frac{1}{x^2}\right) \qquad \int_1^4 \left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)dx$$

$$\sqrt{1 + \frac{1}{16}x^4 + \frac{1}{x^4} - \frac{1}{2}}$$

$$= \frac{1}{4}x^2 + \frac{1}{x^2}$$



$y = \sqrt{r^2 - x^2}$ $\quad y' = \frac{-x}{\sqrt{r^2 - x^2}}$

$$\sqrt{1 + \frac{x^2}{r^2 - x^2}}$$

$$= \frac{r}{\sqrt{r^2 - x^2}}$$

$$\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx$$ 需反三角

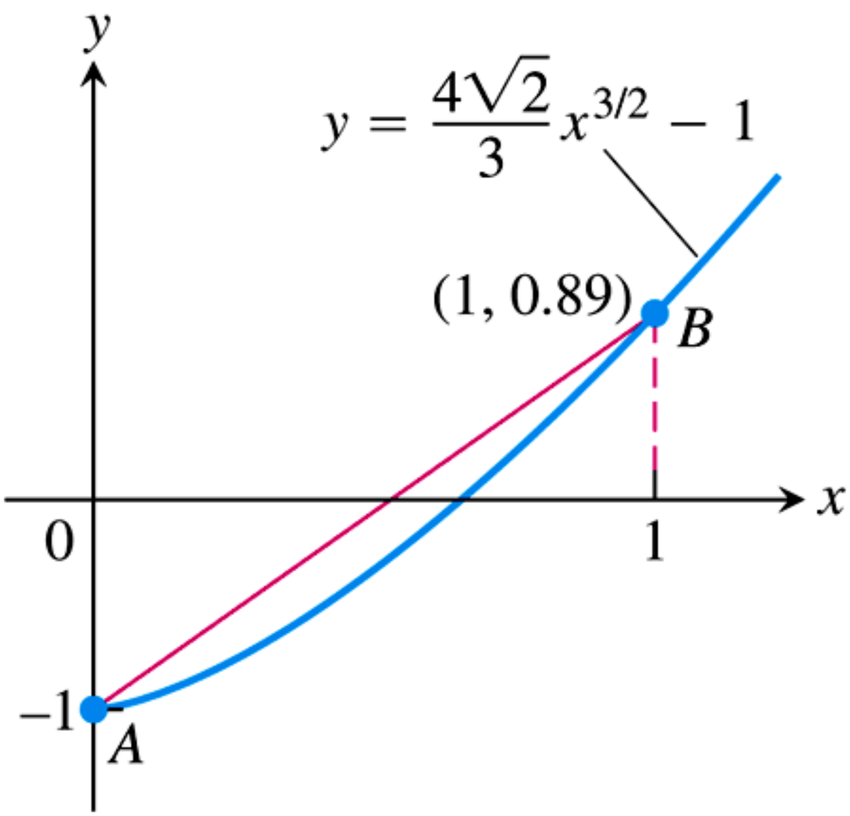


FIGURE 6.24 The length of the curve is slightly larger than the length of the line segment joining points A and B (Example 1).

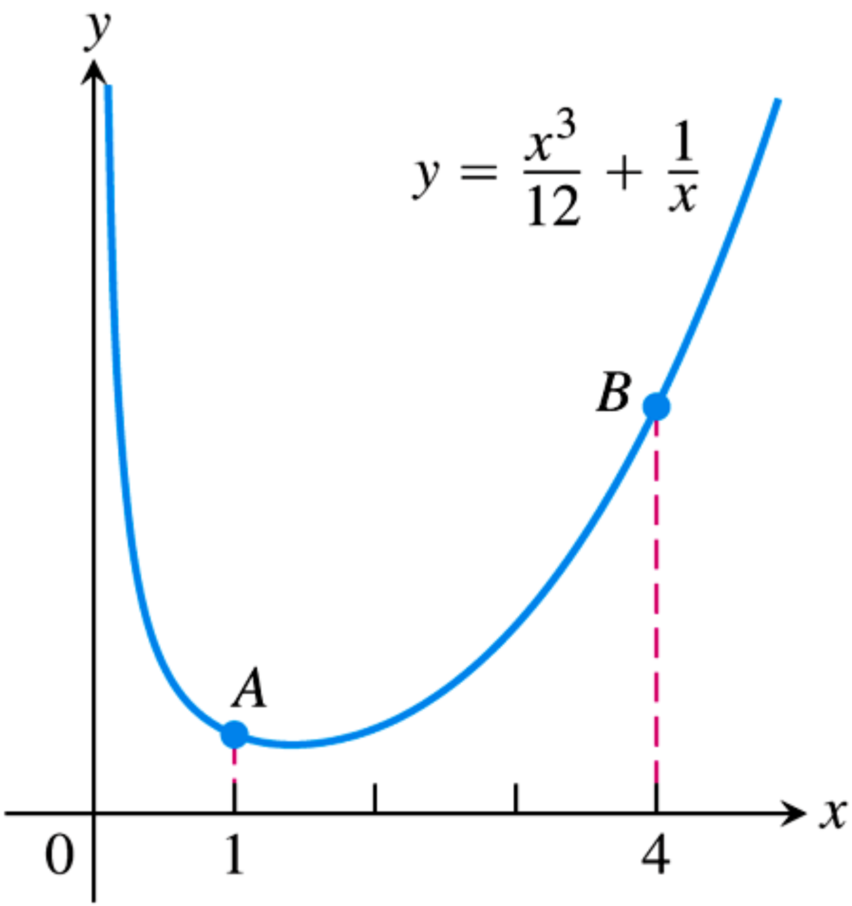


FIGURE 6.25 The curve in Example 2, where $A = (1, 13/12)$ and $B = (4, 67/12)$.

$x = g(y)$

$= \sqrt{(g(y_k)-g(y_{k-1}))^2 + (y_k - y_{k-1})^2}$

$= \sqrt{\left(\frac{g(y_k)-g(y_{k-1})}{y_k - y_{k-1}}\right)^2 + 1}\ \Delta y$

$= \sqrt{g'(c_k)^2+1}\ \Delta y = \int_c^d \sqrt{(g'(y))^2+1}\ dy$

Formula for the Length of $x = g(y), c \leq y \leq d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

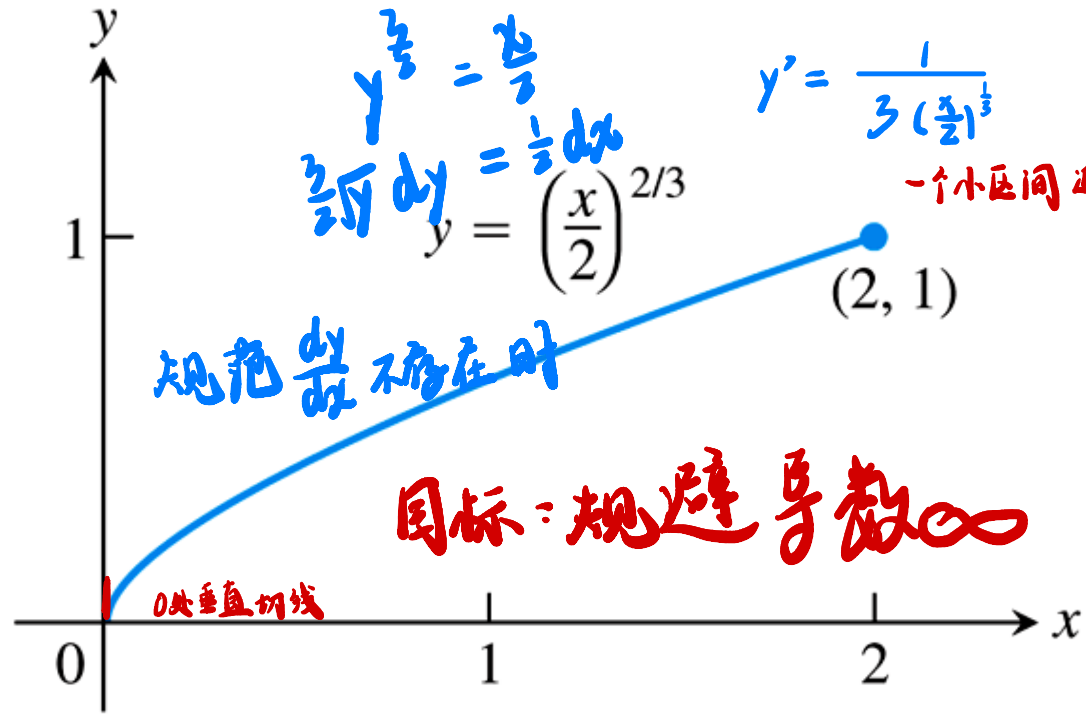
$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{4}$$

x对y求导

先写 $x = f(y)$

$\frac{dx}{dy} = f'(y)$

EXAMPLE 3 Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.



$y^{\frac{3}{2}} = \frac{x}{2}$

$\frac{3}{2}\sqrt{y}\,dy = \frac{1}{2}dx$

$y' = \frac{1}{3\left(\frac{x}{2}\right)^{\frac{1}{3}}}$

一个小区间近似会很大 无界

规范 $\frac{dy}{dx}$ 不存在时

目标：规避导数∞

0处垂直切线

FIGURE 6.26 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 3).

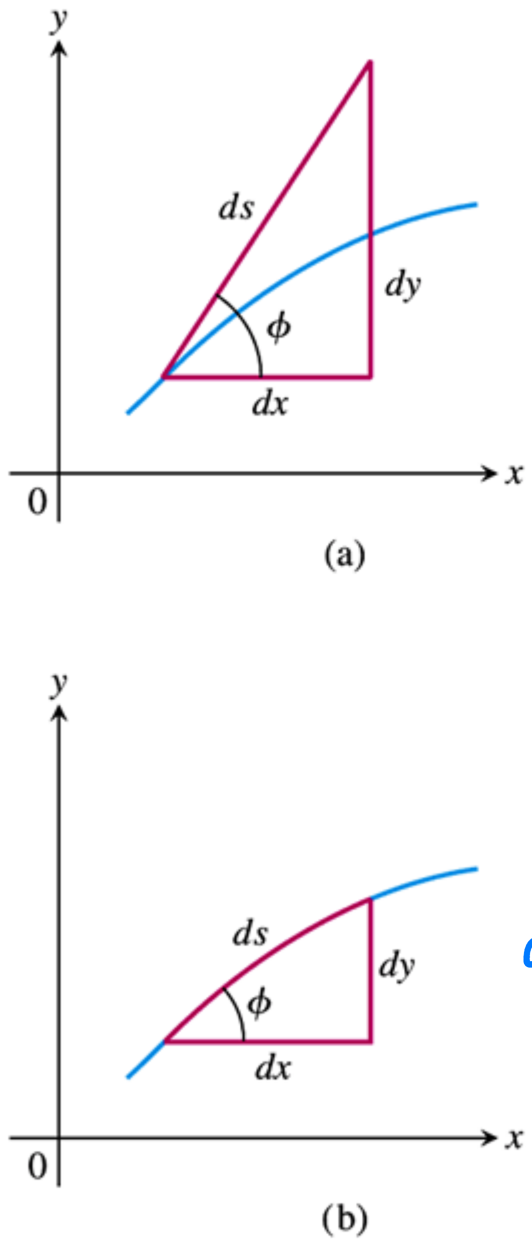
$\frac{dx}{dy} = 3\sqrt{y}$

$\int_0^1 \sqrt{1+9y}\,dy$

$u = 1+9y \quad du = 9dy$

$\int_1^{10} \sqrt{u}\,\frac{1}{9}du$

$= \frac{1}{9} \times \frac{2}{3} u^{\frac{3}{2}}\Big]_1^{10} = \frac{2}{27}(10\sqrt{10}-1)$



ds、dy 都不是精确值
是切线上的变化量

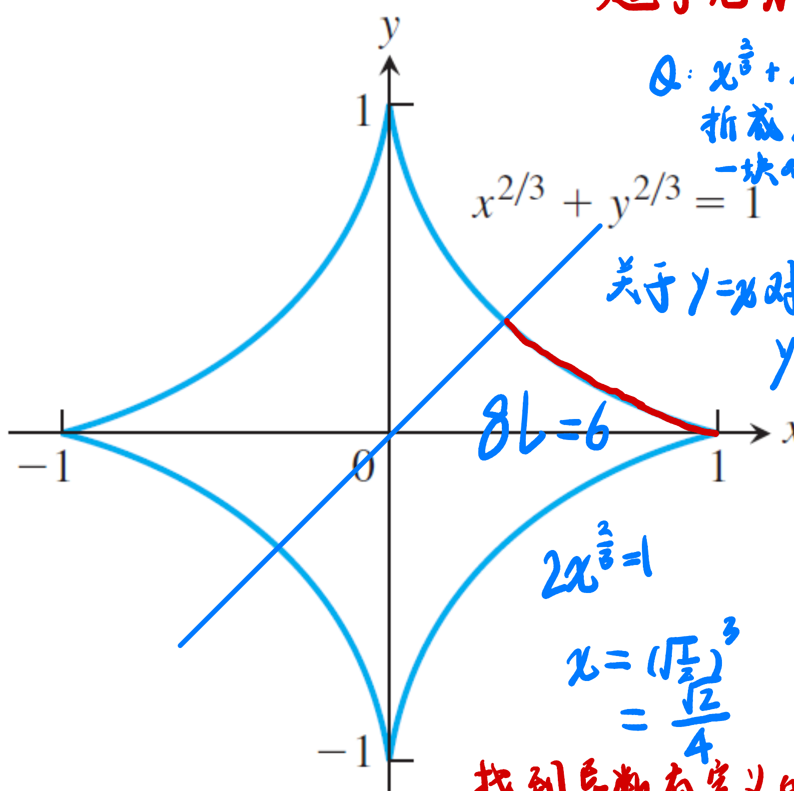
$s=\int ds$

两个公式统一
写法不同

FIGURE 6.27 Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

The length of an astroid The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.

