1) 有理化
2) 重要比例极限 $\frac{\sin x}{x}$, $\frac{\tan x}{x}$
3) 代入
4) 同构类似式子
5) [ ] 扩展成范围

Chapter 2

Limits and Continuity

2.1

Rates of Change and Tangents to Curves

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}$$

Length of time interval h	**Average speed over interval of length h starting at $t_0 = 1$**	**Average speed over interval of length h starting at $t_0 = 2$**
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049	19.60049

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

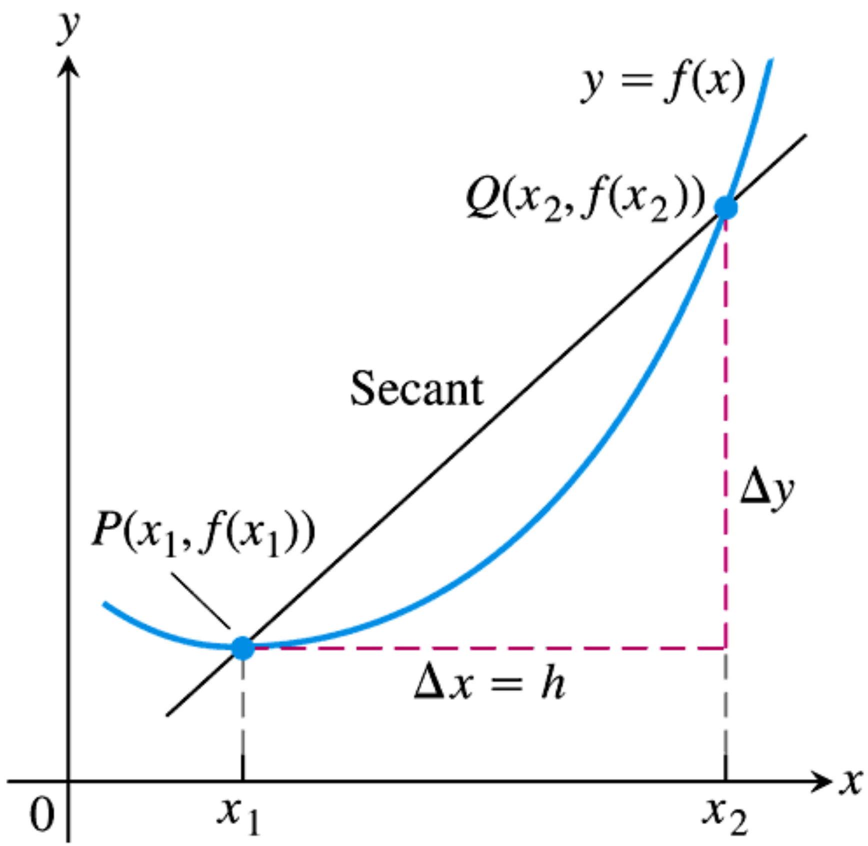


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

To define tangency for general curves, we need an approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). Here is the idea:

1. Start with what we *can* calculate, namely the slope of the secant PQ.
2. Investigate the limiting value of the secant slope as Q approaches P along the curve. (We clarify the *limit* idea in the next section.)
3. If the *limit* exists, take it to be the slope of the curve at P and *define* the tangent to the curve at P to be the line through P with this slope.

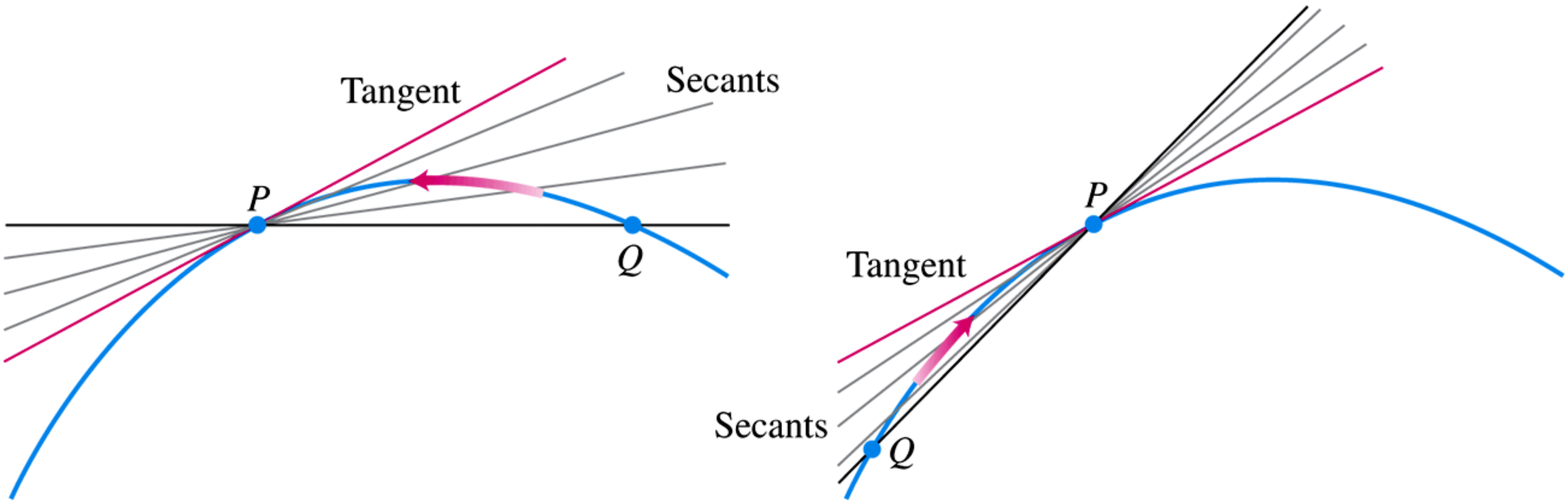


FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

EXAMPLE 3 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

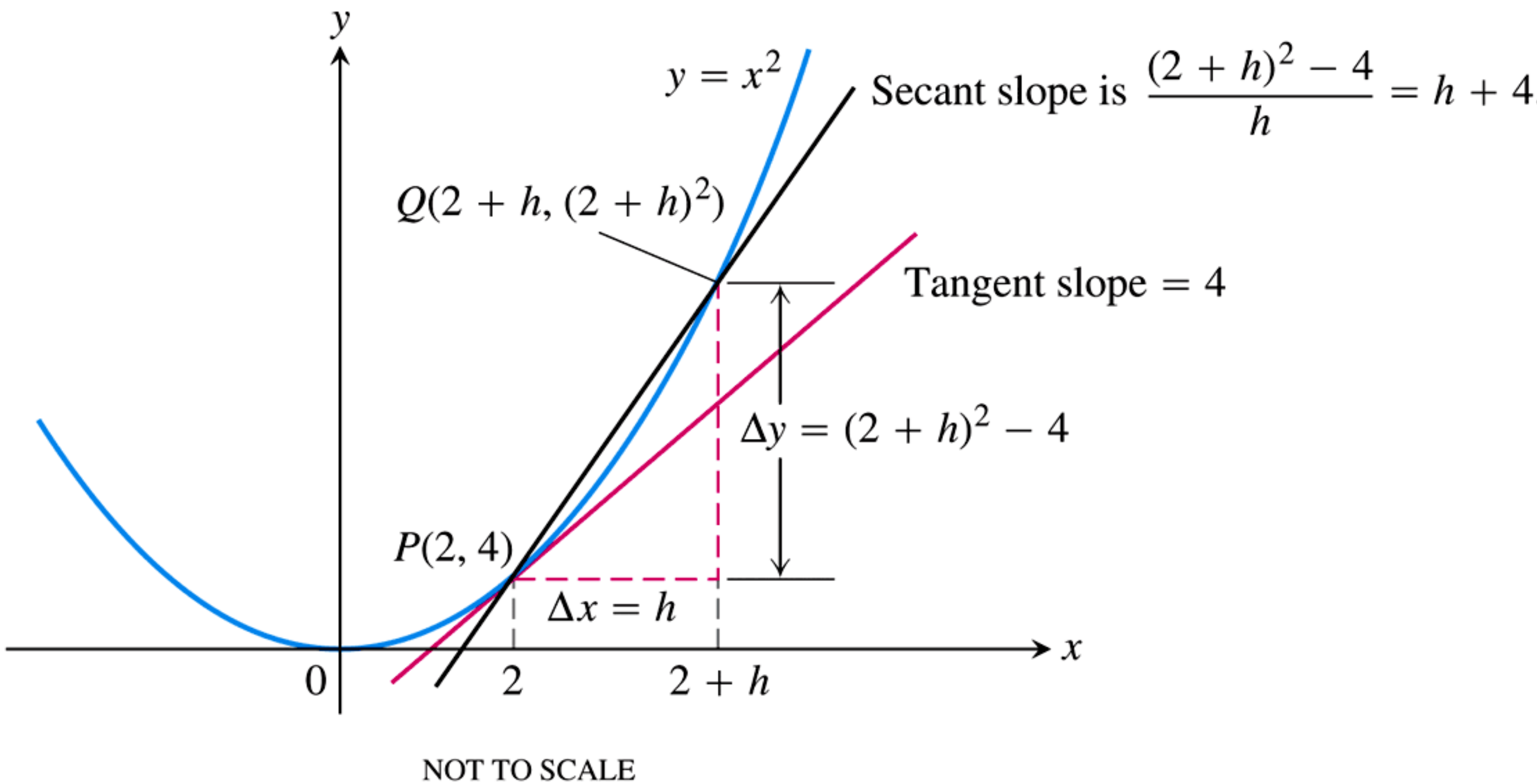


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

2.2

Limit of a Function and Limit Laws

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

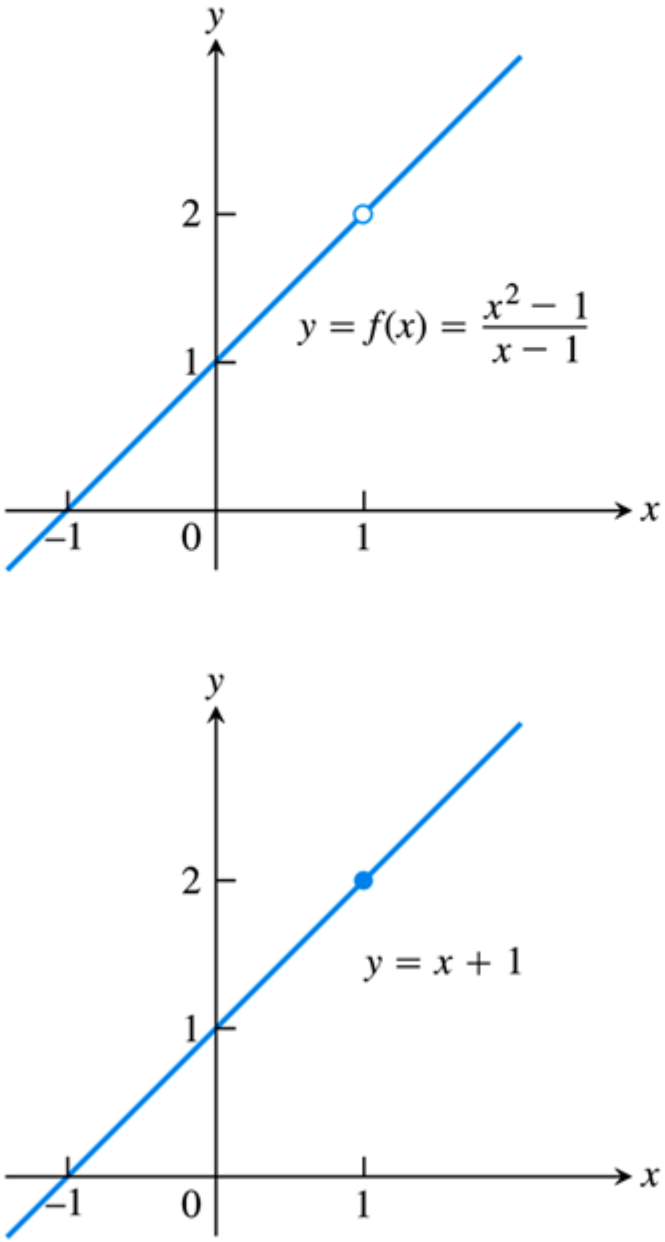
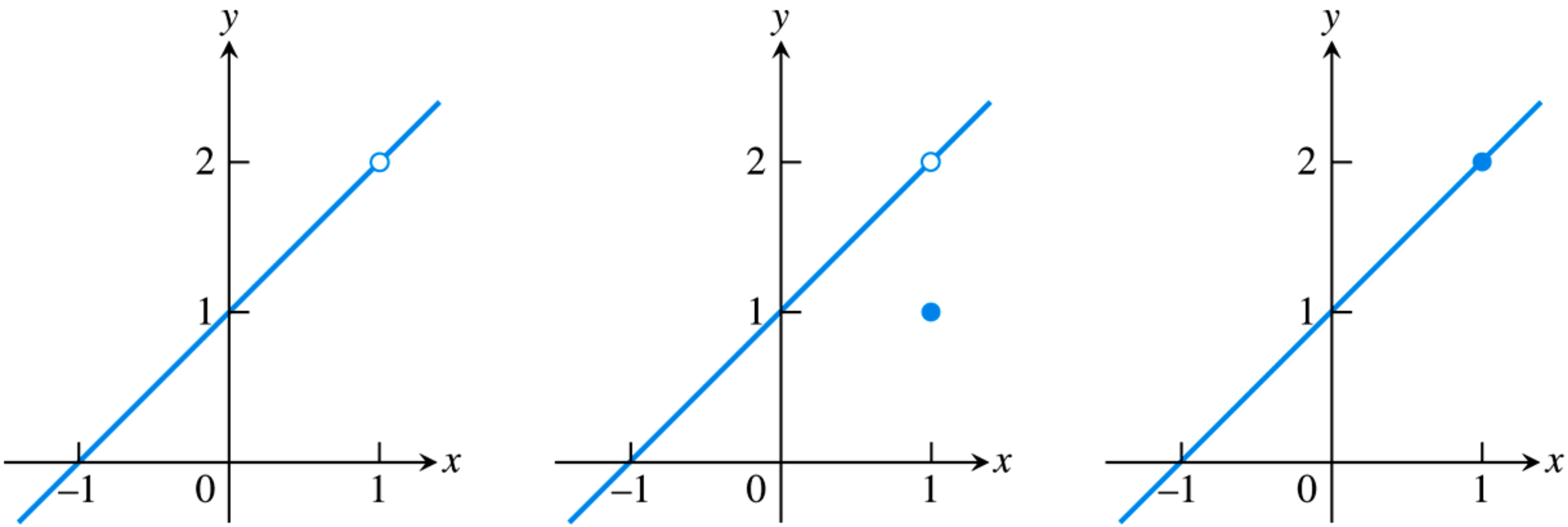


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

EXAMPLE 2 The limit value of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \to 1$ equals its value at $x = 1$. For h, we have $\lim_{x \to 1} h(x) = h(1)$. This equality of limit and function value is of special importance, and we return to it in Section 2.5. ■



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

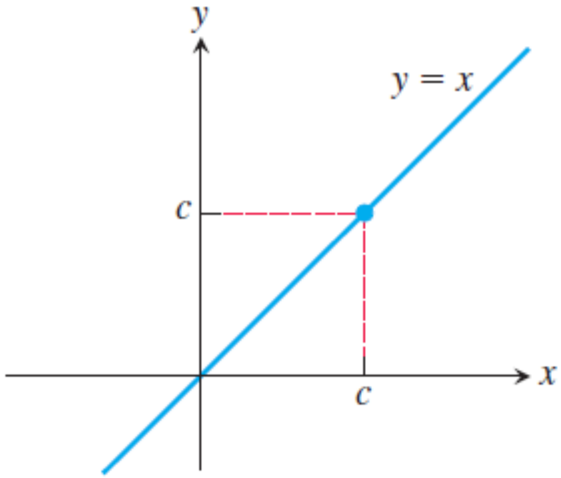
EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

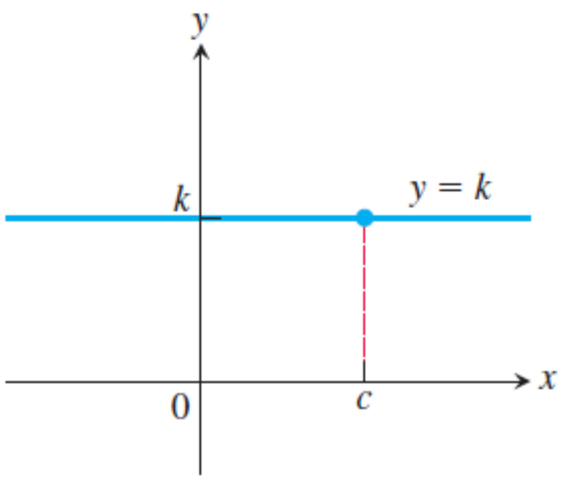
$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$



(a) Identity function



(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points c.

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\dfrac{1}{x}, & x > 0 \end{cases}$

* 有极限：左极限 = 右极限

有无极限与该点函数无关

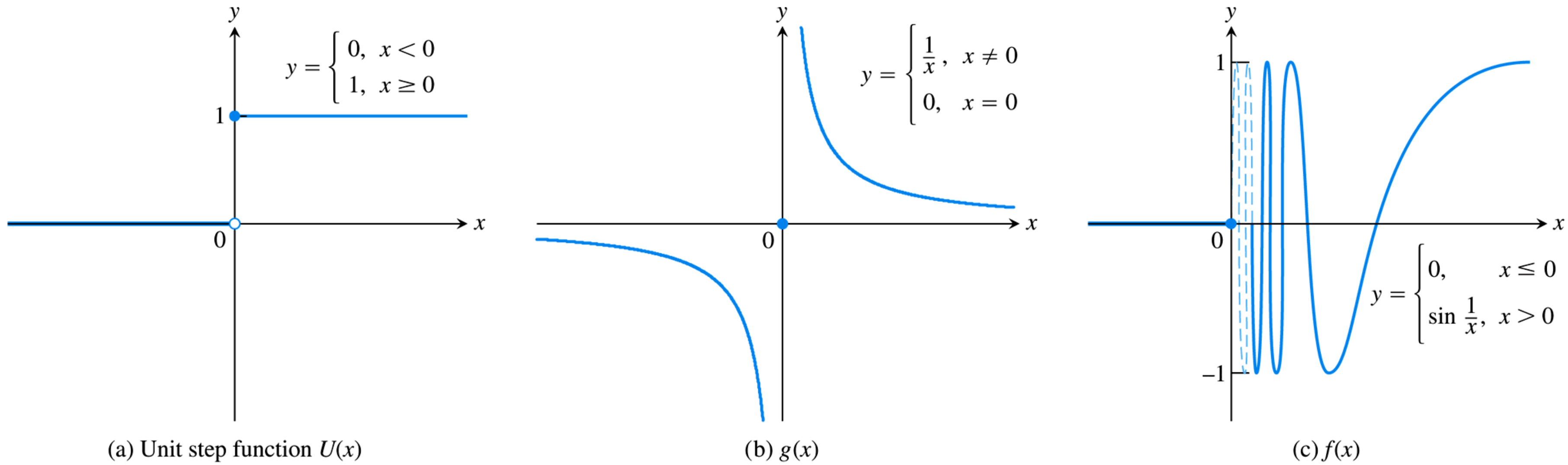


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

多項式

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Identifying Common Factors

It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

EXAMPLE 7 Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

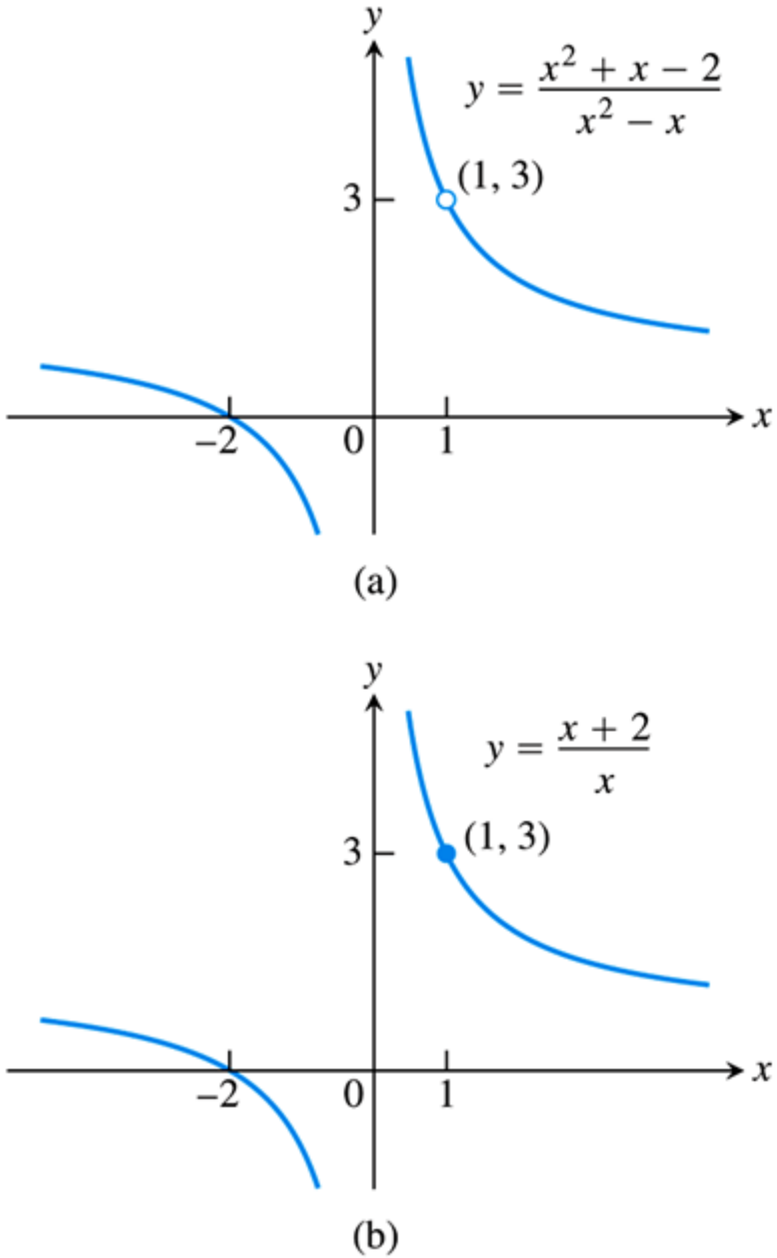


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 7).

