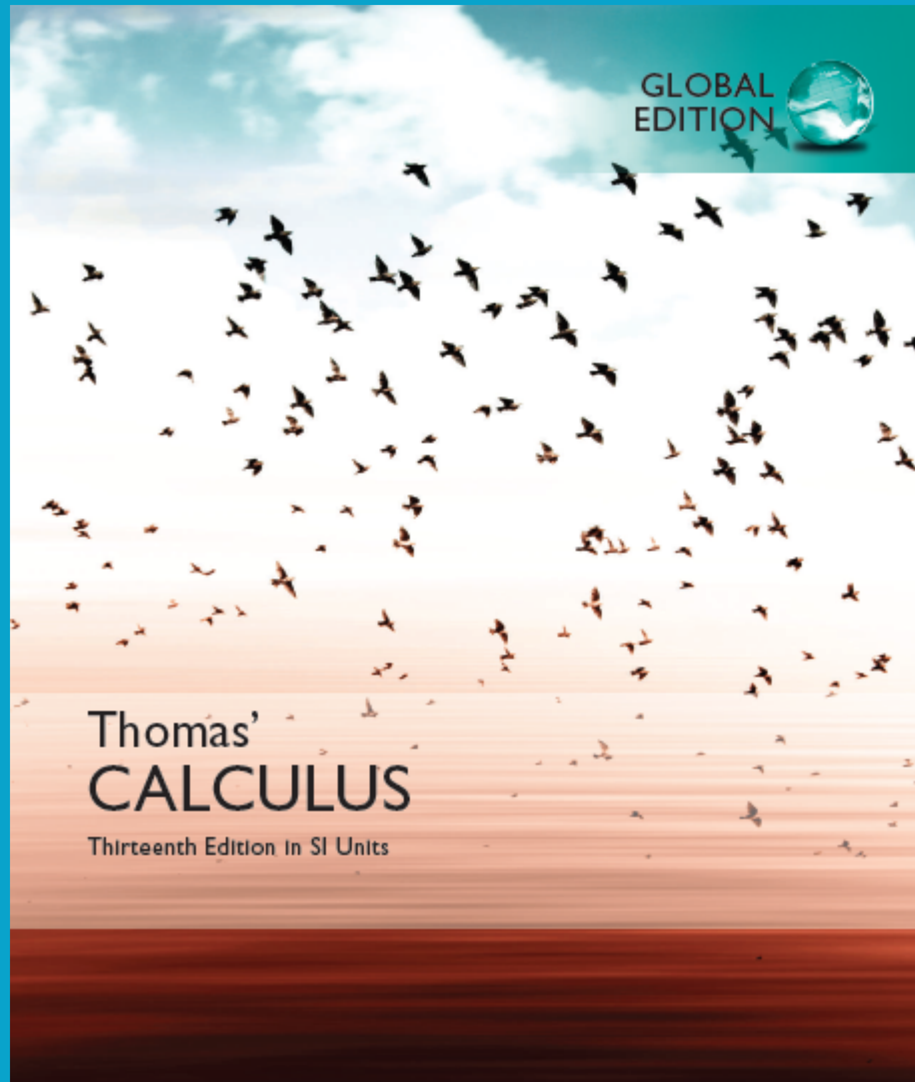
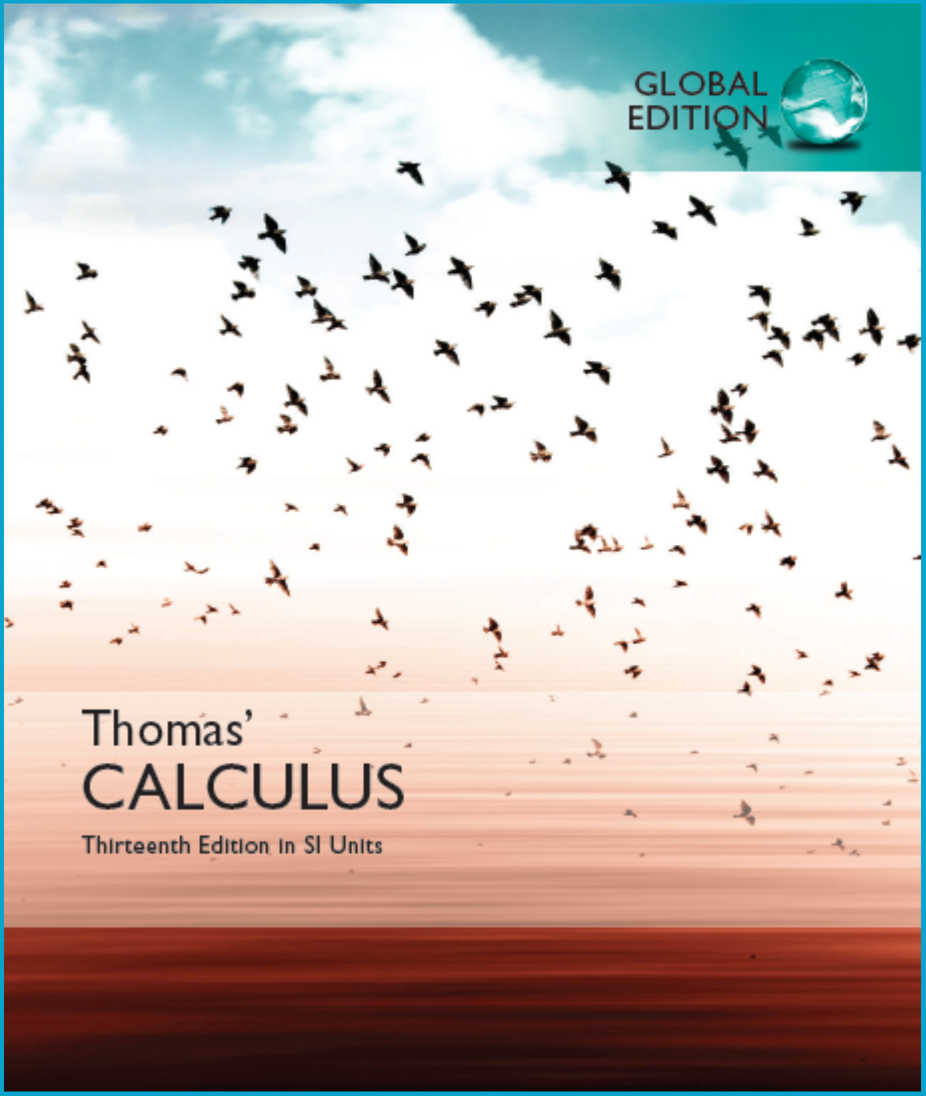


GLOBAL EDITION

Thomas' CALCULUS

Thirteenth Edition in SI Units


GLOBAL EDITION
Thomas'
CALCULUS
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Chapter 8

Techniques of Integration

8.1

Using Basic Integration Formulas

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$ $= -\ln|\csc x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

EXAMPLE 1 Evaluate the integral

* 类型: $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ (一次式 / $\sqrt{\text{二次式}}$)

(1, 2, 6)

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

$w = x^2-3x+1$

$dw = (2x-3)\,dx$

$\int \frac{dw}{\sqrt{w}}$

EXAMPLE 2 Complete the square to evaluate

$$\int \frac{dx}{\sqrt{8x-x^2}}.$$

$8x-x^2 = -(x-4)^2+16$ 底下配方

$\int \frac{1}{\sqrt{16-(x-4)^2}}\,dx$

$u = x-4$

$= \int \frac{1}{\sqrt{16-u^2}}\,du$

$u = 4v$

$du = 4\,dv$

$= \int \frac{1}{4}\frac{1}{\sqrt{1-v^2}}\cdot 4\,dv$

$= \int \frac{dv}{\sqrt{1-v^2}} = \sec^{-1} v + C$

$= \sec^{-1}(\frac{1}{4}x-1) + C$

$\frac{\text{一次}}{\sqrt{\text{二次}}}$

EXAMPLE 3 Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x)\,dx.$$

$= \frac{1}{3}\int \sin 3x \; 3\,dx$

$= -\frac{1}{3}\cos 3x + C$

化归成标准形式

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx \Rightarrow \begin{cases} ① \ a>0 \ \int \frac{px+q}{\sqrt{x^2+bx+c}}dx \\ ② \ a<0 \ \int \frac{px+q}{\sqrt{-x^2+bx+c}}dx \end{cases}$$

① $x^2+bx+c=(x+\frac{b}{2})^2+c-\frac{b^2}{4}$

$u=x+\frac{b}{2}$ $\quad \tilde{c}=c-\frac{b^2}{4}$

$du=dx$ $\quad px+q=p(u-\frac{b}{2})+q$
$=pu+q-\frac{1}{2}pb$
$=pu+\tilde{q}$

$$\int \frac{pu+\tilde{q}}{\sqrt{u^2+\tilde{c}}}du$$

i、$\tilde{c}>0$ $\quad \tilde{c}=s^2$

$$\int \frac{pu+\tilde{q}}{\sqrt{u^2+s^2}}du=\int\frac{\hat{p}v+\hat{q}}{\sqrt{v^2+1}}dv$$

$u=sv$ $\quad du=sdv$

ii、$\tilde{c}<0$ $\quad c=-s^2$

$$=\int\frac{pu+\tilde{q}}{\sqrt{u^2-s^2}}du$$

$$=\int\frac{\hat{p}u+\hat{q}}{\sqrt{v^2-1}}dv$$

② $-x^2+bx+c=-(x-\frac{b}{2})^2+\underline{c+\frac{b^2}{4}}$ ≤ 0 只可>0

$\tilde{c}=c+\frac{b^2}{4}=s^2>0$

$u=x-\frac{b}{2}$ $\quad du=dx$

$$=\int\frac{pu+\tilde{q}}{\sqrt{s^2-u^2}}du$$

$$=\int\frac{\hat{p}v+\hat{q}}{\sqrt{1-v^2}}dv$$

$u=sv$

$du=sdv$

不同于 $\int\frac{v}{v^2+1}dv=\frac{1}{2}\ln(v^2+1)+C$

③ $\int\frac{pv+q}{\sqrt{1-v^2}}dv$

$$=p\int\frac{v}{\sqrt{1-v^2}}dv+q\int\frac{1}{\sqrt{1-v^2}}dv$$

$w=1-v^2$

$dw=-2dv\cdot v$

$$=-\frac{p}{2}\int\frac{1}{\sqrt{w}}dw$$

$$=-p\sqrt{1-v^2}+q\sin^{-1}v+C$$

① $=\int\frac{pv}{\sqrt{v^2+1}}dv+q\int\frac{1}{\sqrt{v^2+1}}dv$ 没有$\sqrt{\ }$才是$\tan^{-1}$

$=p\sqrt{v^2+1}+\cdots$

so $=p\sqrt{v^2+1}+q\ln|\sqrt{v^2+1}+v|+C$

$$\int\frac{1}{\sqrt{v^2+1}}dv=\int\frac{\sec^2\theta}{\sec\theta}d\theta$$

$\theta=\tan^{-1}v$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$v=\tan\theta$ $\quad dv=\sec^2\theta\, d\theta$

写出来看 $\sec^2\theta\, d\theta=dv$

$=\int\sec\theta\, d\theta$

$=\ln|\sec\theta+\tan\theta|+C$

$=\ln|\sqrt{v^2+1}+v|+C$

实际上对付三种情况

$\cos\theta=\sqrt{1-v^2}$ $\quad \in(-\frac{\pi}{2},\frac{\pi}{2})$ $\theta=\sin^{-1}v$ ✓ 或三角(保证唯一)

$(v=\sin\theta)$

有无数个θ不确定

$\sec\theta=\sqrt{1+v^2}$ $\quad \in(-\frac{\pi}{2},\frac{\pi}{2})$ $\theta=\tan^{-1}v$

$(v=\tan\theta)$

$\tan\theta \quad v\ge1$

$-\tan\theta \quad v\le1$

$=\sqrt{v^2-1}$ $\quad \theta=\sec^{-1}v$

$(v=\sec\theta)$

$0\le\theta<\frac{\pi}{2}$ $\quad v\ge1$

$\frac{\pi}{2}<\theta\le\pi$ $\quad v\le-1$

② $\int\frac{pv+q}{\sqrt{v^2-1}}dv$

先写这个在前

$\underline{\theta=\sec^{-1}v}$ 定义好θ

$v=\sec\theta$

$dv=\sec\theta\tan\theta\, d\theta$

$=p\sqrt{v^2-1}+q\int\frac{1}{\sqrt{v^2-1}}dv$

(2)后项)

需确定范围

(1) $\underline{v>1}$ $\quad \theta\in(0,\frac{\pi}{2})$ $\quad$ (2) $v<-1$ $\theta\in(\frac{\pi}{2},\pi)$

$$=\int\frac{\sec\theta\tan\theta}{\tan\theta}d\theta \qquad =\int\frac{\sec\theta\tan\theta}{-\tan\theta}d\theta$$

$=\ln|\sec\theta+\tan\theta|+C$ $\qquad =-\ln|\sec\theta+\tan\theta|+C$

$=\ln(v+\sqrt{v^2-1})+C$ $\qquad =-\ln|v-\sqrt{v^2-1}|+C$

$=\ln|v+\sqrt{v^2-1}|+C$

可以统一

$\ln|v+\sqrt{v^2-1}|+C$

EXAMPLE 4 Find $\displaystyle\int_0^{\pi/4} \frac{dx}{1-\sin x}$.

$$= \int_0^{\pi/4} \frac{1+\sin x}{1-\sin^2 x}\,dx$$

$$= \int_0^{\pi/4} (\sec^2\theta + \sec\theta\tan\theta)\,dx$$

$$= (\tan\theta + \sec\theta)\Big]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2}+1-1 = \sqrt{2}$$

想多生 $1-\cos\theta$ 倍角

另: $u = \frac{\pi}{2} - x$

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{-du}{1-\cos u}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{du}{2\sin^2\frac{u}{2}}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2\frac{u}{2}\,d\frac{u}{2}$$

$$= -\cot\frac{u}{2}\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= (\sqrt{2}+1)-1 = \sqrt{2}$$

EXAMPLE 5 Evaluate $\displaystyle\int \frac{3x^2-7x}{3x+2}\,dx.$ 大除法

$$= \int\left[(x-3) + \frac{6}{3x+2}\right]dx$$

$$= \frac{1}{2}x^2 - 3x + 2\ln|3x+2| + C$$

EXAMPLE 6 Evaluate $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}\,dx.$

$$= 3\int \frac{x}{\sqrt{1-x^2}}\,dx + 2\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= -3\sqrt{1-x^2} + 2\sec^{-1}x + C$$

EXAMPLE 7 Evaluate

逐步变简单

$$\int \frac{dx}{(1+\sqrt{x})^3}.$$

$u = \sqrt{x}$
$u^2 = x$
$2u\,du = dx$

$\int \frac{2u\,du}{(1+u)^3}$
$v = (1+u)$
$dv = du$
$= \int \frac{2(v-1)\,dv}{v^3}$

Solution We might try substituting for the term $\sqrt{x}$, but we quickly realize the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $(1 + \sqrt{x})$, and it turns out this works:

$$\int \frac{dx}{(1+\sqrt{x})^3} = \int \frac{2(u-1)\,du}{u^3}$$

$u = 1 + \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx;$
$dx = 2\sqrt{x}\,du = 2(u-1)\,du$

$$= \int \left(\frac{2}{u^2} - \frac{2}{u^3}\right) du$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C$$

$$= \frac{1-2u}{u^2} + C$$

$$= \frac{1 - 2(1+\sqrt{x})}{(1+\sqrt{x})^2} + C$$

$$= C - \frac{1+2\sqrt{x}}{(1+\sqrt{x})^2}.$$

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

对称

期末：定积分 $\int_{-a}^{a}$ 可能用奇偶性化简

奇函数

函数特性！

EXAMPLE 8 Evaluate $\int_{-\pi/2}^{\pi/2} (x^3 \cos x)\, dx.$

积分特性

Solution No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\pi/2, \pi/2]$. Moreover, the factor x^3 is an odd function, and $\cos x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x\, dx = 0. \qquad \text{Theorem 8, Section 5.6}$$

模型: $\int\sqrt{\frac{a+x}{b-x}}\,dx$ $a.b>0$ $x\in(-a,b)$

$=\int\frac{a+x}{\sqrt{-x^2+(b-a)x+ab}}dx$

$a.b>0$

$\int\sqrt{\frac{x-a}{x-b}}\,dx$ $0<a<b$ $x>b/x<a$

when $x>b$

$=\int\frac{x-a}{\sqrt{x^2-(a+b)x+ab}}dx$

when $x<a$

$=\int\sqrt{\frac{a-x}{b-x}}\,dx$

$=\int\frac{a-x}{\sqrt{x^2-(a+b)x+ab}}dx$

$a.b>0$ $0<a<b$

$\int\sqrt{\frac{x+a}{x+b}}$

$x>-a$

$=\int\frac{x+a}{\sqrt{x^2+(a+b)x+ab}}$

$x<-b$

$=\int\frac{-(x+a)}{\sqrt{x^2+(a+b)x+ab}}$

化归为"3种情况"

$$\int_{-1}^{0}\sqrt{\frac{1+y}{1-y}}\,dy$$

分子分母同乘 $1+y$

实积分 有一个点上下同乘 0 不严格 (得取点逼近)

