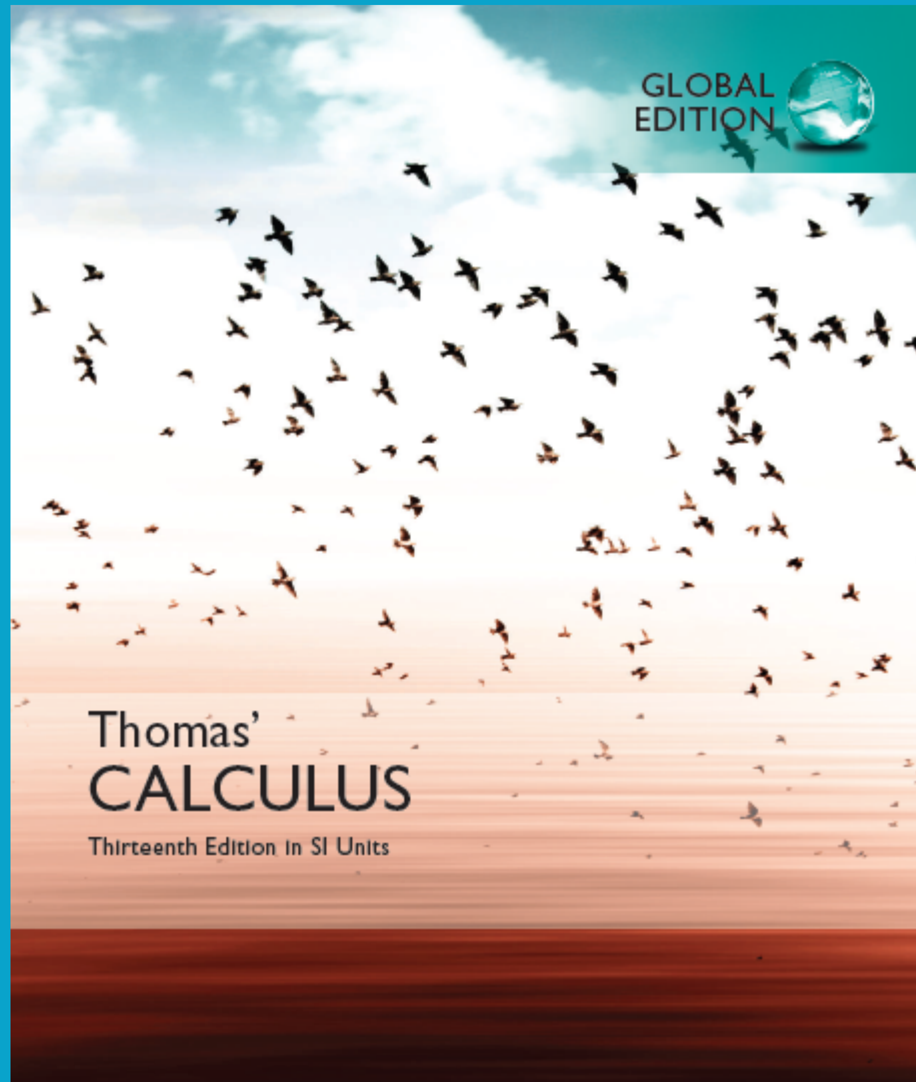

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

连续
导数
三种切线

Chapter 3

Differentiation

3.1

Tangents and the Derivative at a Point

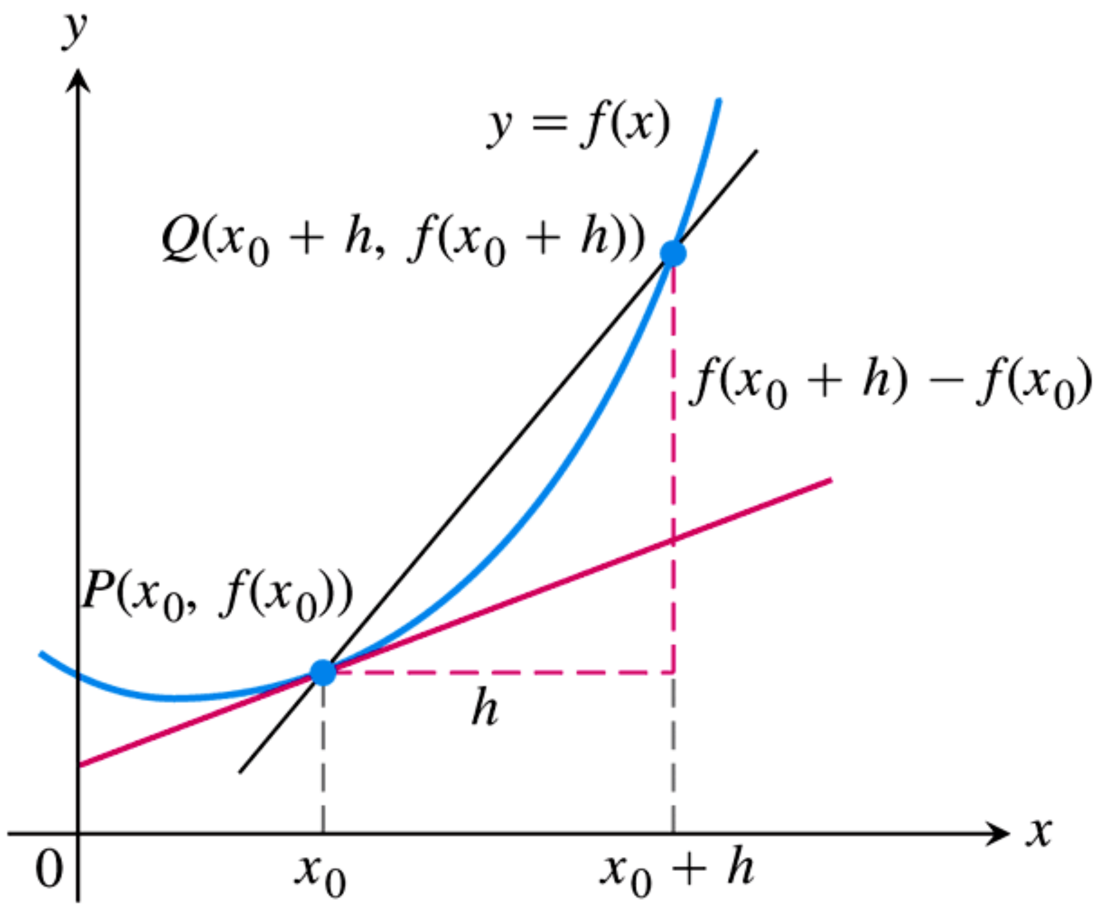


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h\to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h}$$
$$=\lim_{h\to 0}\frac{-1}{a(a+h)}=-\frac{1}{a^2}$$

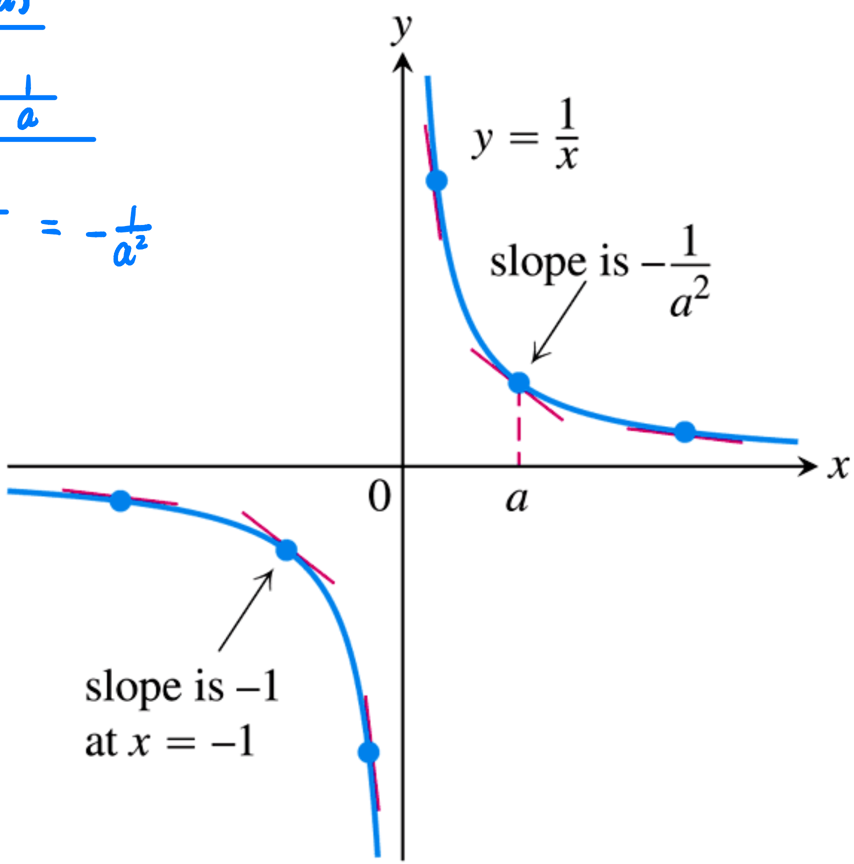


FIGURE 3.2 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

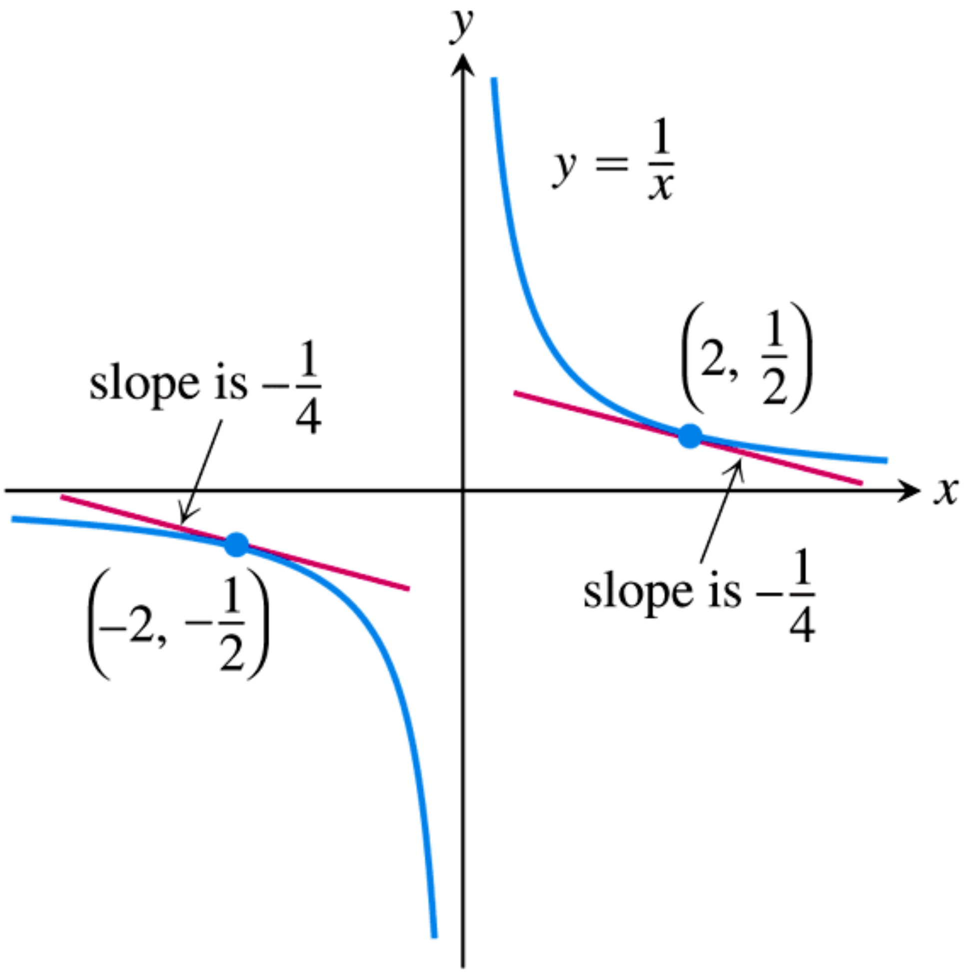


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

DEFINITION The **derivative of a function *f* at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

不确定连续性

* 分段函数相交处 ⇒ 只能用定义

不在定义域

不用求导法则

Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

回归定义

have a tangent at the origin? Give reasons for your answer.

$$2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\,(-1)\frac{1}{x^2}$$

$$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

$$-|h| \le h\sin\frac{1}{h} \le |h|$$

$$0$$

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

每个点有定义

但 0 处不连续 ⇐ 无极限

由此为基础构造

* $f(x) = \begin{cases} x^3\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ 导数连续

We say that a continuous curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if the limit of the difference quotient is ∞ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent at $x = 0$ (see accompanying figure):

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^{1/3}-0}{h}$$

$$= \lim_{h\to 0}\frac{1}{h^{2/3}} = \infty.$$

都要求两侧"极限"相等

垂直切线 — 两边 $\lim_{x\to c^+/c^-}$ 相等，可以为$\pm\infty$

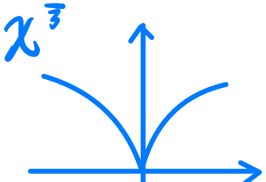
存在导数 — $\lim_{x\to c^+/c^-}$ 相等 但不可为$\pm\infty$

区分

切线斜率

同侧无穷

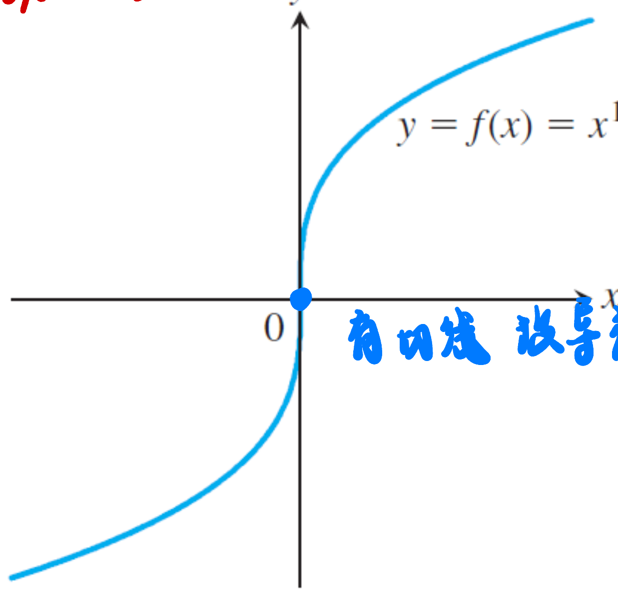
$y=x^{\frac{2}{3}}$



$\lim_{h\to 0}\frac{h^{\frac{2}{3}}-0}{h}$

$=\lim_{h\to 0}\frac{1}{h^{\frac{1}{3}}}$ ×

左右 $-\infty, +\infty$ 极限不存在

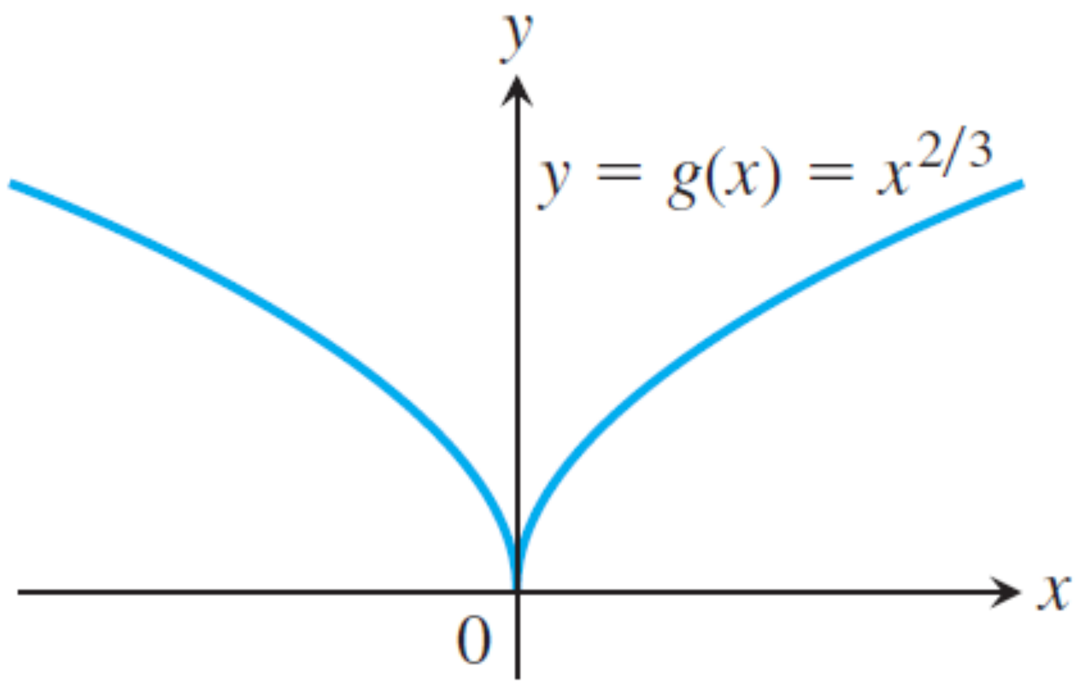


有切线 没导数

VERTICAL TANGENT AT ORIGIN

However, $y = x^{2/3}$ has *no* vertical tangent at $x = 0$ (see next figure):

$$\lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0} \frac{h^{2/3} - 0}{h}$$



