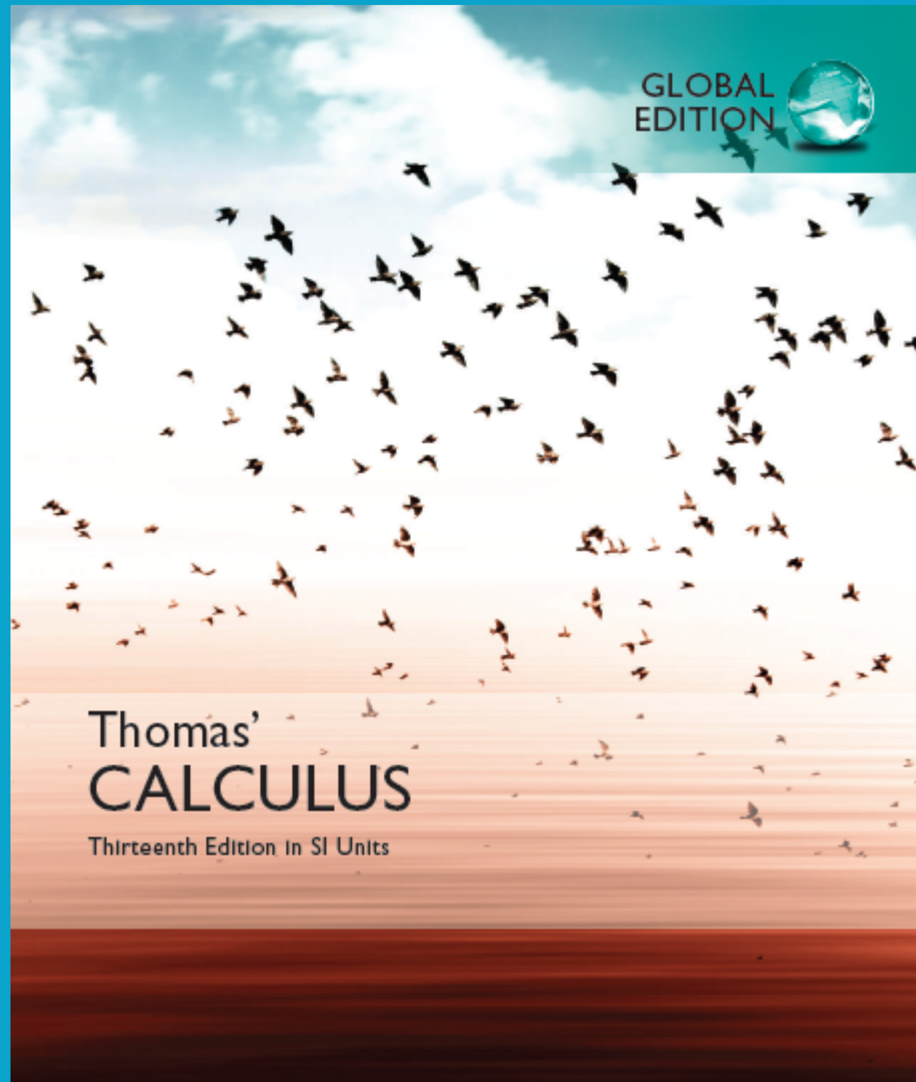


GLOBAL EDITION

Thomas' CALCULUS

Thirteenth Edition in SI Units

Chapter 4

Applications of Derivatives

4.1

Extreme Values of Functions

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

取等号

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as **global** / absolute maxima or minima.

全局极大/极小

local/relative

局部极大/小

含 c 开区间

(不建议在"导数"章看中文书)

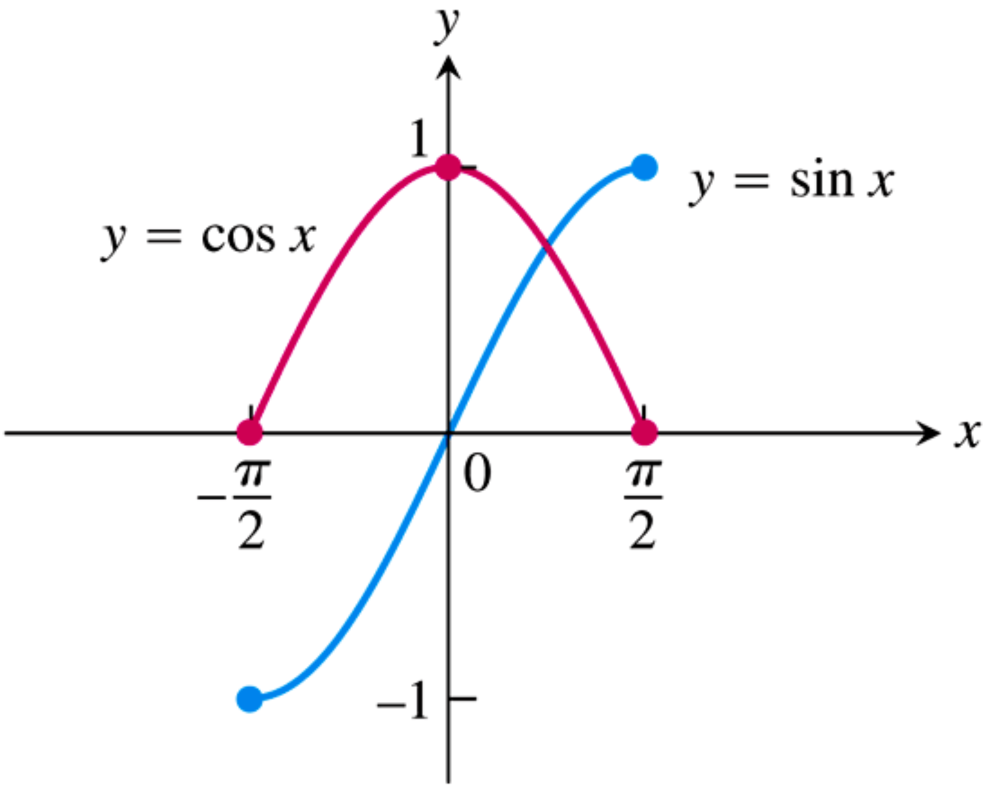


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

改变定义域

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.

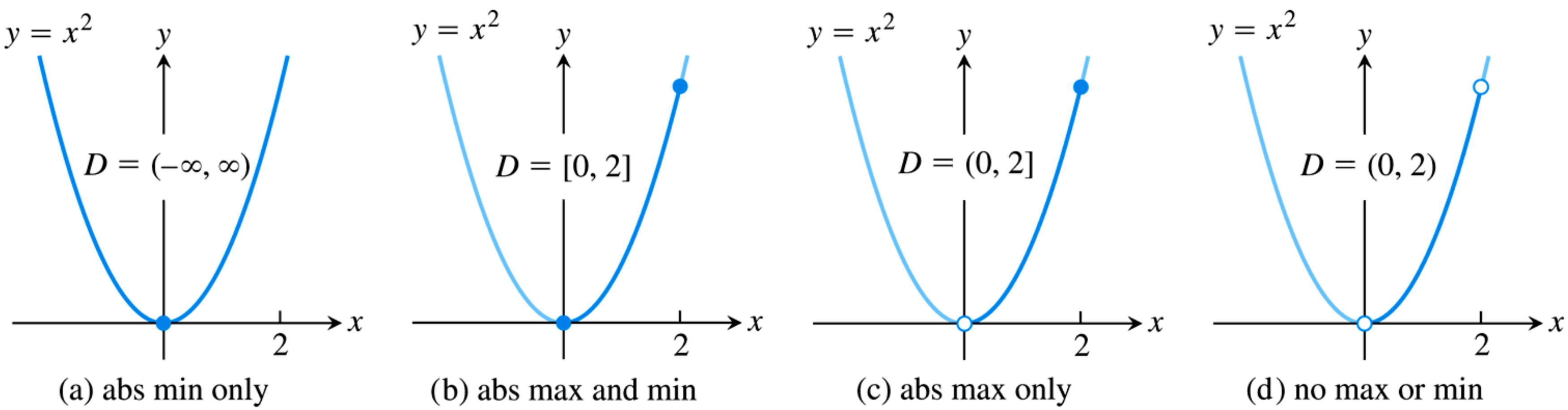
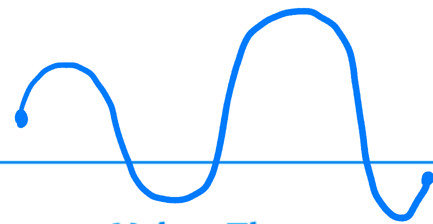


FIGURE 4.2 Graphs for Example 1.



闭区间 一定有全局极大/小
连续

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

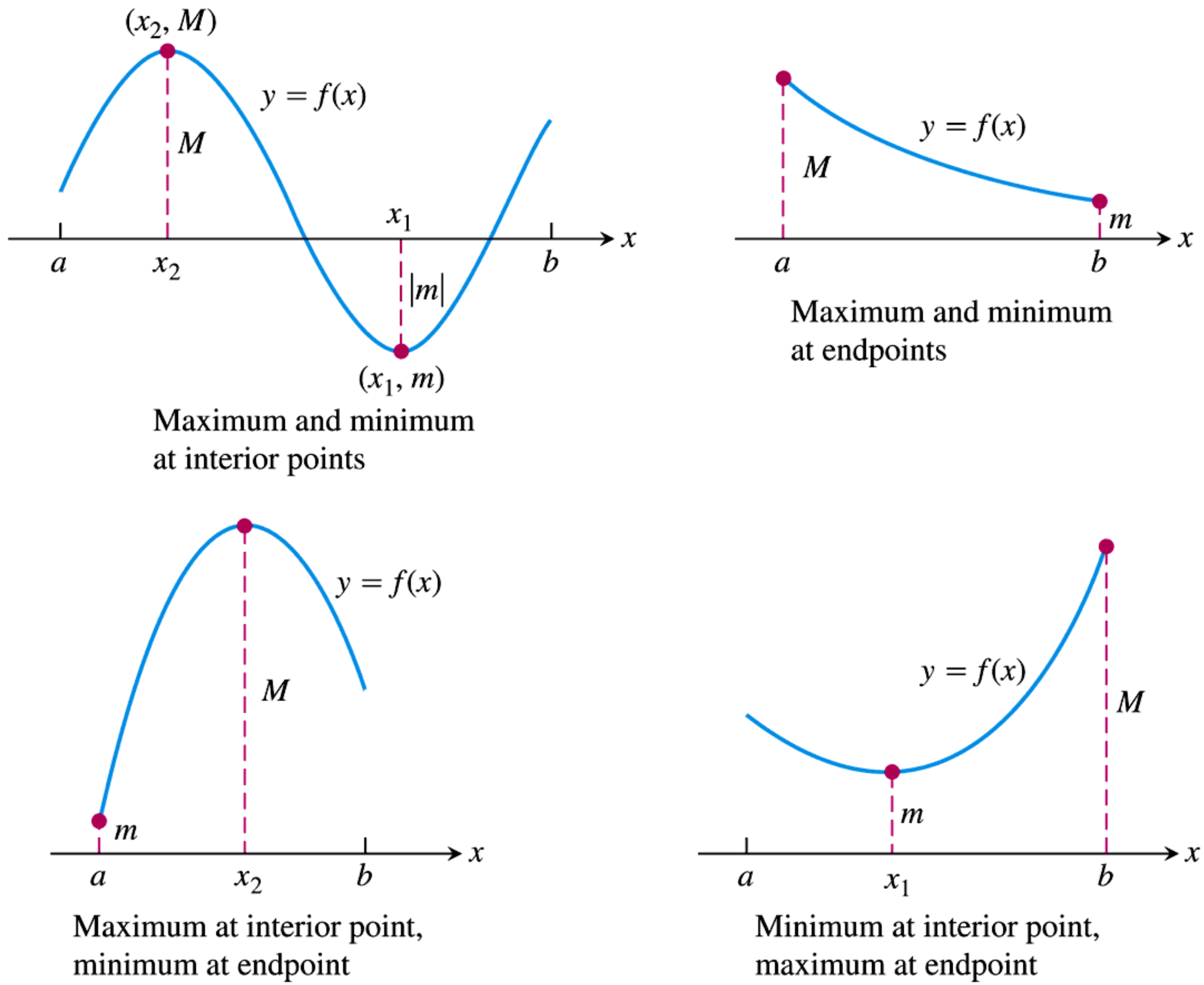
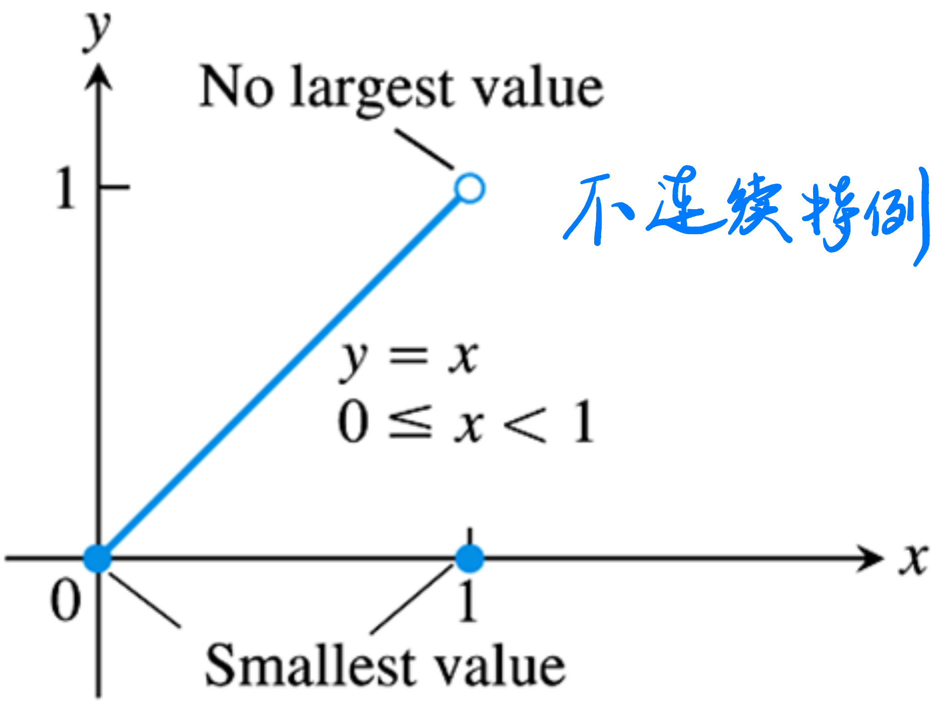


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.



不连续指例

FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of [0, 1] except $x = 1$, yet its graph over [0, 1] does not have a highest point.

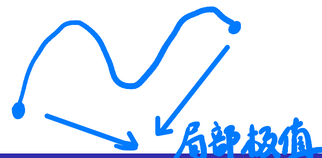
DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

包含c的开区间

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

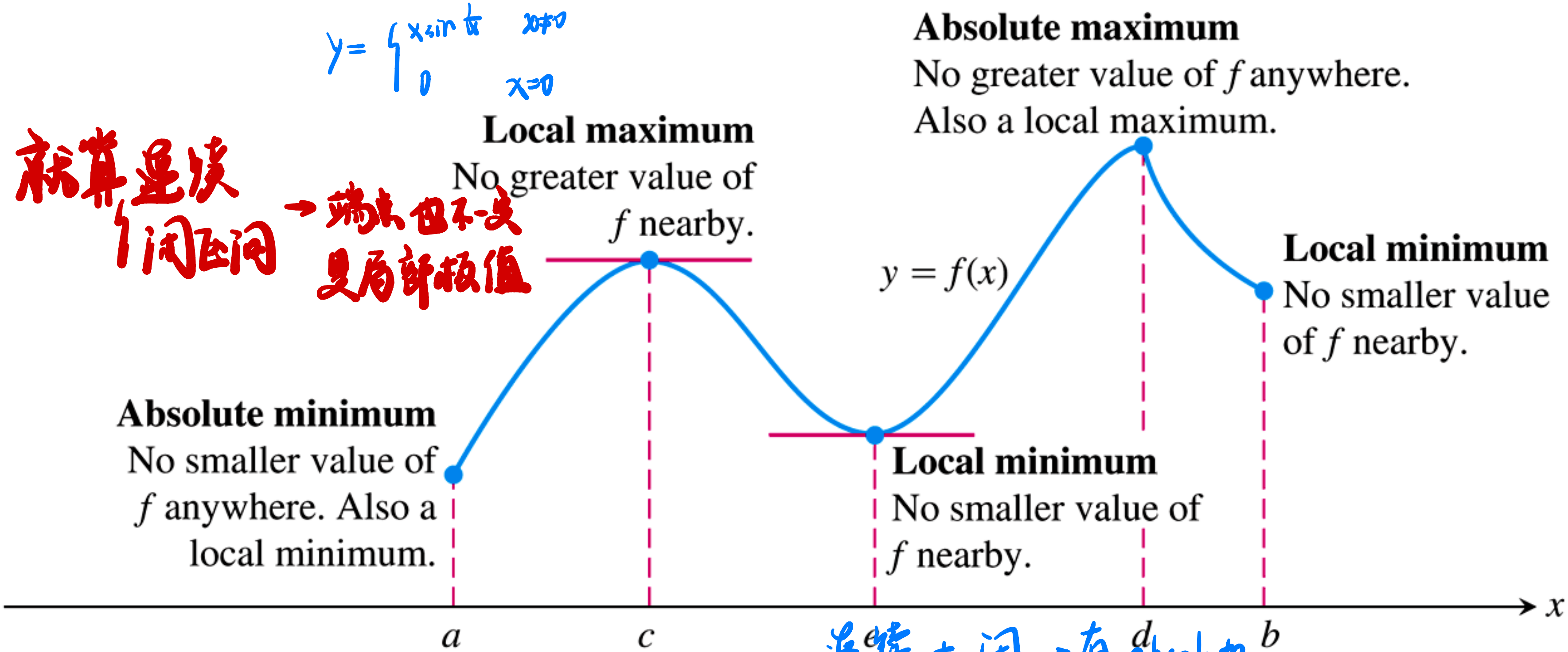
If the domain of f is the closed interval $[a, b]$, then f has a local maximum at the endpoint $x = a$, if $f(x) \leq f(a)$ for all x in some half-open interval $[a, a + \delta)$, $\delta > 0$. Likewise, f has a local maximum at an interior point $x = c$ if $f(x) \leq f(c)$ for all x in some open interval $(c - \delta, c + \delta)$, $\delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \leq f(b)$ for all x in some half-open interval $(b - \delta, b]$, $\delta > 0$. The inequalities are reversed for local minimum values. In Figure 4.5, the function f has local maxima at c and d and local minima at a, e, and b. Local extrema are also called **relative extrema**. Some functions can have infinitely many local extrema, even over a finite interval. One example is the function $f(x) = \sin(1/x)$ on the interval $(0, 1]$. (We graphed this function in Figure 2.40.)

