Thomas Calculus | Chapter 4
Application of Derivatives

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自我介绍

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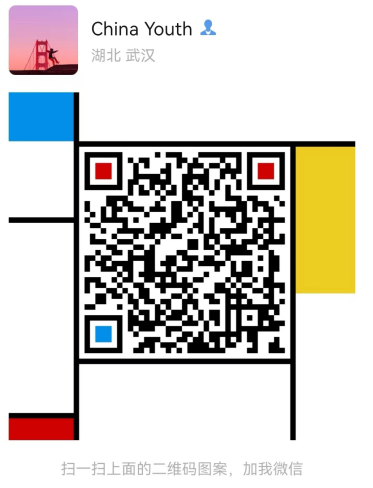


目录

- 基础知识回顾
- 提升训练（习题课，补充题，《同济》高数）

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

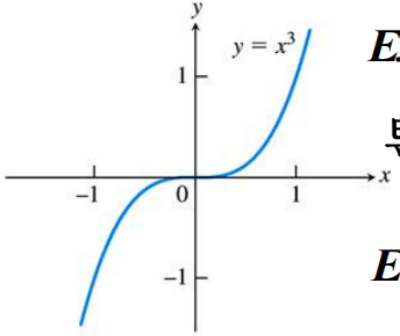
THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.



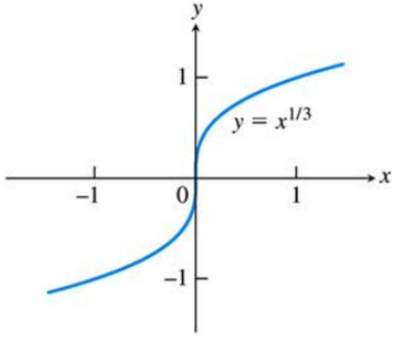
(a)

$Ex.1 \quad y = x^3 \qquad y'\big|_{x=0} = 0$

导数为零的点不一定都是极值点

$Ex.2 \quad y = x^{1/3} \qquad y'\big|_{x=0}$ 不存在

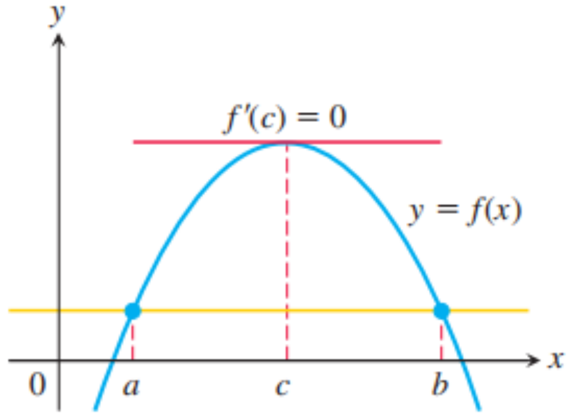
导数不存在的点不一定都是极值点



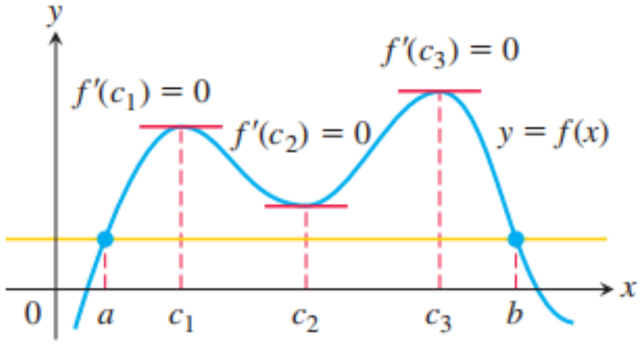
(b)

FIGURE 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

THEOREM 3—Rolle's Theorem Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.



(a)



(b)

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of f crosses the x-axis somewhere in the open interval $(-1, 0)$. (See Figure 4.12.) Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. However, the derivative

$$f'(x) = 3x^2 + 3$$

is never zero (because it is always positive). Therefore, f has no more than one zero. ■

THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

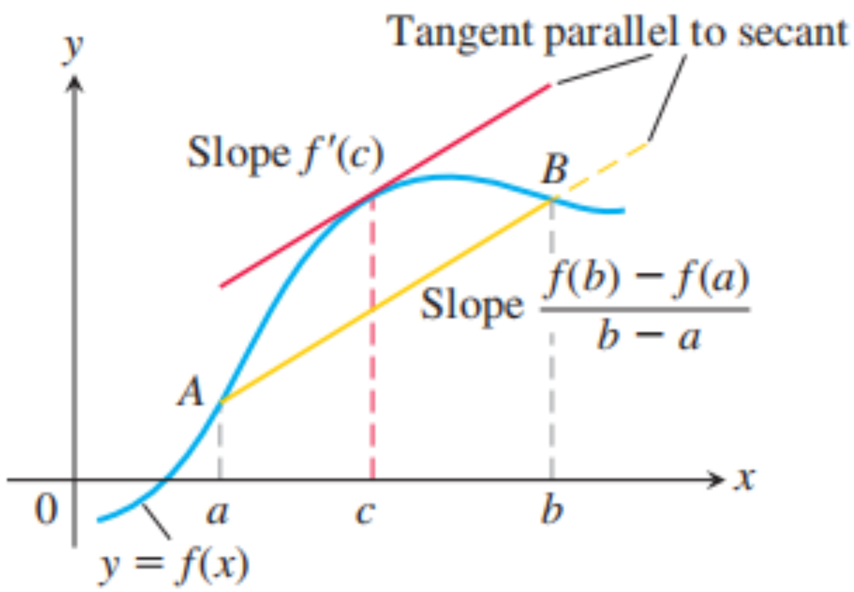
$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.



DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.

2. If $f'' < 0$ on I, the graph of f over I is concave down.

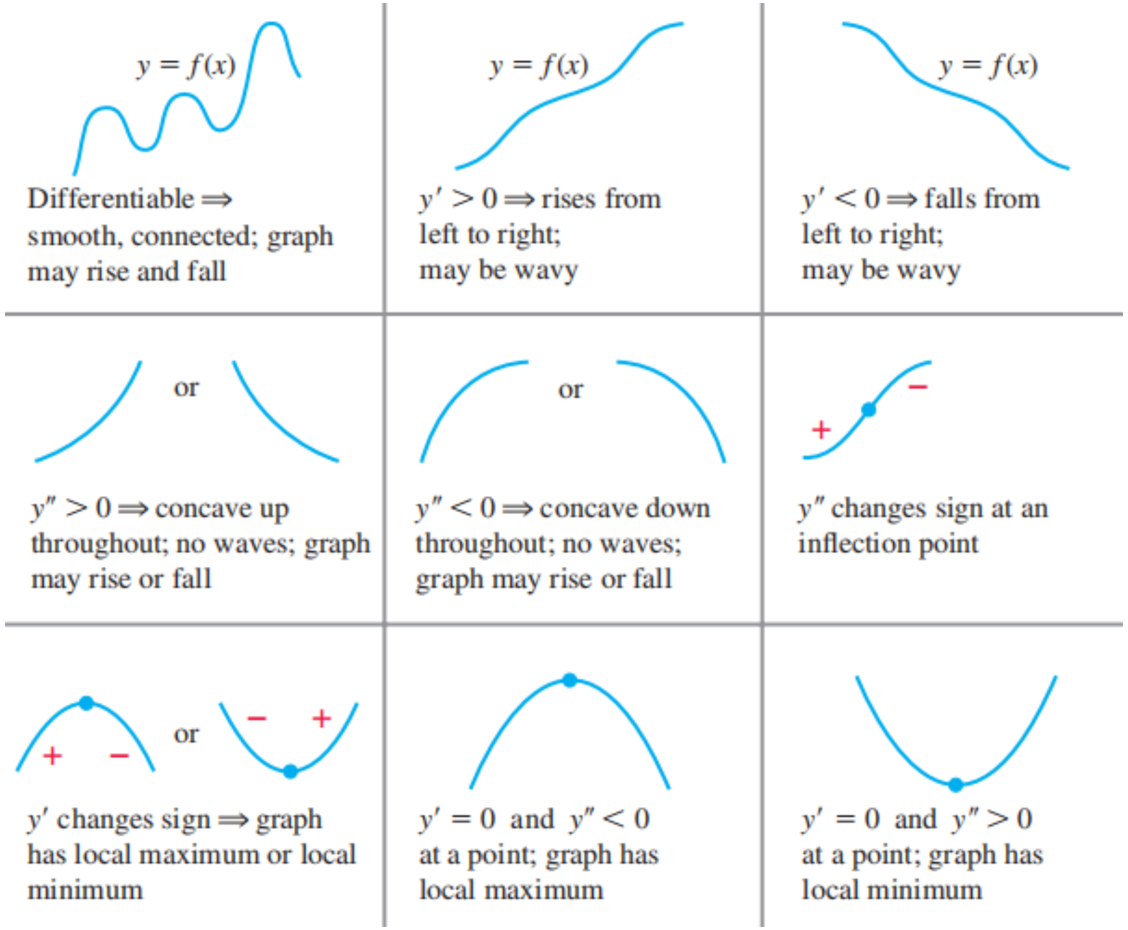
DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

二阶导数为零或不存在的点不一定是拐点。

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

(1) $f(x), g(x)$ are positive and differentiable functions, and $f'(x)g(x) - f(x)g'(x) < 0$, then if $a < x < b$, we know (　　)

