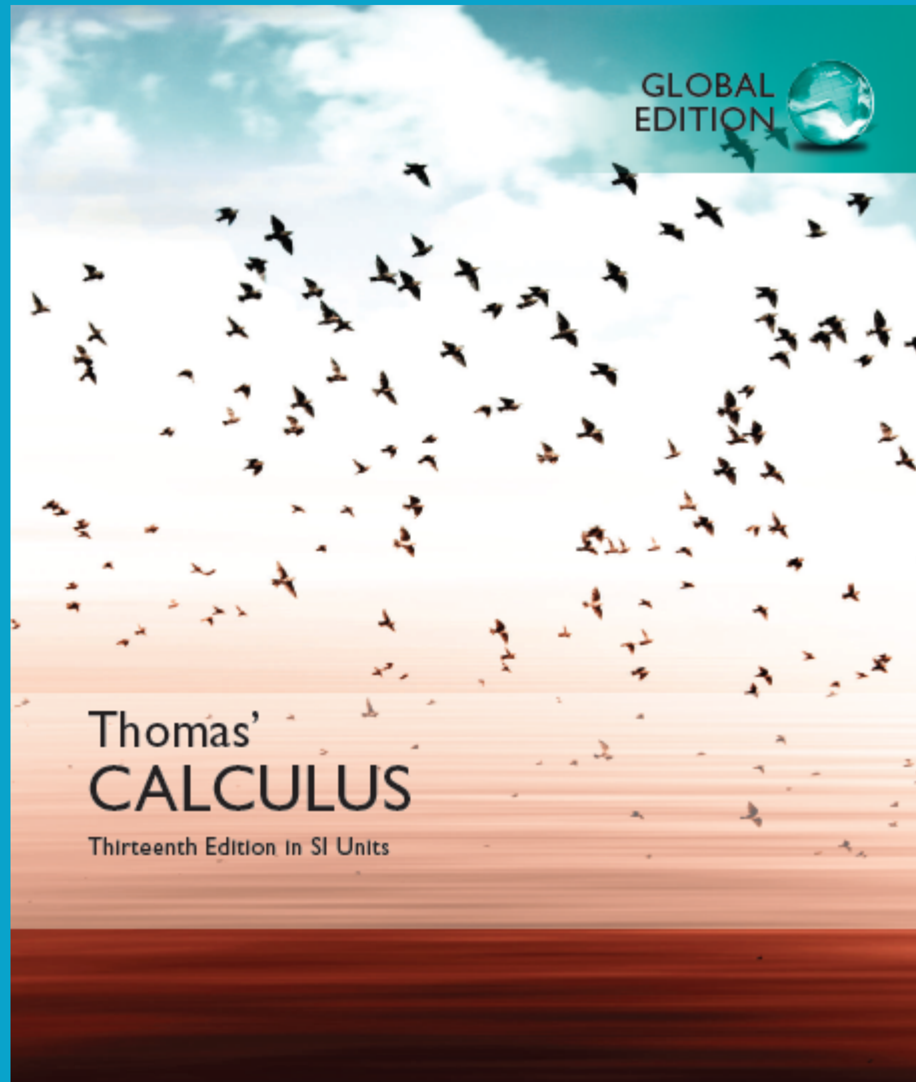


GLOBAL EDITION

Thomas' CALCULUS

Thirteenth Edition in SI Units

Chapter 5

Integrals

5.1

Area and Estimating with Finite Sums

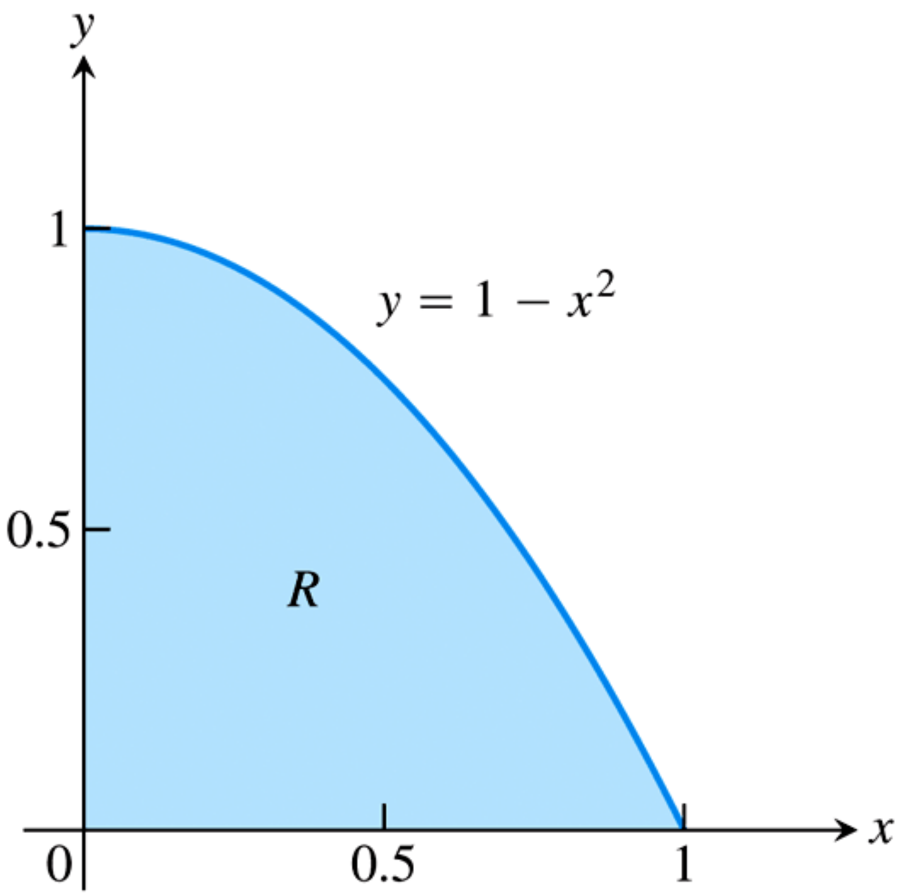


FIGURE 5.1 The area of the region R cannot be found by a simple formula.

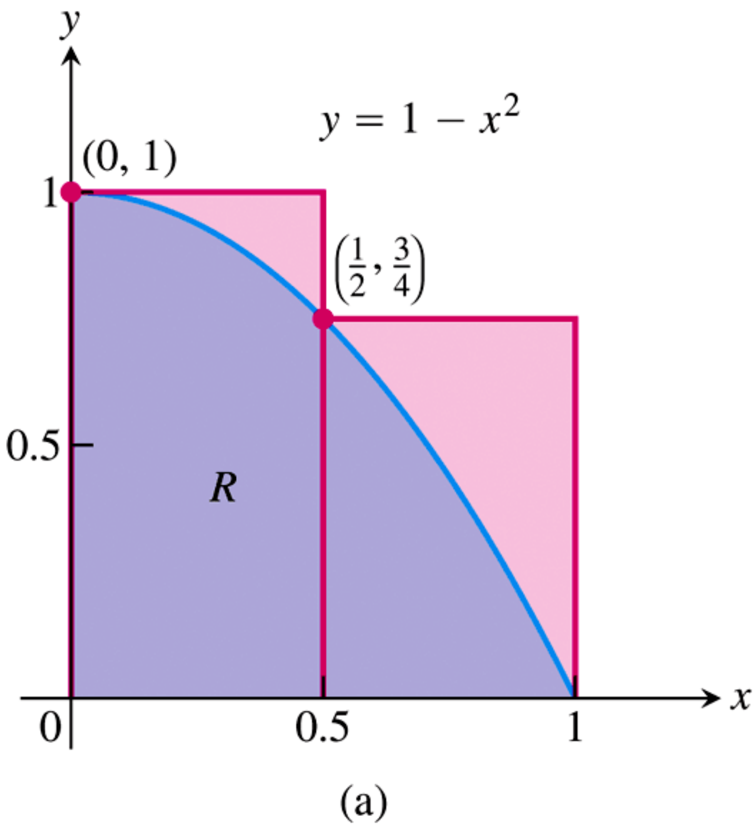


FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

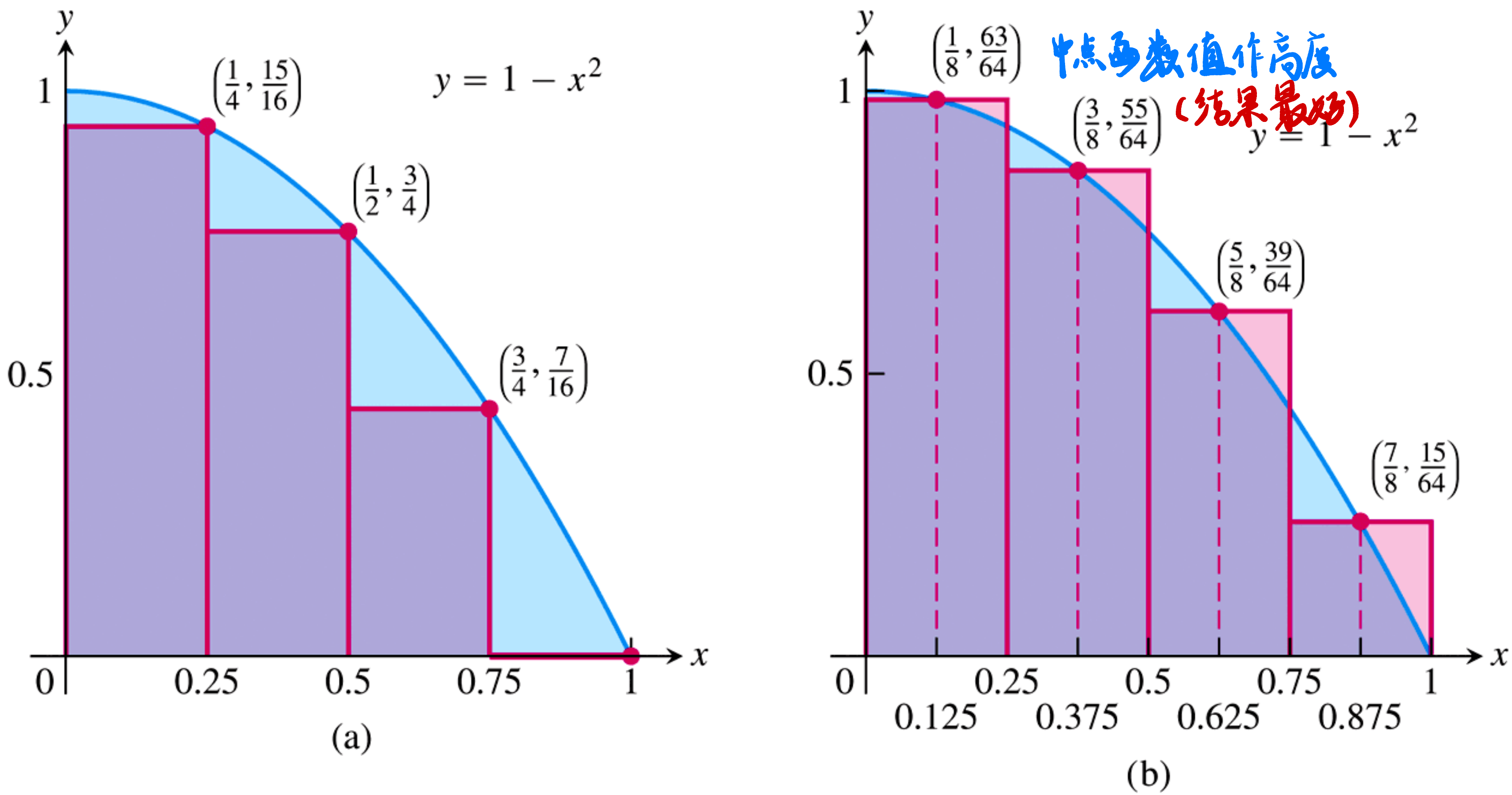
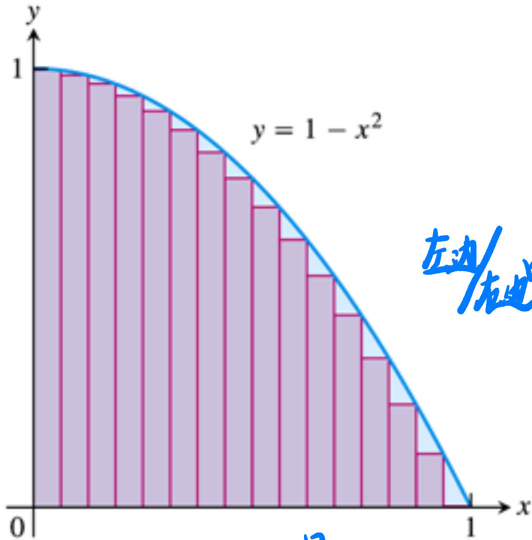
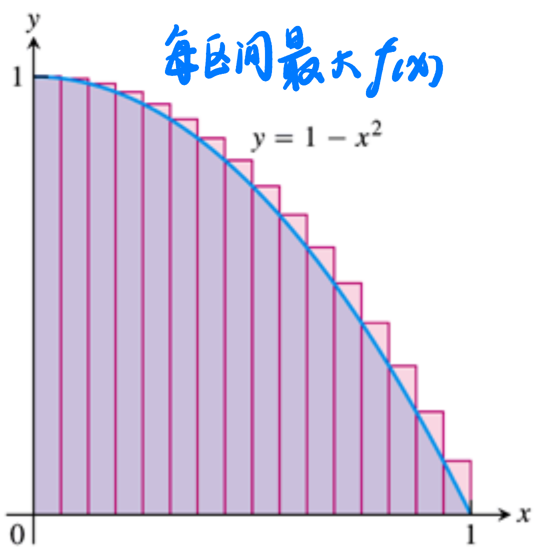


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.