对于端点只要看单边



$y = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

就算连续 { 闭区间 → 端点也不一定是局部极值



连续 + 闭 → 有 absolute max, min

FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

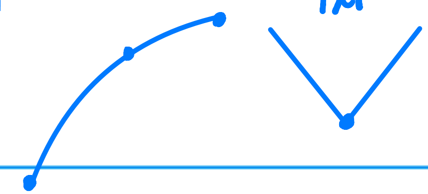
* 边界不一定是局部极值

$\begin{cases} x\sin\frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$ 振荡

距 $x=0$ 在无限近处可能有负值

THEOREM 2—The First Derivative Theorem for Local Extreme Values ② If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then ① 非边界点 $|x|$ ③

$$f'(c) = 0.$$



以特例说明

$$\left\{\begin{array}{l}\text{内部非边界} \\ \text{局部abs} \\ f'\text{有定义}\end{array}\right. \Rightarrow f'(c)=0$$

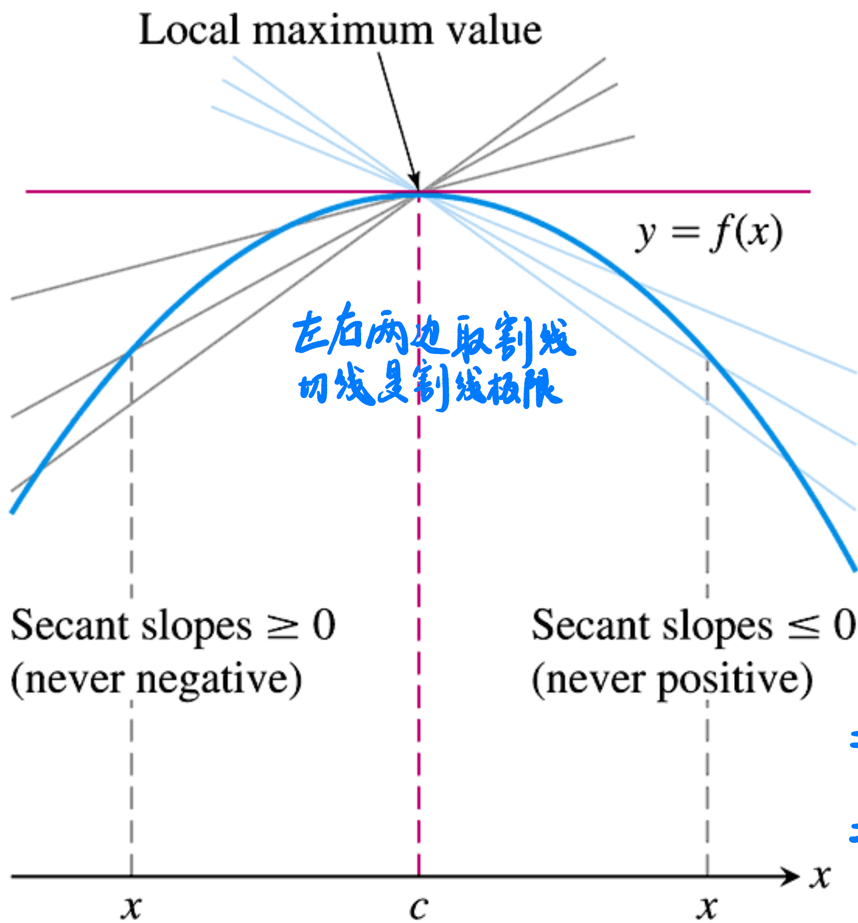


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

既局部极小又局部极大
(中文书可能有全部但没局部)
全局极值一定是局部极值

$$f'(c) = \lim_{h \to 0} \frac{f(h+c) - f(c)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \quad \begin{matrix} \leq 0 \\ > 0 \end{matrix} \quad \leq 0$$

$f'(c)$ (有定义)

$$= \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \quad \begin{matrix} \leq 0 \\ < 0 \end{matrix} \quad \geq 0$$

$$f(x) \leq f(c) \Rightarrow f'(c) = 0$$

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. If we recall that all the domains we consider are intervals or unions of separate intervals, the only places where a function f can possibly have an extreme value (local or global) are

都不一定是局部极值，但满足局部极值一定

1. interior points where $f' = 0$, $y=x^3$ — At $x = c$ and $x = e$ in Fig. 4.5

为其中一种

2. interior points where f' is undefined. — At $x = d$ in Fig. 4.5
3. endpoints of the domain of f. — At $x = a$ and $x = b$ in Fig. 4.5

$\begin{cases} x\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$

局部极值必满足其中一条

If $f(x)$ is a local max / min,
then $(c, f(x))$ is a critical point or an end point.

DEFINITION An interior point of the domain of a function f where f' is zero ① or undefined ② is a **critical point** of f.

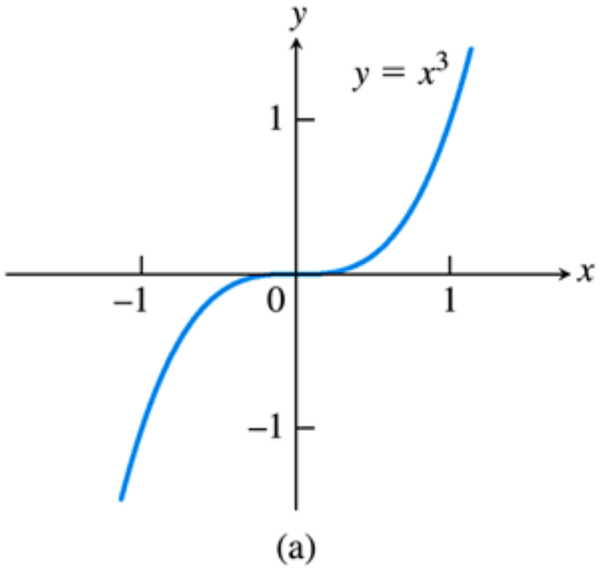
临界点

$f'(x)$ $\begin{cases} 0 \\ \infty \end{cases}$

临界点 $\Leftrightarrow$ $f'(x)=0$ / 无 $f'(x)$

但不一定有最值

$\begin{cases} \text{临界点只要 } f'(x)=0 \text{ / 无 } f'(x) \\ \text{拐点还要凹凸性改变} \end{cases}$



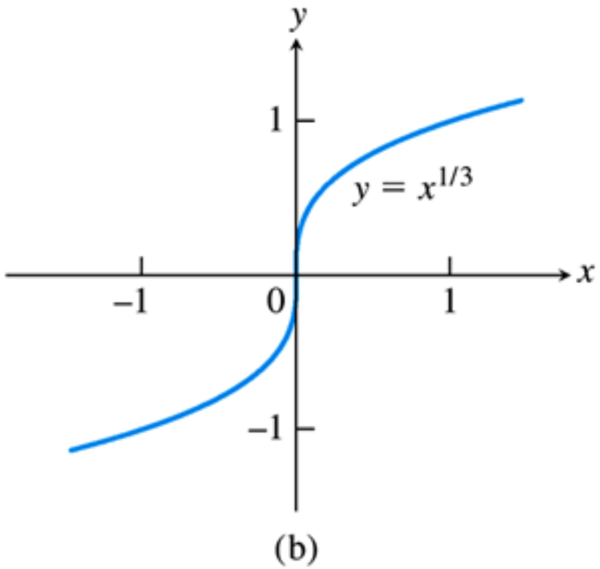


FIGURE 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

①　　　②　不要忘记

1. Evaluate f at all critical points and endpoints.　局部最值只可能在此
2. Take the largest and smallest of these values.

local / global　max/min　value : 值
…　　　　　　　　　point : 点

看英文找定义

中英文表述题目结合

EXAMPLE 3 Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

$g'(t) = 8 - 4t^3 = 0$

$t = \sqrt[3]{2} > 1$ 不在定义域 无临界点

$t = \sqrt[3]{2}$ $f(\sqrt[3]{2}) = 6\sqrt[3]{2}$

$t = -2$ -32

$t = 1$ 7

EXAMPLE 4 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

$y' = \frac{2}{3} \frac{1}{x^{\frac{1}{3}}}$

$x = -2$

$x = 0$

$x = 3$

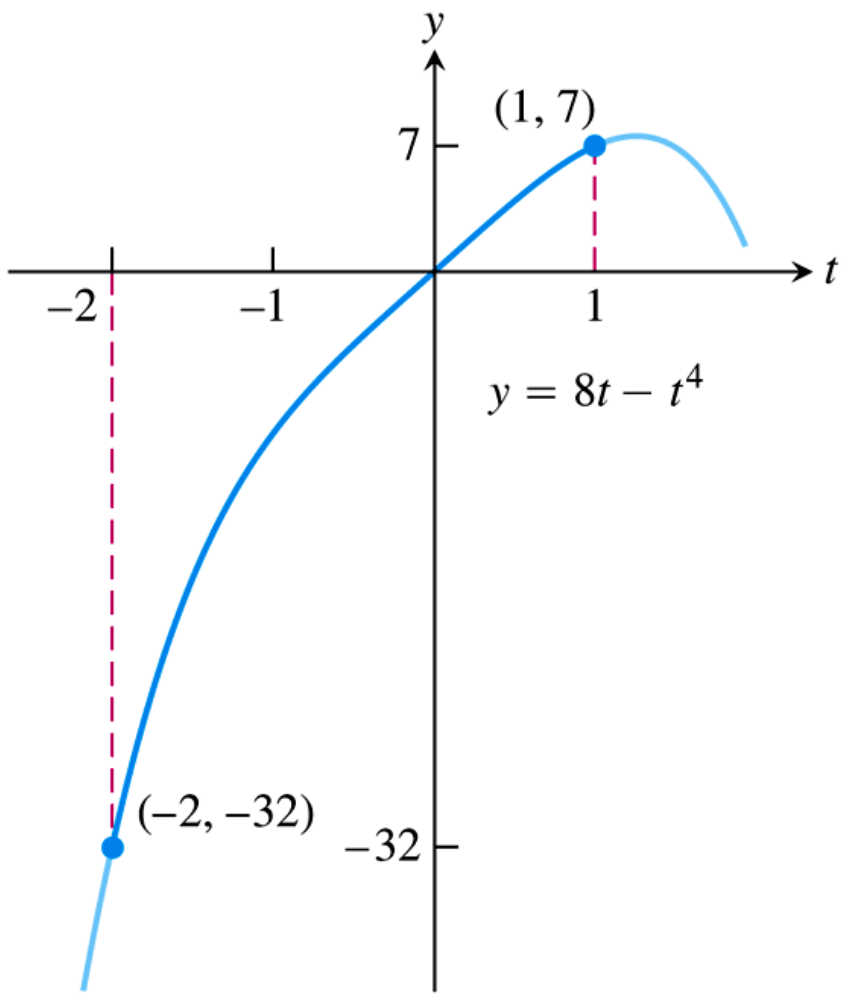


FIGURE 4.8 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

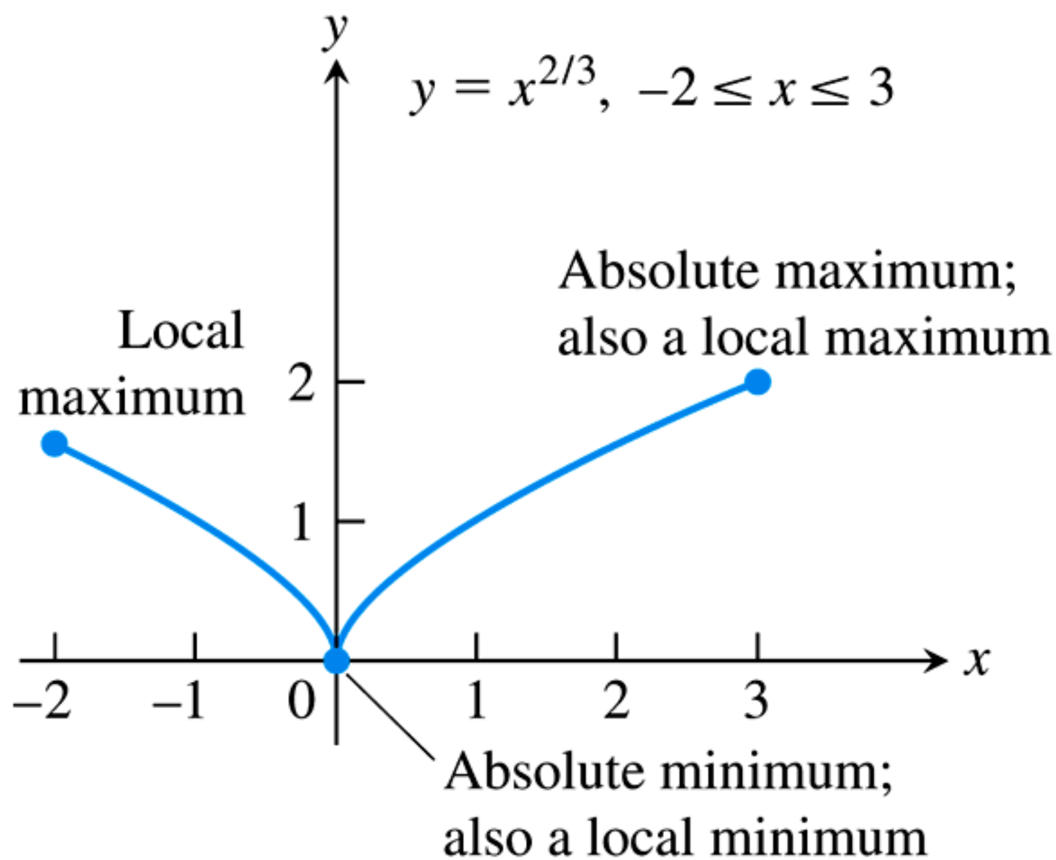


FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

4.2

The Mean Value Theorem

① 闭区间连续

THEOREM 3—Rolle's Theorem Suppose that $y = f(x)$ is continuous over

② 开区间可导

the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

③

① $\Rightarrow$ $\exists$ abs max / min

② if $f(x) = k$ 成立

if $f(x) \neq k$ then abs max $\neq$ abs min

③ $f(a) = f(b) \Rightarrow$ At least one of abs max and abs min occur at an interior point c.

② $\Rightarrow f'(c) = 0$

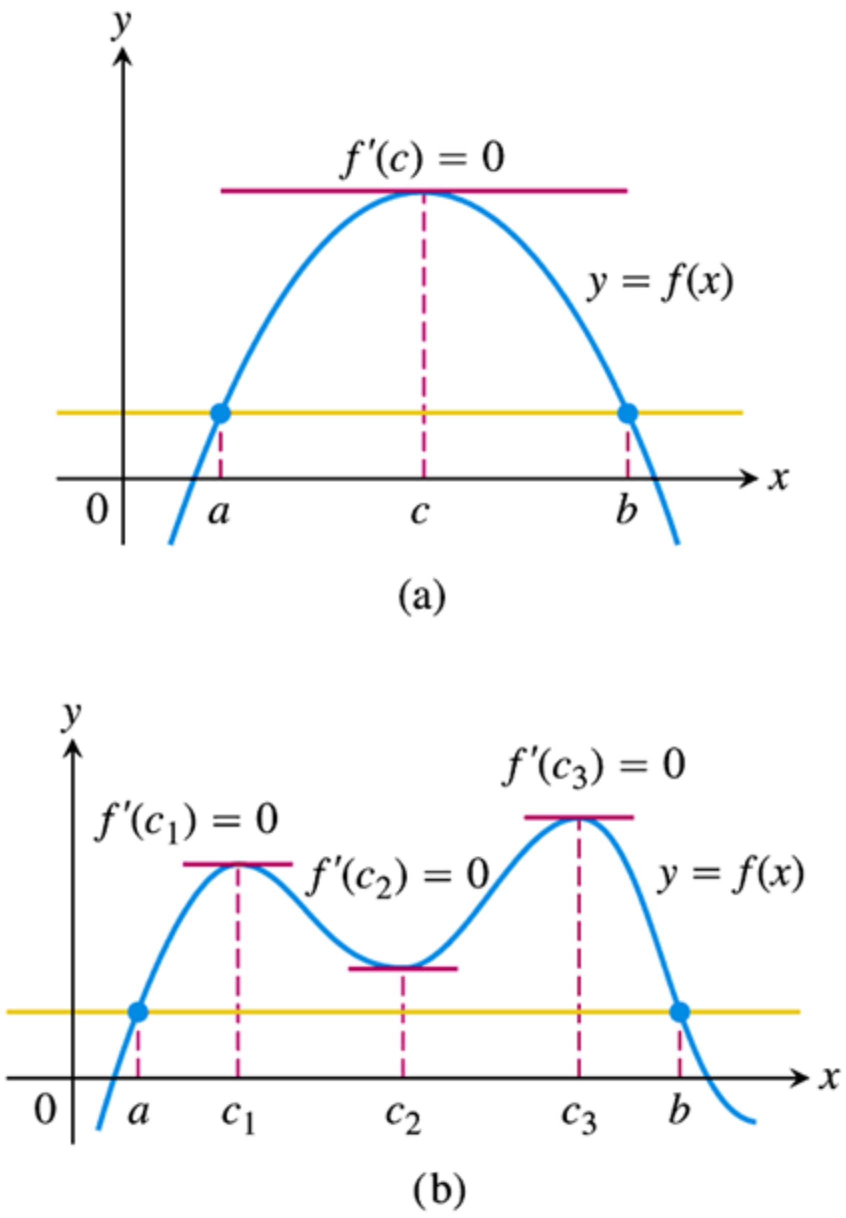
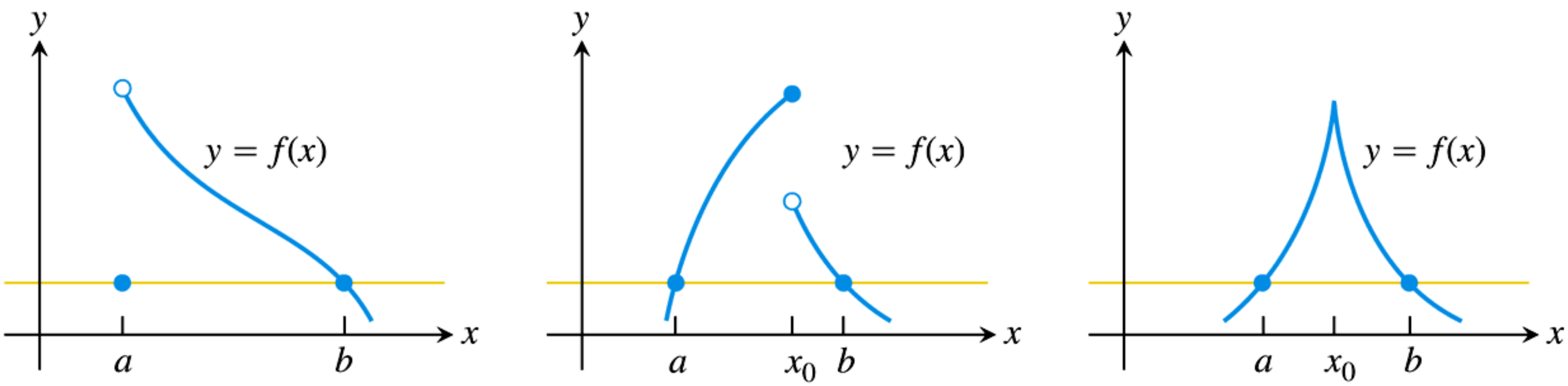


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).



(a) Discontinuous at an endpoint of $[a, b]$

闭区间不连续

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

开区间不可导

FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

$f(0) = 1 > 0$

$f(-1) = -3 < 0$

$\Rightarrow$ At least there exists one root.

Prove by contradiction

Assume that $\exists\, c_1 < c_2$

$f(c_1) = f(c_2) = 0$

$\Rightarrow \exists\, c_1 < \alpha < c_2$

$f'(\alpha) = 0$ $\quad f'(x) = 3x^2 + 3 > 0$

It leads to the contradiction

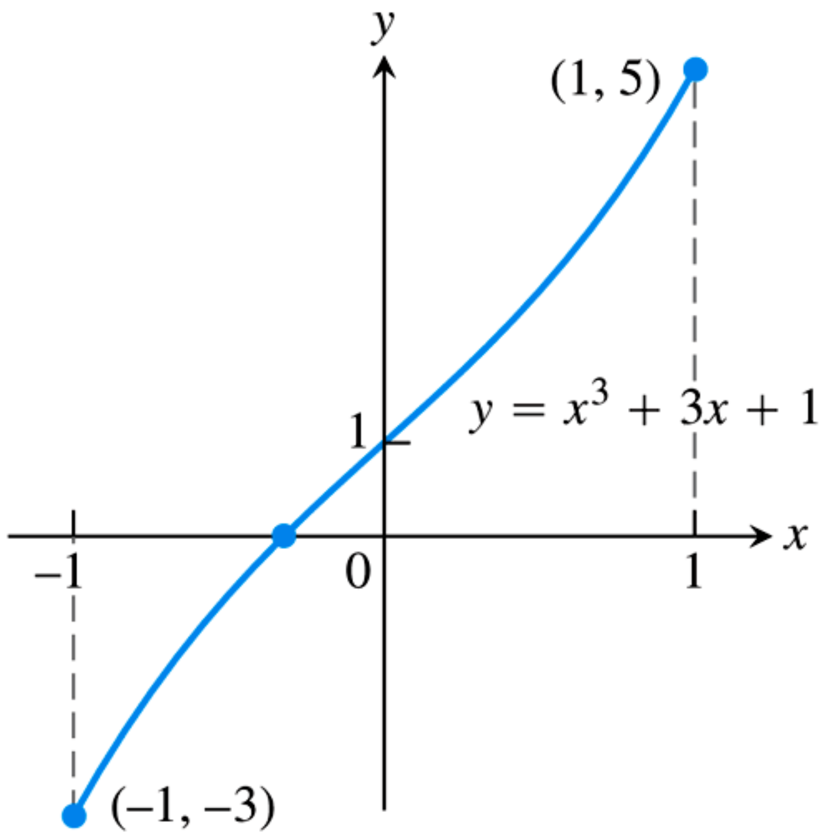


FIGURE 4.12 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 1).