EXAMPLE 8 Estimate the value of $\lim_{x \to 0} \dfrac{\sqrt{x^2 + 100} - 10}{x^2}$.

TABLE 2.3 Computed values of $f(x) = \dfrac{\sqrt{x^2 + 100} - 10}{x^2}$ near $x = 0$

x	$f(x)$	
± 1	0.049876	approaches 0.05?
± 0.5	0.049969	
± 0.1	0.049999	
± 0.01	0.050000	
± 0.0005	0.050000	approaches 0?
± 0.0001	0.000000	
± 0.00001	0.000000	
± 0.000001	0.000000	

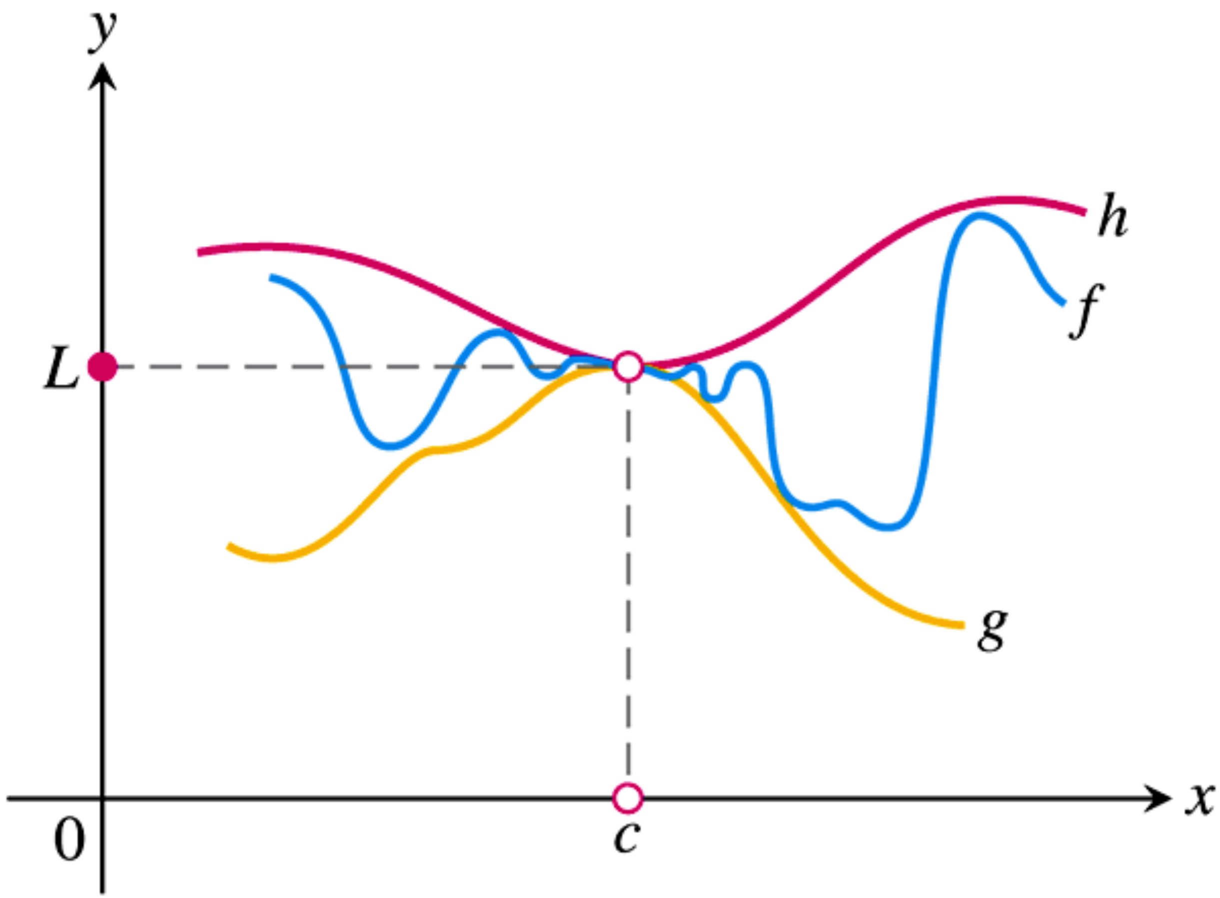


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

①大小关系

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

②

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

EXAMPLE 10 Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \qquad \text{for all } x \ne 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

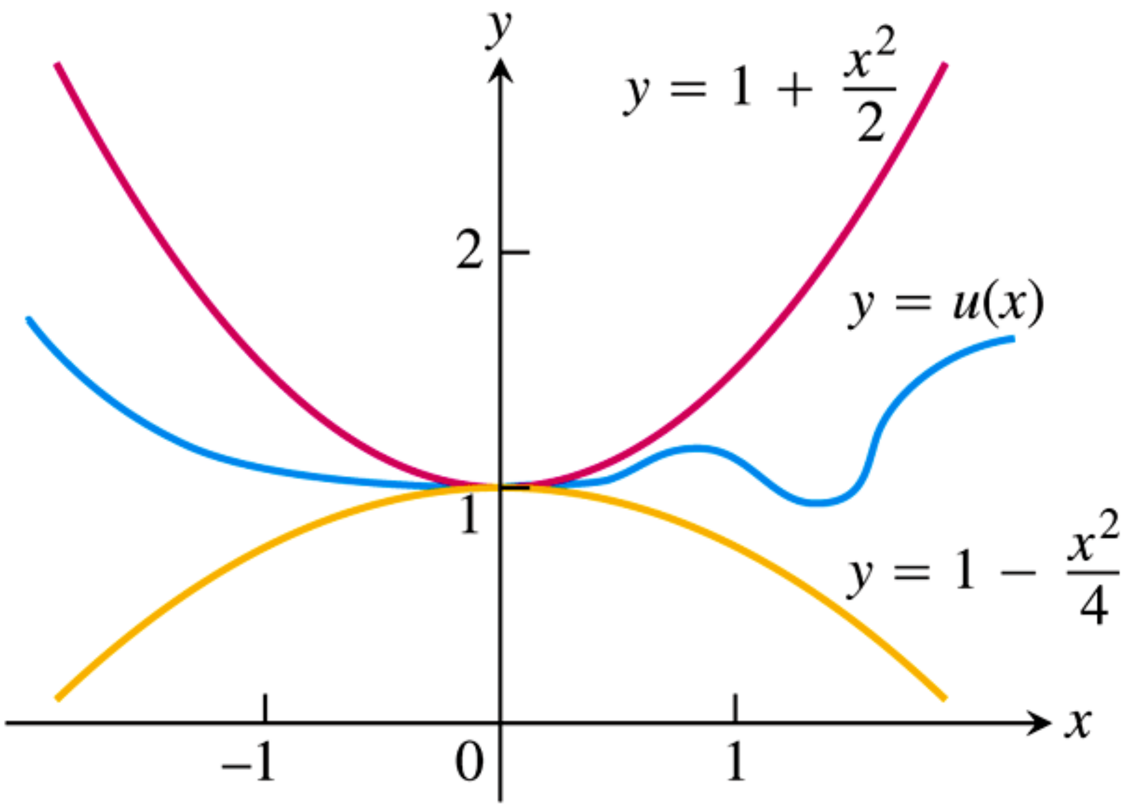


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 10).

EXAMPLE 11 The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta \to 0} \sin \theta = 0$

(b) $\lim_{\theta \to 0} \cos \theta = 1$

(c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

$-|f(x)| \leq f(x) \leq |f(x)|$

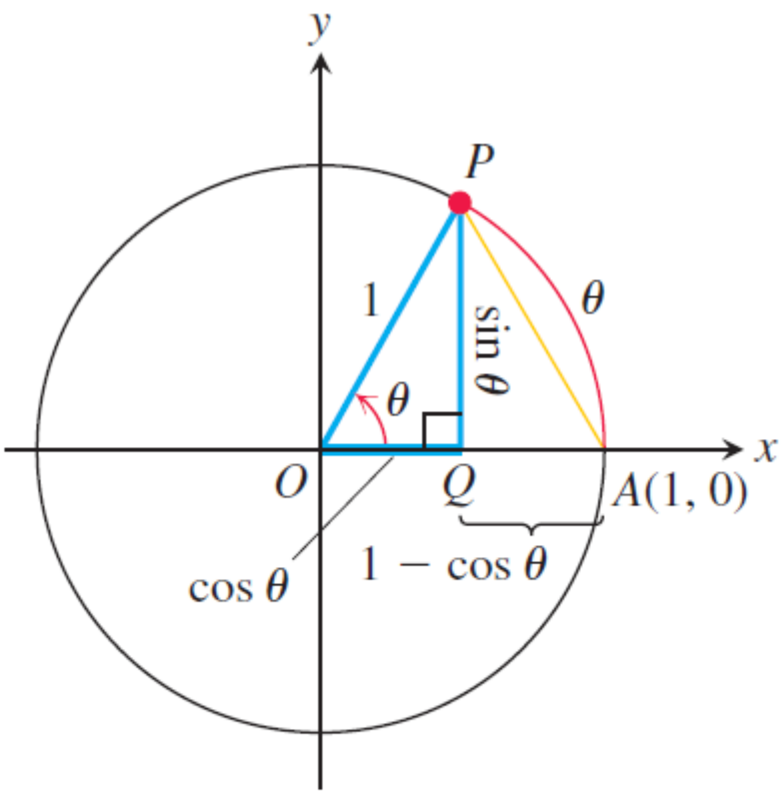
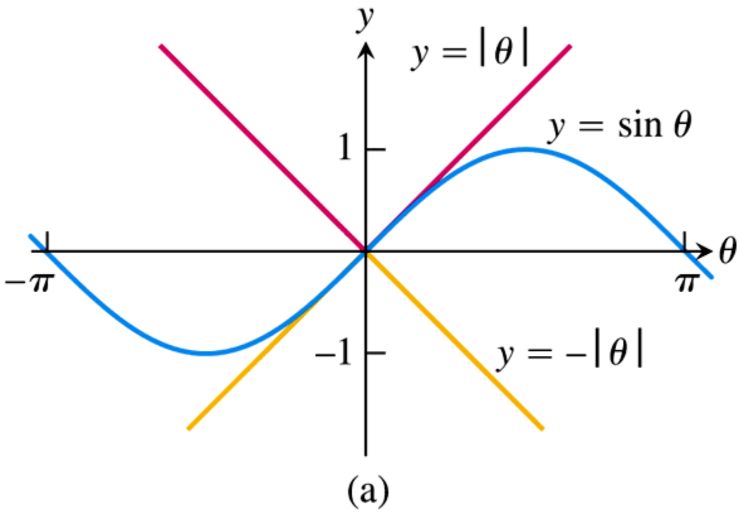


FIGURE 1.47 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2 \theta + (1 - \cos \theta)^2 \leq \theta^2$.



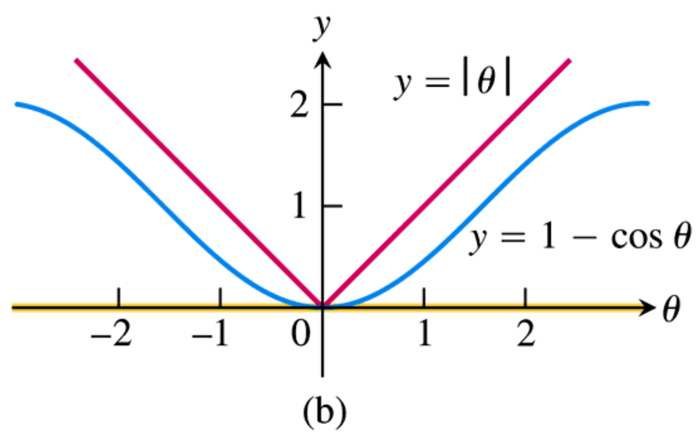


FIGURE 2.14 The Sandwich Theorem confirms the limits in Example 11.

＊讨论极限只看周围的点，

改成< 结论仍不变

THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

极限可以取等

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

2.3

严格定义

The Precise Definition of a Limit

$\lim_{x \to x_0} f(x) = L$

$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$

足够近　　　足够小

DEFINITION Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

离得充分近　$\exists$

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

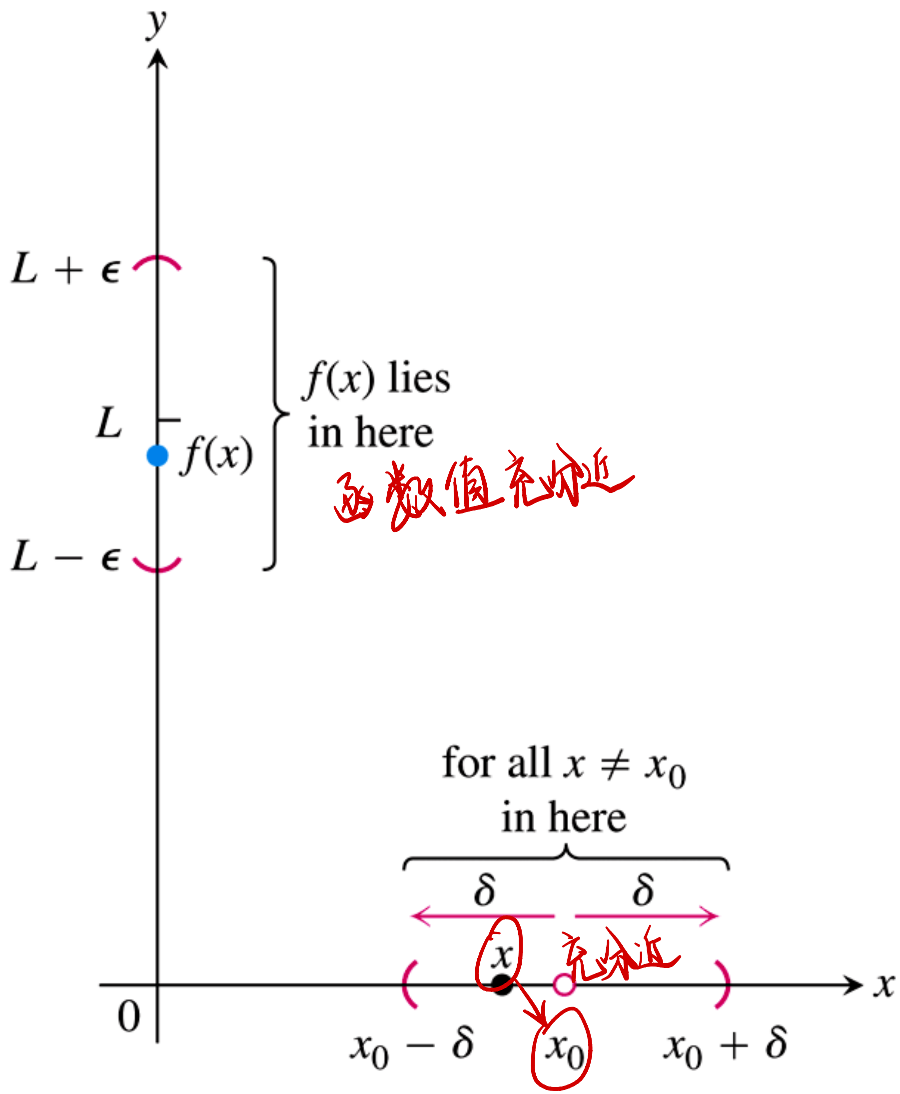


FIGURE 2.17 The relation of δ and ϵ in the definition of limit.

EXAMPLE 2 Show that

$$\lim_{x \to 1} (5x - 3) = 2.$$

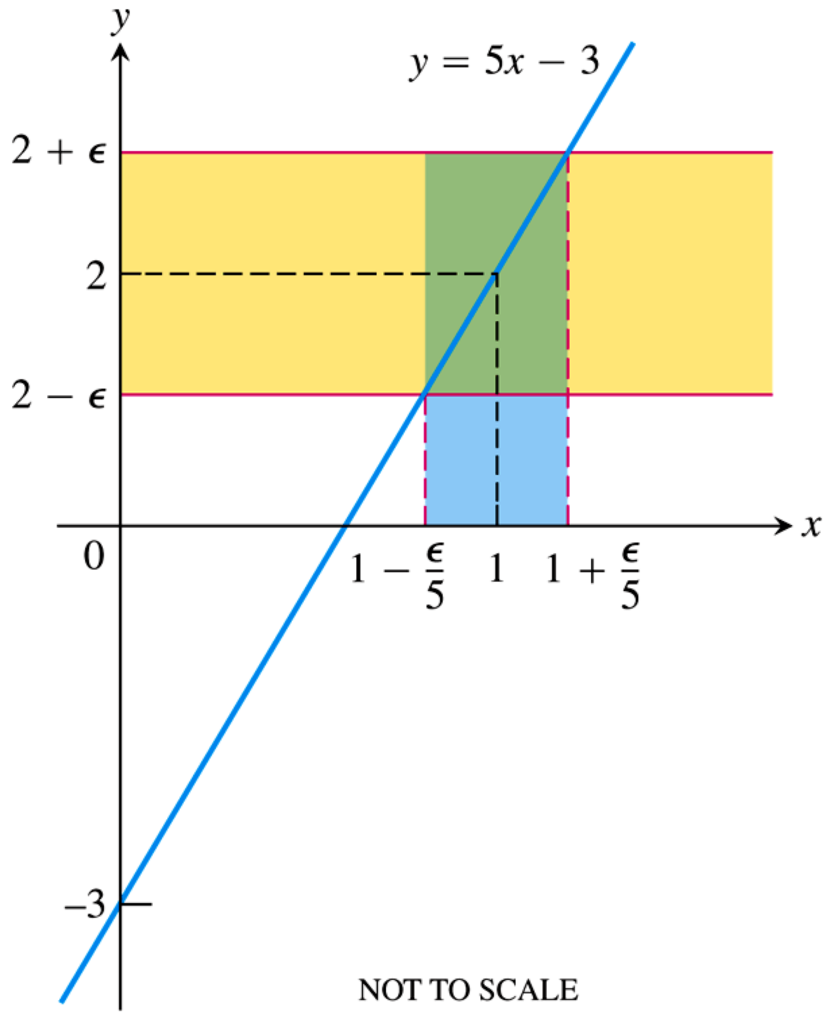


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

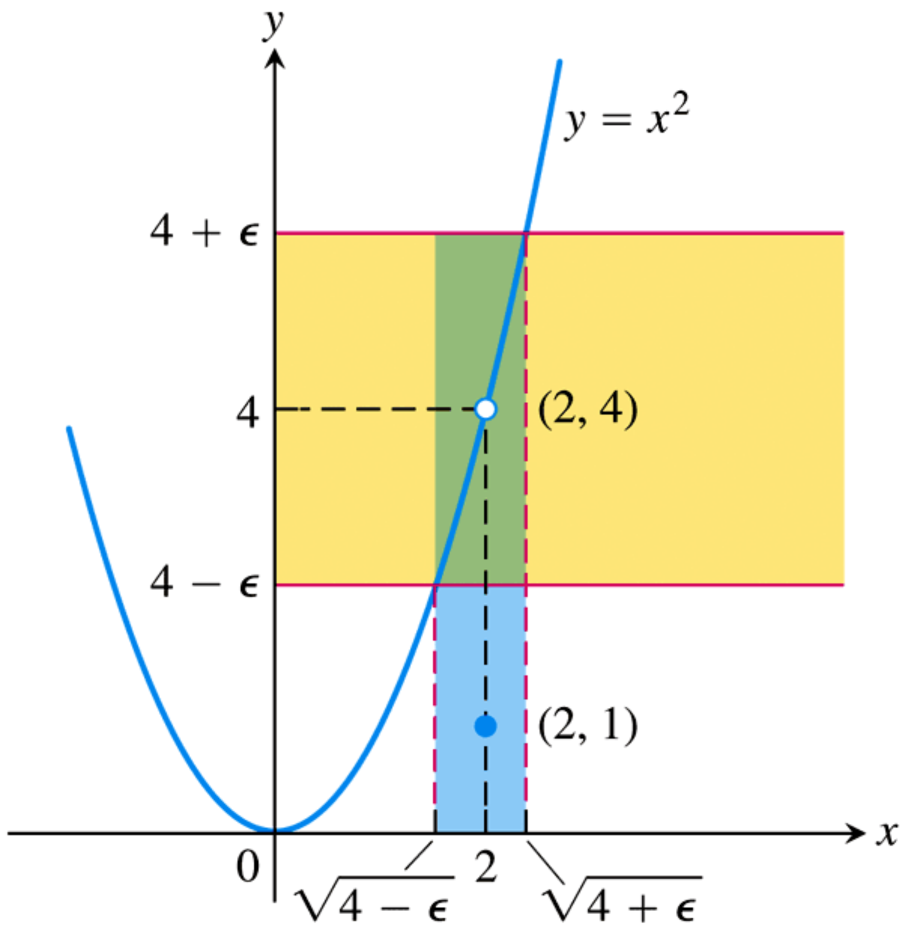
$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