左边/右边函数值作高度

(a) 最小$f(x)$



(b) 每区间最大$f(x)$

$\Delta x = \frac{1}{n}$

$x_k = \frac{k}{n}$

$x_{n-1} = \frac{n-1}{n}$

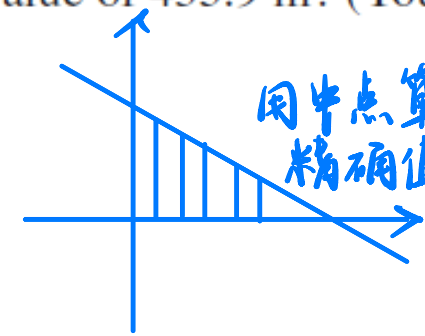
$$S_n = \Delta x f(x_0) + \Delta x f(x_1) + \cdots + \Delta x f(x_{n-1})$$
$$= \Delta x\left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right)$$
$$= \Delta x\left(1 - x_0^2 + 1 - x_1^2 + \cdots + 1 - x_{n-1}^2\right)$$
$$= \frac{1}{n}\left(n - \frac{0^2 + 1^2 + \cdots + (n-1)^2}{n^2}\right)$$
$$= 1 - \frac{1}{n^3} \times \frac{1}{6}(n-1)(2n-1)n$$

FIGURE 5.~~4~~ $\frac{2}{3}$ (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

EXAMPLE 2 The velocity function of a projectile fired straight into the air is $f(t) = 160 - 9.8t$ m/sec. Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m? (You will learn how to compute the exact value easily in Section 5.4.)



用中点算能是
精确值

TABLE 5.2 Travel-distance estimates

Number of subintervals	Length of each subinterval	Upper sum	Lower sum
3	1	450.6	421.2
6	1/2	443.25	428.55
12	1/4	439.57	432.22
24	1/8	437.74	434.06
48	1/16	436.82	434.98
96	1/32	436.36	435.44
192	1/64	436.13	435.67

Average Value of a Nonnegative Continuous Function

The average value of a collection of n numbers $x_1, x_2, \ldots, x_n$ is obtained by adding them together and dividing by n. But what is the average value of a continuous function f on an interval $[a, b]$? Such a function can assume infinitely many values. For example, the temperature at a certain location in a town is a continuous function that goes up and down each day. What does it mean to say that the average temperature in the town over the course of a day is 73 degrees?

When a function is constant, this question is easy to answer. A function with constant value c on an interval $[a, b]$ has average value c. When c is positive, its graph over $[a, b]$ gives a rectangle of height c. The average value of the function can then be interpreted geometrically as the area of this rectangle divided by its width $b - a$ (Figure 5.6a).

What if we want to find the average value of a nonconstant function, such as the function g in Figure 5.6b? We can think of this graph as a snapshot of the height of some water that is sloshing around in a tank between enclosing walls at $x = a$ and $x = b$. As the water moves, its height over each point changes, but its average height remains the same. To get the average height of the water, we let it settle down until it is level and its height is constant. The resulting height c equals the area under the graph of g divided by $b - a$. We are led to *define* the average value of a nonnegative function on an interval $[a, b]$ to be the area under its graph divided by $b - a$. For this definition to be valid, we need a precise understanding of what is meant by the area under a graph. This will be obtained in Section 5.3, but for now we look at an example.

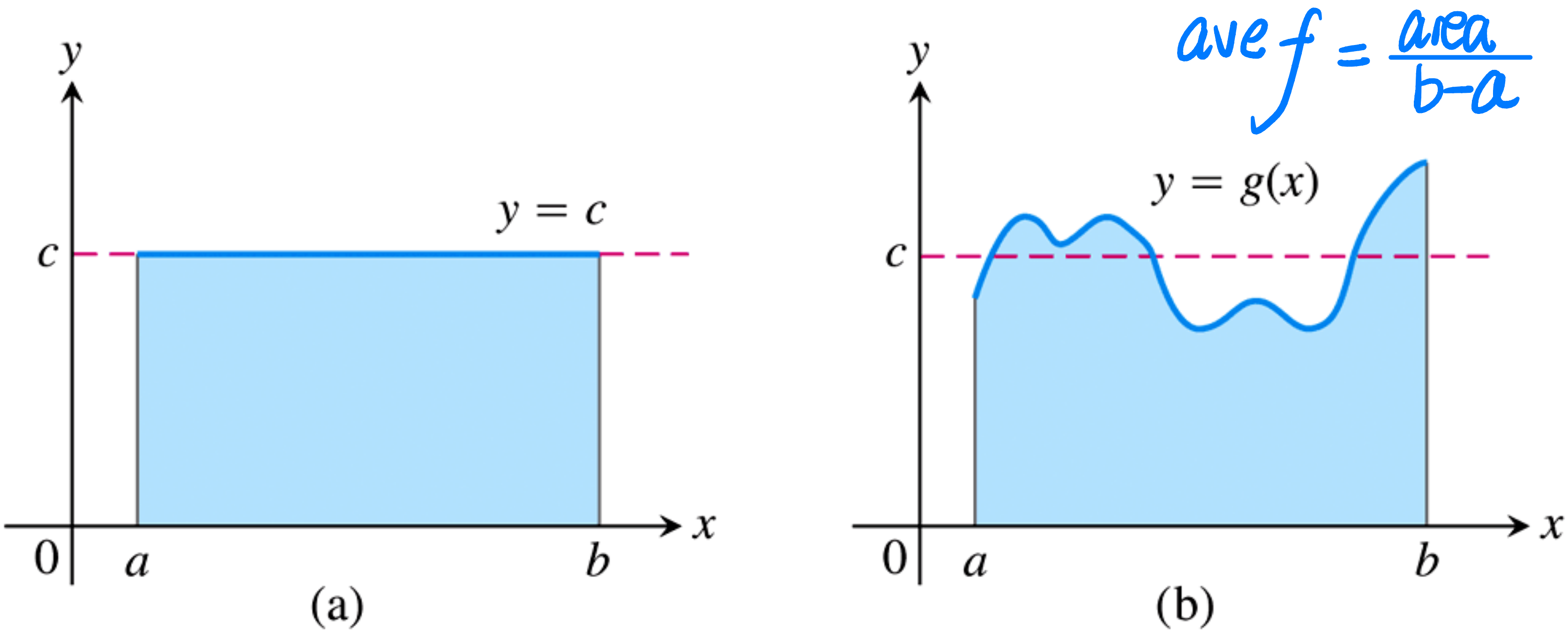
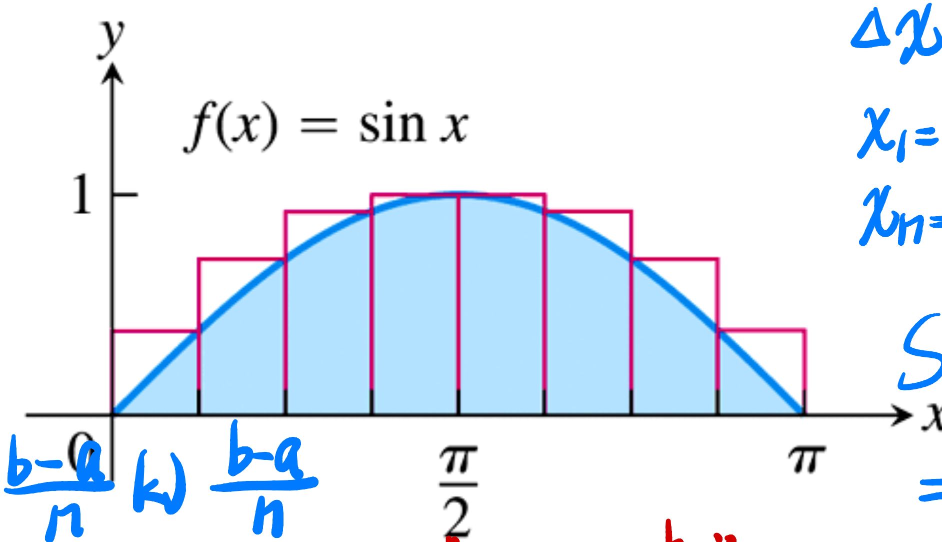


FIGURE 5.6 (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$. (b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b - a$.

EXAMPLE 4 Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.



$\Delta x = \frac{\pi}{n}$

$x_1 = \frac{\pi}{n}$ $x_k = \frac{k\pi}{n}$

$x_n = \pi$

极限不与取左/右端点而改变

$S = \Delta x\,(f(x_1) + \cdots f(x_n))$

$= \frac{\pi}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\pi\right)$

$= \frac{\pi}{n}\,\frac{\cos\frac{\pi}{2n} - \cos[(n+\frac{1}{2})\frac{\pi}{n}]}{2\sin\frac{\pi}{2n}}$

$= \frac{\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}(\cos 0 - \cos\pi)$

$= 2$

$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right)\frac{b-a}{n}$

同 × $\sin\frac{h}{2}$

FIGURE 5.7 Approximating the area under $f(x) = \sin x$ between 0 and π to compute the average value of $\sin x$ over $[0, \pi]$, using eight rectangles (Example 4).

$\frac{2}{\pi}$

TABLE 5.5 Average value of sin x on $0 \leq x \leq \pi$

Number of subintervals	Upper sum estimate
8	0.75342
16	0.69707
32	0.65212
50	0.64657
100	0.64161
1000	0.63712

$$\frac{\cos\frac{h}{2} - \cos(mh + \frac{1}{2}h)}{2\sin\frac{h}{2}}$$

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh$$

$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$$= \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2) \neq 0}$$

$\sin h \sin\frac{h}{2} = \frac{1}{2}(\cos\frac{h}{2} - \cos\frac{3h}{2})$

$\sin 2h \sin\frac{h}{2} = \frac{1}{2}(\cos\frac{3h}{2} - \cos\frac{5h}{2})$

$\vdots$

$\sin mh \sin\frac{h}{2} = \frac{1}{2}(\cos\frac{2m-1}{2}h - \cos\frac{2m+1}{2}h)$

$$\sin h + \cdots + \sin mh = \frac{\cos\frac{h}{2} - \cos(mh + \frac{1}{2}h)}{2\sin\frac{h}{2}}$$

5.2

Sigma Notation and Limits of Finite Sums

The index k ends at $k = n$.

The summation symbol (Greek letter sigma)

$$\sum_{k=1}^{n} a_k$$

a_k is a formula for the kth term.

The index k starts at $k = 1$.

The sum in sigma notation	**The sum written out, one term for each value of *k***	**The value of the sum**
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

Algebra Rules for Finite Sums

1. *Sum Rule*: $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

2. *Difference Rule*: $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

3. *Constant Multiple Rule*: $$\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$$ (Any number c)

4. *Constant Value Rule*: $$\sum_{k=1}^{n} c = n \cdot c$$ (c is any constant value.)

EXAMPLE 3 We demonstrate the use of the algebra rules.

(a) $\sum_{k=1}^{n}(3k - k^2) = 3\sum_{k=1}^{n}k - \sum_{k=1}^{n}k^2$ Difference Rule and Constant Multiple Rule

(b) $\sum_{k=1}^{n}(-a_k) = \sum_{k=1}^{n}(-1)\cdot a_k = -1\cdot\sum_{k=1}^{n}a_k = -\sum_{k=1}^{n}a_k$ Constant Multiple Rule

(c) $\sum_{k=1}^{3}(k + 4) = \sum_{k=1}^{3}k + \sum_{k=1}^{3}4$ Sum Rule

$= (1 + 2 + 3) + (3\cdot 4)$ Constant Value Rule

$= 6 + 12 = 18$

(d) $\sum_{k=1}^{n}\frac{1}{n} = n\cdot\frac{1}{n} = 1$ Constant Value Rule ($1/n$ is constant)

高次累加
可拆成 $n^k-(n-1)^k$
裂项

最高次系数 $\frac{1}{n+1}$

The first n squares: $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The first n cubes: $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$n^5=((n-1)+1)^5$