不定积分 $\int\frac{(1+y)}{\sqrt{1-y^2}}dy$

$=\sin^{-1}y-\sqrt{1-y^2}+C$

让上下都成为 $m\sin^2x+n\cos^2x$

$$\int\frac{dx}{1+\sin^2 x}.$$

* $u=\tan x$

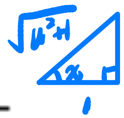
$du=\sec^2x\,dx$

自然多出2次 (不一定要把2次直接凑了)

构造三角形

$u=\tan x$ $du=\sec^2x\,dx$

$=(1+u^2)dx$



$=\int\frac{\frac{1}{1+u^2}du}{1+\frac{u^2}{1+u^2}}$

$=\int\frac{1}{2u^2+1}du$

$u=\frac{1}{\sqrt2}v$ $du=\frac{1}{\sqrt2}dv$

$=\frac{1}{\sqrt2}\int\frac{1}{v^2+1}dv=\frac{1}{\sqrt2}\tan^{-1}v+C$

$=\frac{1}{\sqrt2}\tan^{-1}(\sqrt2\tan x)+C$

次数关联

$$\int\frac{\sqrt{x}}{1+x^3}\,dx$$

$\frac12$ 多出分子 3

$u=x^{\frac32}$ $du=\frac32\sqrt{x}\,dx$

$=\int\frac{\frac23du}{1+u^2}=\frac23\tan^{-1}u+C=\frac23\tan^{-1}x^{\frac32}+C$

万能公式：—— 对不齐次也适用

$$\int \frac{f(\sin x, \cos x)}{g(\sin x, \cos x)}\,dx$$

用 $z = \tan\frac{x}{2}$ 换元

$dz = \sec^2\frac{x}{2}\cdot\frac{1}{2}$ 好处：z 可表示 dx → 一定可做

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2z}{z^2+1}$$

$$\cos x = \frac{2\cos^2\frac{x}{2} - \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}$$

$$= \frac{1-z^2}{z^2+1}$$

8.2

换元法

分步积分 (技巧)

Integration by Parts

$(uv)' = u'v + uv'$ $du = u'dx$ $dv = v'dx$

$\int (uv)'dx = \int u'v\,dx + \int uv'\,dx$

由求导法则倒推

$uv = \int u\,dv + \int v\,du$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Integration by Parts Formula

希望这个积分易求

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

① sinx/cosx · x $\int \cos x \, d(\frac{1}{2}x^2) = \frac{1}{2}x^2\cos x - \int \frac{1}{2}x^2 d(\cos x)$ ×

EXAMPLE 1 Find $\int x\cos x\,dx$.

sinx/cosx 乘多项式都可以

$\int x^n \cos x\,dx$
$=\int x^n d(\sin x)$
$= x^n \sin x - \int \sin x\, d(x^n)$
$= x^n \sin x - \int \underline{nx^{n-1}} \sin x\,dx$

每次次数下降！

拓展：$\int x^2 \sin x\,dx$
$= -\int x^2 d(\cos x)$
$= -x^2\cos x + \int d(x^2)\cos x$
$= -x^2\cos x + 2\int x\cos x$ 再做分步积分

$= \int x\, d(\sin x)$
$= x\sin x - \int \sin x\,dx$
$= x\sin x + \cos x + C$

② lnx · x

EXAMPLE 2 Find $\int \ln x\,dx$.

若 sin 2x → 换元成标准形式

三角能降次优先考虑降次

$\int x\cos^2 x\,dx = \int x\frac{1+\cos 2x}{2}dx$
$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx$

三倍角
* $\int x\cos^3 x\,dx = \int x\frac{1}{4}(\cos 3x + 3\cos x)\,dx$

拓展 $\int x^n \ln x\,dx \quad (n \neq -1)$
$= \int \ln x\, d(\frac{1}{n+1}x^{n+1})$
$= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}dx$
$= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C$

若 $n=-1$ $\int \frac{1}{x}\ln x\,dx$
换元 $= \int \ln x\, d(\ln x)$
$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

$= \ln x \cdot x - \int x\, d(\ln x)$
$= \ln x \cdot x - \int dx$
$= \ln x \cdot x - x + C$

EXAMPLE 3 Evaluate $\int x^2 e^x\,dx$.

③ xe^x

e^x - 隐形底

次数下降

$\int x^n e^x dx$
$= \int x^n d(e^x) = x^n e^x - n\int x^{n-1}e^x dx$

$= \int x^2 d(e^x)$
$= x^2 e^x - \int e^x d(x^2)$
$= x^2 e^x - 2\int e^x x\,dx$
$= x^2 e^x - 2\int d(e^x)\, x$
$= x^2 e^x - 2e^x x + 2\int e^x dx$
$= x^2 e^x - 2e^x x + 2e^x$

$$\cos 3x = \cos(2x+x) = \cos 2x\cos x - \sin 2x\sin x$$
$$= (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x$$
$$= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x$$
$$= 4\cos^3 x - 3\cos x$$

例2 再拓展 $\int x^3(\ln x)^2\,dx$ 对数函数放不进 so 只能放 x^n $\Rightarrow \int x^n(\ln x)^m\,dx$

$n\in R$　　$n=-1$ 换元

$m\in N^*$

↓

每次降一

$$= \int \frac{1}{4}d(x^4)(\ln x)^2$$
$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{4}\int d(\ln x)^2\, x^4$$
$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{4}\int x^4\, 2\ln x\,\frac{1}{x}\,dx$$
$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\int x^3\ln x\,dx$$

例3 再拓展 $\int x^3 2^x\,dx$

$$= \int x^3 e^{x\ln 2}\,dx$$ 换元

$u = x\ln 2$　$du = \ln 2\,dx$　　先写成自然对数 (不然求导易错)

$$= \frac{1}{(\ln 2)^3}\int u^3 e^u \frac{1}{\ln 2}\,du$$

标准例子

EXAMPLE 4 Evaluate

三指动
反对不动

(指数也是动2次)

有时不要先算
但有几个特例记住
可能产生相同式方程

$$\int e^x \cos x\, dx.$$

方程中2个以上不定积分不用加C
只有1个：必须加C

两函数C不一样

$\int e^x \, d(\sin x)$

$= e^x \sin x - \int \sin x \, e^x dx$ 与原式差不多

$= e^x \sin x + \int e^x d(\cos x)$ (两种 e^x 会回去)

$= e^x \sin x + e^x \cos x - \int \cos x \, d(e^x)$

Ans $= \frac{1}{2}(e^x \sin x + e^x \cos x) + C$

EXAMPLE 5 Obtain a formula that expresses the integral

递推公式 下降次数

$$\int \cos^n x\, dx$$

in terms of an integral of a lower power of cos x.

$= \int \cos^{n-1} x \cos x \, dx$

$= \int \cos^{n-1} x \, d(\sin x)$

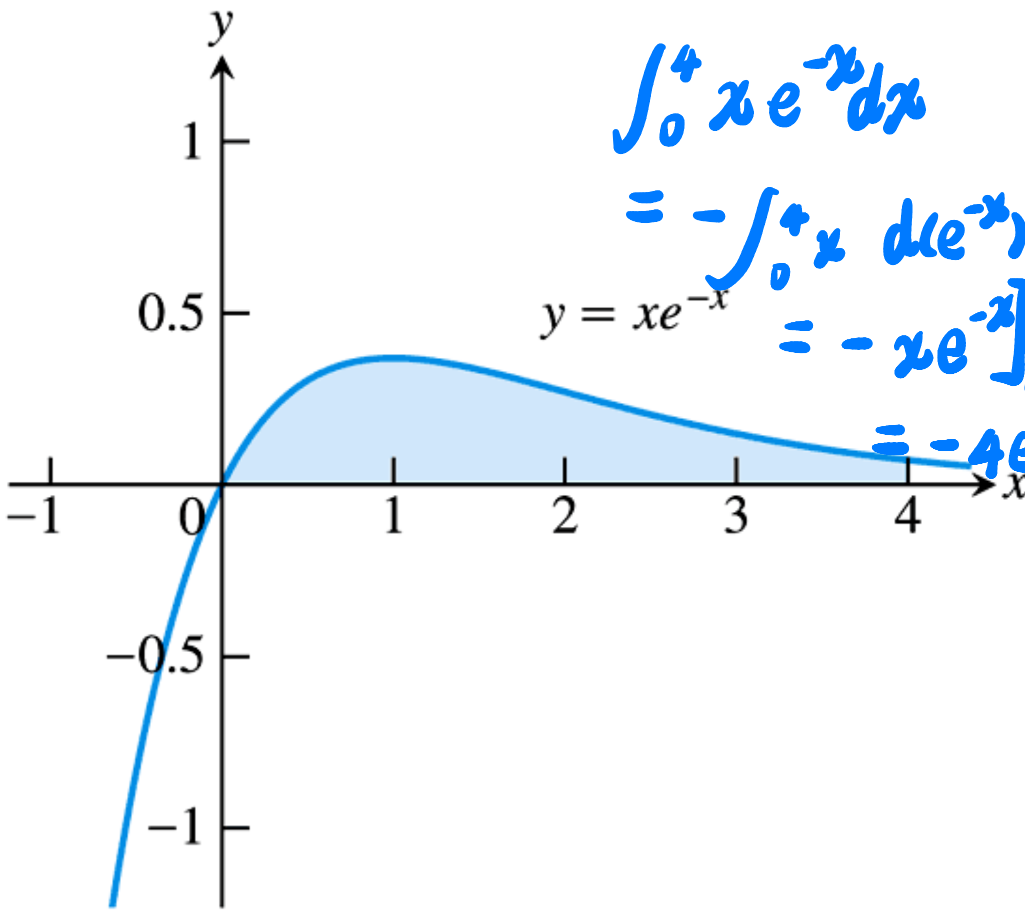
$= \sin x \cos^{n-1} x - \int \sin x \, d(\cos^{n-1} x)$

$= \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x \cdot dx$

$= \sin x \cos^{n-1} x + (n-1)\int (\cos^{n-2} x - \cos^n x)\, dx$

$\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



$$\int_0^4 x e^{-x} dx$$
$$= -\int_0^4 x \, d(e^{-x})$$
$$= -xe^{-x}\Big]_0^4 + \int_0^4 e^{-x} dx$$
$$= -4e^{-4} - e^{-x}\Big]_0^4$$

FIGURE 8.1 The region in Example 6.

EXAMPLE 7 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 3.

EXAMPLE 8 Find the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

for $f(x) = 1$ on $[-\pi, 0)$ and $f(x) = x^3$ on $[0, \pi]$, where n is a positive integer.

Solution The integral is

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx &= \frac{1}{\pi}\int_{-\pi}^{0}\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\, dx \\ &= \frac{1}{n\pi}\sin nx\Big]_{-\pi}^{0} + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\, dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\, dx.\end{aligned}$$

Using tabular integration to find an antiderivative, we have

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\cos nx$
$3x^2$	$(-)$	$\frac{1}{n}\sin nx$
$6x$	$(+)$	$-\frac{1}{n^2}\cos nx$
6	$(-)$	$-\frac{1}{n^3}\sin nx$
0		$\frac{1}{n^4}\cos nx$

直观想法

改造成"导数结果" $d(\ )$

$$\int_0^{\pi/3} x \underbrace{\tan^2 x}_{} \, dx \qquad \tan^2 x = (\sec^2 x - 1)$$

$$= \int_0^{\pi/3} x \sec^2 x \, dx - \int_0^{\pi/3} x \, dx$$

$$\int_0^{\pi/3} x \sec^2 x \, dx = \int_0^{\frac{\pi}{3}} x \, d(\tan x)$$

$$= x \tan x \Big]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan x \, dx$$

$$\int_0^{\frac{\pi}{3}} \tan x \, dx = \ln|\sec x| \Big]_0^{\frac{\pi}{3}}$$

$$\int \sin(\ln x) \, dx$$

$u = \ln x \quad x = e^u \quad dx = e^u du$

$$= \int \sin u \, e^u \, du$$

(循环)

2次分步

$$\int z(\ln z)^2 \, dz$$

$$\int (\ln z)^2 \, d\left(\tfrac{1}{2} z^2\right)$$

$$= (\ln z)^2 \tfrac{1}{2} z^2 - \int \tfrac{1}{2} z^2 \, 2\ln z \, \tfrac{1}{z} \, dz$$

$t=\frac{\pi}{2}-x$ 换元

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

偶时

$$= \begin{cases} \dfrac{\pi}{2}\,\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & n \text{ even} \\[2ex] \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}, & n \text{ odd} \end{cases}$$

$= \frac{1}{n}\underbrace{\sin x\cos^{n-1}x\Big]_0^{\frac{\pi}{2}}}_{\text{一定为0}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}}\cos^{n-2}dx$

$= \frac{n-1}{n}\int_0^{\frac{\pi}{2}}\cos^{n-2}dx$

递推下去

$= \frac{n-1}{n}\,\frac{n-3}{n-2}\int_0^{\frac{\pi}{2}}\cos^{n-4}dx$

n为偶 $= \frac{(n-1)\cdots 1}{n(n-2)\cdots 2}\int_0^{\frac{\pi}{2}}dx = \frac{\pi}{2}\,\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdots n}$

n为奇 $= \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}\underbrace{\int_0^{\frac{\pi}{2}}\cos x\,dx}_{}$

$\sin x\Big]_0^{\frac{\pi}{2}} = 1$

$= \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}$

严格上不可以 计算单位圆面积

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

$= \sqrt{1-x^2}\,x - \int x\,d(\sqrt{1-x^2})$

$= \sqrt{1-x^2}\,x - \int x\,\frac{x}{2\sqrt{1-x^2}}\,(-2dx)$

$= x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$

$= x\sqrt{1-x^2} - \left(\int\sqrt{1-x^2}\,dx - \int\frac{1}{\sqrt{1-x^2}}\,dx\right)$

$x=\pm 1$ → 但定义 (加不可积量和)

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$$\theta = \sin^{-1} x$$

$$\sin\theta = x \quad , \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \; d(\sin\theta)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \; d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} \; d\theta$$

$$= \frac{\theta}{2}\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{4}\sin 2\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

2)

$$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx$$

$$= r^2 \int_{-1}^{1} \sqrt{1-v^2}\, dv$$

$$x = rv \quad dx = dv$$

$$\int f^{-1}(x)\,dx$$

$y = f^{-1}(x)$

$x = f(y) \Rightarrow 1 = f'(y)\frac{dy}{dx} \Rightarrow dx = f'(y)\,dy$

$= \int y f'(y)\,dy$

先由性质写出来

$$\int f^{-1}(x)\,dx = \int \underbrace{y}_{u}\,\underbrace{f'(y)\,dy}_{v'}$$

$v = f(y)$

$v' = f'(y)\,dy$

先用 $f^{-1}(x) = y$，$f(y) = x$ 代换

* 含反函数积分！

$$= yf(y) - \int f(y)\,dy$$

$$= xf^{-1}(x) - \int f(y)\,dy$$

$y = f^{-1}(x), \quad x = f(y)$

$dx = f'(y)\,dy$

Integration by parts with

$u = y,\ dv = f'(y)\,dy$

$\in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = \sin^{-1}x \quad x = \sin y$

$\cos y \geq 0$

$\int \sin^{-1}x\,dx$

$= \int y\,d(\sin y)$

$= y\sin y - \int \sin y\,dy$

$= y\sin y + \cos y + C$

$\int x \sin^{-1}x\,dx$

$= \int \sin y\; y\,d(\sin y)$

$= \int \sin y\; y\cos y\,dy$

$= \frac{1}{2}\int \sin 2y\; y\,dy$ 分步积分

另： 先积 x $\quad x\,dx = d(\frac{1}{2}x^2)$

$= \int \sin^{-1}x\,d(\frac{1}{2}x^2)$

$= \int \frac{1}{2} y\,d(\sin^2 y) = \frac{1}{2}y\sin^2 y - \frac{1}{2}\int \sin^2 y\,dy = x\sin^{-1}x + \sqrt{1-x^2} + C$

8.3

解决三角函数

Trigonometric Integrals

Products of Powers of Sines and Cosines

We begin with integrals of the form:

将 cscx、secx、tanx、cotx 都化成 sinx、cosx

$$\int \sin^m x \cos^n x \, dx,$$

m、n 各有正负

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If ***m* is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

(1) $m=2k+1$ n 可奇可偶

$$\int \sin^3 x \cos^2 x \, dx$$

$$= -\int \sin^2 x \cos^2 x \, d(\cos x)$$

$u=\cos x$

$$= -\int (1-u^2) u^2 \, du$$

$$\int -\sin^{2k} x \cos^n x \, d(\cos x)$$

$u=\cos x$

$$= \int -(1-u^2)^k u^n \, du$$

(2) $m=2k$ $n=2l+1$

$$= \int \sin^m x \cos^{2l} x \cdot d(\sin x)$$

$u=\sin x$

$$= \int u^m (1-u^2)^l \, du$$

m、n 其中有一个为奇

同样方法

(3) $m=2k$, $n=2l$

二倍角

降次

$$= \int \left(\frac{1-\cos 2x}{2}\right)^k \left(\frac{1+\cos 2x}{2}\right)^l dx$$

有常数次、奇数次、偶数次

换元　再次倍角

全是偶降阶变成有奇!