EXAMPLE 5 Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$



$y = xe^x + \sin x$

when $x = 1$ $\quad e + \sin 1$

$|xe^x + \sin x - (e + \sin 1)| < \varepsilon$

函数充分简单才可用

反解

FIGURE 2.23 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

代入x

2.4

One-Sided Limits

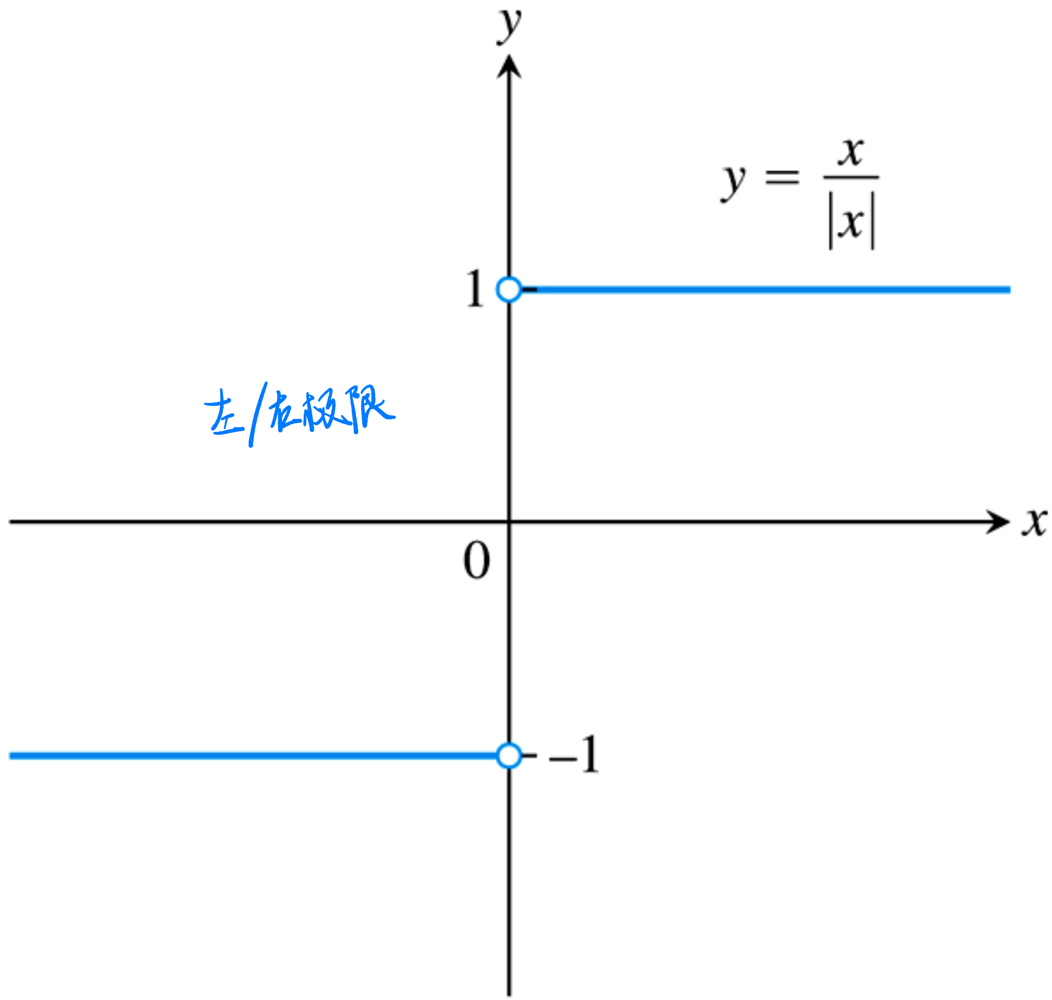
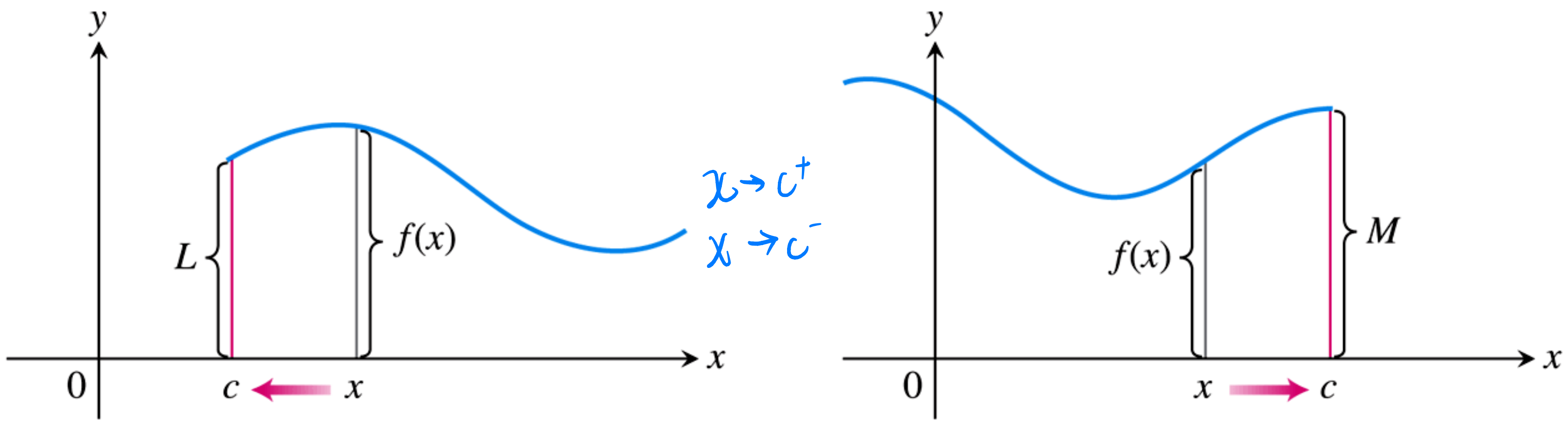


FIGURE 2.24 Different right-hand and left-hand limits at the origin.



$x \to c^+$

$x \to c^-$

(a) $\lim_{x \to c^+} f(x) = L$

比c大、从右逼近

(b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

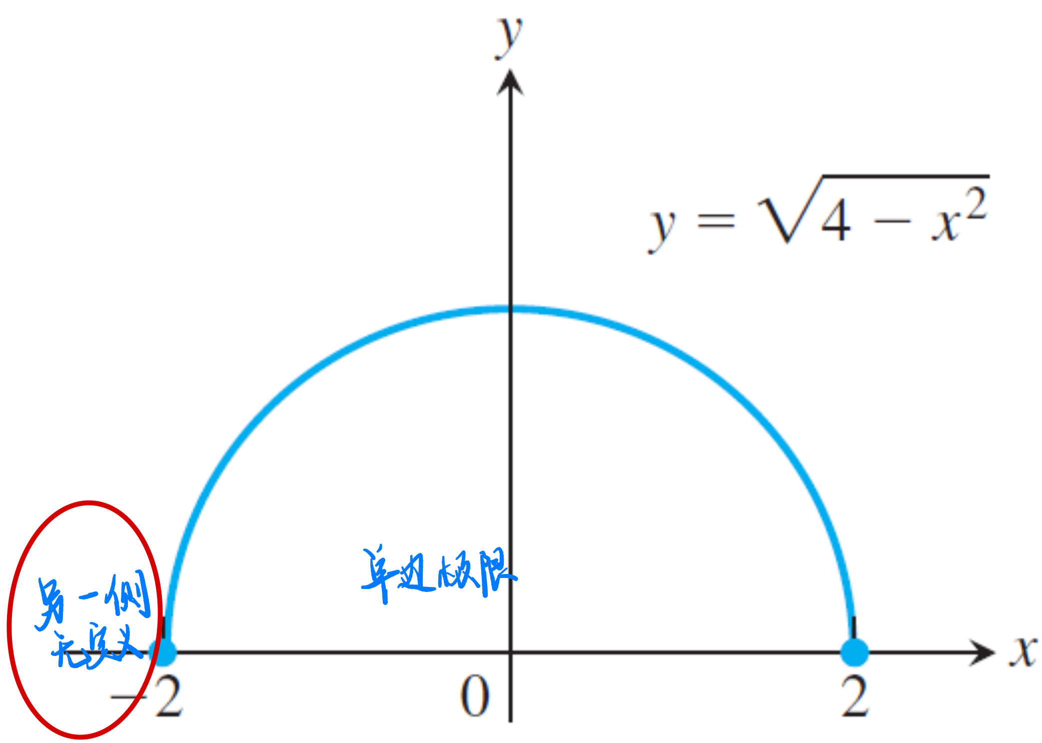


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has right-hand limit 0 at $x = -2$ and left-hand limit 0 at $x = 2$ (Example 1).

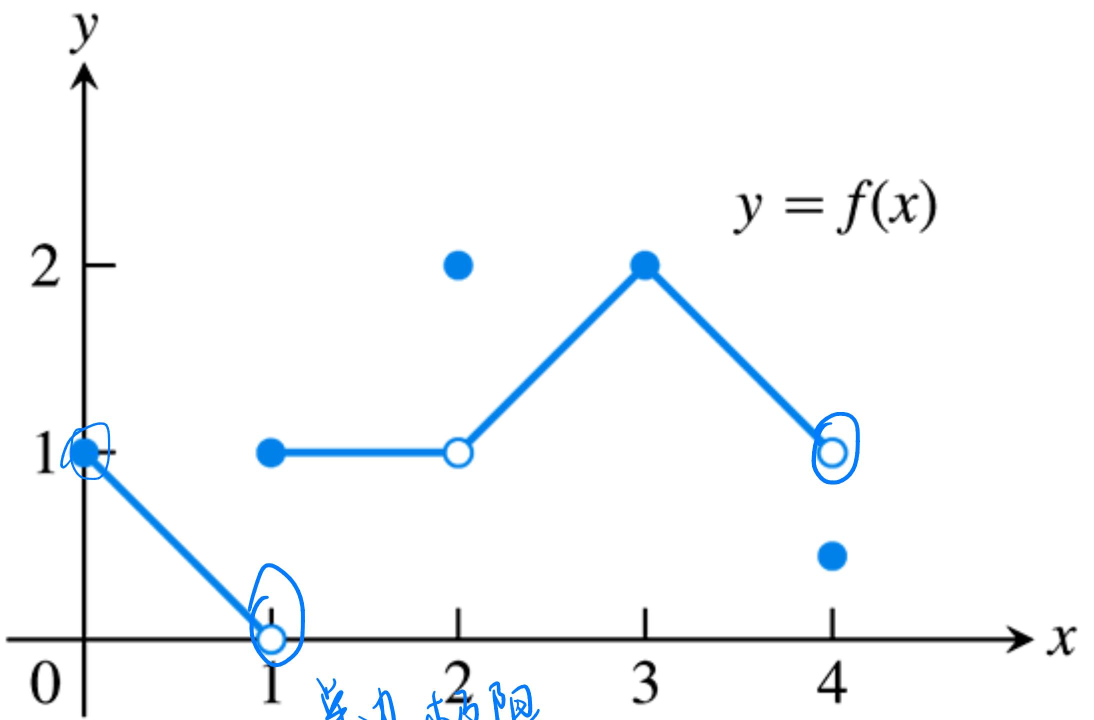
*

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it ① has left-hand and right-hand limits there and ② these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

有左、右
相等

左=右



单边极限

FIGURE 2.27 Graph of the function in Example 2.

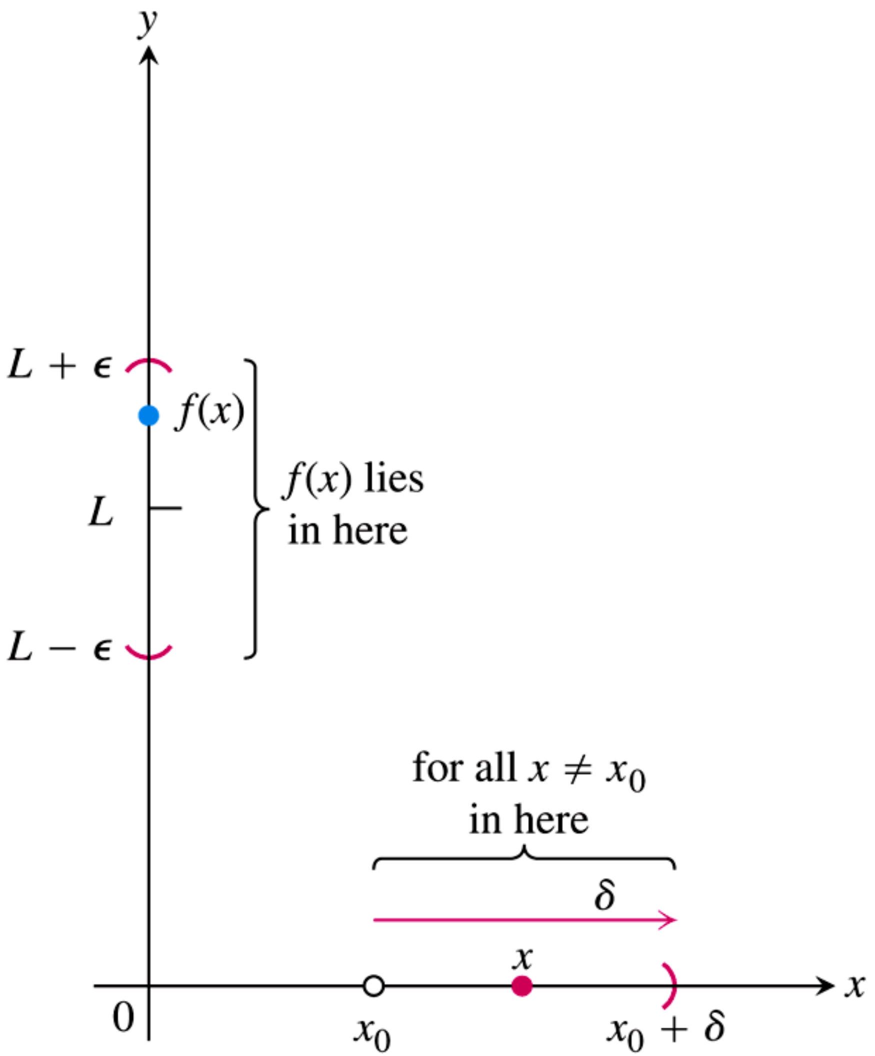


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

DEFINITIONS We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L \qquad \text{(see Figure 2.28)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c < x < c + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L \qquad \text{(see Figure 2.29)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c - \delta < x < c \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

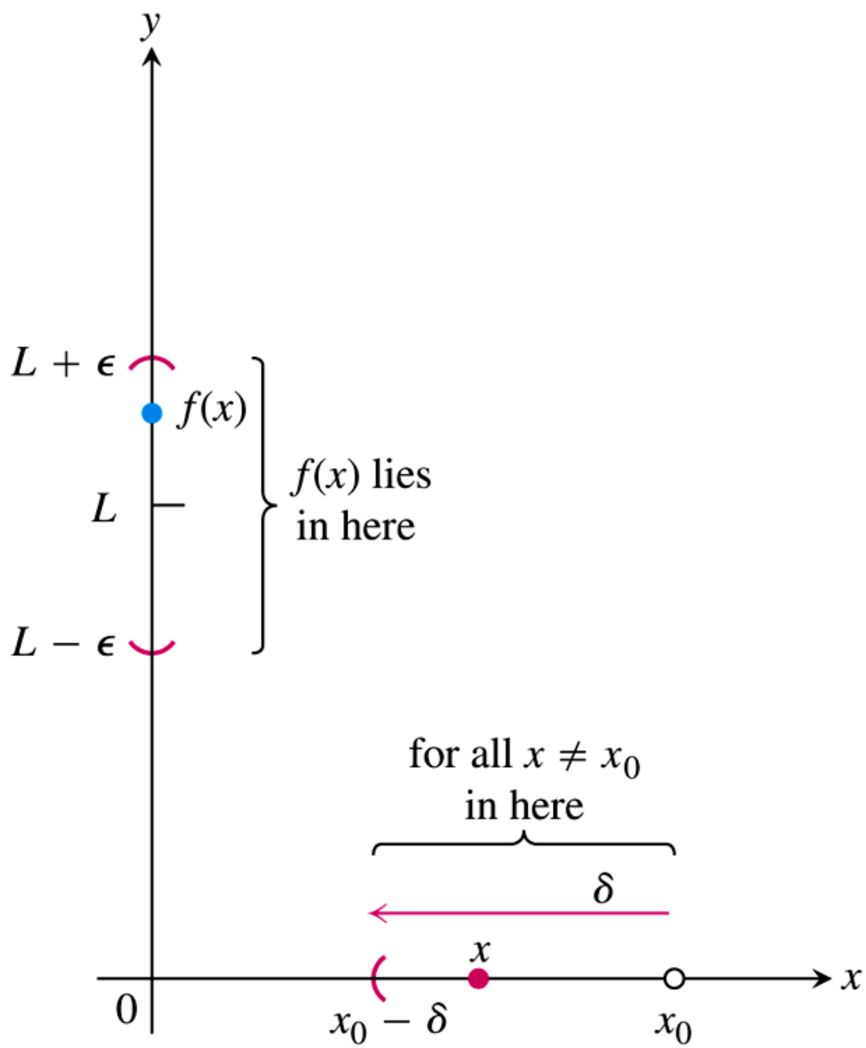


FIGURE 2.29 Intervals associated with the definition of left-hand limit.

EXAMPLE 3 Prove that

$$\lim_{x\to 0^+} \sqrt{x} = 0.$$

*四则运算对单边极限适用

$\lim_{x\to 0^+} x = 0$

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.31).