NO VERTICAL TANGENT AT ORIGIN

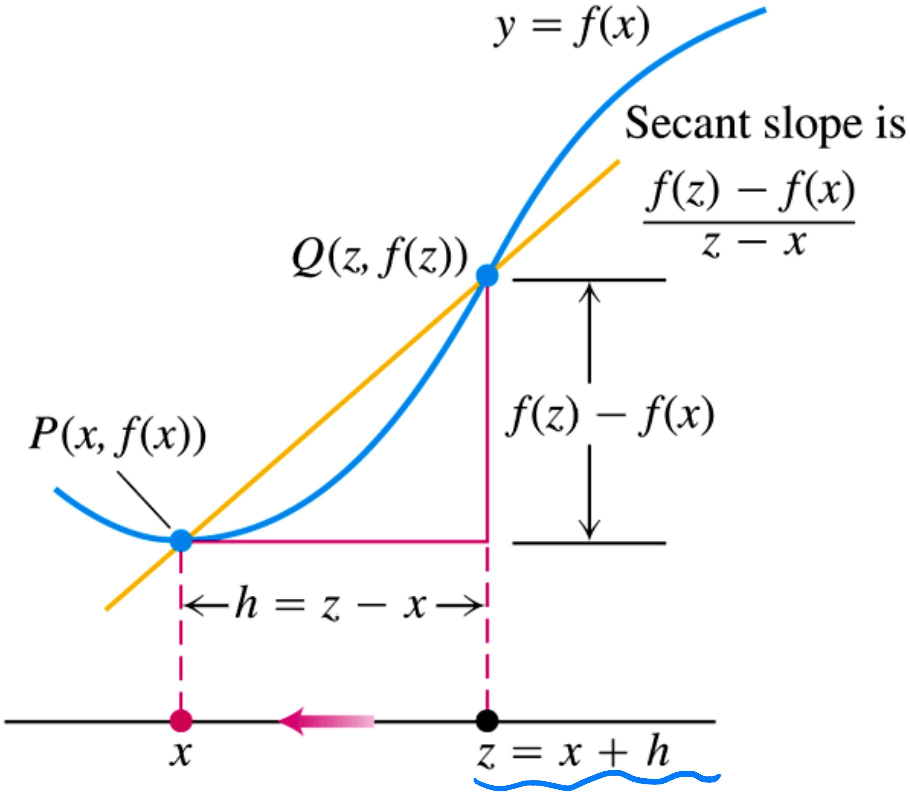
3.2

The Derivative as a Function

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.



Derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

FIGURE 3.4 Two forms for the difference quotient.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

EXAMPLE 2

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

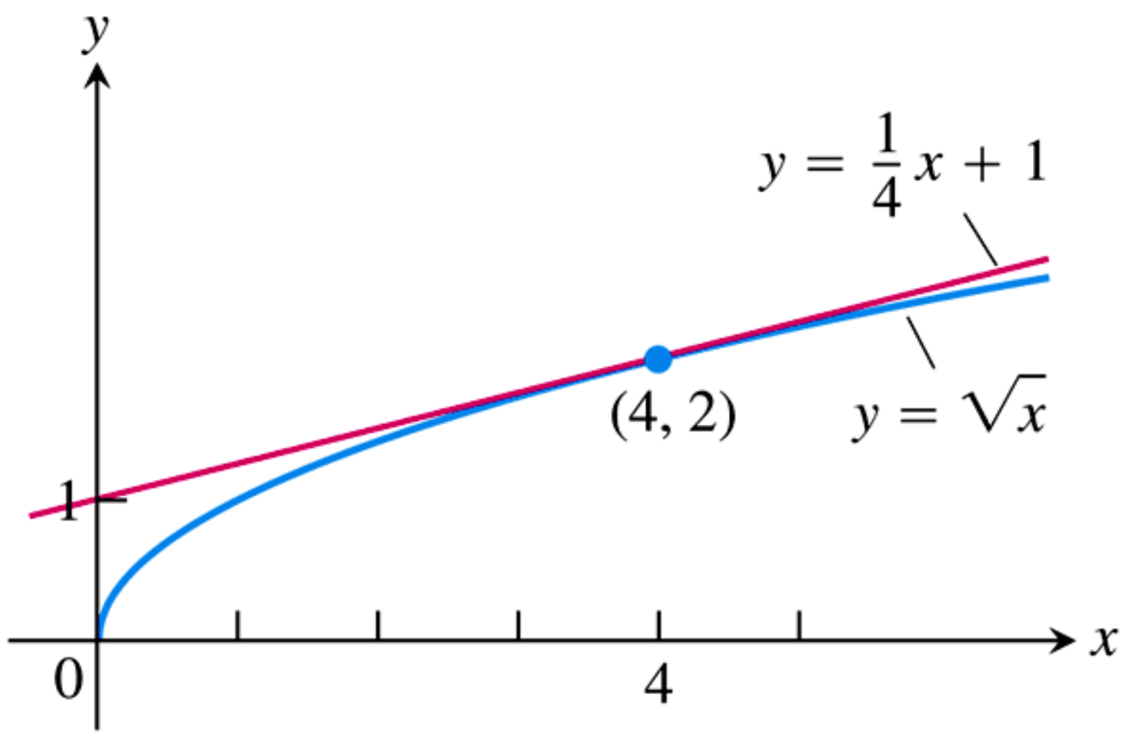


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

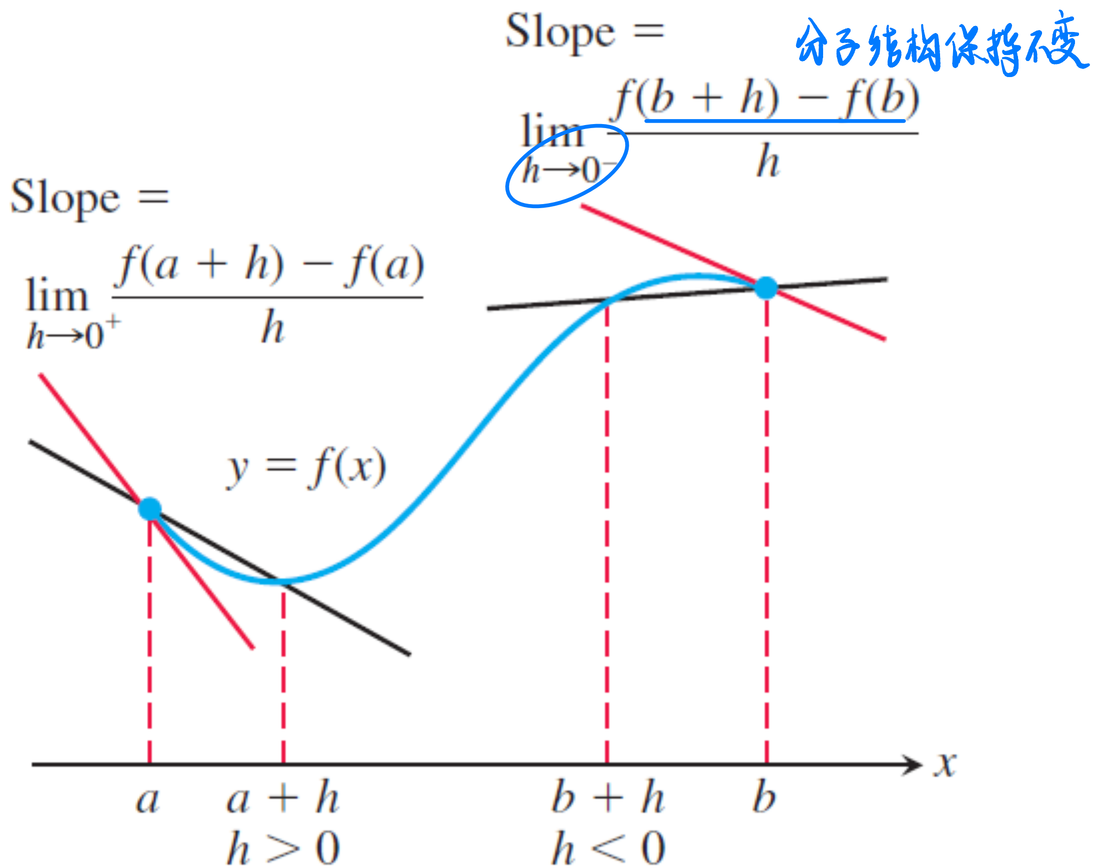


FIGURE 3.7 Derivatives at endpoints of a closed interval are one-sided limits.

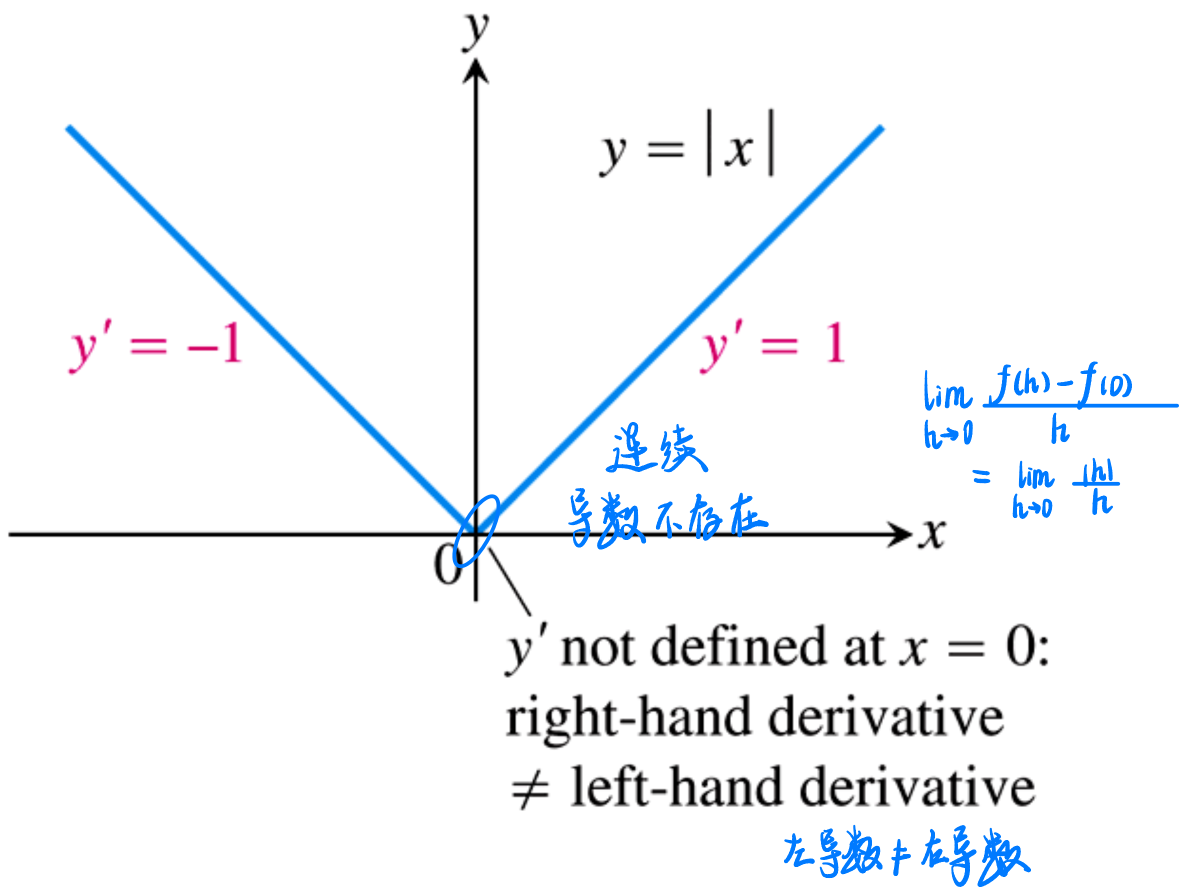
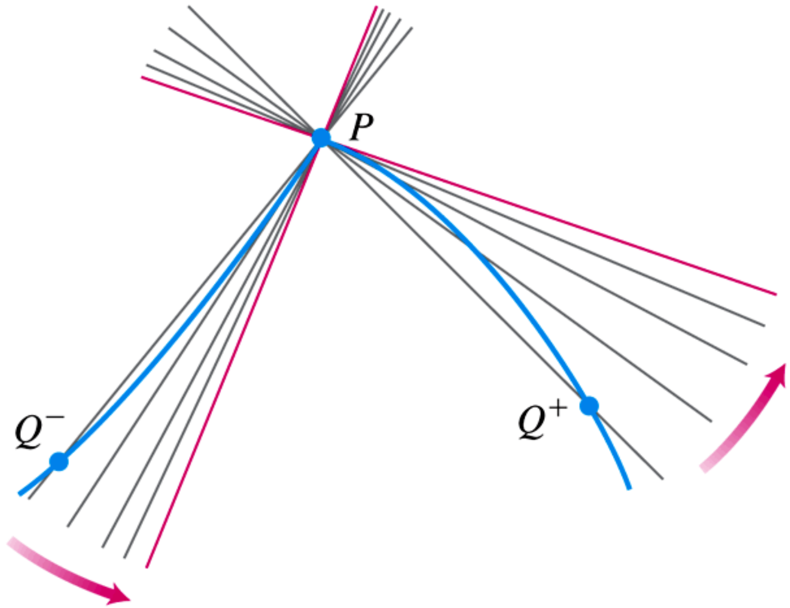
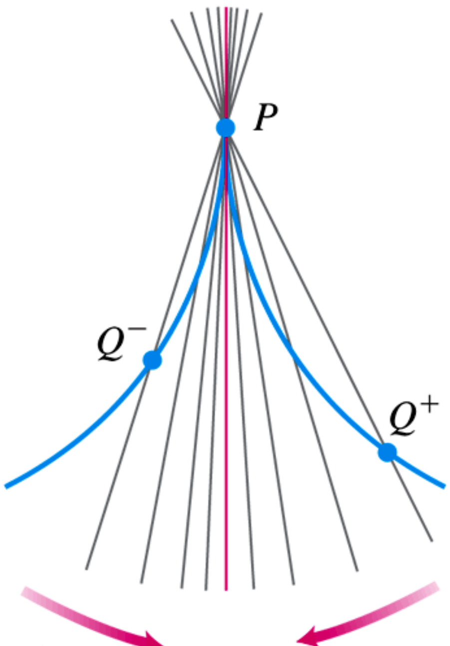


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a 拐角 "corner" (Example 4).

When Does a Function *Not* Have a Derivative at a Point?

A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a finite limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. Thus differentiability is a "smoothness" condition on the graph of f. A function can fail to have a derivative at a point for many reasons, including the existence of points where the graph has

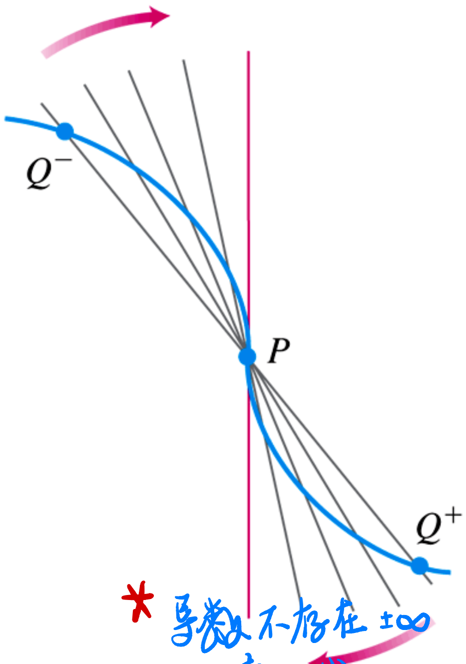




1. a *corner*, where the one-sided derivatives differ.