$= (n-1)^5 + C_5^1(n-1)^4 + C_5^2(n-1)^3 + C_5^3(n-1)^2 + C_5^4(n-1) + 1$

$(n-1)^5 = (n-2)^5 + C_5^1(n-2)^4 + \cdots + C_5^4(n-2) + 1$

$\vdots$

$2^5 = (1+1)^5 = 1 + C_5^1 + C_5^2 + C_5^3 + C_5^4 + 1$

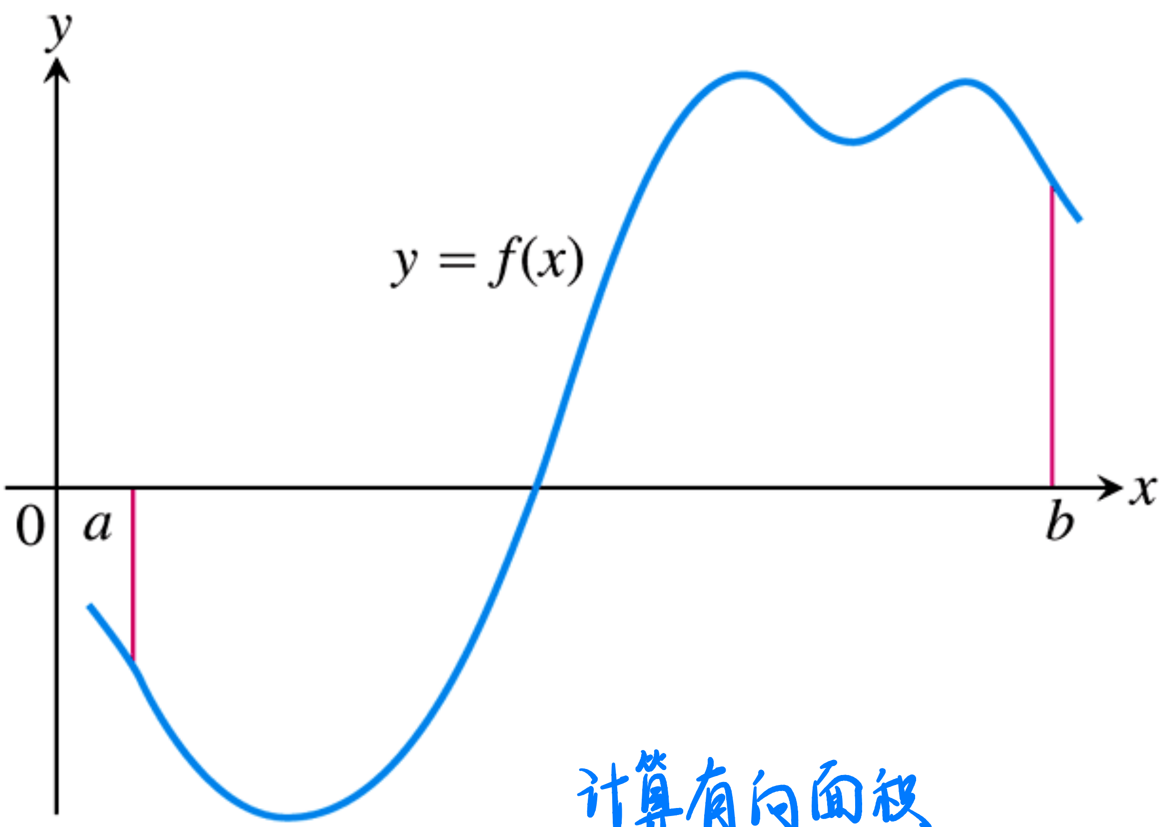
系数一致

用低次累加

5次裂项和 = 4次累加 + 3次累加

和式最高次 $k+1$

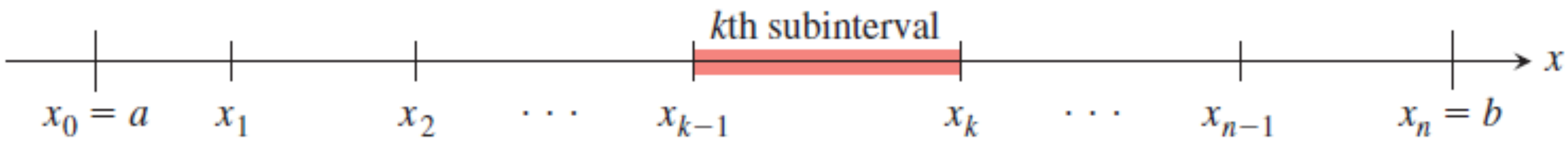
且系数 $\frac{1}{k+1}$



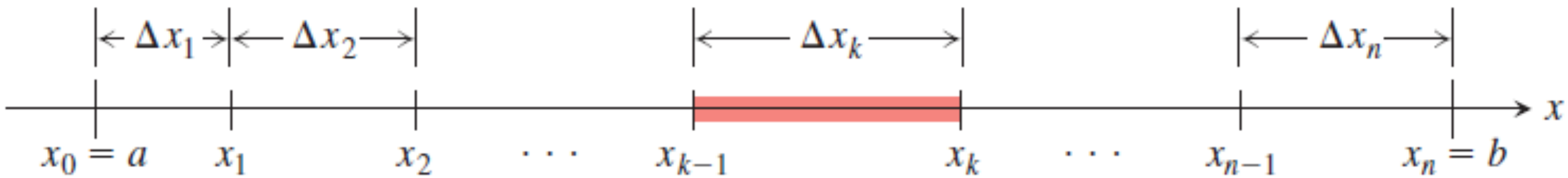
计算有向面积

FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the **kth subinterval** of P is $[x_{k-1}, x_k]$, for k an integer between 1 and n.



The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then the common width Δx is equal to $(b - a)/n$.



不一定等分

In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$ (Figure 5.9).

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k. When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\, \Delta x_k .$$

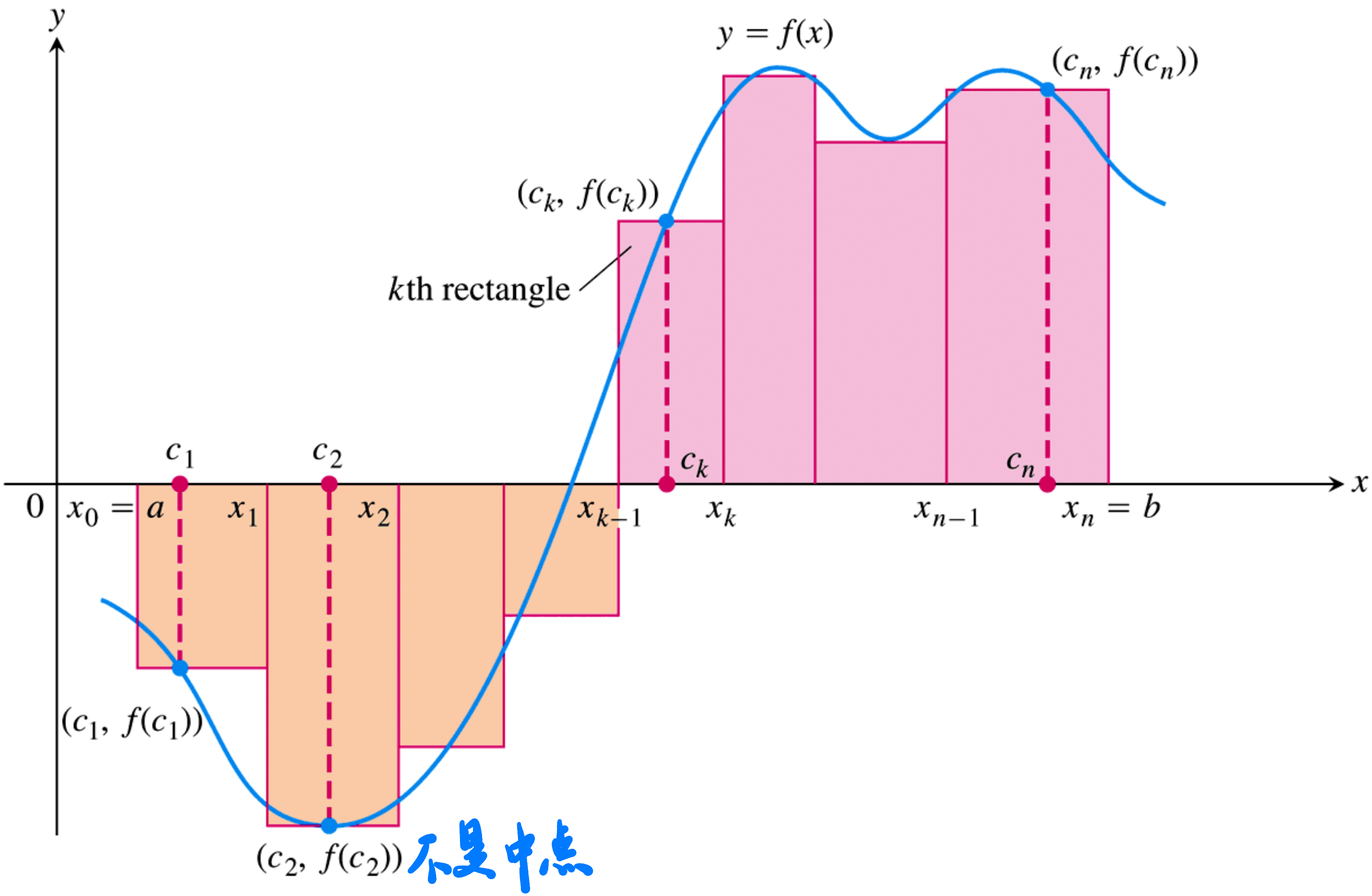


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis. Figure 5.8 has been enlarged to enhance the partition of $[a, b]$ and selection of points c_k that produce the rectangles.

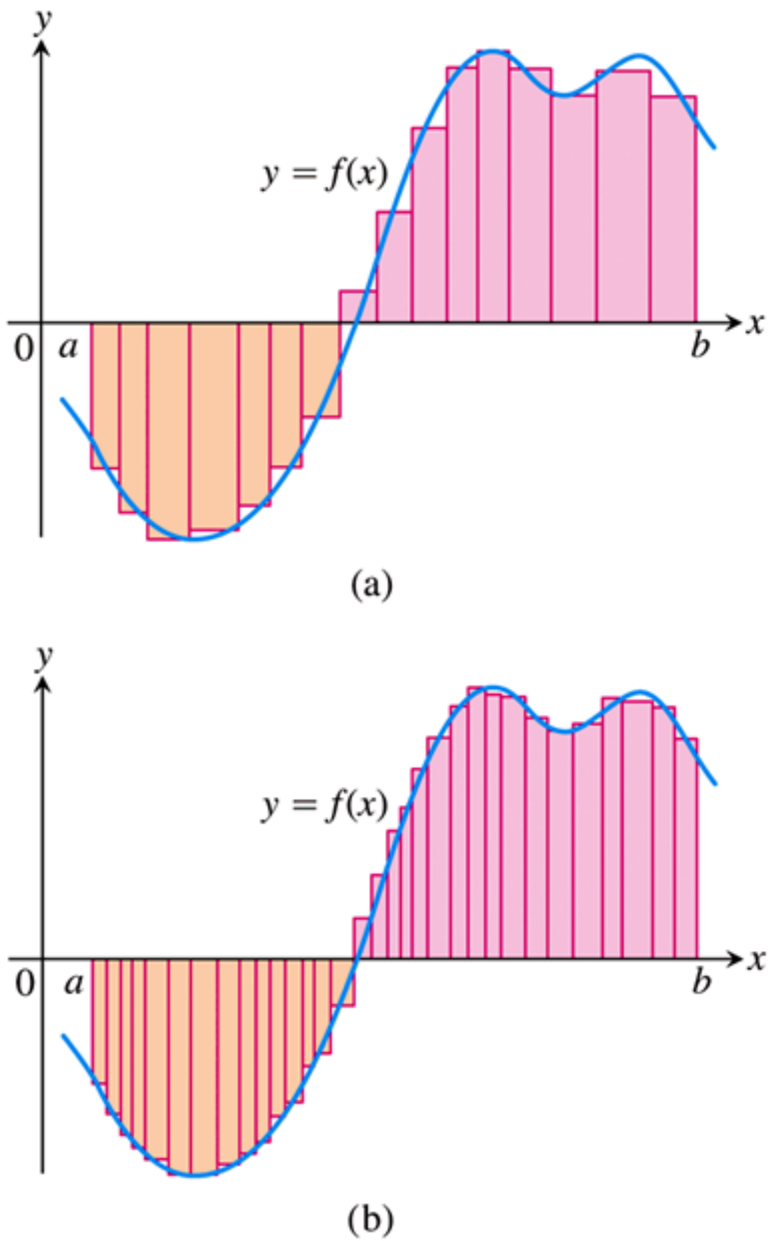


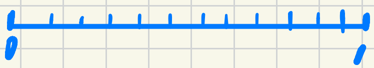
FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)\Delta x \qquad = \lim_{n\to\infty}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x \qquad = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right)\frac{1}{n}$$

用中间值替代



$\Delta x = \frac{1}{n}$ $\quad x_k = \frac{k}{n}$ $\quad c_k = \frac{2k-1}{2n}$

例. $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{1-\left(\frac{1}{n}\right)^2} + \cdots \sqrt{1-\left(\frac{n}{n}\right)^2}\right)$

$\frac{1}{n} \to \Delta x$ $\qquad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\sqrt{1-\left(\frac{k}{n}\right)^2}$

看到 $\frac{k}{n}$ 用 x 代替

$$= \int_0^1 \sqrt{1-x^2}\,dx$$

尽量使 $\begin{cases} b-a=1 \\ \Delta x = \frac{1}{n} \end{cases}$

$$\lim_{n\to\infty} \frac{1}{n\sqrt{2n}}\left(\sqrt{1}+\sqrt{3}+\cdots+\sqrt{2n-1}\right)$$

$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{2k-1}{2n}}$ $\quad$ *无 $\frac{k}{n}$ 必定中点公式

$$= \int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big]_0^1 = \frac{2}{3}$$

* $0\to1$ 始发：必 $\frac{1}{n}$ 后求和 $\begin{cases} \frac{k}{n} \\ \frac{2k-1}{2n} \end{cases}$

5.3

The Definite Integral

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon.$$

$\|P\| = \max\limits_{1\le k\le n} \Delta x_k$

n越大 不一定越来越细

只要分割越来越细一定可以越来越近

不一定均匀网格

Upper limit of integration

The function is the integrand.

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

定积分

前提：

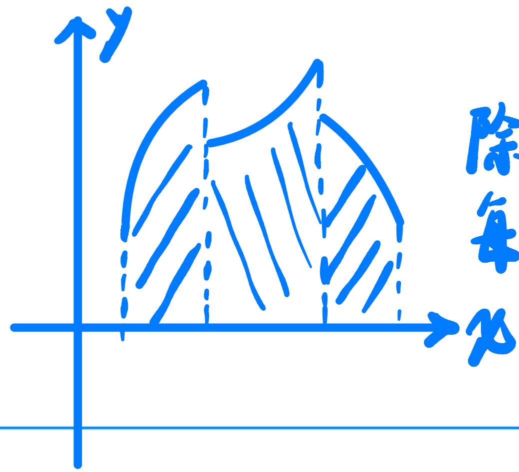
左边可积

就算左侧不可积
右侧可能存在

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} \boxed{f}\left(a + k\frac{(b-a)}{n}\right)\underline{\underline{\left(\frac{b-a}{n}\right)}}$$

等分保证极限

定义决定



除边界
每小块连续

边界两点不影响 的连续
的有限个
跳跃间断

THEOREM 1—Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many ~~jump discontinuities~~ there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

EXAMPLE 1 The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

无论切多细

c_k 可全取有理/无理

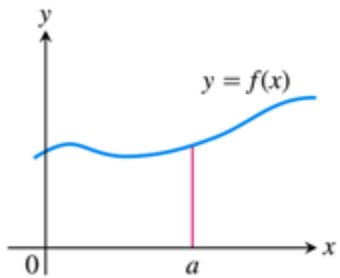
c_k rational $\sum_{k=1}^{n} f(c_k)\Delta x_k = b-a$

c_k irrational $\sum_{k=1}^{n} f(c_k)\Delta x_k = 0$

THEOREM 2 When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules in Table 5.6.