(A) $f(x)g(b) > f(b)g(x)$

(B) $f(x)g(a) > f(a)g(x)$

(C) $f(x)g(x) > f(b)g(b)$

(D) $f(x)g(x) > f(a)g(a)$

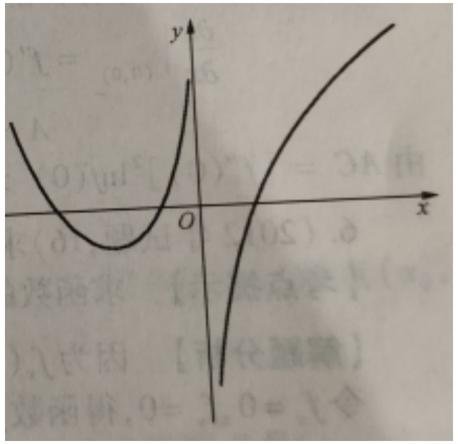
(2) $f(x)$ is continuous on $(-\infty, \infty)$, the figure of $f'(x)$ is shown, then $f(x)$ has (　)

(A) has one local minimum point and two local maximum points

(B) has two local minimum points and one local maximum point

(C) has two local minimum points and two local maximum points

(D) has three local minimum points and one local maximum point



(3) $f(x)$ is differentiable, and $f(x)f'(x) > 0$, then (　　)

(A) $f(1) > f(-1)$　　(B) $f(1) < f(-1)$

(C) $|f(1)| > |f(-1)|$　　(D) $|f(1)| < |f(-1)|$

(4) The function $y = f(x)$ is defined by the equation $y^3 + xy^2 + x^2y + 6 = 0$, find the extreme values of $f(x)$.

(1) The equation $|x|^{\frac{1}{4}} + |x|^{\frac{1}{2}} - \cos x = 0$ in $(-\infty, \infty)$ ( )

A. has no zero root.　　B. has exactly one zero root.

C. has exactly two zero roots.　　D. has infinite zero root.

(2) $f(x)$ is continuous on $[0,1]$, and differentiable on $(0,1)$, $f(0) = f(1) = 0, f(\frac{1}{2}) = 1$. Show that:

a. there exists some $c \in (\frac{1}{2}, 1)$ such that $f(c) = c$;

b. for any value λ, there must exist $\alpha \in (0, c)$, such that $f'(\alpha) - \lambda[f(\alpha) - \alpha] = 1$.

(3) $f(x)$ is continuous on $[0,3]$, and differentiable on $(0,3)$, $f(0) + f(1) + f(3) = 3, f(3) = 1$. Show that there must exist some $c \in (0,3)$ such that $f'(c) = 0$;

(1) If $f'(x_0) = f''(x_0) = 0, f'''(x_0) = a > 0$, then we know (   )
(A) f has local minimum at x_0.
(B) f has local maximum at x_0.
(C) there exists $\delta > 0$, such that f is concave up on the interval $(x_0 - \delta, x_0)$, and concave down on the interval $(x_0, x_0 + \delta)$.
(D) there exists $\delta > 0$, such that f is concave down on the interval $(x_0 - \delta, x_0)$, and concave up on the interval $(x_0, x_0 + \delta)$.

(2) $f(x)$ has continuous second derivative, and $f'(0) = 0$, $\lim_{x \to 0} \frac{f''(x)}{|x|} = 1$, then (   )
(A) f has local maximum at $x = 0$.
(B) f has local minimum at $x = 0$.
(C) f has inflection point $(0, f(0))$.
(D) f does not have local extreme at $x = 0$, and $(0, f(0))$ is not the inflection point.

(1) Let $f''(x)+[f'(x)]^2=x$, $f'(0)=0$, then ( ) （f(x)存在三阶导）

(A) f has local maximum at $x=0$.

(B) f has local minimum at $x=0$.

(C) f has inflection point $(0,f(0))$.

(D) f does not have local extreme at $x=0$, and $(0,f(0))$ is not the inflection point.

(2) Let $f(x)=(x-1)(x-2)^2(x-3)^3(x-4)^4$ has inflection point at ( )

(A) $(1,0)$. (B) $(2,0)$. (C) $(3,0)$. (D) $(4,0)$.

(3) $y=y(x)$ is defined by the equation $y\ln y-x+y=0$, identify the concavity of $y=y(x)$ at the point $(1,1)$.

10. (5 pts) (**Use Rolle's theorem to prove the mean value theorem.**) If the function $f(x)$ is continuous on $[a,b]$, and differentiable on (a,b), prove that there exists a number c in (a,b), such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

十、（5分）使用罗尔定理证明拉格朗日中值定理：如果函数 $f(x)$ 在闭区间 $[a,b]$ 上连续，在开区间 (a,b) 上可微，证明：存在 (a,b) 中的一点 c，使得

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

2. 若实数 $a_n, a_{n-1}, \cdots, a_0$ 满足 $\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + a_0 = 0$，证明方程 $a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0 = 0$ 在 $(0,1)$ 内必有实根.

谢谢大家！