中值定理

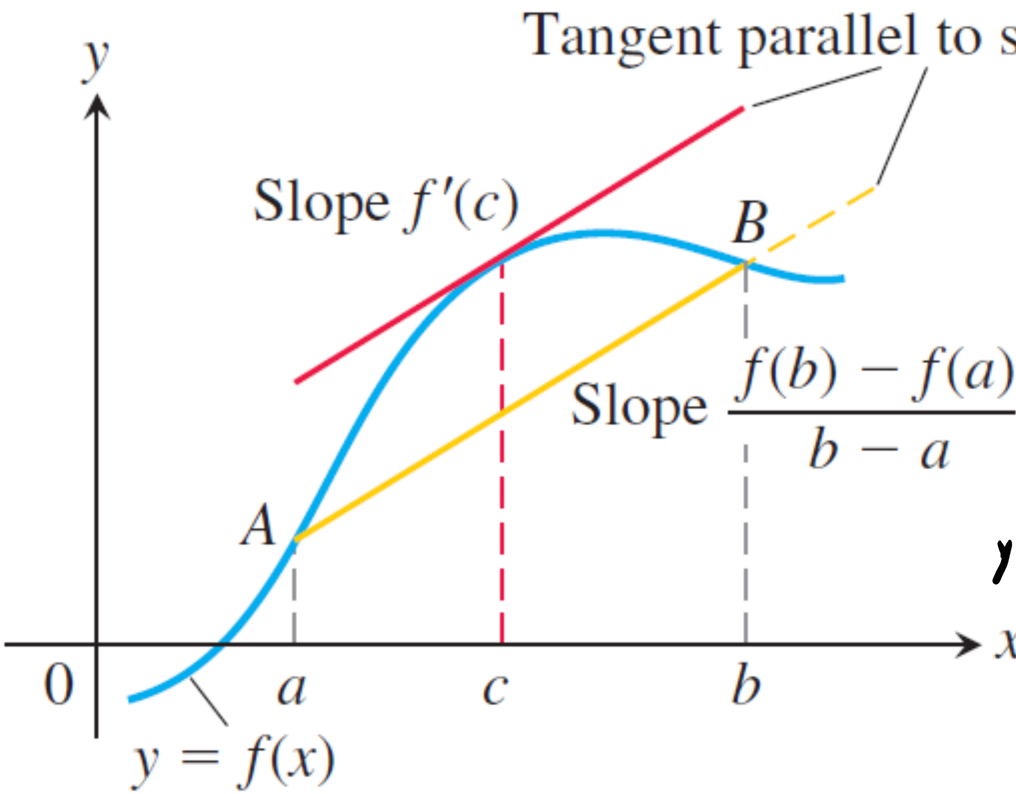
① 闭区间连续

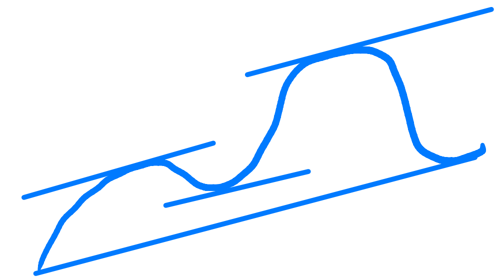
THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

② 开区间可导

when $f(a)=f(b)$ $f'(c)=0$ (罗尔定理)

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$





局部到割线最远
移动平行线至相切

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$$g(a) = g(b) = 0$$

闭区间连续 开区间可导

$$\Rightarrow \exists\, c \in (a, b)$$

$$g'(c) = 0$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between *a* and *b* the curve has at least one tangent parallel to the secant joining *A* and *B*.

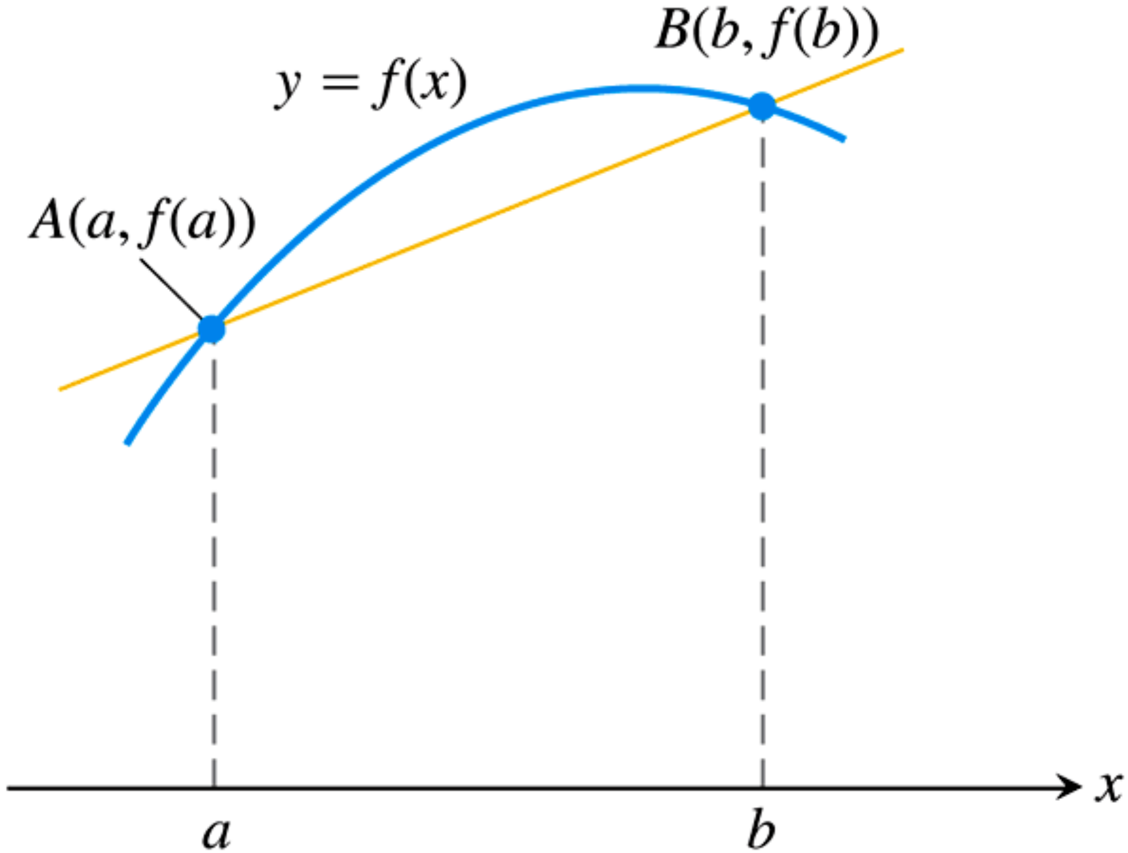


FIGURE 4.14 The graph of f and the chord AB over the interval $[a, b]$.

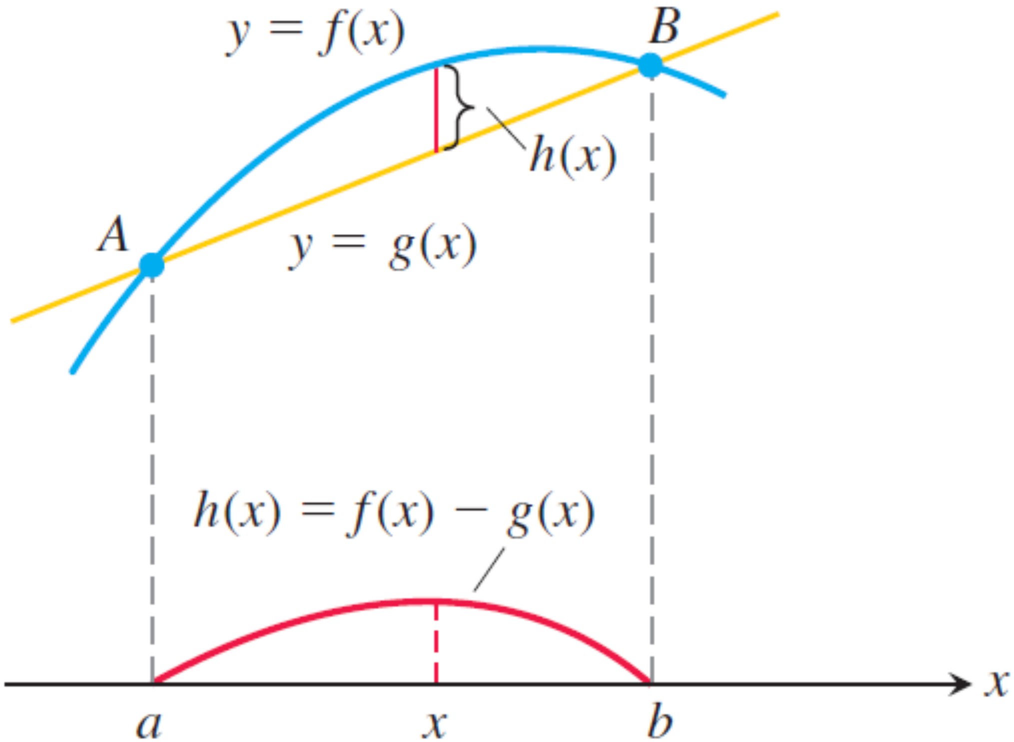


FIGURE 4.15 The secant AB is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

The hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b. One-sided continuity at a and b is enough (Figure 4.16).

只要求开区间可导

且会有左右端点不可导

端点可导：只看单边导数

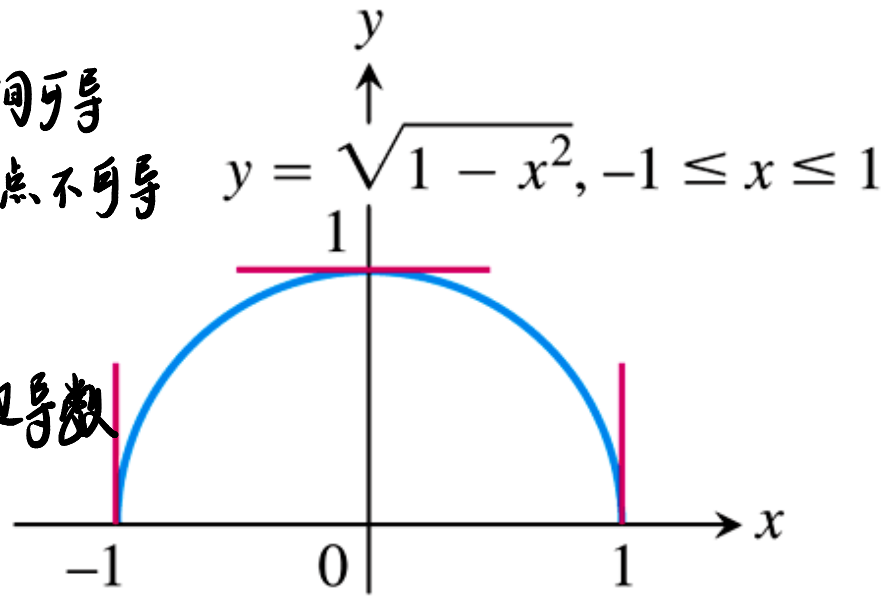
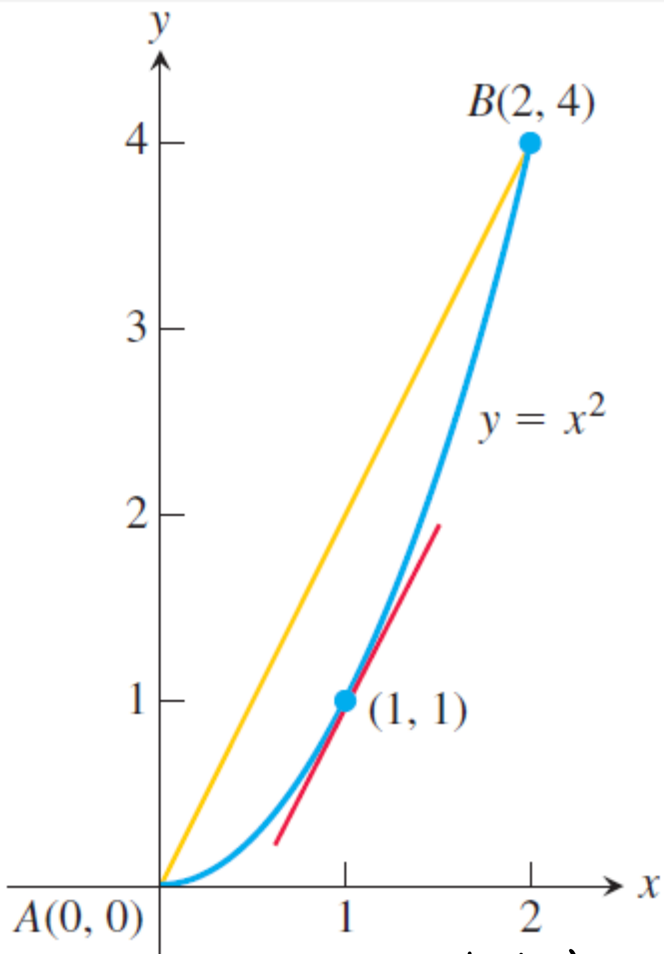


FIGURE 4.16 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

EXAMPLE 2 The function $f(x) = x^2$ (Figure 4.17) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists. ■



中值定义不关心位置，只知道存在

FIGURE 4.17 As we find in Example 2, $c = 1$ is where the tangent is parallel to the secant line.

Prove by contradiction:

Assume that $\exists\, a < c_1 < c_2 < b$

$f(c_1) \neq f(c_2)$ (不常值)

$\Rightarrow \exists\, c_1 < d < c_2$

$f'(d) = \dfrac{f(c_1) - f(c_2)}{c_1 - c_2} \neq 0$

(导数不等于0, 矛盾)

推论

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

$h(x) = f(x) - g(x)$

$h'(x) = 0$

$\Rightarrow h(x) = C$ ✓

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

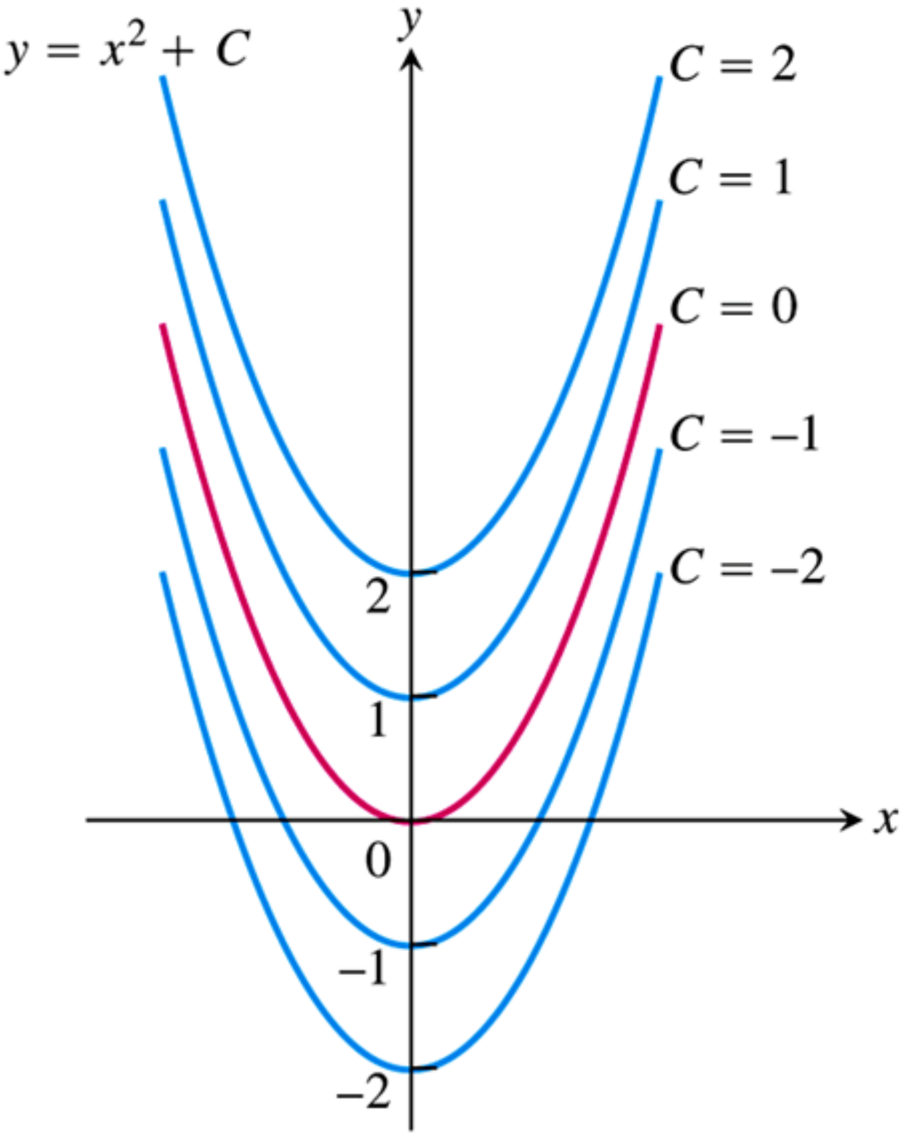


FIGURE 4.19 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

注: $f'(c_1) = \frac{f(b)-f(a)}{b-a}$

$g'(c_2) = \frac{g(b)-g(a)}{b-a}$

不能直接除

把普通函数变参数函数

THEOREM 6—Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

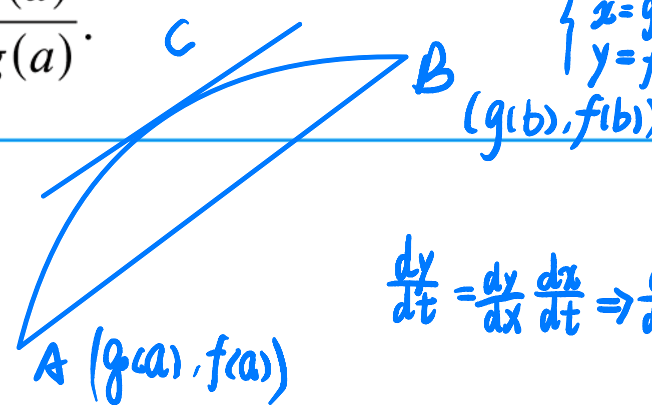
$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

令 $g(x)=x$

即中值定理

$y = h(x)$

$\begin{cases} x = g(t) \\ y = f(t) \end{cases}$



$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

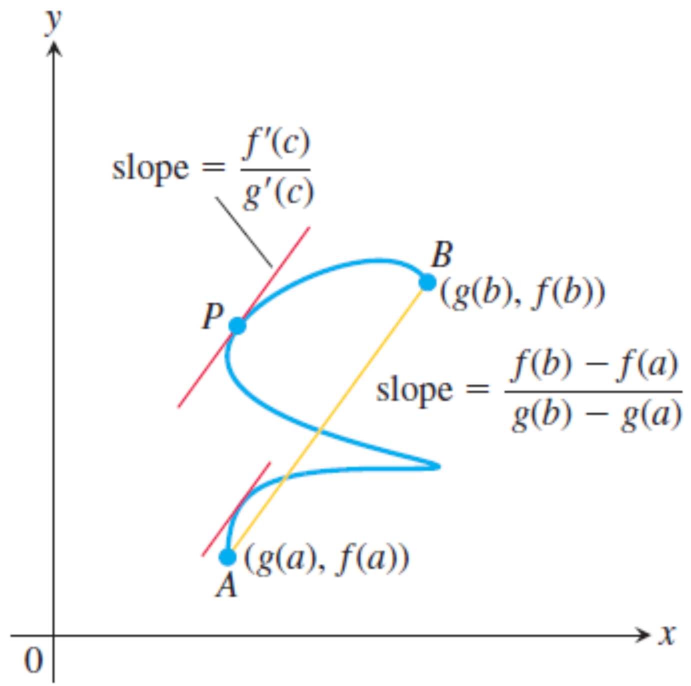


FIGURE 7.20 There is at least one point P on the curve C for which the slope of the tangent to the curve at P is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

$\boxed{\text{使 } x=a/b \quad h(x)=0}$

$$h(x)=f(x)-f(a)+k(x-a)$$

$$f(b)-f(a)+k(b-a)=0$$

$$k=-\frac{f(b)-f(a)}{b-a}$$

构造函数

$\exists\ a<c<b$

证出 Mean Value

$$h'(c)=0$$

$$f'(c)-\frac{f(b)-f(a)}{b-a}=0$$

拉格朗日

用上 g

$$h(x)=f(x)-f(a)+k(g(x)-g(a))$$

$$h(a)=h(b)=0$$

$$f(b)-f(a)+k(g(b)-g(a))=0$$

$$k=-\frac{g(b)-g(a)}{f(b)-f(a)}$$

$\exists\ a<c<b$

$$h'(c)=0$$

$$f'(c)-\frac{g(b)-g(a)}{f(b)-f(a)}g'(c)=0$$

柯西

利用罗尔定理 结果反推

构造型证明

$$\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{f'(\xi)}{g'(\xi)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$f(b)$ $g(b)$

ξ可能是中间的特殊点

此极限应存在

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} = \lim_{x\to 0}\frac{x}{\sin x}\, x\sin\frac{1}{x}$$

$\frac{x}{\sin x}\to 1$, $x\sin\frac{1}{x}\to 0$

但同求导

$$\lim_{x\to 0}\frac{2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)}{\cos x}$$

$\to 0$ 反复振荡

期中不可用

适用范围：右边极限存在

THEOREM 5— L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

限定 $\frac{0}{0}$ so $f(a)=g(a)=0$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

$\frac{\infty}{\infty}$ 若 $\lim\frac{f}{g} = \lim\frac{\frac{1}{g}}{\frac{1}{f}}$

$$\lim\frac{f'}{g^2}\cdot\frac{g^2}{f^2} \neq \lim\frac{f'}{g'}\cdot\lim\frac{g^2}{f^2}$$

例31（2005） 已知函数 $f(x)$ 在 $[0,1]$ 上连续，在 $(0,1)$ 内可导，且 $f(0)=0$，$f(1)=1$. 证明：

(I) 存在 $\xi\in(0,1)$，使得 $f(\xi)=1-\xi$；

(II) 存在两个不同的点 η，$\gamma\in(0,1)$，使得 $f'(\eta)\,f'(\gamma)=1$.