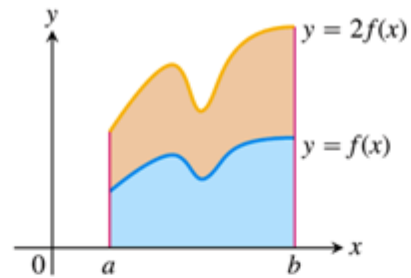
TABLE 5.6 Rules satisfied by definite integrals

1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A definition
2. *Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A definition when $f(a)$ exists
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	与a、b、c排序无关
6. *Max-Min Inequality:* 不要求f(x)连续	If f has maximum value max f and minimum value min f on $[a, b]$, then 在[a,b]上 $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$.	积分介值
7. *Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
	$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special case)



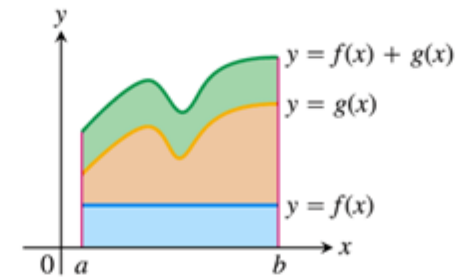
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



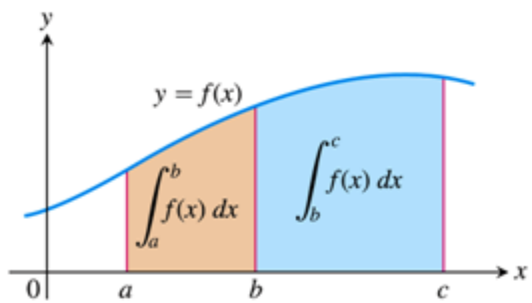
(b) *Constant Multiple:* ($k = 2$)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



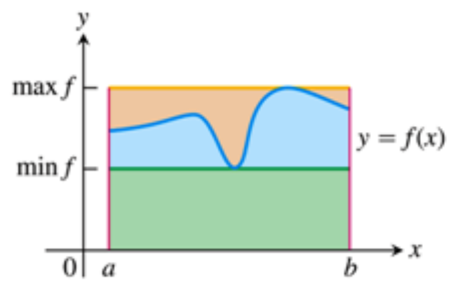
(c) *Sum:* (*areas add*)

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



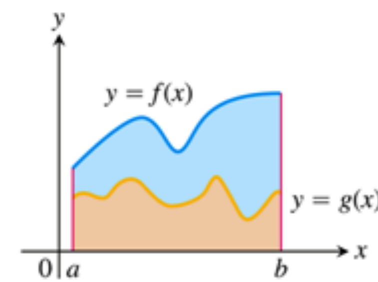
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx$$
$$\le \max f \cdot (b - a)$$



(f) *Domination:*

$$f(x) \ge g(x) \text{ on } [a, b]$$
$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.11 Geometric interpretations of Rules 2–7 in Table 5.6.

EXAMPLE 2 To illustrate some of the rules, we suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $$\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$$ Rule 1

2. $$\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$$ Rules 3 and 4

$$= 2(5) + 3(7) = 31$$

3. $$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$$ Rule 5

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

EXAMPLE 4 Compute $\int_0^b x\, dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

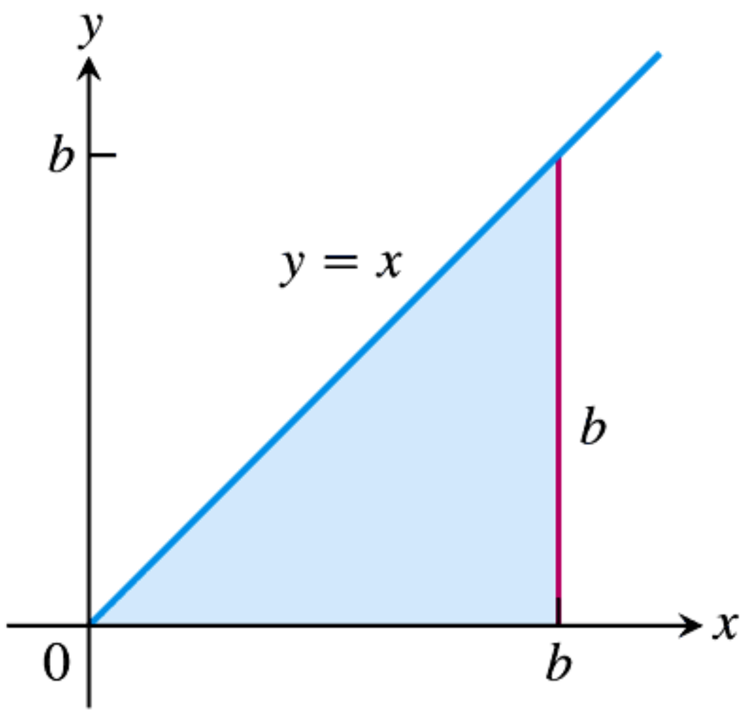


FIGURE 5.12 The region in Example 4 is a triangle.

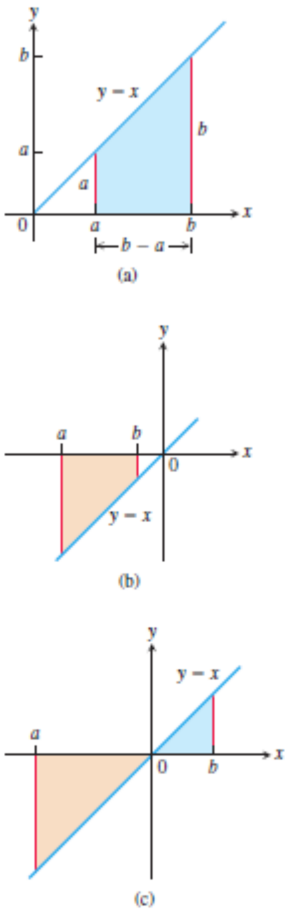


FIGURE 5.13 (a) The area of this trapezoidal region is $A = (b^2 - a^2)/2$. (b) The definite integral in Equation (2) gives the negative of the area of this trapezoidal region. (c) The definite integral in Equation (2) gives the area of the blue triangular region added to the negative of the area of the tan triangular region.

$$\int_a^b f(x)\,dx$$

$$= \lim_{n\to\infty} f(x_k)\,\Delta x_k$$

$$\Delta x = \frac{b-a}{n}$$

$$x_k = a + k\Delta x$$

$$= a + \frac{k}{n}(b-a)$$

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \qquad (1)$$

$$\int_a^b c\,dx = c(b-a), \qquad c \text{ any constant} \qquad (2)$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \qquad a < b \qquad (3)$$

$$\int_0^b x^2\,dx = \frac{1}{3}b^3 :$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}b\right)^m \frac{b}{n}$$

$$= b^{m+1} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^m}{n^{m+1}}$$

$$= b^{m+1} \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^m}{n^{m+1}}$$

$$= b^{m+1} \lim_{n\to\infty} \frac{\frac{1}{m+1} n^{m+1} + \cdots}{n^{m+1}}$$

$$= \frac{1}{m+1} b^{m+1}$$

$$\int_0^b x^n\,dx$$

$$= \frac{1}{m+1} b^{m+1}$$

$$\int_a^b x^m\,dx = \frac{1}{m+1}\left(b^{m+1} - a^{m+1}\right)$$

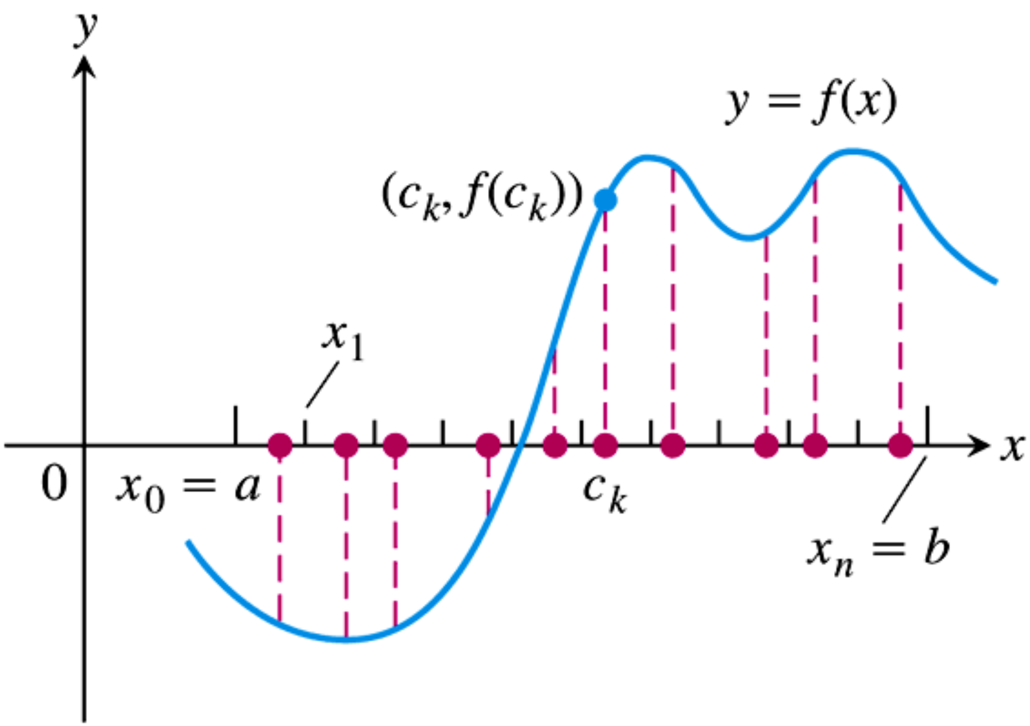


FIGURE 5.14 A sample of values of a function on an interval $[a, b]$.

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

平均值

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXAMPLE 5 Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

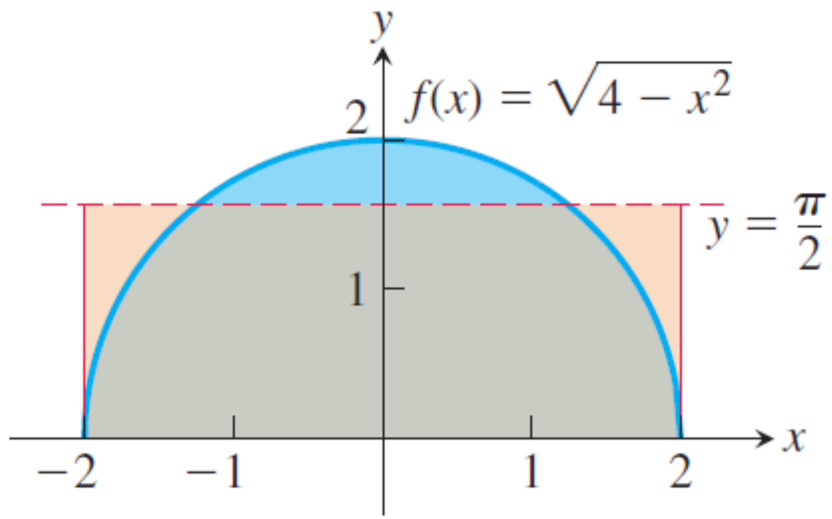


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

5.4

The Fundamental Theorem of Calculus

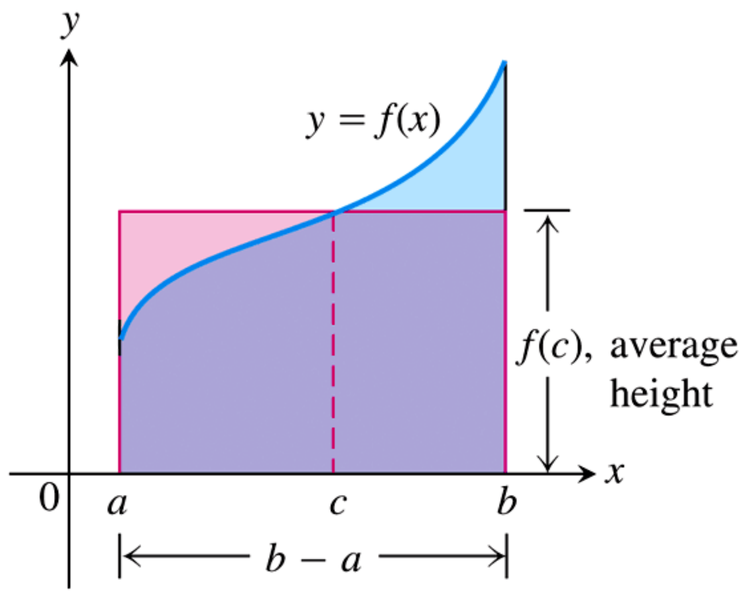


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b - a) = \int_a^b f(x)\,dx.$$

黎曼和定义
只能算简单函数

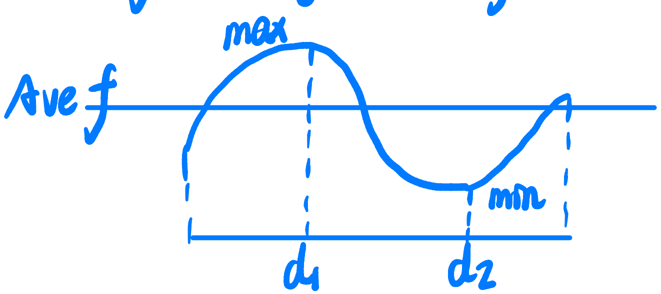
THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