因x、y都趋于无穷



Q: $x^{\frac{2}{3}} + 2y^{\frac{2}{3}} = 1$
拆成2块
一块对x，一块对y

关于y=x对称

$y = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$

$\frac{dy}{dx} = \cancel{\frac{3}{2}}\sqrt{1 - x^{\frac{2}{3}}}\,(-1)\cancel{\frac{2}{3}}\,x^{-\frac{1}{3}}$

$= \frac{-\sqrt{1-x^{\frac{2}{3}}}}{x^{\frac{1}{3}}}$

$8L = 6$

$2x^{\frac{2}{3}} = 1$

$x = (\sqrt{\tfrac{1}{2}})^3 = \frac{\sqrt{2}}{4}$

$\int_{\frac{\sqrt{2}}{4}}^{1} \sqrt{1 + \frac{1-x^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\,dx = \int_{\frac{\sqrt{2}}{4}}^{1} x^{-\frac{1}{3}}\,dx = \frac{3}{8}$

找到导数有定义的部分算

6.4

Areas of Surfaces of Revolution

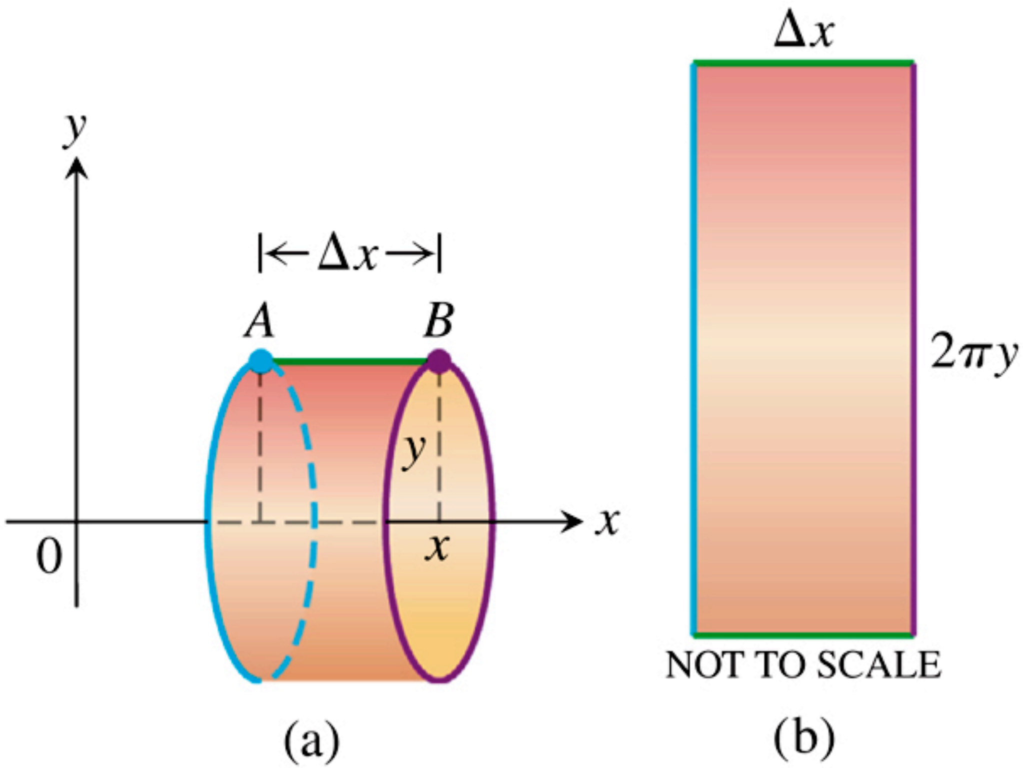


FIGURE 6.28 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x-axis has area $2\pi y \Delta x$. (b) The cut and rolled-out cylindrical surface as a rectangle.

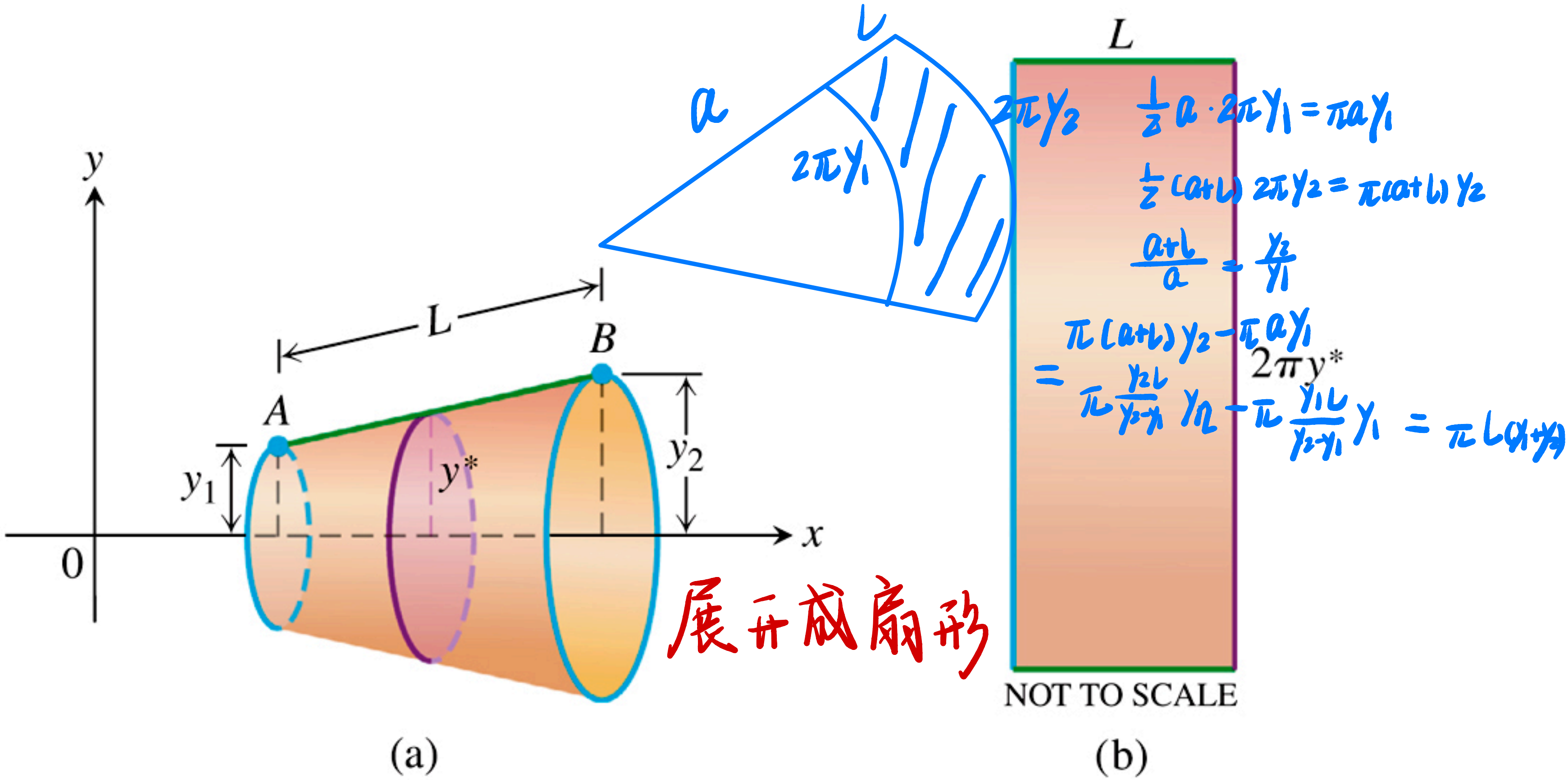
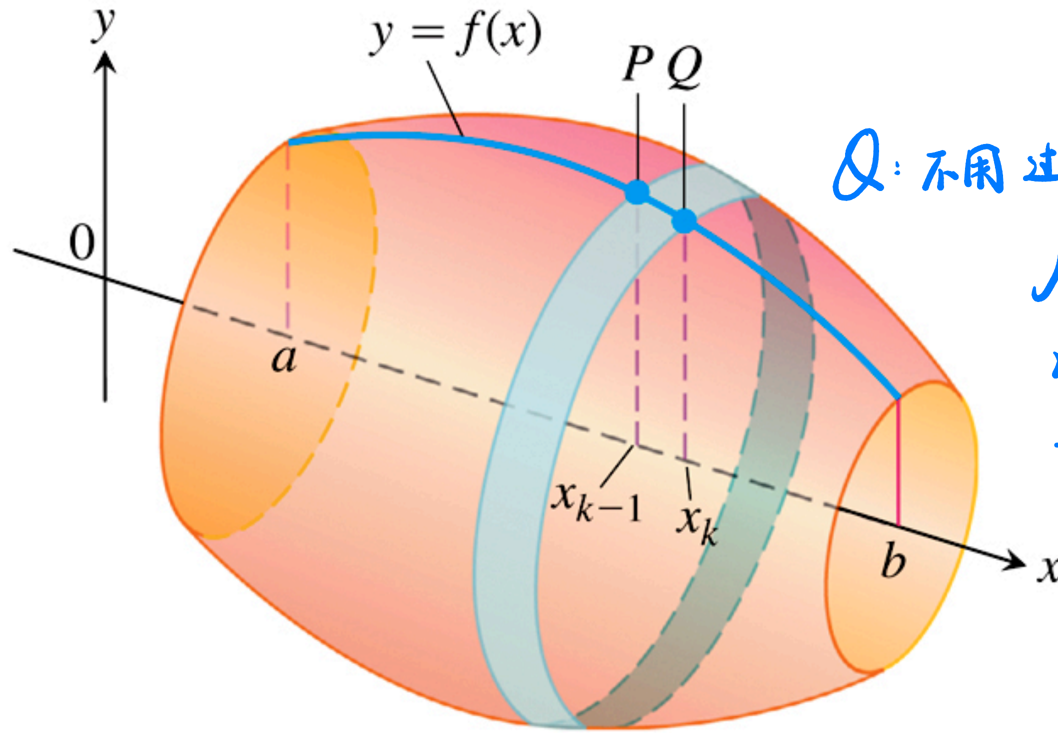


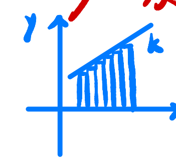
FIGURE 6.29 (a) The frustum of a cone generated by rotating the slanted line segment AB of length L about the x-axis has area $2\pi y^* L$. (b) The area of the rectangle for $y^* = \dfrac{y_1 + y_2}{2}$, the average height of AB above the x-axis.



Q: 不用 过Q平行线作圆柱累加

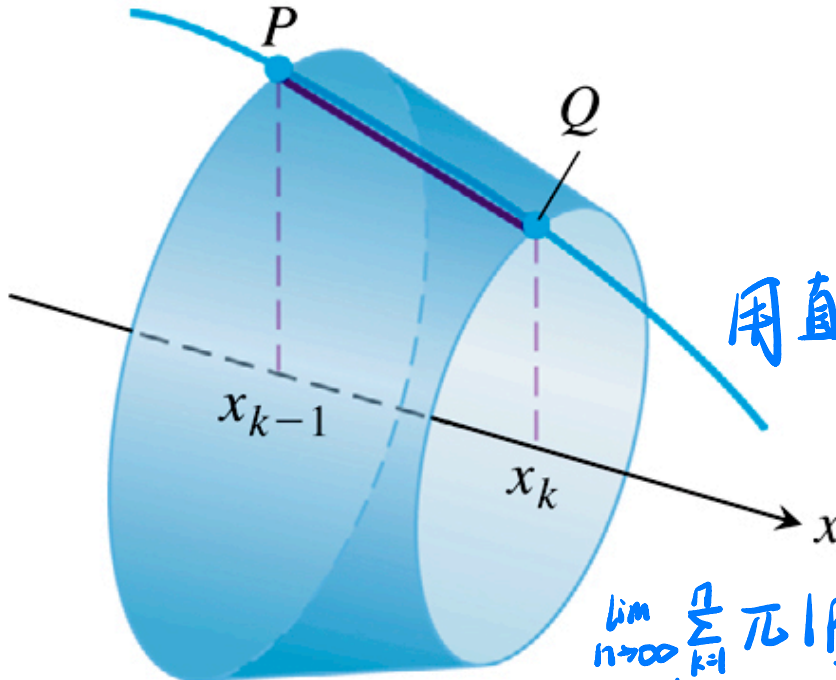
$\int_a^b 2\pi f(x) \sqrt{1+f'(x)^2}\, dx$ 导数越大相差大

少一项 不连续



无论切多细 有比例 $\sqrt{1+k^2}$

FIGURE 6.30 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \le x \le b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.



用直线段近似

圆 × "高"

$$\lim_{n\to\infty}\sum_{k=1}^{n} \pi |PQ| \left(f(x_{k-1}) + f(k)\right)$$

$$|PQ| = \sqrt{1+(f'(c_k))^2}\,\Delta x$$

$$= \int_a^b 2\pi \sqrt{1+(f'(k))^2}\, f(x)\,dx$$

FIGURE 6.31 The line segment joining P and Q sweeps out a frustum of a cone.