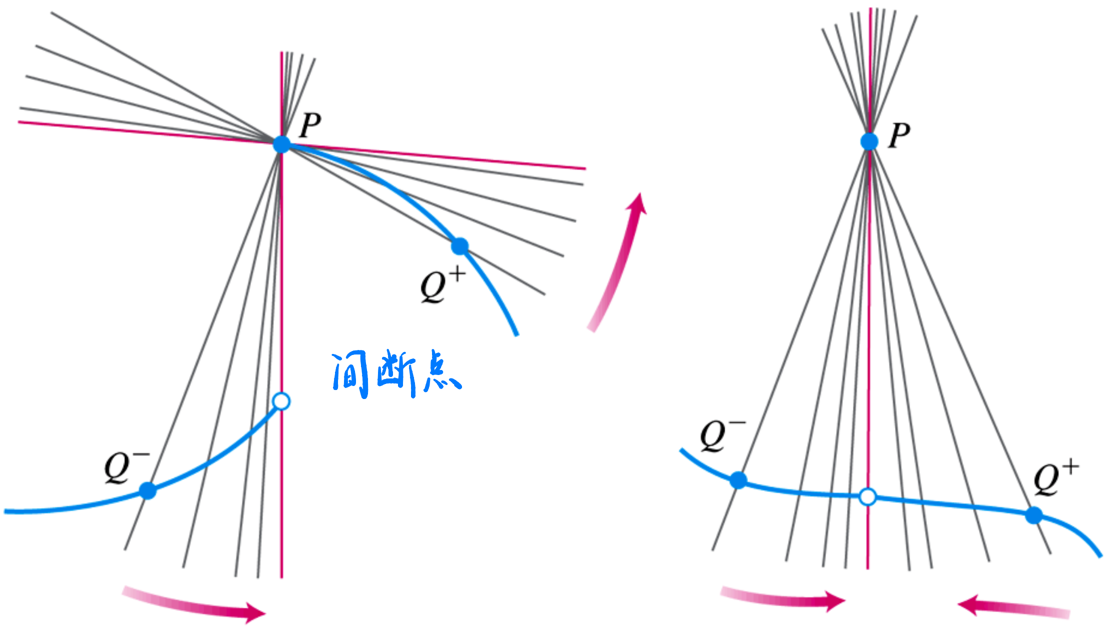
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.

尖端



* 导数不存在 ±∞
但有切线

3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



间断点

4. a *discontinuity* (two examples shown).

Another case in which the derivative may fail to exist occurs when the function's slope is oscillating rapidly near P, as with $f(x) = \sin(1/x)$ near the origin, where it is discontinuous (see Figure 2.31).

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L$ 一点可导 → 该点连续 $\lim_{x \to c} f(x) = f(c)$

$\Updownarrow$

逆否命题 不连续 → 不可导 $\lim_{x \to c} (f(x) - f(c)) = 0$

$$\lim_{x \to c} (f(x) - f(c)) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} (x - c) \right) = 0$$

3.3

Differentiation Rules

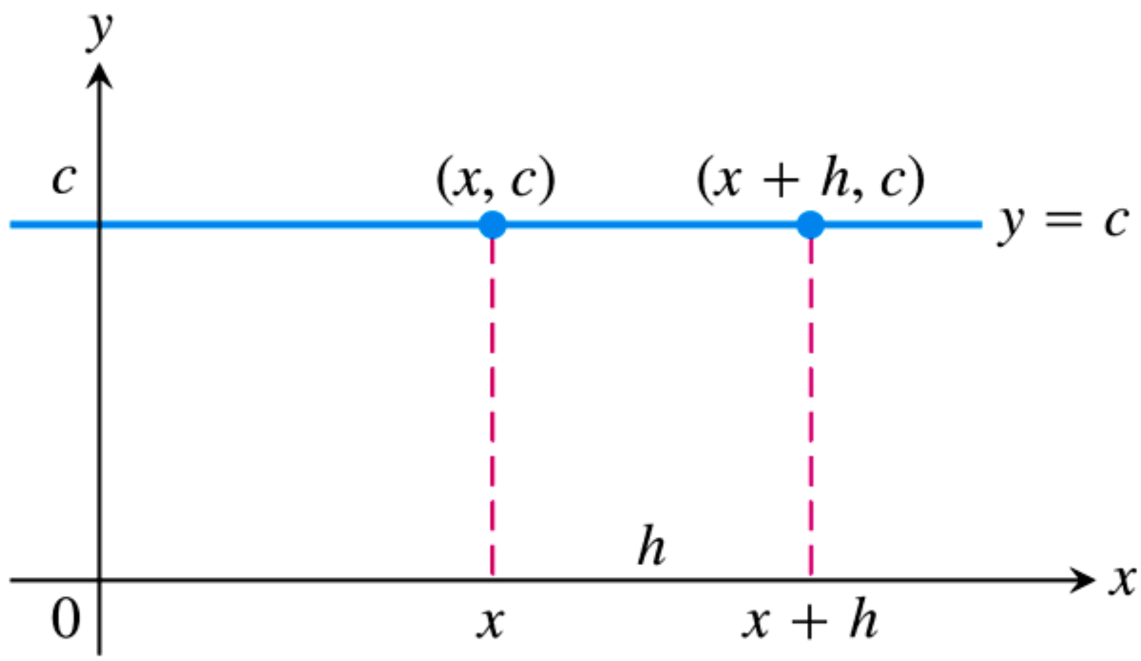


FIGURE 3.9 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Derivative of a Con常数stant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{C_n^1 x^{n-1}h + C_n^2 x^{n-2}h^2 + \cdots + C_n^n h^n}{h}$$

$$= \lim_{h\to 0}\left(C_n^1 x^{n-1} + C_n^2 x^{n-2}h + \cdots + C_n^n h^{n-1}\right)$$

$$= nx^{n-1}$$

① 另: $\lim_{z\to x}\frac{z^n - x^n}{z - x}$

$$= \lim_{z\to x}\left(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}\right)$$

$$= nx^{n-1}$$

*前提: n是正整数

② 负整数也成立

③ 推到有理数 $\frac{1}{n}$ ✓

④ 有理数 $\frac{m}{n}$ 幂次差

$$\lim_{x\to z}\frac{z^{\frac{m}{n}} - x^{\frac{m}{n}}}{z^m - x^m}\cdot\frac{z^m - x^m}{z - x}$$

换元 $z^m = \tilde{z}$ $x^m = \tilde{x}$

$$= \lim_{x\to z}\frac{\tilde{z}^{\frac{1}{n}} - \tilde{x}^{\frac{1}{n}}}{\tilde{z} - \tilde{x}}\cdot\frac{z^m - x^m}{z - x}$$

$$= \left(\frac{1}{n}\tilde{x}^{\frac{1}{n}-1}\right)\times m x^{m-1}$$

$$= \frac{m}{n}x^{\frac{m}{n}-m+m-1}$$

$$= \frac{m}{n}x^{\frac{m}{n}-1}$$

由易到难

一次变一步

⑤ 负有理数

$f(x) = x^{-\frac{m}{n}}$

$$\lim_{x\to z}\frac{z^{-\frac{m}{n}} - x^{-\frac{m}{n}}}{z - x} = \lim_{x\to z}\frac{\frac{x^{\frac{m}{n}} - z^{\frac{m}{n}}}{z^{\frac{m}{n}}x^{\frac{m}{n}}}}{z - x} = -\frac{m}{n}x^{\frac{m}{n}-1}\frac{1}{x^{\frac{2m}{n}}}$$

$$= -\frac{m}{n}x^{-\frac{m}{n}-1}$$

⑥ 无理数

RAMP

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

EXAMPLE 2

(a) The derivative formula

$$\frac{d(3x^2)}{dx}$$

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.10).

(b) Negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

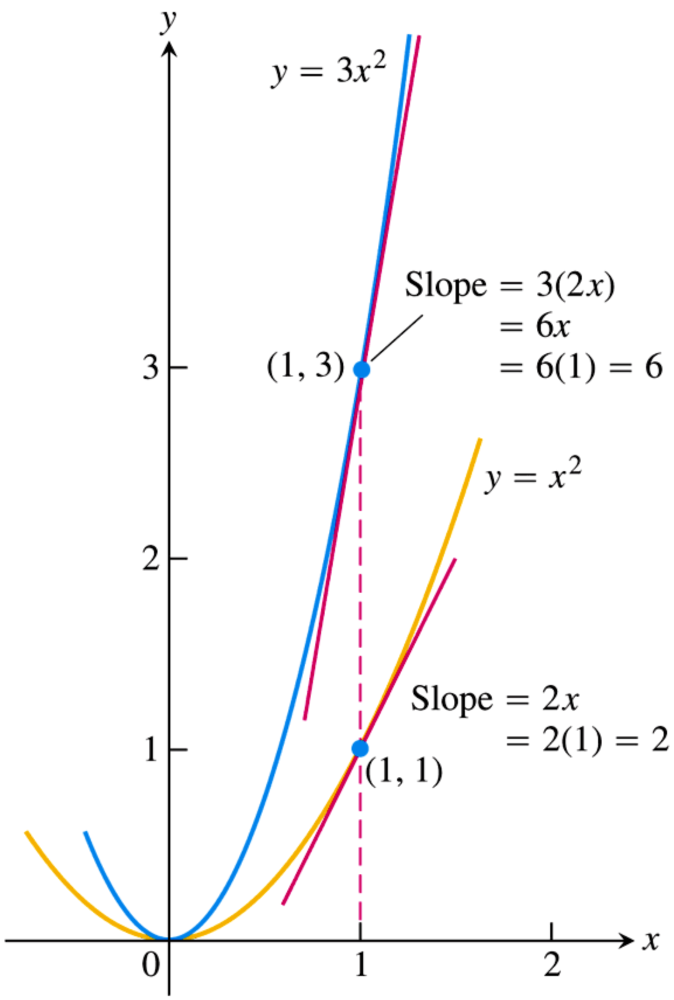


FIGURE 3.10 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinate triples the slope (Example 2).

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

$$\lim_{h \to 0} \frac{[u(x+h) + v(x+h)] - (u(x) + v(x))}{h}$$

EXAMPLE 4 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

$$\frac{dy}{dx} = 4x^3 - 4x$$

$$= 4x(x-1)(x+1)$$

$0, \pm 1$

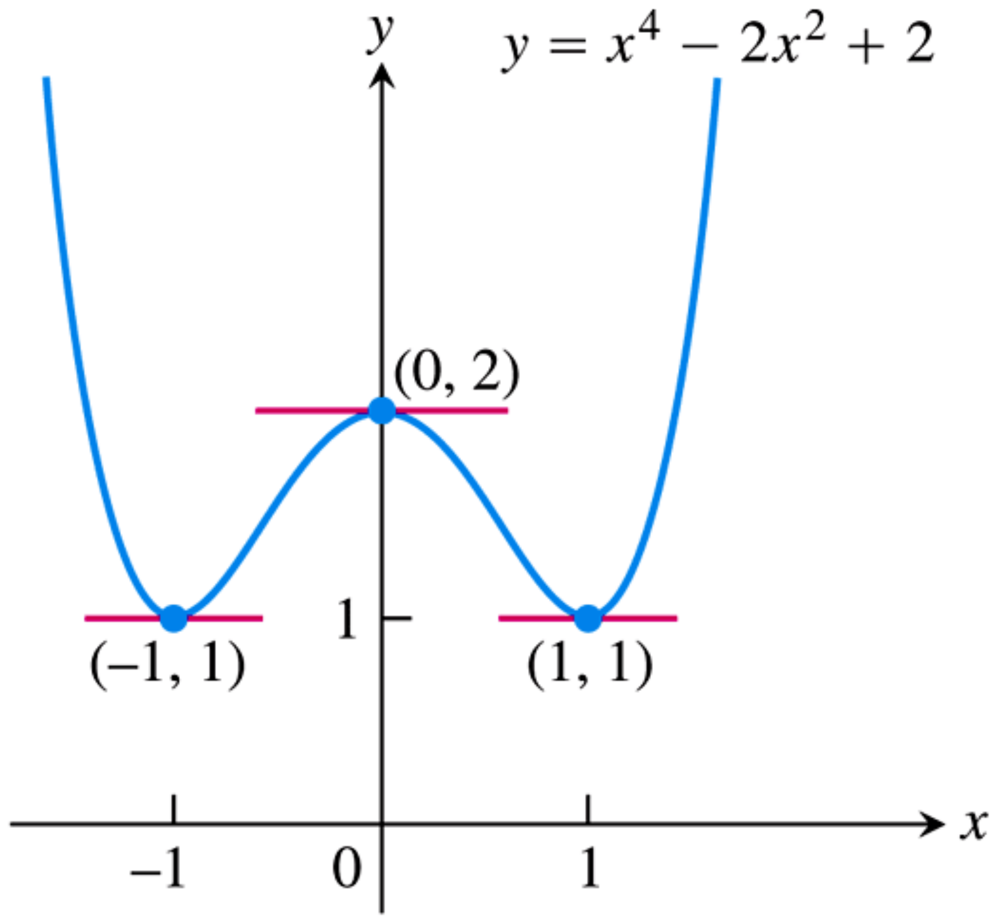


FIGURE 3.11 The curve in Example 4 and its horizontal tangents.

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

定义

在内部用减式拆也相同

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

有公因子

$$= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0}\left(u(x+h)v' + v(x+h)u'\right)$$

$$= uv' + u'v$$

$$(uvw)' = u'vw + uv'w + uvw'$$

看成 $[(uv)\cdot w]'$

n个函数相乘有n项 每项仅一个导数

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

$$\lim_{h \to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x+h)} + \dfrac{u(x)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$

加一项减一项

$\boxed{\text{乘一项除一项}} \neq 0$

准没错 (一步一步拆解)

$$= \lim_{h \to 0} \frac{u'}{v(x+h)} + \lim_{h \to 0} u(x) \frac{v(x) - v(x+h)}{v(x)\,v(x+h)\,h}$$

$$= \frac{u'}{v(x)} + \frac{-v'u(x)}{v(x)^2} = \frac{vu' - v'u}{v^2}$$

Second- and Higher-Order Derivatives

$\frac{dy}{dx}$ y' $\frac{d}{dx}$ 算子

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = \underline{y''} = D^2(f)(x) = D_{\underset{\text{变量}}{x}}^{2} f(x).$$

求偏导会多变量

The symbol D^2 means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x. The names continue as you imagine, with

n阶导数

“事不过三” $$y^{(\underline{n})} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the ***n*th derivative** of y with respect to x for any positive integer n.

① $|x|\geq 1$

$\sqrt[n]{|x|^{3n}} \leq \sqrt[n]{1+|x|^{3n}} \leq \sqrt[n]{2|x|^{3n}}$

$= |x|^3 \qquad = |x|^3$

$\sqrt[n]{2} = 2^{\frac{1}{n}}$

夹逼定理

+自然不等式

② $|x|<1$

$\sqrt[n]{1} \leq \sqrt[n]{1+|x|^{3n}} \leq \sqrt[n]{2}$

$= 1 \qquad = 1$

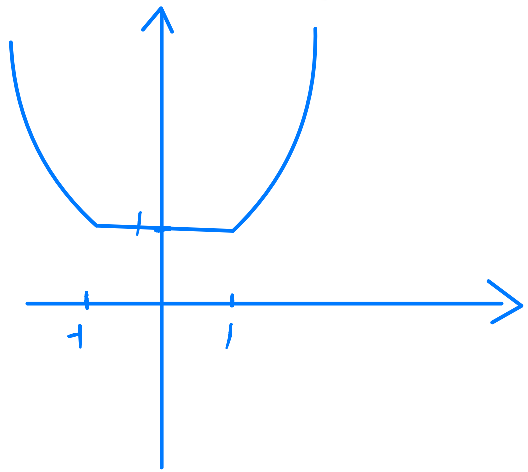
例4（2005） 设函数 $f(x) = \lim\limits_{n\to\infty} \sqrt[n]{1+|x|^{3n}}$，则 $f(x)$ 在 $(-\infty,+\infty)$ 内

(A) 处处可导.

(B) 恰有一个不可导点.

(C) 恰有两个不可导点.

(D) 至少有三个不可导点.