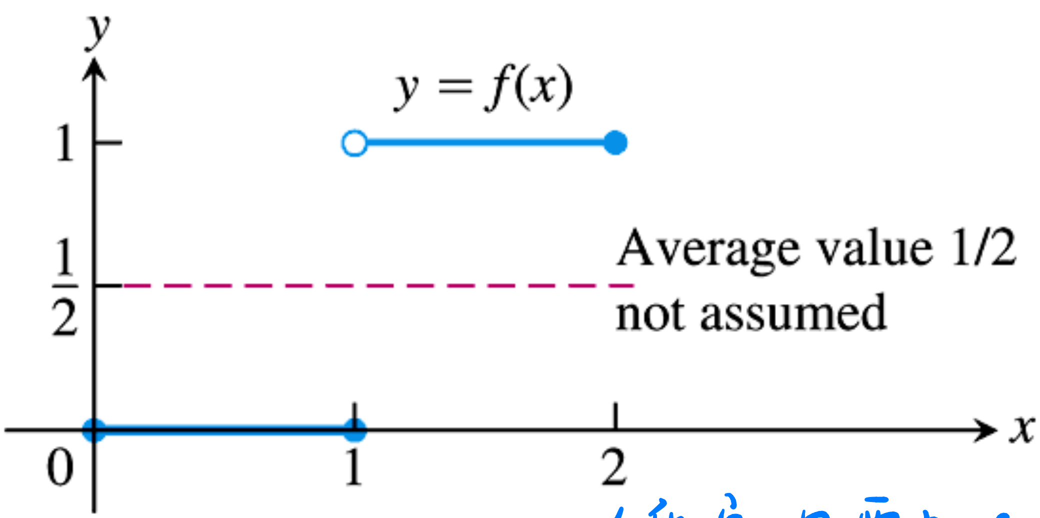
定积分中值定理

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

注：条件
闭区间连续！

$\min f \le \mathrm{Ave}\, f \le \max f$





介质定理要求连续性

FIGURE 5.17 A discontinuous function need not assume its average value.

EXAMPLE 1 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$. 用定理!

$$F(x) = \int_a^x f(t)\,dt.$$

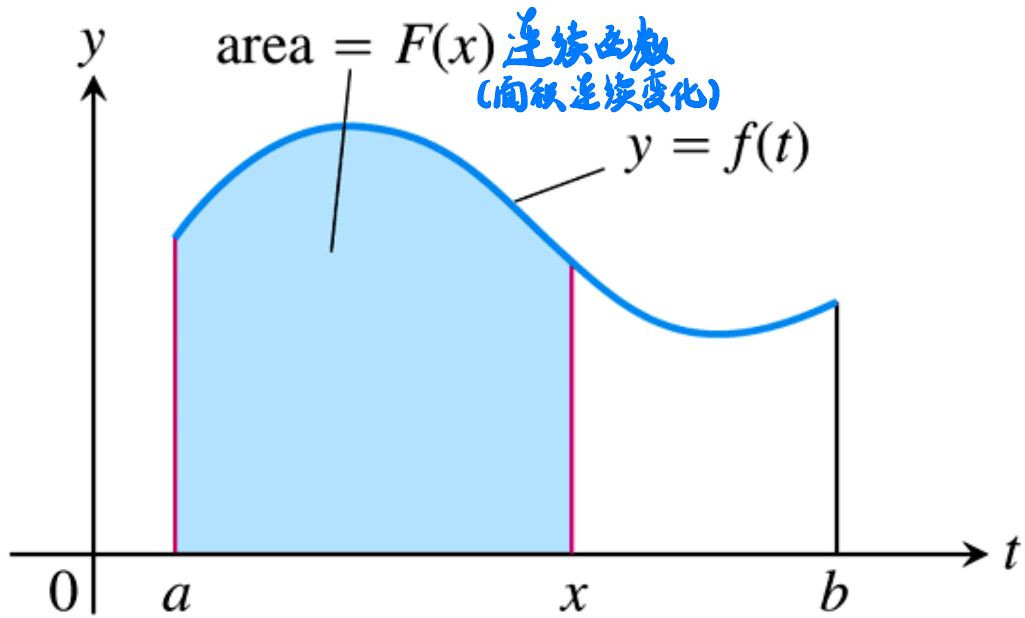
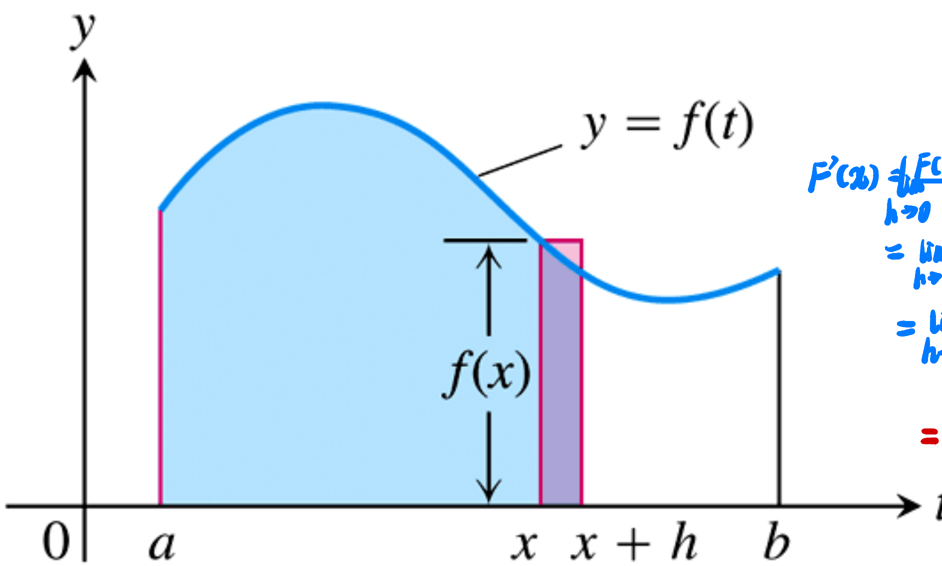


FIGURE 5.18 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.



$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0} \frac{\int_0^{x+h} f(t)\,dt - \int_0^{x} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} f(c)$$

不写x写t

c是在x, x+h中的一个值

中值定理

FIGURE 5.19 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 ① If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

有f(x)可以构造出原函数

EXAMPLE 2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\, dt$

$\frac{dy}{dx} = x^3+1$

(b) $y = \int_x^5 3t \sin t\, dt$

$y = -\int_5^x 3t\sin t$

$\frac{dy}{dx} = -3x\sin x$

(c) $y = \int_1^{x^2} \cos t\, dt$

换元 $y = \int_1^u \cos t\, dt$

$u = x^2$ 代入的是u

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \cos x^2 \cdot 2x$

写慢些

也会改变!

(d) $y = \int_{1+3x^2}^4 \frac{1}{2+t}\, dt$

也视为函数!

$y = -\int_4^{3x^2+1} \frac{1}{2+t}\, dt$

$u = 3x^2+1$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= -\frac{1}{3x^2+3}\cdot 6x$

要用微积分基本定理
下限换常数

Leibniz's Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

易错：

①求导 ②代入什么 ③分别求导

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

$$y = \int_{u(x)}^{v(x)} f(t)\,dt = \underbrace{\int_a^{v(x)} f(t)\,dt}_{W} - \underbrace{\int_a^{u(x)} f(t)\,dt}_{S}$$

$$\frac{dW}{dx} = \frac{dW}{dv}\frac{dv}{dx} = f(v(x))\,v'(x)$$

$$\frac{dy}{dx} = f(v(x))\,v'(x) - f(u(x))\,u'(x)$$

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

找出原函数

$G(x) = \int_a^x f(t)\,dt$

$G'(x) = f(x)$

$G(a) = 0 \quad G(b) = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx$

表示式积分上限不变
只改变量(坐标)

2个不同原函数

$G'(x) = f(x)$
$F'(x) = f(x)$ $\Rightarrow F(x) = G(x) + C$ 数值改变

$F(b) - F(a) = G(b) - G(a)$
$= \int_a^b f(x)\,dx$

EXAMPLE 3 We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

(a) $$\int_0^\pi \cos x\, dx = \sin x \Big]_0^\pi$$ 只要找到特殊的原函数

$$\frac{d}{dx}\sin x = \cos x$$

$f(x)\Big]_a^b = f(b) - f(a)$

$$= \sin \pi - \sin 0 = 0 - 0 = 0$$

$(\cos x + x\tan x)\Big]_a^b$

多项用( )

(b) $$\int_{-\pi/4}^0 \sec x \tan x\, dx = \sec x \Big]_{-\pi/4}^0$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

(c) $$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$

$$\frac{d}{dx}\left(x^{3/2} + \frac{4}{x}\right) = \frac{3}{2}x^{1/2} - \frac{4}{x^2}$$

$$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$

$$= [8 + 1] - [5] = 4$$

The Relationship Between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

先积分再微分

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

类似"反函数"

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, replacing b by x and x by t in Equation (6) gives

$$\int_a^x F'(t)\,dt = \underline{F(x) - F(a)},$$

但变成差式

so that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are "inverses" of each other. The Fundamental Theorem also says that every continuous function f has an antiderivative F. It shows the importance of finding antiderivatives in order to evaluate definite integrals easily. Furthermore, it says that the differential equation $dy/dx = f(x)$ has a solution (namely, any of the functions $y = F(x) + C$) for every continuous function f.

THEOREM 5—The Net Change Theorem The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\, dx. \tag{6}$$

微基定理 P2

EXAMPLE 7 Figure 5.21 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$. 0

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

负的加绝对值

$$\left|\int_0^{\pi} \sin x \, dx\right| + \left|\int_{\pi}^{2\pi} \sin x \, dx\right|$$

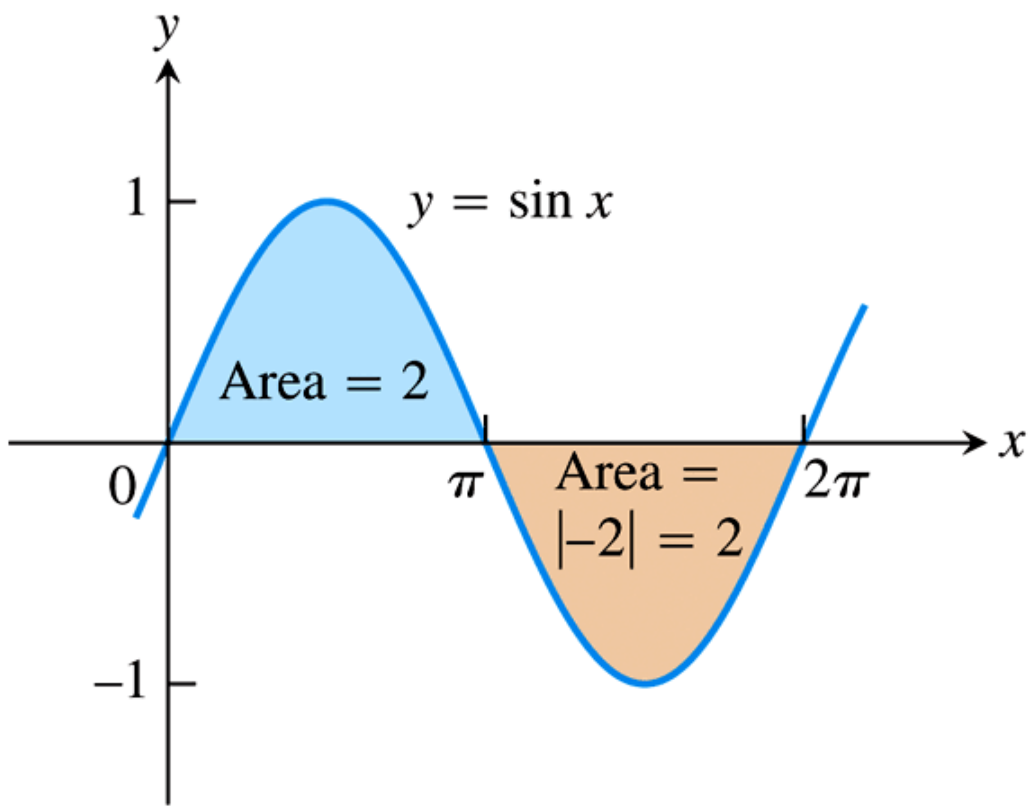


FIGURE 5.21 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

由零点分区间
求绝对值

EXAMPLE 8 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

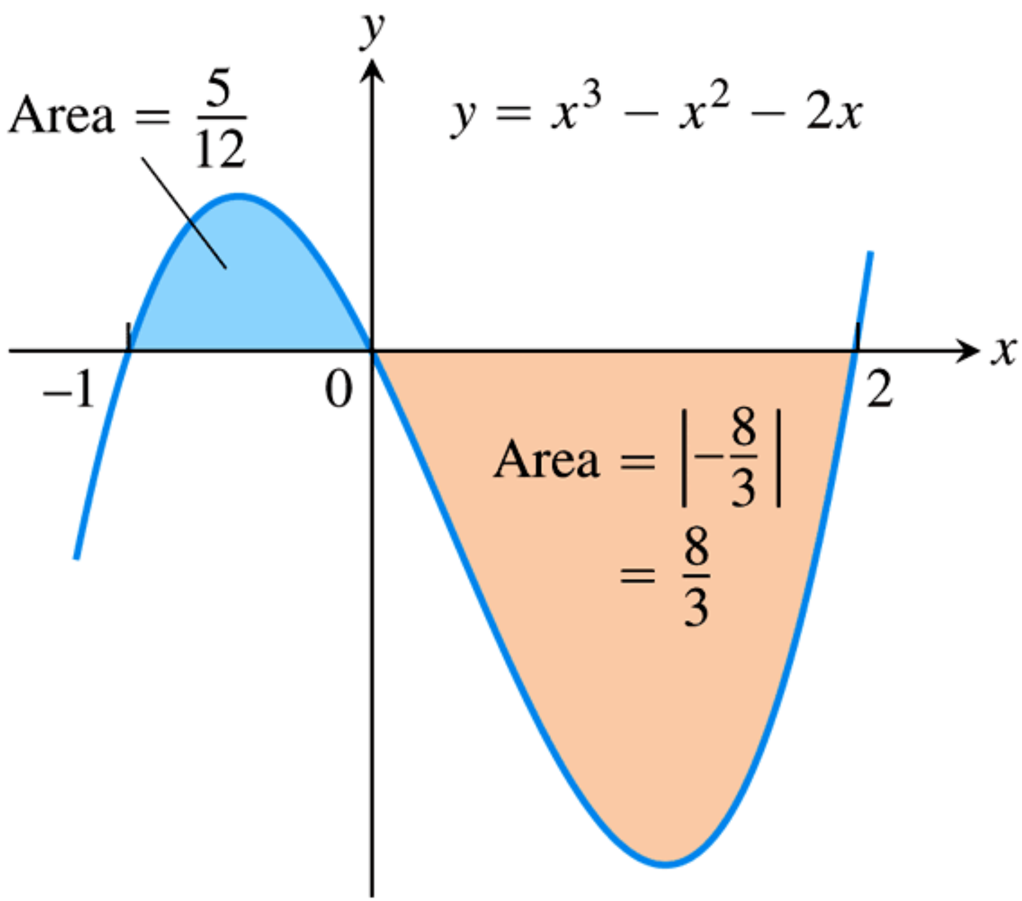


FIGURE 5.22 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

$(\cos h + \cos 2h + \cdots \cos mh)\,\frac{\sin\frac{h}{2}}{\sin\frac{h}{2}}$ * 乘一项再消