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

EXAMPLE 2 Evaluate

$$\int \cos^5 x \, dx.$$

另：倍角公式

$$= \int \frac{1-\cos 2x}{2}\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1+\cos 2x - \underline{\cos^2 2x} - \cos^3 2x)\, dx$$

↗奇数次

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

$\frac{1+\cos 4x}{2}$ 再次倍角

法一

$$= \int (1-\cos^2 x)\cos^4 x \, dx$$

$$= \int \cos^4 x \, dx - \int \cos^6 x \, dx$$

递推公式 最终只有 $\sin x$, $\cos x$

EXAMPLE 4 Evaluate

$2x \in [0, \frac{\pi}{2}]$

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

$$= \int_0^{\pi/4} \sqrt{2}\, \cos 2x\, dx$$

EXAMPLE 5 Evaluate

"能被拆的

$\sec^2$

$\cos^2$

$$\int \tan^4 x\, dx. = \int (\sec^2 x - 1)^2\, dx$$

$$= \int (\sec^4 x - 2\sec^2 x + 1)\, dx$$

$$= \int (u^2+1)\, du - 2\tan x + x$$

$$\int \sec^4 x\, dx$$

$$= \int \sec^2 x\, \sec^2 x\, dx$$

$u = \tan x$

$$= \int (u^2+1)\, du$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x\, dx.$$

EXAMPLE 7 Evaluate

$$\int \tan^4 x \sec^4 x \, dx.$$

$= \int \tan^4 x \sec^2 x \, dx \sec^2 x$

$u = \tan x$

$= \int u^4 (1+u^2) \, du$

$$\int \frac{\sin^m x}{\cos^n x}\,dx$$

(1) $n \geq m > 0$

$$= \int \frac{\sin^m x}{\cos^m x}\,\frac{1}{\cos^{n-m} x}\,dx$$

$$= \int \tan^m x\,\sec^{n-m} x\,dx$$

(2) $n < m$ 降次

$$\int \frac{\sin^7 x}{\cos^3 x}\,dx$$

$$= \int \frac{\sin^6 x\,\sin x}{\cos^3 x}\,dx$$

$$= \int \frac{(1-\cos^2 x)^3 \sin x}{\cos^3 x}\,dx$$

$$= \int \frac{(1-3\cos^2 x+3\cos^4 x-\cos^6 x)\sin x}{\cos^3 x}\,dx$$

要么成为 $\dfrac{\sin^m x}{\cos^n x}$ ($n \geq m$) 或 $\sin^m x \cos^n x$

$$\int \tan^m x\,\sec^n x\,dx$$

(1) n为偶 (i) $n=2k$ $k \geq 1$

$$= \int \tan^m x\,\sec^{2k-2} x\,\sec^2 x\,dx$$

$u = \tan x$ 导出来全为 $\sec^2 x$

$$= \int u^m (1+u^2)^{k-1}\,du$$ 不用配凑

(ii) $n=0$

$$\int \tan^m x\,dx = \begin{cases} C + \ln|\sec x| & m=1 \\ \int \tan^{m-2} x\,(\sec^2 x+1)\,dx & m \geq 2 \end{cases}$$

$\tan^{2k} x = (\sec^2 x - 1)^k$

$\tan^{2k+1} x = (\sec^2 x - 1)^k \tan x$

(2) n为奇

$$\int \sec^3 x\,\tan^5 x\,dx$$

$$= \int \sec^2 x\,\tan^4 x\,\sec x\tan x\,dx$$

$u = \sec x$

$$= \int u^2 (u^2-1)^2\,du$$

$n=2k+1 \quad m=2l+1$

$$=\int \tan^{2l}x \sec^{2k}x \sec x\tan x\,dx$$

$u=\sec x$

$$=\int (u^2-1)^l u^{2k}\,du$$

(3) $n=2k+1 \quad m=2l$

$$\int \tan^{2l}x \sec^{2k+1}x\,dx$$

$$=\int (\sec^2 x-1)^l \sec^{2k+1}x\,dx$$

只剩下 $\sec^n x$

每一项都是 $\sec x$ 奇数次

对 $\int \sec^3 x\,dx$

$$=\int \sec x \sec^2 x\,dx$$

乘法较除法好处：可分步积分

$$=\int \sec x\,d(\tan x)$$

$$=\sec x\tan x-\int \tan x\,d(\sec x)$$

$$=\sec x\tan x-\int \tan^2 x \sec x\,dx$$

$$=\sec x\tan x-\int \sec^3 x\,dx+\int \sec x\,dx$$

递推回去

是否可以推到一般情况

$$\int \sec^{2k+1}x\,dx \quad k\geq 1$$

$$=\int \sec^{2k-1}x \sec^2 x\,dx$$

$$=\int \sec^{2k-1}x\,d(\tan x)$$

$$=\sec^{2k-1}x\tan x-\int \tan x\,d(\sec^{2k-1}x)$$

$$=\sec^{2k-1}x\tan x-(2k-1)\int \tan x\sec^{2k-2}x\cdot\sec x\cdot\tan x\,dx$$

$$=\sec^{2k-1}x\tan x-(2k-1)\int \sec^{2k-1}x(\sec^2 x-1)\,dx$$

$$=\sec^{2k-1}x\tan x-(2k-1)\int \sec^{2k+1}x\,dx+(2k-1)\int \sec^{2k-1}x\,dx$$

$$A_{2k+1}=\int \sec^{2k+1}x\,dx$$

$$A_{2k+1}=\frac{1}{2k}\tan x\sec^{2k-1}x+\frac{2k-1}{2k}A_{2k-1}$$

递推降阶到一次

$$\int \frac{1}{\sin^3 x \cos^2 x} dx = \int \frac{\sin x}{\sin^4 x \cos^2 x} dx$$

有奇则可产出 d( )

$$= -\int \frac{1}{(1-u^2)u^2} du \quad u=\cos x$$

$$\int \frac{1}{\sin^4 x \cos^2 x} dx$$

$$= \int \csc^4 x \sec^2 x \, dx$$

$$= \int (1+\cot^2 x)^2 (1+\tan^2 x) dx$$

$$\int \frac{1}{\sin^m x \cos^n x} dx$$

m、n至少有一奇时

(1) $m=2k+1$ $\quad u=\cos x$

$$\int \frac{\sin x}{\sin^{2k+2} x \cos^n x} dx = -\int \frac{1}{(1-u^2)^{k+1} u^n} du$$

(2) $n=2k+1$ $\quad u=\sin x$

$$\int \frac{\cos x}{\sin^m x \cos^{2k+2} x} dx = \int \frac{1}{u^m (1-u^2)^{k+1}} du$$

(3) $m=2k$ $\quad n=2l$

$$\int \csc^{2k} x \sec^{2l} x \, dx$$

$$= \int (1+\cot^2 x)^k (1+\tan^2 x)^l dx$$

只会产出 $\cot x$ / $\tan x$ 的偶数次方

与万能公式非常契合

$$*\int \frac{dx}{1+\cos x} = \tan\frac{x}{2} + C$$

$$\int \frac{dx}{1+\sin x} = \int \frac{d(x-\frac{\pi}{2})}{1+\cos(x-\frac{\pi}{2})} = \tan\frac{x-\frac{\pi}{2}}{2} + C$$

知乎 三角

积化和差

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \tag{5}$$

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

$$\int_0^{\pi/6} \sqrt{1+\cancel{\sin x}}\,dx$$

$2\sin\frac{x}{2}\cos\frac{x}{2}$

$$\int_0^{\frac{\pi}{6}} \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right) dx$$

另： $u=\frac{\pi}{2}-x$

好算一些 $\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{1+\cos x}\,dx$

$$\int_{\pi/2}^{3\pi/4} \sqrt{1-\sin 2x}\,dx$$

$$\int \frac{\sec^3 x}{\tan x}\,dx$$

先化成标准形式

看到奇次先凑乘

$$= \int \frac{\sin x}{\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{-d(\cos x)}{\sin^2 x\cos^2 x}$$

$u=\cos x$

$$= -\int \frac{du}{(1-u^2)u^2}$$

$$= \int \frac{du}{u^2-1} \quad \int\frac{du}{u^2}$$

$$= \int \frac{1}{2}\left(\frac{1}{u+1}+\frac{1}{u-1}\right)du + u^{-1}$$

$$= \frac{1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1| + u^{-1}$$

代回

* $\int x\cos^3 x\,dx$

分步积分

三倍角公式

8.4

三角换元法

Trigonometric Substitutions

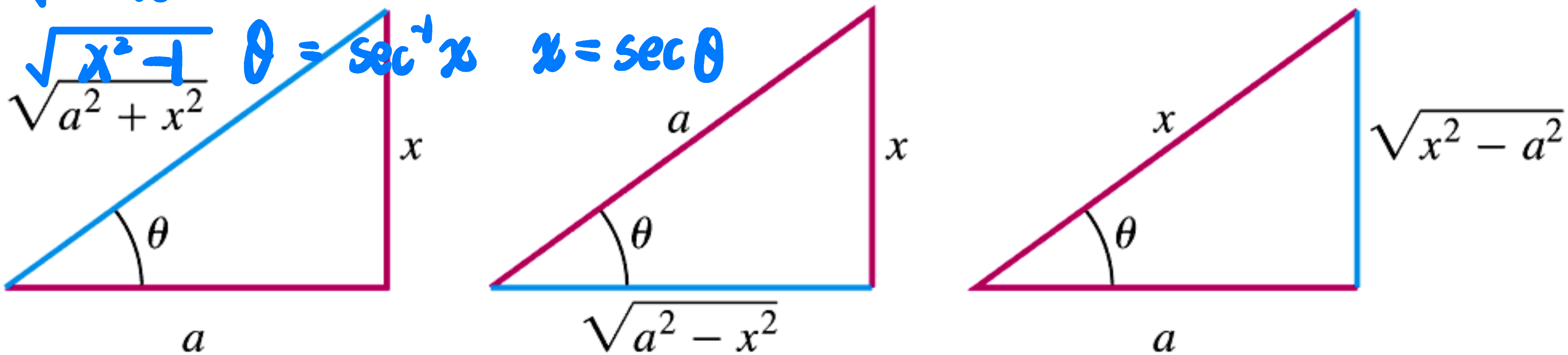
先写反三角

$\sqrt{1-x^2}$ $\theta = \sin^{-1} x$ $x = \sin\theta$

取值范围

$\sqrt{1+x^2}$ $\theta = \tan^{-1} x$ $x = \tan\theta$

$\sqrt{x^2-1}$ $\theta = \sec^{-1} x$ $x = \sec\theta$



$x = a\tan\theta$ $\qquad$ $x = a\sin\theta$ $\qquad$ $x = a\sec\theta$

$\sqrt{a^2+x^2} = a|\sec\theta|$ $\qquad$ $\sqrt{a^2-x^2} = a|\cos\theta|$ $\qquad$ $\sqrt{x^2-a^2} = a|\tan\theta|$

FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

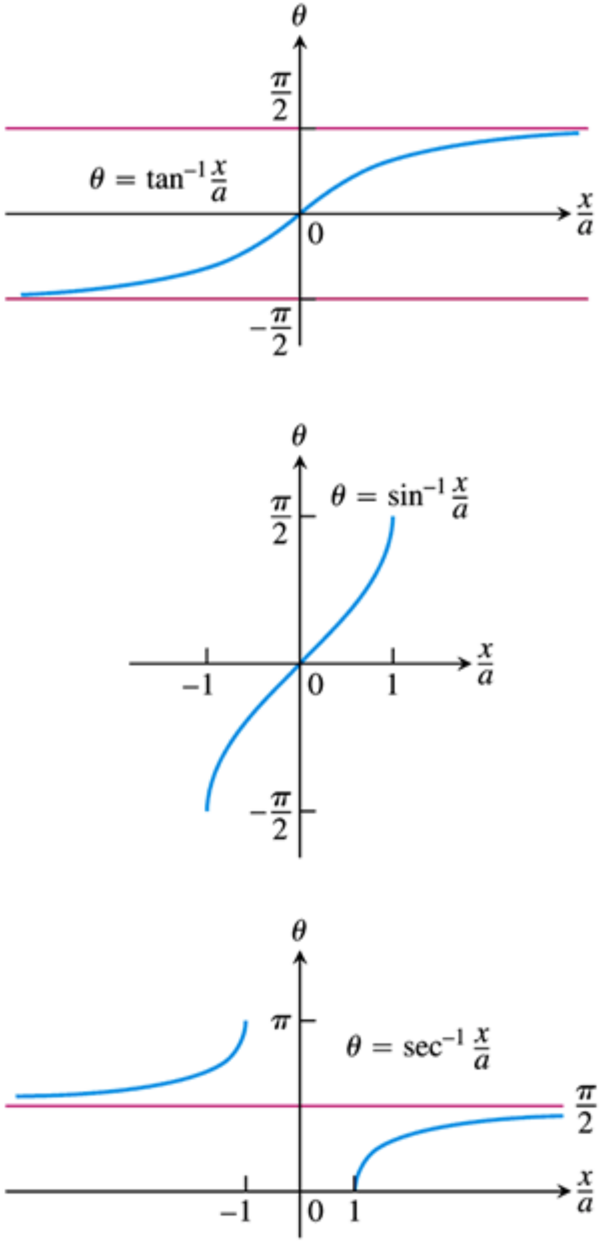


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

Procedure For a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

EXAMPLE 1 Evaluate

$$\int \frac{dx}{\sqrt{4+x^2}}.$$

当心 换元后范围

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx$$

$\theta = \tan^{-1}x$ $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$x = \tan\theta$

同一个θ范围

$\sec\theta = \sqrt{1+x^2}$

$$= \int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta$$

$$= \int \tan^3\theta\,\sec\theta\,d\theta$$

$u = \sec\theta$

$du = \tan\theta\,\sec\theta\,d\theta$

$$= \int (u^2-1)\,du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\sec^3\theta - \sec\theta + C$$

$$= \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 - \sqrt{1+x^2} + C$$

知道一个三角函数

↑ 其他也可求

再三确实

是否和"想象中一样"

$x = 2u$ $dx = 2du$

$$= \int \frac{\cancel{2}du}{\cancel{2}\sqrt{1+u^2}}$$

范围

有√!

