Tutorial 02 for Calculus I Sect. 2.5, 3.1-3.3

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Review of Sect. 2.5 - 3.3

1. Section 2.5:

   Definitions of continuity at a point and continuous function;

   The types of discontinuities;

   The Intermediate Value Theorem for continuous functions.

2. Section 2.6:

   Definitions of limits involving infinity; Asymptotes.

3. section 3.1-3.2: The Formula for the Derivative, The Derivative as a Function.

4. Section 3.3: Derivatives of constant c, power x^n, multiple cu, sum $u+v$, product uv and quotient $\frac{u}{v}$; Second- and higher-order derivatives.

Theorem (Composites of continuous functions)

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c. P93

Remark: Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem? Give reasons for your answer.P99(64)

The types of discontinuities:

1. The first types of discontinuities: $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist.

(1)removable: $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c} f(x)$,but f(c) doesn't exist or $f(c) \neq \lim_{x \to c} f(x)$.

(2)a jump discontinuity: $\lim_{x \to c^+} f(x), \lim_{x \to c^-} f(x), f(c)$ exist,
but $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$.

2.The second types of discontinuities: $\lim_{x \to c^+} f(x)$ or $\lim_{x \to c^-} f(x)$ doesn't exist.

Example: (1) infinite discontinuity: $f(x) = \frac{1}{x^2}$ at x=0, $f(x) = \tan x$ at $x = \frac{\pi}{2}$. P91

(2)oscillating discontinuity: $f(x) = \sin \frac{1}{x}$ at $x = 0$. P91

Theorem (Intermediate Value Theorem for continuous functions)

If f is a continuous function on a closed interval $[a, b]$*, and if* y_0 *is any value between* $f(a)$ *and* $f(b)$*, then* $y_0 = f(c)$ *for some c in* $[a, b]$*. P95*

Remark: The zero point Theorem: if f ia a continuous function on a closed interval $[a, b]$, $f(a)(b) < 0$, then $f(c) = 0$ for some c in (a, b). P95

Review of Sect. 2.6

1. Horizontal asymptotes:A line $y = b$ is a horizontal asymptote of the function $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

2. Oblique asymptotes: If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique asymptote.

Remark:A line $y = kx + b$ is a oblique asymptote if

either $\lim_{x\to\infty} \frac{f(x)}{x} = k \ (k \neq 0, \infty)$, $\lim_{x\to\infty} [f(x) - kx] = b$

or $\lim_{x\to-\infty} \frac{f(x)}{x} = k \ (k \neq 0, \infty)$, $\lim_{x\to-\infty} [f(x) - kx] = b$.

3. Vertical asymptotes: if a line $x = a$ is a vertical asymptote of the function $y = f(x)$ if either $\lim_{x\to a^+} = \pm\infty$ or $\lim_{x\to a^-} = \pm\infty$.

Review of Sect. 3.1-3.2

1. The Formula for the Derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{z\to x}\frac{f(z)-f(x)}{z-x}$$ P124-125

2.A function f is differentiable on a closed interval $[a, b]$ if it is differentiable on the interval (a, b) and if the limits

$$f'_+(a) = \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$$ right-hand derivative at a

$$f'_-(b) = \lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h}$$ left-hand derivative at b

exist at the endpoints. P126

Remark: If $f'(a) = f'_+(a) = f'_-(a) = \infty$ or $f'(a) = f'_+(a) = f'_-(a) = -\infty$, then $f(x)$ has a vertical tangent at $x = a$.

Theorem (Differentiability implies continuity)

If f has a derivative at $x = c$, then f is continuous at $x = c$. P128

Remark: If f has the discontinuity at a point, then it cannot be differentiable there. P128

Example: $f(x) = |x|$ is continuous at every x, but has no derivative at $x = 0$. P126; The function $y = \lfloor x \rfloor$ is not continuous at every integer $x = n$, so it fails to be differentiable at every integer $x = n$. P128

Review of Sect. 3.3

Derivative Product Rule, Derivative Quotient Rule:

If u and v are differentiable at x, then so are their product uv, and quotient $\frac{u}{v}$ (if $v(x) \neq 0$).

$$\frac{d}{dx}(uv) = u\frac{du}{dx} + v\frac{dv}{dx}$$

$$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Remark: (2015,18) $u(x), v(x)$ is differentiable, using definition to prove $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$.

Homework of Section 2.5

专题一: 函数的连续性.

例: Find the limits.Are the functions continuous at the point being approached ?

(1) $\lim_{y\to 1} \sec(y\sec^2 y - \tan^2 y - 1)$ (书本33)

找公因式

(2) $\lim_{x\to 0} \sin\sqrt{\frac{\cos^2 x - \cos x}{x}}$ (书本37)

(3) $\lim_{x\to 0} \sec(\frac{\pi(\sin 2x - \sin x)}{3x})$ (书本38)

延伸: (1) Find a such that the following function is continuous:

$$f(x) = \begin{cases} \frac{1-\cos\sqrt{x}}{ax}, & x > 0 \\ 1, & x \leq 0 \end{cases}.$$

Homework of Section 2.5

专题一: 函数的连续性.