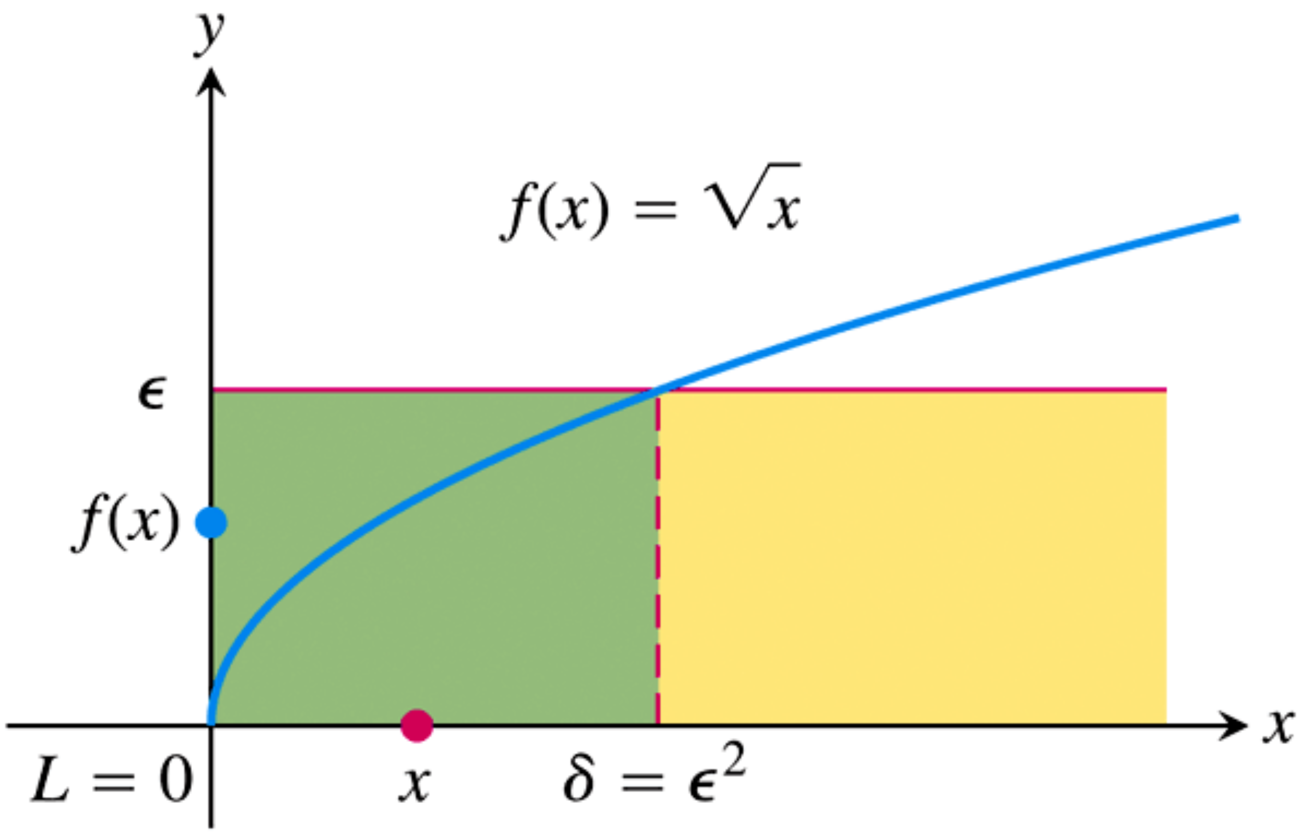


FIGURE 2.30 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

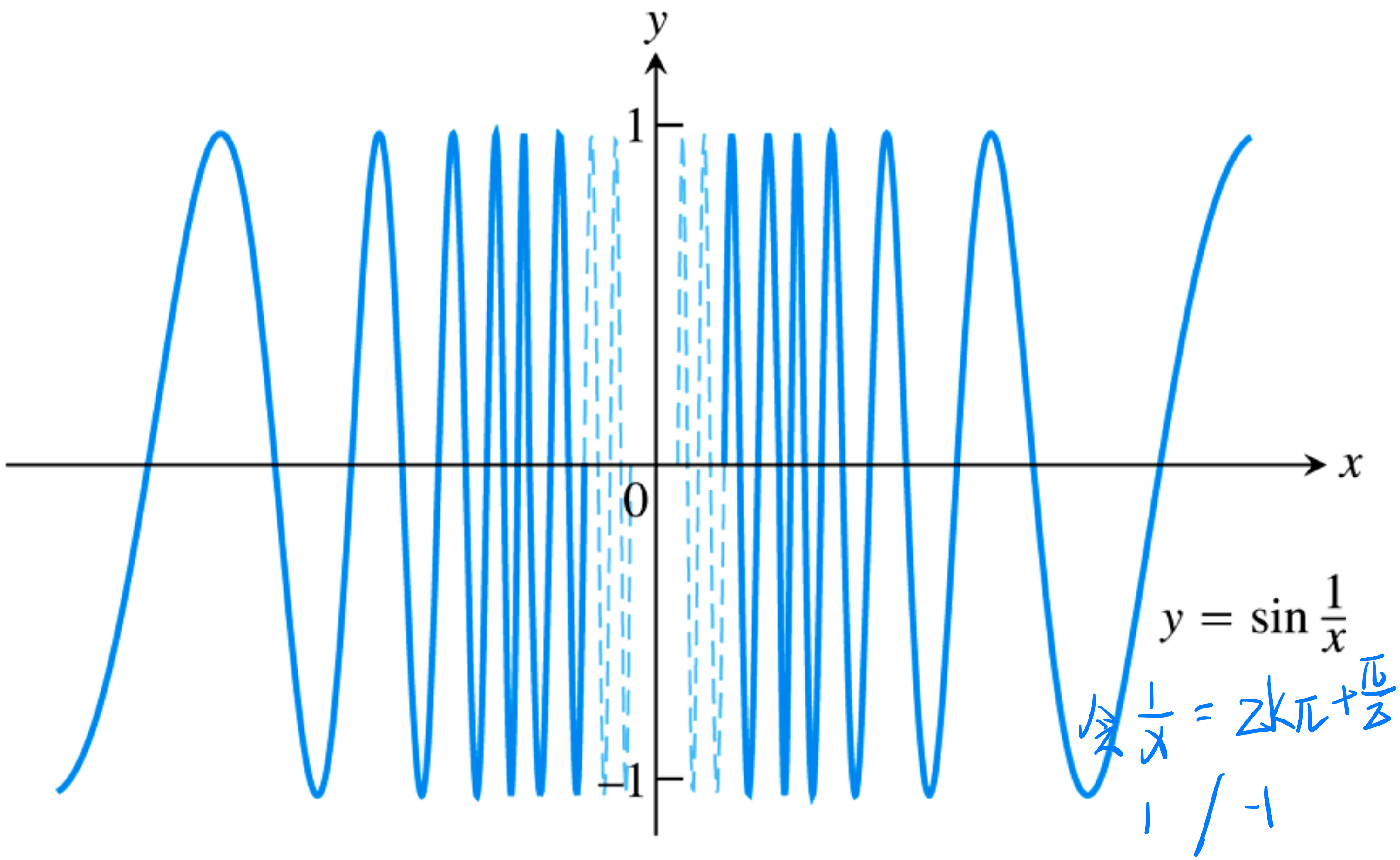


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

$\lim\limits_{x \to 0^+} \frac{\sin\theta}{\theta}$ 单位圆

$0 < x < \frac{\pi}{2}$

$\sin\theta < \theta < \tan\theta$

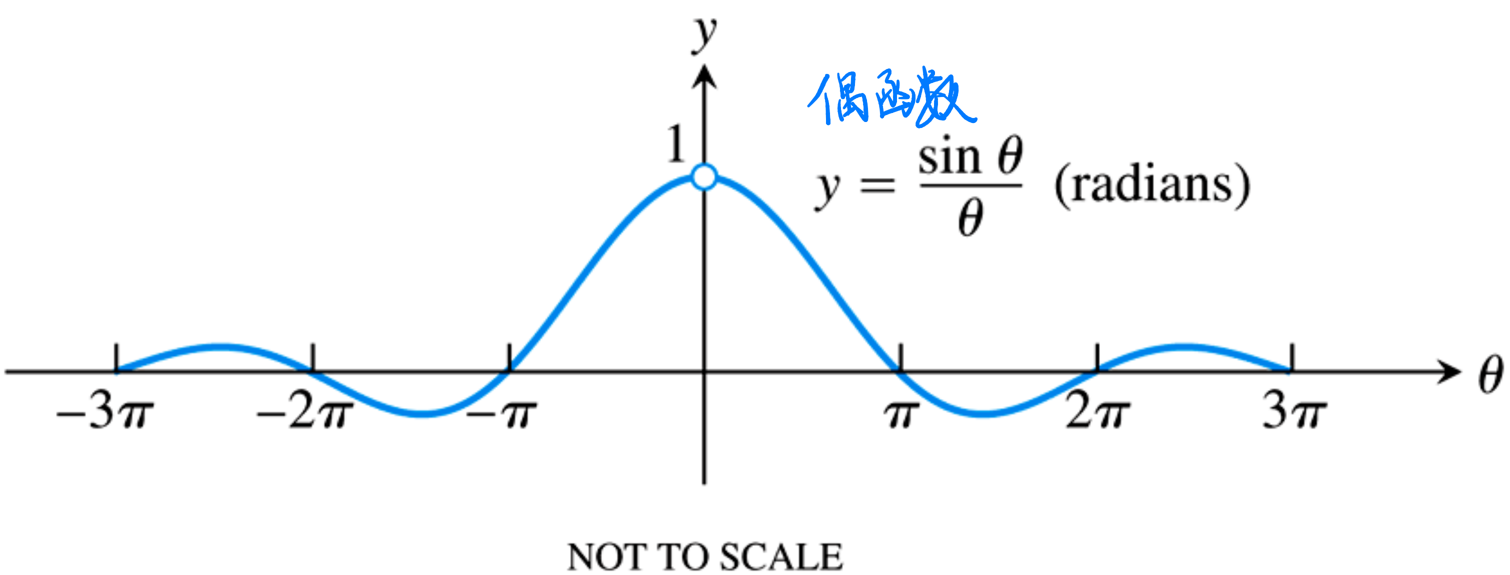


FIGURE 2.32 The graph of $f(\theta) = (\sin\theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

***

THEOREM 7—Limit of the Ratio sin θ/θ as $\theta \to 0$

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1} \qquad (\theta \text{ in radians}) \qquad (1)$$

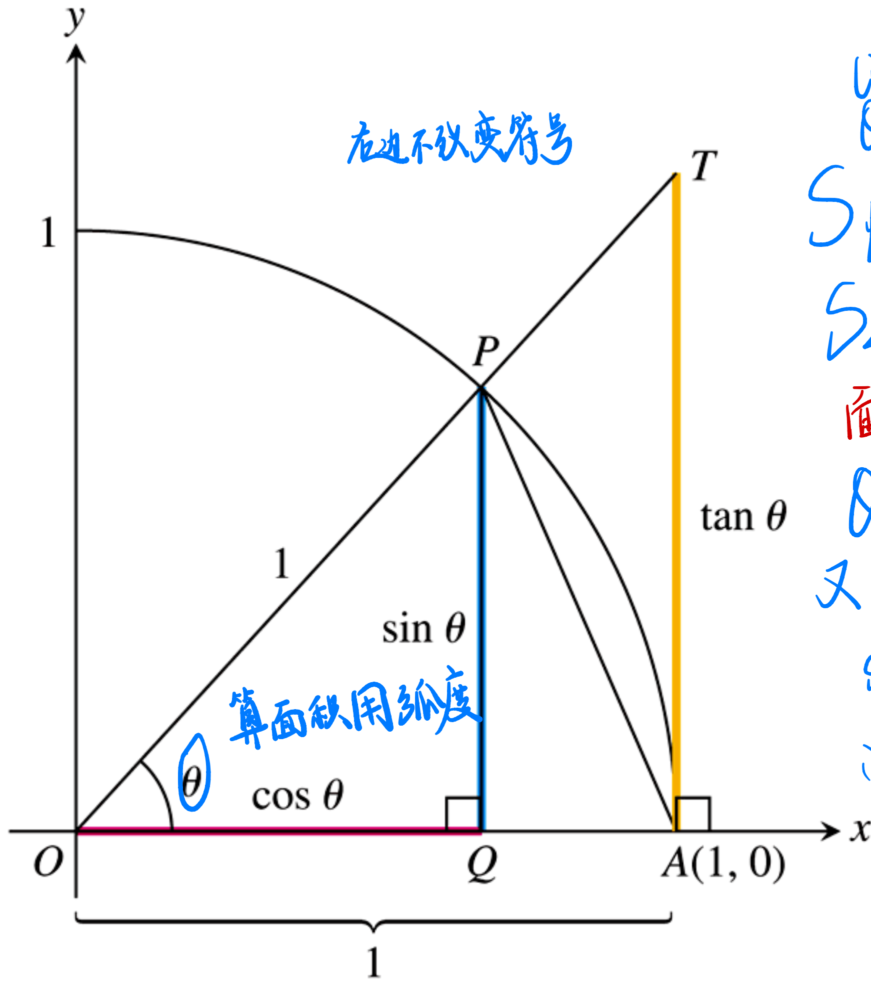


FIGURE 2.33 The figure for the proof of Theorem 7. By definition, $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

$\lim_{\theta \to 0^+} \frac{\theta}{\sin\theta}$

$S_{POA} = \frac{1}{2}\theta$

$S_{\triangle TOA} = \frac{1}{2}\tan\theta$

面积

$\theta < \tan\theta$

又 $\sin\theta < \theta$

$\sin\theta < \theta < \tan\theta$

$\therefore 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

夹逼 1

改变：$\theta^\circ = \frac{\theta\pi}{180}$ rad

$\lim_{\theta \to 0} \frac{\sin\theta^\circ}{\theta} = \frac{\pi}{180} \lim_{\theta \to 0} \frac{\sin\frac{\theta\pi}{180}}{\frac{\theta\pi}{180}} = \frac{\pi}{180}$ 换元

角度转化

$\theta^\circ = \frac{\theta\pi}{180}$

不建议用等价无穷小 $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

利泰勒展开

$$\lim_{x\to 0}\frac{x\sin x}{\tan^2 3x}=\lim_{x\to 0}\frac{\frac{x\sin x}{x\cdot x}}{\frac{\tan^2 3x}{x^2}}=\lim_{x\to 0}\frac{1}{\frac{\tan^2 3x}{(3x)^2}\times 9}=\frac{1}{9}$$

$$\lim_{x\to 0}\frac{\tan x}{x}=1$$

配出"等阶"形式

EXAMPLE 5 Show that (a) $\lim_{h\to 0}\frac{\cos h-1}{h}=0$ and (b) $\lim_{x\to 0}\frac{\sin 2x}{5x}=\frac{2}{5}$.

$$\lim_{h\to 0}\frac{-2\sin^2\frac{h}{2}}{\frac{h}{2}\times 2}=\lim_{h\to 0}-\sin\frac{h}{2}=0$$

$$\cos 2A+1=2\cos^2 A \iff \cos\theta+1=2\cos^2\frac{\theta}{2}$$

$$1-2\cos 2A=2\sin^2 A \iff 1-2\cos\theta=2\sin^2\frac{\theta}{2}$$

$$\lim_{x\to 0}\frac{x^2\sin x-x\tan x+x^6}{\tan^2 3x-x\sin x\tan 2x}=\lim_{x\to 0}\frac{\sin x-\frac{\tan x}{x}+x^2}{\frac{\tan^2 3x}{x^2}-\frac{\sin x}{x}\tan 2x}=\frac{0-1+0}{3^2-1\times 0}=-\frac{1}{8}$$

像: $\frac{x^3-x^2+x^6}{(3x)^2-2x^3}$ 除以 x^2

⇒ 知道次数最低的是什么

最低的项不能完全抵消

$$\lim_{x\to 0}\frac{\sin(\sin(\sin x))}{x}$$ 同结论

$$=\lim_{x\to 0}\frac{\sin(\sin(\sin x))}{\sin(\sin x)}\cdot\frac{\sin(\sin x)}{x}$$

$$=\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}\cdot\frac{\sin x}{x}$$

$$=1$$

$$\lim_{\boxed{x\to 1}}\frac{\sin(\sin x)}{x}=\sin(\sin 1)$$

2.5

Continuity

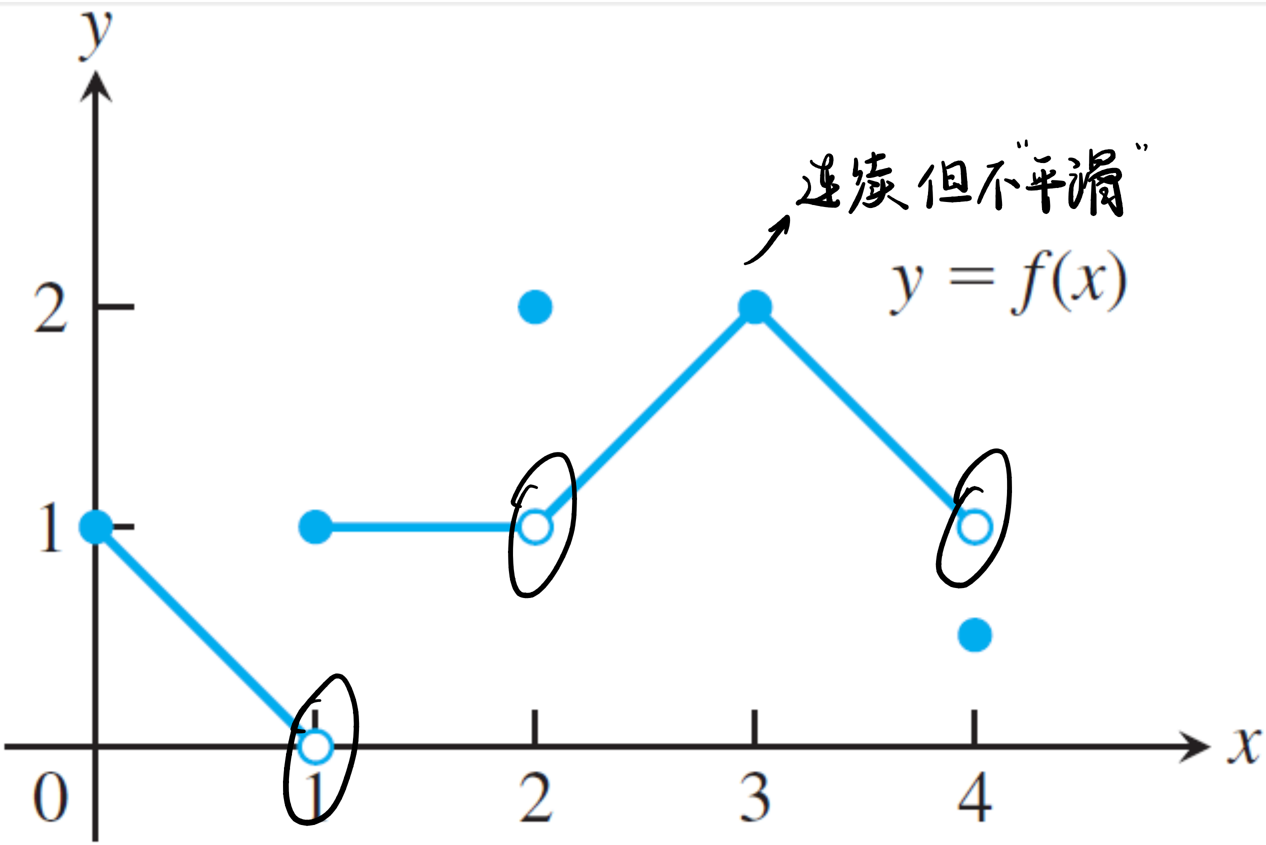


FIGURE 2.35 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

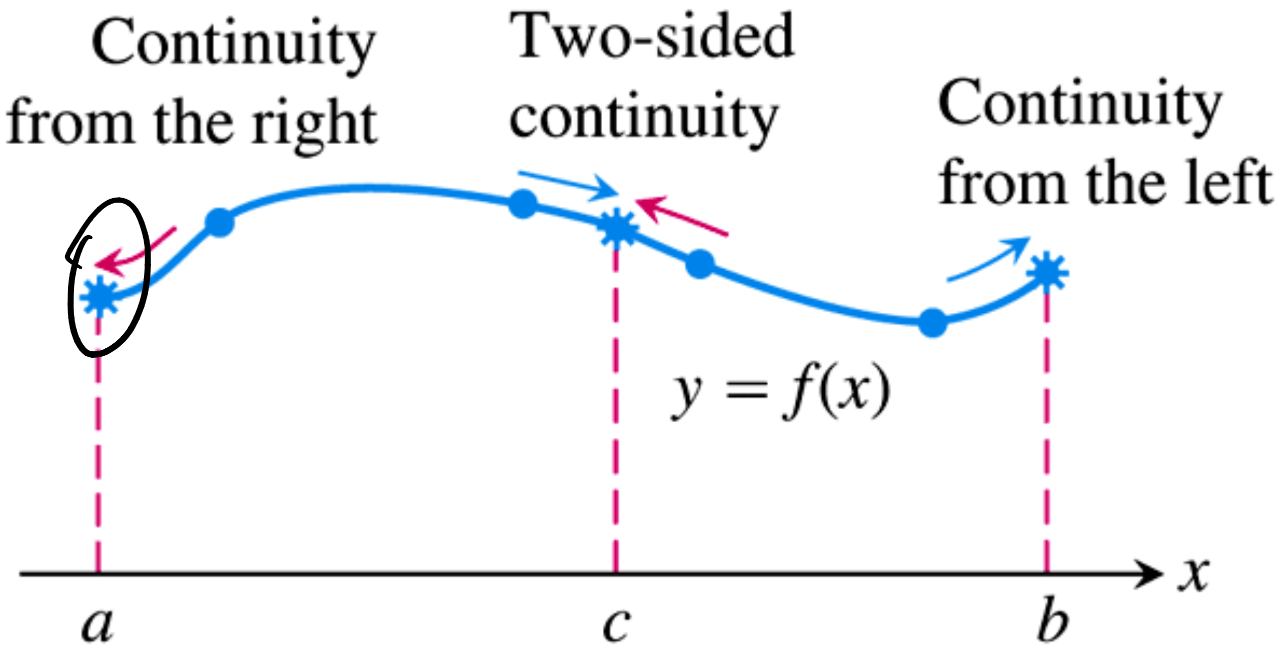


FIGURE 2.36 Continuity at points a, b, and c.

DEFINITIONS Let c be a real number on the x-axis.

连续的3个条件 $f(c)$

The function f is **continuous at c** if ①极限存在 ②有定义

$$\lim_{x \to c} f(x) = f(c).$$ ③极限=函数值

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c).$$

From Theorem 6, it follows immediately that a function f is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.36). We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior points of the interval.