3.4

The Derivative as a Rate of Change

DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

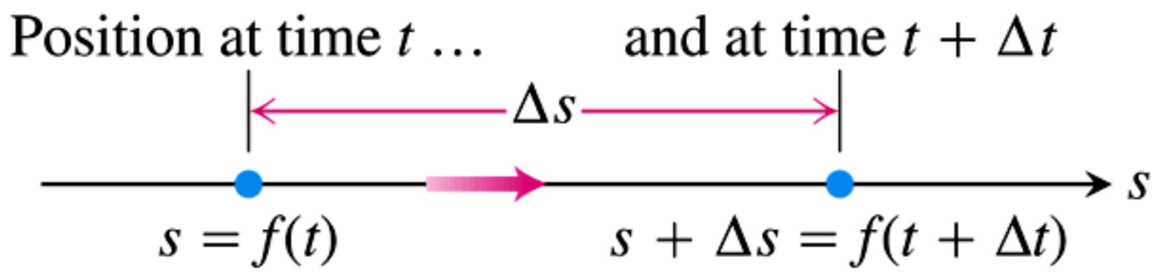


FIGURE 3.12 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$. Here the coordinate line is horizontal.

DEFINITION **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

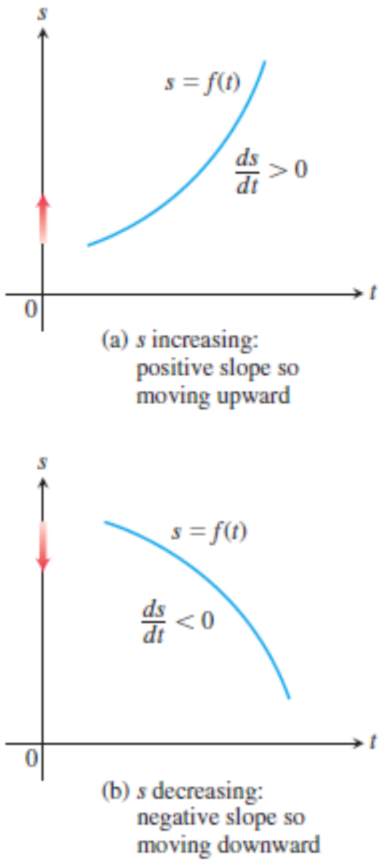


FIGURE 3.13 For motion $s = f(t)$ along a straight line (the vertical axis), $v = ds/dt$ is (a) positive when s increases and (b) negative when s decreases.

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed}_{\geq 0} = |v(t)| = \left|\frac{ds}{dt}\right|$$

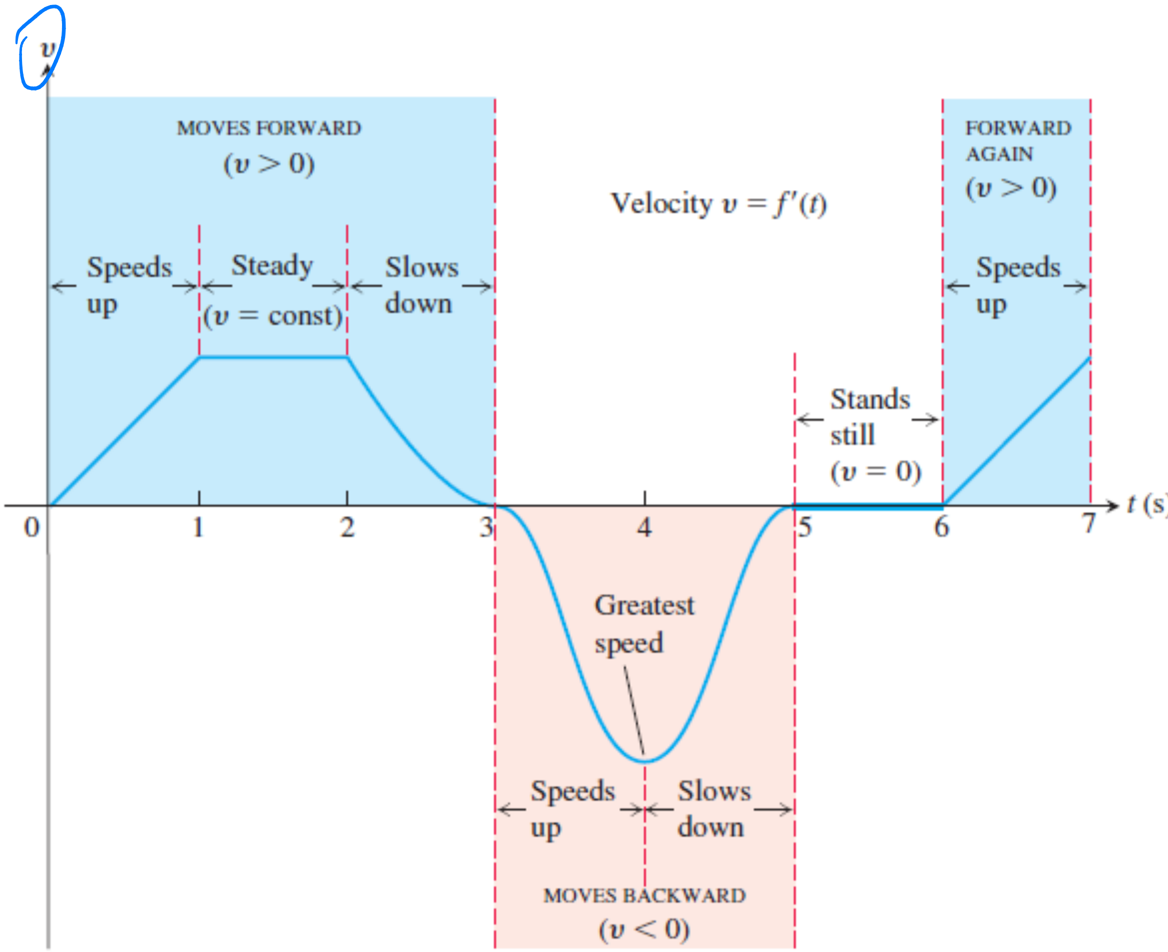


FIGURE 3.14 The velocity graph of a particle moving along a horizontal line, discussed in Example 2.

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3 s}{dt^3}.$$

EXAMPLE 3 Figure 3.15 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

(a) How many meters does the ball fall in the first 3 sec?

(b) What is its velocity, speed, and acceleration when $t = 3$?

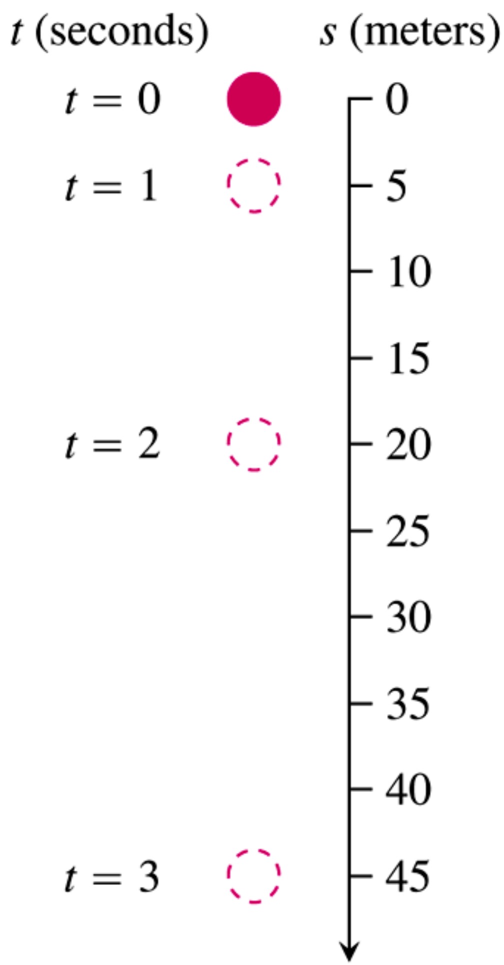


FIGURE 3.15 A ball bearing falling from rest (Example 3).

3.5

Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

三角函数和差角引入：单位圆+余弦定理

$$\frac{d}{dx}(\sin x) = \cos x.$$

sin 与 cos $\frac{\pi}{2}$ 关系

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

定义只能定义一个表达式

另一式只能证明等价

$$= \lim_{h\to 0}\frac{\cancel{\sin x}\cos h+\cos x\sin h-\cancel{\sin x}}{h}$$

不能就当成0 $=1-2\sin^2\frac{h}{2}$ $2\cos^2\frac{h}{2}-1=\cos 2h$

$$= \lim_{h\to 0}\cos x\frac{\sin h}{h}+\lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}$$

∵上下都有0

$$= \cos x$$

$$= \lim_{h\to 0}\frac{\sin x\left(-2\sin^2\frac{h}{2}\right)}{h} = 0$$

上下都有0时不可以提前消去

只有极限是常数时可算出来

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine function:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\
&= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{limit } 1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x. && \text{Example 5a and Theorem 7, Section 2.4}
\end{aligned}$$

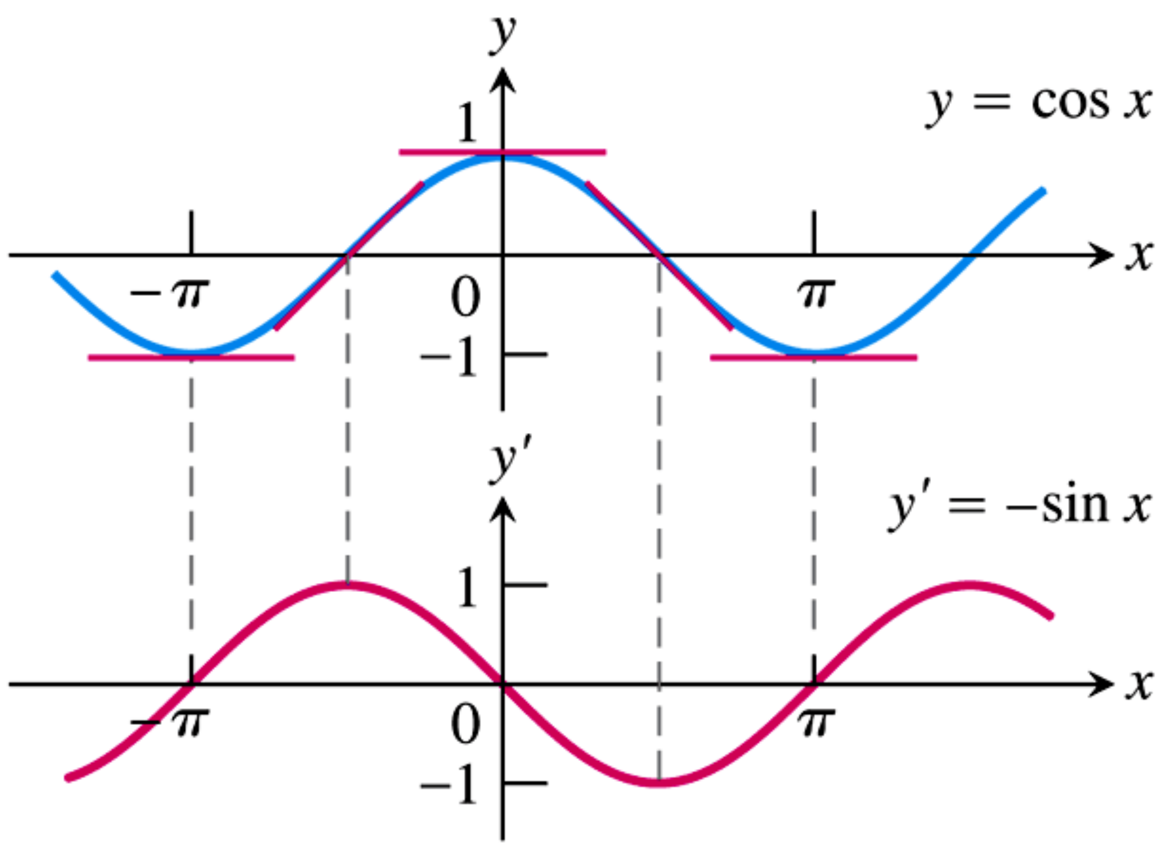


FIGURE 3.20 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

$\lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x(\cos h-1)-\sin x\sin h}{h}$

$= \cos x \lim_{h\to 0} \frac{\cos h-1}{h} - \sin x$

$= -\sin x$

$\lim_{h\to 0} \frac{\cos h-1}{h} \frac{\cos h+1}{\cos h+1}$

$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h+1)}$

$= 0$

EXAMPLE 3 A weight hanging from a spring (Figure 3.21) is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t?

$v = s' = -5\sin t$

$a = v' = -5\cos t$

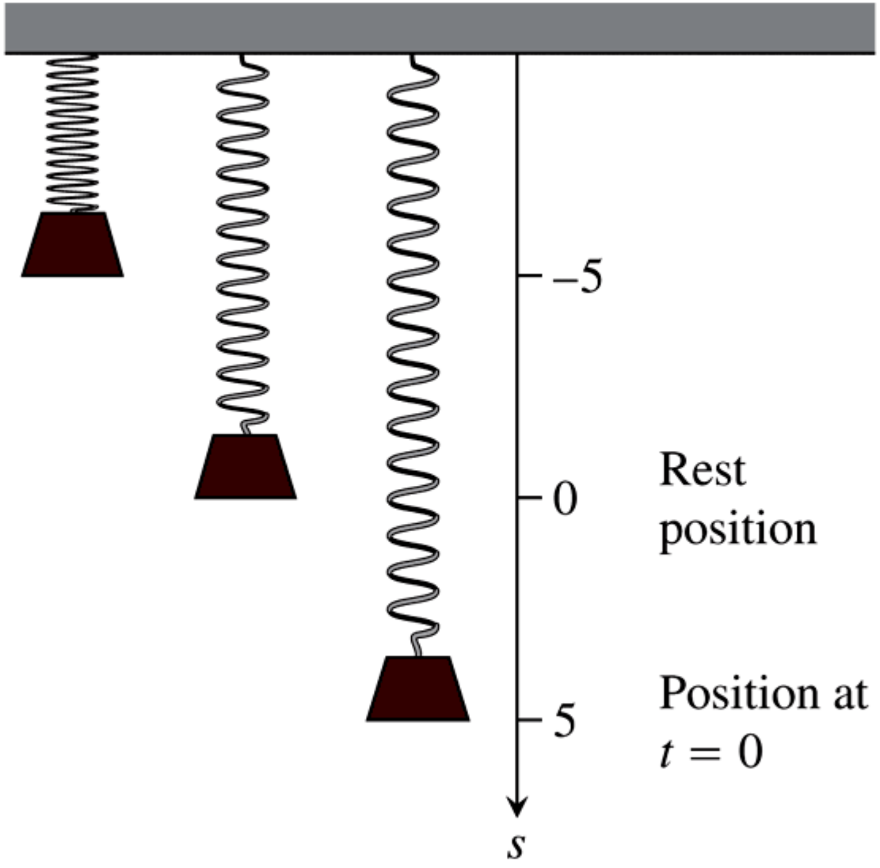


FIGURE 3.21 A weight hanging from a vertical spring and then displaced oscillates above and below its rest position (Example 3).