$\frac{1}{2}\left(\sin(-\frac{h}{2}) + \sin\frac{3h}{2} + \sin(-\frac{3h}{2}) + \sin\frac{5h}{2}\right) \cdots$ 构造 $\frac{h}{2}$

$= \frac{\left[\sin(-\frac{h}{2}) + \sin(mh+\frac{h}{2})\right]}{}$

★ Let $f(n) = \sum_{m=1}^{n} 2\sin\frac{h}{2} \int_0^{m} \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,dx$, here $\lfloor x+1\rfloor$ is the largest integer which is less than or equal to $x+1$. Evaluate $f(2021)$.

$\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,dx$

$45^2 - 2^2 = 43\times 47$

$m = 1, 43, 47, 2021$

Ans $= 1+43+47+2021$

$0\le x<1 \quad \lfloor x+1\rfloor = 1$

$1\le x<2 \quad \lfloor x+1\rfloor = 2$

$\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m} = \int_0^1 \cos\frac{2n\pi}{m}\,dx + \int_1^2 \cos\frac{2n\pi}{m}\times 2\,dx \cdots$

$= \cos\frac{2n\pi\cdot 1}{m} + \cos\frac{2n\pi\cdot 2}{m} + \cdots + \cos\frac{2n\pi\cdot m}{m}$

公式 ①

$= \cos h + \cos 2h + \cdots \cos mh$

$= \frac{-\sin\frac{h}{2} + \sin(m+\frac{1}{2})h}{2\sin\frac{h}{2}}$ when $\sin\frac{h}{2} \ne 0$

$h = \frac{2n\pi}{m}$

$= 0$ 只是不满足这个式子

② $\sin\frac{h}{2} = 0$

$\frac{h}{2} = k\pi$

$h = 2k\pi \Leftrightarrow h = \frac{2n\pi}{m}$

$k = \frac{n}{m} \Rightarrow m \mid n$

$\int_0^m \cos\frac{2\pi n\lfloor x+1\rfloor}{m}\,dx = m$

5.5

Indefinite Integrals and the Substitution Method

代替

換元法

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

$F'(x) = f(x)$

$(F(g(x)))' = F'(g(x))\,g'(x)$

$= f(g(x))\,g'(x)$

$F(g(x)) + C = \int f(g(x))g'(x)\,dx$

$\parallel$

$F(u) + C = \int f(u)\,du$

$u = g(x)$

$du = g'(x)\,dx$ 换元

$\int f(u)\,du = \int f(g(x))g'(x)\,dx$

The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

把相对复杂积分变简单

EXAMPLE 2 Find $\int \sqrt{2x+1}\,dx.$

$\left((2x+1)^{\frac{3}{2}}\right)' = \frac{3}{2}\sqrt{2x+1}\times 2$

$u = 2x+1$

$du = 2dx$

$\frac{1}{2}du = dx$

$\int \sqrt{u}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$

EXAMPLE 3 Find $\int \sec^2(5x+1)\cdot 5\,dx$

$u = 5x+1$

$\frac{du}{dx} = 5$

$du = 5dx$

$\int \sec^2 u \cdot du$

$= \tan u + C$

不定积分加C

恢复到原变量 $\tan(5x+1) + C$

EXAMPLE 6 Evaluate $\int x\sqrt{2x+1}\,dx.$

不要根式×其他式

$u = 2x+1$

$du = 2dx$

$dx = \frac{1}{2}du$

保证弄出dx

$\int \frac{u-1}{2}\sqrt{u}\,\frac{1}{2}du$

$= \frac{1}{4}\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$

$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C$

$= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C$

记得换回来

保证$\sqrt{\ }$内结构简单（其他复合结构）

$$\int \frac{18\tan^2 x \sec^2 x}{(2+\tan^3 x)^2}\,dx$$

$u = \tan x$

$\underline{du} = \sec^2 x\, dx$

$$\int \frac{18u^2\,du}{(2+u^3)^2}$$

$v = 2 + u^3$

$dv = 3u^2\,du$

$$\int \frac{6\,dv}{v^2}$$

$$= 6\int v^{-2}\,dv$$

$$= -6 \times v^{-1} + C$$

$$= -6 \times \frac{1}{(2+u^3)} + C$$

$$= \frac{-6}{2+\tan^3 x} + C$$

多次换元 换完变简单就行

5.6

Definite Integral Substitutions and the Area Between Curves

定积分换元

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b \underbrace{f(g(x))}_{u} \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

积分上下限变化 *

但不会换自变量回去

$$F'(x) = f(x)$$

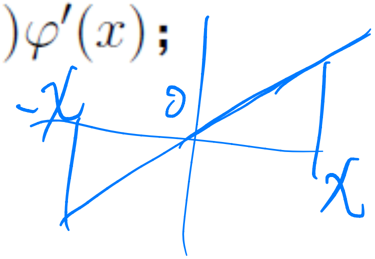
$$(F(g(x)))' = f(g(x)) \cdot g'(x)$$

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

上下限不是 a, b

- $\left(\int_{\varphi(x)}^{\psi(x)} f(t)\,dt\right)' = f(\psi(x))\psi'(x) - f(\varphi(x))\varphi'(x)$;



- $\left(\int_{\varphi(x)}^{\psi(x)} f(x,t)\,dt\right)'$; 特殊能做

- 设 $f(x)$ 连续，则

  画图感受

  $F(x) = \int_0^{-x} f(t)\,dt = \int_0^{x} f(-u)\,(-du) = \int_0^x f(u)\,du = \int_0^x f(t)\,d(t)$

  换元后改变上下限

  $u=-t$, $du=-dt$

  只是换符号

  (1) 若 $f(x)$ 是奇函数，则 $F(x) = \int_0^x f(t)\,dt$ 是偶函数；

  (2) 若 $f(x)$ 是偶函数，则 $F(x) = \int_0^x f(t)\,dt$ 是奇函数；

  $f(x)+f(-x)=0$

- (1) 可导的偶函数的导函数为奇函数，而可导的奇函数的导函数为偶函数；

  $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

  $f'(-x) = \lim_{h\to 0}\frac{-f(x+h)+f(x)}{h}$, $h=-\tilde{h}$, $= \lim_{\tilde{h}\to 0}\frac{f(x)-f(x+\tilde{h})}{\tilde{h}} = f'(x)$

  $f(x)$ odd 得 $F(x)$ even

  $G(x) = F(x)+C$

  (2) 奇函数的原函数都是偶函数，而偶函数的原函数之一为奇函数；

  $G(-x) = F(-x)+C = G(x)$

  (3) 可导的周期函数的导函数仍为周期函数（但周期函数的原函数并不一定是周期函数）

  $f(x+P)=f(x)$

  $f'(x+P) = \lim_{h\to 0}\frac{f(x+P+h)-f(x+P)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$

EXAMPLE 1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

$u = x^3+1$

$du = 3x^2\,dx$

$\cancel{\int_{-1}^{1}}\sqrt{u}\,du$

when $x=1$ $u=2$

$x=-1$ $u=0$

变上下限 $\int_0^2 \sqrt{u}\,du$

$= \frac{2}{3}\sqrt{2}\times 2 - \frac{2}{3}\times 0\times 0$

$= \frac{4}{3}$

EXAMPLE 2 We use the method of transforming the limits of integration.

(a) $$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta = \int_1^0 u \cdot (-du)$$

$\to \cot\theta = 1$

Let $u = \cot\theta$, $du = -\csc^2\theta \, d\theta$,
$-du = \csc^2\theta \, d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin\theta} \frac{1}{\sin^2\theta} d\theta$$

$$= -\int_1^0 u \, du$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$\int_{\frac{\sqrt{2}}{2}}^{1} u^{-3} \, du$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

$$= -\frac{1}{2} u^{2} \Big]_{\frac{\sqrt{2}}{2}}^{1}$$

$u = \csc\theta$

$du = -\csc\theta \cot\theta \, d\theta$

$$\int_{\sqrt{2}}^{1} u(-du) = \int_1^{\sqrt{2}} u \, du = \frac{1}{2} u^2 \Big]_1^{\sqrt{2}}$$

(b) $\int_{-\pi/4}^{\pi/4} \tan x\, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\, dx$

奇函数

$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$

换元后映射到

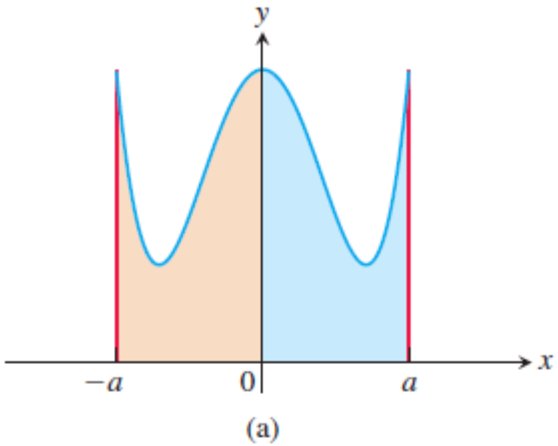
同一个点上

面积为0

Let $u = \cos x$, $du = -\sin x\, dx$.
When $x = -\pi/4$, $u = \sqrt{2}/2$.
When $x = \pi/4$, $u = \sqrt{2}/2$.

$= 0$

Zero width interval



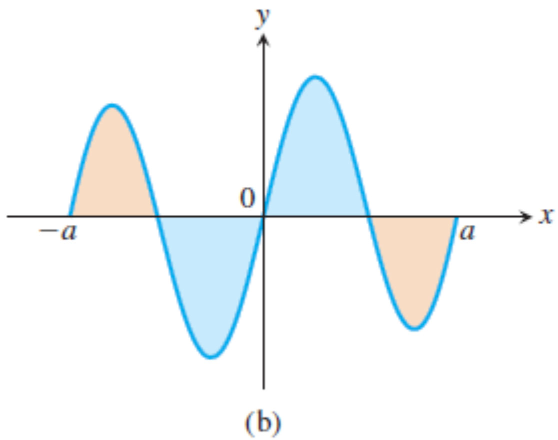


FIGURE 5.23 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, 偶 then $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(b) If f is odd, 奇 then $$\int_{-a}^{a} f(x)\,dx = 0.$$