延伸: (2) Determine whether the following statements are true or false?

a. If $f^2(x)$ is continuous, then $f(x)$ is continuous. ✗

b. If $f^3(x)$ is continuous, then $f(x)$ is continuous. ✓

then: c. If $f(x)$ is continuous, then $|f(x)|$ is continuous. ✓

d. If $|f(x)|$ is continuous, then $f(x)$ is continuous. ✗

(3) Let $f(x)$ and $g(x)$ are continuous at x_0, prove
$\varphi(x) = \max\{f(x), g(x)\}$ and $\psi(x) = \min\{f(x), g(x)\}$ are also continuous at x_0. ✓

专题一: 函数的连续性.

延伸:

(4) a. Using the fact the every nonempty interval of real numbers contains both rational and irrational numbers to show that the function:

$$f(x)=\begin{cases}1, & \text{if x is rational}\\ 0, & \text{if x is irrational}\end{cases}$$ is discontinuous at every point.

b. Is f is right-continuous or left-continuous at any point? (书本61)

专题二: 可去间断点, 跳跃间断点, 无穷断点, 震荡点.

例: At which points the functions fail to be continuous? At which points, if any, are the discontinuities removable? Not removable?. (书本30)

$$f(x)=\begin{cases}\frac{x^3-8}{x^2-4}, & \text{x}\neq 2,\text{x}\neq -2\\ 3, & \text{x=2}\\ 4, & \text{x= -2}\end{cases}$$

延伸: (1) Let $f(x)=\frac{e^{1/x}-1}{e^{1/x}+1}$,the $f(x)$ has a ( ) at $x=0$.

(A) removable discontinuity; (B) jump discontinuity;

(C) infinite discontinuity; (B) continuity.

专题二: 可去间断点, 跳跃间断点, 无穷断点, 震荡点.

延伸: (2) Find all removable discontinuities of the function $f(x) = \frac{x(x+1)}{\sin(\pi x)}$.

有极限 不连续

(3) At which points do the function $f(x) = \lim\limits_{n\to\infty} \frac{1-x^{2n}}{1+x^{2n}} x$ fail to be continuous? At which points, if any, are the discontinuous removable? Not removable? Give reasons for your answers.

有lim 不等

jump

Homework of Section 2.5

专题三: 中值定理或者零点定理.

例: A fixed point theorem

Suppose that a function f is continuous on the closed interval $[0,1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0,1]$. Show that there must exist a number c in $[0,1]$ such that $f(c) = c$. (书本67)

延伸: (1) Suppose $f(x)$ is continuous in $[0,2a]$ with $f(0) = f(2a)$. Prove that there exist a point $x_0 \in [0,a]$ such that $f(x_0) = f(x_0 + a)$.

(2) Prove that any polynomial of order 3 has at least one real root. What about polynomial of order $2k+1$?

专题三: 中值定理或者零点定理.

延伸: (3) Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in R$. Suppose that $f(1) < 0, f(2) > 0, f(3) < 0$. Prove that the equation $f(x) = 0$ has 4 real roots.

$-\frac{a+b}{2}$

(4) Show that the function $F(x) = (x-a)^2 \cdot (x-b)^2 + x$ takes on the value $\frac{a+b}{2}$ for some value of x.

(5) Show that the function $F(x) = (x-1)(\sin x - 1)$ has the value of $\frac{1}{2}$ for some value of x.

(6) Show that the equation $x = a\sin x + b, a > 0, b > 0$ has at least one positive solution and it is under $a + b$.

专题四: 无穷极限和渐近线.

例1: Find the limits.

(1) $\lim\limits_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}$ (书本36) (2) $\lim\limits_{x\to-2^+}\frac{x^2-3x+2}{x^3-4x}$ (书本58)

(3) $\lim\limits_{x\to\infty}(\sqrt{x^2+25}-\sqrt{x^2-1})$ (书本81)

例2: Graph the rational functions $y=\frac{x^2-4}{x-1}$. Include the graphs and equations of the asymptotes. (书本101)

延伸: (1) Find the limit.

a. $\lim\limits_{x\to\infty}\frac{\cos x-1}{x}$ b. $\lim\limits_{x\to\infty}\frac{3x^2+5}{5x+3}\sin\frac{2}{x}$ c. $\lim\limits_{x\to-\infty}\frac{\sqrt{4x^2+x-1}+x+1}{\sqrt{x^2+\sin x}}$

d. $\lim\limits_{x\to-\infty}(x+\sqrt{x^2+x+4})$ e. $\lim\limits_{x\to1}(1-x)\tan\frac{\pi x}{2}$

(2) Use limits to determine the equations for all asymptotes.

a. $y=\frac{x^3+x+1}{(x-1)(x+2)}$

Homework of Section 3.1-3.2

专题五: 导数的定义.