* 端点只要单边连续

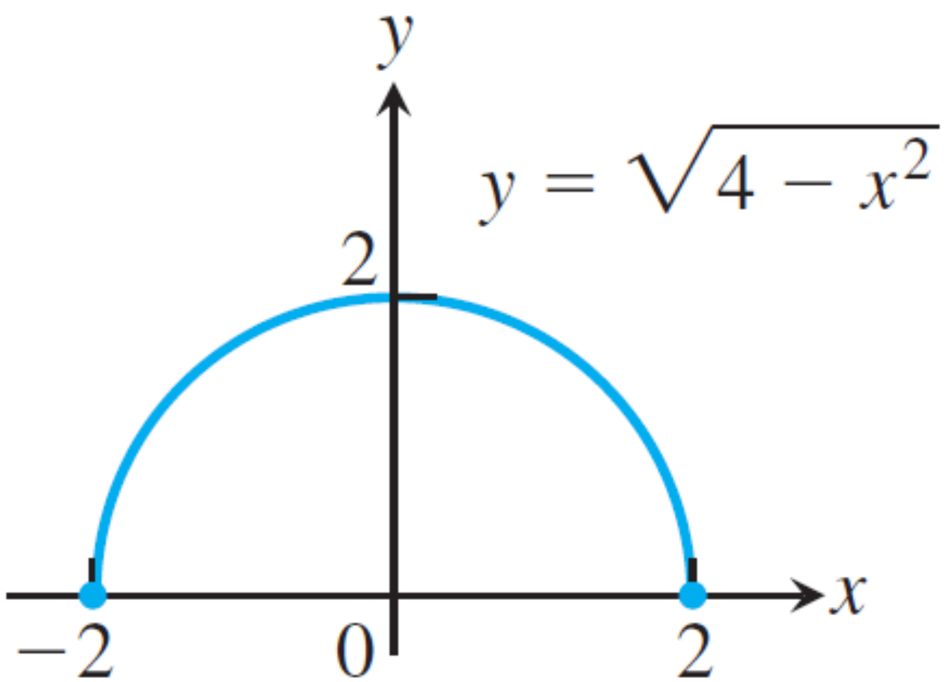


FIGURE 2.37 A function that is continuous over its domain (Example 2).

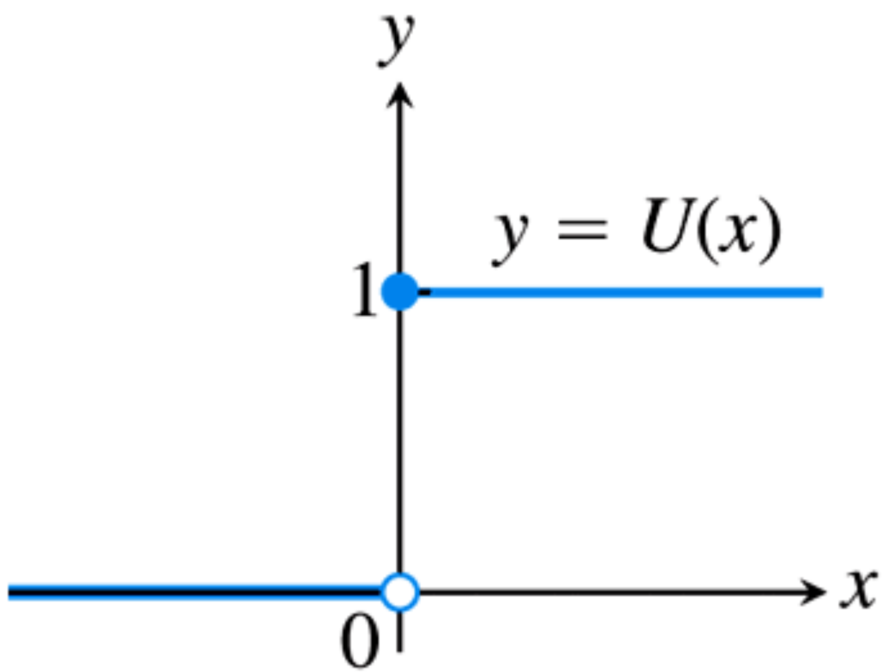


FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

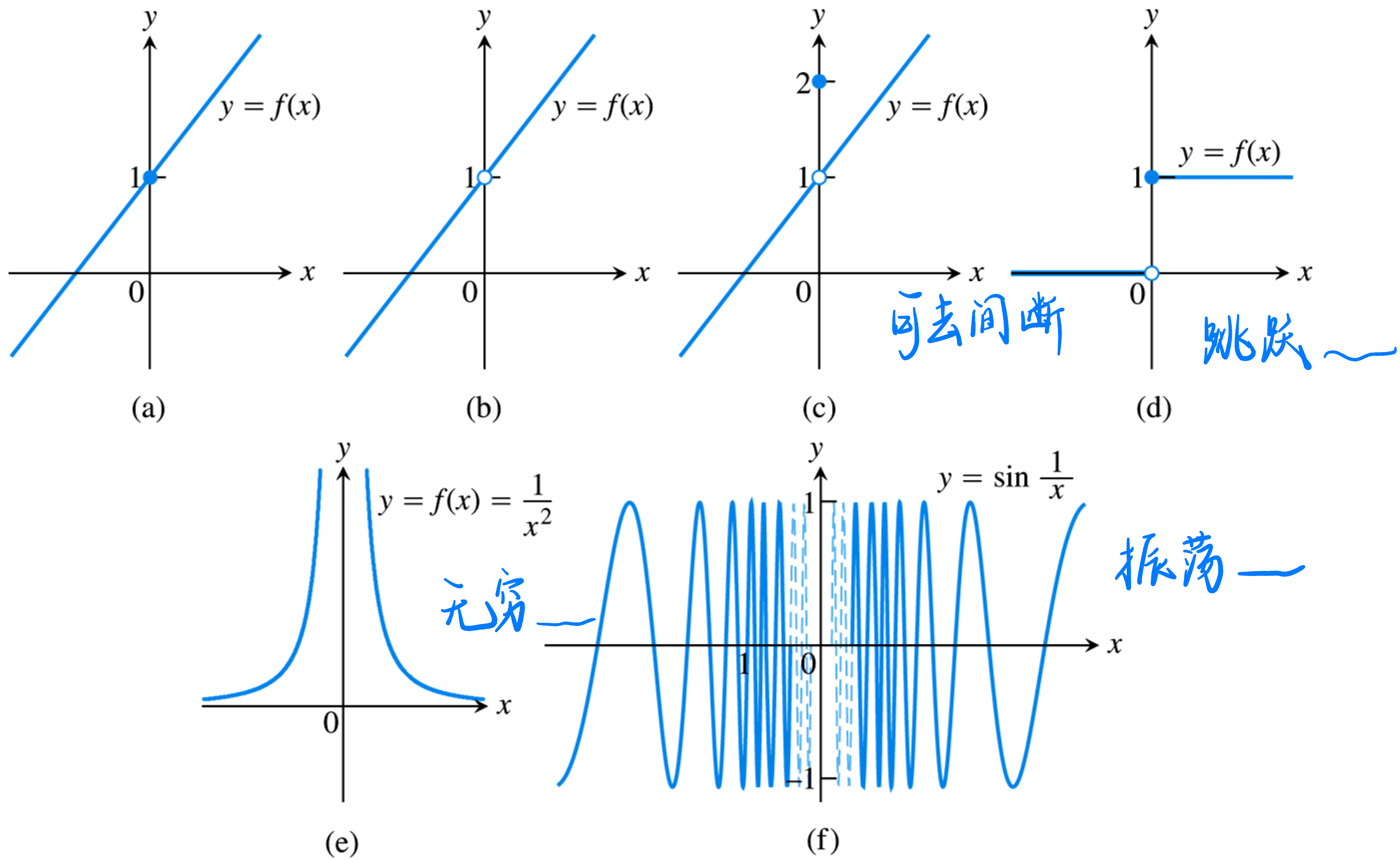
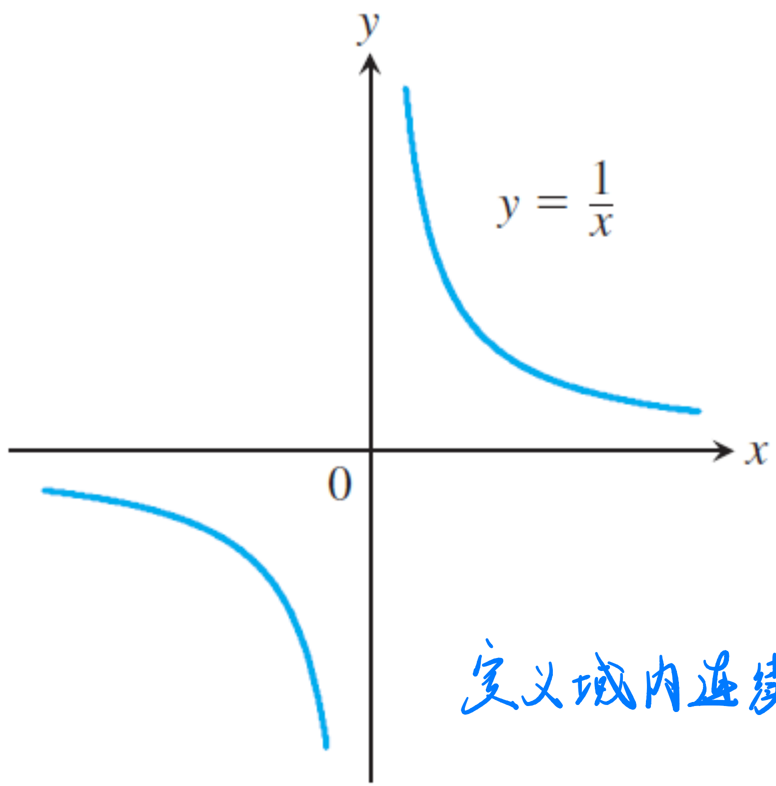


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Figure 2.40 displays several common types of discontinuities. The function in Figure 2.40a is continuous at $x = 0$. The function in Figure 2.40b would be continuous if it had $f(0) = 1$. The function in Figure 2.40c would be continuous if $f(0)$ were 1 instead of 2. The discontinuity in Figure 2.40c is **removable**. The function has a limit as $x \rightarrow 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.40d through f are more serious: $\lim_{x \to 0} f(x)$ does not exist, and there is no way to improve the situation by changing f at 0. The step function in Figure 2.40d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.40e has an **infinite discontinuity**. The function in Figure 2.40f has an **oscillating discontinuity**: It oscillates too much to have a limit as $x \rightarrow 0$.

没有极限



定义域内连续函数

FIGURE 2.41 The function $y = 1/x$ is continuous over its natural domain. It has a point of discontinuity at the origin, so it is discontinuous on any interval containing $x = 0$ (Example 5).

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

连续函数复合仍连续

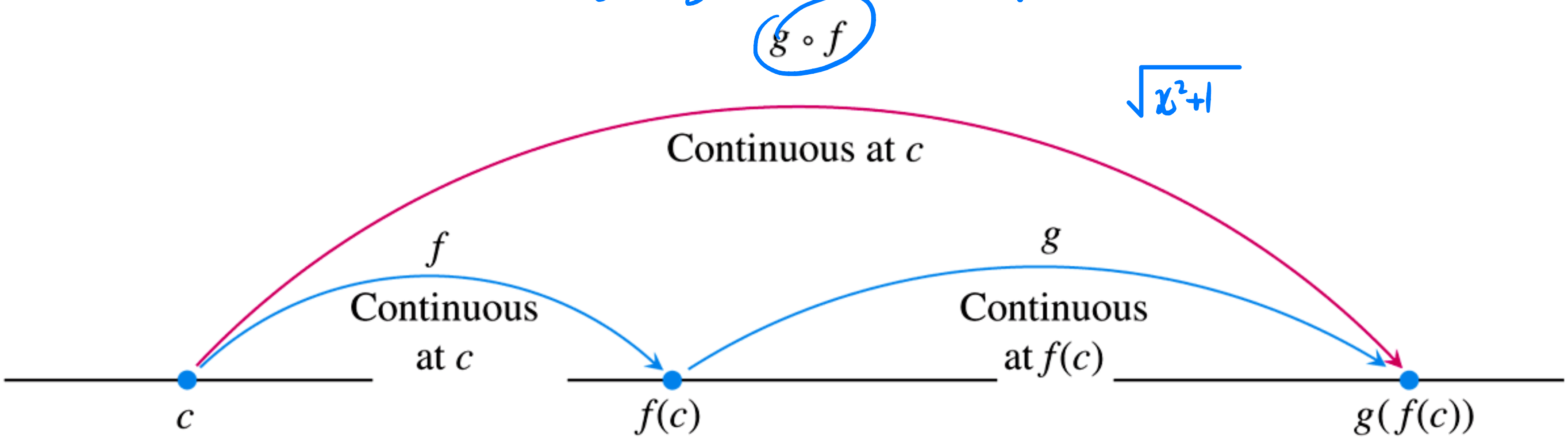


FIGURE 2.42 Composites of continuous functions are continuous.

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

在其取值上连续

EXAMPLE 8 Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$

(b) $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ X

(d) $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

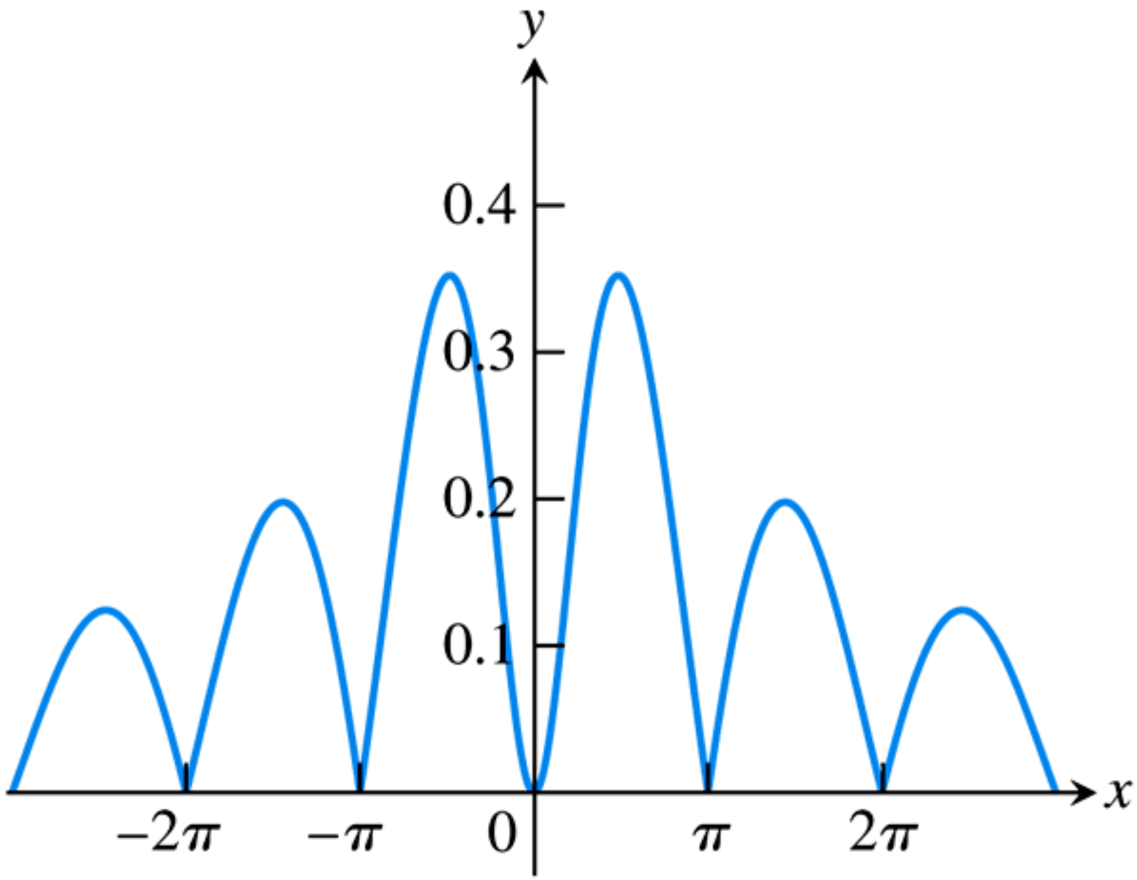


FIGURE 2.43 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

弱

外连续

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

内不一定

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

"结合律"

举反例

介值定理 ← 零点存在定理

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

① 连续函数

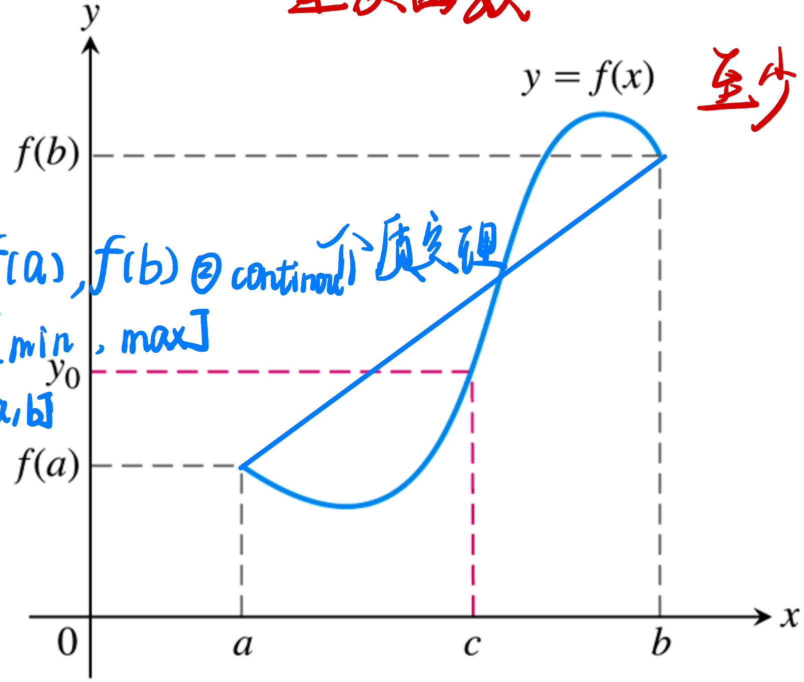
至少切到一次

① a, b $f(a), f(b)$ ② continuous 介质定理

③ $y_0 \in [min, max]$

$\exists c \in [a, b]$

$y_0 = f(c)$



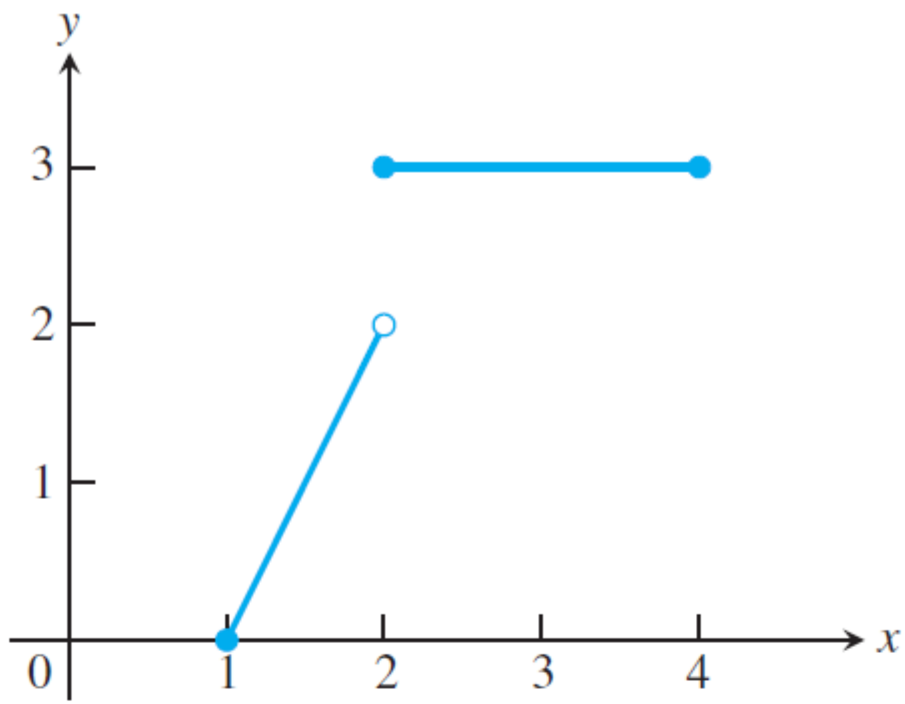


FIGURE 2.44 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

A Consequence for Graphing: Connectedness Theorem 11 implies that the graph of a function continuous on an interval cannot have any breaks over the interval. It will be **connected**—a single, unbroken curve. It will not have jumps like the graph of the greatest integer function (Figure 2.39), or separate branches like the graph of $1/x$ (Figure 2.41).

A Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** of the equation or **zero** of the function f. The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.

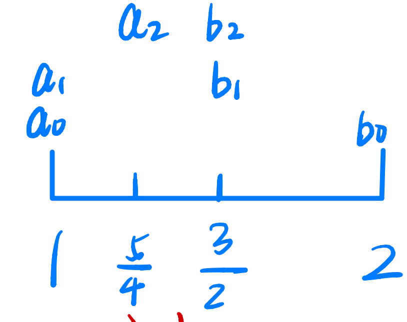
In practical terms, when we see the graph of a continuous function cross the horizontal axis on a computer screen, we know it is not stepping across. There really is a point where the function's value is zero.

EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

$f(1) = -1$

$f(2) = 5$

$f\left(\frac{3}{2}\right) > 0$



二分法

最后给近似解

$\frac{a_n + b_n}{2}$

$\left(\frac{1}{2}\right)^{n+1} < 10^{-4}$

$2^{n+1} > 10^{4}$

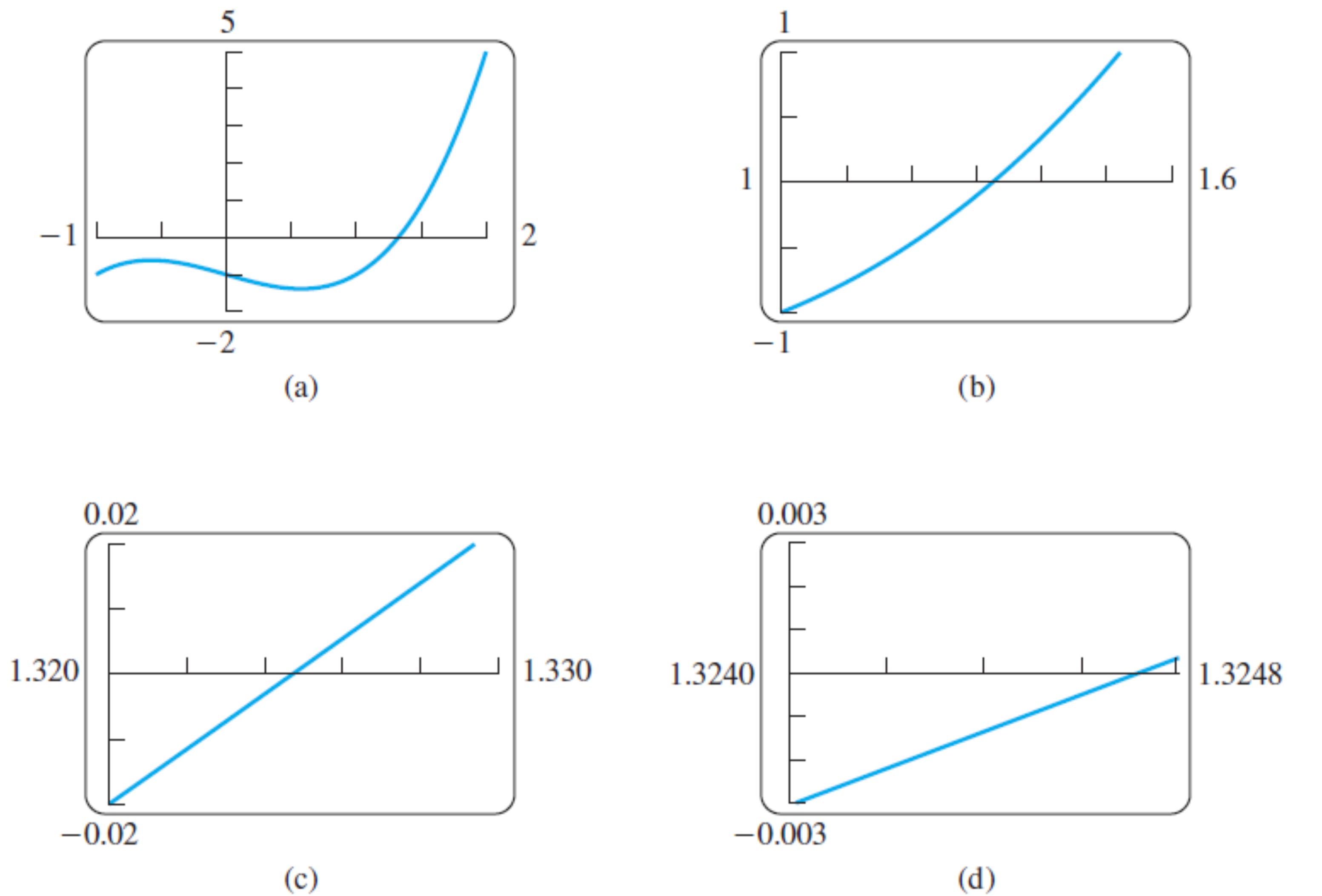


FIGURE 2.45 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$ (Example 10).

Continuous Extension to a Point

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f, to include $x = c$, creating a new function that is continuous at $x = c$. For example, the function $y = f(x) = (\sin x)/x$ is continuous at every point except $x = 0$, since the origin is not in its domain. Since $y = (\sin x)/x$ has a finite limit as $x \to 0$ (Theorem 7), we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function

$$F(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

极限存在

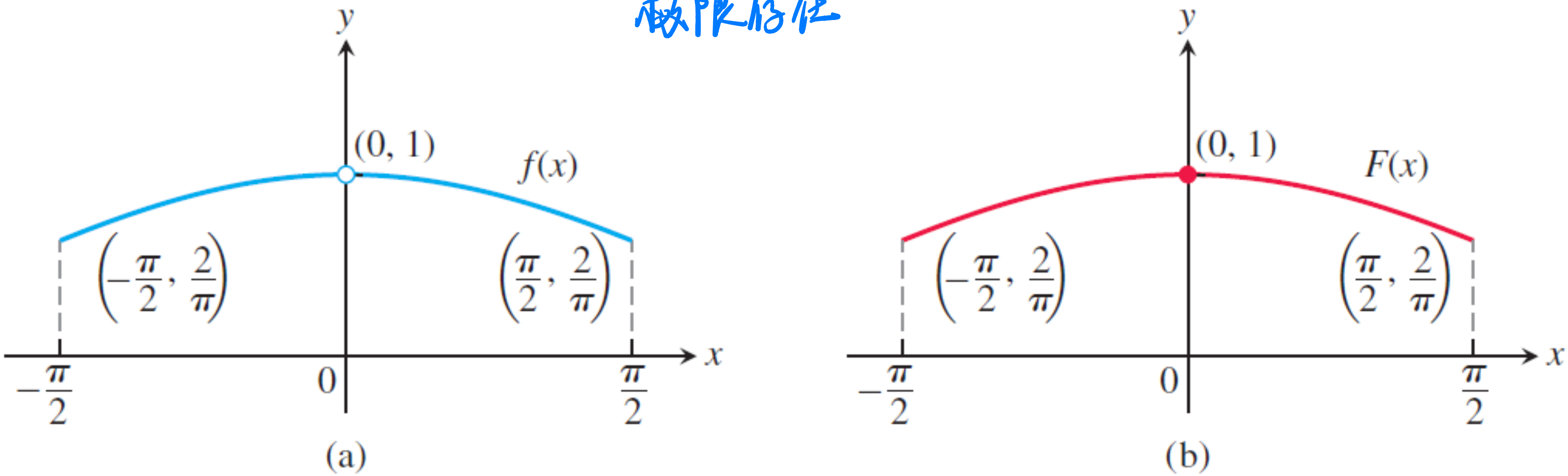


FIGURE 2.47 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \le x \le \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x\to 0} f(x)$.

EXAMPLE 12 Show that

$$\cfrac{(x+3)\cancel{(x-2)}}{}$$

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

$$\cancel{(x-2)}(x+2)$$

has a continuous extension to $x = 2$, and find that extension.

$$\lim_{x \to 2} \ldots = \frac{5}{4}$$

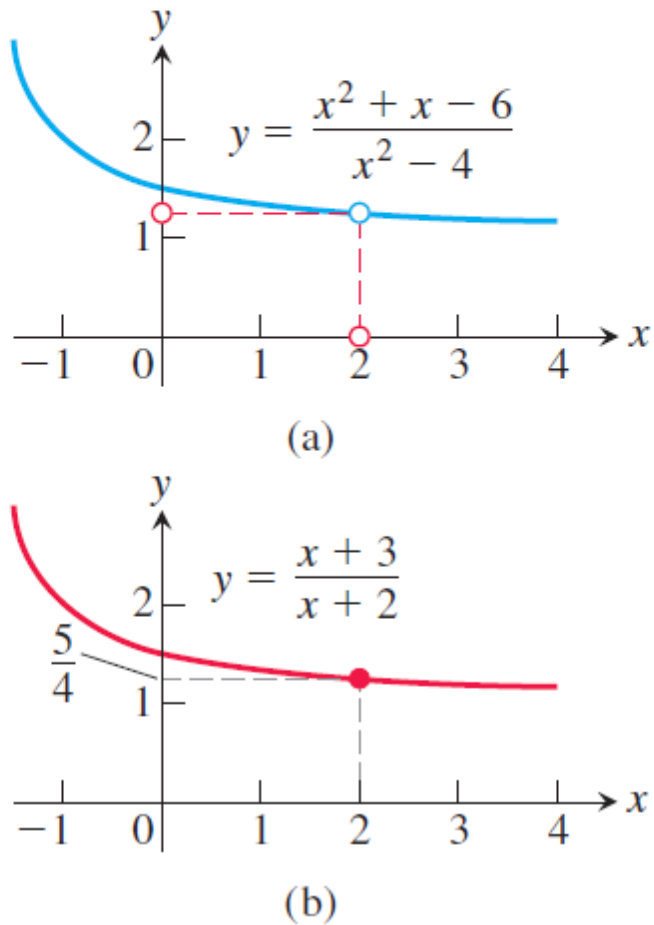


FIGURE 2.48 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 12).

A function discontinuous at every point

a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

极小的区间
总有点 0/1

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

b. Is f right-continuous or left-continuous at any point?

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$
$= 1 - 2\sin^2\frac{\theta}{2}$

$\frac{1}{a \cdot 2} = b$
$\therefore ab = \frac{1}{2}$

$\lim_{x\to 0} \frac{1-\cos\sqrt{x}}{x} = \lim_{x\to 0} \frac{2\sin^2\frac{\sqrt{x}}{2}}{x}$
$= \lim_{x\to 0} 2 \frac{\sin^2\frac{\sqrt{x}}{2}}{(\frac{\sqrt{x}}{2})^2 \times 4} = \frac{1}{2}$

二倍角公式

例17（2017） 若函数$f(x) = \begin{cases} \frac{1-\cos\sqrt{x}}{ax}, & x > 0, \\ b, & x \le 0, \end{cases}$在$x = 0$处连续，则

$\begin{cases} 1+\cos\theta = 2\cos^2\frac{\theta}{2} \\ 1-\cos\theta = 2\sin^2\frac{\theta}{2} \end{cases}$

(A) $ab = \frac{1}{2}$. A (B) $ab = -\frac{1}{2}$.
(C) $ab = 0$. (D) $ab = 2$.

2.6

无穷

Limits Involving Infinity; Asymptotes of Graphs

渐近线

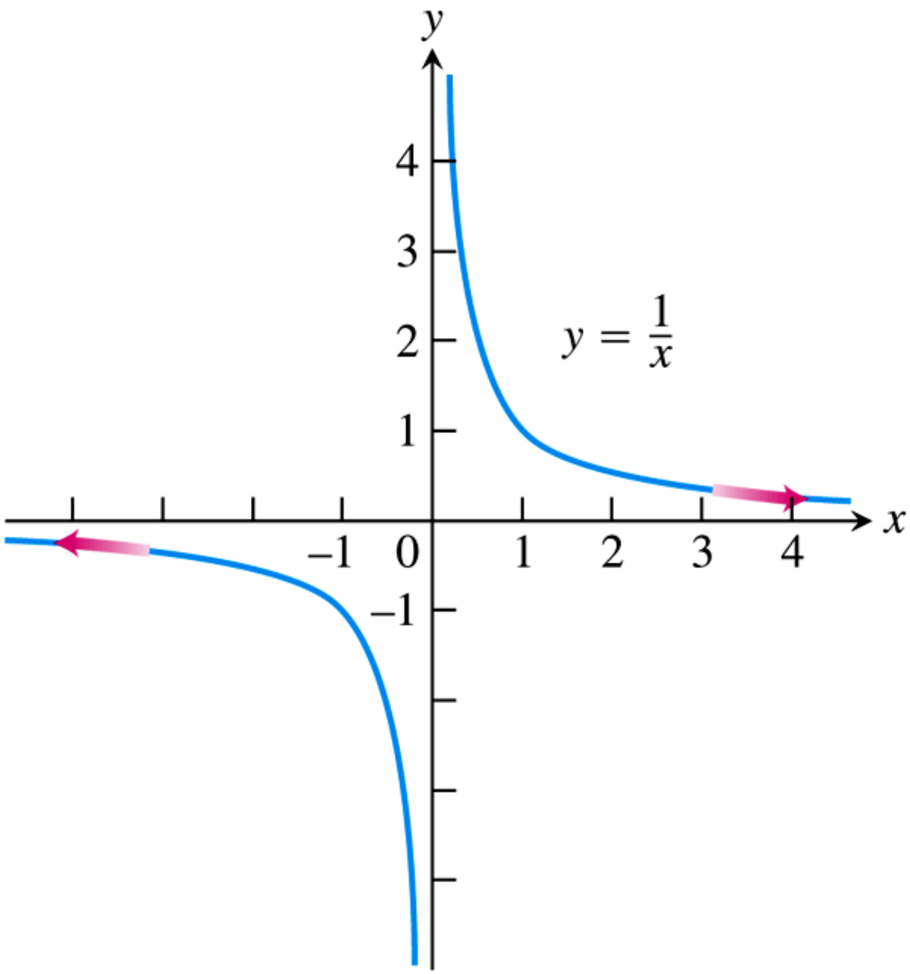


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

默认$+\infty$

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Longrightarrow \qquad |f(x) - L| < \epsilon.$$ $x > M \Rightarrow |f(x) - L| < \epsilon$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Longrightarrow \qquad |f(x) - L| < \epsilon.$$ $x < N \Rightarrow |f(x) - L| < \epsilon$

EXAMPLE 1 Show that

(a) $\lim_{x\to\infty} \dfrac{1}{x} = 0$ (b) $\lim_{x\to-\infty} \dfrac{1}{x} = 0.$

$x > \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$

$\frac{1}{x} < \varepsilon$

$x > \frac{1}{\varepsilon}$

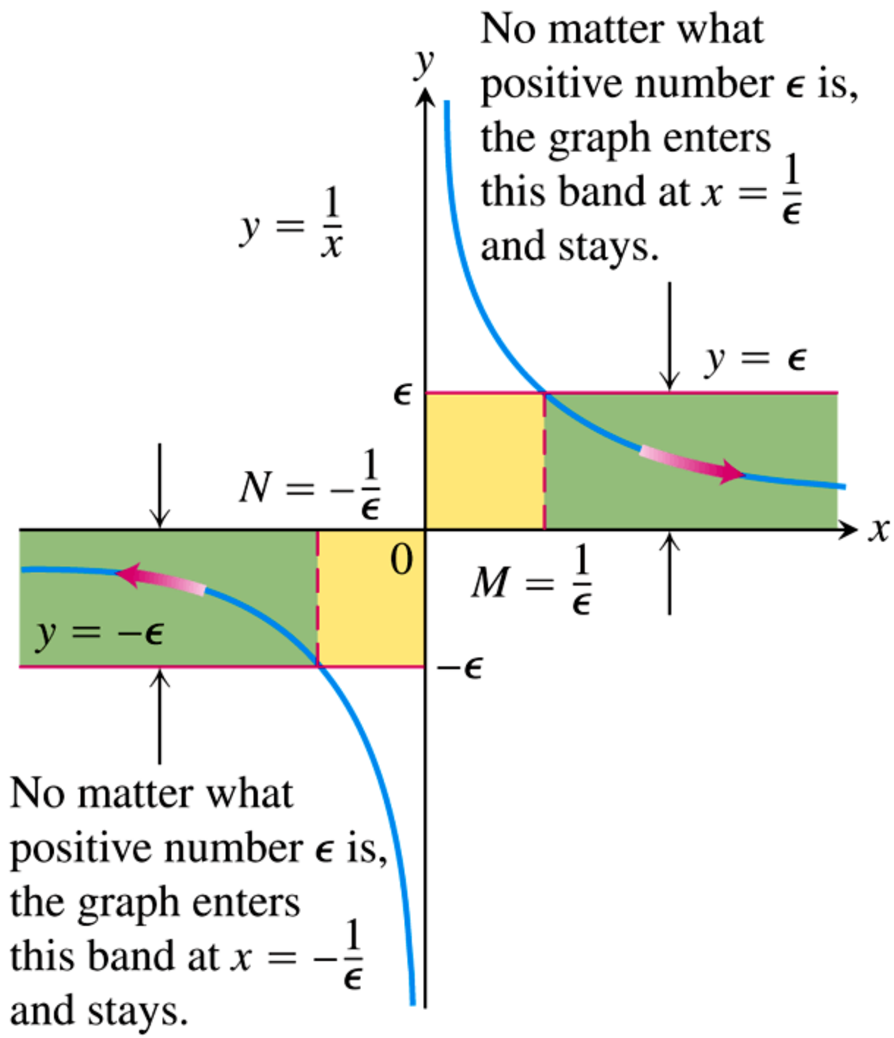


FIGURE 2.50 The geometry behind the argument in Example 1.

四则运算

THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x \to c}$ by $\lim_{x \to \infty}$ or $\lim_{x \to -\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

(a) $$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$$ Divide numerator and denominator by x^2.

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$ See Fig. 2.51.

(b) $$\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$ Divide numerator and denominator by x^3.

不致符号 ~~0:?~~

$$= \frac{0 + 0}{2 - 0} = 0$$ 多注意、正负 See Fig. 2.52. ■

$$\lim_{x\to \substack{+\infty \\ -\infty}} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

$n < m$ $L = 0$

$n = m$ $L = \frac{a_n}{b_m}$

$n > m$ $\frac{a_n x^{n-m} + \cdots}{b_m + \cdots}$ 不存在L

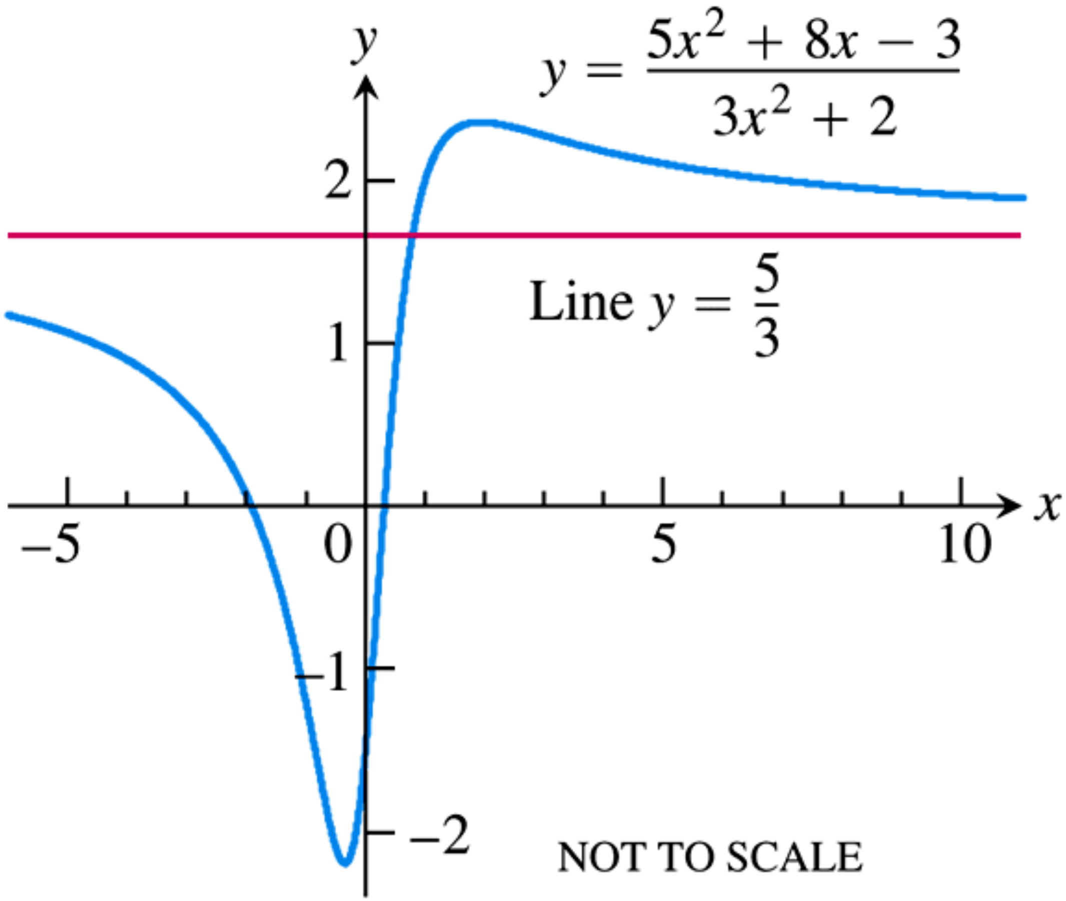


FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

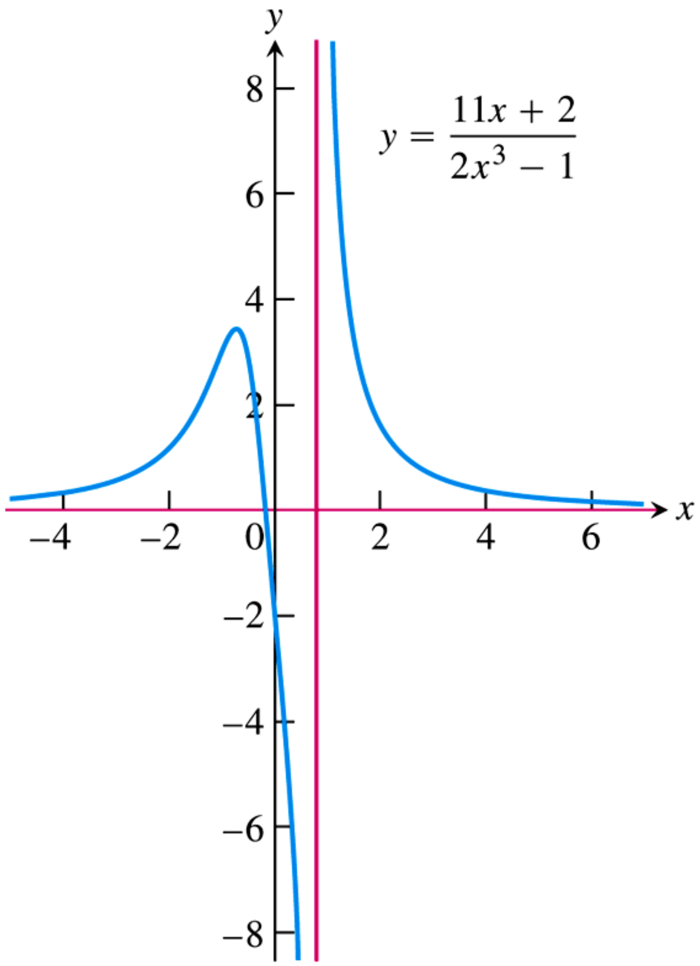


FIGURE 2.52 The graph of the function in Example 3b. The graph approaches the x-axis as $|x|$ increases.