4.3

Monotonic Functions and the First Derivative Test

A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

与推论1，证法相同

Assume $\exists\ a < c_1 < c_2 < b$

$f(c_1) \geq f(c_2)$

$\Rightarrow \exists\ c_1 < d < c_2$

$f'(d) = \dfrac{f(c_1) - f(c_2)}{c_1 - c_2} \leq 0$

矛盾

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

严格 ↑ / ↓

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

反过来不成立 $y = x^3$

EXAMPLE 1 Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

$$f'(x) = 3x^2 - 12$$
$$= 3(x+2)(x-2)$$

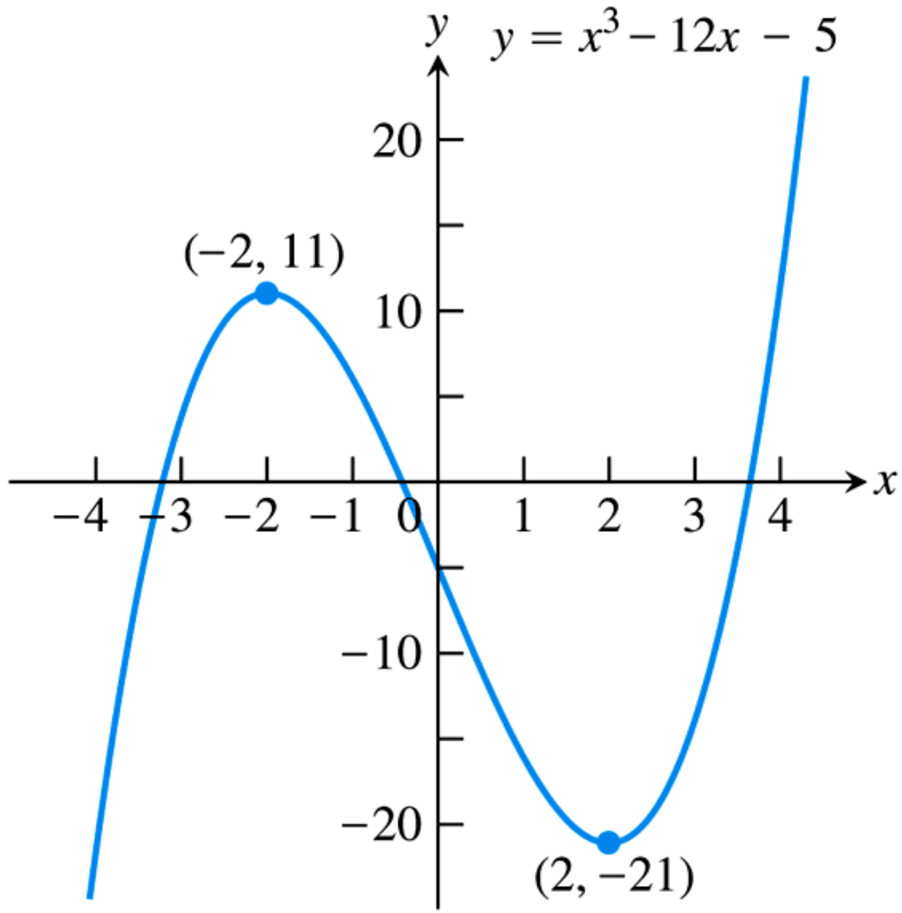


FIGURE 4.20 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

$f'(x)=0$ 但不一定变号

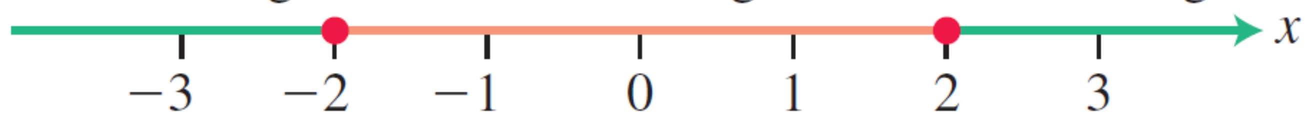
$f(x)$

$\boxed{\text{相邻零点内}}$ $f'(x)$ 符号不变

代入一个值计算 $+-$

sample point

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



画图

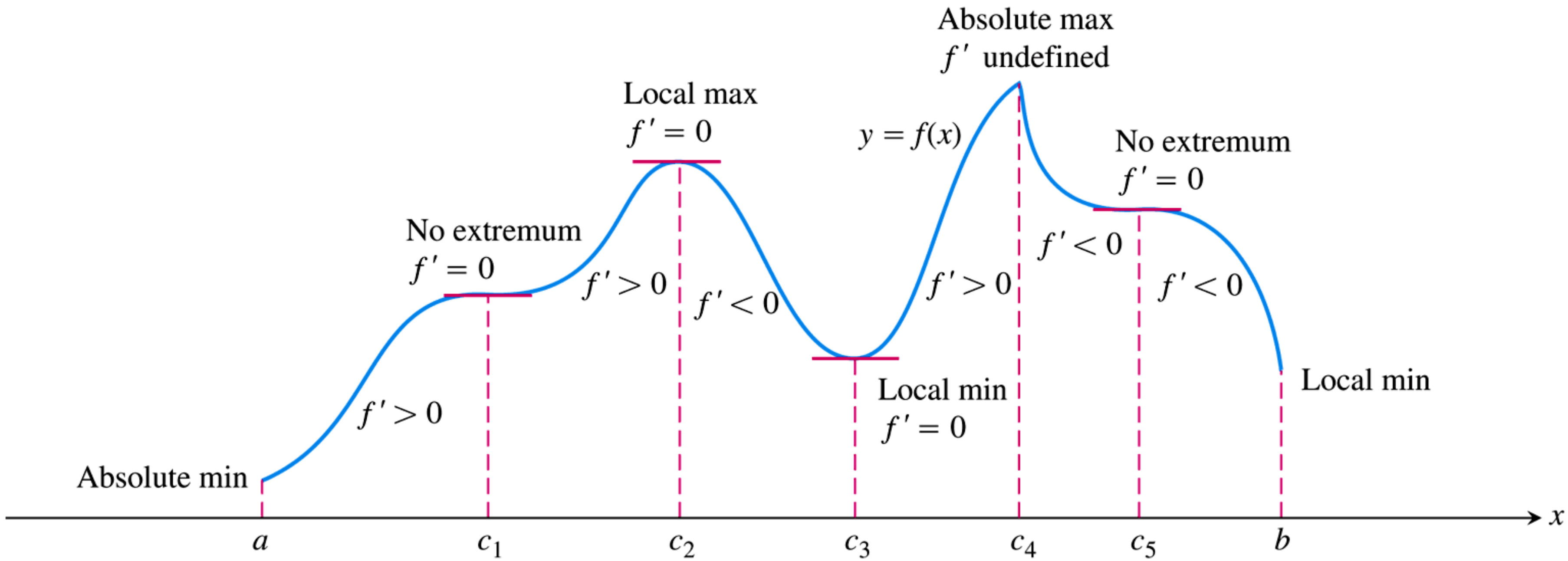


FIGURE 4.21 The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

EXAMPLE 2 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

$$f'(x) = \frac{4}{3}\,\frac{x-1}{x^{\frac{2}{3}}}$$

f': $-$ | 0 | $-$ | 1 | $+$

$(-\infty, 0)$ <u>and</u> $(0, 1)$ decreasing → 尽管不要用并集

$(1, +\infty)$ increasing

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

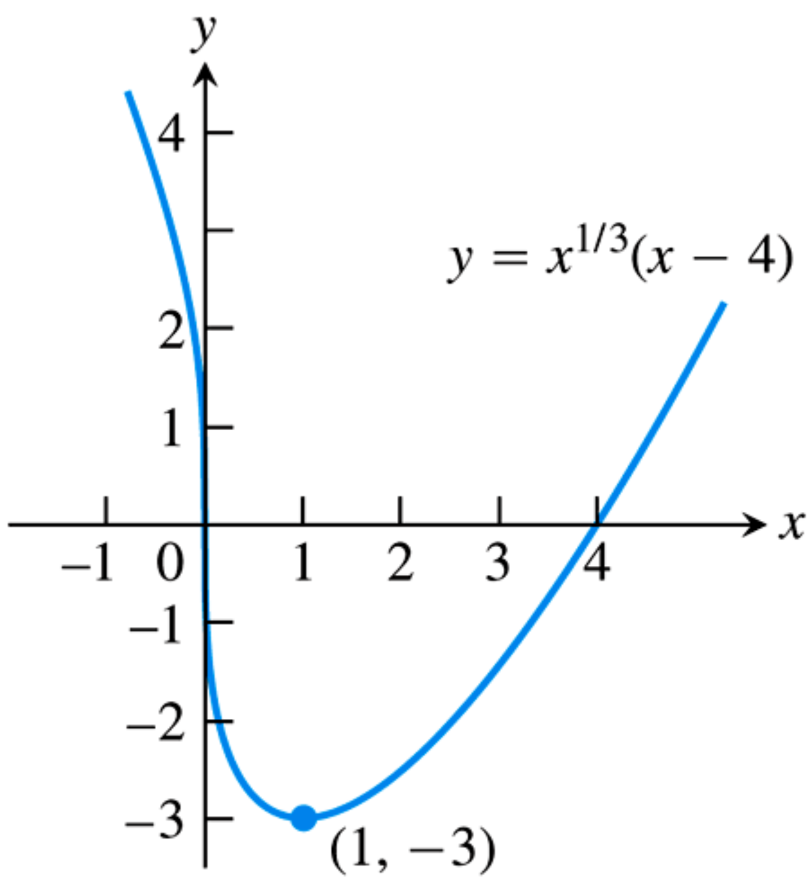


FIGURE 4.22 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

4.4

凹度

Concavity and Curve Sketching

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

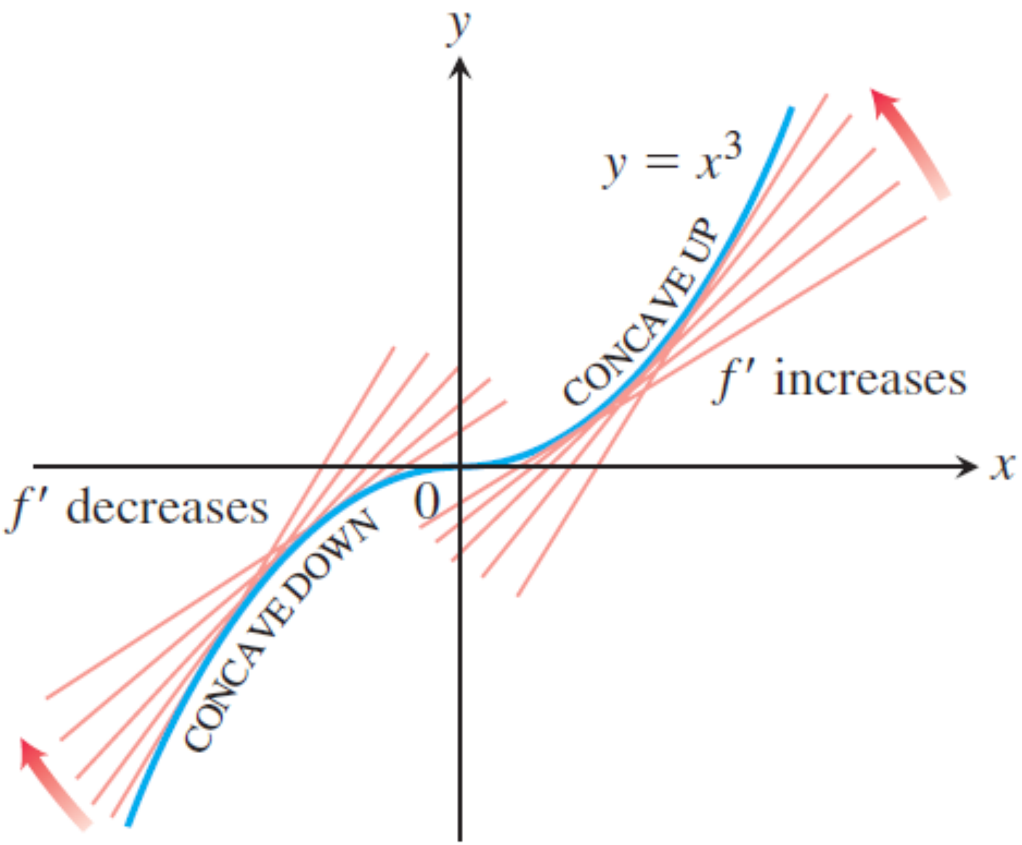


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

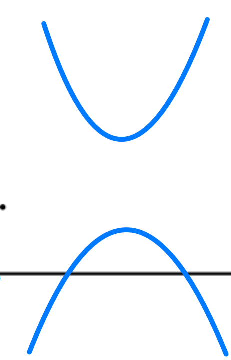
concave {up / down 只与 $f''(x)$ +/- 有关

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

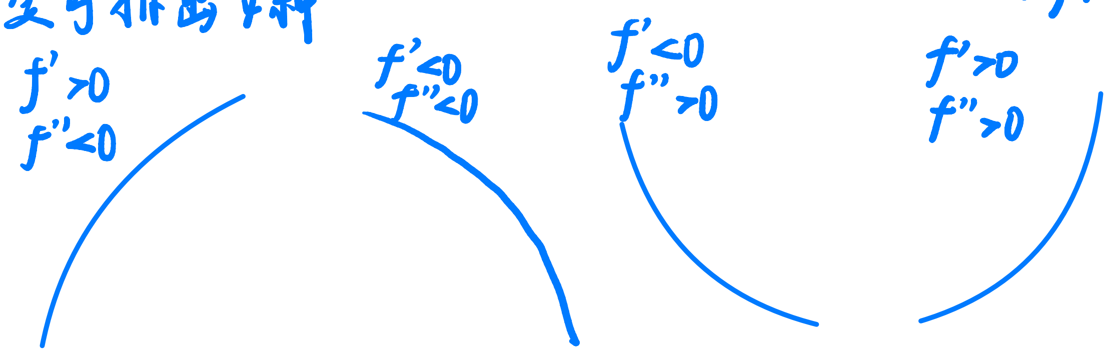
凹

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.



记抛物线向上/下

一定可拆出4种



分开成哪些区间

EXAMPLE 1

(a) The curve $y = x^3$ (Figure 4.24) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.25) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

$y'' = -\sin x$

f'' $-$ $+$

0 π 2π

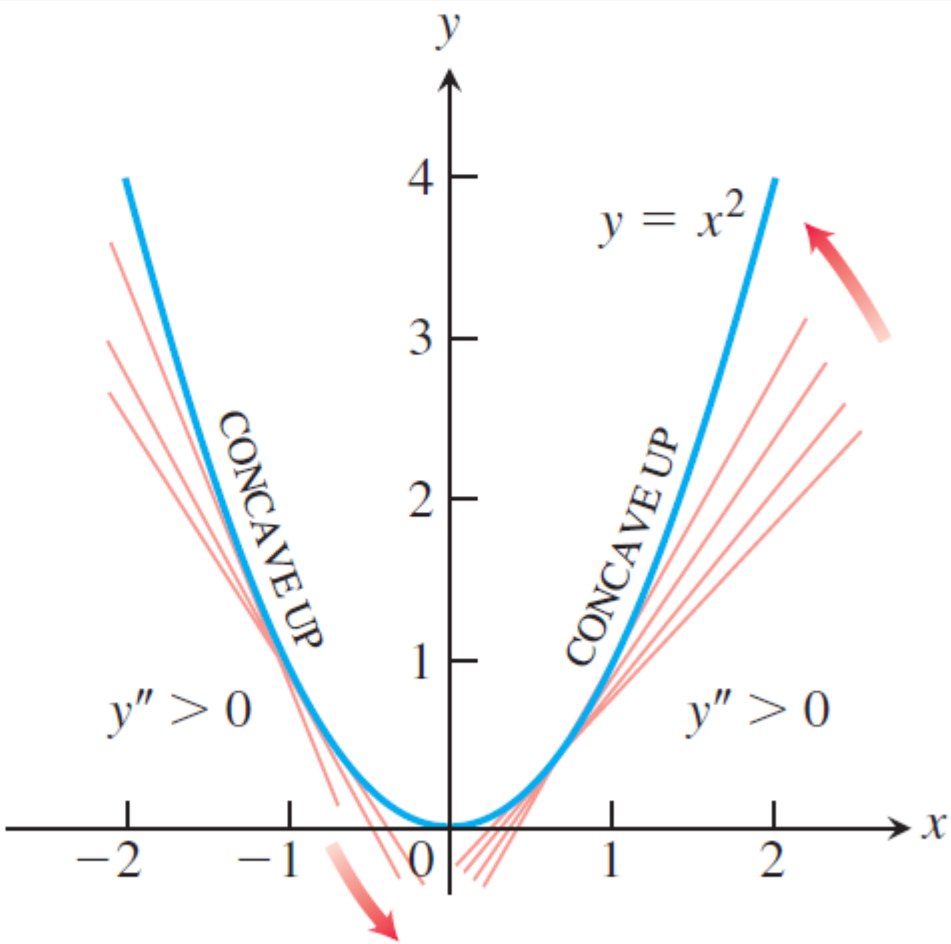


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

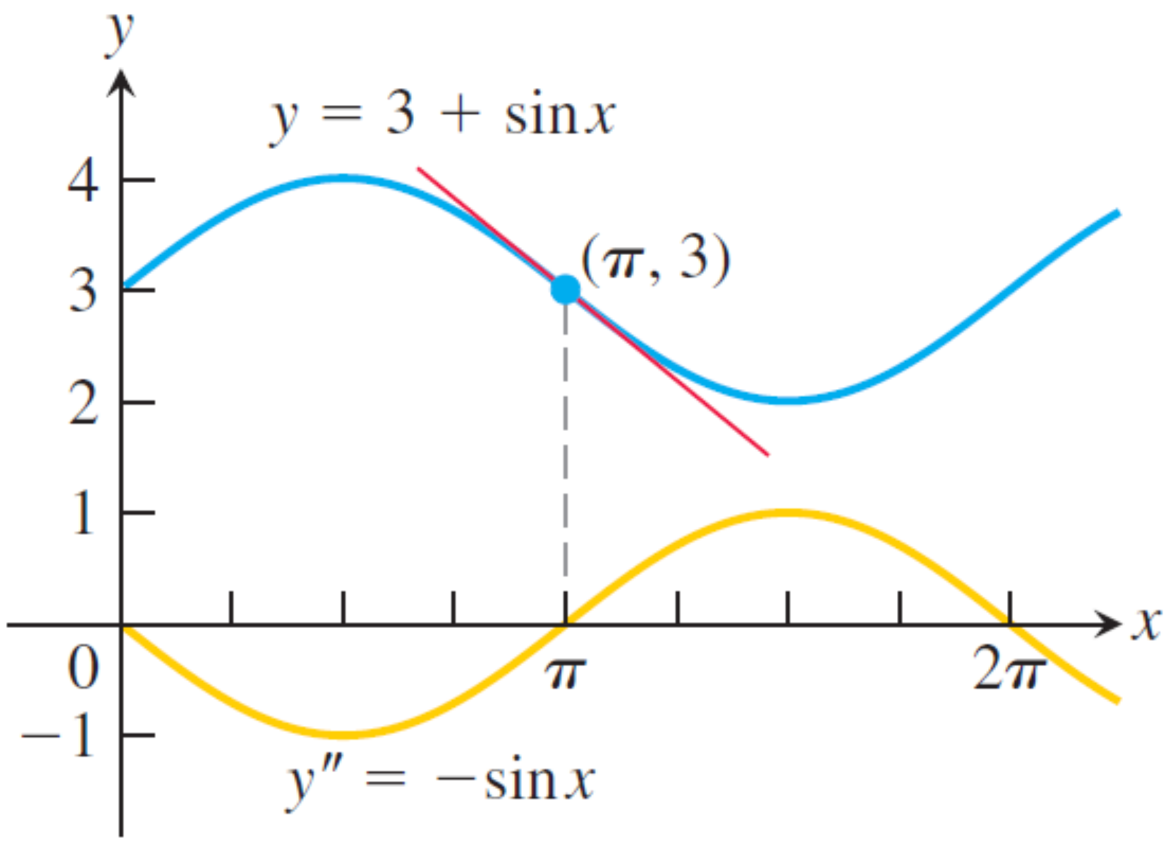


FIGURE 4.26 Using the sign of y'' to determine the concavity of y (Example 2).