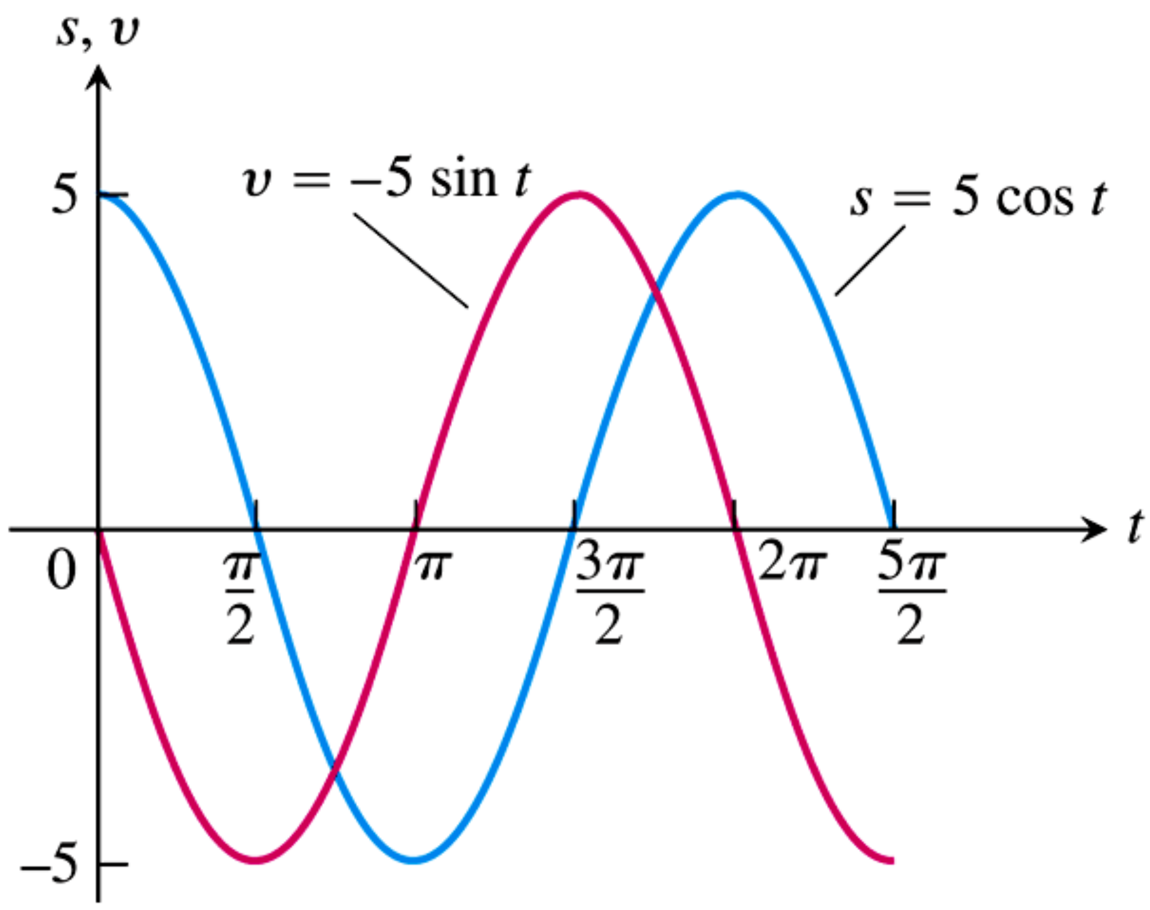
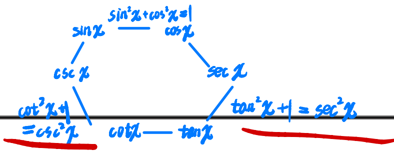


FIGURE 3.22 The graphs of the position and velocity of the weight in Example 3.

$\cot x = \frac{\cos x}{\sin x}$

$\sec x = \frac{1}{\cos x}$

$\csc x = \frac{1}{\sin x}$



The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$\left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x$

$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$

$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$

$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$

c开头加"—"
对应改变

3.6

The Chain Rule

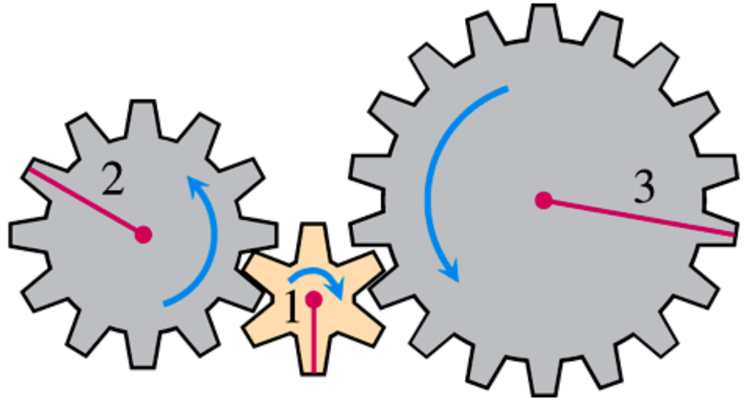


FIGURE 3.23 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

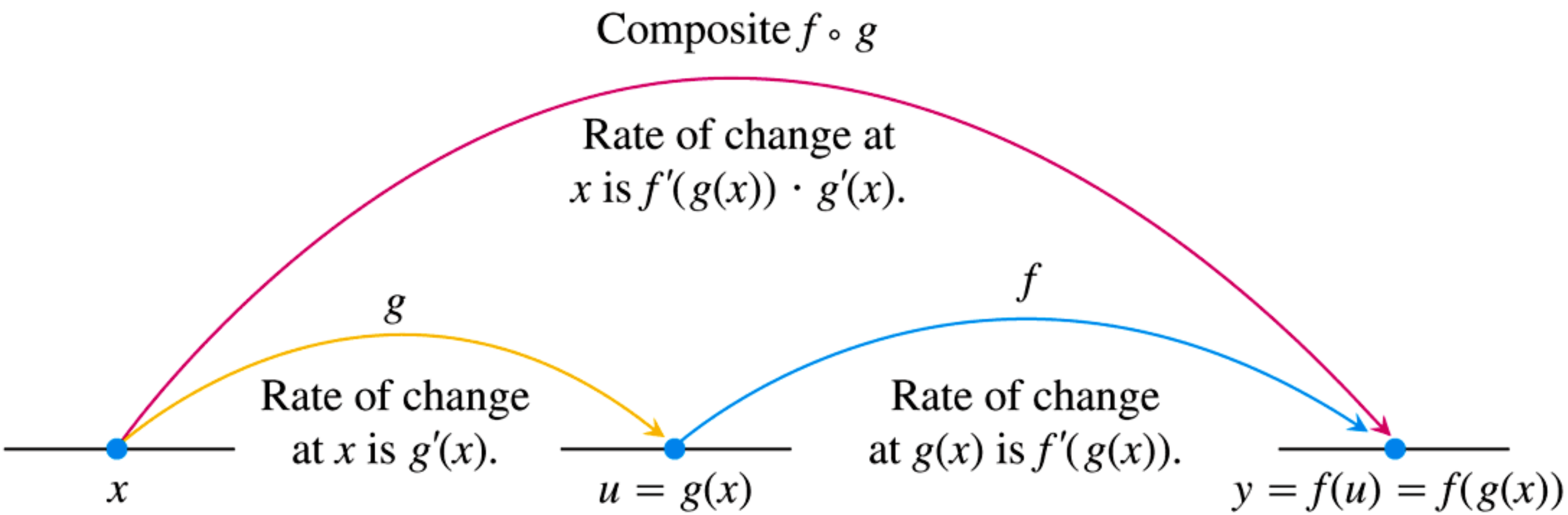


FIGURE 3.24 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

$$\lim_{z \to x} \frac{z^{\frac{m}{n}} - x^{\frac{m}{n}}}{z^n - x^n} \times \frac{z^n - x^n}{z - x}$$

乘一项除一项保证$\neq 0$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

取 $x \sin$式 振荡 ∴x附近无数点函数值相同

EXAMPLE 4 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 8 The formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180^\circ = \pi$ radians, $x^\circ = \pi x/180$ radians where x° is the size of the angle measured in degrees.

By the Chain Rule,

$$\frac{d}{dx}\sin(x^\circ) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x^\circ).$$

See Figure 3.25. Similarly, the derivative of $\cos(x^\circ)$ is $-(\pi/180)\sin(x^\circ)$.

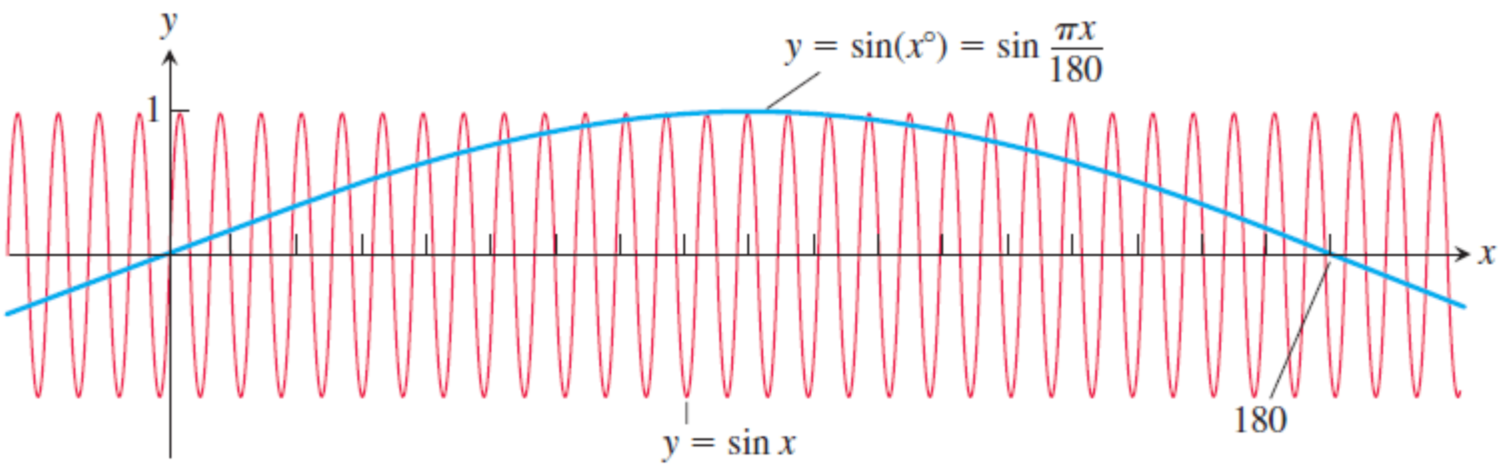
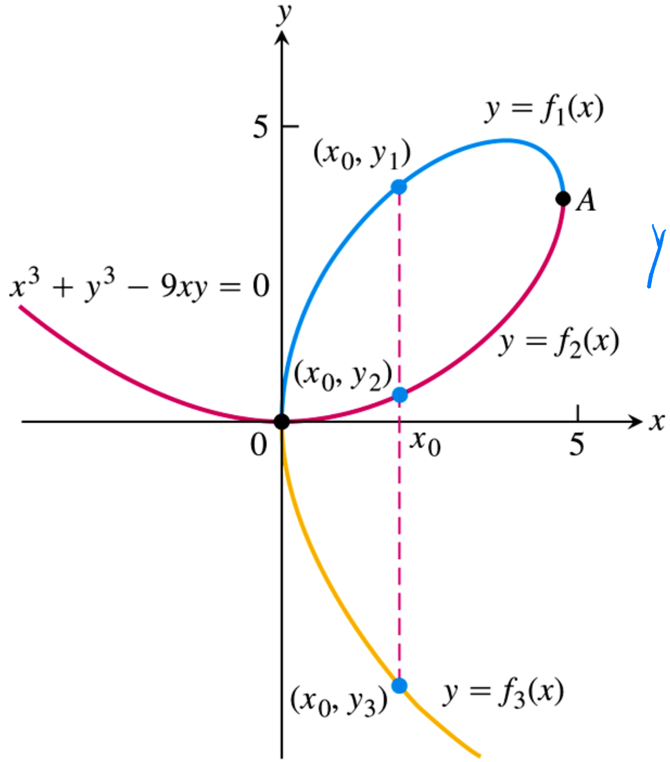


FIGURE 3.25 The function $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 8).

3.7

隐函数

Implicit Differentiation



y存在并唯一

FIGURE 3.26 方程的三个根 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

$2y\frac{dy}{dx} = 1$

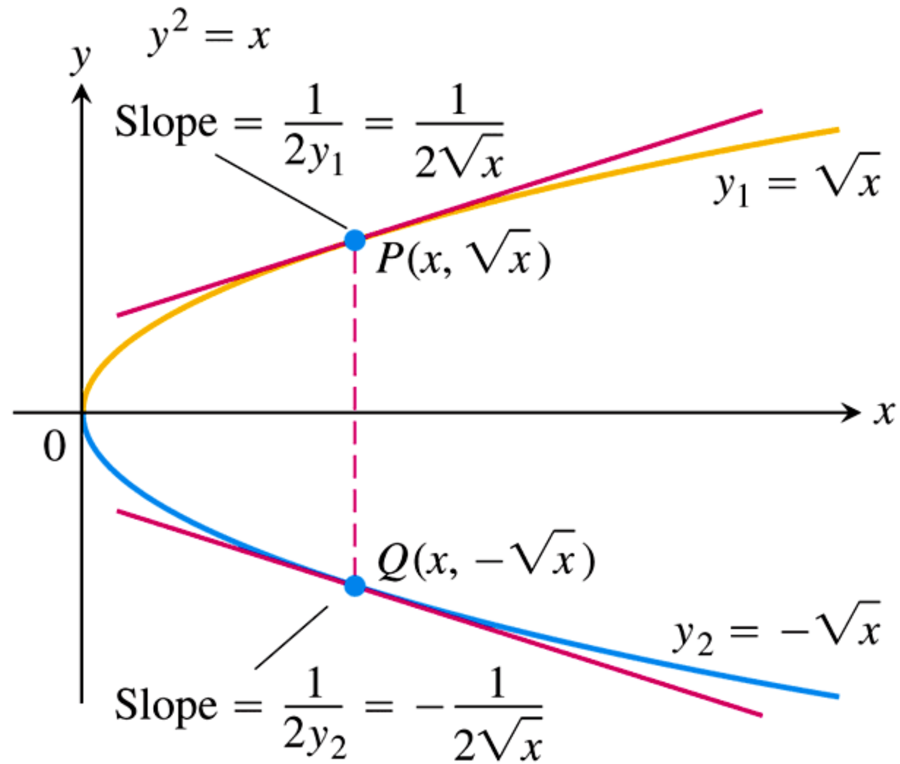
$\frac{dy}{dx} = \frac{1}{2y}$

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

$\frac{d(x^2+y^2)}{dx} = 0$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{y}$



目标是$\frac{dy}{dx}$
中间可变换

FIGURE 3.27 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x > 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

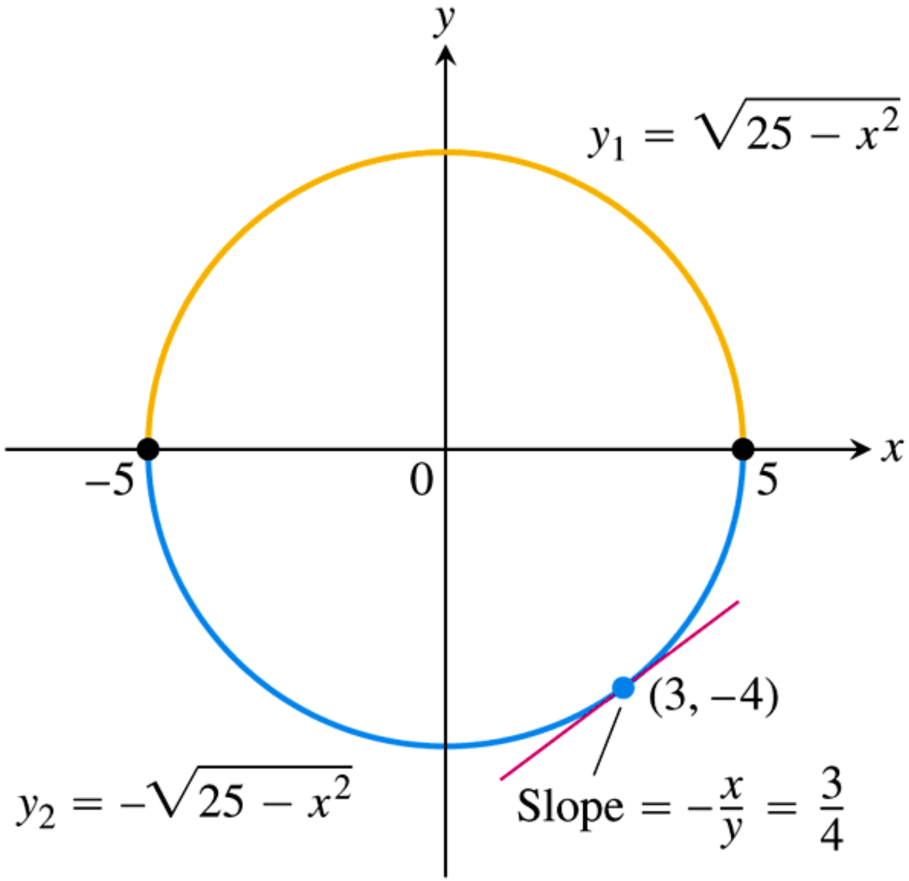


FIGURE 3.28 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

真题

EXAMPLE 3 Find dy/dx if $y^2 = x^2 + \sin xy$ (Figure 3.29).

$$\frac{dy^2}{dx} = \frac{d(x^2+\sin xy)}{dx}$$

$$2y\frac{dy}{dx} = 2x + \cos xy \cdot \frac{dxy}{dx}$$

$$2y\frac{dy}{dx} = 2x + \cos xy\left(y+\frac{dy}{dx}x\right)$$

$$\frac{dy}{dx} = \frac{2x+y\cos xy}{2y-x\cos xy}$$

$y=x$

第一象限内 $y=\sqrt{x^2+\sin xy}$

$$\lim_{x\to+\infty}(f(x)-x) = \lim_{x\to+\infty}\left(\sqrt{x^2+\sin xf(x)}-x\right) = \lim_{x\to+\infty}\frac{\left(\sqrt{x^2+\sin xf(x)}-x\right)\left(\sqrt{x^2+\sin xf(x)}+x\right)}{\cdots} = \lim_{x\to+\infty}\frac{\sin(xf(x))}{\sqrt{x^2+\sin xf(x)}+x}$$

$$-\frac{1}{\sqrt{x^2-1}+x} \leq 原式 \leq \frac{1}{\sqrt{x^2-1}+x}$$

0

EXAMPLE 5 Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.31).