DEFINITION A line $y = b$ is a 水平的 **horizontal** 渐近线 **asymptote** of the graph of a function $y = f(x)$ if either

最多2条

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

$x \to +\infty \quad L = 1$

$x \to -\infty \quad L = -1$

EXAMPLE 5 Find **(a)** $\lim_{x \to \infty} \sin(1/x)$ and **(b)** $\lim_{x \to \pm\infty} x \sin(1/x)$.

$t = \frac{1}{x}$

$= \lim_{t \to 0} \sin t$

$= \lim_{x \to \pm\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}}$

$= 1$

EXAMPLE 6 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

$-1 \le \sin x \le 1$

$-\frac{1}{|x|} \le \frac{\sin x}{x} \le \frac{1}{|x|}$ 偶函数 一起讨论

$x \to \pm\infty \quad \frac{\sin x}{x} = 0$

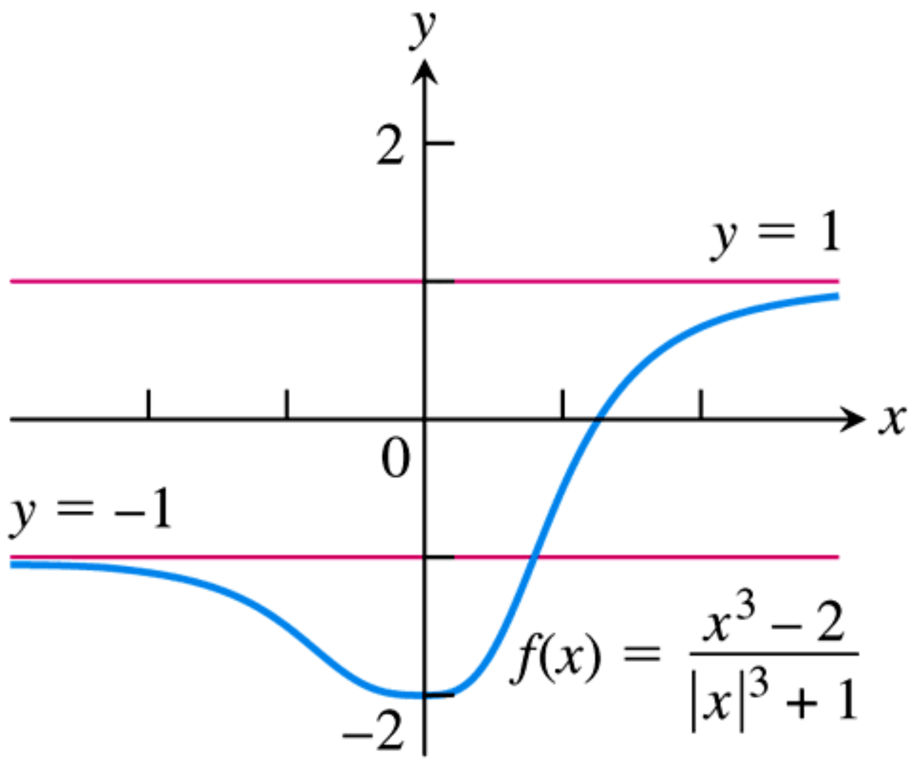


FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

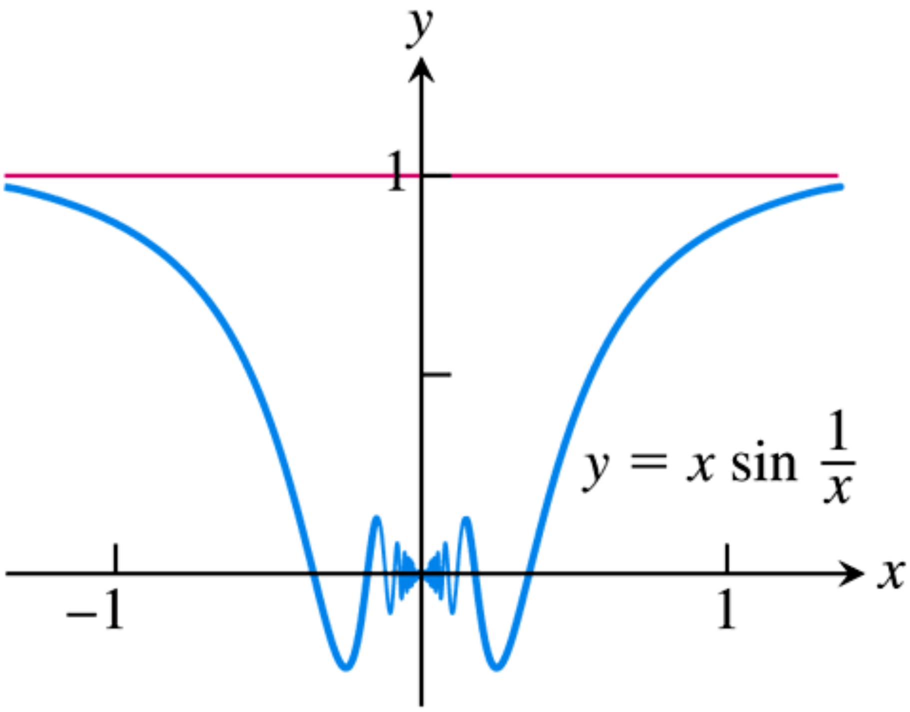


FIGURE 2.54 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 5b).

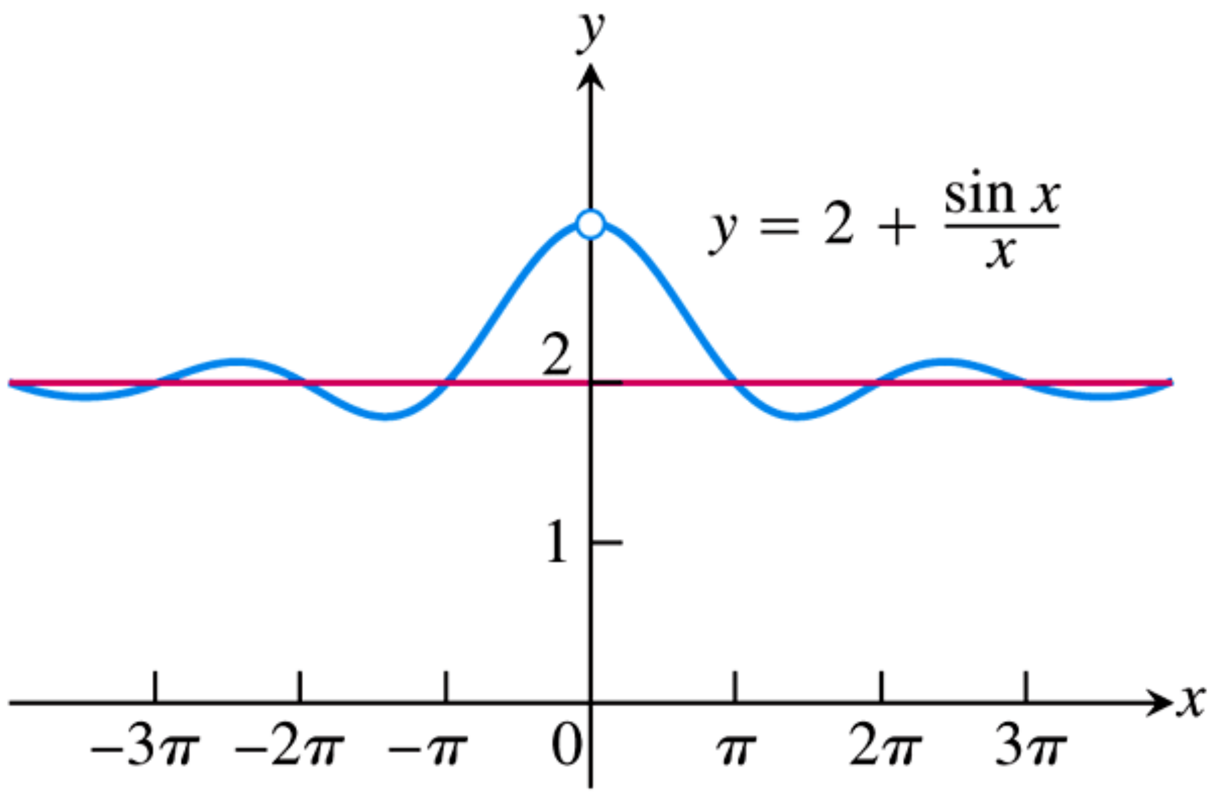


FIGURE 2.55 A curve may cross one of its asymptotes infinitely often (Example 6).

EXAMPLE 7

Find $\lim_{x\to 0^+} x\left\lfloor \frac{1}{x} \right\rfloor$.

拆出范围

夹逼 ⌊ ⌋

$\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \le \frac{1}{x}$

$1 - x < x\left\lfloor \frac{1}{x} \right\rfloor \le 1$

$x \to 0^+$ $\quad \to 1$

EXAMPLE 8

Find $\lim_{x\to\infty} \left(x - \sqrt{x^2 + 16}\right)$.

$\lim_{x\to\infty} (x - \sqrt{x^2+16x})$

$= \lim_{x\to\infty} \frac{-16x}{x+\sqrt{x^2+16x}}$

$= \lim_{x\to\infty} \frac{-16}{1+\sqrt{1+\frac{16}{x}}}$

$= -8$

$\lim_{x\to-\infty} (x + \sqrt{x^2-16x})$

$= \lim_{x\to-\infty} \frac{16x}{x-\sqrt{x^2-16x}}$

$= \lim_{x\to-\infty} \frac{-16}{-1-\sqrt{1-\frac{16}{x}}}$ 符号易错 对于√

$= 8$

* 建议 $t=-x$ 换元 负的换成正的

$\stackrel{t=-x}{=} \lim_{t\to\infty} \frac{-16t}{-t-\sqrt{t^2+16t}} = \lim_{x\to\infty} \frac{-16}{-1-\sqrt{1+\frac{16}{t}}}$

$= \lim_{x\to\infty} \frac{(x-\sqrt{x^2+16})(x+\sqrt{x^2+16})}{x+\sqrt{x^2+16}}$

$= \lim_{x\to\infty} \frac{-16}{x+\sqrt{x^2+16}} = 0$

除以负数易错

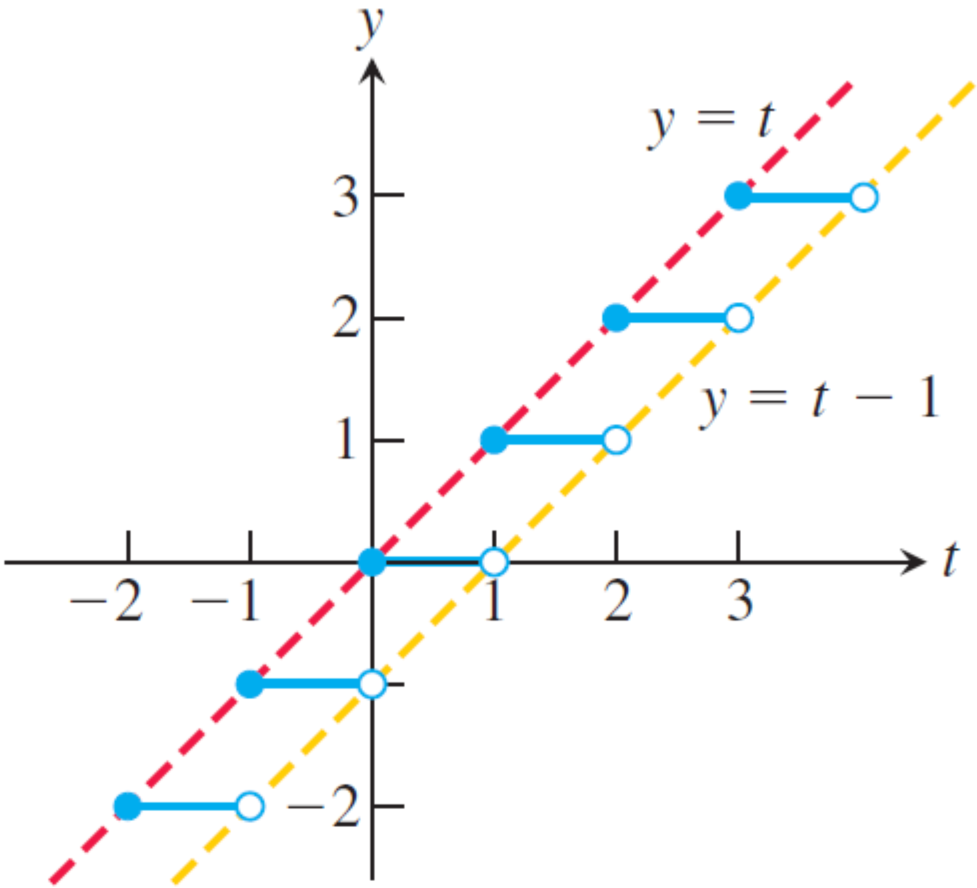


FIGURE 2.56 The graph of the greatest integer function $y = \lfloor t \rfloor$ is sandwiched between $y = t - 1$ and $y = t$.

斜 Oblique Asymptotes

$\exists a, b \quad \lim_{\substack{x \to +\infty \\ -\infty}} (f(x) - (ax+b)) = 0$

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$. 分母

EXAMPLE 9 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

$$= \frac{x^2 - 4 + 1}{2(x-2)}$$

$$= \frac{1}{2}(x+2) + \frac{1}{2(x-2)}$$

$$= \frac{1}{2}x + 1 + \frac{1}{2(x-2)}$$

$x \to +\infty / -\infty \quad \frac{1}{2(x-2)}$

$$y = \frac{x^3 + 3x^2 + 1}{x^2 + x - 1}$$

$$\begin{array}{r} x + 2 \\ x^2+x-1 \overline{\big) x^3 + 3x^2 + 1} \\ x^3 + x^2 - x \\ \hline 2x^2 + x + 1 \\ 2x^2 + 2x - 2 \\ \hline -x + 3 \end{array}$$

$$= x + 2 + \frac{-x+3}{x^2+x-1}$$

$x \to \pm\infty \quad y \to x + 2$

余式最高次 < 除式最高次

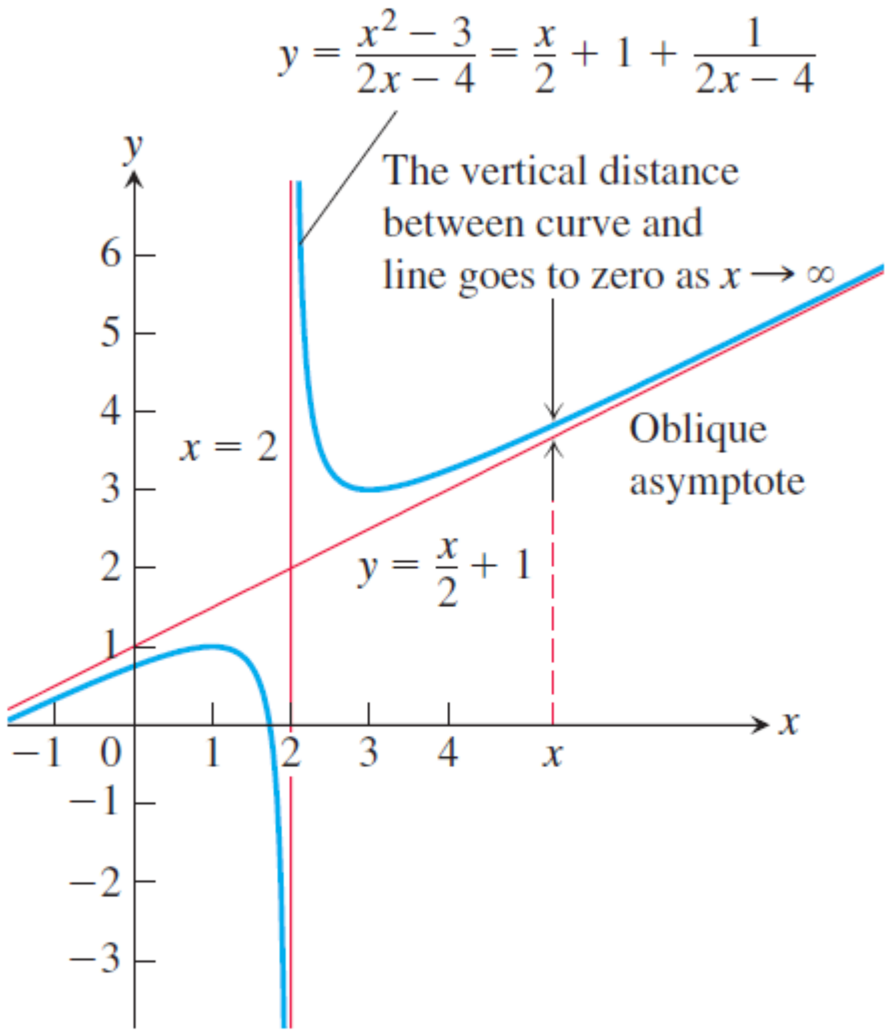


FIGURE 2.57 The graph of the function in Example 9 has an oblique asymptote.

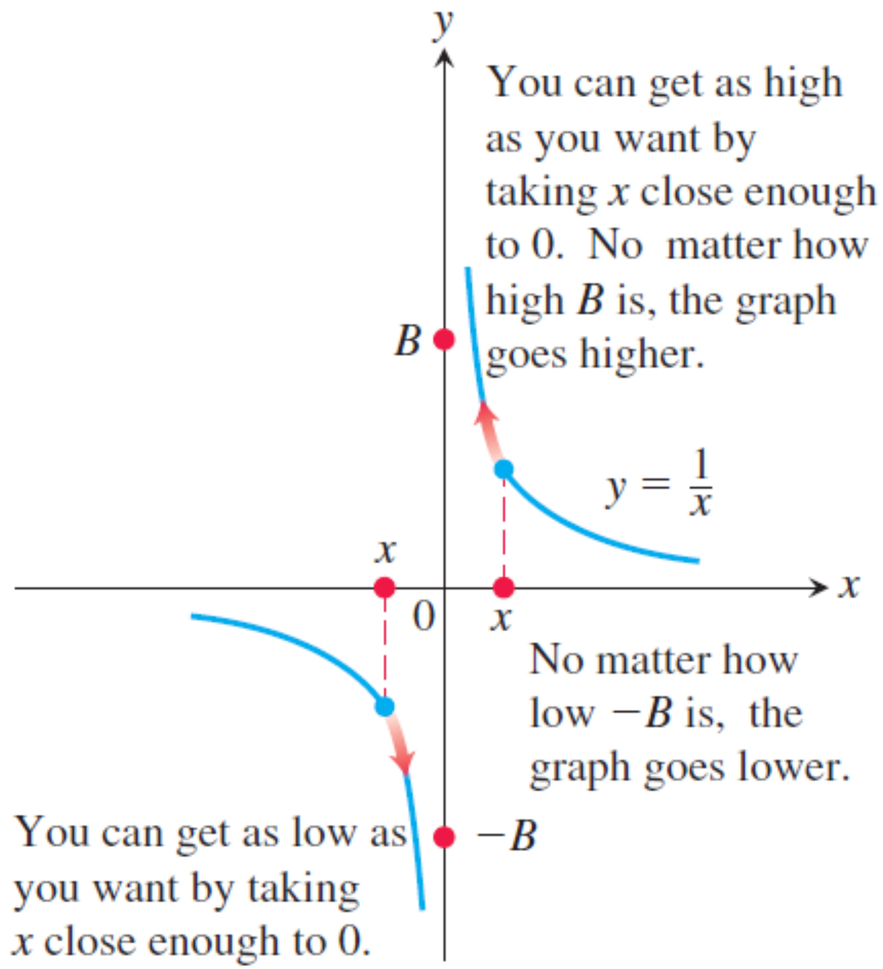


FIGURE 2.58 One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

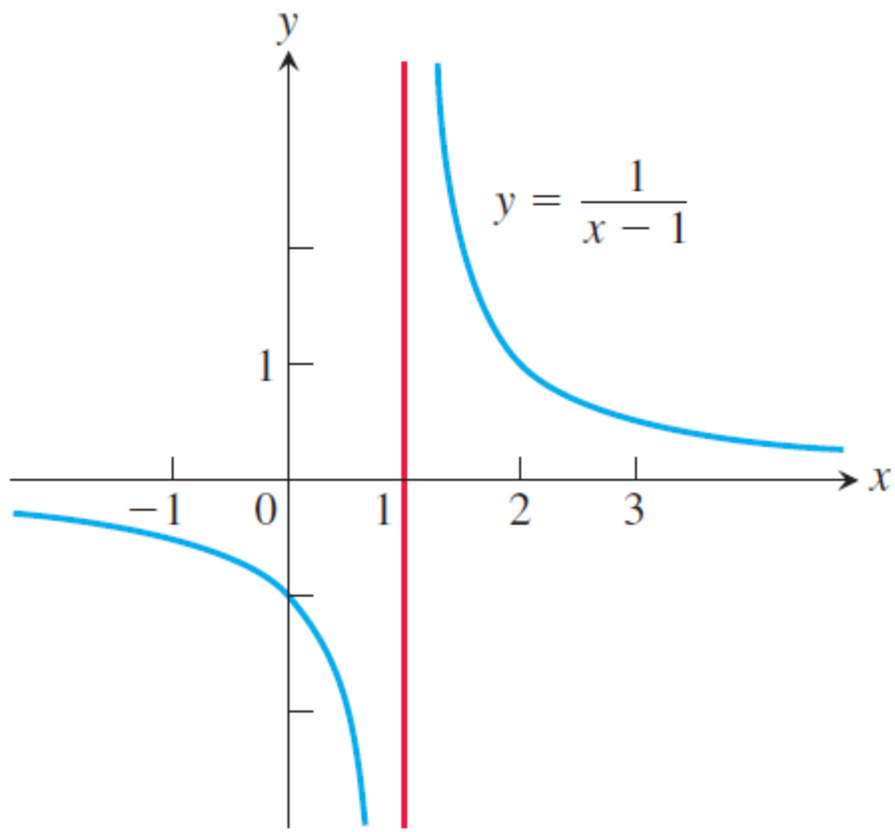


FIGURE 2.59 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 10).