$\theta = \tan^{-1}u$ $\in (-\frac{\pi}{2}, \frac{\pi}{2})$

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$$= \int \frac{\sec^2\theta\,du}{\sec\theta}$$

$$= \int \sec\theta\,du$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right| + C$$

$$= \ln\left(\sqrt{4+x^2}+x\right) + C$$

忘写C

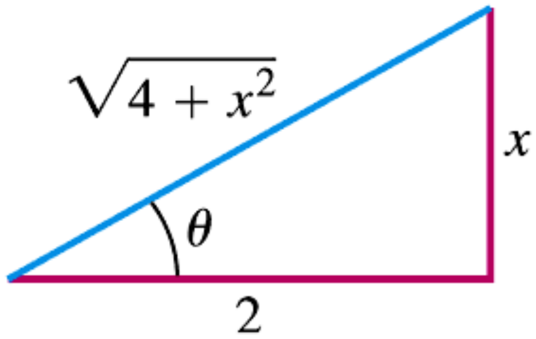


FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4 + x^2}}{2}.$$

EXAMPLE 3 Evaluate

先换元变标准形式

$x=3u \quad dx=3du$

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$$

$$=\int \frac{9u^2\,d(3u)}{3\sqrt{1-u^2}}$$

$\theta=\sin^{-1}u \quad \underline{u=\sin\theta}$ 好处是

$du=\cos\theta\,d\theta$

$\cos\theta$ 在 $(-\frac{\theta}{2},\frac{\theta}{2})>0$

让 $\sqrt{1-u^2}$ 符号不动

$$=9\int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos\theta}$$

$$=9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

二倍角

$$=\frac{9}{2}(\theta-\frac{1}{2}\sin 2\theta)+C$$

$$=\frac{9}{2}(\sin^{-1}\frac{x}{3}-\frac{1}{2}\times 2u\sqrt{1-u^2})+C$$

$$=\frac{9}{2}\sin^{-1}\frac{x}{3}-\frac{x}{3}\sqrt{1-\frac{x^2}{9}}+C$$

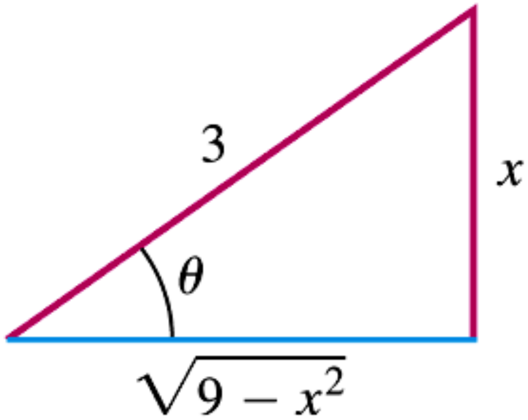


FIGURE 8.5 Reference triangle for $x = 3 \sin \theta$ (Example 2):

$$\sin \theta = \frac{x}{3}$$

and

$$\cos \theta = \frac{\sqrt{9 - x^2}}{3}.$$

EXAMPLE 4 Evaluate

$$\int \frac{dx}{\sqrt{25x^2-4}}, \quad x > \frac{2}{5}.$$

$$= \int \frac{dx}{5\sqrt{x^2-\frac{4}{25}}}$$

$x = \frac{2}{5}u \quad dx = \frac{2}{5}du \quad u > 1$

$$= \int \frac{\frac{2}{5}du}{5\times\frac{2}{5}\sqrt{u^2-1}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2-1}}$$

$\theta = \sec^{-1}u \quad u = \sec\theta$

$\theta \in (0, \frac{\pi}{2}) \quad du = \sec\theta\tan\theta\, d\theta$

$$= \frac{1}{5}\int \frac{\sec\theta\tan\theta\, d\theta}{\tan\theta}$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|u + \sqrt{u^2-1}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{5}{2}x + \sqrt{\frac{25}{4}x^2-1}\right| + C$$

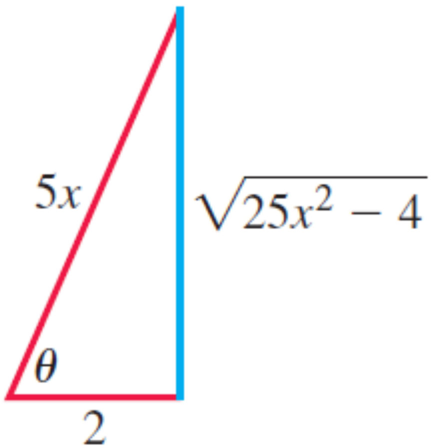


FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 4).

$x \geq 2$

$$\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$$

同乘分子 → 见8.1

$$= \int \frac{x-2}{\sqrt{x^2-3x+2}}\,dx$$

$$= \int \frac{x-2}{\sqrt{(x-\frac{3}{2})^2-\frac{1}{4}}}\,dx$$

$u = x - \frac{3}{2} \geq \frac{1}{2}$

$$= \int \frac{u-\frac{1}{2}}{\sqrt{u^2-\frac{1}{4}}}\,du$$

$u = \frac{1}{2}v \quad du = \frac{1}{2}dv$

$$= \int \frac{\frac{1}{2}v-\frac{1}{2}}{\frac{1}{2}\sqrt{v^2-1}}\,\frac{1}{2}dv$$

$$= \frac{1}{2}\int \frac{v-1}{\sqrt{v^2-1}}\,dv$$

$$\int \sqrt{\frac{x}{1-x^3}}\,dx$$

整合以数

定义域 $0 \leq x < 1$

找 $u = x^{\frac{3}{2}}$

$x = u^{\frac{2}{3}}$

$du = \frac{3}{2}x^{\frac{1}{2}}dx$

看分子是否可以被表示

$$= \int \frac{\frac{2}{3}du}{\sqrt{1-u^2}}$$

$$= \frac{2}{3}\sin^{-1}u + C$$

$$= \frac{2}{3}\sin^{-1}x^{\frac{3}{2}} + C$$

8.5

Integration of Rational Functions by Partial Fractions

部分分式

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. Example 3 of this section illustrates such a case.
- *We must know the factors of $g(x)$.* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

* 分子最高次 < 分母最高次 (分子次数高 则用大除法)
分解因式

$$\frac{x^7+5x^3+1}{x^2+x-1} =$$

$$x^5-x^4+2x^3-3x^2+10x-13$$

$$x^2+x-1 \,\big)\, x^7+5x^3+1$$

$$x^7+x^6-x^5$$

$$-x^6+x^5$$

$$-x^6-x^5+x^4$$

$$2x^5-x^4$$

$$+2x^4-2x^3$$

$$-3x^4+2x^3$$

$$-3x^3+3x^2$$

$$10x^3-3x^2$$

$$10x^3+10x^2-10x$$

$$-13x^2+10x+1$$

$$+(23x-12)$$

易错：除到后面会忘了原式的低级项

Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x - r)} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

2. 复数域上多项式的因式分解定理

可分解且彻底

复数域上多项式因式分解定理 每个次数 ≥ 1的复系数多项式在复数域上都可以唯一地分解成一次因式的乘积,其标准分解式为:

$$f(x)=a_n(x-\alpha_1)^{l_1}(x-\alpha_2)^{l_2}\cdots(x-\alpha_s)^{l_s}$$

其中$\alpha_1,\alpha_2,\cdots,\alpha_s$是互异的复数,$a_n$为$f(x)$的首项系数,$l_1,l_2,\cdots,l_s$是正整数,且$l_1+l_2+\cdots+l_s=n=\partial(f(x))$.

$(x-b)(x-\bar{b}) \mid f(x)$

$x^2-(b+\bar{b})+b\bar{b} \mid f(x)$

实系数

定理 (复根成对定理) 如果α是实系数多项式$f(x)$的一个复根,那么α的共轭数$\bar{\alpha}$也是$f(x)$的根,并且α与$\bar{\alpha}$有同一重数. 换句话说, 实系数多项式的虚数根两两成对.

$(x-b)(x-\bar{b})$　　且共轭所占次数相同

$f(b)=0 \quad f(\bar{b})=0$

$a_0+a_1b+\cdots+a_nb^n=0$

$a_0+a_1\bar{b}+\cdots+a_n\bar{b}^n=0$

$x^4+1=(x^2+1)^2-2x^2$

$=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$

实数域上多项式因式分解定理 每个次数 ≥ 1 的实系数多项式在实数域上都可以唯一地分解成一次因式和二次不可约因式的乘积. 其标准分解式为:

线性无关的积 → 唯一

$$f(x)=a_n(x-c_1)^{l_1}\cdots(x-c_s)^{l_s}$$

$$(x^2+p_1x+q_1)^{k_1}\cdots(x^2+p_rx+q_r)^{k_r}$$

根是与x轴交点

$$g(x) = a_n (x-c_1)^{l_1} \cdots (x-c_s)^{l_s} (x^2+p_1x+q_1)^{k_1} \cdots (x^2+p_rx+q_r)^{k_r}$$

$$\frac{f(x)}{g(x)} = \frac{A_{11}}{x-c_1} + \frac{A_{12}}{(x-c_1)^2} + \cdots + \frac{A_{1l_1}}{(x-c_1)^{l_1}}$$

$$+ \frac{A_{21}}{x-c_2} + \cdots + \frac{A_{2l_2}}{(x-c_2)^{l_2}}$$

$$\vdots$$

$$+ \frac{A_{s1}}{x-c_s} + \cdots + \frac{A_{sl_s}}{(x-c_s)^{l_s}}$$

$$+ \frac{B_{11}x+C_{11}}{x^2-p_1x+q_1} + \cdots + \frac{B_{1k_1}x+C_{1k_1}}{(x^2-p_1x+q_1)^{k_1}}$$

$$+ \cdots$$

$$+ \frac{B_{r1}x+C_{r1}}{x^2-p_rx+q_r} + \cdots + \frac{B_{rk_r}x+C_{rk_r}}{(x^2-p_rx+q_r)^{k_r}}$$

线性代数
证明

只要处理以下积分：

① $\int \frac{1}{x-c}dx$ ✓

② $\int \frac{1}{(x-c)^n}dx \quad n\geq 2$ ✓

③ $\int \frac{Ax+B}{x^2+px+q}dx$

④ $\int \frac{Ax+B}{(x^2+px+q)^r}dx \quad r\geq 2$

③ $\int \frac{Ax+B}{x^2+px+q}dx$

$$x^2+px+q = (x+\frac{p}{2})^2 + \underbrace{q-\frac{p^2}{4}}_{>0}$$

$u=x+\frac{p}{2}$

$$= \int \frac{Au+\tilde{B}}{u^2+d^2}$$

$$= \int \frac{\hat{A}v+\hat{B}}{v^2+1}$$ (前换元，后 $\tan^{-1}$)

④ $\int \frac{Ax+B}{(x^2+px+q)^r}dx$ 同③处理分母

$$x^2+px+q = (x+\frac{p}{2})^2 + \underbrace{q-\frac{p^2}{4}}_{=s^2} > 0$$

$u=x+\frac{p}{2}$

$$= \int \frac{Au+\hat{B}}{(u^2+s^2)^r}du$$

$u=sv \quad du=sdv$

$$= \int \frac{\bar{A}v+\bar{B}}{(v^2+1)^r}dv$$

$$= \bar{A}\int \frac{v}{(v^2+1)^r}dv + \bar{B}\int \frac{1}{(v^2+1)^r}dv$$

分子可被构造！ ✓

$w=v^2+1$

$dw=2vdv$

(1) $\tan^{-1} v = \theta$

$v = \tan\theta$

$$= \int \frac{\sec^2\theta}{\sec^{2k}\theta} d\theta$$

$$= \int \cos^{2(k-1)}\theta \, d\theta$$

高阶难处理

其中 $\int \frac{1}{(v^2+1)^{k+1}} dv = \frac{1}{2k} \frac{v}{(v^2+1)^k} + \frac{2k-1}{2k} \cdot \int \frac{1}{(v^2+1)^k} dv$

整合变成降阶公式

$$A_k = \int \frac{1}{(v^2+1)^k} dv$$

$$A_{k+1} = \frac{1}{2k} \frac{v}{(v^2+1)^k} + \frac{2k-1}{2k} A_k$$

(2) $\int \frac{1}{(v^2+1)^k} dv$

$$= \frac{v}{(v^2+1)^k} - \int v \, d\frac{1}{(v^2+1)^k}$$

$$= \frac{v}{(v^2+1)^k} + k\int \frac{2v^2}{(v^2+1)^{k+1}} dv$$

类似裂项

$$= \frac{v}{(v^2+1)^k} + 2k\left(\int \frac{1}{(v^2+1)^k} - \frac{1}{(v^2+1)^{k+1}}\right) dv$$

EXAMPLE 1 Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx.$$

分式 $= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$

$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$

让 $x=1$ $8A=6$

$x=-1$ $-4B=-2$

$x=-3$ $8C=-2$

EXAMPLE 3 Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

先做大除法

$(x-3)(x+1)$

$$\begin{array}{r} 2x \\ x^2-2x-3 \overline{)\, 2x^3-4x^2-x-3} \\ 2x^3-4x^2-6x \\ \hline 5x-3 \end{array}$$

$= \int 2x\,dx + \int \frac{5x-3}{(x-3)(x+1)}\,dx = \frac{3}{x-3} + \frac{2}{x+1}$

EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x + 4}{(x^2+1)(x-1)^2}\,dx.$$

$x=i$ $-2i+4=(Ci+D)(-2i)$

$C=2$ $D=1$

$A=-2$

$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$ 不可约则为线性式

$-2x+4 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

$x=1$ $B=1$ $A+C=0$ 3次系数为0

同样的想法：不过取复数 i

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+C)\,x(x^2+1) + (Ex+F)x$$

$x=0$ $A=1$ $B=-1$

$x=i$ $D=-1$

$x=1$ $C=0$ $F=0$

若 $F \neq 0$ 则要算

$$\int \frac{1}{x^2+1}\,dx = \frac{x}{x^2+1} + \int \frac{2x^2}{(x^2+1)^2}\,dx$$

$$= \frac{x}{x^2+1} + \int \frac{2(x^2+1)-2}{(x^2+1)^2}\,dx$$

$$2\int \frac{1}{(x^2+1)^2}\,dx = \frac{x}{x^2+1} + \int \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{1}{x^2+1}\,dx + C$$

Heaviside Method

1. *Write the quotient with g(x) factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

EXAMPLE 7 Use the Heaviside Method to evaluate

$$\int \frac{x + 4}{x^3 + 3x^2 - 10x}\,dx.$$

变成整式多项式

$$\int \frac{1}{(x^{1/3}-1)\sqrt{x}}\,dx$$

$u = x^{\frac{1}{6}}$

取公倍数

$$\int \frac{6u^5}{(u^2-1)u^3}\,du$$

$$=6\int (1+\frac{1}{u^2-1})\,du$$

$$-\frac{1}{2}(\frac{1}{1+u}+\frac{1}{1-u})$$

$$\int \frac{1}{x\sqrt{x+9}}\,dx$$

$u=\sqrt{x+9}$

$u^2-9=x$

$2u\,du=dx$

$$\int \frac{1}{\cancel{u}(u^2-9)}\,2\cancel{u}\,du$$

$$=2\int -\frac{1}{6}(\frac{1}{u+3}+\frac{1}{3-u})\,du$$

$$\int \frac{1}{x(x^4+1)}\,dx$$

扩次

$$\int \frac{x}{x^2(x^4+1)}\,dx$$

$u=x^2$

$$\int \frac{\tan^{-1} x}{x^2}\,dx$$

$u=\tan^{-1}x$

$\tan u = x$

$dx = \sec^2 u\,du$

$$\int \frac{u}{\tan^2 x}\sec^2 u\,du$$

$$=\int u\csc^2 u\,du$$

$$=-\int u\,d\cot u$$ 综合

$$=-u\cot u+\int \cot u\,du$$

(4) 另解:

$$-\int \tan^{-1}x \, d\left(\frac{1}{x}\right) = -\frac{1}{x}\tan^{-1}x + \int \frac{1}{x} d(\tan^{-1}x)$$

$$= -\frac{1}{x}\tan^{-1}x + \int \frac{1}{x(1+x^2)} dx$$

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$$

* $$\int \frac{1}{x(x^k+c)} dx = \int \frac{x^{k-1}}{x^k(x^k+c)} dx$$ 构造

$$\int \frac{1}{x(\sqrt{x}+1)} dx$$ 也适用

The Substitution $z = \tan(x/2)$

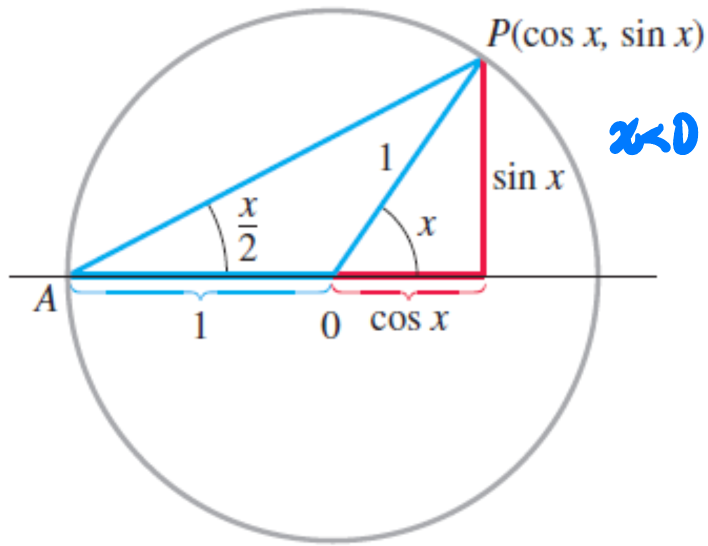
The substitution

$$z = \tan\frac{x}{2} \qquad (1)$$

→ 万能

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



$x<0$ 无意义

we can read the relation

$$\boxed{\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.}$$

we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1$$

$$\cos x = \frac{1 - z^2}{1 + z^2},$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,\frac{\sin(x/2)}{\cos(x/2)}\cdot\cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2}\cdot\frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2z}{1 + z^2}.$$