$f'(x)$ 存在

有切线

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

凹凸性发生改变 拐点

(1) vertical tangent
$\Rightarrow f''(c)$ not exist
(2) not a vertical tangent
(i) $f''(c)$ not exist $x^{\frac{5}{3}}$
(ii) 存在凹凸性变化

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

反之不成立

$y = x^{\frac{4}{3}}$

$y' = \frac{4}{3}x^{\frac{1}{3}}$

$y'' = \frac{4}{9}\frac{1}{x^{\frac{2}{3}}}$ 0时不存在 0左右 $y''>0$

与 $y=x^{\frac{5}{3}}$ 不同

$y = x^4$

$y'' = 12x^2$ 0时 $=0$ 0左右 >0

但正负也要改变

EXAMPLE 3 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.27. ■

Here is an example showing that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

EXAMPLE 4 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.28). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. ■

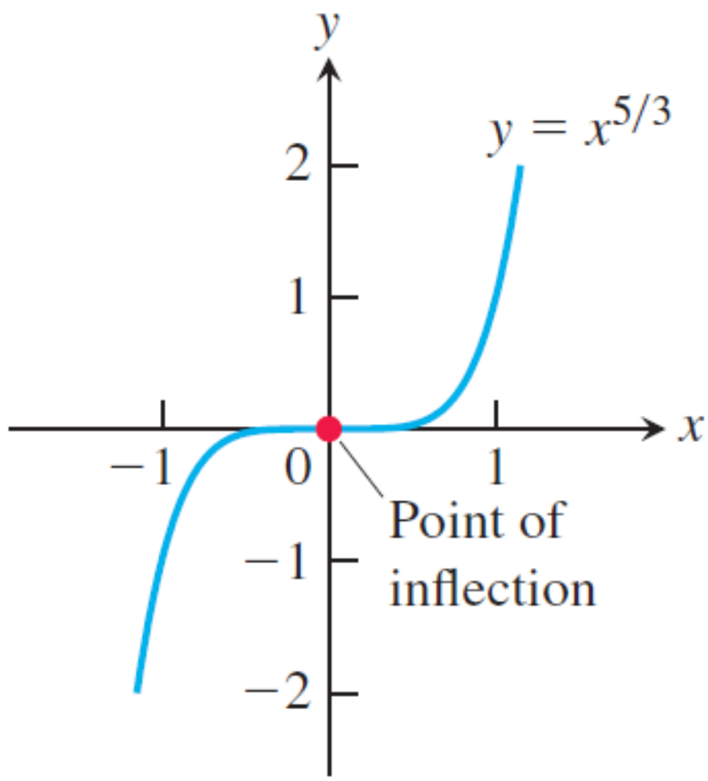


FIGURE 4.27 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 3).

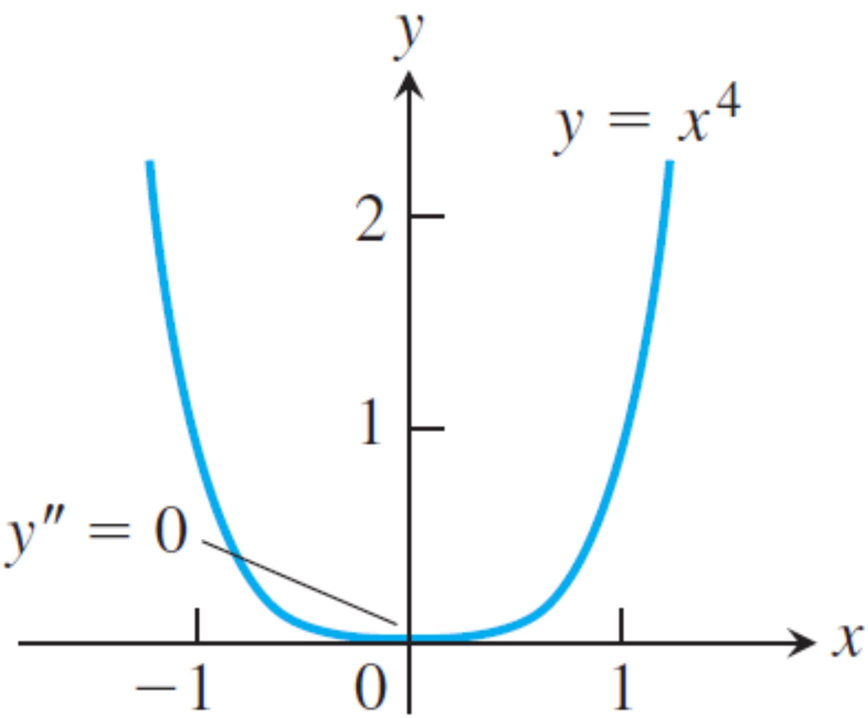


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

EXAMPLE 5 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.29.

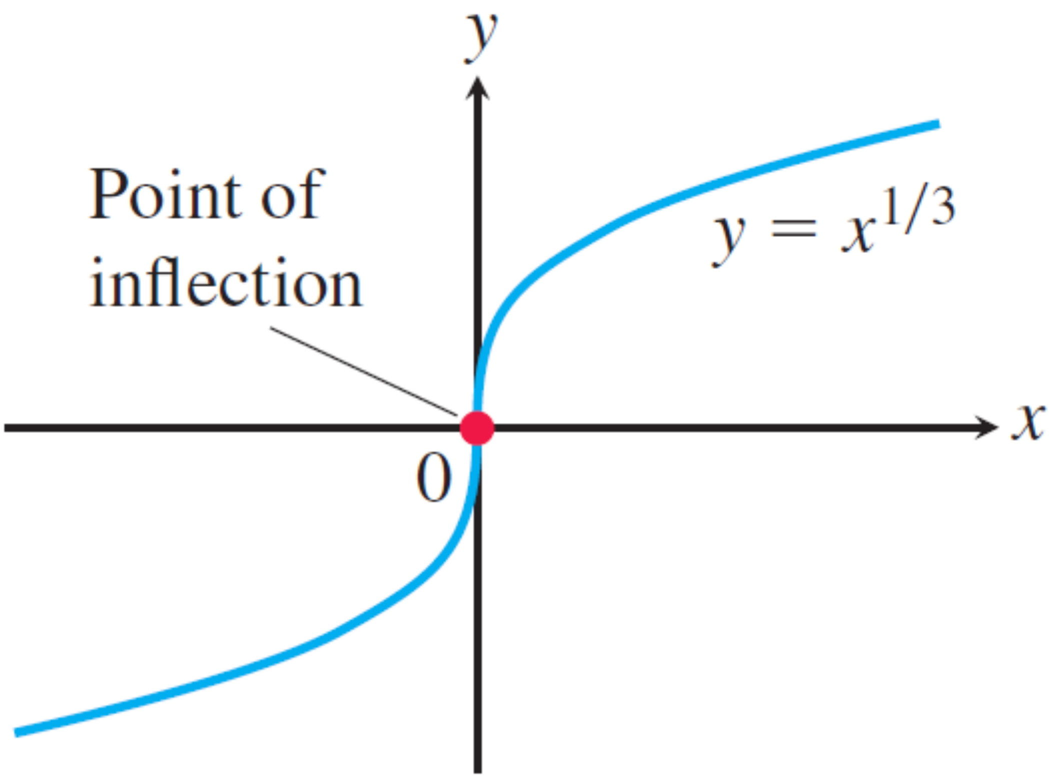


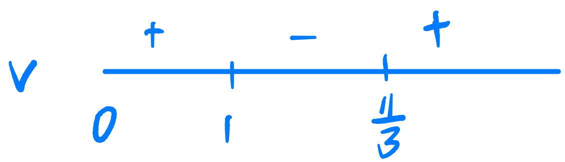
FIGURE 4.29 A point of inflection where y' and y'' fail to exist (Example 5).

EXAMPLE 6 A particle is moving along a horizontal coordinate line (positive to the right) with position function

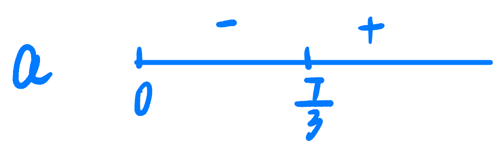
$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

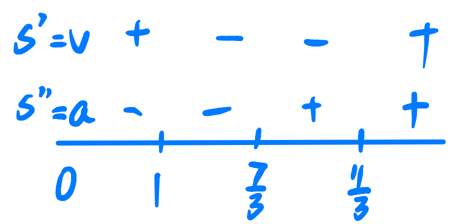
Find the velocity and acceleration, and describe the motion of the particle.

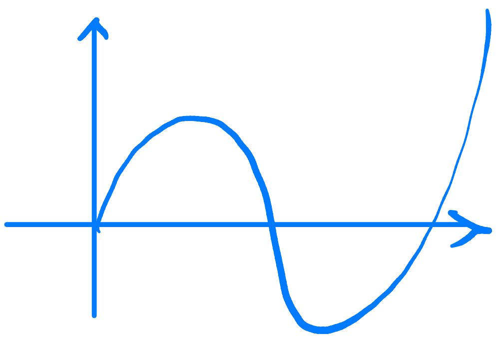
$$v = 6t^2 - 28t + 22$$
$$= (6t - 22)(t - 1)$$



$$a = 12t - 28$$







Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

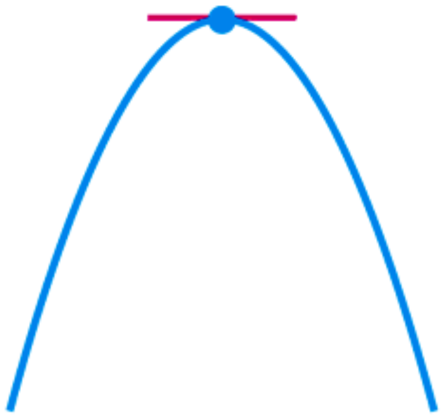
Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

$f'(c)=0$ $f''(c)$ 正负

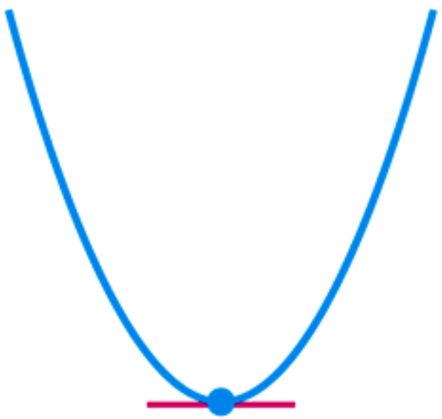
判局部极值

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$
$\Rightarrow$ local max



$f' = 0, f'' > 0$
$\Rightarrow$ local min

必考作图题

EXAMPLE 7 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ $x=0 / x=3$

using the following steps.

$f' = 0$ / 不存在

$f''(x) = 12x^2 - 24x = 12x(x-2)$ $x=0 / x=2$

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

$f'' = 0$ / 不存在

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

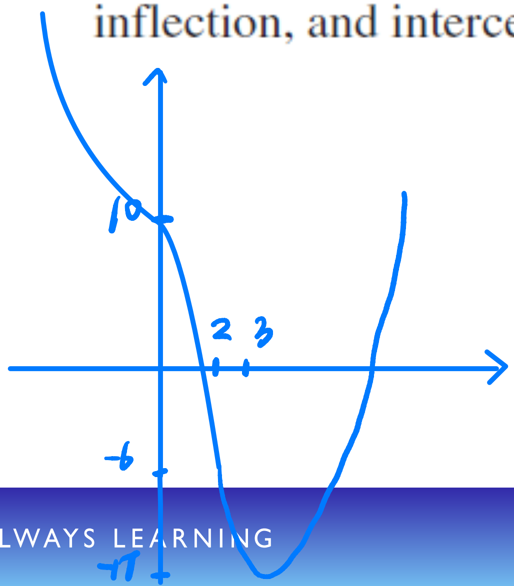
(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

得到该图

f': $-$ $-$ $-$ $+$

f'': $+$ $-$ $+$ $+$ (0, 2, 3)

4种形状



Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

$\boldsymbol{x < 0}$	$\boldsymbol{0 < x < 2}$	$\boldsymbol{2 < x < 3}$	$\boldsymbol{3 < x}$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

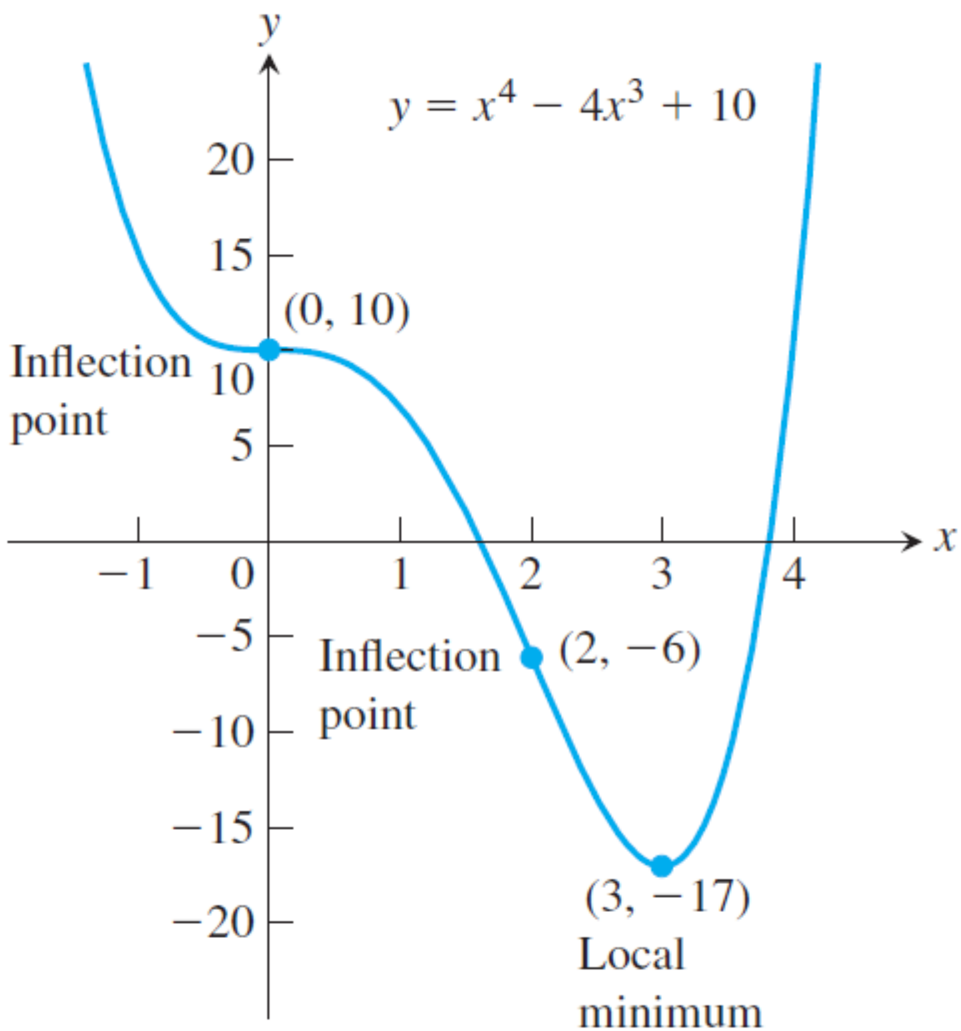


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

Procedure for Graphing $y = f(x)$

对称性 与渐近线

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.

$f'(x)=0$ / 不存在

但正负改变

4. Find where the curve is increasing and where it is decreasing.
5. Find the points of $\boxed{\text{inflection}}$, if any occur, and determine the concavity of the curve.

$f''(x)=0$ / 不存在

渐近线

6. Identify any asymptotes that may exist
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

画图会考有理函数

*有理函数可能产生渐近线

不奇不偶

EXAMPLE 8 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

水平 $\lim\limits_{\substack{x\to+\infty \\ -\infty}} \dfrac{(x+1)^2}{1+x^2} = \lim\limits_{x\to\infty} \dfrac{1+\frac{2}{x}+\frac{1}{x^2}}{\frac{1}{x^2}+1} = 1$

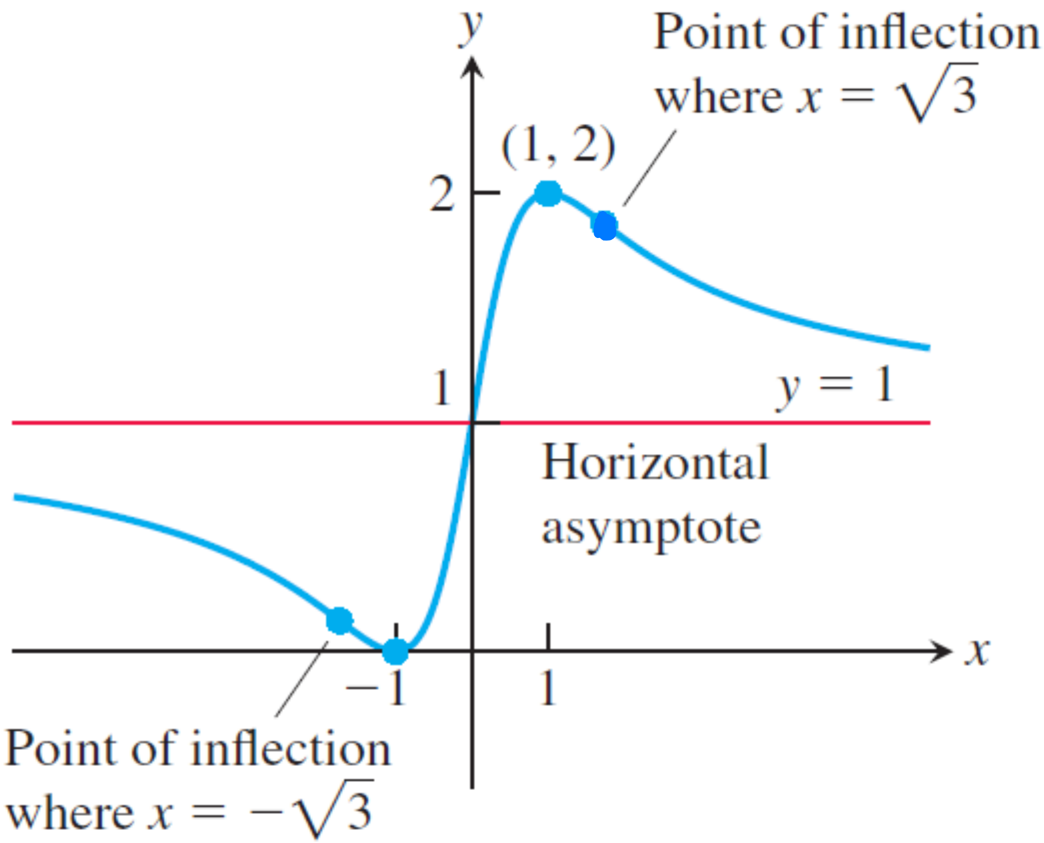
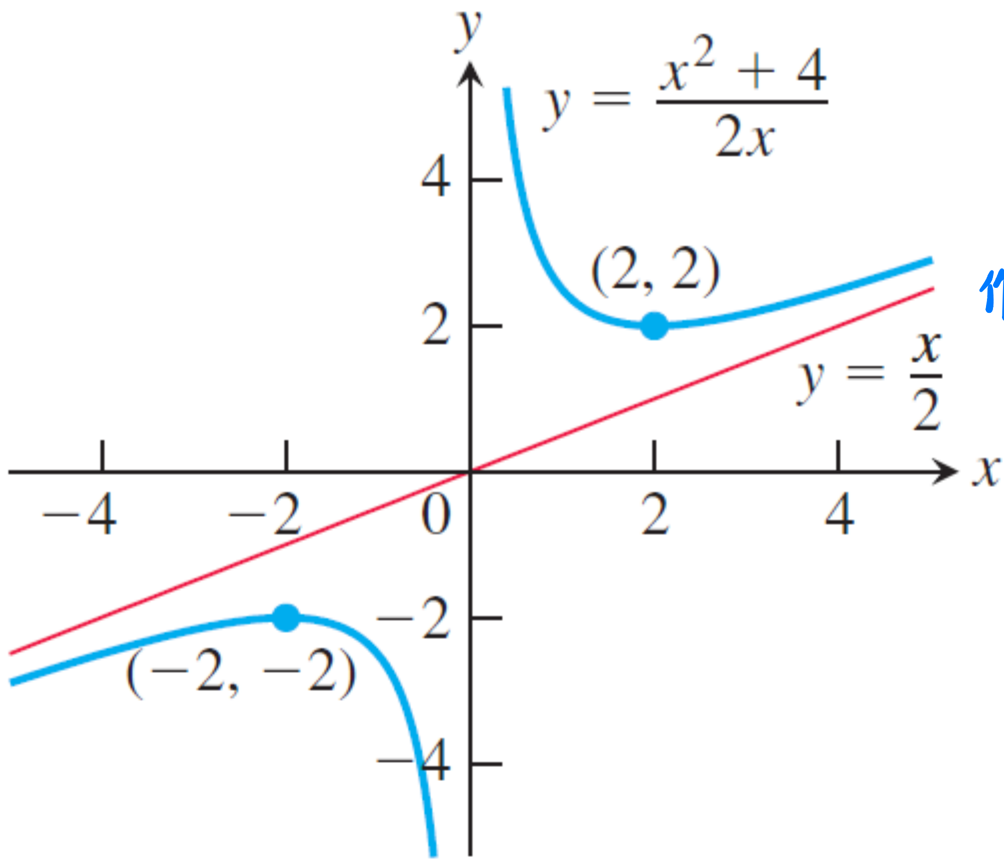


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 8).