$f(x)$ odd

$f(-x) = -f(x)$

$\int_{-a}^{a} f(x)\,dx$

$= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

令 $u = -x$

$x = a \quad u = a$

$= \int_{a}^{0} f(-u)(-du)$

$= -\int_{0}^{a} f(u)\,du$

$= -\int_{0}^{a} f(x)\,dx$

ALWAYS LEARNING

PEARSON

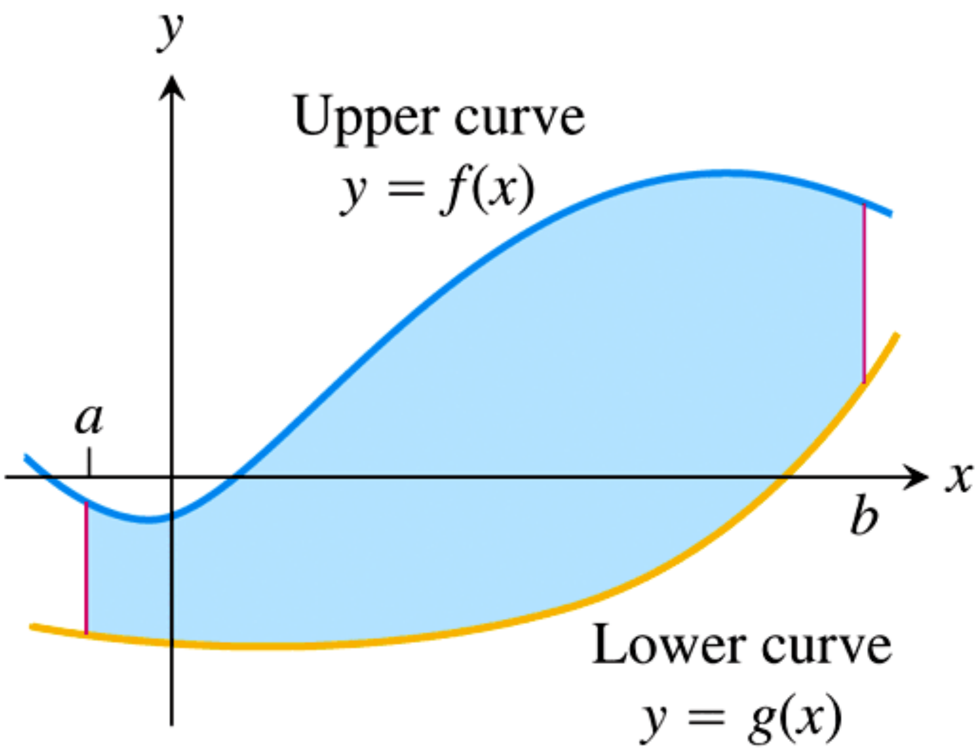


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

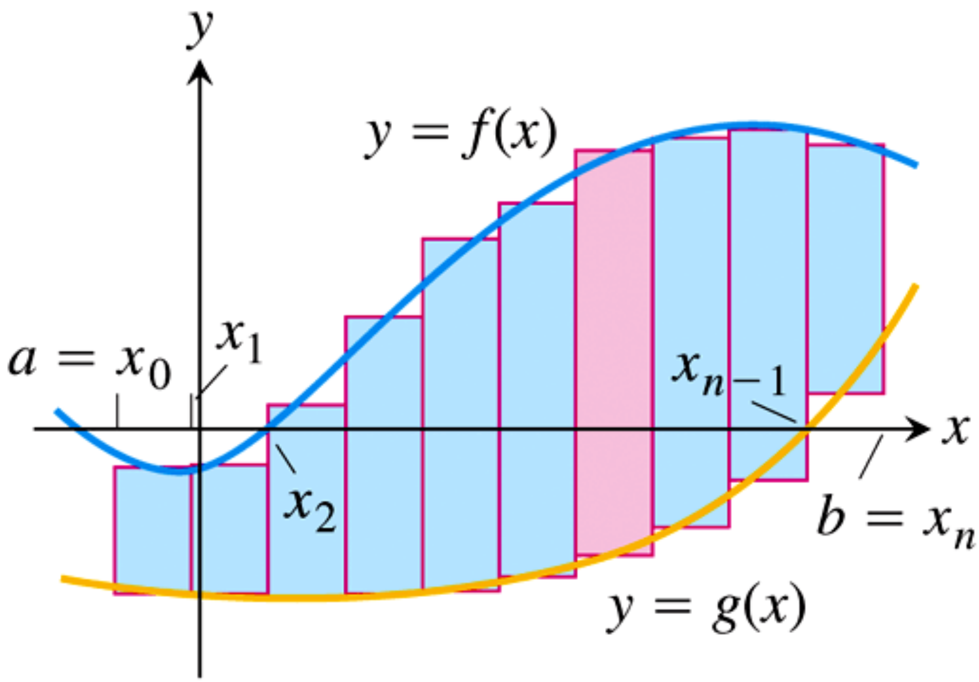


FIGURE 5.26 We approximate the region with rectangles perpendicular to the x-axis.

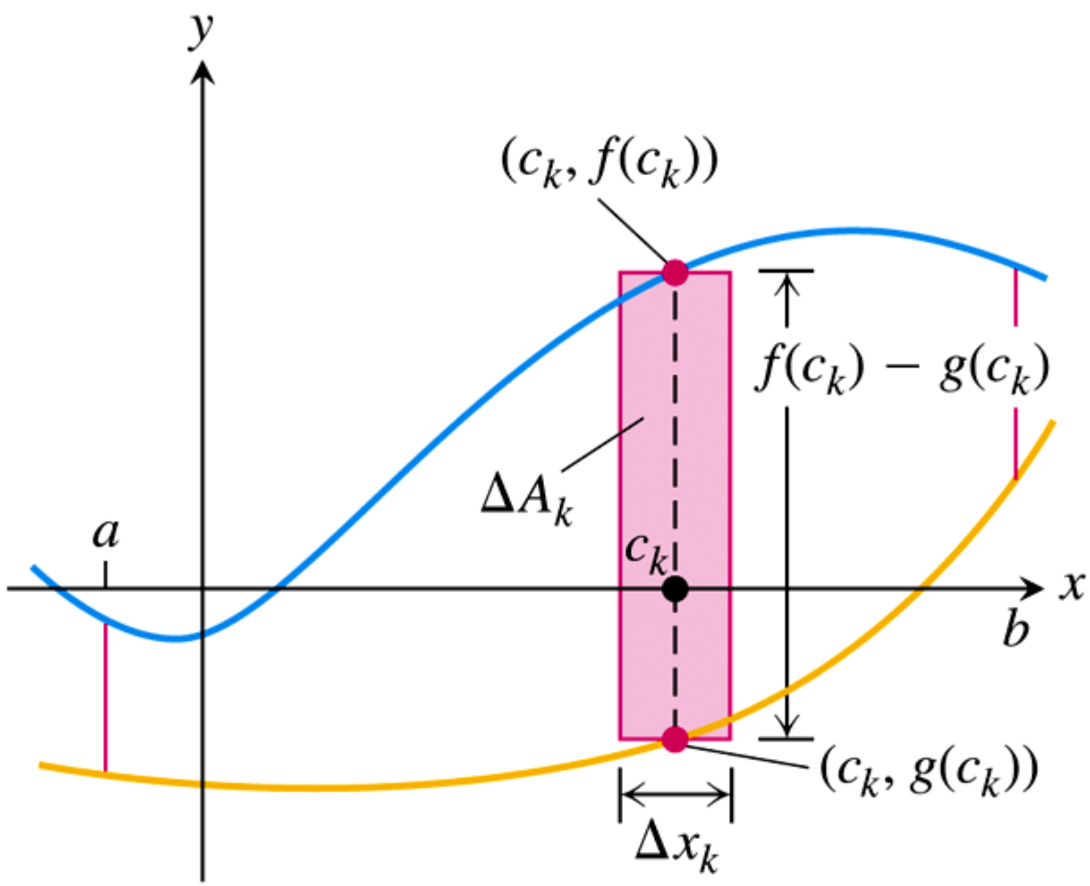


FIGURE 5.27 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

DEFINITION If f and g are continuous with ~~$f(x) \geq g(x)$~~ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE 4 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

EXAMPLE 5 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

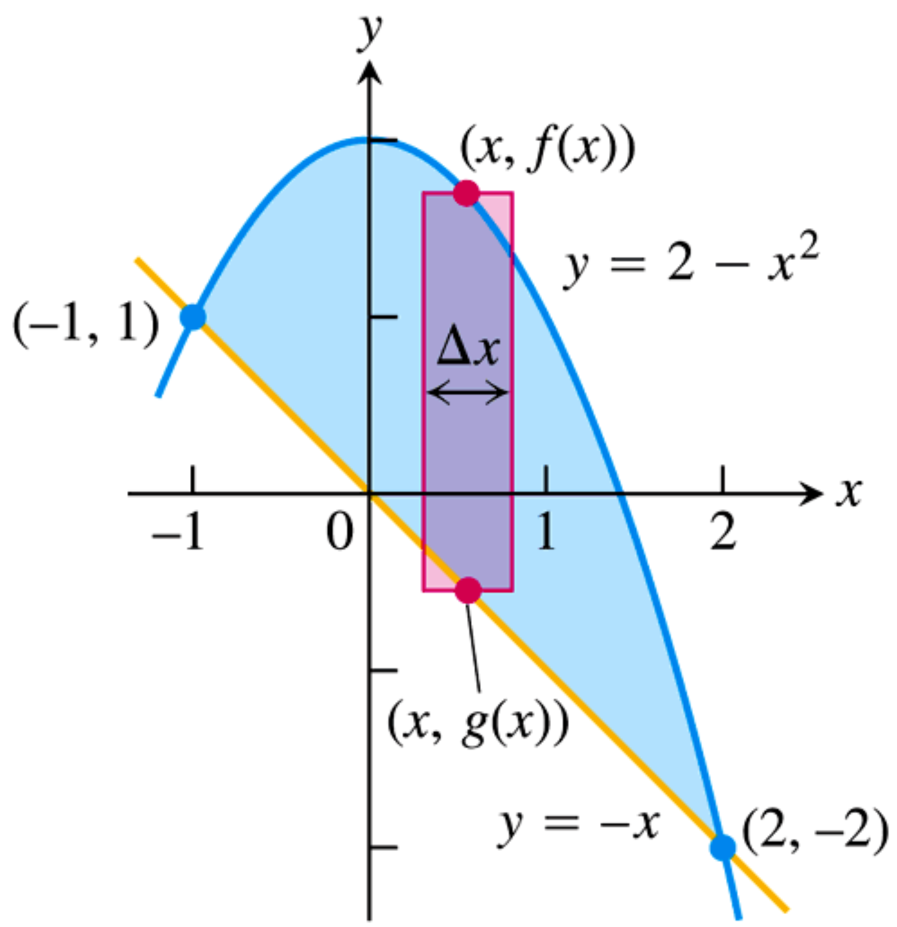


FIGURE 5.28 The region in Example 4 with a typical approximating rectangle.

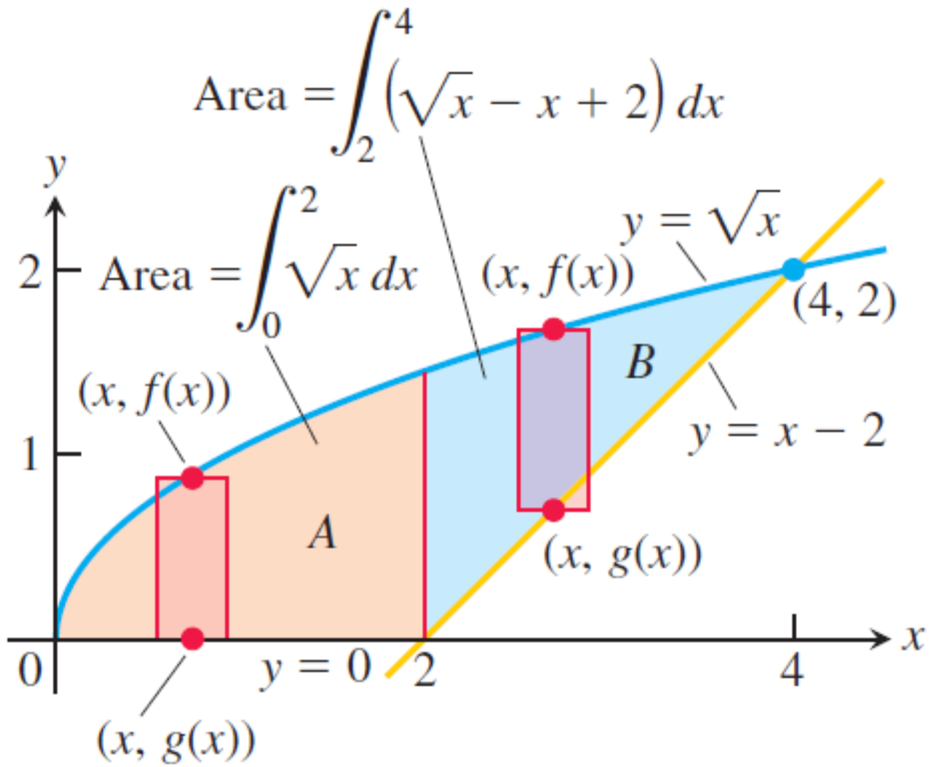
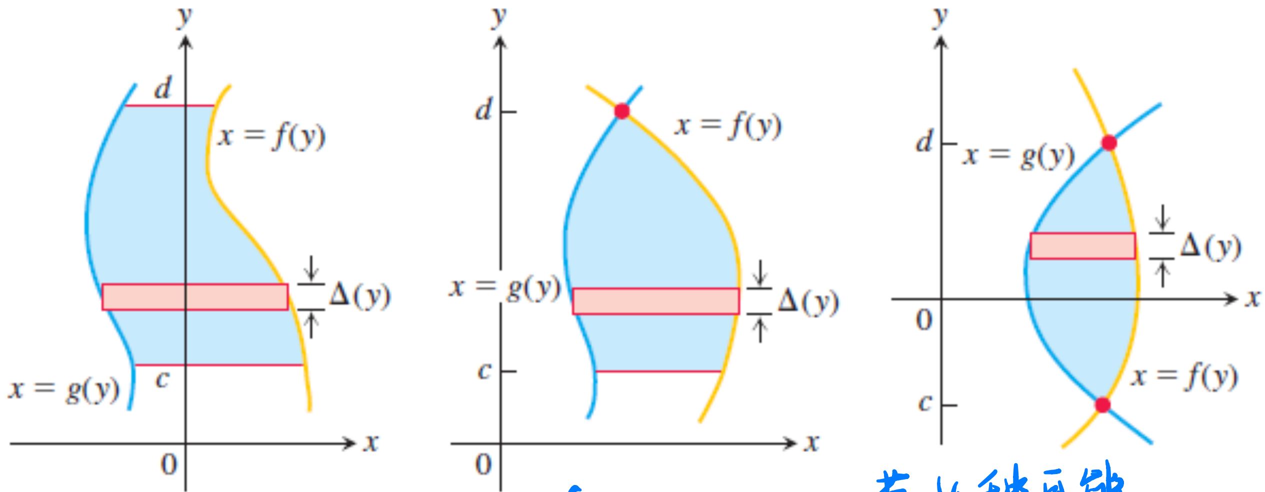


FIGURE 5.28 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

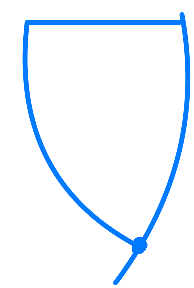
Integration with Respect to *y*

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these:

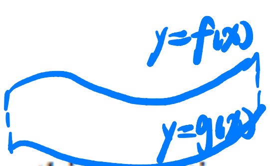


+



x=f(y), x=g(y) 共4种可能

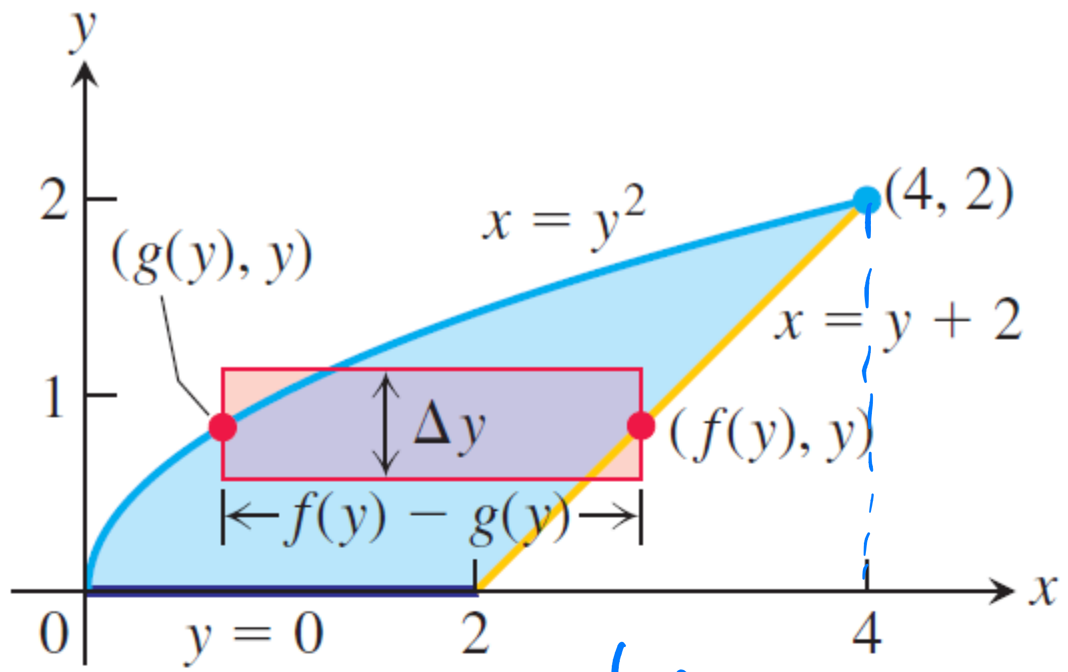
use the formula



$\int_a^b (f(x)-g(x))\,dx$

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.



$\int (y^2 - (y+2))\, dy$

FIGURE 5.29 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

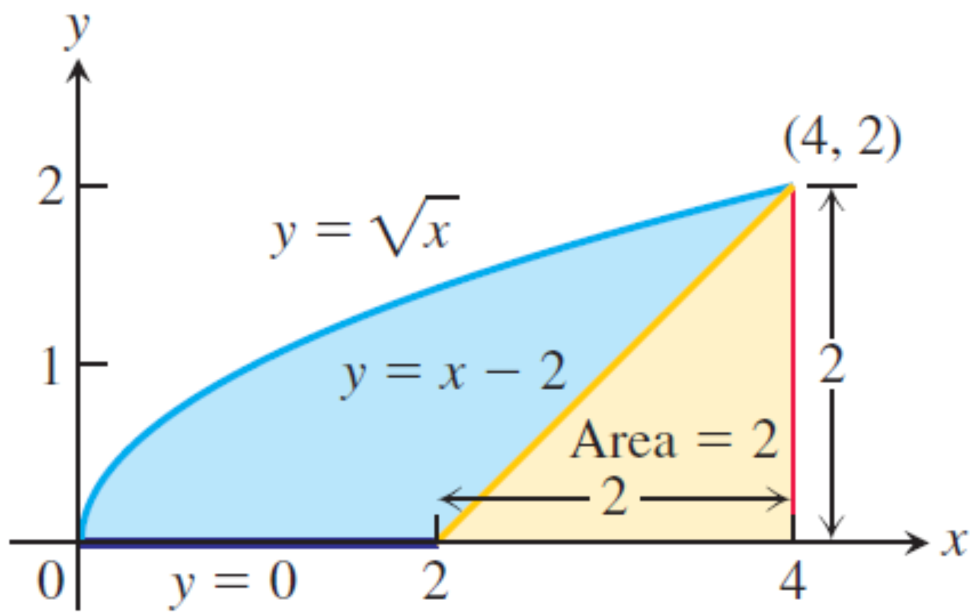


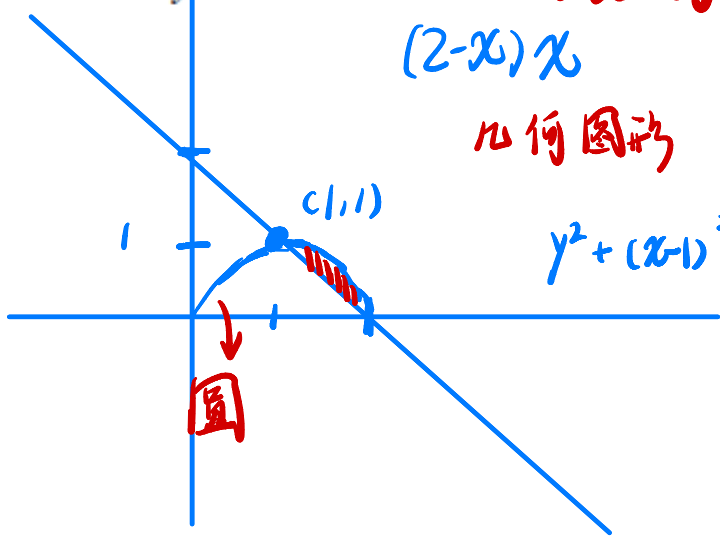
FIGURE 5.30 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

EXAMPLE 7 Find the area of the region bounded below by the line $y = 2 - x$ and above by the curve $y = \sqrt{2x - x^2}$. 函数特殊性质

$(2-x)x$ $x=1$

几何图形

$y^2 + (x-1)^2 = 1^2$



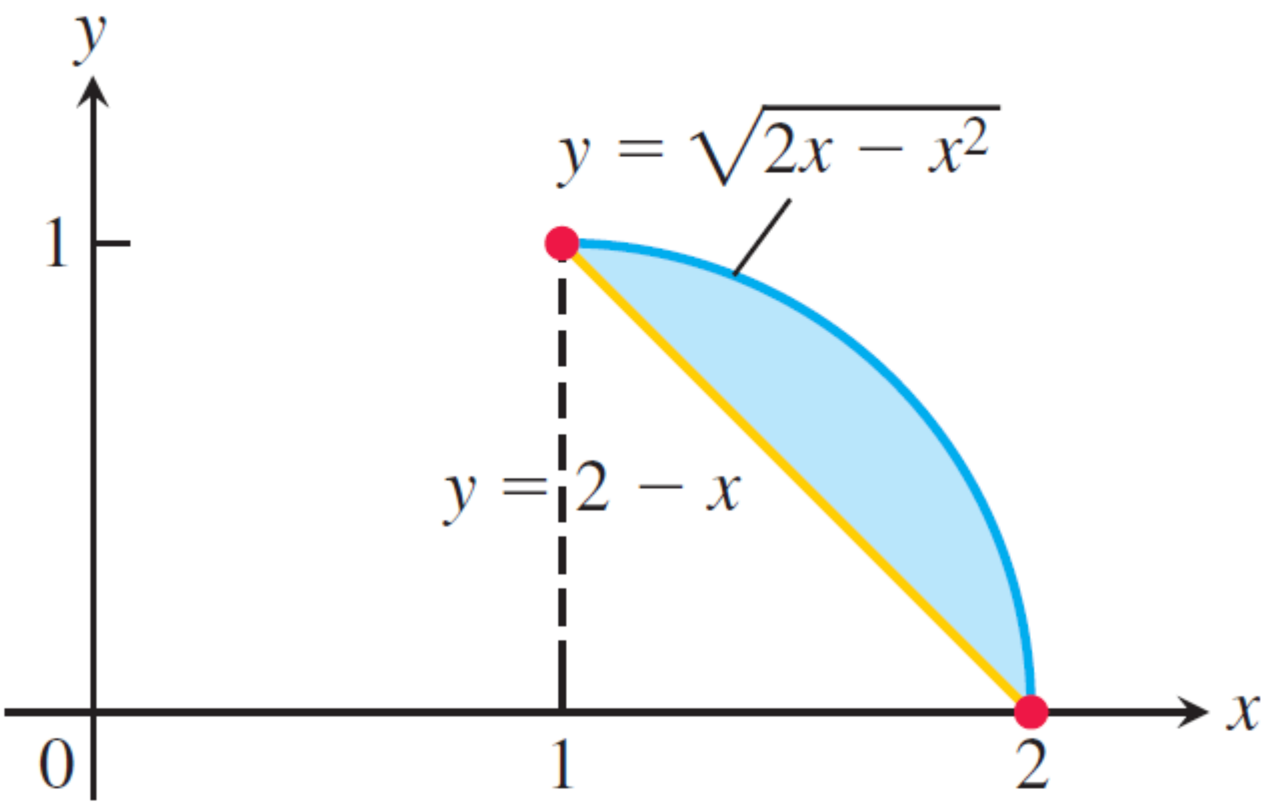
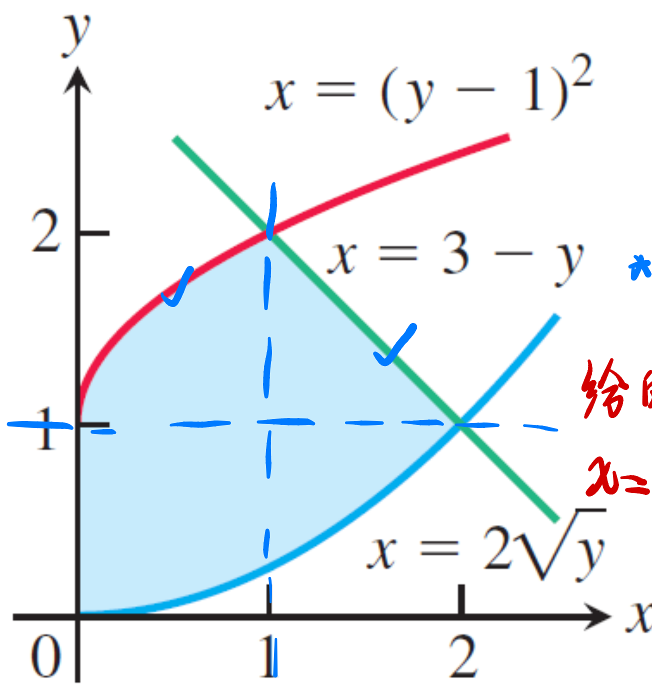


FIGURE 5.31 The region described by the curves in Example 7.

Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



*另：横切

给的是 $x=f(y)$

If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\,dx}{f(x) + f(a-x)}$$

令 $u = a - x$ (只能这么试换元)

$du = -dx$

$$\int_a^0 \frac{f(a-u)\,(-du)}{f(a-u)+f(u)} = \int_0^a \frac{f(a-u)\,du}{f(a-u)+f(u)}$$

$$= \int_0^a \frac{f(a-x)\,dx}{f(a-x)+f(x)}$$ 只变符号

$$2\,Ans = \int_0^a \frac{f(a-x)+f(x)}{f(a-x)+f(x)}\,dx = a$$

$$Ans = \frac{a}{2}$$

例5（2012） 设 $I_k = \int_0^{k\pi} e^{x^2} \sin x \, dx \quad (k = 1, 2, 3)$，则有

(A) $I_1 < I_2 < I_3$.

(B) $I_3 < I_2 < I_1$.

(C) $I_2 < I_3 < I_1$.

(D) $I_2 < I_1 < I_3$. 直观比较

$u = x - \pi$

$$I_1 = \int_0^{\pi} e^{x^2} \sin x \, dx$$

$$I_2 = \int_{\pi}^{2\pi} e^{x^2} \sin x \, dx < 0 = -\int_0^{\pi} e^{(u+\pi)^2} \sin u \, dx$$

$u = x - 2\pi$ 指数比较

$$I_3 = \int_{2\pi}^{3\pi} e^{x^2} \sin x \, dx = \int_0^{\pi} e^{(x+2\pi)^2} \sin u \, du$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx$$

$= \sin(\frac{\pi}{2} - x)$

$$\frac{f(x)}{f(x) + f(\frac{\pi}{2} - x)}$$

$$= \frac{\pi}{4}$$

x 视为常数

Find $\frac{dy}{dx}$ if

都有x函数

$$y = \int_{x^2+1}^{2x^2+3} t \tan\sqrt{x+t}\,dt.$$

使被积函数无x

① − ②

$u=x+t \rightarrow t=u-x$

$du=dt$

$t=x^2+1 \quad u=x^2+x+1$

$t=2x^2+3 \quad u=2x^2+x+3$

$$= \int_{x^2+x+1}^{2x^2+x+3} (u-x)\tan\sqrt{u}\,du$$

$$= \int_{x^2+x+1}^{2x^2+x+3} u\tan\sqrt{u}\,du - x\int_{x^2+x+1}^{2x^2+x+3} \tan\sqrt{u}\,du$$

$$\frac{dy}{dx} = \text{①}\left[(2x^2+x+3)\tan\sqrt{2x^2+x+3}\,(4x+1) - (x^2+x+1)\tan\sqrt{x^2+x+1}\,(2x+1)\right)$$

$$\text{②} = \int_{x^2+x+1}^{2x^2+x+3} \tan\sqrt{u}\,du + x\left(\tan\sqrt{2x^2+x+3}\,(4x+1) - \tan\sqrt{x^2+x+1}\,(2x+1)\right)$$