$\tan x = \frac{2z}{1-z^2}$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1+z^2}.$$

期末考
万能公式考得少
(但考研多)

有理函数多项式
(但公式可能复杂)

对 $\int \frac{f(\sin\theta, \cos\theta)}{g(\sin\theta, \cos\theta)}\,d\theta$

需知道分母因式

(tan⁻¹ 内很复杂)

✓ 只能用万能公式

$z = \tan\frac{x}{2}$

$$\int \frac{1}{2+\sin x}\,dx \qquad \frac{2}{1+z^2}dz = dx$$

$$= \int \frac{1}{2+\frac{2z}{1+z^2}} \frac{2}{1+z^2}\,dz$$

$$= \int \frac{1}{z^2+z+1}\,dz$$

$$= \int \frac{1}{(z+\frac{1}{2})^2+\frac{3}{4}}\,dz$$

$$= \int \frac{1}{u^2+\frac{3}{4}}\,du$$

$$= \int \frac{1}{\frac{3}{4}(v^2+1)} \frac{\sqrt{3}}{2}\,dv$$

$$\int \frac{dt}{\sin t - \cos t}$$

$\sqrt{2}\sin(t-\frac{\pi}{4})$

$$\int \csc u\,du$$

$$\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta + \sin\theta}$$

$\theta = 2u$

$$\int \frac{\cos\theta}{\sin\theta(\cos\theta+1)}\,d\theta = \int \frac{2\cos^2 u - 1}{2\cos^2 u \, 2\sin u} 2\,du$$

有奇数次的好做

尝试增添一次

分成两项 $\int \frac{1}{\cos u \sin u}\,du = \int \frac{\cos u}{\cos^2 u \sin u}\,du = \int \frac{du}{u \ldots}$

$v = \sin u$

8.6

Integral Tables and Computer Algebra Systems

EXAMPLE 1 Find

$$\int x(2x + 5)^{-1}\, dx.$$

Solution We use Formula 24 at the back of the book (not 22, which requires $n \neq -1$):

$$\int x(ax + b)^{-1}\, dx = \frac{x}{a} - \frac{b}{a^2} \ln |ax + b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x + 5)^{-1}\, dx = \frac{x}{2} - \frac{5}{4} \ln |2x + 5| + C.$$

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x - 4}}.$$

Solution We use Formula 29b:

$$\int \frac{dx}{x\sqrt{ax - b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax - b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x - 4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x - 4}{4}} + C = \tan^{-1} \sqrt{\frac{x - 2}{2}} + C.$$

EXAMPLE 3 Find

$$\int x \sin^{-1} x \, dx.$$

Solution We begin by using Formula 106:

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} \, dx}{\sqrt{1 - a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2 \, dx}{\sqrt{1 - x^2}}.$$

Next we use Formula 49 to find the integral on the right:

$$\int \frac{x^2}{\sqrt{a^2 - x^2}} dx = \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) - \frac{1}{2} x \sqrt{a^2 - x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2 \, dx}{\sqrt{1 - x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C.$$

The combined result is

$$\begin{aligned} \int x \sin^{-1} x \, dx &= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \left(\frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1 - x^2} + C \right) \\ &= \left(\frac{x^2}{2} - \frac{1}{4} \right) \sin^{-1} x + \frac{1}{4} x \sqrt{1 - x^2} + C'. \end{aligned}$$

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying reduction formulas like

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \tag{1}$$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \tag{2}$$

$$\int \sin^n x \cos^m x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\,dx \qquad (n \neq -m). \tag{3}$$

By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly. The next example illustrates this procedure.

EXAMPLE 4 Find

$$\int \tan^5 x \, dx.$$

Solution We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx = \frac{1}{2} \tan^2 x + \ln |\cos x| + C.$$

The combined result is

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C'.$$ ■

As their form suggests, reduction formulas are derived using integration by parts. (See Example 5 in Section 8.2.)

EXAMPLE 5 Suppose that you want to evaluate the indefinite integral of the function

$$f(x) = x^2\sqrt{a^2 + x^2}.$$

Using Maple, you first define or name the function:

$$> f{:=}\ x\text{^}2 * \text{sqrt}\,(a\text{^}2 + x\text{^}2);$$

Then you use the integrate command on f, identifying the variable of integration:

$$> \text{int}(f, x);$$

Maple returns the answer

$$\frac{1}{4}x(a^2 + x^2)^{3/2} - \frac{1}{8}a^2x\sqrt{a^2 + x^2} - \frac{1}{8}a^4 \ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want to see if the answer can be simplified, enter

$$> \text{simplify}(\%);$$

Maple returns

$$\frac{1}{8}a^2x\sqrt{a^2 + x^2} + \frac{1}{4}x^3\sqrt{a^2 + x^2} - \frac{1}{8}a^4 \ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want the definite integral for $0 \le x \le \pi/2$, you can use the format

$$> \text{int}(f, x = 0..\text{Pi}/2);$$

Maple will return the expression

$$\frac{1}{64}\pi(4a^2 + \pi^2)^{(3/2)} - \frac{1}{32}a^2\pi\sqrt{4a^2 + \pi^2} + \frac{1}{8}a^4 \ln(2)$$
$$- \frac{1}{8}a^4 \ln\left(\pi + \sqrt{4a^2 + \pi^2}\right) + \frac{1}{16}a^4 \ln(a^2).$$

You can also find the definite integral for a particular value of the constant a:

$$> a{:=}\ 1;$$
$$> \text{int}(f, x = 0..1);$$

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} + \frac{1}{8}\ln\left(\sqrt{2} - 1\right).$$

8、6 不考

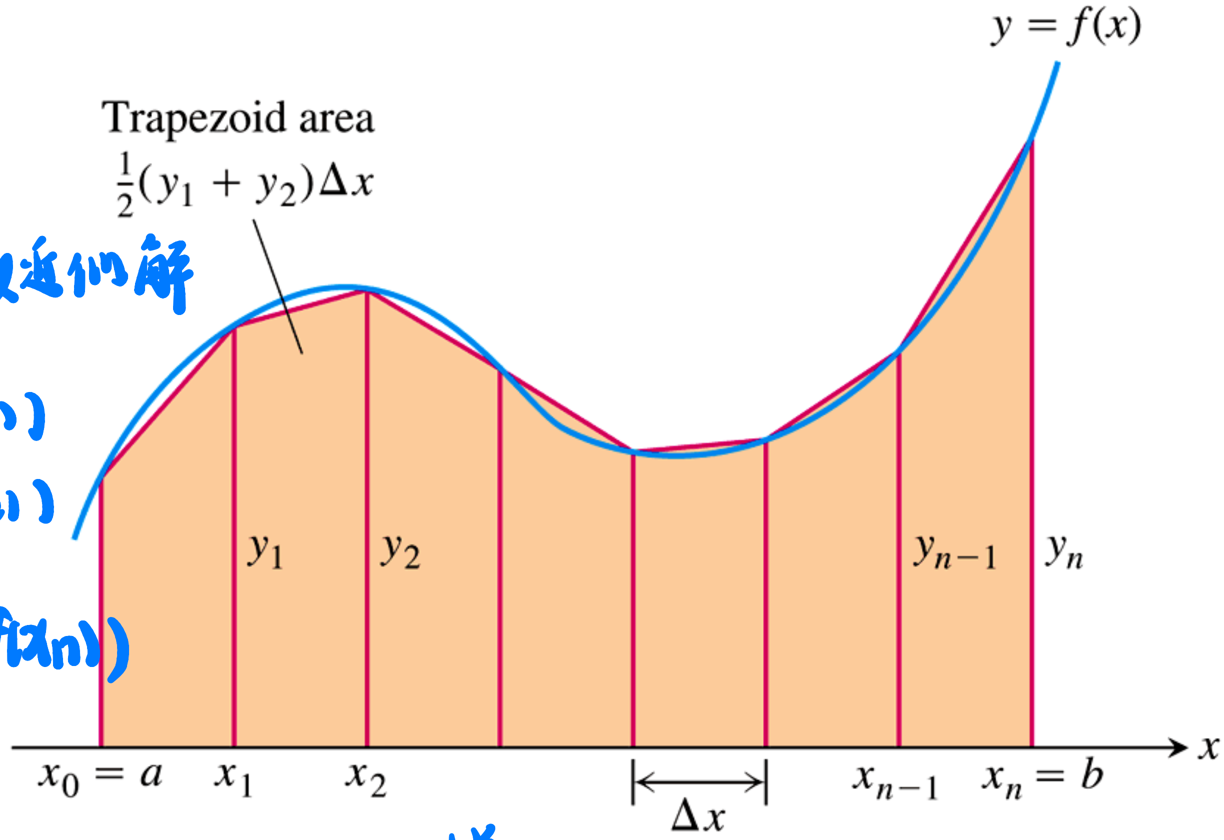
8.7

数字积分

Numerical Integration

做近似解

$\frac{\Delta x}{2}(f(x_0)+f(x_1))$
$+\frac{\Delta x}{2}(f(x_1)+f(x_2))$
$\cdots$
$+\frac{\Delta x}{2}(f(x_{n-1})+f(x_n))$



梯形

FIGURE 8.7 The Trapezoidal Rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the areas of the trapezoids made by joining the ends of the segments to the x-axis.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$
where $\Delta x = (b - a)/n$.

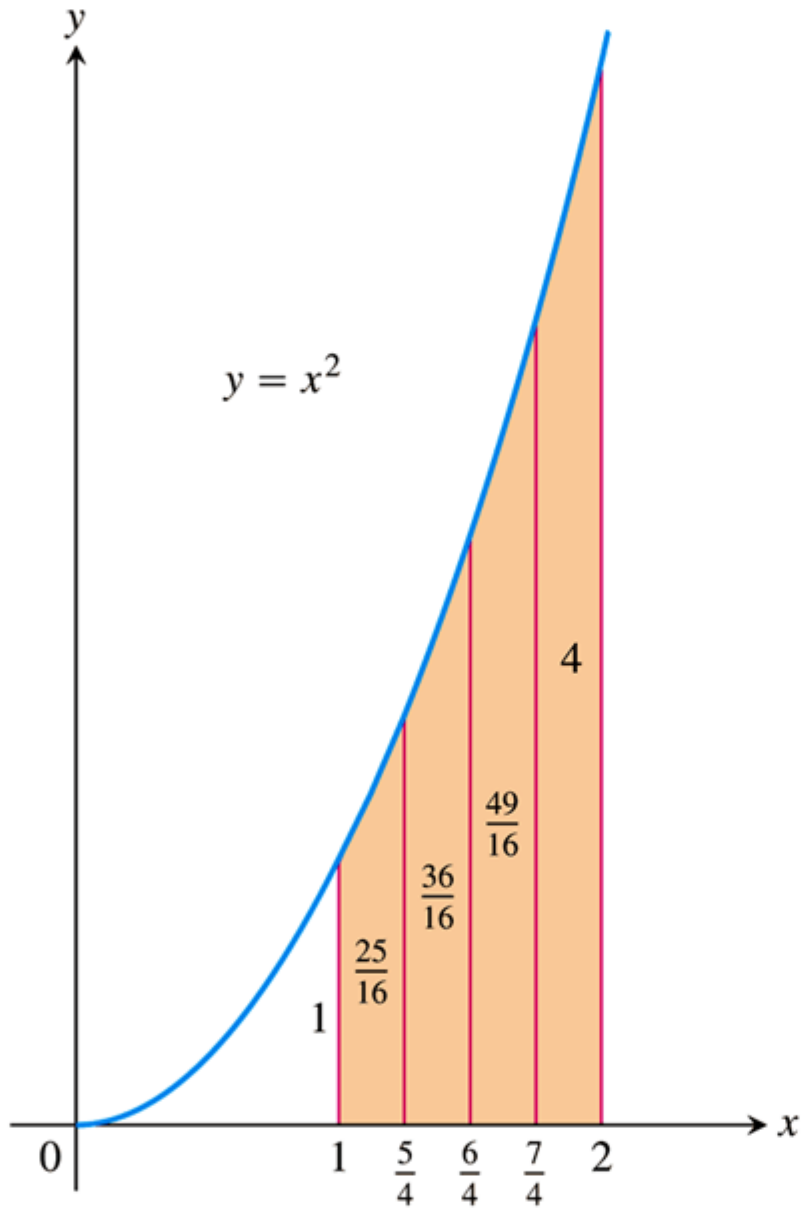


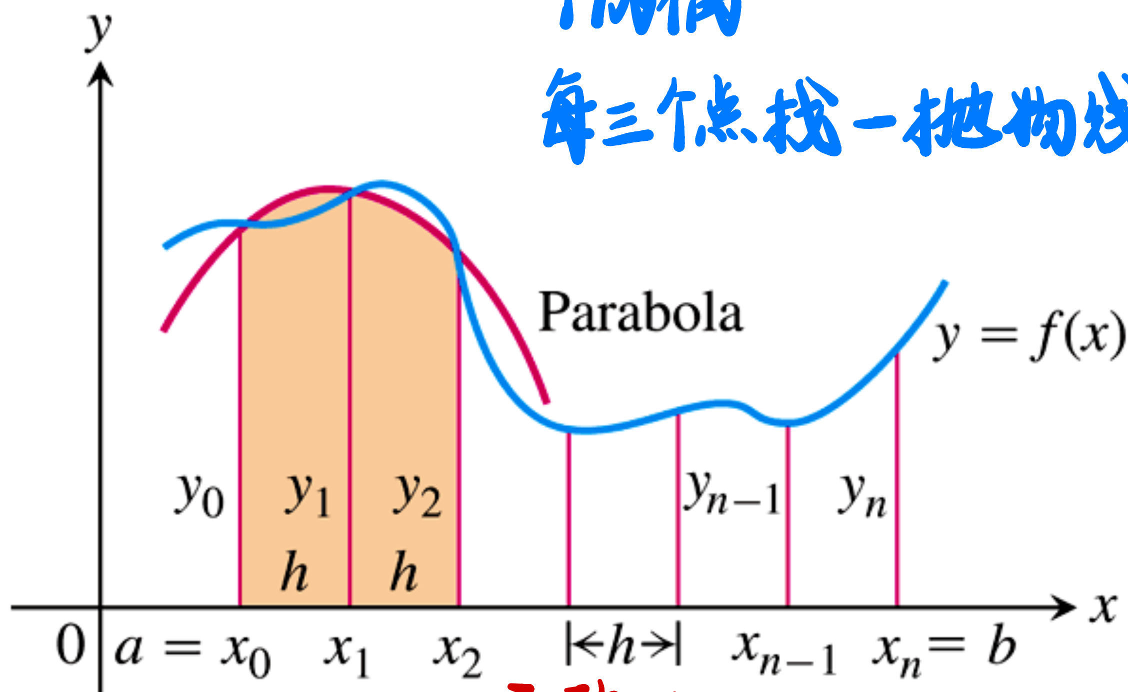
FIGURE 8.8 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

TABLE 8.2

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

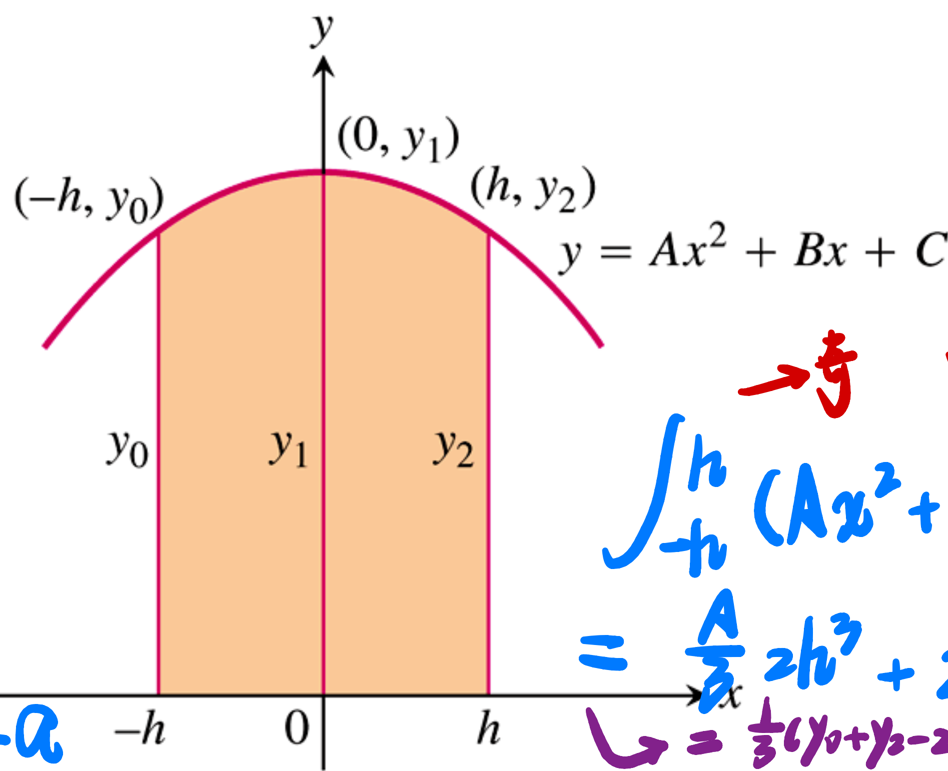
n为偶

每三个点找一抛物线



平移使中点在x=0处

FIGURE 8.9 Simpson's Rule approximates short stretches of the curve with parabolas.