$$\frac{d(x^3+y^3-9xy)}{dx} = 0$$

$$3x^2 + \frac{d}{dx}y^3 - \frac{d}{dx}9xy = 0$$

$$\frac{d}{dx}y^3 = \frac{d(y^3)}{dy}\frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

$$\frac{d}{dx}(9xy) = 9\left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} - 9\left(y+x\frac{dy}{dx}\right) = 0$$

$$(3y^2-9x)\frac{dy}{dx} = -3x^2+9y$$

$$\left.\frac{dy}{dx}\right|_{(2,4)} = -\left.\frac{x^2-3y}{y^2-3x}\right|_{(2,4)} = \frac{4}{5}$$

$$\left.\frac{dy}{dx}\right|_{(2,4)} = -\left.\frac{x^2-3y}{y^2-3x}\right|_{(2,4)}$$

复合函数求导易错

相乘

$$\frac{d(9xy)}{dx} = 9y + 9x\frac{dy}{dx}$$

之前表达式 $y=f(x)$

y 依赖 x，一定最后只有 x

每一段上 y 表达式不同

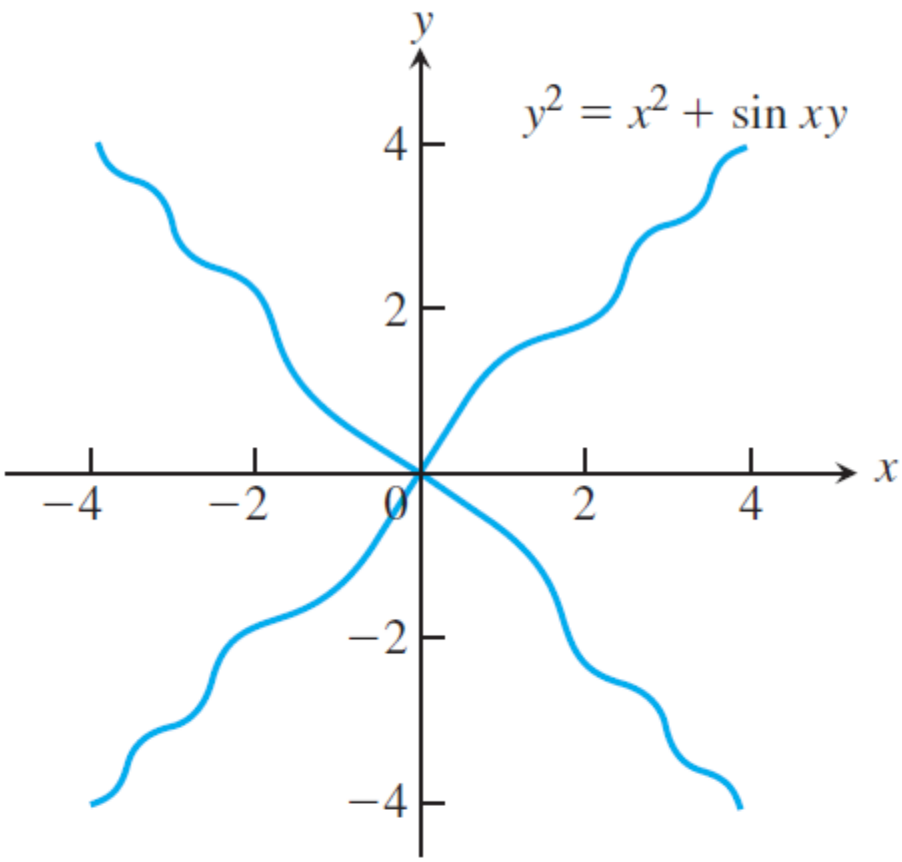


FIGURE 3.29 The graph of the equation in Example 3.

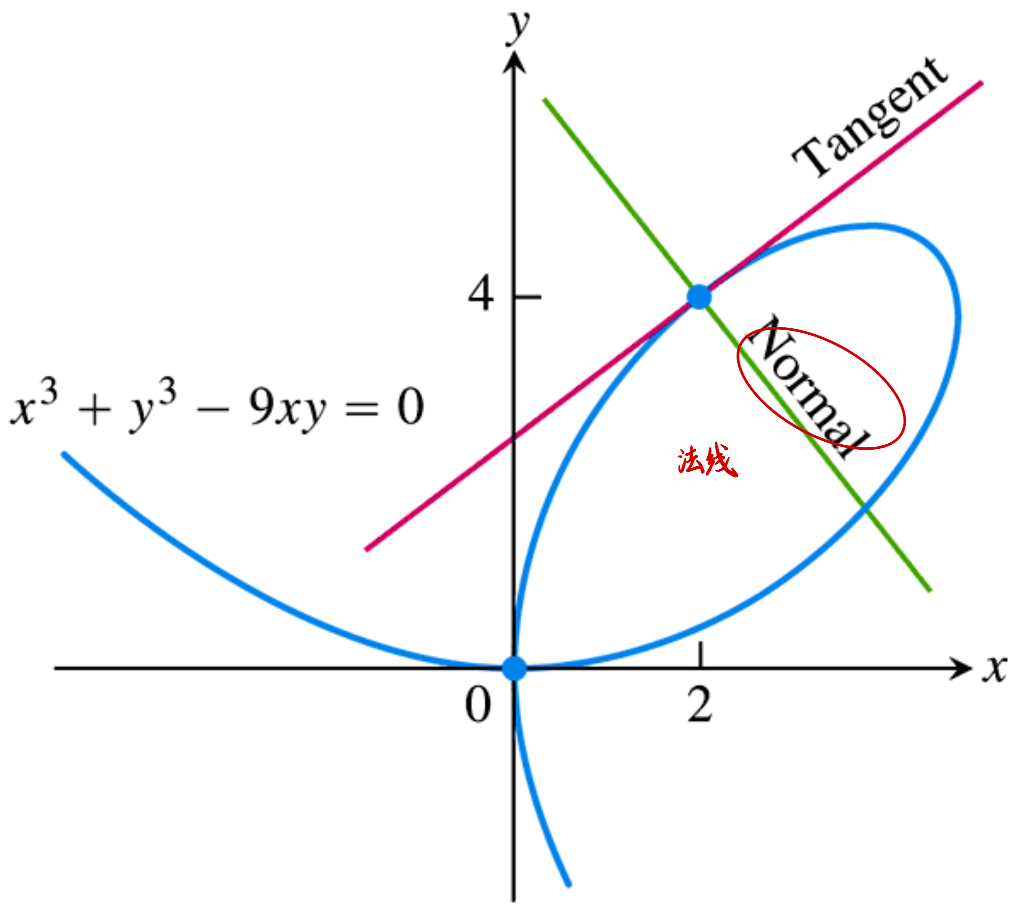


FIGURE 3.31 Example 5 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

3.8

Related Rates

圆锥的 $\frac{dV}{dt}$

EXAMPLE 1 Water runs into a conical tank at the rate of $0.25\ \text{m}^3/\text{min}$. The tank stands point down and has a height of 3 m and a base radius of 1.5 m. How fast is the water level rising when the water is 1.8 m deep?

$\frac{dy}{dt}$

隐函数求导简化证明

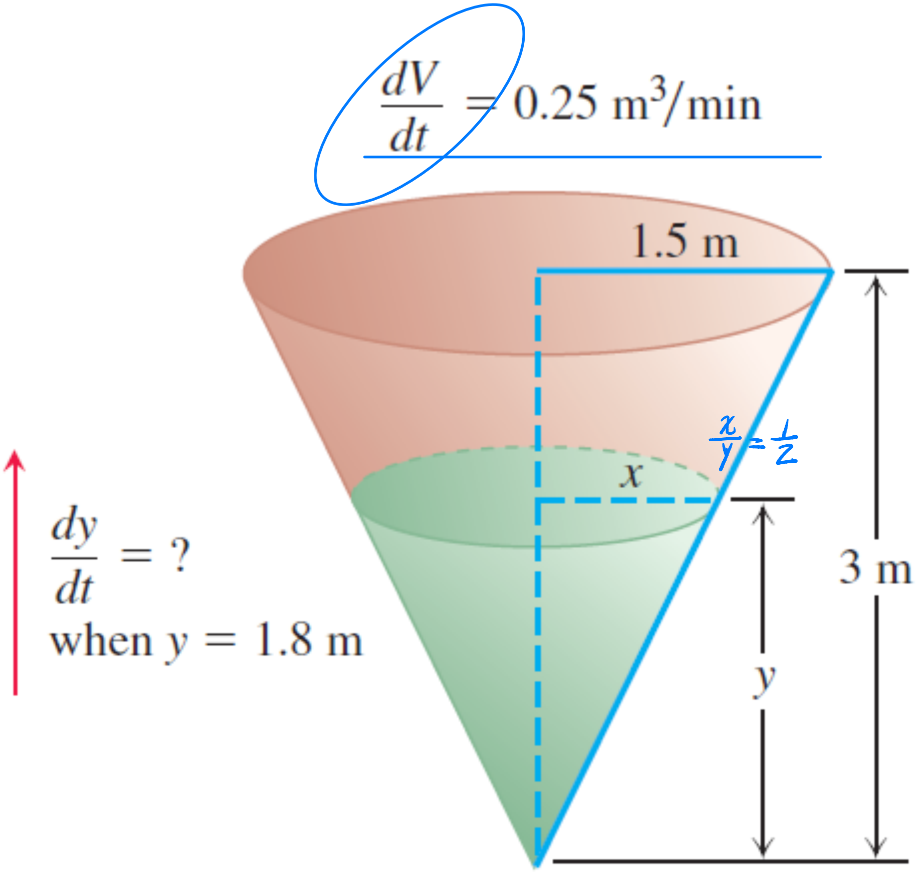
$y = x^{\frac{n}{m}}$

$y^m = x^n$ (整数已证)

$\frac{d\,y^m}{dx} = n x^{n-1}$

$= m y^{m-1} \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{n}{m} \frac{x^{n-1}}{(x^{\frac{n}{m}})^{m-1}} = \frac{n}{m} x^{\frac{n}{m}-1}$



$V = \frac{1}{3}\pi x^2 y$

$= \frac{1}{12}\pi y^3$

$\frac{dV}{dt} = \frac{1}{12}\pi\, 3y^2 \frac{dy}{dt}$

$y = 1.8 \quad \frac{dy}{dt}$

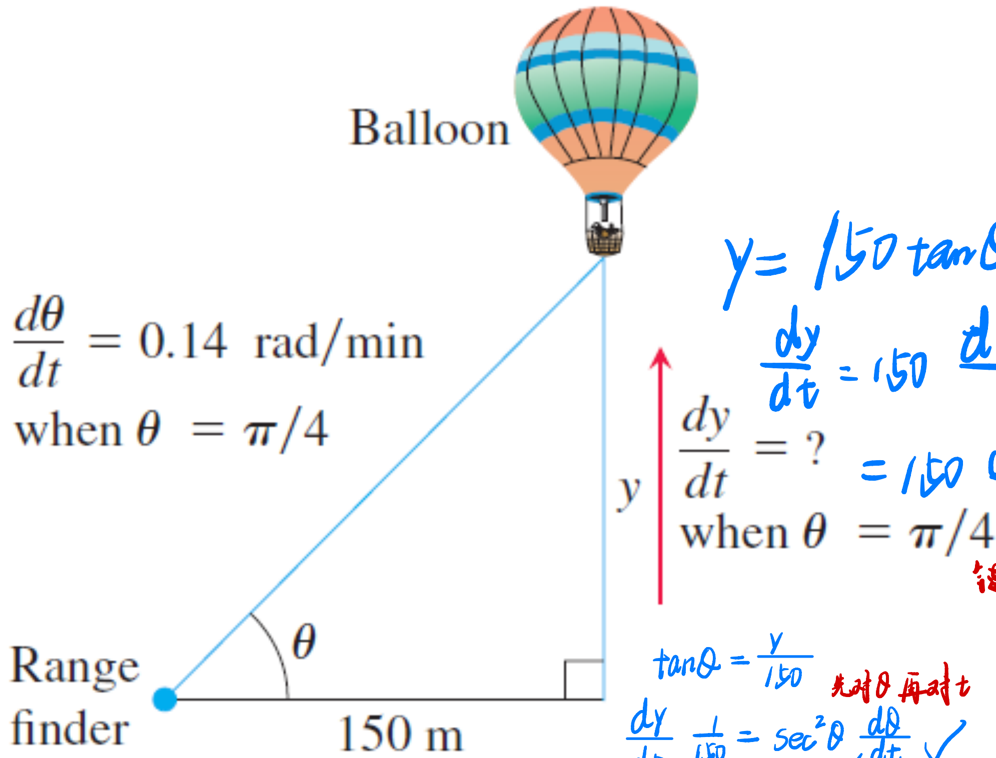
FIGURE 3.32 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t*. Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

物理含义



$y = 150 \tan\theta$

$\frac{dy}{dt} = 150 \frac{d \tan\theta}{dt}$ 写清

$= 150 \frac{d \tan\theta}{d\theta} \frac{d\theta}{dt}$

链式

$\tan\theta = \frac{y}{150}$ 先对θ 再对t

$\frac{dy}{dt} \frac{1}{150} = \sec^2\theta \frac{d\theta}{dt}$ ✓

$\theta = \pi/4$ ✓

42m/min.

FIGURE 3.33 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

EXAMPLE 3 A police cruiser, approaching a right-angled intersection 十字路口 from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 km north of the intersection and the car is 0.8 km to the east, the police determine with radar that the distance between them and the car is increasing at 30 km/h. If the cruiser is moving at 100 km/h at the instant of measurement, what is the speed of the car?

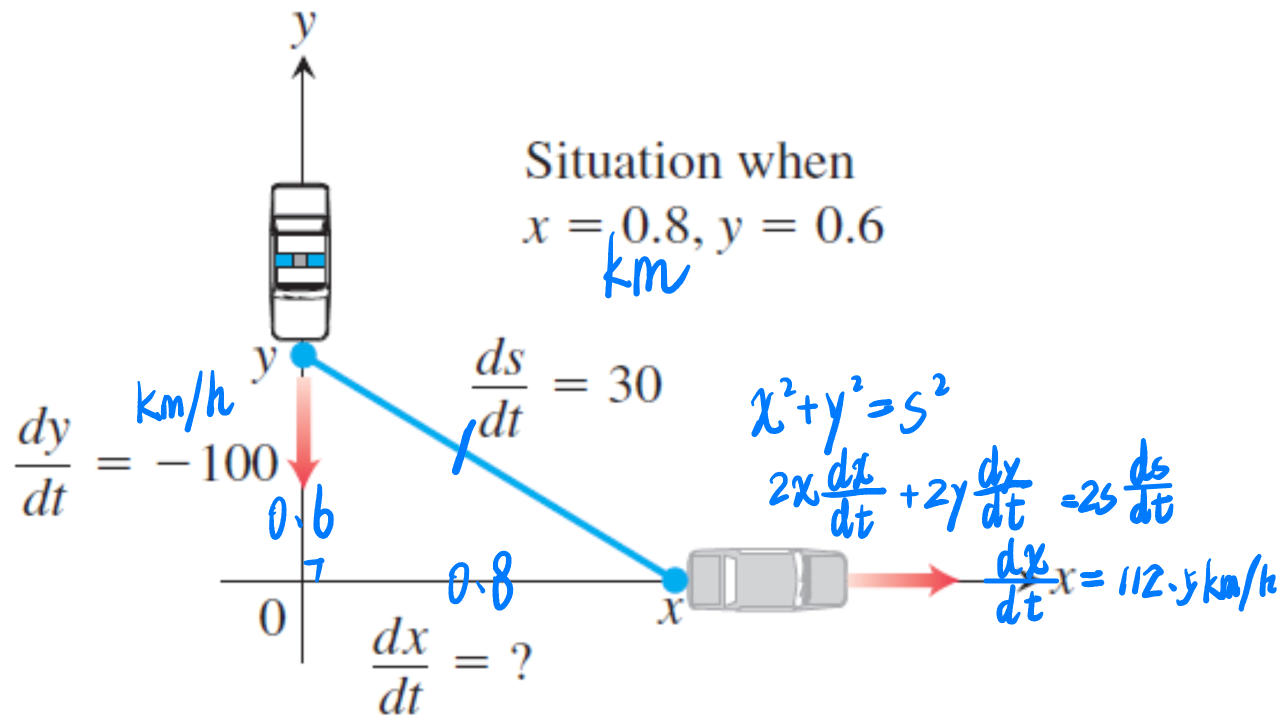


FIGURE 3.34 The speed of the car is related to the speed of the police cruiser 巡逻车 and the rate of change of the distance s between them (Example 3).