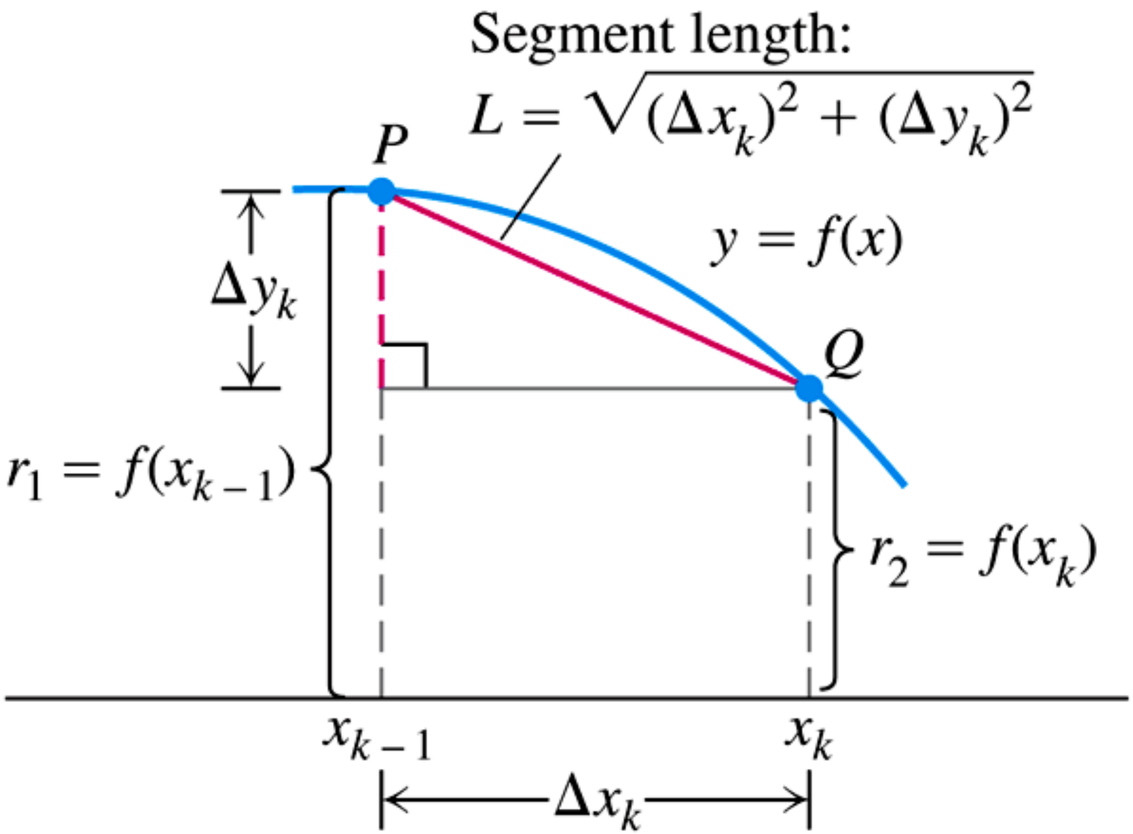


FIGURE 6.32 Dimensions associated with the arc and line segment PQ.

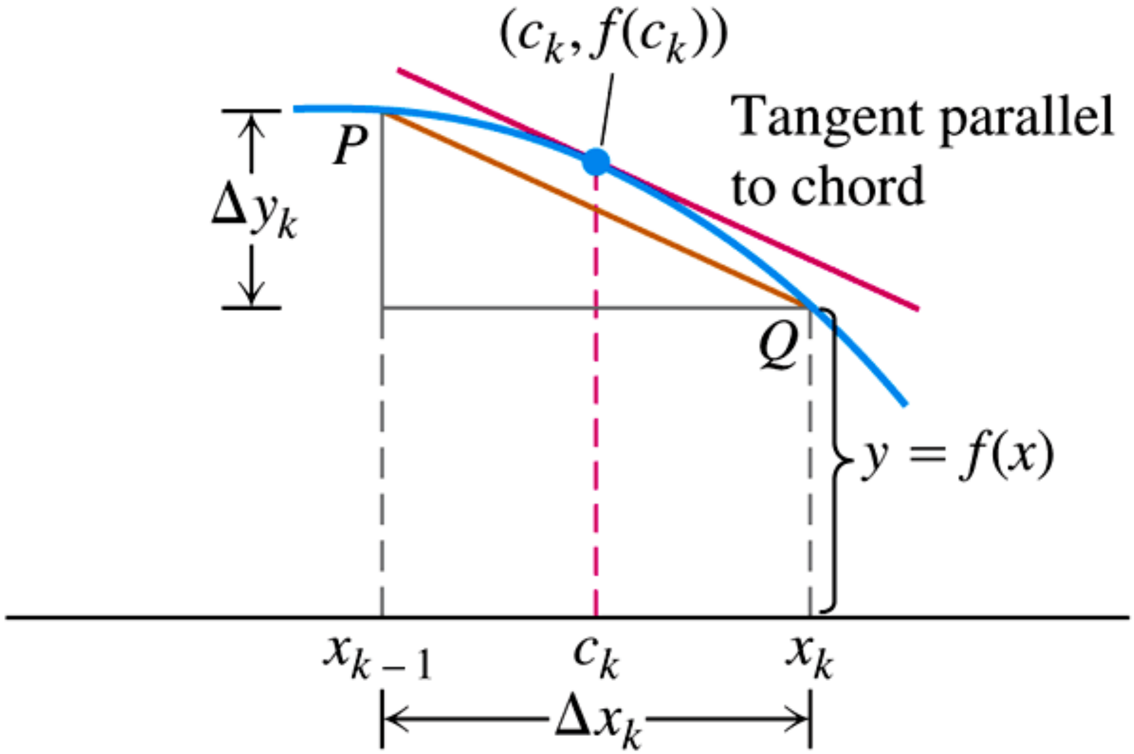


FIGURE 6.33 If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ.

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \underline{2\pi f(x)} \sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis (Figure 6.34).

$$y' = \frac{1}{\sqrt{x}}$$

$$\sqrt{1+y'^2} = \sqrt{1+\frac{1}{x}}$$

$$\int_1^2 2\pi f(x) \sqrt{1+\frac{1}{x}}\, dx$$

$$= 4\pi \int_1^2 \sqrt{1+x}\, dx$$

$$= 4\pi \int_2^3 \sqrt{u}\, du \qquad u = 1+x$$

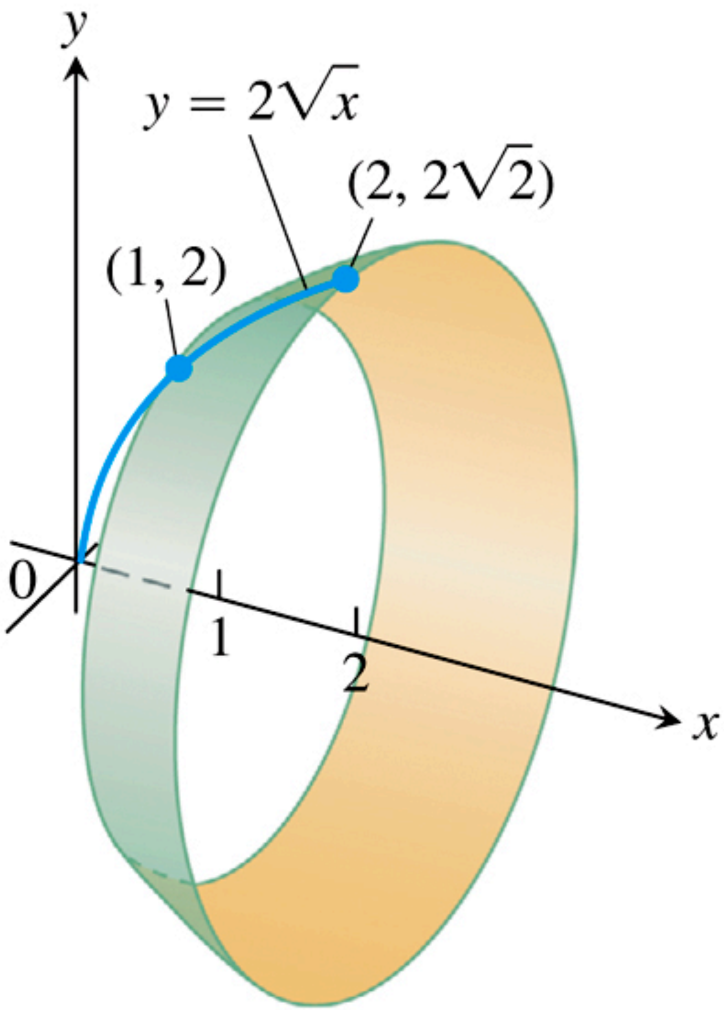


FIGURE 6.34 In Example 1 we calculate the area of this surface.

Surface Area for Revolution About the *y*-Axis

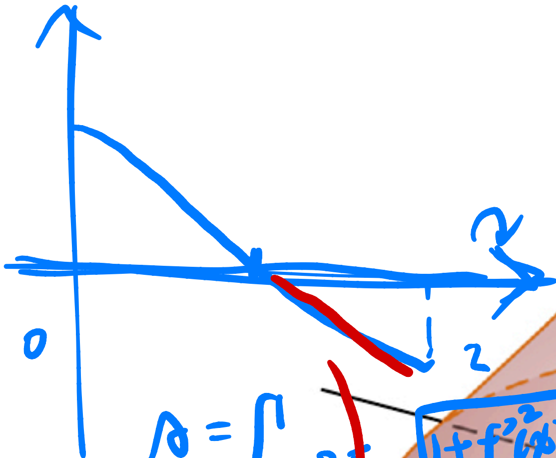
If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

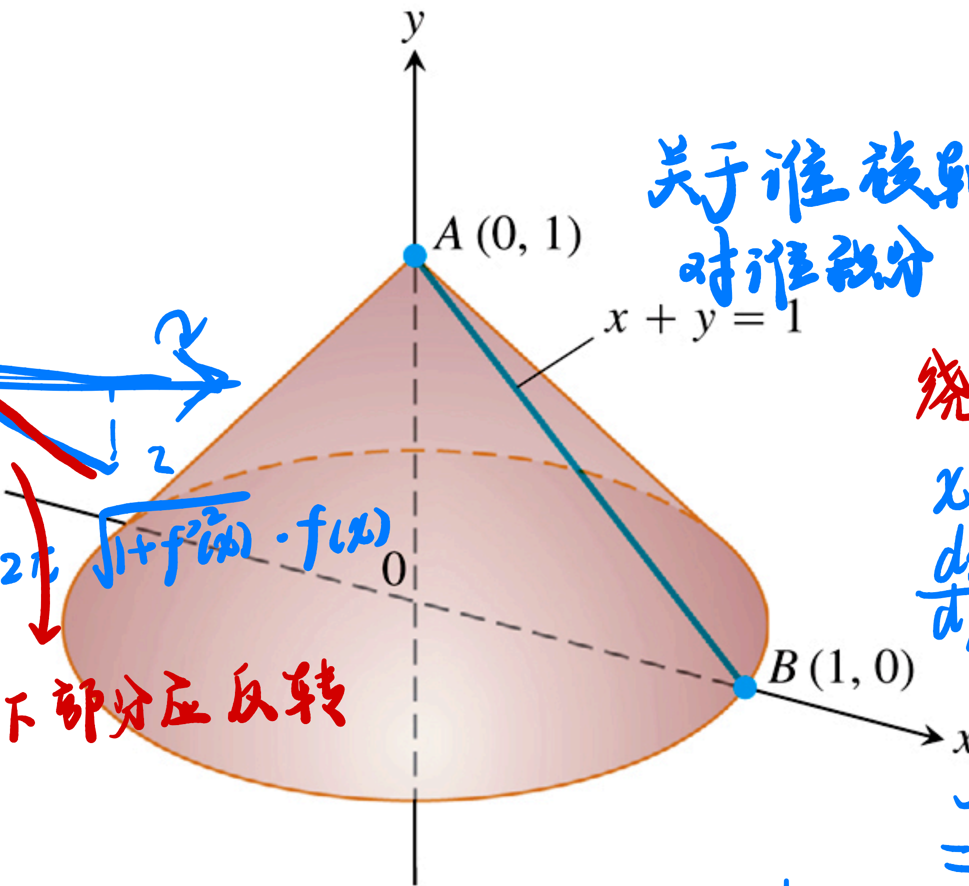
$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \underline{\int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy.} \qquad (4)$$

$x \to y$

到转轴距离

EXAMPLE 2 The line segment $x = 1 - y, 0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Figure 6.35. Find its lateral surface area (which excludes the base area).





关于谁旋轴
对谁积分

绕y轴

$x = 1 - y$

$\frac{dx}{dy} = -1$

$A = \int_0^1 2\pi \sqrt{1+f'^2(x)} \cdot f(x)$

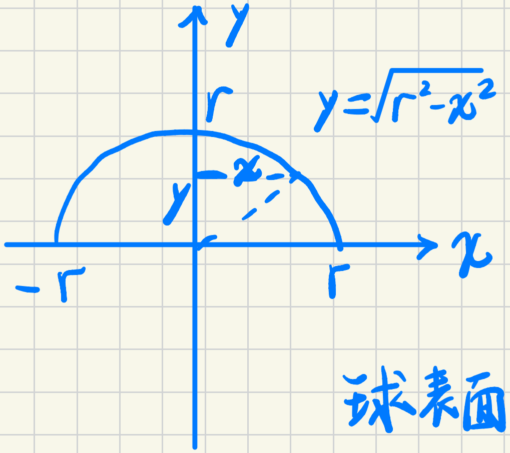
轴下部分应反转

$\int_0^1 2\pi f(y) \sqrt{1+1}\, dy$

$= \sqrt{2}\pi$

$S = \pi r l$

FIGURE 6.35 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).



$$\int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$=\int_{-r}^{r} 2\pi r\,dx$$

球表面积 $= 4\pi r^2$

$$2\pi\sqrt{r^2-x^2}\sqrt{1+f'(x)}\,dx$$

6.5

Work and Fluid Forces

DEFINITION The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\, dx. \qquad (2)$$

Hooke's Law for Springs: $F = kx$

EXAMPLE 2 Find the work required to compress a spring from its natural length of 30 cm to a length of 20 cm if the force constant is $k = 240$ N/m.

$$\int_0^{0.1} 240x\,dx = 120x^2\Big]_0^{0.1} = 1.2\text{J}$$

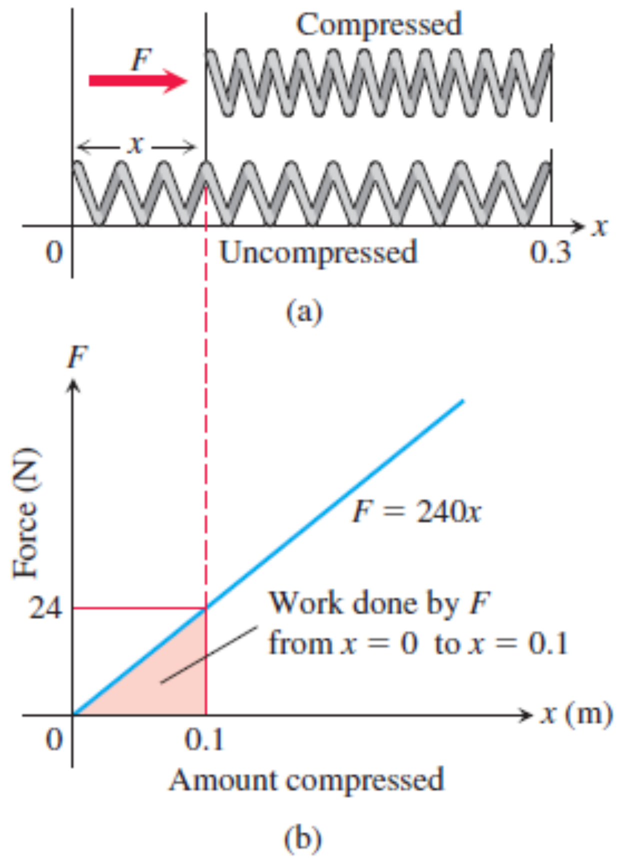


FIGURE 6.36 The force F needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

EXAMPLE 3 A spring has a natural length of 1 m. A force of 24 N holds the spring stretched to a total length of 1.8 m.