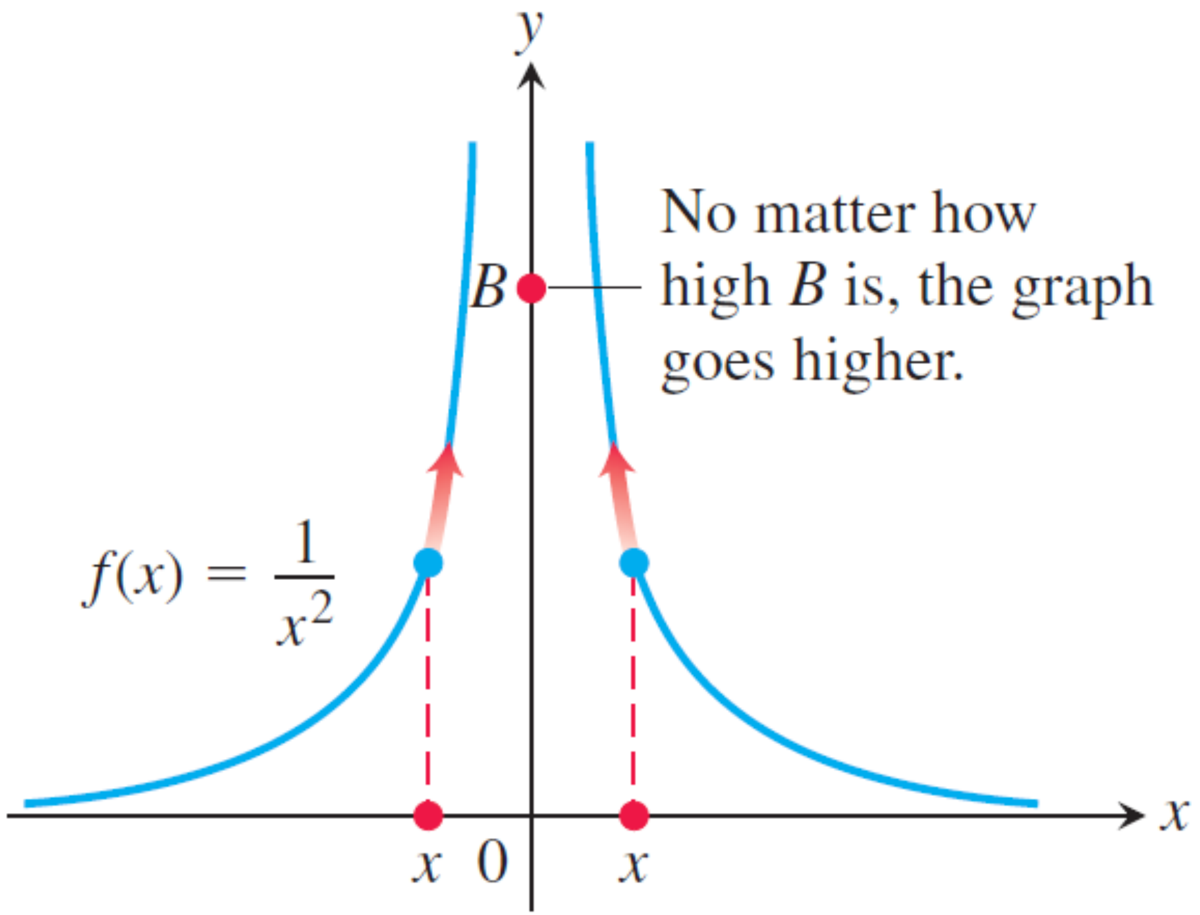


FIGURE 2.60 The graph of $f(x)$ in Example 11 approaches infinity as $x \to 0$.

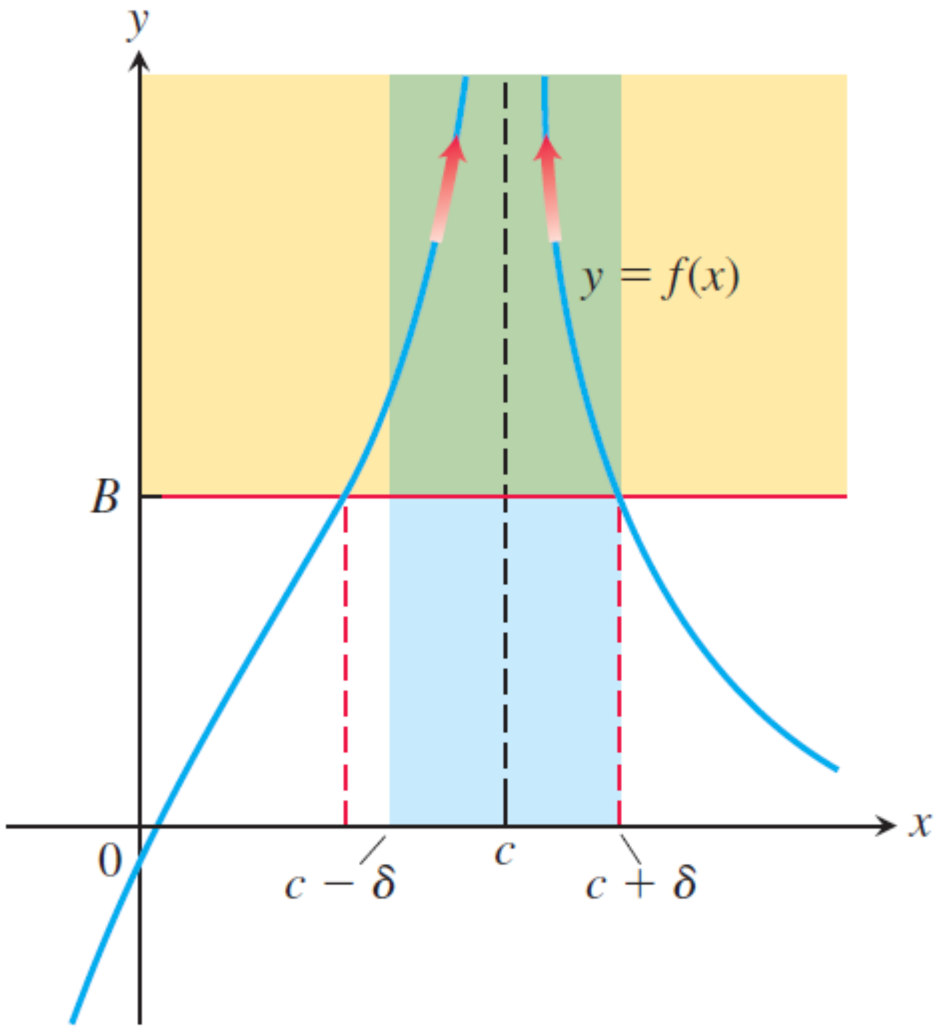


FIGURE 2.61 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

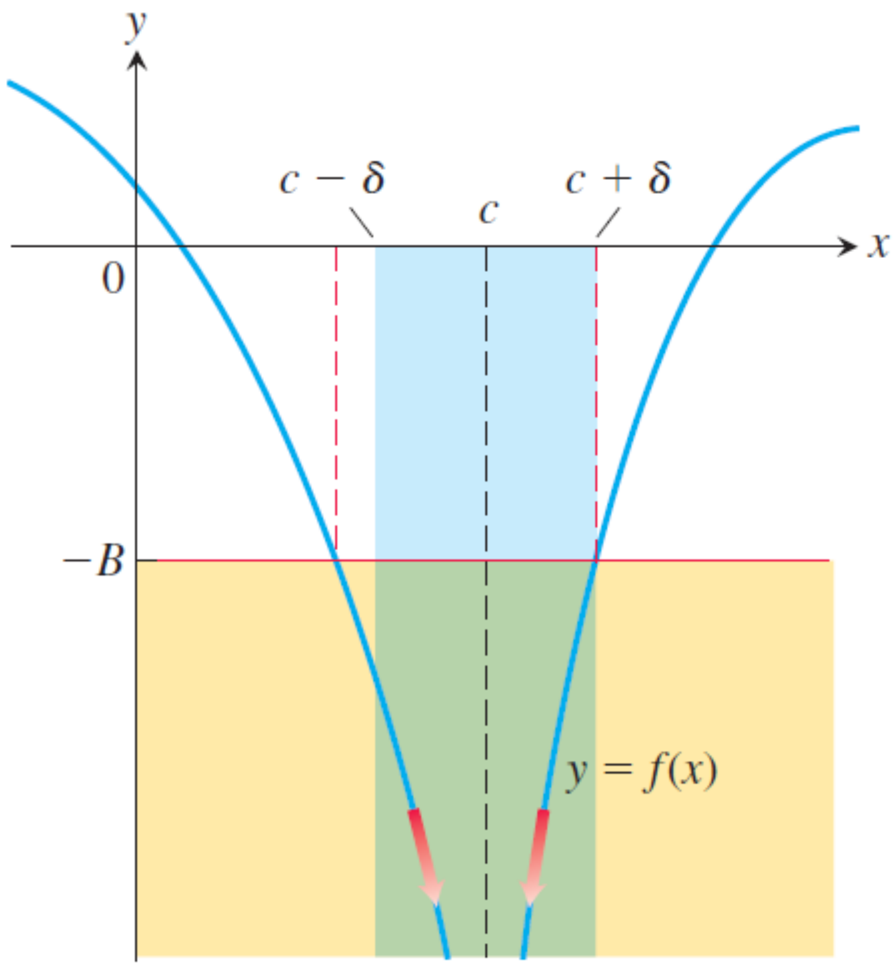


FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITIONS

1. We say that ***f*(*x*) approaches infinity as *x* approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

充分近　　　　想多大就多大

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that ***f*(*x*) approaches minus infinity as *x* approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$

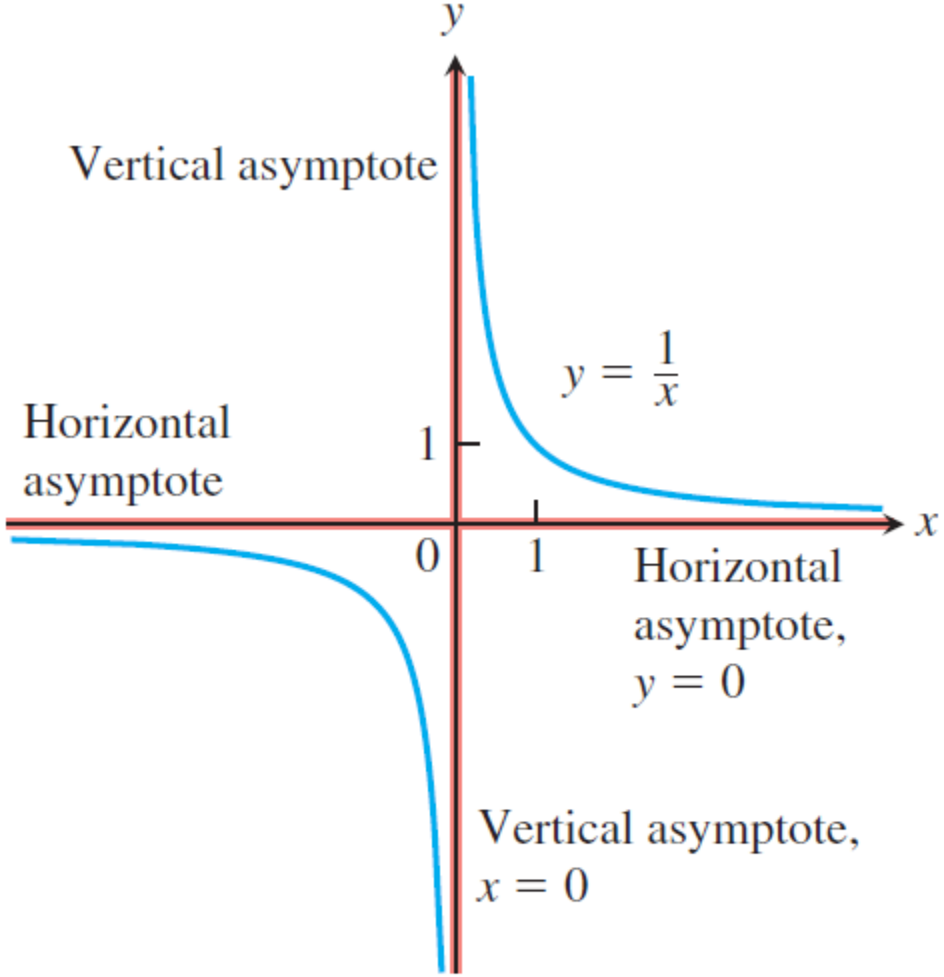


FIGURE 2.63 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

垂直　　　$x=0$

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

可以相交

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty .$$

只要单边极限

只要一侧无穷渐近
就垂直渐近线

渐近线可以
相交

只要越来越近就行

EXAMPLE 16 Find the horizontal and vertical asymptotes of the graph of

$\lim_{x \to \pm\infty} -\frac{8}{x^2-4} = 0 \Rightarrow y=0$

$$f(x) = -\frac{8}{x^2 - 4}.$$

$-\frac{8}{(x+2)(x-2)}$ $x=\pm 2$

$\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots b_0}$ ① horizontal $m \geq n$

② oblique $n = m+1$ 线性函数

EXAMPLE 17 The curves

$$y = \sec x = \frac{1}{\cos x} \qquad \text{and} \qquad y = \tan x = \frac{\sin x}{\cos x}$$

$\frac{\pi}{2}$的奇数倍

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$

水平/斜 最多2条

垂直 无数条

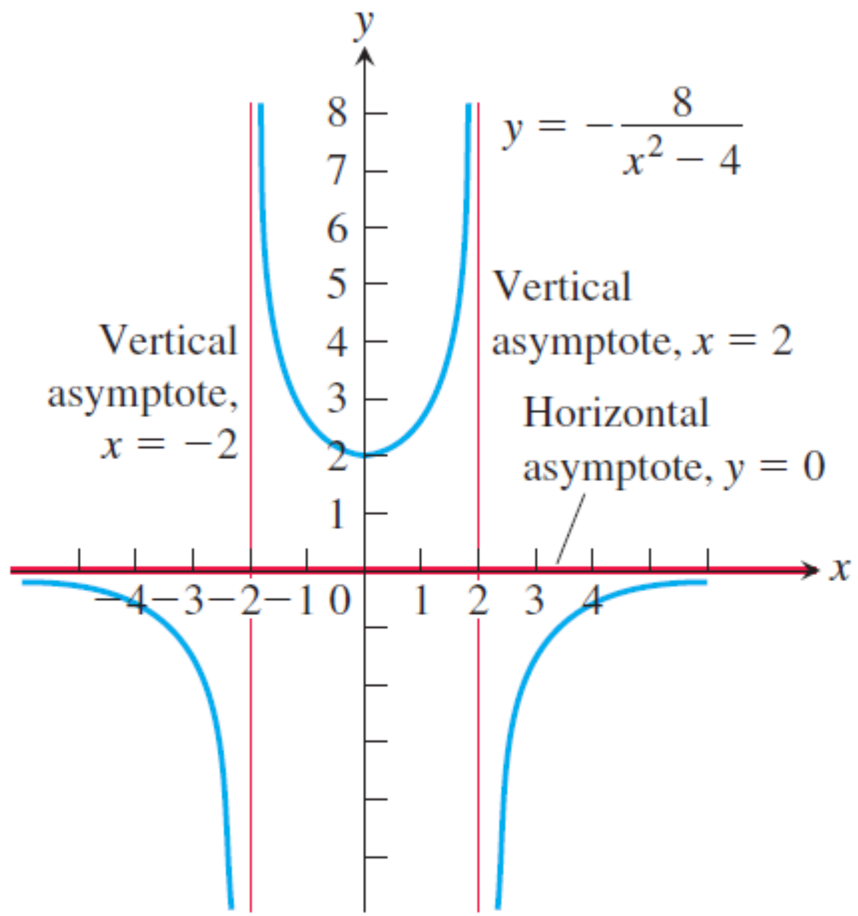


FIGURE 2.65 Graph of the function in Example 16. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

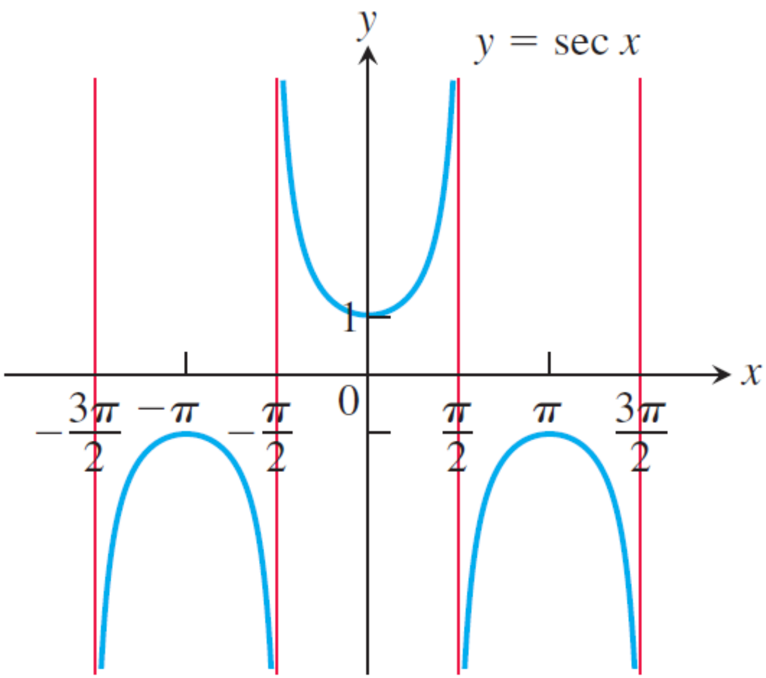


FIGURE 2.66 The graphs of sec x and tan x have infinitely many vertical asymptotes (Example 17).

作差

↑

例24（2014） 下列曲线中有渐近线的是

C

不能渐近 $\lim\limits_{x\to+\infty}\left((x+\sin x)-x\right)$ $\lim\limits_{x\to+\infty}\sin\frac{1}{x}=0$

(A) $y = x + \sin x$. (B) $y = x^2 + \sin x$.

(C) $y = x + \sin\frac{1}{x}$. $\lim\limits_{\substack{x\to+\infty\\ -\infty}}\left((x+\sin\frac{1}{x})-x\right) = \lim\limits_{x\to+\infty}\sin\frac{1}{x}$ (D) $y = x^2 + \sin\frac{1}{x}$.

斜：① $x\to\pm\infty$
定义 ② 或直线
③ $\to 0$

* $\lim\limits_{x\to\pm\infty}\left(f(x)-(ax+b)\right)=0$

$\Updownarrow$

$\lim\limits_{x\to\pm\infty}\left(f(x)-ax\right)=b$

$\Rightarrow \lim\limits_{x\to\pm\infty}\dfrac{f(x)-ax}{x}=0$

$\Rightarrow$ ① $\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}=a$ 如果极限存在

② 再求 b

若有一个不存在 $\Rightarrow$ 无斜渐近线