$h = \frac{b-a}{n}$

→奇 中间项元

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx$$

$$= \frac{A}{3} 2h^3 + 2Ch$$

$$= \frac{1}{3}(y_0 + y_2 - 2y_1)h + 2y_1 h$$

$$= \frac{1}{3}(y_0 + y_2 + 4y_1)h$$

$Ah^2 - Bh + C = y_0$

$C = y_1$

$Ah^2 + Bh + C = y_2$

$2Ah^2 + 2C = y_0 + y_2$

$A = \frac{y_0 + y_2 - 2y_1}{2h^2}$

FIGURE 8.10 By integrating from $-h$ to h, we find the shaded area to be

不依赖x的取值 $$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Simpson's Rule 比梯形公式更准些

To approximate $\int_a^b f(x)\,dx$, use

n为偶数

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ldots,\ x_{n-1} = a + (n-1)\Delta x,\ x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

TABLE 8.3

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

EXAMPLE 2 Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution Partition [0, 2] into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.3). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

2个抛物线近似

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right)$$

$$= 32\frac{1}{12}.$$

This estimate differs from the exact value (32) by only 1/12, a percentage error of less than three-tenths of one percent, and this was with just four subintervals.

THEOREM 1—Error Estimates in the Trapezoidal and Simpson's Rules If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

全局最大

梯形公式为线性近似

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}. \qquad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

辛普森用二次函数近似

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}. \qquad \text{Simpson's Rule}$$

1) 只要记住 n^2, n^4

2) 会用两个公式

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(f(a)+f(b))(b-a)$$

$f(x)$ 为常数/线性常数 完全相等

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

常数、一次、两次、三次都相等

$$\int_a^b f(x)\,dx \approx f(a)(b-a)$$

仅常数相等 误差 $\frac{1}{n}$

$$\approx f\left(\frac{a+b}{2}\right)(b-a)$$

常数、一次相等 误差 $\frac{1}{n^2}$

EXAMPLE 5 As we saw in Chapter 7, the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\, dx.$$

Table 8.4 shows T and S values for approximations of $\int_1^2 (1/x)\, dx$ using various values of n. Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule.

TABLE 8.4 Trapezoidal Rule approximations (T_n) and Simpson's Rule approximations (S_n) of $\ln 2 = \int_1^2 (1/x)\,dx$

n	T_n	|Error| less than . . .		S_n	|Error| less than . . .	
10	0.6937714032	0.0006242227	$\frac{1}{4}$	0.6931502307	0.0000030502	$\frac{1}{16}$
20	0.6933033818	0.0001562013		0.6931473747	0.0000001942	
30	0.6932166154	0.0000694349		0.6931472190	0.0000000385	
40	0.6931862400	0.0000390595		0.6931471927	0.0000000122	
50	0.6931721793	0.0000249988		0.6931471856	0.0000000050	
100	0.6931534305	0.0000062500		0.6931471809	0.0000000004	

EXAMPLE 6 A town wants to drain and fill a polluted swamp (Figure 8.11). The swamp averages 1.5 m deep. About how many cubic meters of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 1.5. To estimate the area, we use Simpson's Rule with $\Delta x = 6$ m and the y's equal to the distances measured across the swamp, as shown in Figure 8.11.

$$\begin{aligned} S &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{6}{3}(44 + 148 + 46 + 64 + 24 + 36 + 4) = 732 \end{aligned}$$

The volume is about $(732)(1.5) = 1098\ \text{m}^3$. ■

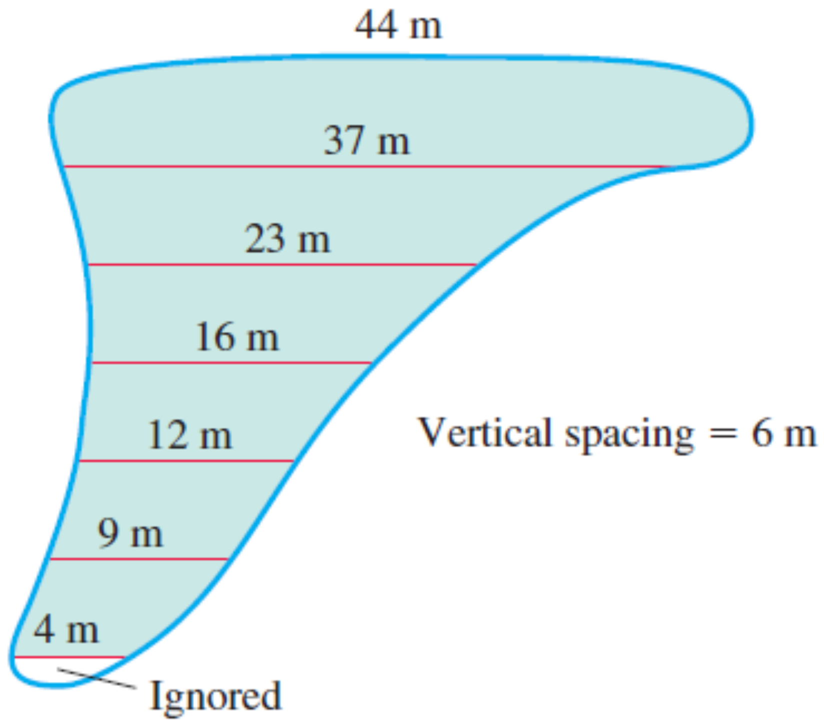


FIGURE 8.11 The dimensions of the swamp in Example 6.

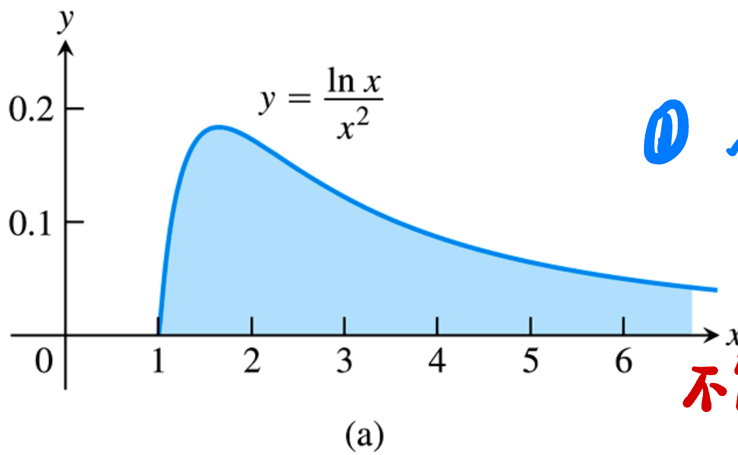
8.8

反常积分

Improper Integrals

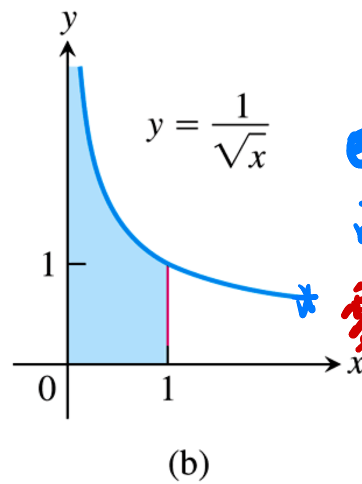
广义积分

① 积分区域(长度)无穷



不能分成n份
(黎曼和)

② 小区间上
函数值无界



* 黎曼和算会是无穷
(不能用)

FIGURE 8.12 Are the areas under these infinite curves finite? We will see that the answer is yes for both curves.

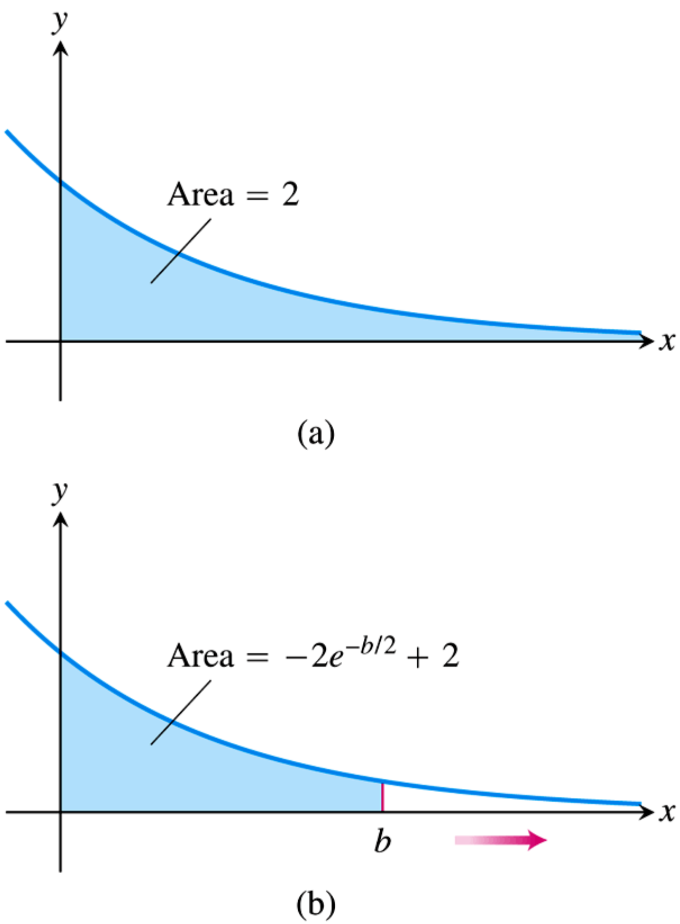


FIGURE 8.13 (a) The area in the first quadrant under the curve $y = e^{-x/2}$
(b) The area is an improper integral of the first type.

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then 存在，则积分收敛

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then 两个都收敛原积分才收敛

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number. 这种情况有必要拆 收敛

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**. 发散

$$\int_{-\infty}^{\infty} f(x)\,dx \cancel{=} \lim_{a\to+\infty}\int_{-a}^{a} f(x)\,dx$$

只要某一侧有问题
则一定要拆

$$\int_{-\infty}^{+\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{+\infty} x\,dx$$

2个都发散　　　定义上不同

$$= \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_{a}^{b} f(x)\,dx$$

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

$$\int_1^{+\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to+\infty}\int_1^b \frac{\ln x}{x^2}\,dx$$

$$\int \frac{\ln x}{x^2}\,dx = \int \ln x \, d\left(-\frac{1}{x}\right) = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$$

$$= -\frac{\ln x + 1}{x} + C$$

$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2} = \lim_{b\to\infty}\left(1 - \frac{\ln b + 1}{b}\right) = \lim_{b\to\infty}\left(1 - \frac{\frac{1}{b}}{1}\right)$$

$$= 1$$

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

$$= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{+\infty} \frac{dx}{1+x^2}$$

$$= \lim_{b\to+\infty}\int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to+\infty} \tan^{-1} x \Big]_0^b$$

$$= \lim_{b\to+\infty} \tan^{-1} b = \frac{\pi}{2}$$

偶函数 SD $= \pi$

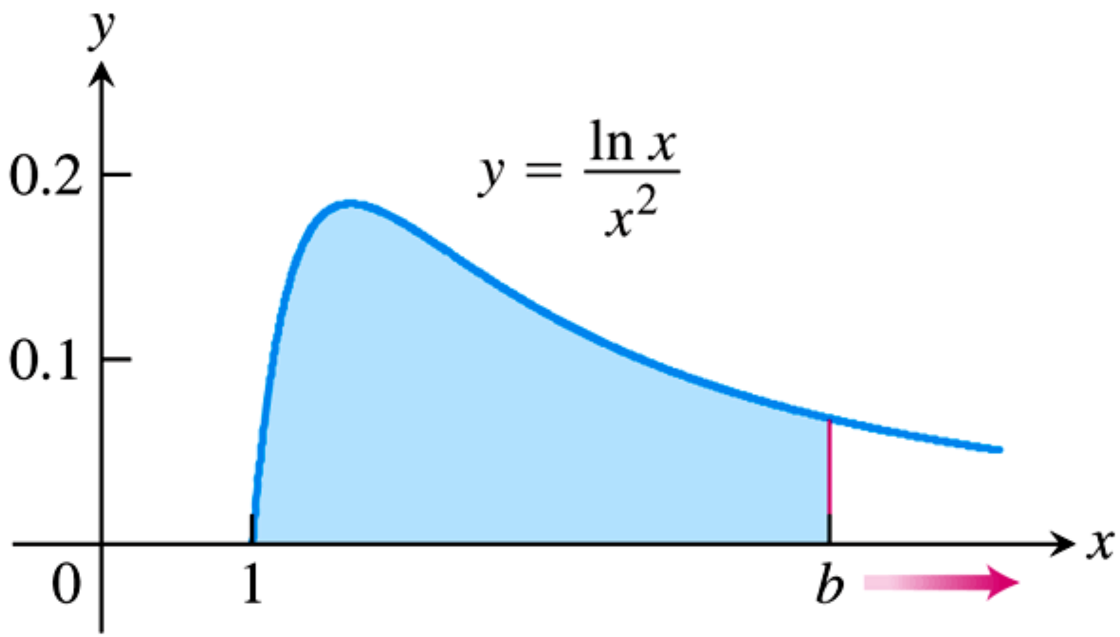


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

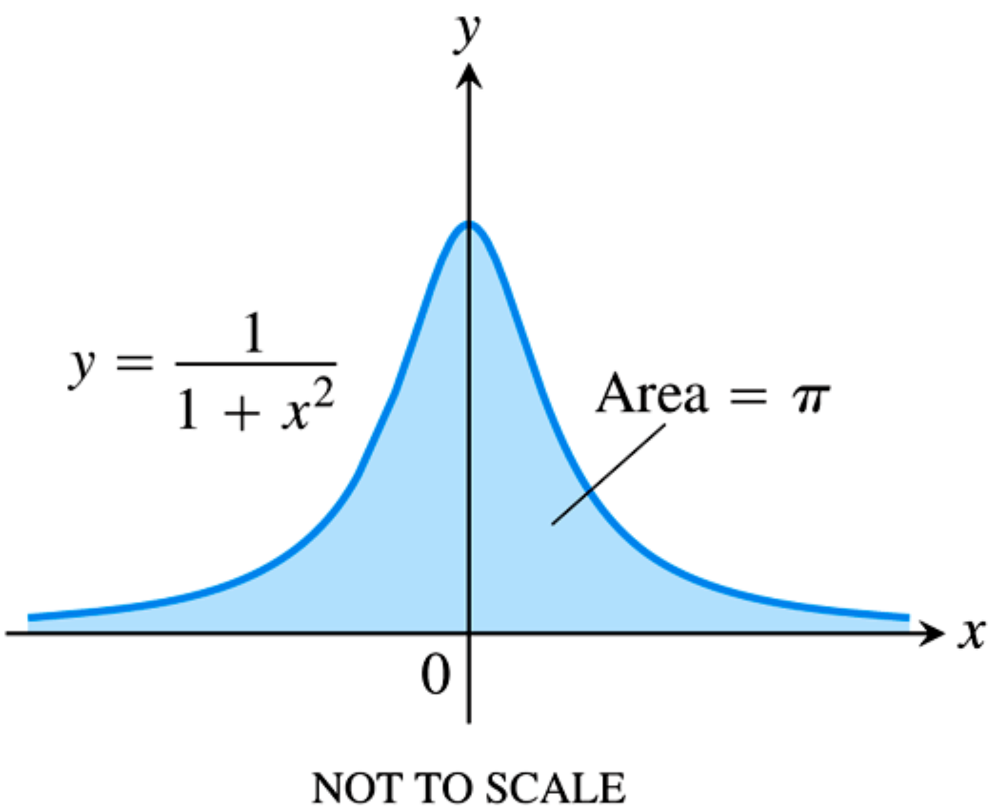


FIGURE 8.15 The area under this curve is finite (Example 2).

EXAMPLE 3 For what values of p does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

$\int_1^\infty \frac{1}{x^p}dx \quad p>0$

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}(b^{-p+1}-1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right).$$

Thus, 分类讨论

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right] = \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$

because

$$\lim_{b\to\infty}\frac{1}{b^{p-1}} = \begin{cases} 0, & p>1 \\ \infty, & p<1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p>1$ and it diverges if $p<1$.