(a) Find the force constant k. $\frac{24}{0.8} = 30$

伸长2m $\int_0^2 kx\,dx$

(b) How much work will it take to stretch the spring 2 m beyond its natural length?

(c) How far will a 45-N force stretch the spring?

$\frac{45}{30} = 1.5\text{m}$

$L_{总} = 2.5\text{m}$

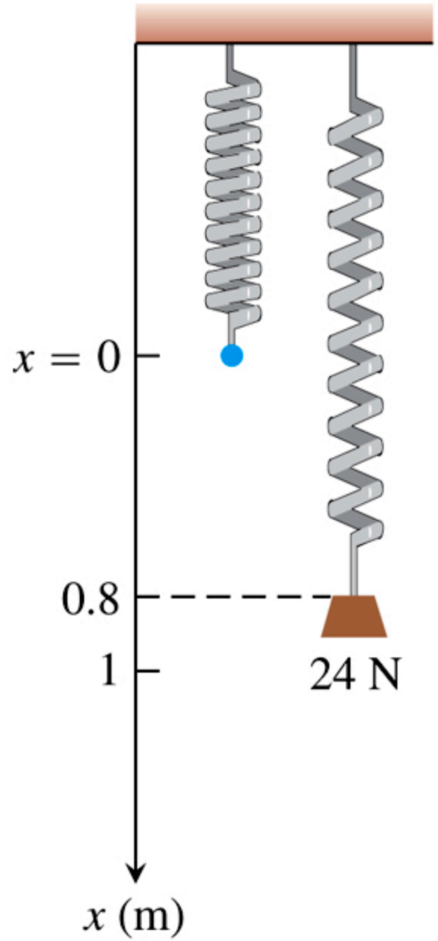
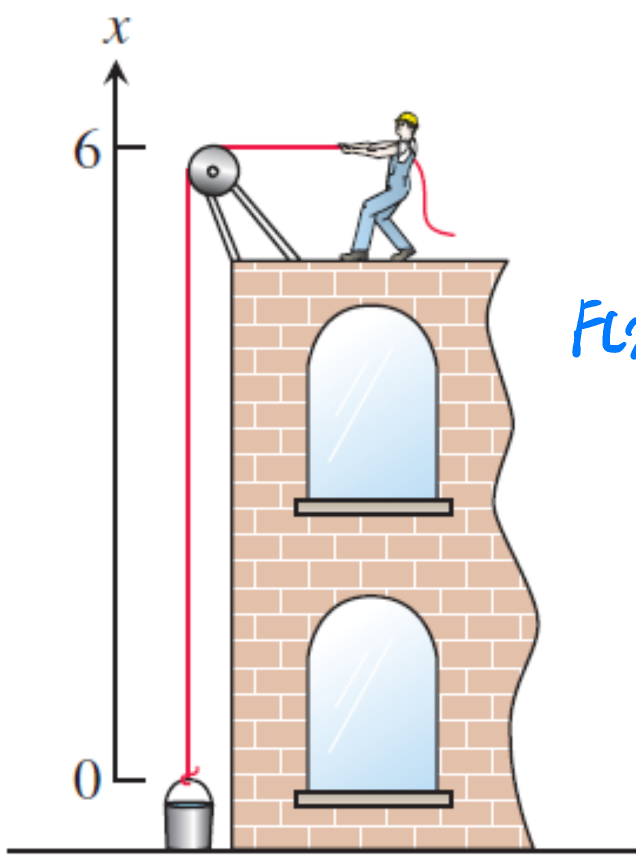


FIGURE 6.37 A 24-N weight stretches this spring 0.8 m beyond its unstressed length (Example 3).

EXAMPLE 4 A 2-kg bucket is lifted from the ground into the air by pulling in 6 m of rope at a constant speed (Figure 6.38). The rope weighs 0.1 kg/m. How much work was spent lifting the bucket and rope?



$F(x) = 2 + 0.1(6-x)g$

$\int_0^6 F(x)\,dx\,g$

改: ρ 变化

$\int_0^x \rho(t)\,dt = \int_0^x t\,dt = \frac{1}{2}x^2$

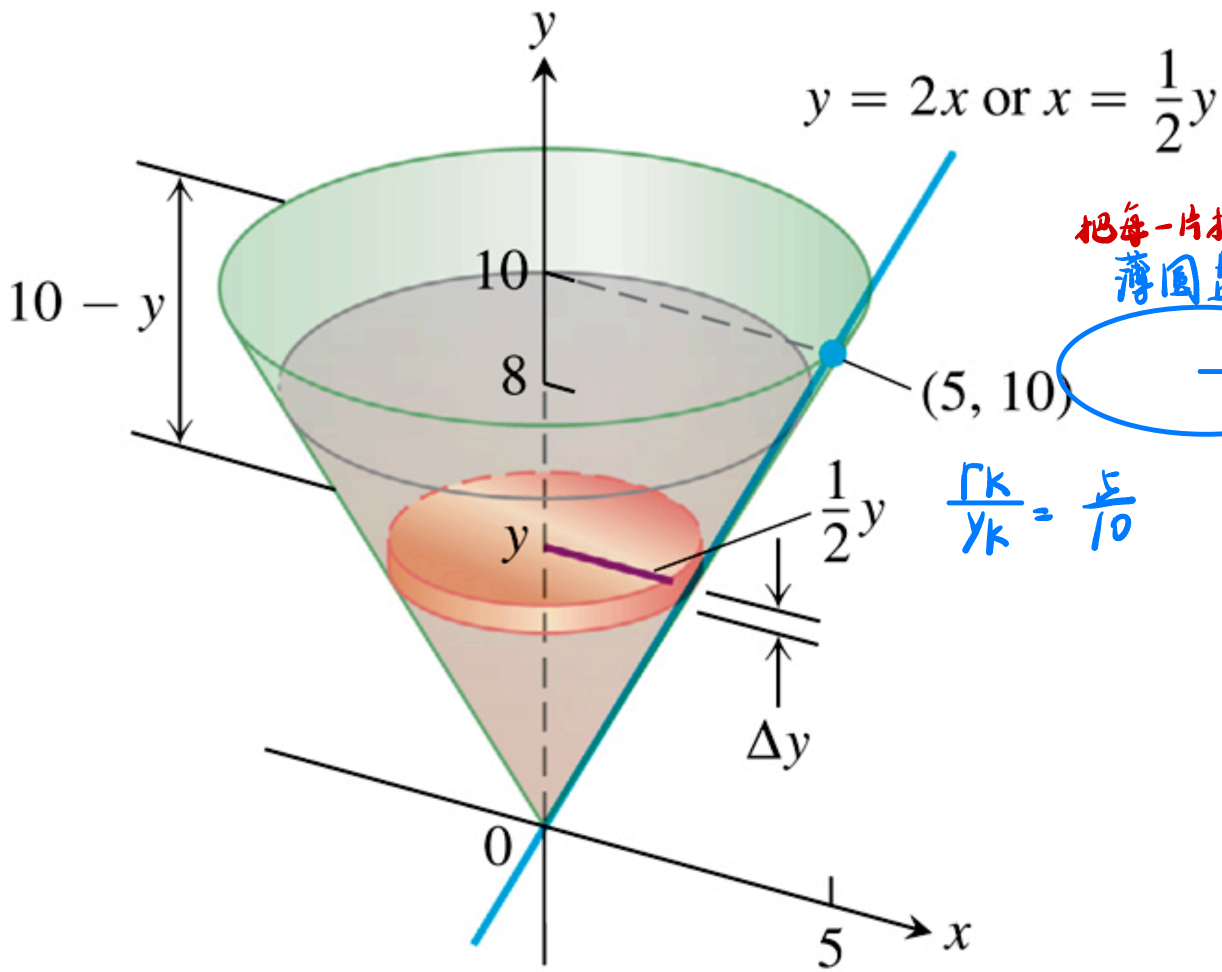
$F(x) = 2 + \frac{1}{2}(6-x)^2$

第: 分成小段求W

足够细 ρ 不变

FIGURE 6.38 Lifting the bucket in Example 4.

EXAMPLE 5 The conical tank in Figure 6.39 is filled to within 2 m of the top with olive oil weighing 0.9 g/cm^3 or 8820 N/m^3. How much work does it take to pump the oil to the rim of the tank?



把每一片抽上来

薄圆盘 $\rho g \pi r_k^2 \Delta y_k$

r_k

Δy_k

$\frac{r_k}{y_k} = \frac{5}{10}$

$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{4} \rho g \pi y_k^2 \Delta y_k$

$= \frac{1}{4}\int_0^8 \rho g \pi y^2 \, dy$

FIGURE 6.39 The olive oil and tank in Example 5.

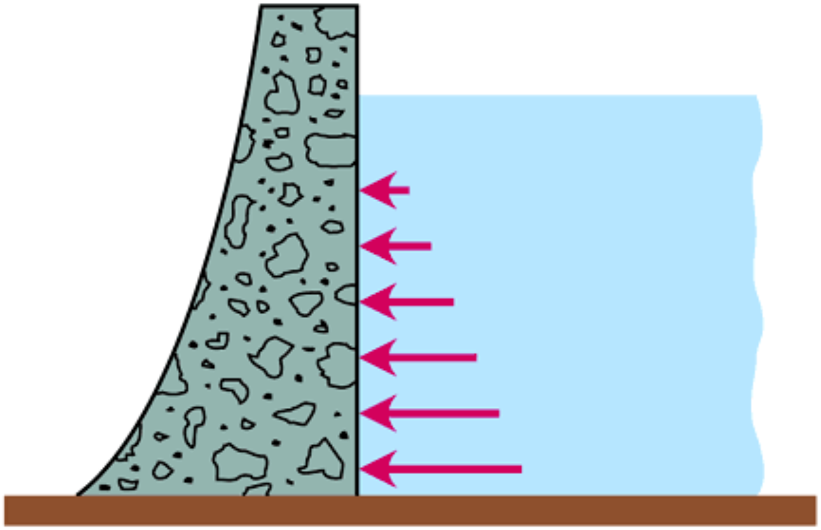


FIGURE 6.40 To withstand the increasing pressure, dams are built thicker as they go down.

Weight-density

A fluid's weight-density w is its weight per unit volume. Typical values (N/m^3) are listed below.

Gasoline	6600
Mercury	133,000
Milk	10,100
Molasses	15,700
Olive oil	8820
Seawater	10,050
Freshwater	9800

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h:

$$p = wh. \tag{4}$$

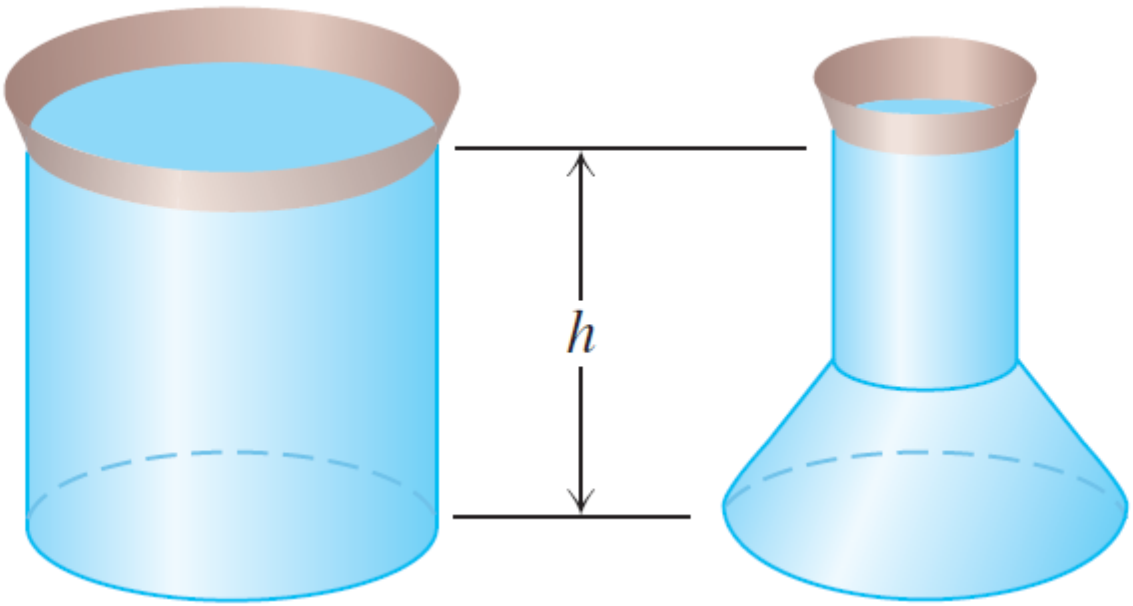


FIGURE 6.41 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

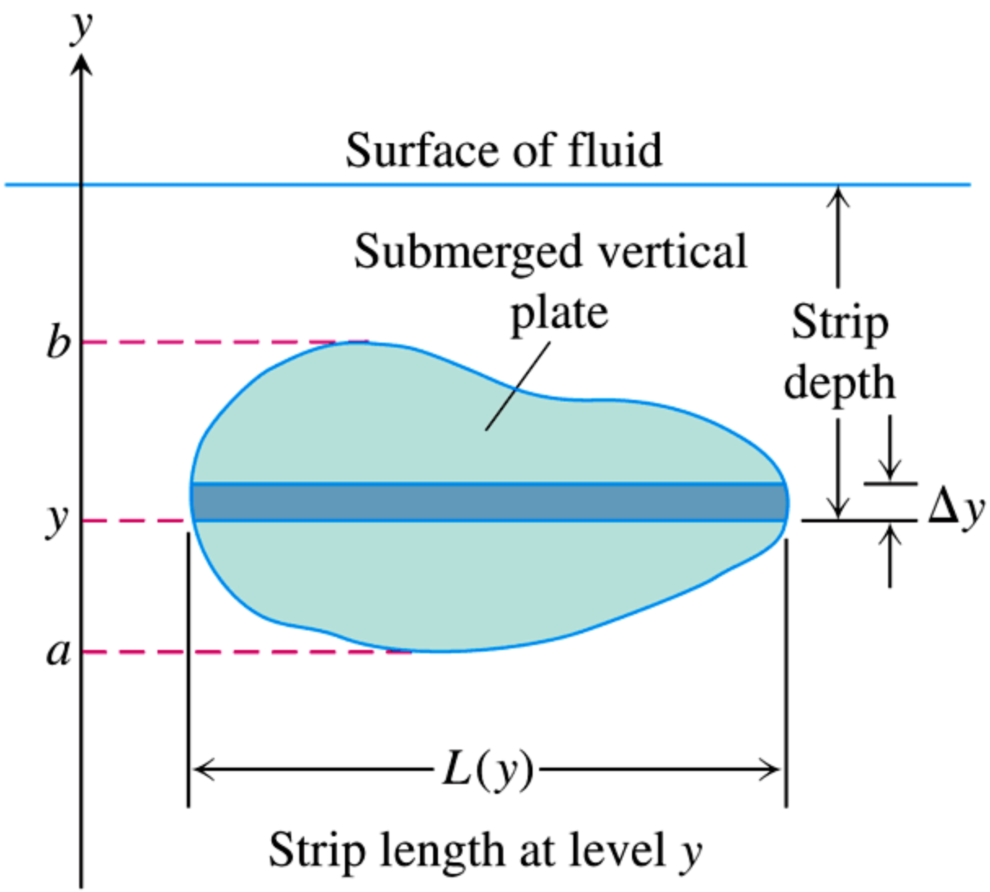


FIGURE 6.42 The force exerted by a fluid against one side of a thin, flat horizontal strip is about $\Delta F =$ pressure $\times$ area $= w \times$ (strip depth) $\times L(y)\,\Delta y$.

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \qquad (5)$$

$$F \approx \sum_{k=1}^{n} (w \cdot (\text{strip depth})_k \cdot L(y_k))\,\Delta y_k. \qquad (6)$$

The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\, dy. \qquad (7)$$

EXAMPLE 6 A flat isosceles right-triangular plate with base 2 m and height 1 m is submerged vertically, base up, 0.6 m below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

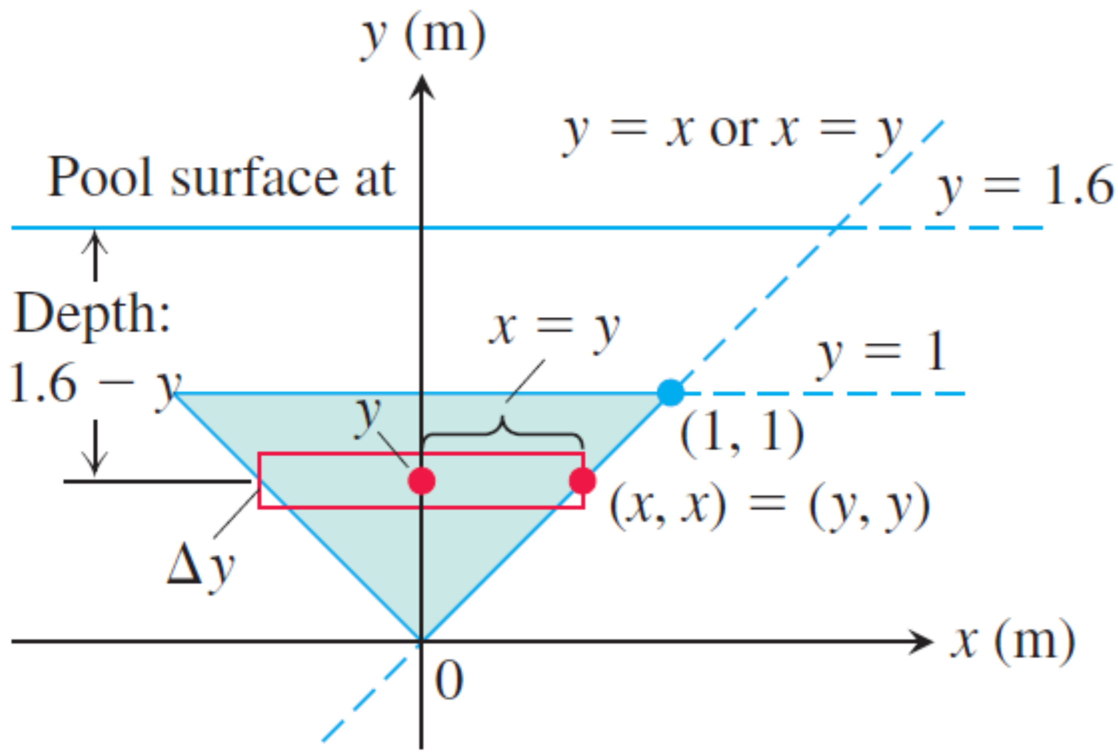


FIGURE 6.43 To find the force on one side of the submerged plate in Example 6, we can use a coordinate system like the one here.

6.6

Moments and Centers of Mass

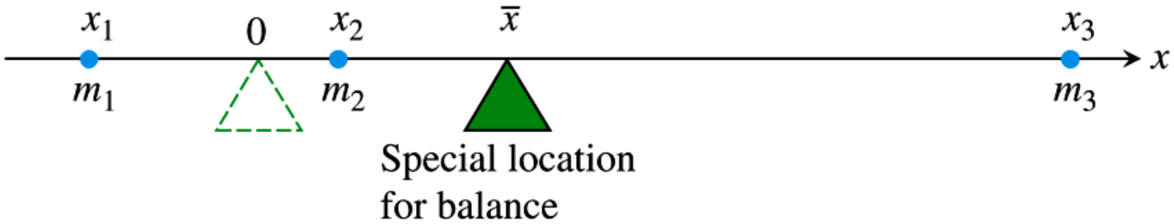
Masses Along a Line

We develop our mathematical model in stages. The first stage is to imagine masses m_1, m_2, and m_3 on a rigid x-axis supported by a fulcrum at the origin.

The resulting system might balance, or it might not, depending on how large the masses are and how they are arranged along the x-axis.

FIGURE 6.44 A wrench gliding on ice turning about its center of mass as the center glides in a vertical line. (*Source: PSSC Physics*, 2nd ed., Reprinted by permission of Education Development Center, Inc.)

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is

$$\text{Torque of } m_k \text{ about } \bar{x} = \begin{pmatrix}\text{signed distance}\\ \text{of } m_k \text{ from } \bar{x}\end{pmatrix}\begin{pmatrix}\text{downward}\\ \text{force}\end{pmatrix}$$

$$= (x_k - \bar{x})m_k g.$$

所有质点关于原点的力矩

The torque of each mass about the fulcrum in this special location is

$$\text{Torque of } m_k \text{ about } \bar{x} = \begin{pmatrix} \text{signed distance} \\ \text{of } m_k \text{ from } \bar{x} \end{pmatrix} \begin{pmatrix} \text{downward} \\ \text{force} \end{pmatrix}$$
$$= (x_k - \bar{x}) m_k g.$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\bar{x}$:

$$\sum (x_k - \bar{x}) m_k g = 0 \qquad \text{Sum of the torques equals zero.}$$
$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k}. \qquad \text{Solved for } \bar{x}$$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}. \tag{2}$$

The point $\bar{x}$ is called the system's **center of mass**.

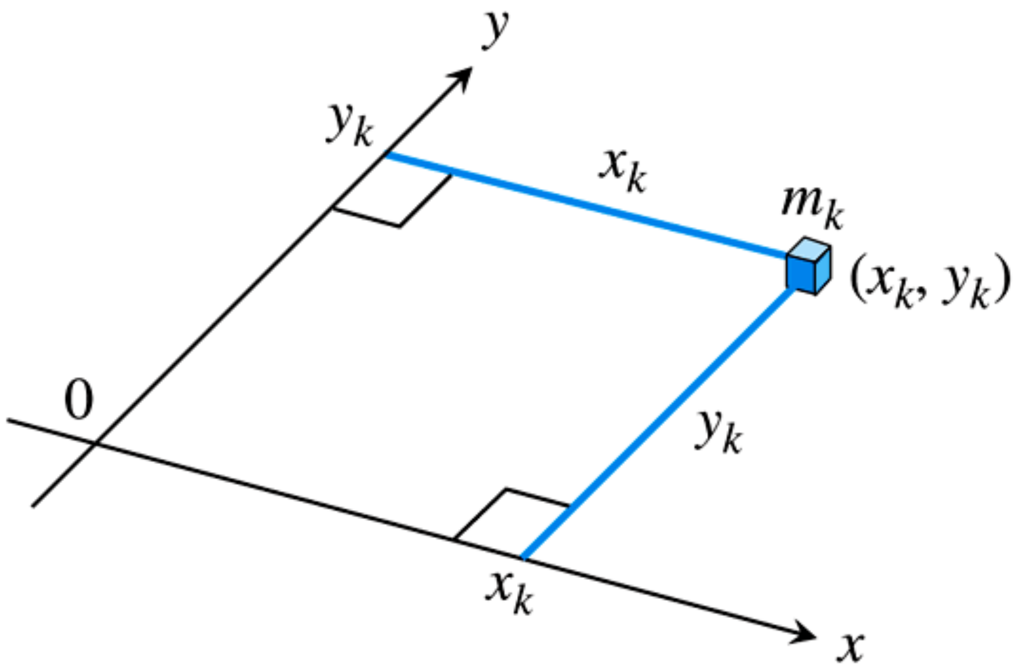


FIGURE 6.45 Each mass m_k has a moment about each axis.

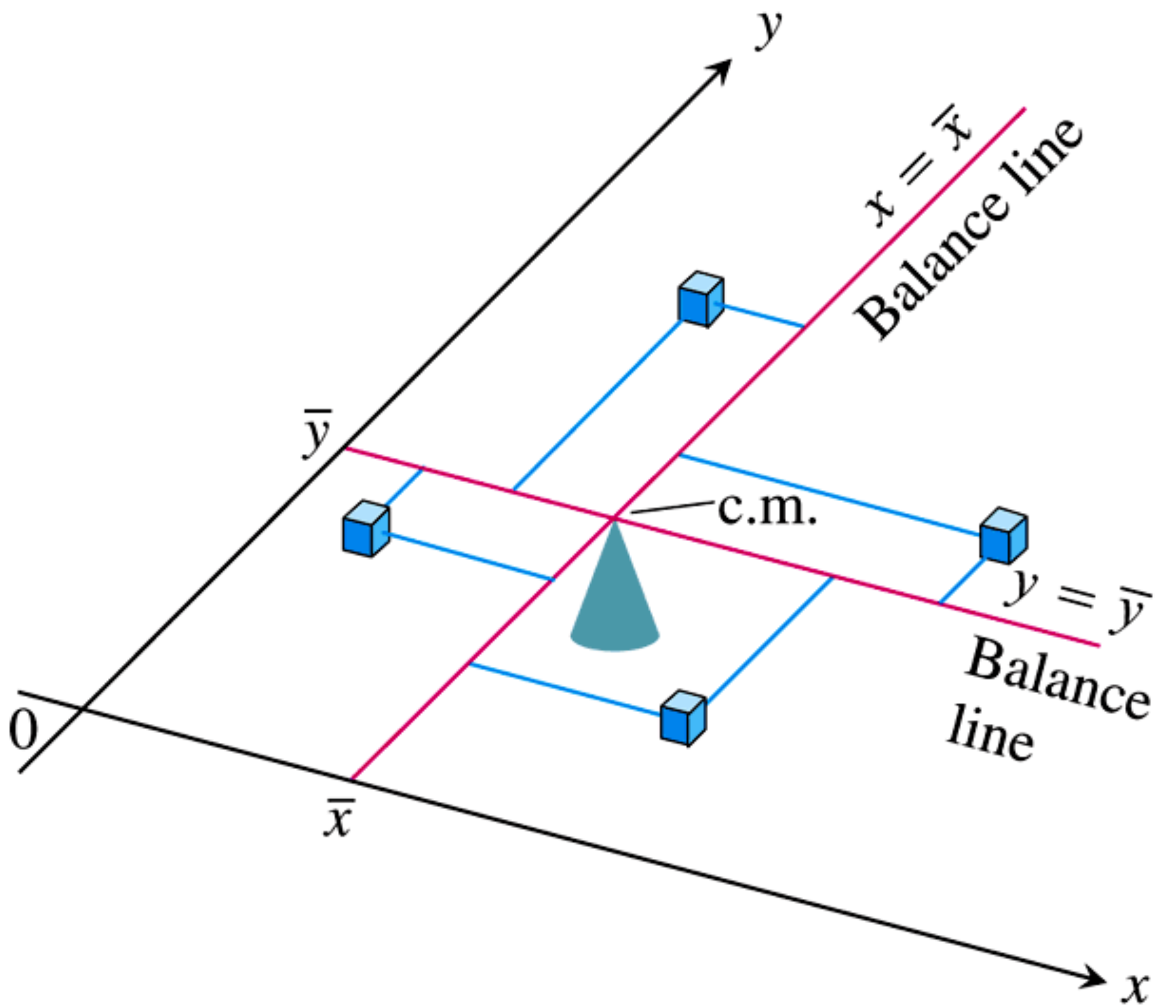


FIGURE 6.46 A two-dimensional array of masses balances on its center of mass.