$= 1 + \dfrac{2x}{1+x^2}$

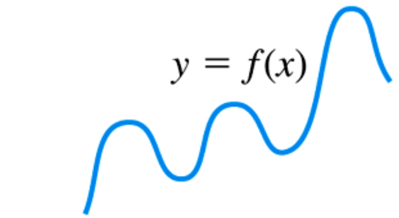
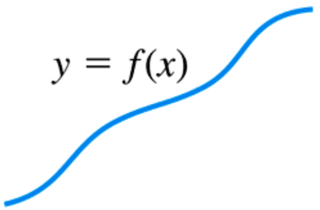
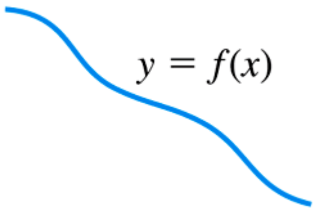
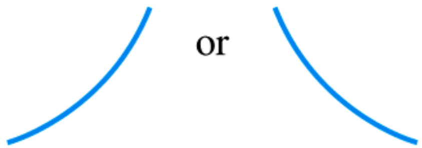
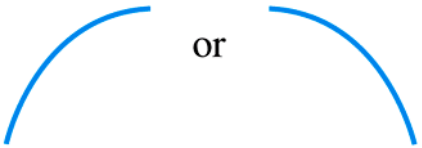
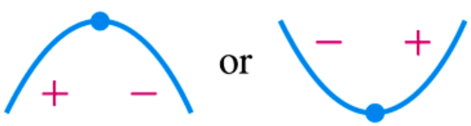
EXAMPLE 9 Sketch the graph of $f(x) = \dfrac{x^2 + 4}{2x}$.

$y = \frac{x}{2} + \frac{2}{x}$



作图 渐近线 画成虚线

FIGURE 4.32 The graph of $y = \dfrac{x^2 + 4}{2x}$ (Example 9).

 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy 波动的	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 y'' changes sign at an inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

例20（2007） 设函数 $f(x)$ 在 $(0,+\infty)$ 内具有二阶导数，且 $f''(x)>0$，令 $u_n=f(n)$ $(n=1,2,\cdots)$，则下列结论正确的是

刻画出 $f''(x)$ 全部情况

(A) 若 $u_1>u_2$，则 $\{u_n\}$ 必收敛. (B) 若 $u_1>u_2$，则 $\{u_n\}$ 必发散.

(C) 若 $u_1<u_2$，则 $\{u_n\}$ 必收敛. (D) 若 $u_1<u_2$，则 $\{u_n\}$ 必发散.

例17（2017） 已知函数 $y(x)$ 由方程 $x^3+y^3-3x+3y-2=0$ 确定，求 $y(x)$ 的极值. 确定极大/极小

方程 —— 隐函数求导

若关于 y 式符号不定

$\frac{dy}{dx}=\frac{(y^2-4)(1-x^2)}{y^2+1}$ 既有 x 也有 y

做法：回去求二阶导 乘回去

回代 $(y^2+1)\frac{dy}{dx}=(y^2-4)(1-x^2)$ $\left.\frac{dy}{dx}\right|_{x=\ ,y=}$

$\frac{d^2y}{dx^2}(y^2+1)+2y\left(\frac{dy}{dx}\right)^2=\cdots$ 一阶导为0

再做代入

有 $\frac{dy}{dx}=0$

$3x^2+3y^2\frac{dy}{dx}-3+3\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-(x^2-1)}{y^2+1}$ 对任何一点 导数存在 分母≠0

$x=1/-1$ $f'(x)=0$

$\frac{dy}{dx}$ $-$ $+$ $-$ （-1，1）

$x=1$ $y^3+3y-4=0$ $y=1$ $(1,1)$

$y-1\ \overline{)\,y^3+3y-4}$ y^2+y+4 y^3-y^2 y^2+3y-4 y^2-y $4y-4$

$x=-1$ $y^3+3y=0$ $y=0$ $(-1,0)$

4.5

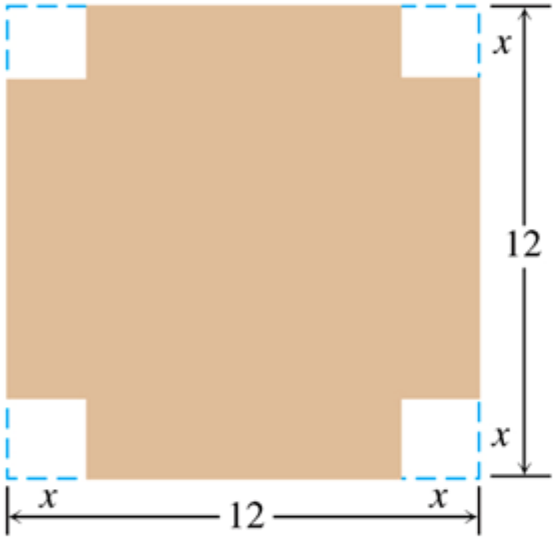
Applied Optimization

应用优化

Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

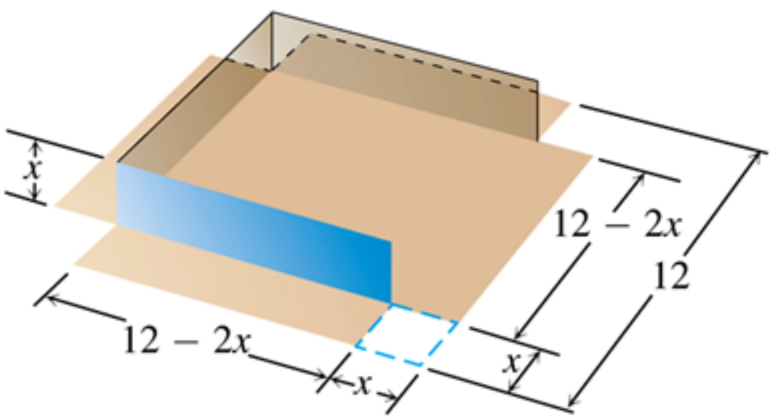
EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



(a)

$$\frac{dV}{dx} = 12(x-2)(x-6)$$

范围



(b)

FIGURE 4.34 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

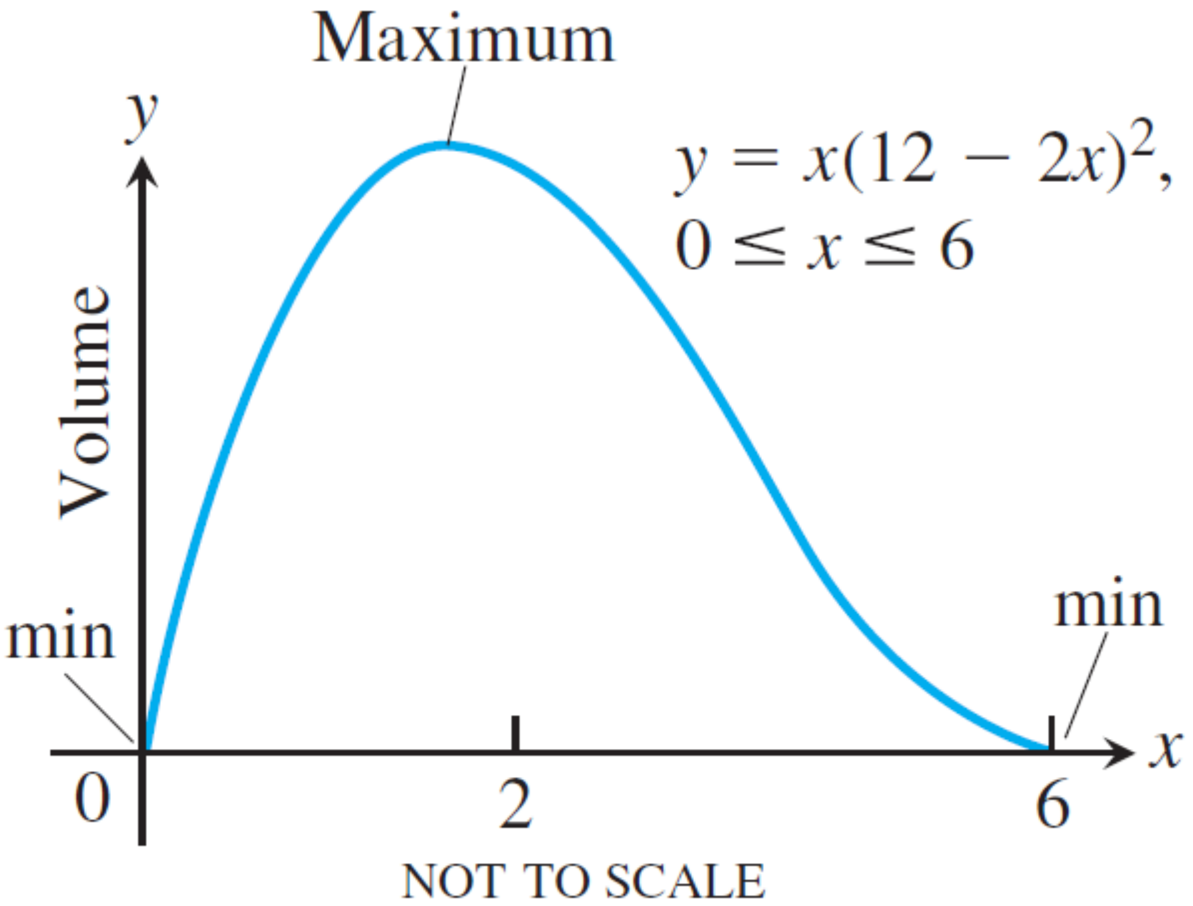


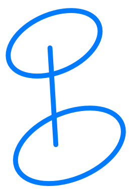
FIGURE 4.35 The volume of the box in Figure 4.34 graphed as a function of x.

注意自变量范围

直角 圆的

EXAMPLE 2 You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.36). What dimensions will use the least material?

柱面



$\pi r^2 h = 1$

$A = 2\pi r + 2\pi r h$

$= 2\pi r^2 + \frac{2}{r}$

$\frac{dA}{dr} = 4\pi r - \frac{2}{r^2}$

$= \frac{4\pi r^3 - 2}{r^2} \quad r \neq 0$

$= 0$

$\Rightarrow r = \sqrt[3]{\frac{1}{2\pi}} \qquad h = 2r$

$-$ $+$

$0 \qquad \frac{1}{\sqrt[3]{2\pi}}$

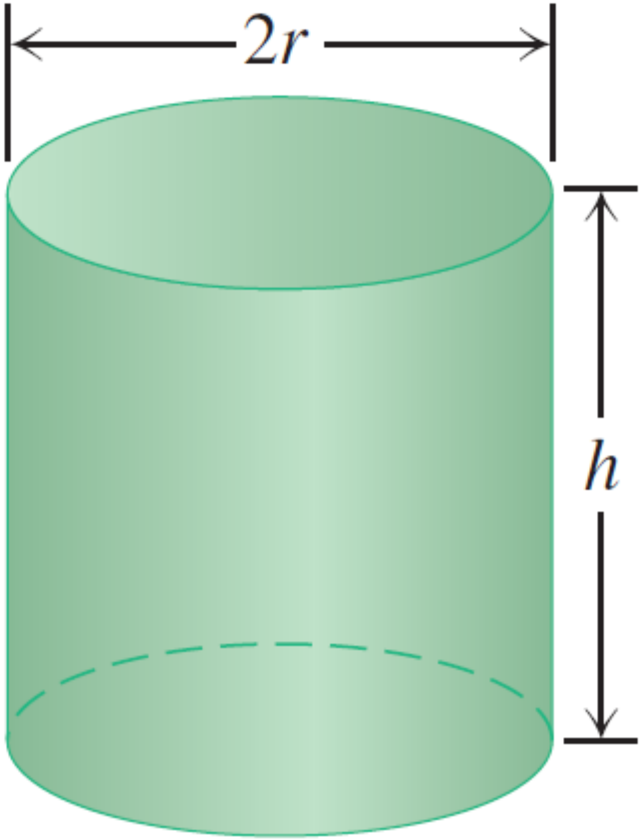


FIGURE 4.36 This one-liter can uses the least material when $h = 2r$ (Example 2).

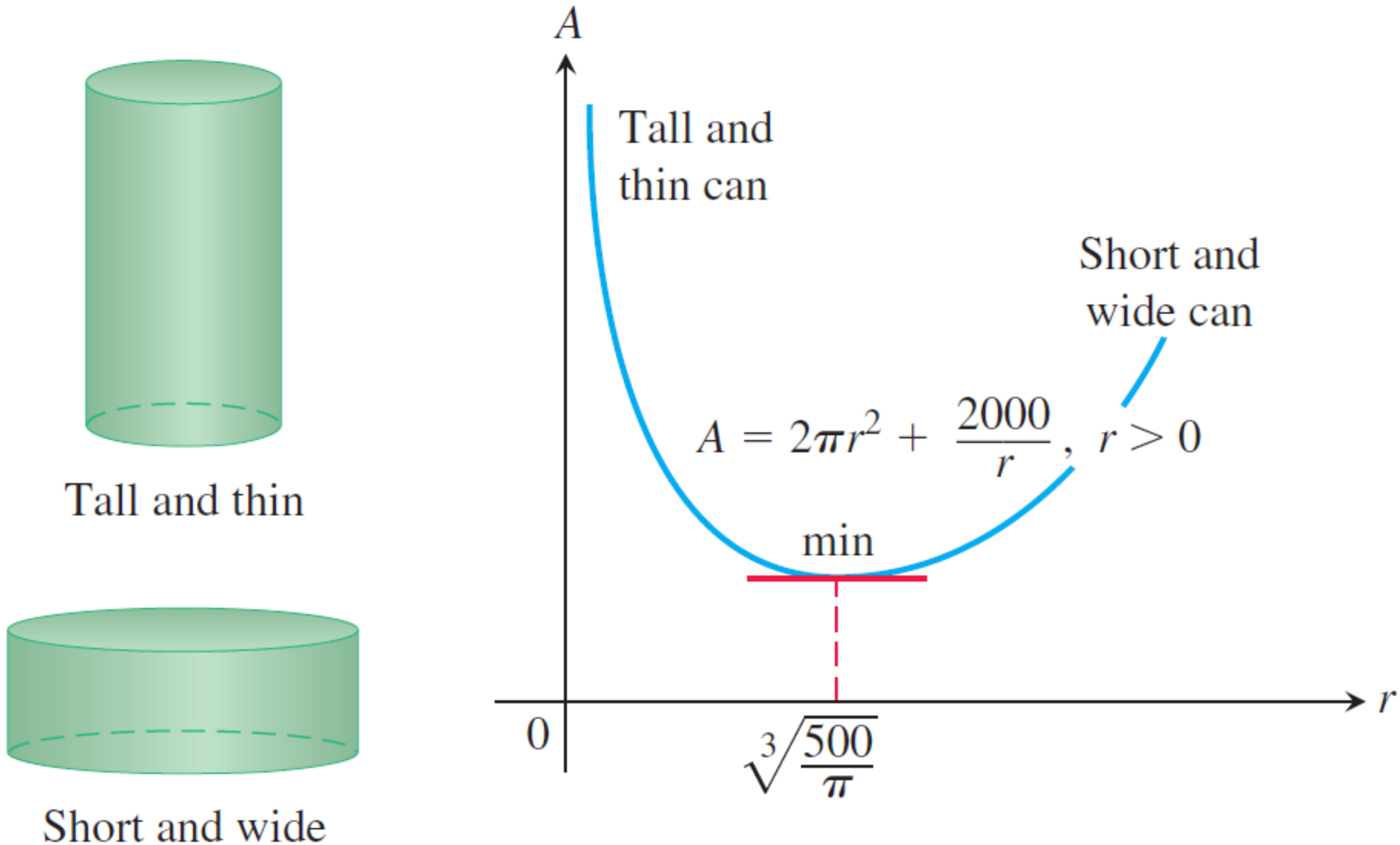
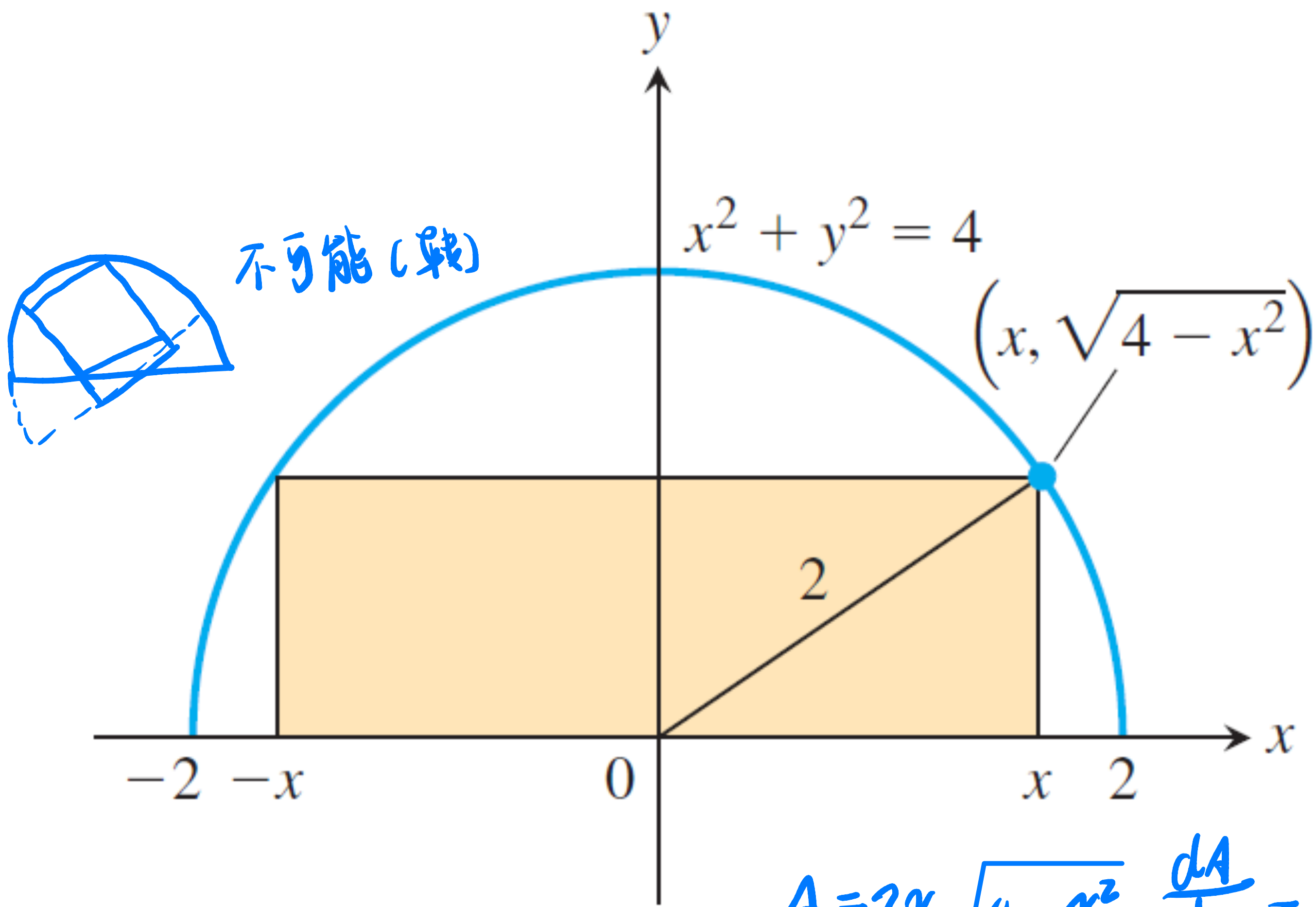


FIGURE 4.37 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

长方形 嵌入 半圆

EXAMPLE 3 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

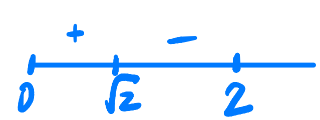


$A = 2x\sqrt{4 - x^2}$ $\frac{dA}{dx} = 2\sqrt{4-x^2} + 2x\frac{-2x}{2\sqrt{4-x^2}}$

$= \frac{2(4-2x^2)}{\sqrt{4-x^2}}$

$x = \sqrt{2}$ $x \in (0, 2)$

FIGURE 4.38 The rectangle inscribed in the semicircle in Example 3.