If $p=1$, the integral also diverges:

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x}$$

$$= \lim_{b\to\infty}\int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \ln x\bigg]_1^b$$

$$= \lim_{b\to\infty}(\ln b - \ln 1) = \infty.$$

$$\int_0^1 \frac{1}{x^p}dx \quad p>0$$

>1 / <1 一个发散

$=1$ 必发散

$$a \to 0^+ \quad \lim_{a\to 0^+}\int_a^1 \frac{1}{x^p}dx$$

☆ 常用例子

$$= \lim_{a\to 0^+} \frac{x^{-p+1}}{-p+1}\Big]_a^1$$

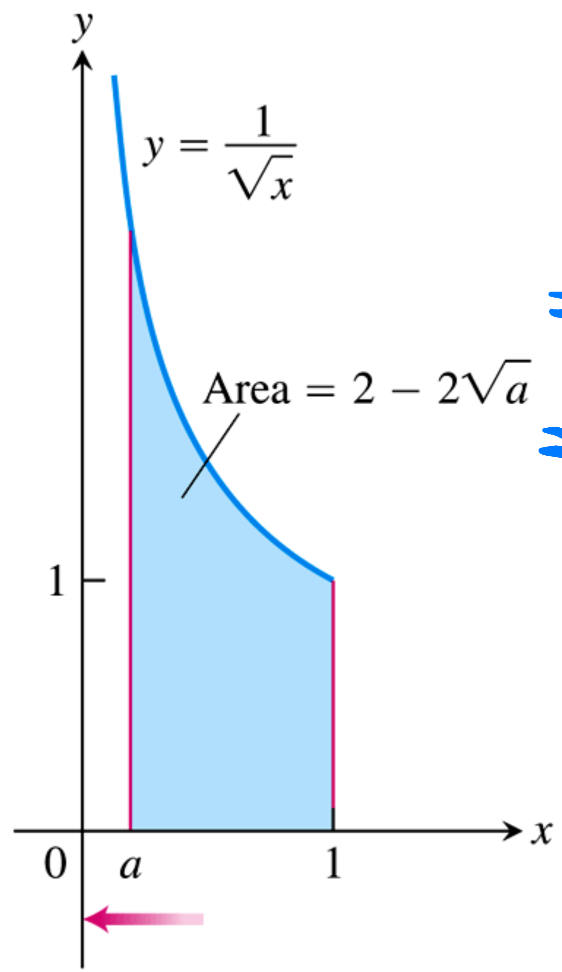
$$= \lim_{a\to 0^+} \frac{1}{1-p} + \frac{1}{(p-1)a^{p-1}}$$

$p>1$ 发散

$p=1 \quad = \lim_{a\to 0^+}(\ln 1 - \ln a) = +\infty$

发散

$p<1 \quad = \frac{1}{1-p}$



$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+} 2\sqrt{x}\Big]_a^1$$

$$= \lim_{a\to 0^+} 2(1-\sqrt{a})$$

FIGURE 8.16 The area under this curve is an example of an improper integral of the second kind.

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then 中间有间断点一定要分开

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

仍2个收敛才收敛

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

EXAMPLE 4 Investigate the convergence of

$u=1-x$

$$\int_0^1 \frac{1}{1-x}\,dx.$$

$= \int_0^1 \frac{1}{u}\,du$

发散(同前)

*Evaluate
可能会是反常积分发散
提高警惕

EXAMPLE 5 Evaluate

含题来与 $3(x-1)^{\frac{1}{3}}\Big]_0^3$ 等 $x=1$ 无定义 但不符合定义!

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

$= \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$

$= \lim_{b\to1^-} 3(x-1)^{\frac{1}{3}}\Big]_0^b + \lim_{a\to1^+} 3(x-1)^{\frac{1}{3}}\Big]_a^3$

*有无定义处一定要拆!

求不出来

EXAMPLE 6 Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

$$\int_0^2 \frac{1}{(x-1)(x-3)}\,dx$$

$x=1$ 无定义

$$= \int_0^1 \frac{1}{(x-1)(x-3)}\,dx + \int_1^2 \frac{1}{(x-1)(x-3)}\,dx$$

$$\lim_{b\to 1^-} \frac{1}{2}\Big(\ln|x-1| - \ln|x-3|\Big)\Big]_0^b$$

$\ln 0$ 发散

先积再用极限代入求

(一例)

一个积分发散则整个发散

但仅一个得化简先判断

- 不定积分 —— 一定是技巧
- 定积
  - 奇偶性
  - 反常积分 * 要指瑕点

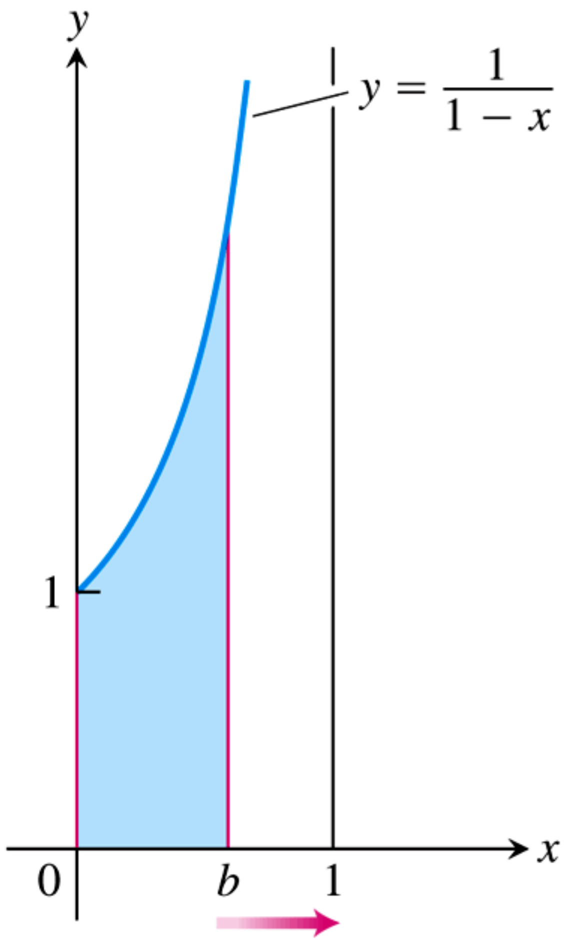


FIGURE 8.17 The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

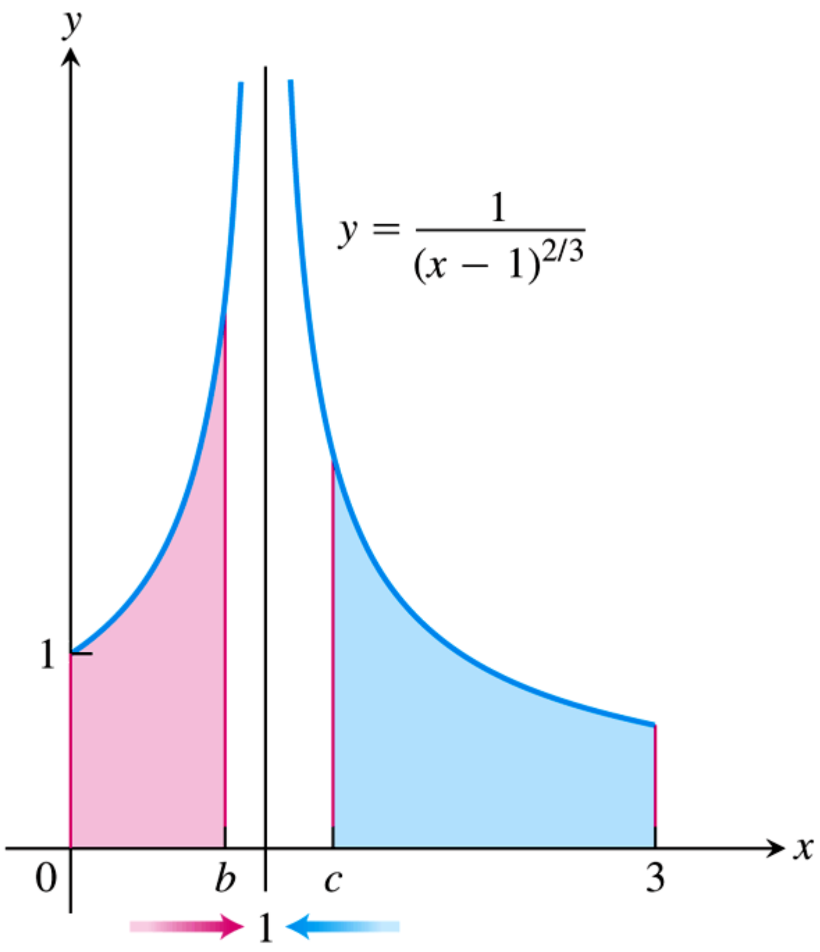
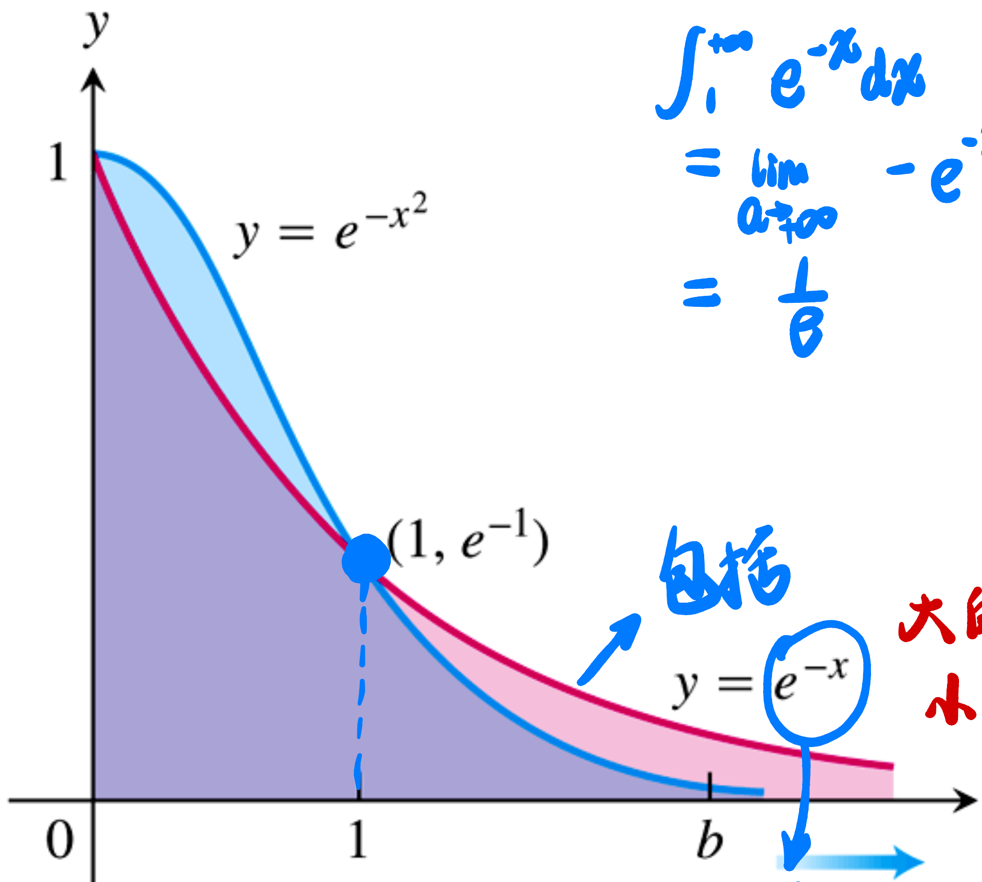


FIGURE 8.18 Example 5 shows that the area under the curve exists (so it is a real number).



$$\int_1^{+\infty} e^{-x}\,dx = \lim_{a\to+\infty} -e^{-x}\Big]_1^a = \frac{1}{e}$$

包括

大的收敛
小必收敛

(单调有界定理)

找相对好积的函数

FIGURE 8.19 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 6).

比较判别法直接形式 ①连续

THEOREM 2—Direct Comparison Test Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

②恒不等关系

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges. 大收小必收

2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges. 小散大必散

* 第一类、第二类都可用

(只是写了一部分)

EXAMPLE 7

$\frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$

(a) $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$

$\int_1^{\infty} \frac{1}{x^2}\,dx$ 收敛

(p积分)

$\frac{1}{\sqrt{x^2-0.1}} > \frac{1}{x}$

(b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}}\,dx$

$\int_1^{\infty} \frac{1}{x}\,dx$ 发散

猜收敛

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$

找 $\frac{1}{\sqrt{x}}$ 大收→小收

$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{x}} = \lim_{a\to 0^+} 2\sqrt{x}\Big]_a^{\frac{\pi}{2}}$

比较判别法极限形式

① $f, g > 0$

THEOREM 3—Limit Comparison Test If the positive functions f and g are continuous on $[a, \infty)$, and if

② 连续

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad \boxed{0 < L < \infty},$$

③

一侧

then

$$\int_a^{\infty} f(x)\,dx \quad \text{and} \quad \int_a^{\infty} g(x)\,dx$$

both converge or both diverge.

必充分大

$$\frac{1}{2}L < \frac{f(x)}{g(x)} < 2L$$

用直接比较

$$\frac{1}{2}L\, g(x) < f(x) < 2L\, g(x)$$

EXAMPLE 8 Show that

$$\int_1^{\infty} \frac{dx}{1 + x^2}$$

converges by comparison with $\int_1^{\infty} (1/x^2)\,dx$. Find and compare the two integral values.

EXAMPLE 9 Investigate the convergence of $\displaystyle\int_1^{\infty} \frac{1 - e^{-x}}{x}\,dx.$

$$\lim_{x \to +\infty} \frac{\frac{1-e^{-x}}{x}}{\frac{1}{x}} = 1$$

发散

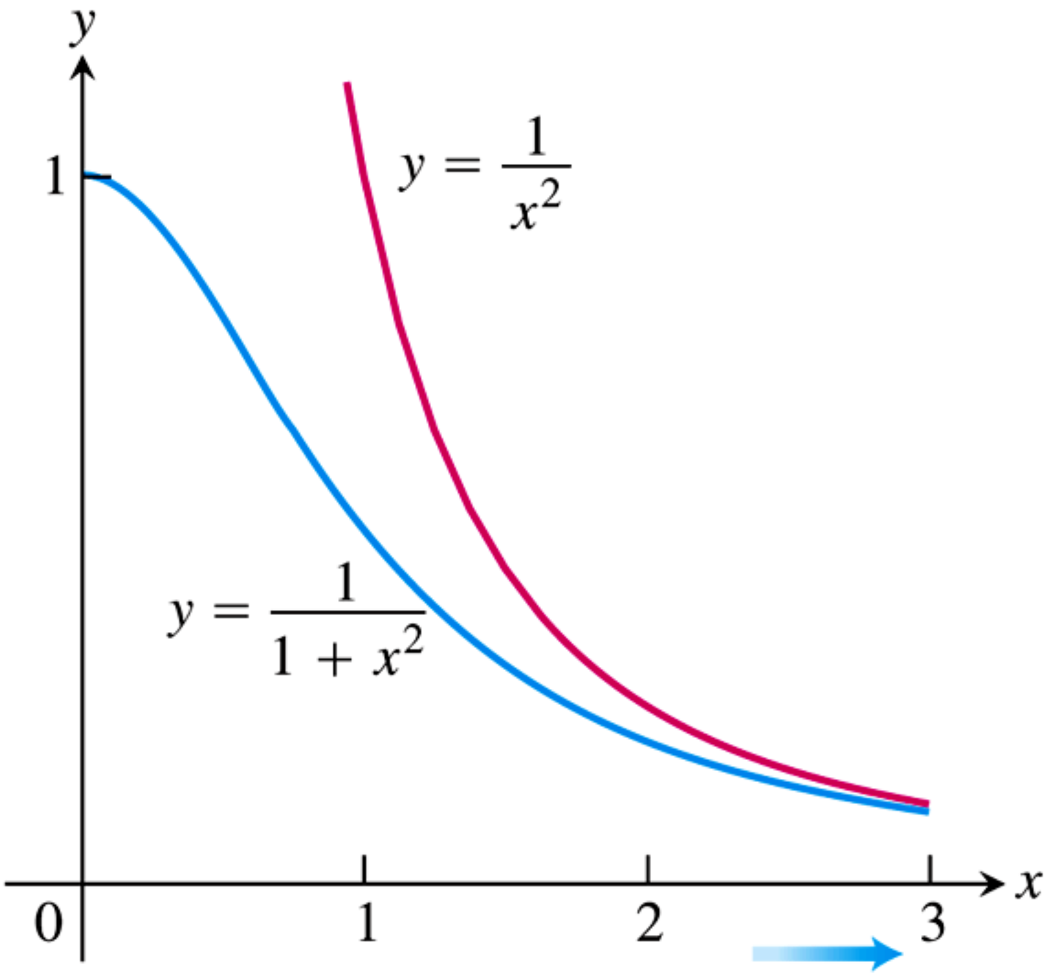


FIGURE 8.20 The functions in Example 8.