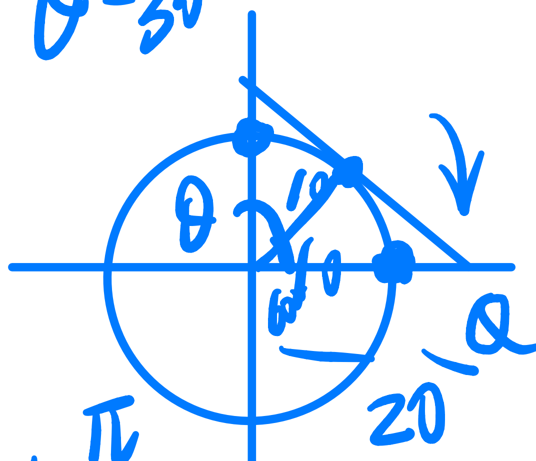
$\frac{\pi}{2} \to 30s$

EXAMPLE 4 A particle P moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is (0, 10) on the y-axis, and its final destination is the point (10, 0) on the x-axis. Once the particle is in motion, the tangent line at P intersects the x-axis at a point Q (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point Q moving along the x-axis when it is 20 m from the center of the circle?

$\frac{ds}{dt}$

$\theta = 30^\circ$

$R = 10$



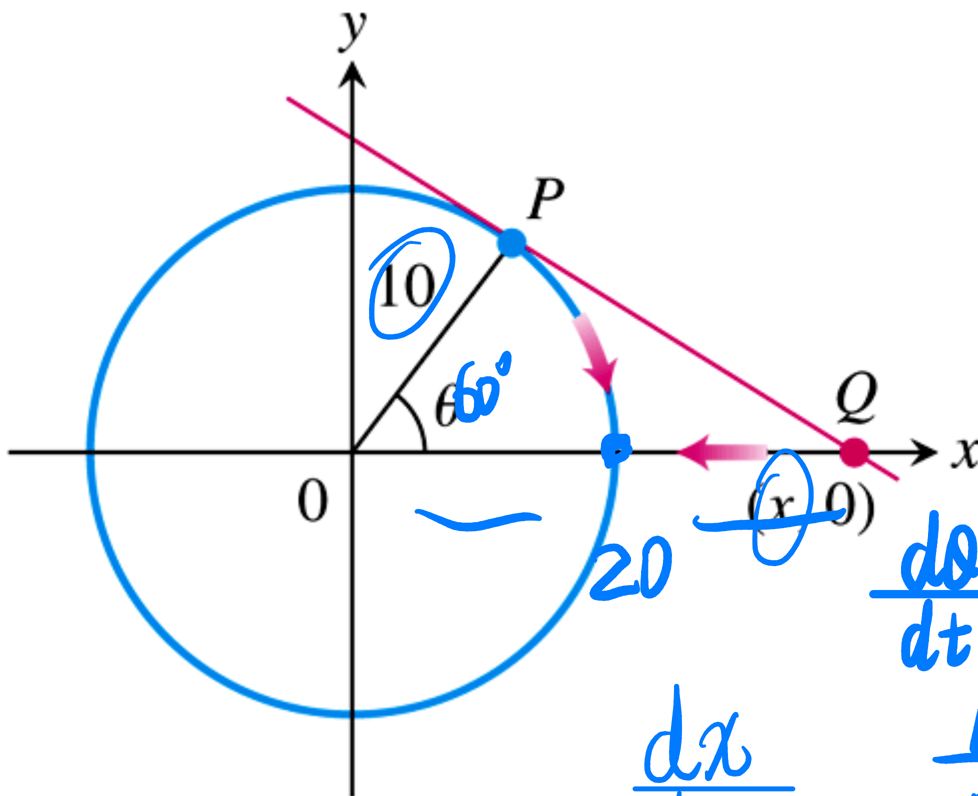
$10\left(-\frac{1}{s^2}\right)\frac{ds}{dt}$

$= \cos\theta \frac{d\theta}{dt}$

$\frac{10}{s} = \cos(90^\circ - \theta)$

$= \sin\theta$

$-\frac{1}{40}\frac{ds}{dt} = \frac{\sqrt{3}}{2} \times \frac{\pi}{3}$



$$\frac{d\theta}{dt} = \omega = \frac{\frac{\pi}{2}}{30} = \frac{\pi}{60}\ \text{rad/s}$$

$$\frac{dx}{dt}$$

$$\frac{10}{x} = \cos\theta$$

$$10 = x\cos\theta$$

$$0 = \frac{dx}{dt}\cos\theta + x(-\sin\theta)\frac{d\theta}{dt}$$

FIGURE 3.35 The particle P travels clockwise along the circle (Example 4).

顺时针的

EXAMPLE 6 Figure 3.37a shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 1.5 m above the ground in the hand M of a worker. Suppose the pulley is 7.5 m above ground, the rope is 13.5 m long, and the worker is walking rapidly away from the vertical line PW at the rate of 1.2 m/s. How fast is the weight being raised when the worker's hand is 6.3 m away from PW?

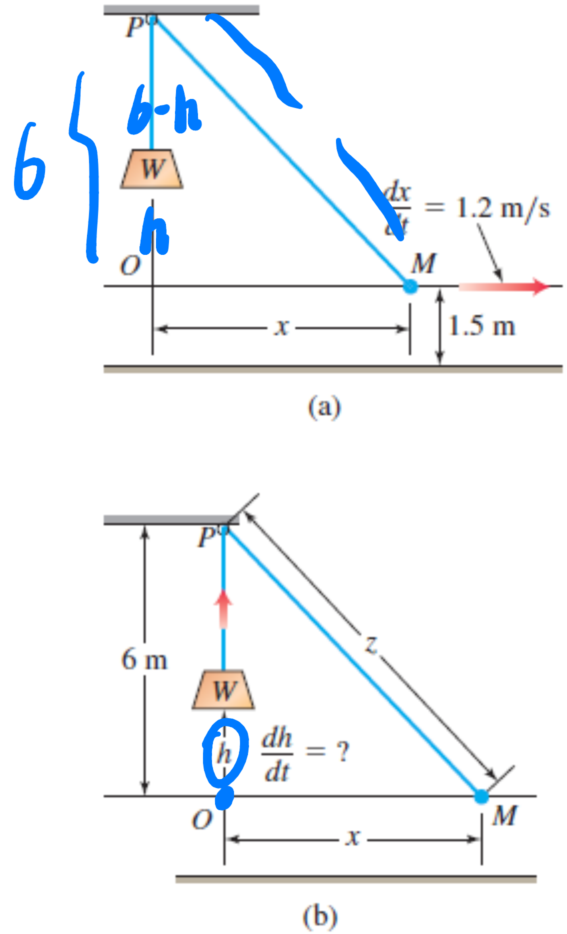


FIGURE 3.37 A worker at M walks to the right, pulling the weight W upward as the rope moves through the pulley P (Example 6).

勾股 利用Rt△

$$6^2+x^2=z^2$$

$$\frac{dx}{dt}=1.2\,m/s \quad x=6.3$$

$$6-h+z=13.5$$

$$z-h=7.5$$

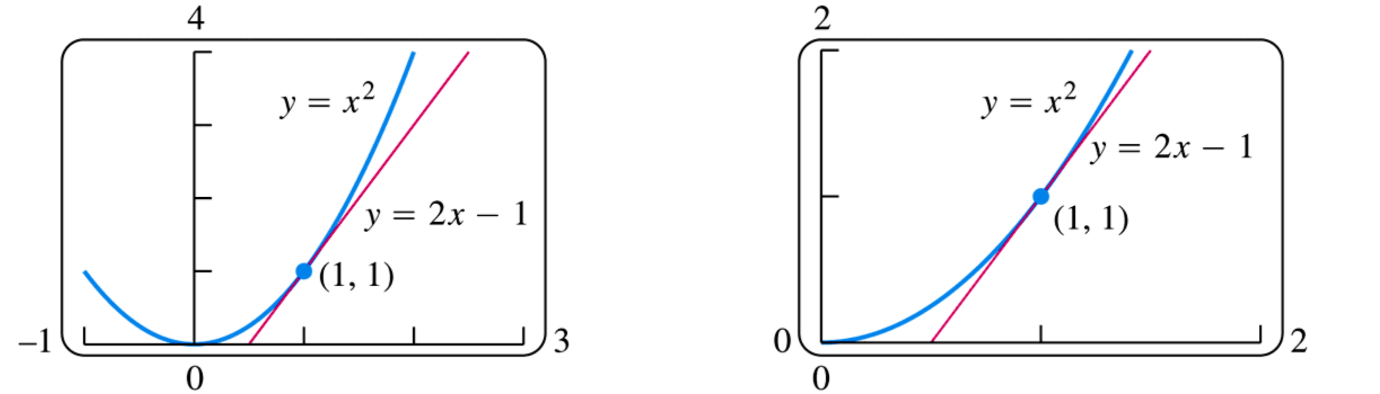
$$\frac{dz}{dt}=\frac{dh}{dt}$$

$$\frac{dx}{dt}\cdot x=\frac{dz}{dt}z$$

3.9

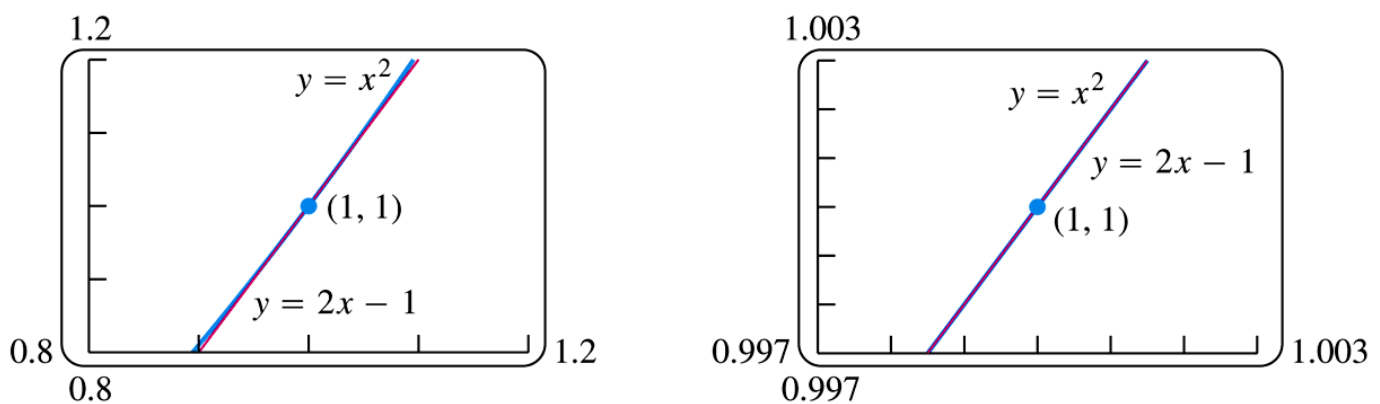
线性化　　　　微分

Linearization and Differentials



$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).

Tangent and curve very close near (1, 1).



Tangent and curve very close throughout entire x-interval shown.

Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

以直代曲

FIGURE 3.38 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

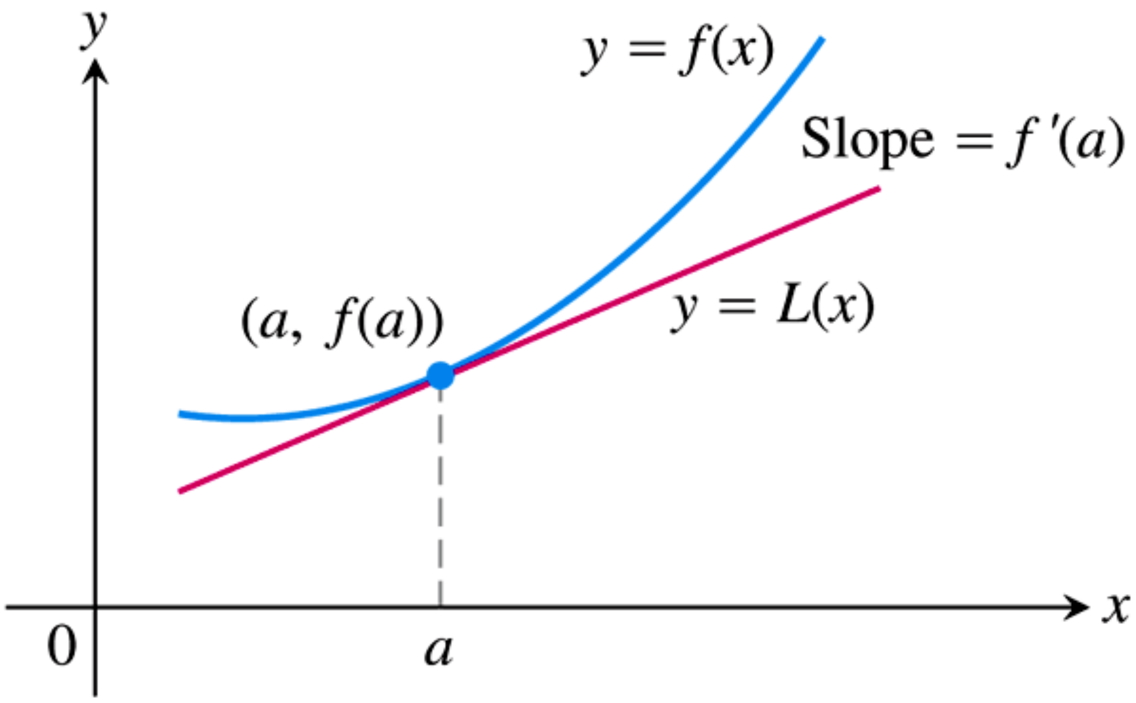


FIGURE 3.39 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

Linearization

切线方程

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$ 近似化

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

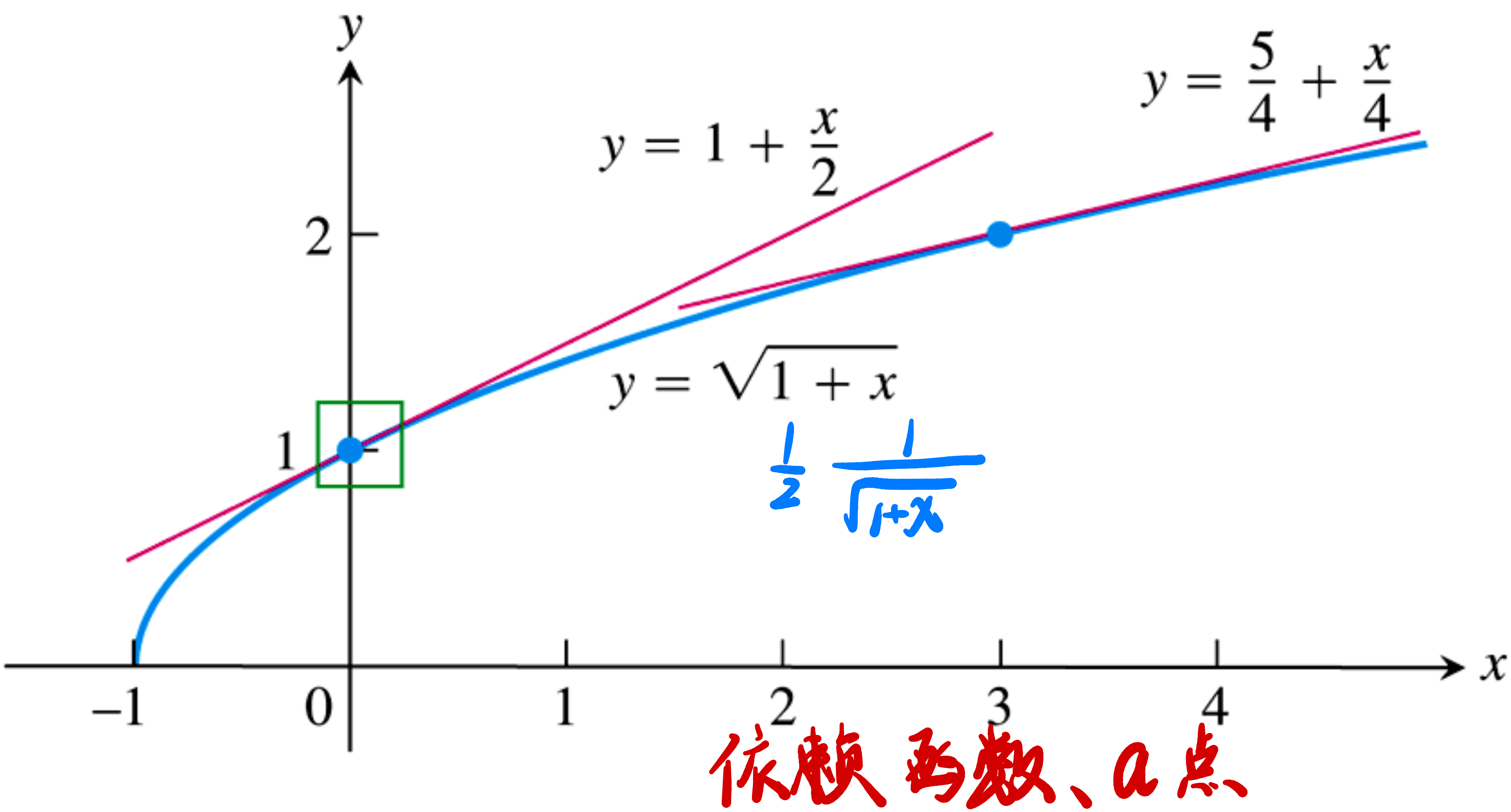


FIGURE 3.40 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.41 shows a magnified view of the small window about 1 on the y-axis.

