Tutorial 09 for Calculus I

Sect. 6.6, 7.1-7.3

Sun Lulu

sunll@sustech.edu.cn

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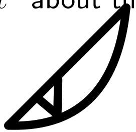
补充作业9: Chapter 5, 23; Chapter 6, 4,8,9

1. $f(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $f'(x) \le 0$, $F(x) = \frac{1}{x-a}\int_a^x f(t)\,dt$. Show that $F'(x) \le 0, x \in (a,b)$.

2. Let $S = \{(x,y) | -3 \le x \le 3, 0 \le y \le x^3 - 4x + 15\}$, find the volume of the solid generated by revolving S about the y-axis.

3. Find the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about the line $y = x$.



补充作业9: Chapter 5, 23; Chapter 6, 4,8,9

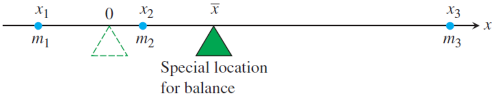
4. Let D_1 be the region on the plane bounded by $y = 2x^2, y = 0, x = 2$ and $x = a$. Let D_2 be the region on the plane bounded by $y = 2x^2, y = 0$ and $x = a$. Here $0 < a < 2$.

(1) Find the volume V_1 of the solid generated by revolving D_1 about the x-axis, and the volume V_2 of the solid generated by revolving D_2 about the y-axis.

(2) Find a such that $V_1 + V_2$ reaches its maximum.

专题一: 求质心 Sect. 6.6

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is

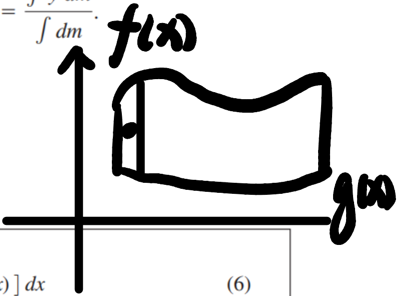
$$\begin{aligned}\text{Torque of } m_k \text{ about } \bar{x} &= \begin{pmatrix}\text{signed distance}\\ \text{of } m_k \text{ from } \bar{x}\end{pmatrix}\begin{pmatrix}\text{downward}\\ \text{force}\end{pmatrix}\\ &= (x_k - \bar{x})m_k g.\end{aligned}$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\bar{x}$:

$$\sum (x_k - \bar{x})m_k g = 0 \qquad \text{Sum of the torques equals zero.}$$

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k}. \qquad \text{Solved for } \bar{x}$$

$$\bar{x} = \frac{\int \tilde{x}\, dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int \tilde{y}\, dm}{\int dm}.$$



These moments give the formulas

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{M}\int_a^b \delta x\,[\,f(x) - g(x)\,]\,dx && (6) \\ \bar{y} &= \frac{1}{M}\int_a^b \frac{\delta}{2}\,[\,f^2(x) - g^2(x)\,]\,dx && (7) \end{aligned}}$$

专题一: 求质心 Sect. 6.6

THEOREM 1 Pappus's Theorem for Volumes If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \tag{9}$$

THEOREM 2—Pappus's Theorem for Surface Areas If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length L of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

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$$S = 2\pi\rho L. \tag{11}$$

In Exercises 1–12, find the center of mass of a thin plate of constant density δ covering the given region.

6. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$

专题一: 求质心 Section 6.6

39. The area of the region R enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the x-axis is $(1/2)\pi ab$, and the volume of the ellipsoid generated by revolving R about the x-axis is $(4/3)\pi ab^2$. Find the centroid of R. Notice that the location is independent of a.

专题二: 反函数的导数 Sect. 7.1

One-to-One Functions

1. A function $f(x)$ is one-to-one on a domain D if $\forall x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, or if $\forall f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. P380

2. The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$: $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$ or $\frac{df^{-1}}{dx}\Big|_{x=b} = \frac{1}{\frac{df}{dx}\big|_{x=f^{-1}(b)}}$. P384

专题二: 反函数的导数 Section 7.1

例: Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of $\frac{df^{-1}}{dx}$ at the point $x = 0 = f(5)$. (书本42)

延伸: Let $f(x) = \int_{-1}^{x} \sqrt{1 - e^t}\, dt$, f^{-1} is the inverse function of $f(x)$, then find $\frac{dx}{dy}\big|_{y=0}$.

2. Equivalence of the washer and shell methods for finding volume

Let f be differentiable and increasing on the interval $a \leq x \leq b$, with $a > 0$, and suppose that f has a differentiable inverse, f^{-1}.

Revolve about the y-axis the region bounded by the graph of f and lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume have identical values: $\int_{f(a)}^{f(b)} \pi((f^{-1}(y))^2 - a^2)dy = \int_a^b 2\pi x(f(b) - f(x))dx$.