Masses Distributed over a Plane Region

Suppose that we have a finite collection of masses located in the plane, with mass m_k at the point (x_k, y_k) (see Figure 6.45). The mass of the system is

$$\text{System mass:} \qquad M = \sum m_k .$$

Each mass m_k has a moment about each axis. Its moment about the x-axis is $m_k y_k$, and its moment about the y-axis is $m_k x_k$. The moments of the entire system about the two axes are

$$\text{Moment about } x\text{-axis:} \qquad M_x = \sum m_k y_k ,$$

$$\text{Moment about } y\text{-axis:} \qquad M_y = \sum m_k x_k .$$

The x-coordinate of the system's center of mass is defined to be

对y轴力矩 → 算对y轴距离

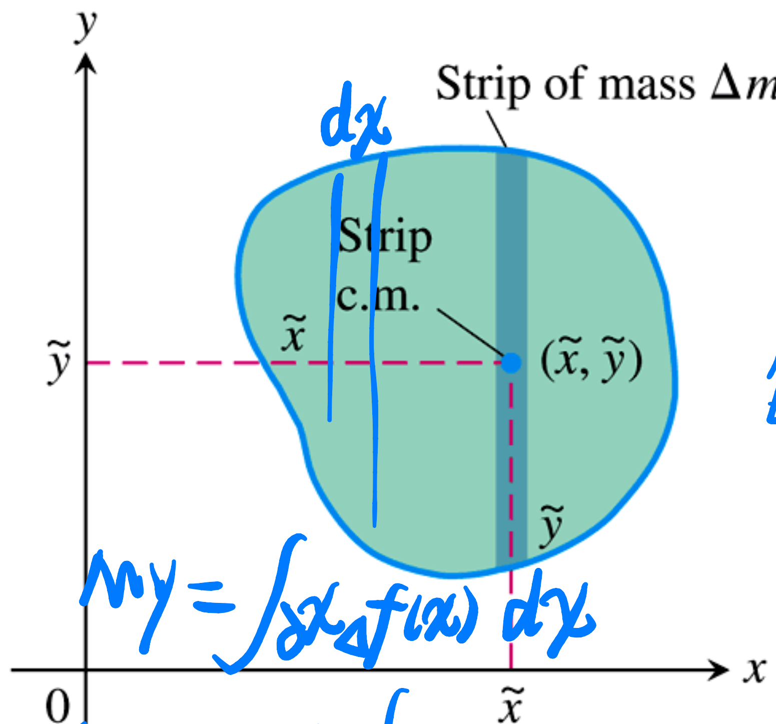
$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}. \tag{3}$$

With this choice of $\bar{x}$, as in the one-dimensional case, the system balances about the line $x = \bar{x}$ (Figure 6.46).

The y-coordinate of the system's center of mass is defined to be

$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}. \tag{4}$$

With this choice of $\bar{y}$, the system balances about the line $y = \bar{y}$ as well. The torques exerted by the masses about the line $y = \bar{y}$ cancel out. Thus, as far as balance is concerned, the system behaves as if all its mass were at the single point $(\bar{x}, \bar{y})$. We call this point the system's **center of mass**.



每个细条对y轴力矩

$M_y = \int \delta \tilde{x} f(x)\, dx$

$M_x = \int \delta (f(x) - g(x))\, dx \cdot \frac{f(x)+g(x)}{2}$

FIGURE 6.47 A plate cut into thin strips parallel to the y-axis. The moment exerted by a typical strip about each axis is the moment its mass Δm would exert if concentrated at the strip's center of mass $(\tilde{x}, \tilde{y})$.

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x}\,\Delta m}{\sum \Delta m} \qquad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y}\,\Delta m}{\sum \Delta m}$$

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm}$$

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the *xy*-Plane

$$\text{Moment about the } x\text{-axis:} \quad M_x = \int \tilde{y}\, dm$$

$$\text{Moment about the } y\text{-axis:} \quad M_y = \int \tilde{x}\, dm$$

$$\text{Mass:} \quad M = \int dm \tag{5}$$

$$\text{Center of mass:} \quad \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

$x_k \to x$
$\Delta x_k \to dx$

EXAMPLE 1 The triangular plate shown in Figure 6.48 has a constant density of $\delta = 3\ \mathrm{g/cm^2}$. Find

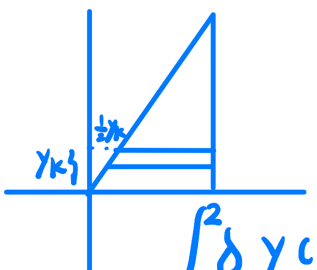
$$\lim_{n\to\infty} \sum_{n=1}^{k} x_k \, \delta f(x_k) \Delta x_k$$

(a) the plate's moment M_y about the y-axis. $= \delta \int_0^1 x f(x)\,dx = \frac{2}{3}\delta$

(b) the plate's mass M. $\int_0^1 \delta f(x)\,dx$

(c) the x-coordinate of the plate's center of mass (c.m.).

$$\bar{x} = \frac{M_y}{M} = \frac{2}{3}$$



$$\int_0^2 \delta\, y (1-\tfrac{1}{2}y)\,dy$$

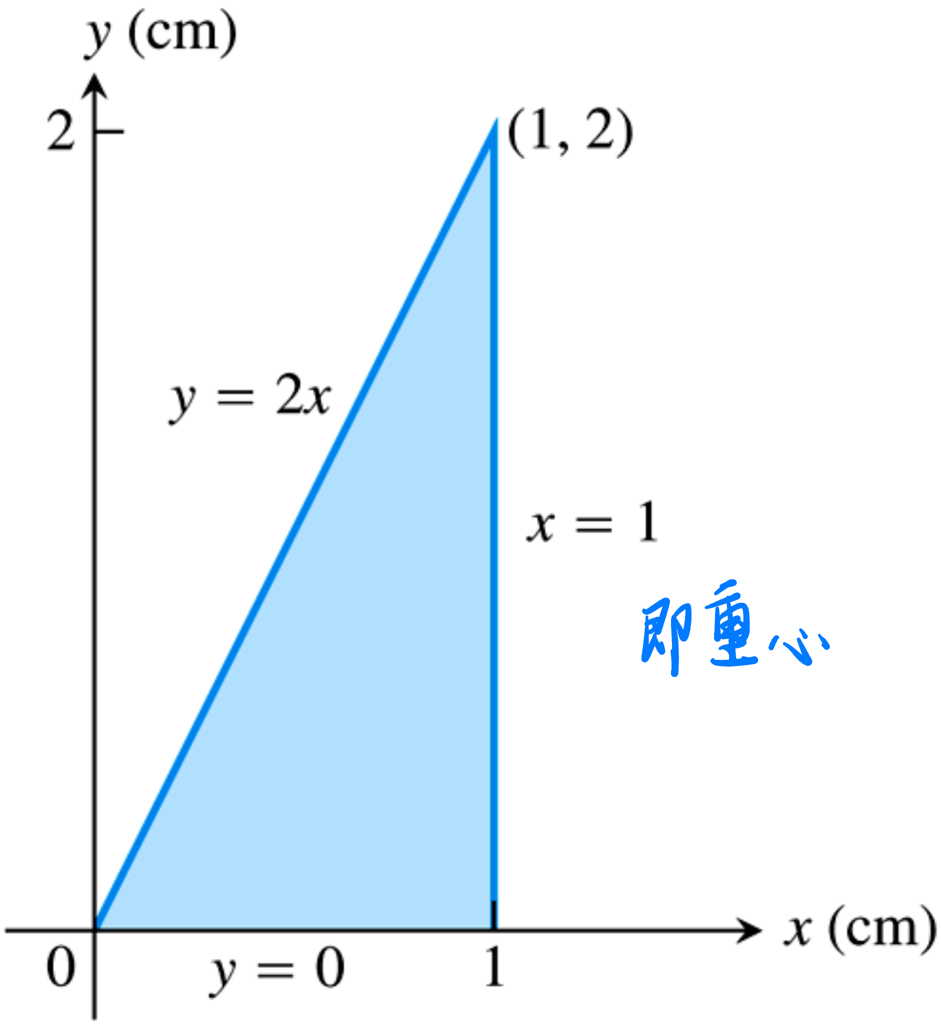
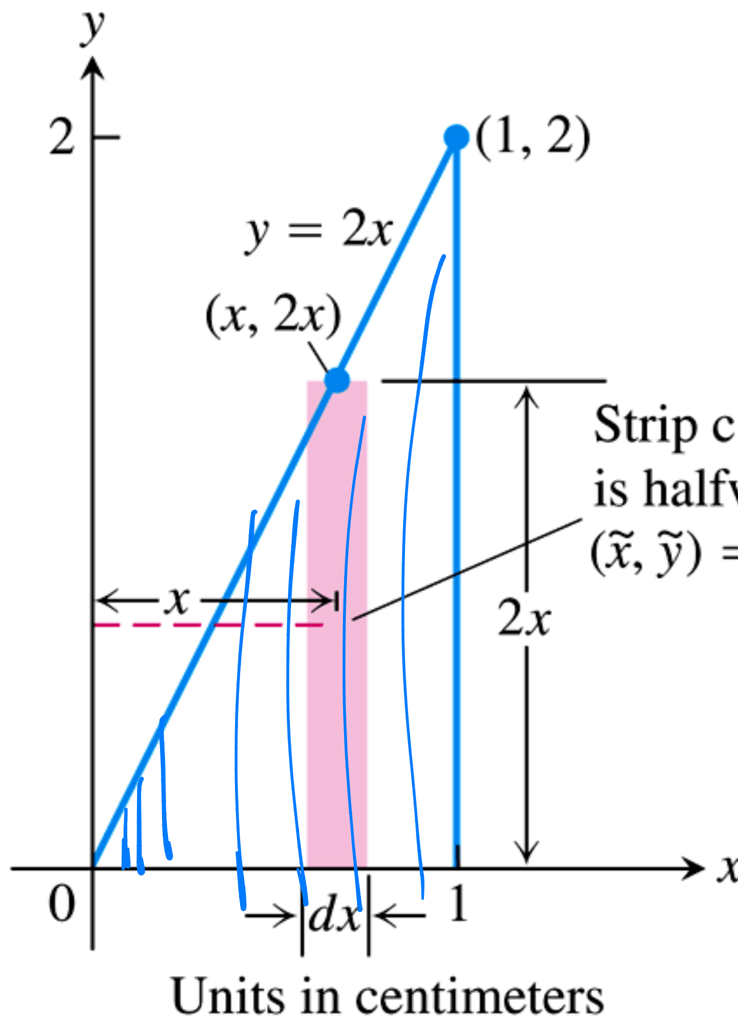


FIGURE 6.48 The plate in Example 1.



另：

等效距离到中心

Δm

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2}f(x_k)\,\delta f(x_k)\Delta x_k$$

若密度与x有关

FIGURE 6.49 Modeling the plate in Example 1 with vertical strips.

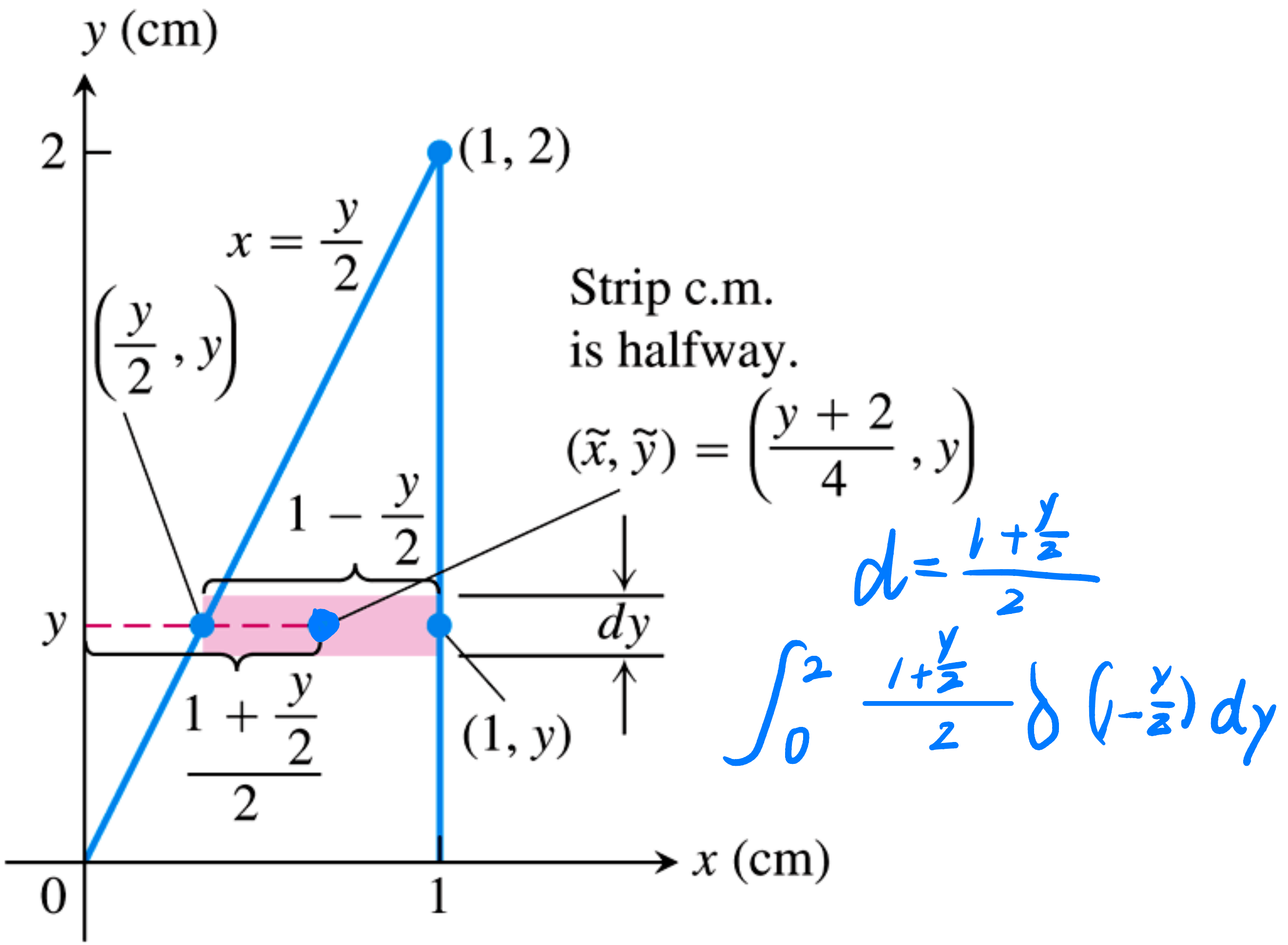
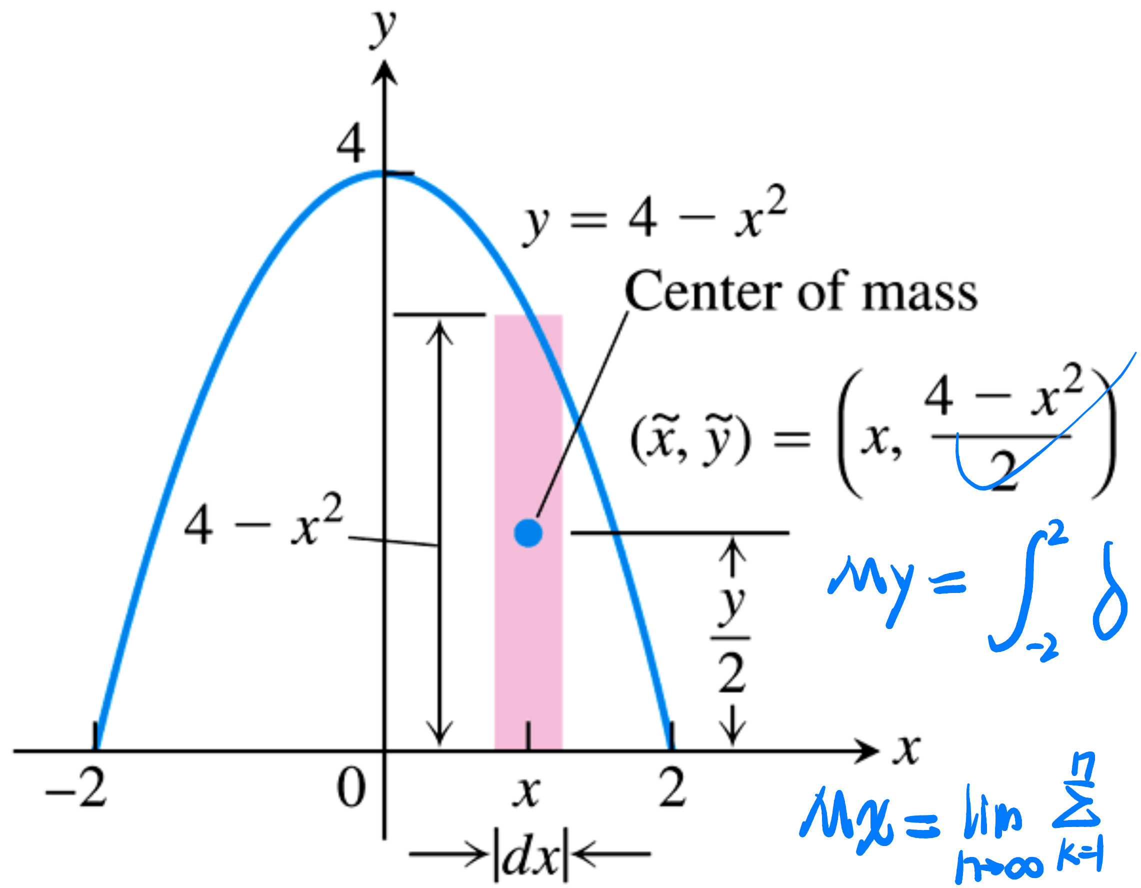