例1: (1) Determine if the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{x}\neq 0 \\ 0, & \text{x}=0 \end{cases}$ is continuous at the origin.

(2) Find $f'(0)$ and $f'(x)$. $\lim\limits_{x\to 0} \cos\frac{1}{x}$ 不存在

(3) Determine if the function $f'(x)$ is continuous at the origin.(书本35)

例2: What do the graph of $y = \sqrt{|4-x|}$ appear to have vertical tangent? (书本48)

例3: Let $f(x)$ be a function satisfying $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$. (书本58)

专题五: 导数的定义. 严格定义

延伸: (1) When ( ), $f(x)$ is differentiable at $x=0$.

A. $\lim\limits_{x\to 0}\frac{f(x^2)-f(0)}{x^2-0}$ exists. B. $\lim\limits_{x\to 0}\frac{f(x^3)-f(0)}{x^3-0}$ exists.

C. $f'_-(0)$ and $f'_+(0)$ exist. D. $\lim\limits_{x\to 0}\frac{f(\sin x)}{x}$ exists.

(2) Which of the following functions is not differentiable at $x=0$?

A. $f(x)=|x|\sin|x|$ B. $f(x)=|x|\sin\sqrt{|x|}$.

C. $f(x)=\cos|x|$. D. $f(x)=\cos\sqrt{|x|}$.

(3) $f(x)$ is continuous at $x=0$, the wrong statement is (D).

A. If $\lim\limits_{x\to 0}\frac{f(x)}{x}=A$ exists, then $f(0)=0$.

拆分导数

B. If $\lim\limits_{x\to 0}\frac{f(x)+f(-x)}{x}=A$ exists, then $f(0)=0$.

找特例

C. If $\lim\limits_{x\to 0}\frac{f(x)}{x}=A$ exists, then $f'(0)=A$.

D. If $\lim\limits_{x\to 0}\frac{f(x)-f(-x)}{x}=A$ exists, then $f'(0)=\frac{A}{2}$.

$\frac{f(x)-f(0)}{x}+\frac{f(-x)-f(0)}{-x}$ 分开不一定存在 $\Rightarrow f(x)=|x|$

专题五: 导数的定义.

延伸: (4) Assume $f(0)=0$. Determine if the following statement is correct or not. If so, prove it. If not, give a counter-example.

a. $\lim\limits_{h\to 0}\frac{f(1-\cos h)}{h}$ exists, then f is differentiable at $x=0$.

b. $\lim\limits_{h\to 0}\frac{f(2h)-f(h)}{h}$ exists, then f is differentiable at $x=0$.

不可能拆
举反例

(5) Assume that $f'(a)$ exists, compute $\lim\limits_{h\to 0}\frac{f(a+h)-f(a-2h)}{h}$.

(6) Assume $f(x)$ is differentiable at $x=0$ with $f(0)=0$, compute: $\lim\limits_{x\to 0}\frac{x^2f(x)-2f(x^3)}{x^3}$.

(7) Assume $y=f(x)$ and $y=\sin x$ have the same tangent at the origin. Compute: $\lim\limits_{x\to\infty}\sqrt{xf(\frac{2}{x})}$.

$=$ 专题五: 导数的定义.

延伸:

(8) Determine if the following function is continuous at $x = 0$ and if it is differentiable at $x0$.

变化 夹逼

$$f(x) = \begin{cases} x, & \text{x≤0;} \\ \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n}. \end{cases}$$

$x \to 0$ 反推 右, 左

(9) Suppose that $g(x)$ is differentiable at $x = 0$ and $g(0) = 0, g'(0) = 0$. Compute $f'(0)$ for $f(x)$ is defined below:

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x}, & \text{x≠0;} \\ 0, & x = 0. \end{cases}$$

Homework of Section 3.1-3.2

= 专题五: 导数的定义

延伸:

(10) $f(x)$ is defined on $(-\infty, \infty)$, $f(x) \neq 0$, $f'(0) = 1$, and we have $f(x+y) = f(x)f(y)$ for all $x, y \in (-\infty, \infty)$. Show that $f(x)$ is differentiable and $f'(x) = f(x)$.

$f(1)=0$

(11) $f(x)$ is defined on $(-\infty, 0) \cup (0, \infty)$, and we have $f(xy) = f(x) + f(y)$ for all $x \neq 0, y \neq 0$. Show that if $f'(1)$ exists, then $f'(x)$ exists.

$f(1) = f(x) + f(\frac{1}{x})$

$\frac{f(1+\Delta x)}{\Delta x}$

$\frac{f(x+\Delta x) - f(x)}{\Delta x}$

专题六: 求导法则.

例1: $w = \dfrac{q^2+3}{(q-1)^3+(q+1)^3}$, find w', w''.(书本40)