To prove this equality, define $W(t) = \int_{f(a)}^{f(t)} \pi((f^{-1}(y))^2 - a^2)dy$, $S(t) = \int_a^t 2\pi x(f(t) - f(x))dx$. Then show that the functions W and S agree at a point of $[a, b]$ and have identical derivatives on $[a, b]$. As you saw in Section 4.7, Exercise 90, this will guarantee $W(t) = S(t)$ for all t in $[a, b]$. In particular, $W(b) = S(b)$. (书本60)

专题三：指数和对数函数 Sect. 7.2-7.3

1. The Derivative of $y = \ln x$: $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, u > 0$;

$\Rightarrow \frac{d}{dx}\ln|x| = \frac{1}{x}, x \neq 0$; $\quad \frac{d}{dx}\ln(bx) = \frac{1}{x}, \quad bx > 0$. P390

2. The Integral $\int \frac{1}{u} du$: If u is a differentiable function that is never zero,

$\int \frac{1}{u} du = \ln|u| + C \Rightarrow \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$. P392

$\int \tan u du = \ln|\sec u| + C$; $\quad \int \sec u du = \ln|\sec u + \tan u| + C$;

$\int \cot u du = \ln|\sin u| + C$; $\quad \int \csc u du = -\ln|\csc u + \cot u| + C$. P394

3. The Derivative of $e^x, a^x, \log_a u$: If u is any differentiable function of x, then $\frac{d}{dx}e^u = e^u \frac{du}{dx}$; $\frac{d}{dx}a^u = a^u \ln a \ \frac{du}{dx}, a > 0$; $\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \ \frac{du}{dx}$.

4. The Integral e^x, a^u: $\int e^u \ du = e^u + C$, P398 $\int a^u \ du = \frac{a^u}{\ln a} + C$.

5. The Number e as a Limit The number e can be calculated as the limit: $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to\infty}(1+\frac{1}{x})^x$. P401

例: Evaluate the limits:

(1) $\lim_{x\to 0}(1-x)^{\frac{1}{x}}$ (2) $\lim_{x\to 0}(1+2x)^{\frac{1}{x}}$

(3) $\lim_{x\to\infty}(\frac{1+x}{x})^{2x}$ (4) $\lim_{x\to\infty}(1-\frac{1}{x})^{kx}$

延伸: $\lim_{x\to\infty}[\frac{x^2}{(x-a)(x+b)}]^x$

考题一: 指数和对数的积分

例: Evaluate the integrals: $\int \frac{dx}{2\sqrt{x}+2x}$. (书本53) $\int \frac{dx}{1+e^x}$ (书本50)

延伸:

$f(x)$ is the monotonous and differentiable function on $[0, \frac{\pi}{4}]$, and $\int_0^{f(x)} f^{-1}(t)\, dt = \int_0^x t\frac{\cos t-\sin t}{\sin t+\cos t}\, dt$, f^{-1} is the inverse function of $f(x)$, find $f(x)$.

考题二: 指数和对数的求导

例: Find the y':

(1) $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$ (书本67)

(2) $y = \ln(\frac{e^\theta}{1+e^\theta})$ (书本19)

(3) $y = \log_5 \sqrt{(\frac{7x}{3x+2})^{\ln 5}}$ (书本74)

(4) $y = \sin(x^x)$ (书本117)

(5) $y = (\sin x)^x$ (书本115)

延伸: 1. Let $y = (1 + \sin x)^x$, find d$y|_{x=\pi}$.

2. $f(x)$ is differentiable, and $y = f(x)$ is defined by the equation $y - x = e^{x(1-y)}$, find $\lim_{n\to\infty} n(f(\frac{1}{n}) - 1)$.

3. $f(x)$ is differentiable, and $y = f(x)$ is defined by the equation $\int_0^{x+y} e^{-t^2}\, dt = \int_0^x x\sin(t^2)\, dt$, find $\frac{dy}{dx}|_{x=0}$.

1. (1) Find the area of the region enclosed by the curves $y = x^2 - 2x$, $y = 0$, $x = 1$, and $x = 3$. (22年期末)
(2) Find the volume of the solid generated by revolving the region in (1) about the y-axis.

2. Find the area of the surface generated by revolving the curve $4y = x^2 (1 \leq y \leq 3)$ about the y-axis.(21年期末)

3. The region is bounded by the x-axis, the curve
$$f(x) = \begin{cases} \frac{\tan^2 x}{x}, & 0 < x \leq \frac{\pi}{4} \\ 0, & x = 0 \end{cases}$$, and the line $x = \frac{\pi}{4}$. Find the volume of the solid generated by revolving the region about the y-axis.(21年期中)

4. The region D is enclosed by the curve $y = \ln\sqrt{x-1}$, the straight line $x = 5$, and the x-axis.

(1) Find the area of the region D.

(2) Find the volumes generated by revolving the region D about the line $x = 5$. (20年期末)

5. The graph of the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ is an astroid. Find the area of the surface generated by revolving the curve about the x-axis.(19年期末)

6. Find the area of the region enclosed by the curve $y = |x^2 - 4|$ and $y = \frac{x^2}{2} + 4$. (19年期末)