FIGURE 6.50 Modeling the plate in Example 1 with horizontal strips.

EXAMPLE 2 Find the center of mass of a thin plate covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x-axis (Figure 6.51). Assume the density of the plate at the point (x, y) is $\delta = 2x^2$, which is twice the square of the distance from the point to the y-axis.

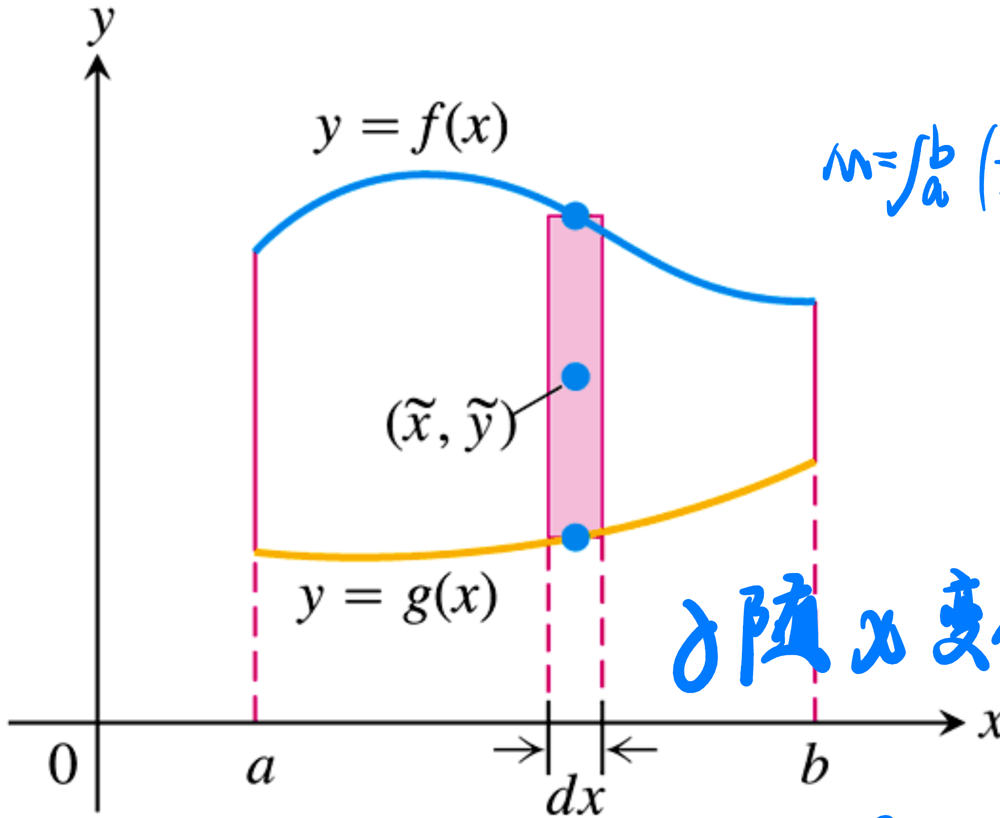


$$M_y = \int_{-2}^{2} \delta\, f(x)\, x\, dx$$

$$M_x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{4 - x_k^2}{2}\, \delta\, f(x_k)\, \Delta x_k$$

$$= \int_{-2}^{2} \frac{4 - x^2}{2}\, \delta\, f(x)\, dx$$

FIGURE 6.51 Modeling the plate in Example 2 with vertical strips.

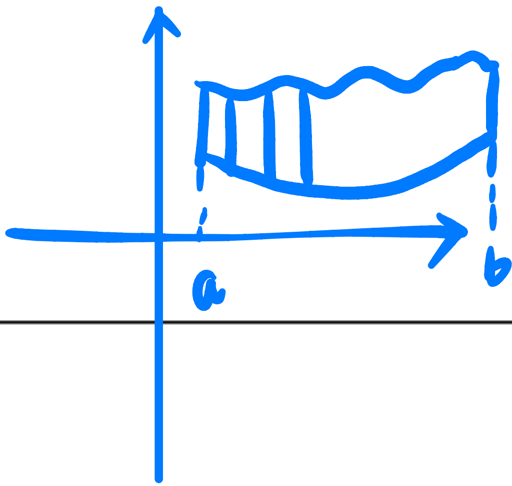


$m=\int_a^b (f(x)-g(x))\,dx$

y随x变化

对y轴力矩可视为在中间

FIGURE 6.52 Modeling the plate bounded by two curves with vertical strips. The strip c.m. is halfway, so $\tilde{y} = \frac{1}{2}[f(x) + g(x)]$.



δ变化

$$m = \int_a^b (f(x) - g(x))\,\delta(x)\,dx$$

$$M_x = \int_a^b (f(x) - g(x))\,\underbrace{\tfrac{1}{2}(f(x)+g(x))}_{h_{com}}\,\delta(x)\,dx$$

算 y_{com}

$$y = \frac{M_x}{m} \qquad M_y = \int_a^b \delta(x)\,x\,(f(x) - g(x))\,dx$$

这就是距离

算 x_{com}

$$\bar{x} = \frac{1}{M}\int_a^b \delta x\,[f(x) - g(x)]\,dx \tag{6}$$

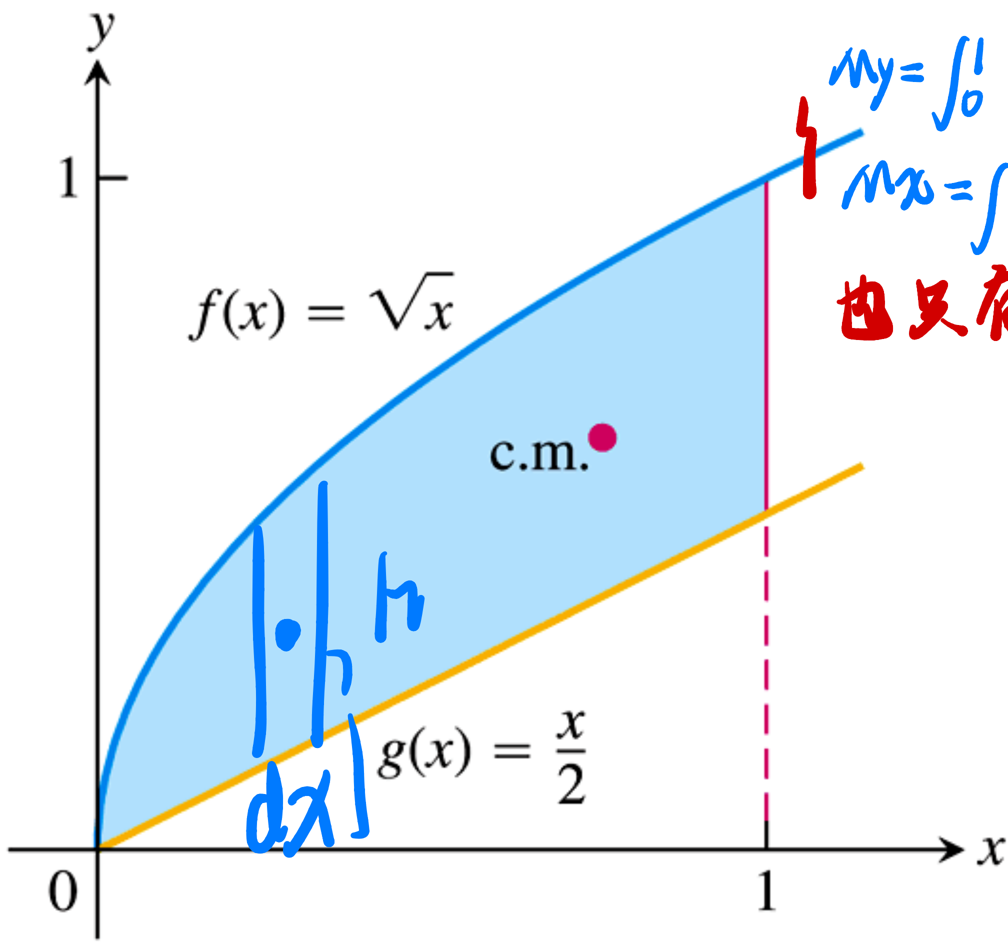
$$\bar{y} = \frac{1}{M}\int_a^b \frac{\delta}{2}\,[f^2(x) - g^2(x)]\,dx \tag{7}$$

因此：

算质心只要以 ρ变 决定积分方向

方向切成条

EXAMPLE 3 Find the center of mass for the thin plate bounded by the curves $g(x) = x/2$ and $f(x) = \sqrt{x}$, $0 \le x \le 1$ (Figure 6.53), using Equations (6) and (7) with the density function $\delta(x) = x^2$.



$M_y = \int_0^1 x\left(\sqrt{x} - \frac{x}{2}\right) x^2 \, dx$

$M_x = \int_0^1 \frac{1}{2}\left(\sqrt{x}^2 - \left(\frac{x}{2}\right)^2\right) x^2 \, dx$

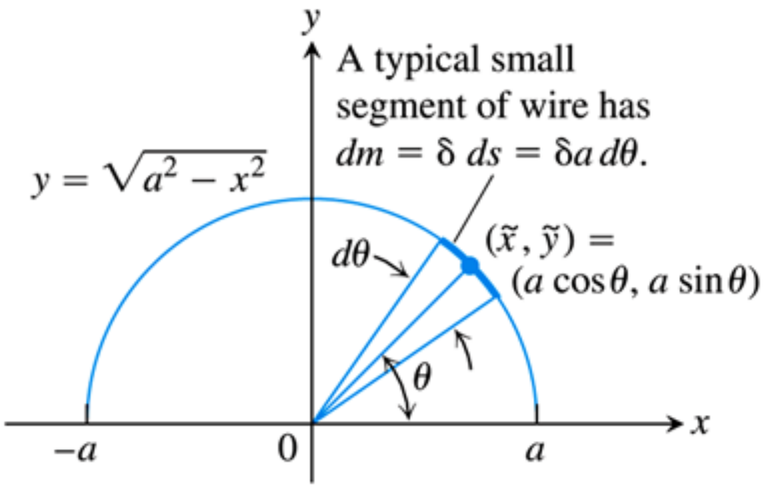
也只有力臂不同

FIGURE 6.53 The region in Example 3.

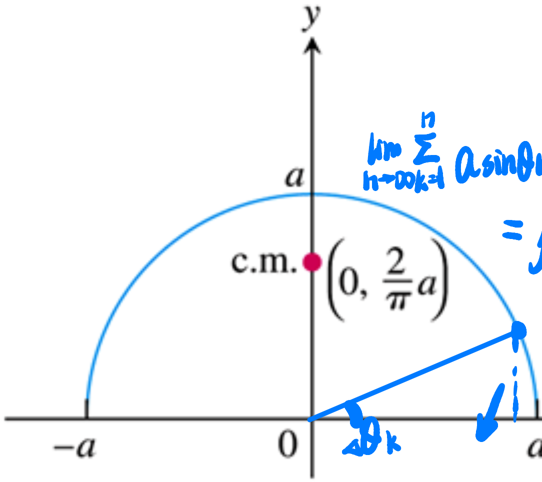
EXAMPLE 4 Find the center of mass (centroid) of a thin wire of constant density δ shaped like a semicircle of radius a.

