SUSTech

Final Exam for Calculus I in Fall Semester, 2018

1. (10 pts) Determine whether the following statements are true or false? No justification is necessary.

   (1) If $f(2) > 0$ and $f(4) < 0$, then there exists a number c between 2 and 4 such that $f(c) = 0$.

   (2) If $f(x) > 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

   (3) If $h(x) \leq f(x) \leq g(x)$ and $\lim_{x\to+\infty}(g(x) - h(x)) = 0$, then $\lim_{x\to+\infty} f(x)$ exists.

2. (10 pts) Express

$$\lim_{n\to\infty} \frac{1}{n\sqrt{2n}}(\sqrt{1} + \sqrt{3} + \cdots + \sqrt{2n-1})$$

   as a definite integral, then evaluate this integral.

3. (15 pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

   (1) If $a = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin x}{1+x^2} \cos^4 x \, dx$, $b = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin^3 x + \cos^4 x\right) dx$,

   $c = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^2 \sin^3 x - \cos^4 x\right) dx$, then ____________________

   ~~(A) $b < c < a$;~~ (B) $a < c < b$; ~~(C) $b < a < c$;~~ (D) $c < a < b$.

   (2) Let the function $f(x)$ be positive and continuous on $[a, b]$. Then the number of roots of the equation $\int_a^x f(t)\,dt + \int_b^x f(t)\,dt = 0$ in (a, b) is ____________________

   (A) 0; (B) 1; (C) 2; (D) 3.

   (3) Among the improper integrals below, ____________________ is convergent.

   ~~(A)~~ $\int_0^{+\infty} \frac{1}{\sqrt{1+x}} dx$; (B) $\int_1^{+\infty} \frac{\ln x}{x+x^2} dx$;

   ~~(C)~~ $\int_0^1 \frac{1}{\sqrt{x}\sin x} dx$; ~~(D)~~ $\int_1^2 \frac{1}{x(\ln x)^2} dx$.

4. (10 pts) Let $f(x) = \int_1^{x^2} (x^2 - t)e^{-t^2} dt$. Identify the open intervals on which f is increasing and decreasing.

5. (10 pts) For what values of a and b is

$$\lim_{x \to 0} \left(\frac{\tan(2x)}{x^3} + \frac{a}{x^2} + \frac{\sin(bx)}{x} \right) = 0\,?$$

6. (10 pts) Evaluate the following limits:

   (1) $\displaystyle \lim_{x \to 1} \frac{x - \sin x}{1 - \sec x}$;

   (2) $\displaystyle \lim_{x \to 1} x^{\frac{x}{1-x}}$.

7. (10 pts)

   (1) Find the derivative, $h'(x)$ of the function

   $$h(x) = \begin{cases} x^{4/3} \sin\left(\frac{1}{x^2}\right), & x \neq 0; \\ 0, & x = 0 \end{cases}$$

   for all $-\infty < x < \infty$.

   (2) Is the derivative $h'(x)$ at $x = 0$ continuous?

8. (10 pts) Find the derivative of the following functions:

   (1) $f(x) = \left(\frac{\sin x}{x}\right)^{x^2}$, $0 < x < \frac{\pi}{2}$;

   (2) $f(x) = \left(\frac{(x+2)(x-1)}{(x-2)(x+3)}\right)^5$, $x > 2$.

$-x^2 = x-1$

9. (10 pts) The graphs of $y = x(1-x)$ and $y = 2x - 1$ $(x > 0)$ intersect at one point $x = r$. Use Newton's method to estimate the value of r starting with $x_0 = 1$ and find x_2. $\frac{13}{21}$

$-4x$

$y' = (6-2x)^2 + x \cdot 2(6-2x)(-2)$

$= (6-2x)(6-6x) = 12(3-x)(1-x)$

$= 12(x-3)(x-1)$

10. (15 pts)

    (1) For $y = x(6-2x)^2$, identify the coordinates of any local and absolute extreme points and inflection points $(2, 8)$

    $y'' = 12(2x-4) = 24(x-2)$

    (2) Sketch the graph of the function. (Please identify some specific points, such as local maximum and minimum points, inflection points, and intercepts.)

11. (10 pts) Find the length of the curve $y = \ln \frac{e^x - 1}{e^x + 1}$ from $x = \ln 2$ to $x = \ln 3$.

12. (10 pts) Find the volume of the solid generated by revolving the region bounded by $y = \frac{1}{\sqrt{1+x^2}}$, $y = 0$, $x = -\frac{\sqrt{3}}{3}$, and $x = \sqrt{3}$, about the x-axis.

13. (10 pts) For what value of a does $\displaystyle \int_1^{+\infty} \left(\frac{ax}{x^2+1} - \frac{1}{2x} \right) dx$ converges ? Evaluate the corresponding integral.

14. (20 pts) Evaluate the integrals.

    (1) $\displaystyle \int \frac{\sqrt{x-2}}{x+1}\, dx$, $x > 2$;

(2) $\displaystyle\int x \tan^2 x \, dx$;

(3) $\displaystyle\int_1^{\sqrt{2}} \frac{1}{x\sqrt{x^4-1}} \, dx$;

(4) $\displaystyle\int x \cos^3 x \, dx$. ()乘法漏抄

15. (10 pts) If $f(x)$ is continuous with $f(x) = x \sin x + \displaystyle\int_0^{\frac{\pi}{4}} f(2x) \, dx$. Find the integral $\displaystyle\int_0^{\frac{\pi}{2}} f(x) \, dx$. 换积分上下限出错

16. (10 pts) Solve the differential equation:

$$\frac{dy}{dx} = xy + 3x - 2y - 6.$$

17. (10 pts) The Bernoulli equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$, where $n > 1$, can be transformed into the linear equation using the substitution $u = y^{1-n}$.

Solve the equation $x^2y' + 2xy = y^3$.

18. (10 pts) Suppose that the function $f(x)$ is defined on $(-\infty, +\infty)$, and satisfies the following properties:

(1) $f(a+b) = f(a) \cdot f(b)$ for any $a, b \in (-\infty, +\infty)$;

(2) $f(0) = 1$;

(3) f is differentiable at $x = 0$.

Show that $f'(x) = f'(0) \cdot f(x)$ for any $x \in (-\infty, +\infty)$.