EXAMPLE 4 The speed of light depends on the medium through which it travels, and is generally slower in denser media.

Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum. Describe the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where its speed is c_2.

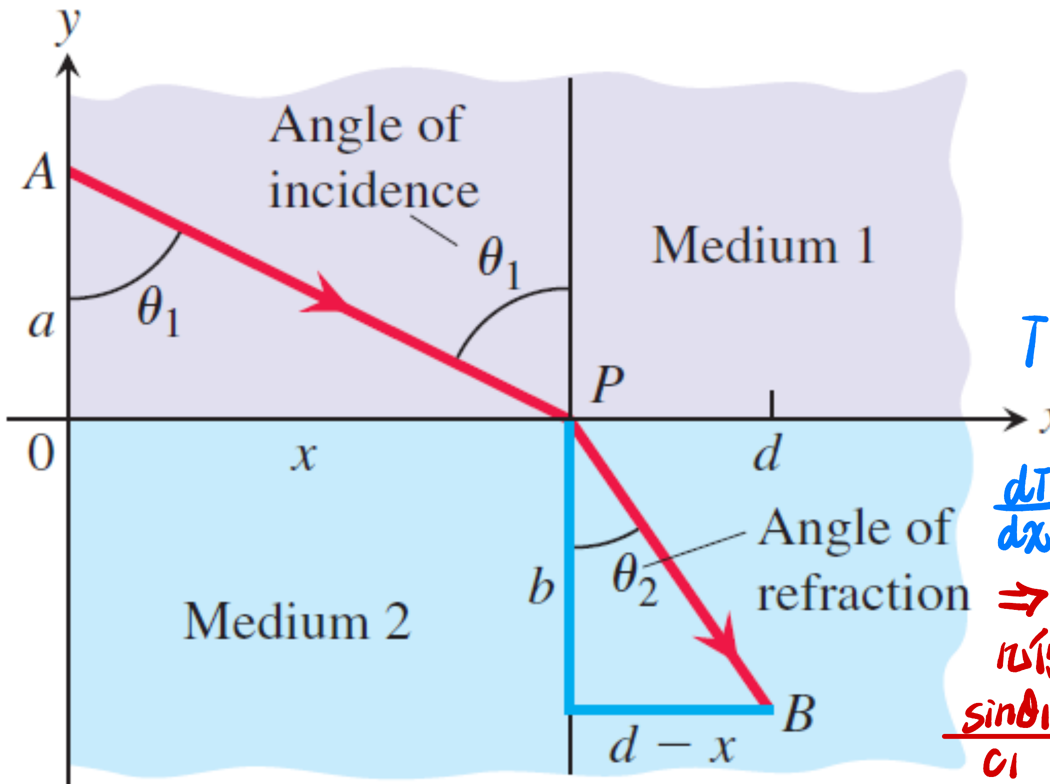


FIGURE 4.39 A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

$$T = \frac{\sqrt{a^2+x^2}}{c_1} + \frac{\sqrt{b^2+(d-x)^2}}{c_2}$$

$$\frac{dT}{dx} = \frac{x}{c_1\sqrt{a^2+x^2}} - \frac{d-x}{c_2\sqrt{b^2+(d-x)^2}} = 0$$

$\Rightarrow$

几何意义

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$$

折射定律
(此时总时间最短)

二分法找近似解

存在：介值定理

$$g(x) = \frac{dT}{dx}$$

$$g(0) = -\frac{d}{c_2\sqrt{d^2+b^2}} < 0$$

$$g(d) = \frac{d}{c_1\sqrt{d^2+a^2}} > 0$$

唯一：单调性

$$g'(x) = \left(\frac{x}{c_1\sqrt{a^2+x^2}}\right)' + \left(\frac{x-d}{c_2\sqrt{b^2+(x-d)^2}}\right)'$$

换元 $x \to x-d$

只要证第一个>0

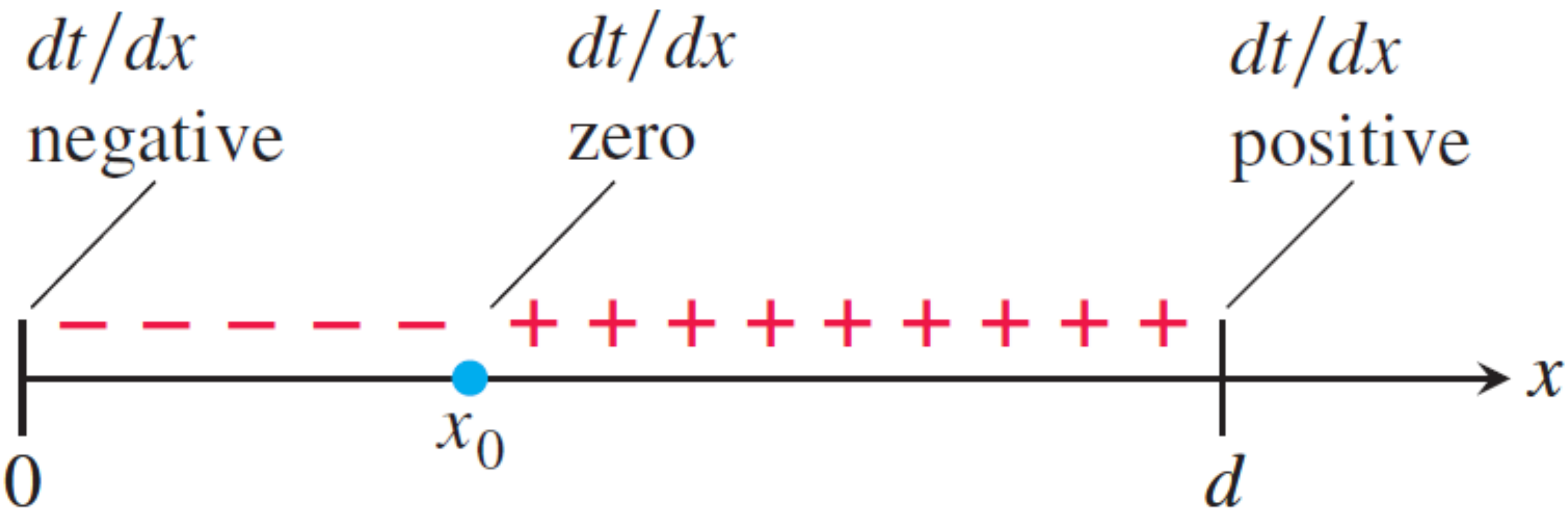
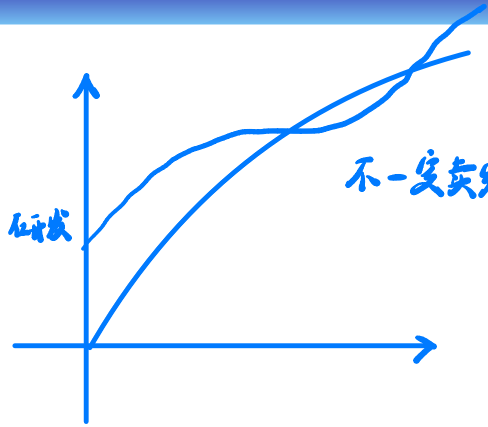


FIGURE 4.40 The sign pattern of dt/dx in Example 4.

At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.41).



不一定卖得越多越好

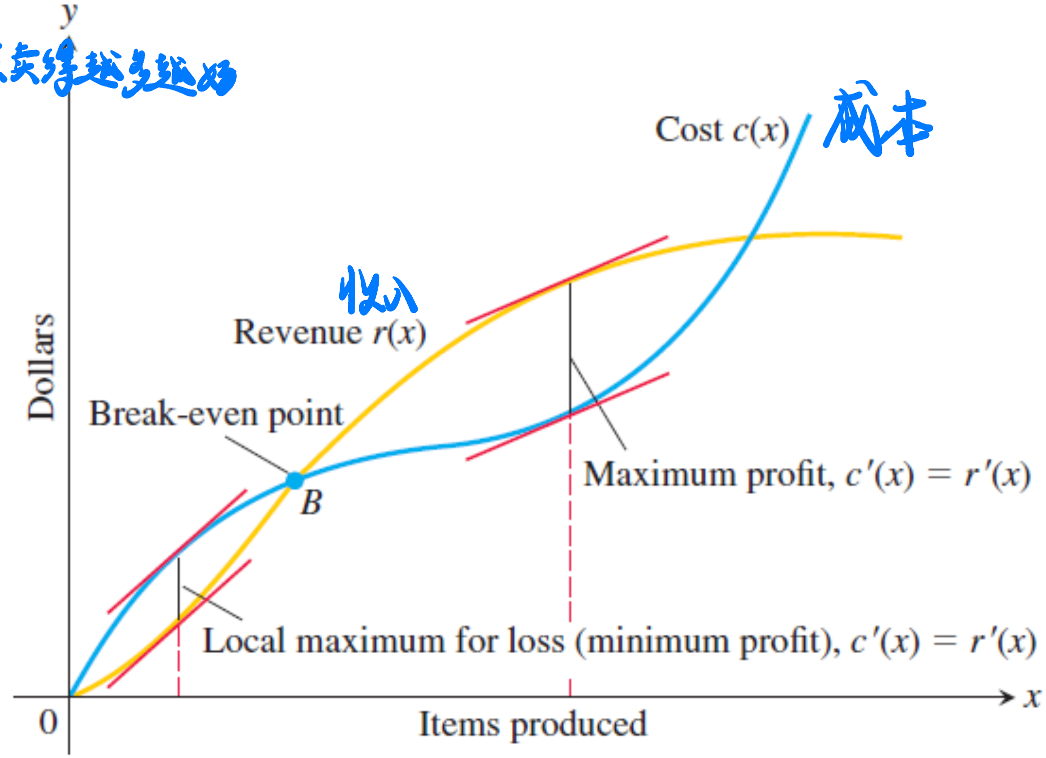


FIGURE 4.41 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.

EXAMPLE 5 Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

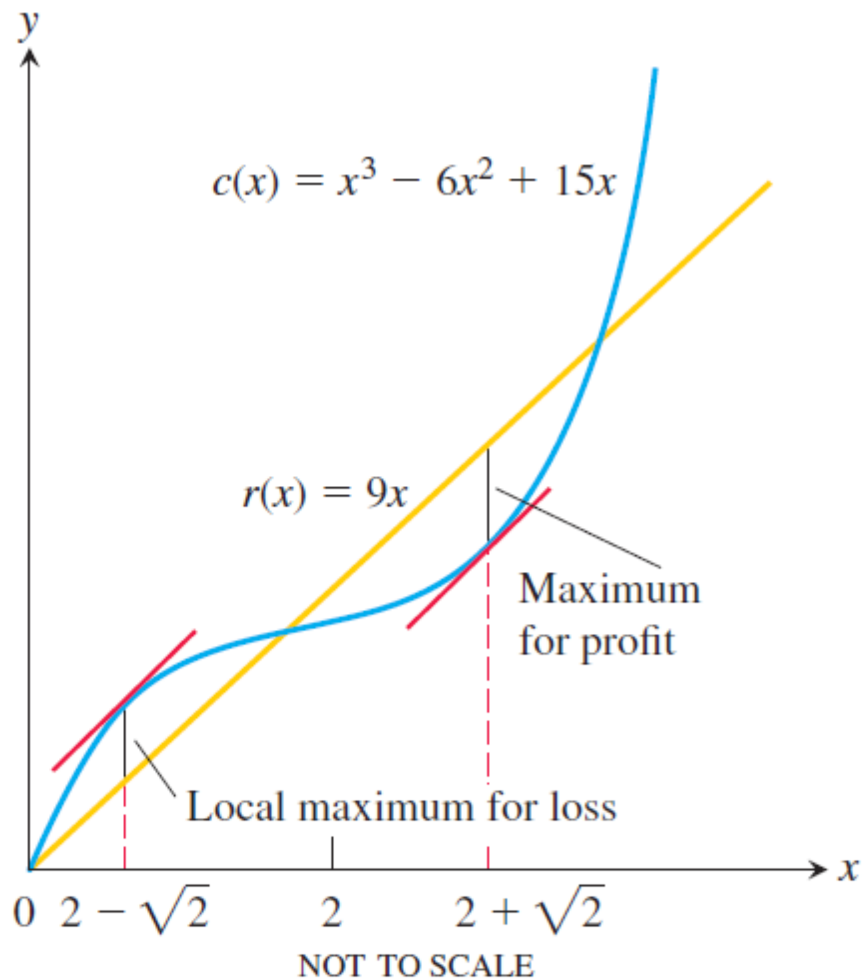
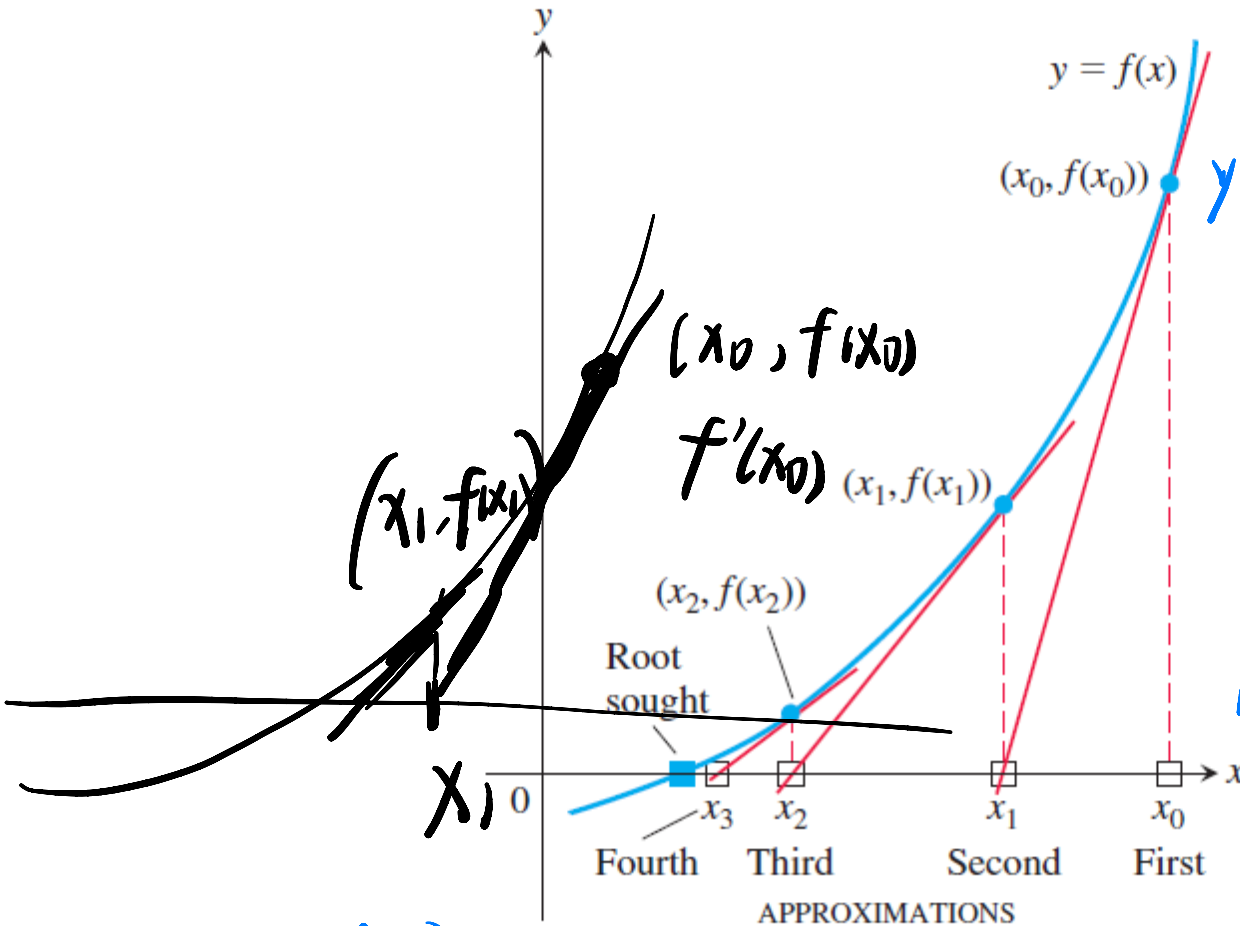


FIGURE 4.42 The cost and revenue curves for Example 5.

4.6

Newton's Method



$y - f(x_0) = f'(x_0)(x - x_0)$

$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$

取点作切线

比二分法收敛得快

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|} \to C$$

加速收敛 10^{-2} 10^{-4} 10^{-8}

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

FIGURE 4.44 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's Method

求非线性方程的根

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0. \tag{1}$$

收敛 $L = L - \frac{f(L)}{f'(L)}$

$f(L) = 0$

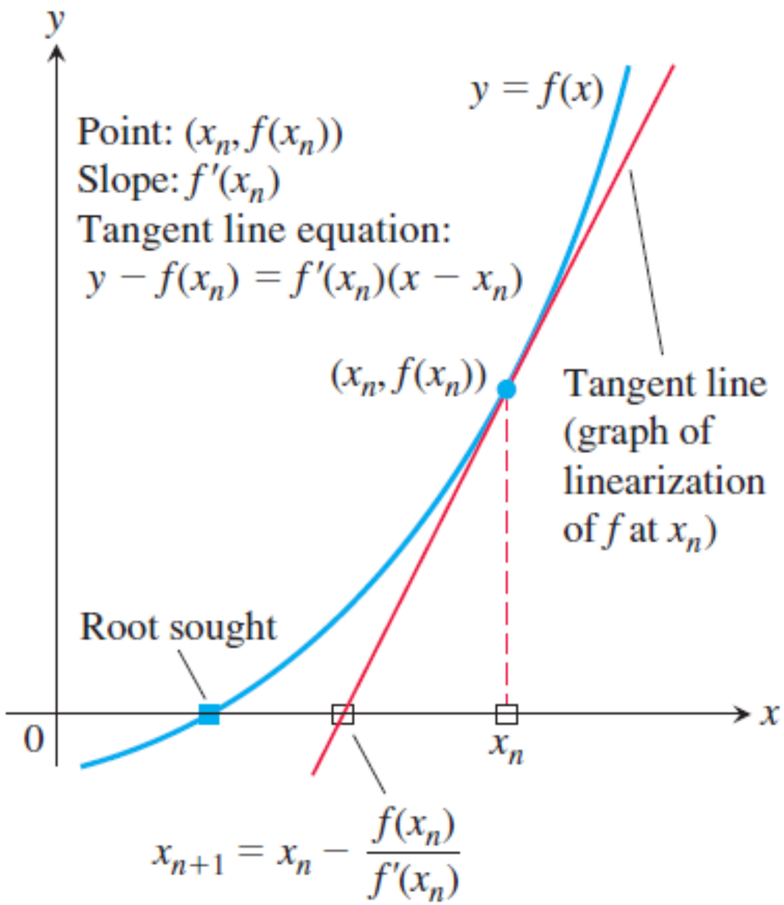


FIGURE 4.45 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

EXAMPLE 2 Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

$y = x^3 - x - 1$

$y' = 3x^2 - 1$

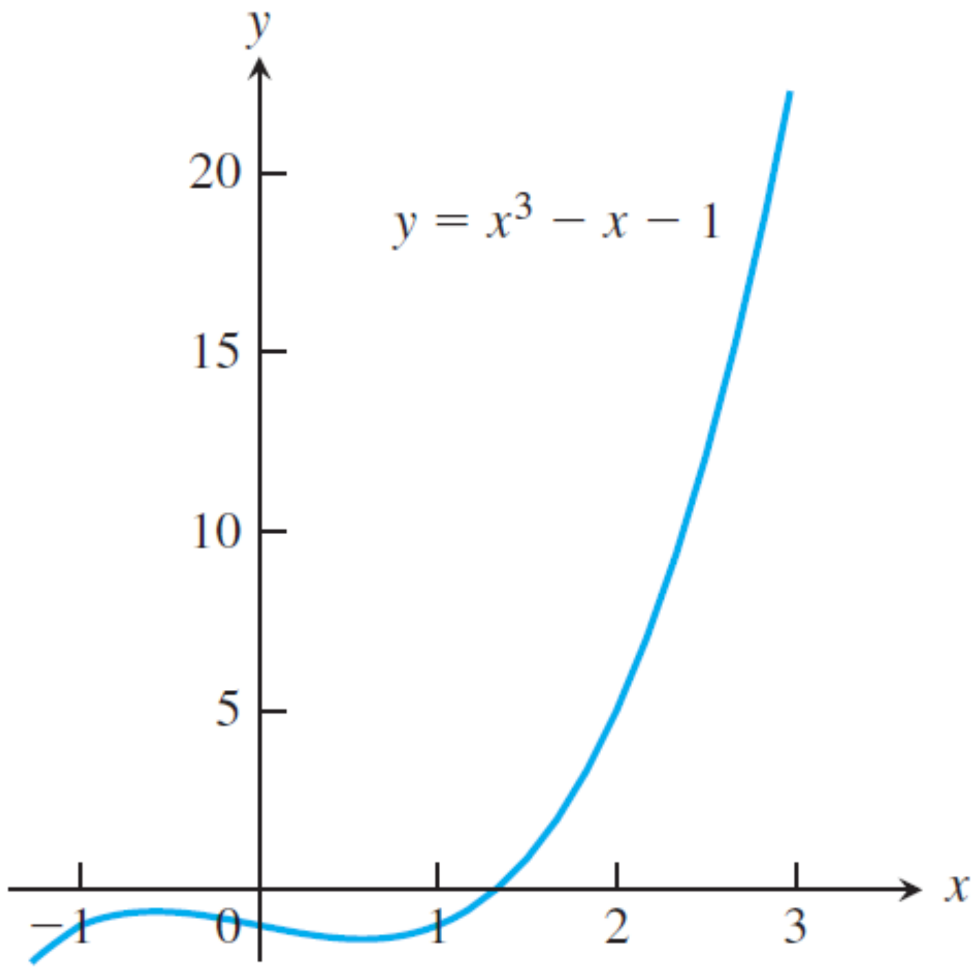


FIGURE 4.46 The graph of $f(x) = x^3 - x - 1$ crosses the x-axis once; this is the root we want to find (Example 2).