质心

半圆



(a)



(b)

描述弧

$$\lim_{n\to\infty}\sum_{k=1}^{n} a\sin\theta_k \cdot a\,\Delta\theta_k = \int_0^{\pi} a^2 \sin\theta\, d\theta = 2a^2$$

$$M = \pi a$$

$$\bar{y} = \frac{M_x}{M} = \frac{2}{\pi}a$$

$$\int_0^{\pi} a\sin\theta\; a\,d\theta$$

FIGURE 6.54 The semicircular wire in Example 4. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

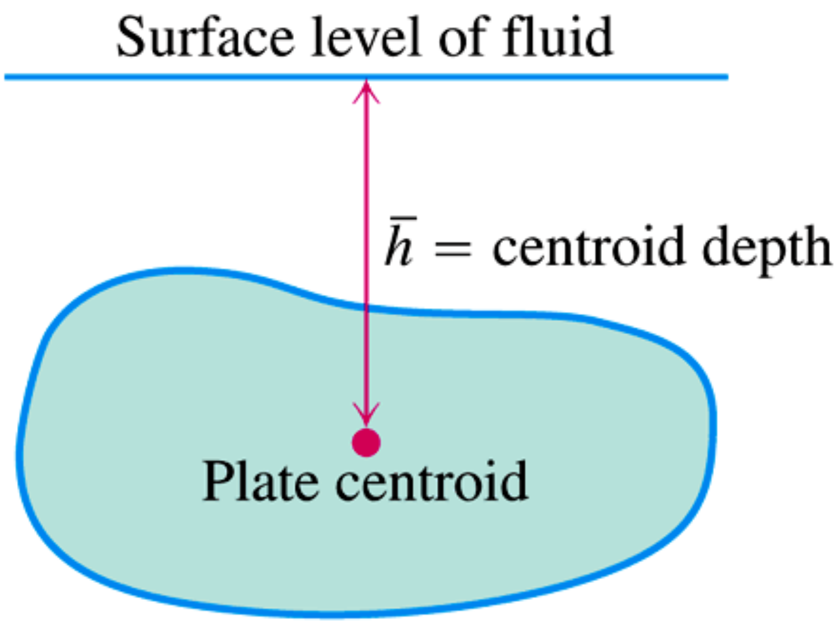
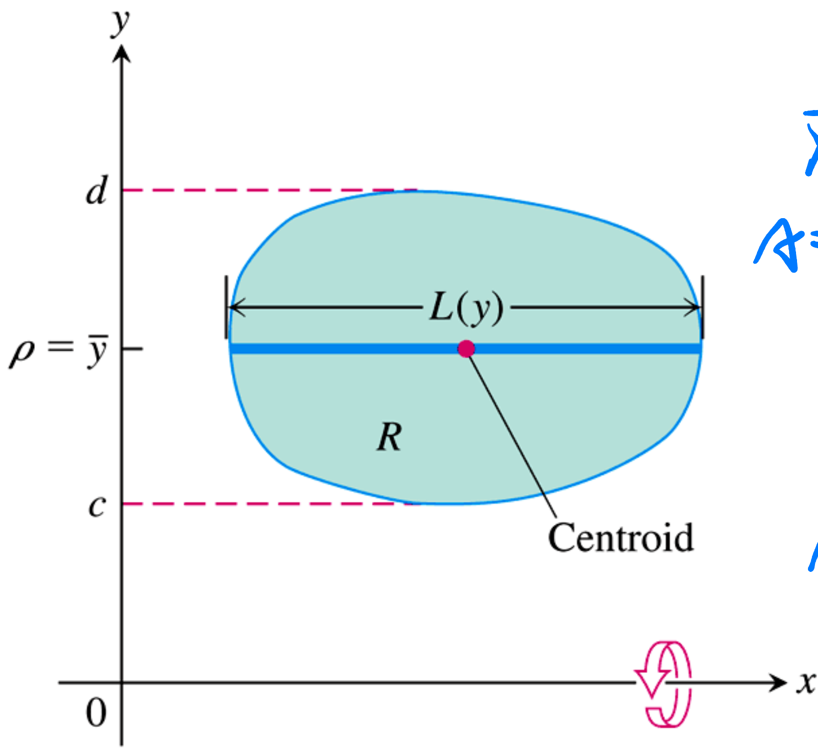


FIGURE 6.55 The force against one side of the plate is $w \cdot \bar{h} \cdot$ plate area.

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w, the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A. \tag{8}$$



$\bar{y} = \dfrac{M_x}{m}$

$A = \int_c^d L(y)\,dy \Rightarrow m = \int_c^d \delta L(y)\,dy$

以质心距离代每个点

$V = \underline{2\pi\rho} A$

$V = \int_c^d 2\pi L(y)\, y\, dy$ (旋转体体积)

$M_x = \int_c^d \delta L(y)\, y\, dy$

$2\pi\rho A$ 去 $A = V$

成立

真正只要求 V 一个积分

FIGURE 6.56 The region R is to be revolved (once) about the x-axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

THEOREM 1 Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

规则形状才有用

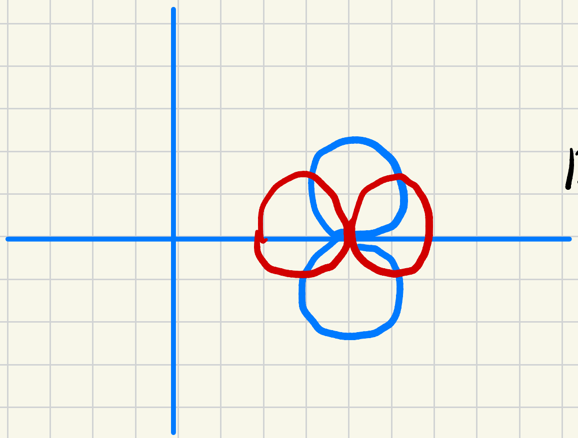
$$V = 2\pi \overset{2个\int}{\rho} \overset{1个\int}{A}. \tag{9}$$

知道质心位置才能用

实际上求V只要1个∫

$A = 2\pi\rho l$

$V = 2\pi\rho A$



闭定理

2组旋转 V不变

(ρA不变)

EXAMPLE 6 Find the volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center (Figure 6.57).

EXAMPLE 7 Locate the centroid of a semicircular region of radius a.

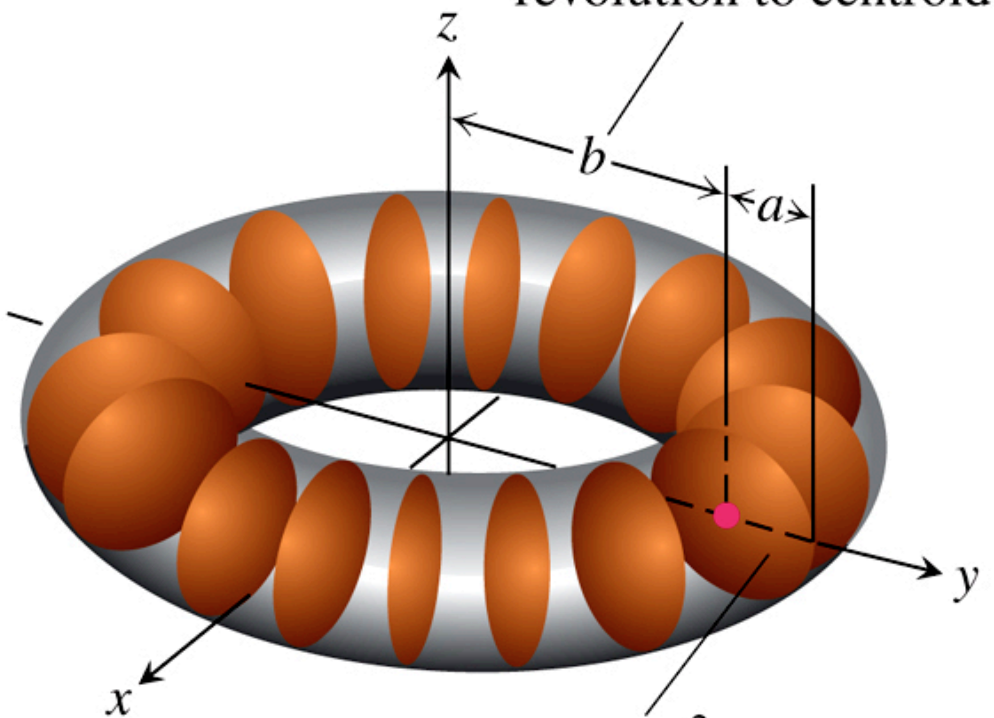


FIGURE 6.57 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 6).

转2π成一个完整体积

只可用半圆面积

$$\frac{4}{3}\pi a^3 = 2\pi y^* \frac{1}{2}\pi a^2$$

$$y^* = \frac{4}{3\pi} a$$

逆用定理

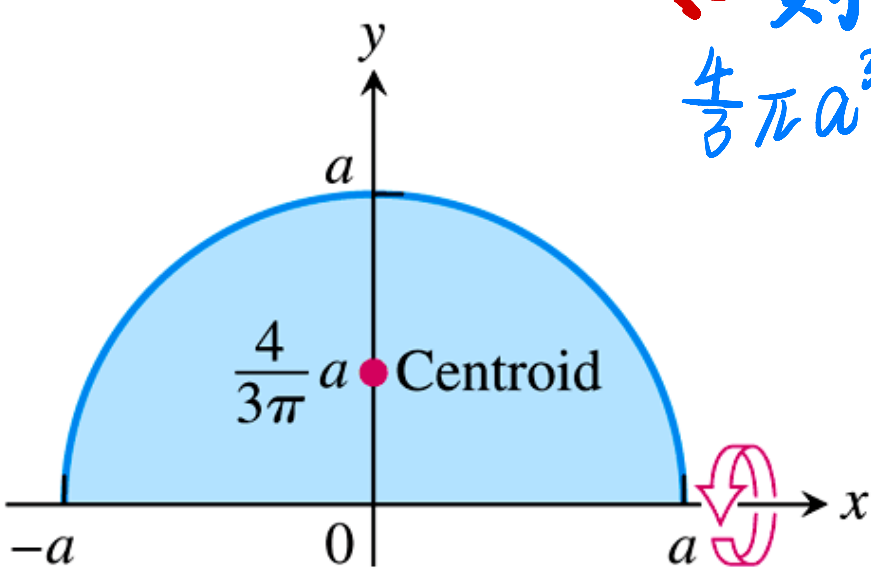


FIGURE 6.58 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 7).

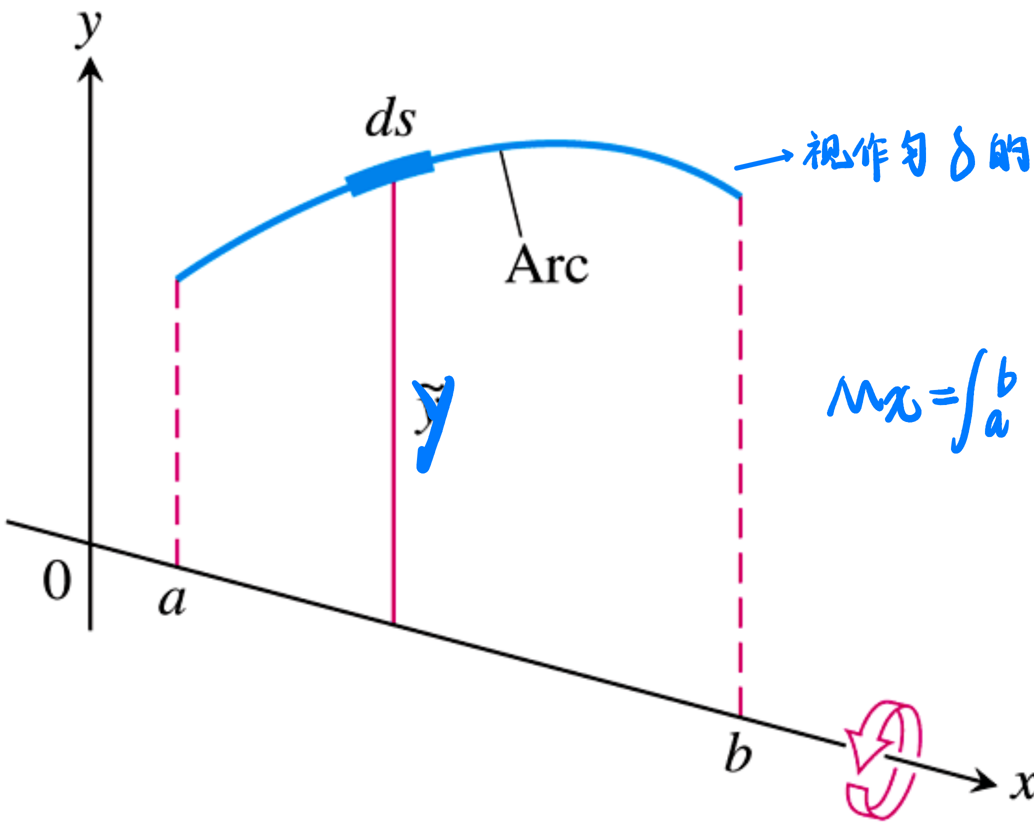
THEOREM 2 **Pappus's Theorem for Surface Areas**

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \tag{11}$$

$\downarrow$ 线质心

$\rho = \frac{M_x}{M}$



→ 视作匀 δ 的绳子

$$M_x = \int_a^b \delta f(x) \sqrt{1 + (f'(x))^2}\, dx$$

FIGURE 6.59 Figure for proving Pappus's Theorem for surface area. The arc length differential *ds* is given by Equation (6) in Section 6.3.

求上半圓周

$$\int f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = 2\pi\rho l$$

$$\overset{2}{\cancel{4}}\cancel{\pi} r^{\cancel{2}} = \cancel{2\pi}\rho \times \pi \cancel{r}$$

$$\frac{2r}{\pi} = \rho$$