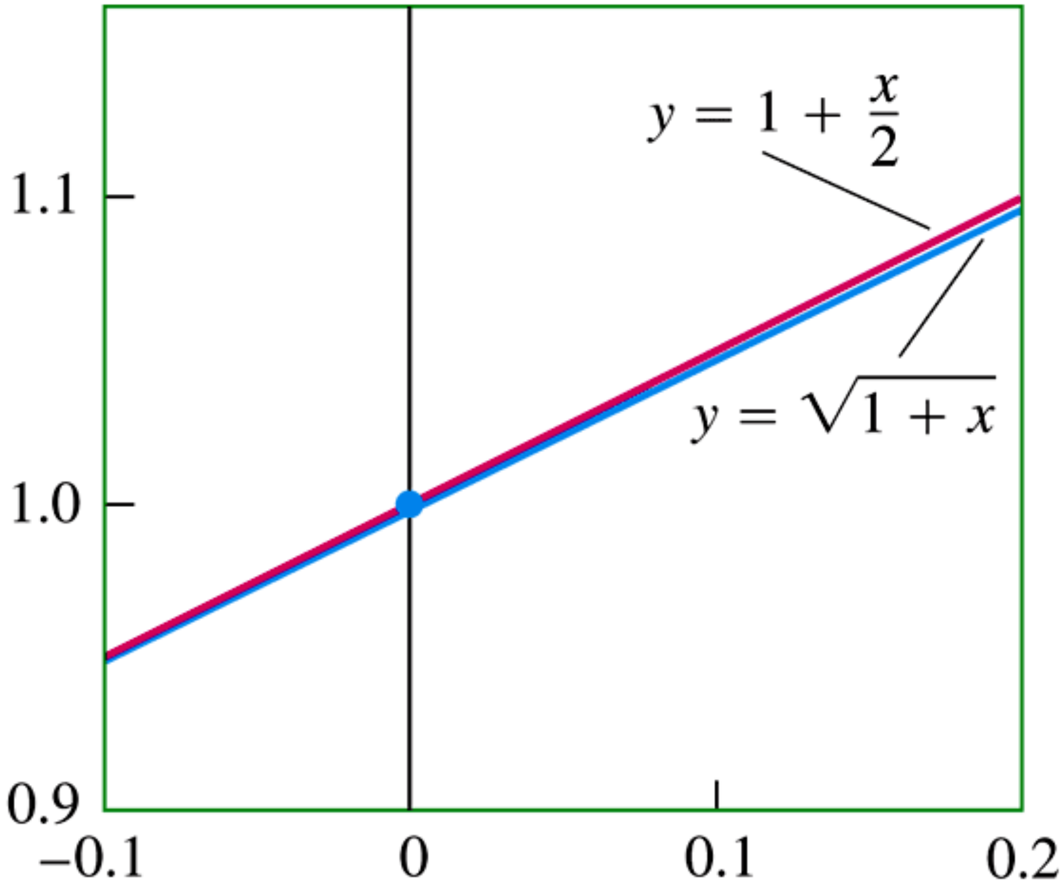


FIGURE 3.41 Magnified view of the window in Figure 3.40.

$\sin\left(\frac{\pi}{6} + 0.01\right)$

$f(x) = \sin x \quad a = \frac{\pi}{6}$

$f'(x) = \cos x = \frac{\sqrt{3}}{2}$

$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$

$f\left(\frac{\pi}{6} + 0.01\right) \approx L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2} \times 0.01$

Approximation	True value	$\lvert$ True value $-$ approximation $\rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$

$7.97^{\frac{1}{3}} \quad f(x) = x^{\frac{1}{3}} \quad f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

when $x = 8 \quad f'(x) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

$L(x) = f(8) + f'(8)(x-8)$

$= 2 + \frac{1}{12} \times (x-8)$

$f(7.97) \approx L(7.97) = 2 - \frac{1}{400} = \frac{799}{400}$

$= 1.9975$

An important linear approximation for roots and powers is

$x \to 0$

$$\boxed{(1 + x)^k \approx 1 + kx} \qquad (x \text{ near } 0; \text{ any number } k)$$

$f'(x) = k(1+x)^{k-1}$

(Exercise 15). This approximation, good for values of x sufficiently close to zero, has broad application. For example, when x is small,

$(0,1)$ $x=0$ $f'(x)=k$

$L(x) = 1+kx$

$$\sqrt{1 + x} \approx 1 + \frac{1}{2}x \qquad k = 1/2$$

$1+(-x)(-1) = 1+x$

$$\frac{1}{1 - x} = (1 - x)^{-1} \approx 1 + (-1)(-x) = 1 + x \qquad k = -1; \text{ replace } x \text{ by } -x.$$

$$\sqrt[3]{1 + 5x^4} = (1 + 5x^4)^{1/3} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4 \qquad k = 1/3; \text{ replace } x \text{ by } 5x^4.$$

线性近似 在0处求导 $y=1$ (不好)

不是线性近似 但是用结论很好逼近

$$\frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \qquad k = -1/2; \text{ replace } x \text{ by } -x^2.$$

$(1-x^2)^{-\frac{1}{2}} = 1+\frac{1}{2}x^2$

* $\frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$ 令$x=0.01$ $\approx (1+\frac{1}{2}\times 0.01^2)m$

微分

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential *dx*** is an independent variable. The **differential *dy*** is

独立变量

切线上变化 自变量

$$dy = f'(x)\,dx.$$

改变的极小量=Δx

Δy 函数真正变化

$\frac{dy}{dx} = f'(x)$ "看上去像"

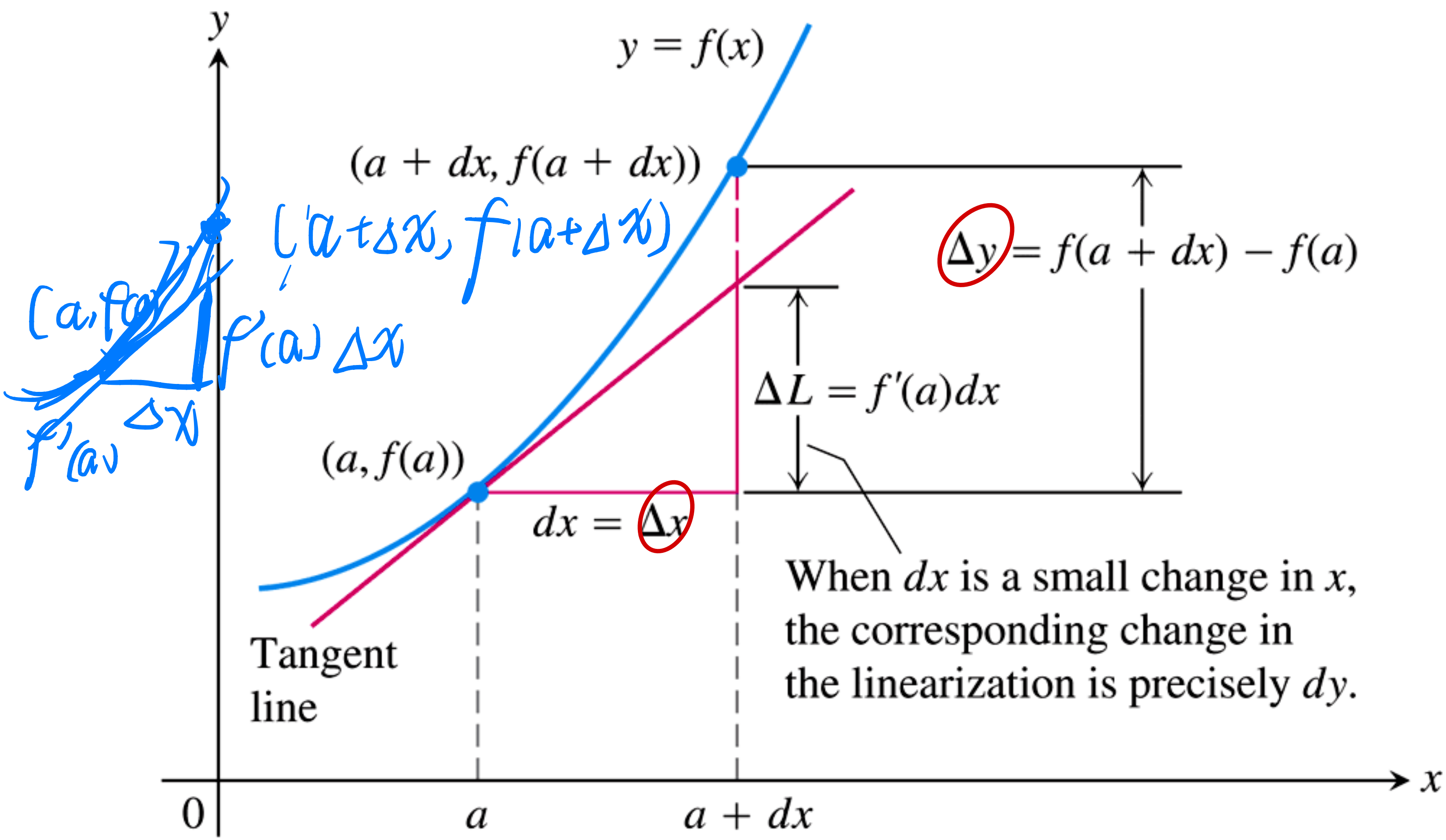


FIGURE 3.43 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

EXAMPLE 5 We can use the Chain Rule and other differentiation rules to find differentials of functions.

$dy = f'(x)\,dx$

(a) $d(\tan 2x) = \sec^2(2x)\, d(2x) = 2\sec^2 2x\, dx$

(b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$ ■

先求导后再加一个dx

EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.44). Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

$A = \pi r^2$

$dA = f'(A)\,dr$

$= 2\pi r \cdot dr = 2\pi$

$\Delta A = f(10.1) - f(10) = 2.01\pi$

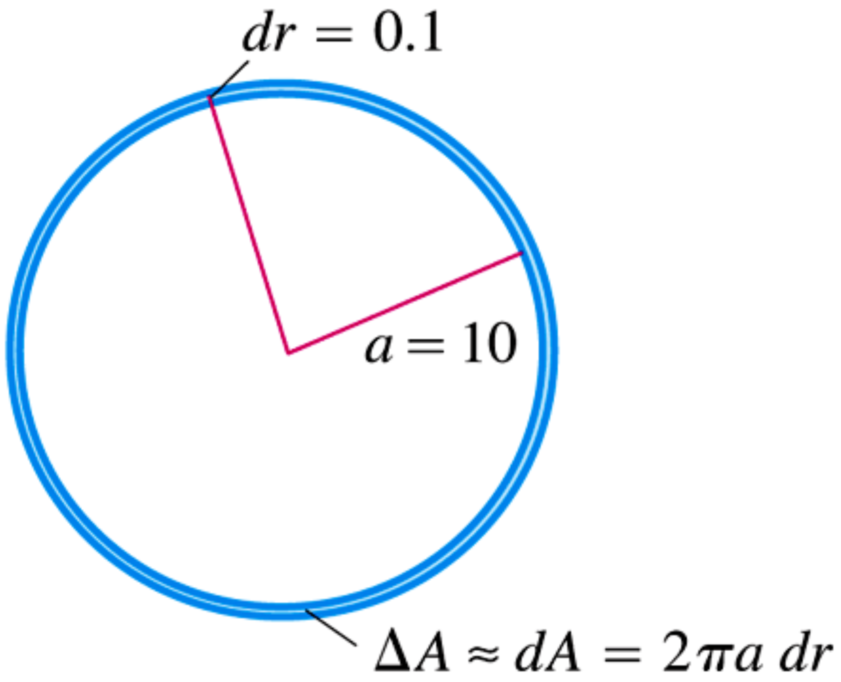


FIGURE 3.44 When dr is small compared with a, the differential dA gives the estimate $A(a + dr) = \pi a^2 + dA$ (Example 6).

写法上有区别

EXAMPLE 7 Use differentials to estimate

(a) $7.97^{1/3}$

(b) $\sin(\pi/6 + 0.01)$.

写法上不同变成:

$dy = f'(x)dx$

$\downarrow$

Δy

$f(x+\Delta x) = \Delta y + f(x)$

(a) $y = f(x) = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$dy = \frac{1}{3}x^{-\frac{2}{3}}dx$ $f'(x)\Delta x$

$f(7.97) - f(8) = \Delta y \approx dy = \frac{1}{3} \times \frac{1}{4} \times (-0.03)$

$f(7.97) \approx 2 - \frac{1}{400} = 1.9975$

(b) $y = f(x) = \sin x$

$f'(x) = \cos x$

$dy = \cos x\, dx$

$\Delta y \approx dy = \cos\frac{\pi}{6} \times 0.01$

$\Delta y = f(\frac{\pi}{6} + 0.01) - f(\frac{\pi}{6})$

准

...

Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of x. We have two ways to describe the change in f as x changes from a to $a + \Delta x$:

The true change:	$\Delta f = f(a + \Delta x) - f(a)$
The differential estimate:	$df = f'(a)\,\Delta x.$

How well does df approximate Δf?

We measure the approximation error by subtracting df from Δf:

$$\begin{aligned}
\text{Approximation error} &= \Delta f - df \\
&= \Delta f - f'(a)\Delta x \\
&= \underbrace{f(a + \Delta x) - f(a)}_{\Delta f} - f'(a)\Delta x \\
&= \underbrace{\left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)\right)}_{\text{Call this part } \epsilon.} \cdot \Delta x \\
&= \epsilon \cdot \Delta x.
\end{aligned}$$

割k-切k

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$\Delta x = dx$

$\Delta y = (f'(a) + \epsilon)\Delta x$

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \qquad (1)$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

$dy = f'(x)\,dx$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$u = g(x) \Rightarrow \Delta u = (g'(x) + \epsilon)\Delta x$

$y = f(x) \Rightarrow \Delta y = (f'(u) + \epsilon_1)\Delta u$

$g(x)$ 连续函数

$\Delta x \to 0 \quad \epsilon_1 \to 0 \quad \epsilon_2 \to 0$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\Delta y = (f'(g(x)) + \epsilon_1)(g'(x) + \epsilon_2)\Delta x$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (f'(g(x)) + \epsilon_1)(g'(x) + \epsilon_2)$$

$$= f'(g(x)) \cdot g'(x)$$

练习：哪个在0可导 T₁ (A) $|x|\sqrt{\sin x+2}$ (B) $|x|+\sqrt{\sin x+2}$

(C) $|x|\sin x$ (D) $|x|+\sin x$

定义

加法用导数四则

A: $\lim\limits_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim\limits_{h\to 0^+}\frac{h\sqrt{\sin h+2}}{h}=\sqrt{2}$

$\lim\limits_{h\to 0^-}\frac{-h\sqrt{\sin h+2}}{h}=-\sqrt{2}$

$f(x)=\sqrt{x+\sqrt{x+\sqrt{x}}}$

	True	**Estimated**
Absolute change	$\Delta f = f(a+dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$