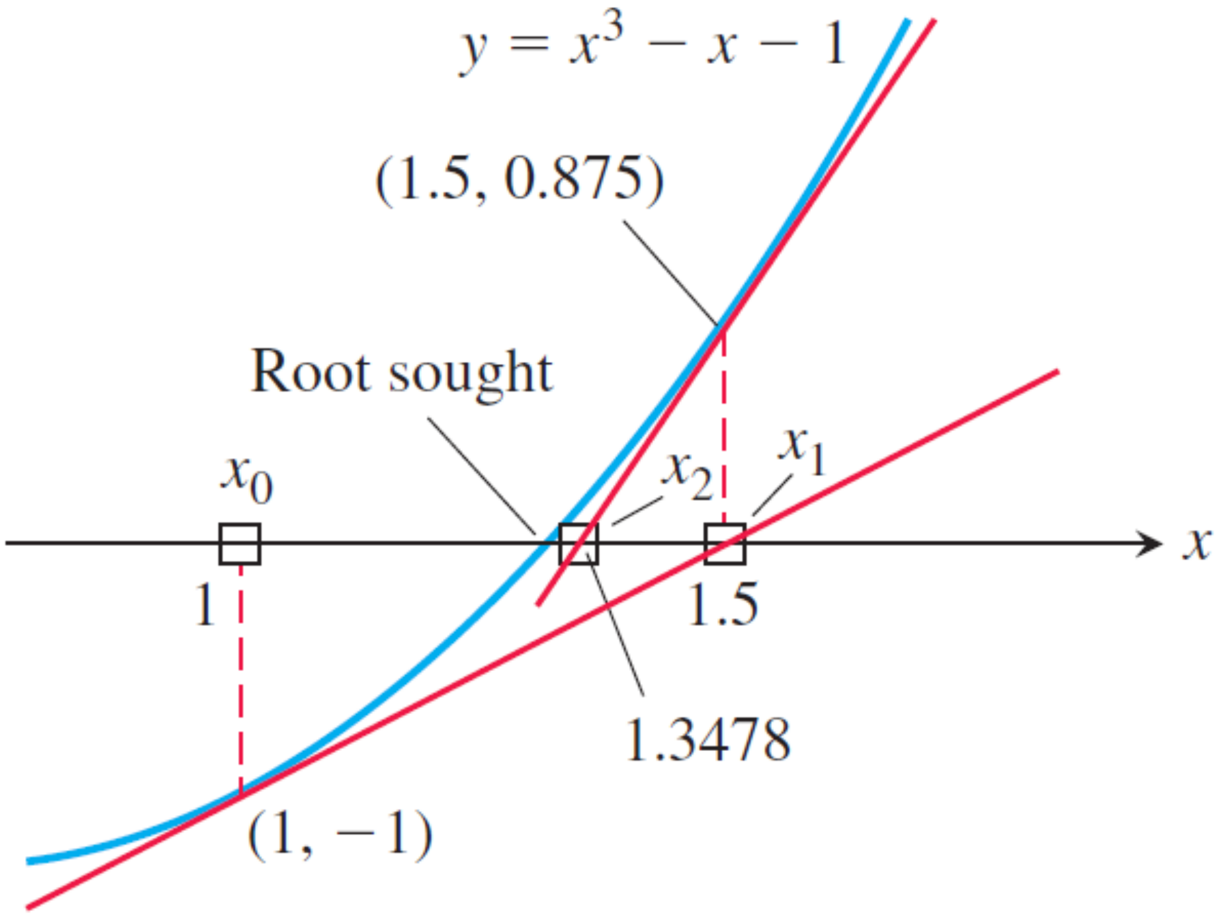


FIGURE 4.47 The first three x-values in Table 4.1 (four decimal places).

TABLE 4.1 The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	-1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	-1.8672E-13	4.2646 32999	1.3247 17957

$x_1^2 + x_2^2 + \cdots + x_{100}^2 = 10000$

$x_1 x_2 + x_{99} = 10$

$\vdots$

$f_1(x_1, \cdots, x_n) = 0$

$f_n(x_1, \cdots, x_n) = 0$

$\vec{F} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \vec{0}$

$x_{n+1} = x_n - \frac{f(n)}{f'(n)}$

$\vec{x}_{n+1} = \begin{pmatrix} (x_1)_{n+1} \\ \vdots \\ (x_n)_{n+1} \end{pmatrix}$

亚克比行列式

$Ax=b$ 线性代数 → 叠代以求非线性方程组

↓

矩阵的性质

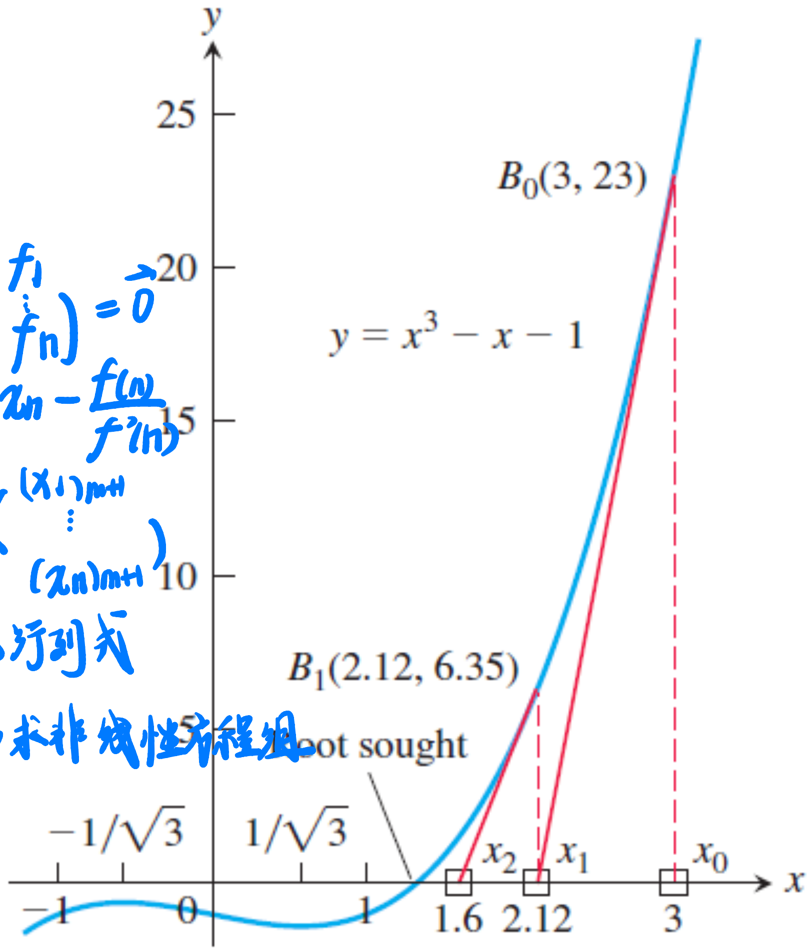
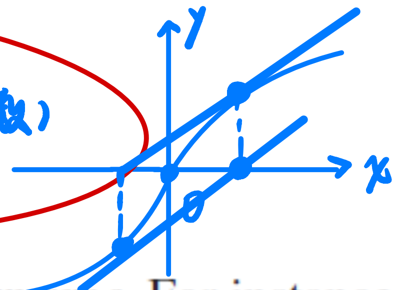


FIGURE 4.48 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root in Example 2.

$$f(x) = \begin{cases} -\sqrt{-x} & x<0 \\ \sqrt{x} & x \geq 0 \end{cases}$$ (奇函数)



$r=0$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$f'(x) = \begin{cases} \frac{1}{2\sqrt{-x}} \\ \frac{1}{2\sqrt{x}} \end{cases}$$ (奇函数求导为偶函数)

Newton's method does not always converge. For instance, if

特例

无法用牛顿法

$$f(x) = \begin{cases} -\sqrt{r-x}, & x < r \\ \sqrt{x-r}, & x \geq r, \end{cases}$$

the graph will be like the one in Figure 4.49. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

$x_0 > 0$

$$x_1 = x_0 - \frac{\sqrt{x_0}}{\frac{1}{2\sqrt{x_0}}} = -x_0$$

$$x_2 = x_1 - \frac{-\sqrt{-x_1}}{\frac{1}{2\sqrt{-x_1}}} = -x_1 = x_0$$ 反复振荡

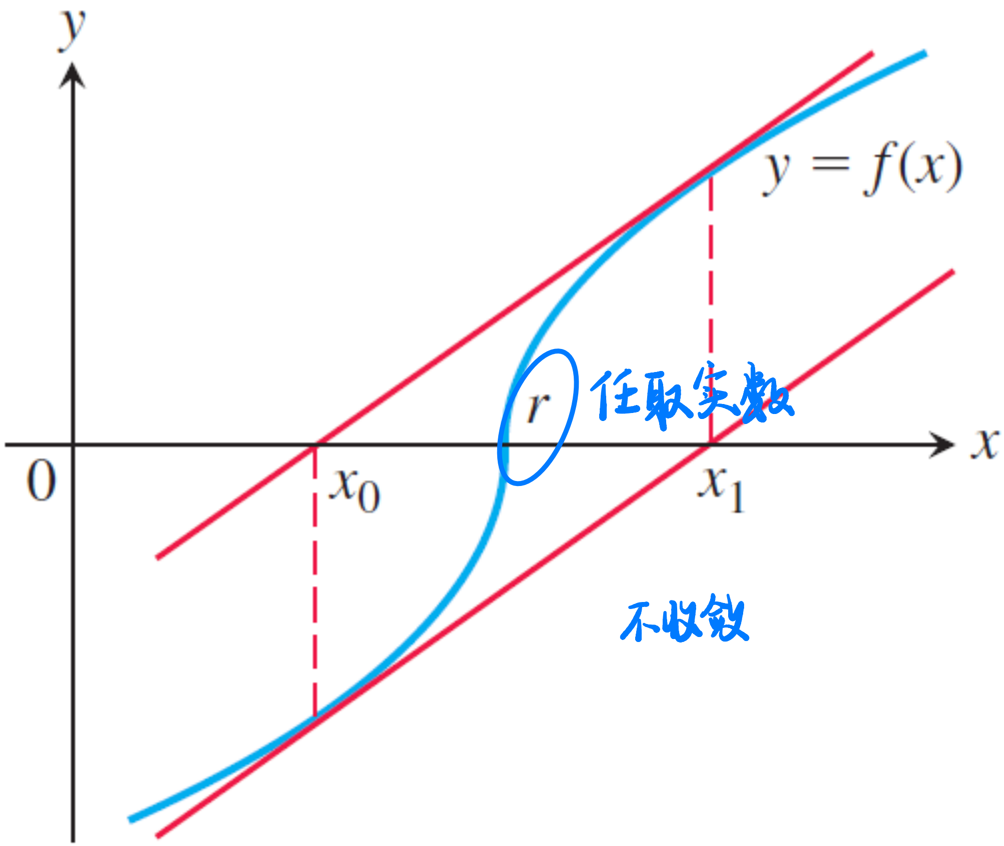
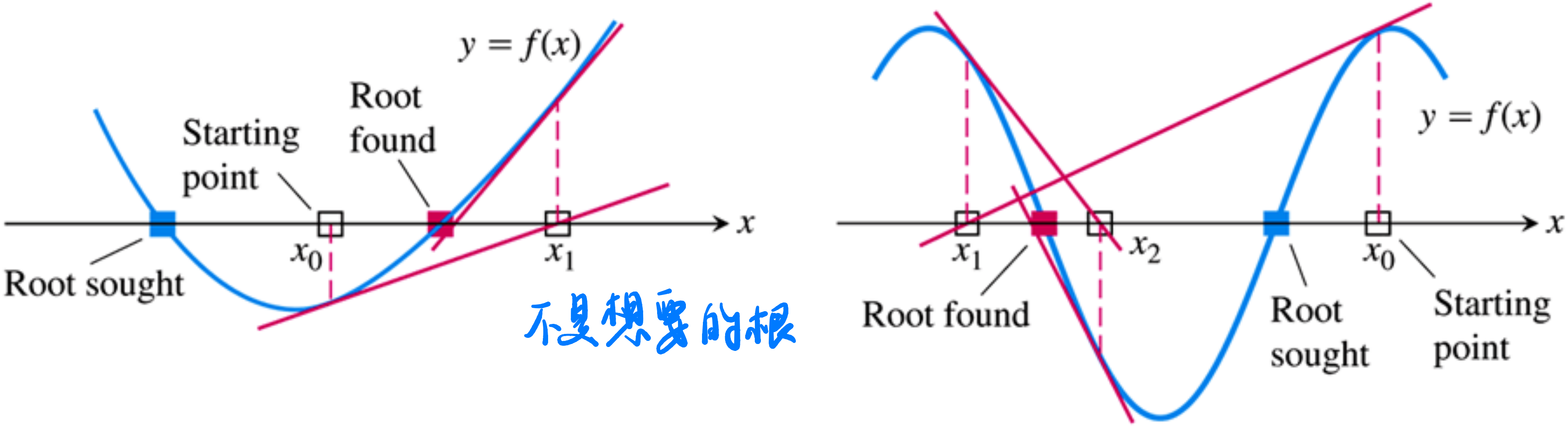


FIGURE 4.49 Newton's method fails to converge. You go from x_0 to x_1 and back to x_0, never getting any closer to r.

趋向

When Newton's method converges to a root, it may not be the root you have in mind. Figure 4.50 shows two ways this can happen.



不是想要的根

FIGURE 4.50 If you start too far away, Newton's method may miss the root you want.

一开始离想要的根近不一定得到此根

Sufficient conditions for the convergence of Newton's method

(1) $f(a) \cdot f(b) < 0$; 存在一根

(2) $f'(x) \neq 0$ for all $x \in (a, b)$; 增减性不变

凹凸性不变

(3) $f''(x) \neq 0$ for all $x \in (a, b)$;

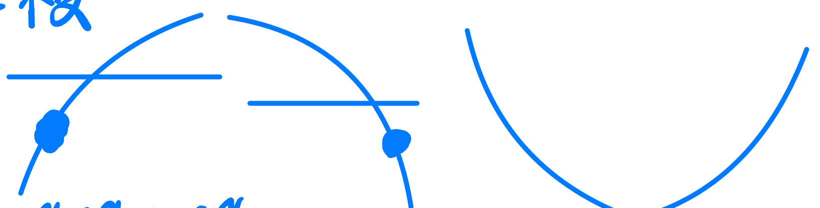
(4) $\underline{f(x_0) f''(x_0) > 0}$.

(证明：中值定理)

充分不必要

不满足仍可能收敛

(非线性方程组)



$x_0 < x_1 < x_2 < \cdots < x_*$

*二分法慢但一定能求到想要的解

适用范围小(单个方程)

4.7

不定积分

Antiderivatives

(原函数)

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + \boxed{C}$$

任意的

where C is an arbitrary constant.

EXAMPLE 2 Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

$F(x) = x^3 - 2$

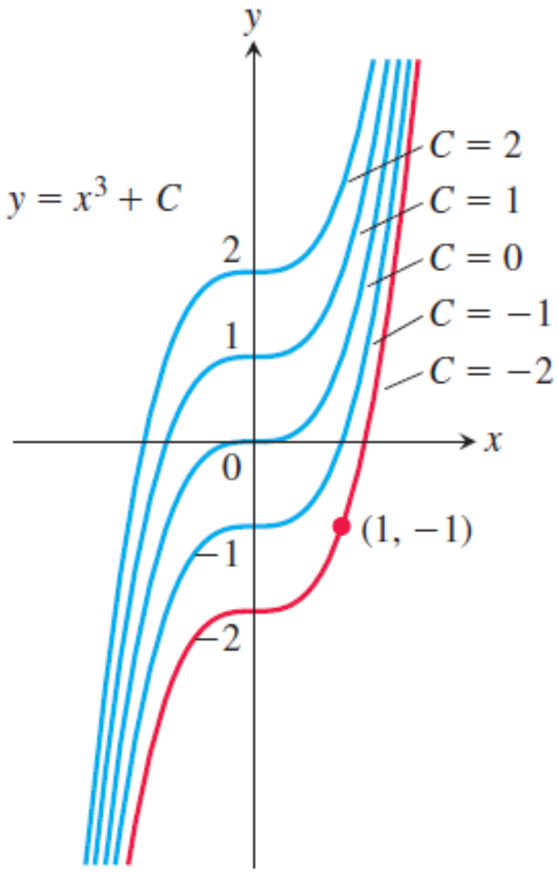


FIGURE 4.51 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$
3. cos kx	$\frac{1}{k}\sin kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

三角函数求导

TABLE 4.3 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C, \quad k$ a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C,$
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

EXAMPLE 4 Find the general antiderivative of

$3x^{\frac{1}{2}}$

$$f(x) = 3\sqrt{x} + \sin 2x.$$

$2x^{\frac{3}{2}} - \frac{1}{2}\cos 2x + C$

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

不定：原函数的集合

定：值

$$\int \frac{x}{\sqrt{5-x^2}}\,dx$$

$$(\sqrt{5-x^2})' = \cancel{\tfrac{1}{2}}\,\frac{\cancel{-2}x}{\sqrt{5-x^2}}$$

$$= -\sqrt{5-x^2} + C$$

$$\int \cot^2 x\,dx$$

$$(\cot x)' = -\csc^2 x$$

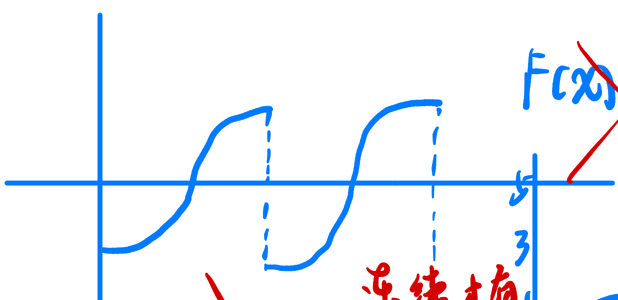
$$(-\cot x)' = \csc^2 x$$

$$\begin{cases} \tan^2 x + 1 = \sec^2 x \\ \cot^2 x + 1 = \csc^2 x \end{cases}$$

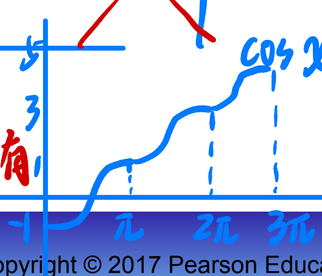
$$\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx = -\cot x - x + C$$

$$\int |\sin x|\,dx = \begin{cases} \sin x & 2k\pi \le x \le (2k+1)\pi \\ -\sin x & (2k+1)\pi < x \le (2k+2)\pi \end{cases}$$

$$\cancel{F(x) =} \begin{cases} -\cos x & 2k\pi \,\cancel{\le}\!< x \le (2k+1)\pi \\ \cos x & (2k+1)\pi < x \le (2k+2)\pi \end{cases}$$



连续才有导数



$$F(x) = \begin{cases} -\cos x + 4k, & 2k\pi < x \le (2k+1)\pi \\ \cos x + 2 + 4k, & (2k+1)\pi < x \le (2k+2)\pi \end{cases}$$

$$\int |\sin x|\,dx = F(x) + C$$