TABLE 8.5

b	$\displaystyle\int_1^b \frac{1-e^{-x}}{x}\,dx$
2	0.5226637569
5	1.3912002736
10	2.0832053156
100	4.3857862516
1000	6.6883713446
10000	8.9909564376
100000	11.2935415306

$$\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t} \qquad \lim_{t\to 0^+} \frac{\frac{1}{\sqrt{t}+\sin t}}{\frac{1}{\sqrt{t}}}$$

收敛

$$\int_0^{\pi} \frac{1}{\sqrt{t}} = 2\sqrt{t}\Big]_0^{\pi} \qquad = \lim_{t\to 0^+} \frac{\sqrt{t}}{\sqrt{t}+\sin t} = \lim_{t\to 0^+} \frac{\frac{1}{2\sqrt{t}}}{\frac{1}{2\sqrt{t}}+\cos t} = \lim_{t\to 0^+} \frac{1}{1+2\sqrt{t}\cos t} = 1$$

$$\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$$

$$\lim_{x\to 1^-} \frac{\frac{1}{1-x^2}}{\frac{1}{1-x}} = \lim_{x\to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

$t = 1-x$

$$\int_0^1 \frac{1}{1-x}dx = \int_0^1 \frac{1}{t}\,dt = \ln|t| + C$$

$$\int_{-1}^{1} \ln|x|\,dx = 2\int_0^1 \ln x\,dx = \lim_{a\to 0^+} 2\int_a^1 \ln x\,dx = \lim_{a\to 0^+} 2\times(x\ln x - x)\Big]_a^1$$

$a\to 0^+$

$\ln x$ 的乘式

不一定导致发散

$$\lim_{a\to 0^+} a\ln a = \frac{\ln a}{\frac{1}{a}} = \frac{\frac{1}{a}}{-\frac{1}{a^2}} = -a = 0$$

收敛

$$\int_0^1 \frac{dt}{t-\sin t} \qquad \lim_{t\to 0^+} \frac{\frac{1}{t-\sin t}}{\frac{1}{t^3}} = \lim_{t\to 0^+} \frac{t^3}{t-\sin t} = \lim_{t\to 0^+} \frac{3t^2}{1-\cos t} = 6$$

$\int_0^1 \frac{1}{t^3}dt$ 发散

(i) Use the substitution $x = \dfrac{1}{1-u}$, where $0 < u < 1$, to find in terms of x the integral

$$\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}\,\mathrm{d}x \quad (\text{where } x > 1).$$

(ii) Find in terms of x the integral

$$\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\,\mathrm{d}x \quad (\text{where } x > 2).$$

(iii) Show that

$$\int_2^{\infty} \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}\,\mathrm{d}x = \frac{1}{3}\pi.$$

$dx = -\frac{1}{t^2}\,dt$

倒数换元 $\frac{1}{x-1} = t$

$x = \frac{1+t}{t}$

拿出

来系数

$x-2 = \frac{1-t}{t}$

$3x-2 = \frac{3+t}{t}$

$$\int_1^0 \frac{1}{\frac{1}{t}\sqrt{\frac{1-t}{t}}\sqrt{\frac{3+t}{t}}}\left(-\frac{1}{t^2}\right)dt$$

$$= \int_0^1 \frac{1}{\sqrt{1-t}\sqrt{3+t}}\,dt$$

$$= \int_0^1 \frac{1}{\sqrt{-t^2-2t+3}}\,dt$$

$u = t+1$

$$\int_1^2 \frac{1}{\sqrt{-u^2+4}}\,du$$

$u = 2v$

$$\int_{\frac{1}{2}}^1 \frac{1}{\cancel{2}\sqrt{1-v^2}}\cdot \cancel{2}dv$$

$$\int_{\frac{1}{2}}^1 \sin^{-1} v$$

$\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

T2、$u = x-2$

$$= \int \frac{1}{u^{\frac{3}{2}}(u+3)^{\frac{1}{2}}} du$$

$u = 3v$

$$= \int \frac{1}{3\sqrt{3}\, v^{\frac{3}{2}} \sqrt{3}\, (v+1)^{\frac{1}{2}}} 3dv$$

$$= \int \frac{1}{3 v^{\frac{3}{2}}(v+1)^{\frac{1}{2}}} dv$$

$\theta = \tan^{-1}\sqrt{v}$

$\theta \in (0, \frac{\pi}{2})$

$v = \tan^2\theta$

v 与 $v+1 \to \tan^2\theta$

$v-1 \to \sec^2\theta$

$$= \int \frac{1}{3\tan^3\theta \sec\theta} 2\tan\theta \sec^2\theta \, d\theta$$

$dv = 2\tan\theta \sec^2\theta \, d\theta$

$$= \frac{1}{3} \int \frac{2\sec\theta}{\tan^2\theta} d\theta$$

三角先写出来看什么类型

$$\int \frac{\sec\theta}{\tan^2\theta} d\theta = \int \frac{\cos\theta}{\sin^2\theta} d\theta$$

$$= \int \frac{d(\sin\theta)}{\sin^2\theta}$$

8.9

Probability

DEFINITION A **random variable** is a function X that assigns a numerical value to each outcome in a sample space.

Value of X	0	1	2	3
Frequency	1	3	3	1
$P(X)$	1/8	3/8	3/8	1/8

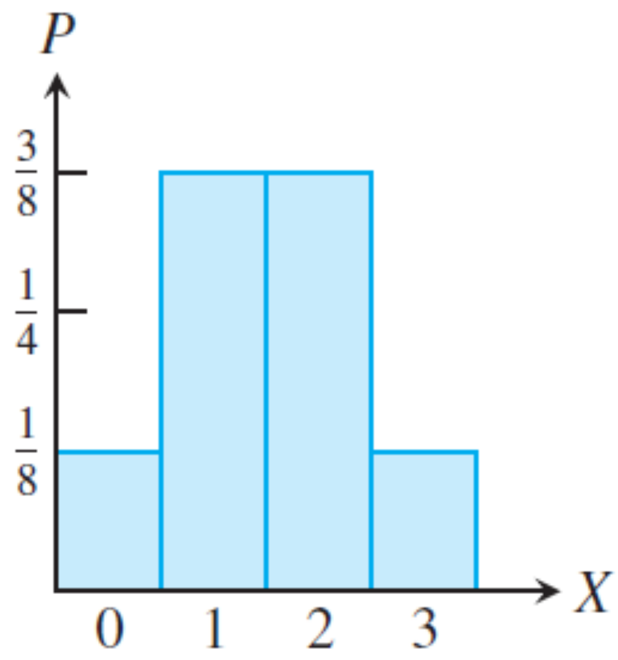


FIGURE 8.21 Probability bar graph for the random variable X when tossing a fair coin three times.

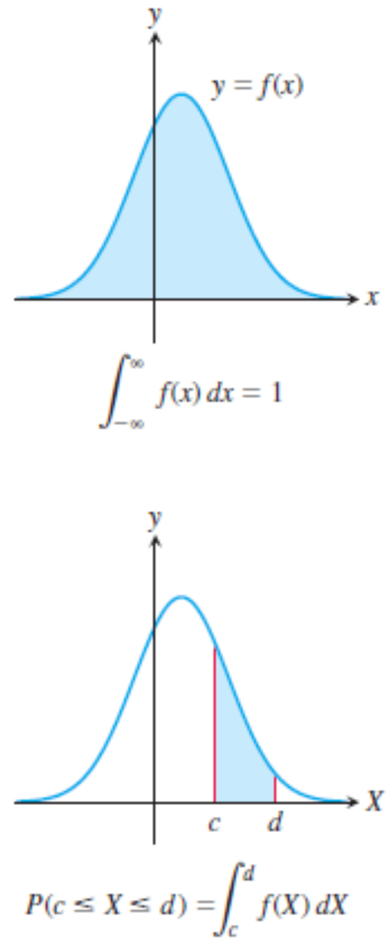


FIGURE 8.22 A probability density function for the continuous random variable *X*.

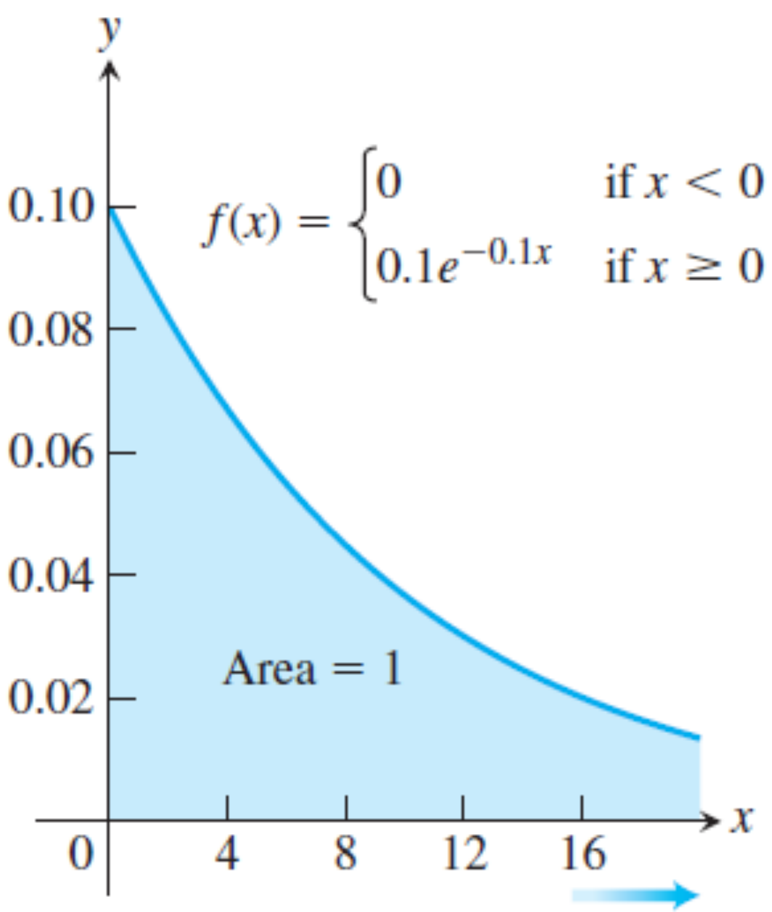


FIGURE 8.23 An exponentially decreasing probability density function.

DEFINITION The **expected value** or **mean** of a continuous random variable X with probability density function f is the number

$$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} X f(X)\, dx.$$

Exponential Density Function for a Random Variable X with Mean μ

$$f(X) = \begin{cases} 0 & \text{if } X < 0 \\ \mu^{-1} e^{-X/\mu} & \text{if } X \geq 0 \end{cases}$$

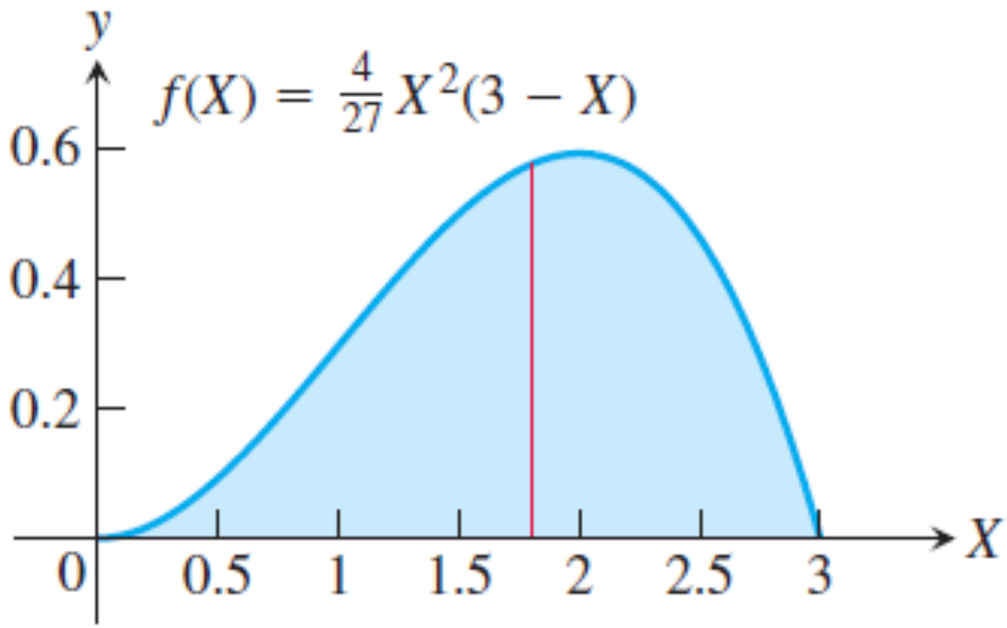


FIGURE 8.24 The expected value of a random variable with this probability density function is $\mu = 1.8$ (Example 8).

DEFINITION The **median** of a continuous random variable X with probability density function f is the number m for which

$$\int_{-\infty}^{m} f(X)\,dX = \frac{1}{2} \quad \text{and} \quad \int_{m}^{\infty} f(X)\,dX = \frac{1}{2}.$$

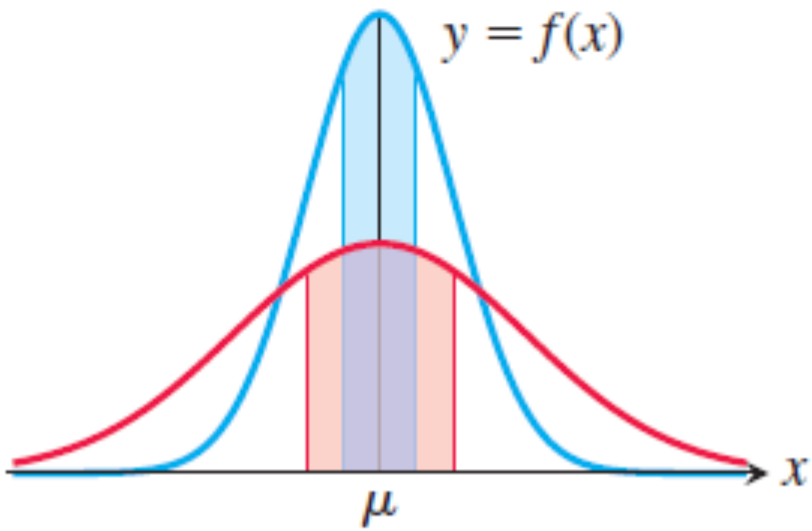


FIGURE 8.25 Probability density functions with the same mean can have different spreads in relation to the mean. The blue and red regions under the curves have equal area.

DEFINITIONS The **variance** of a random variable X with probability density function f is the expected value of $(X - \mu)^2$:

$$\operatorname{Var}(X) = \int_{-\infty}^{\infty} (X - \mu)^2 f(X)\, dX$$

The **standard deviation** of X is

$$\sigma_X = \sqrt{\operatorname{Var}(X)} = \sqrt{\int_{-\infty}^{\infty} (X - \mu)^2 f(X)\, dX}.$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

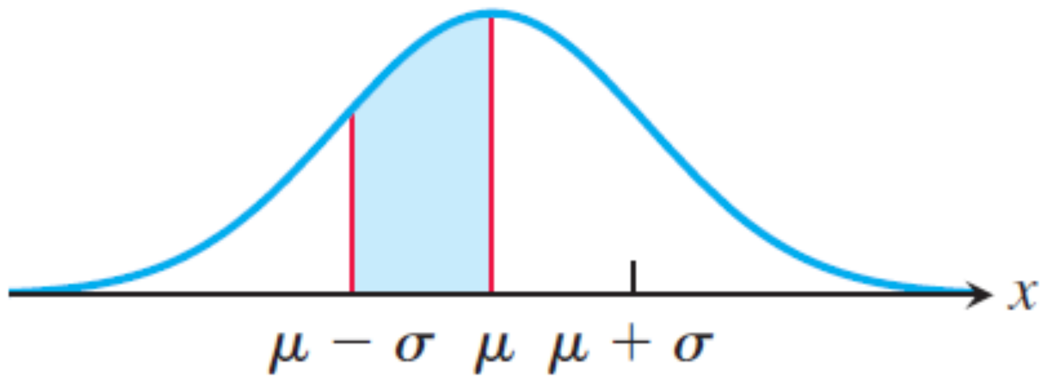


FIGURE 8.26 The normal probability density function with mean μ and standard deviation σ.

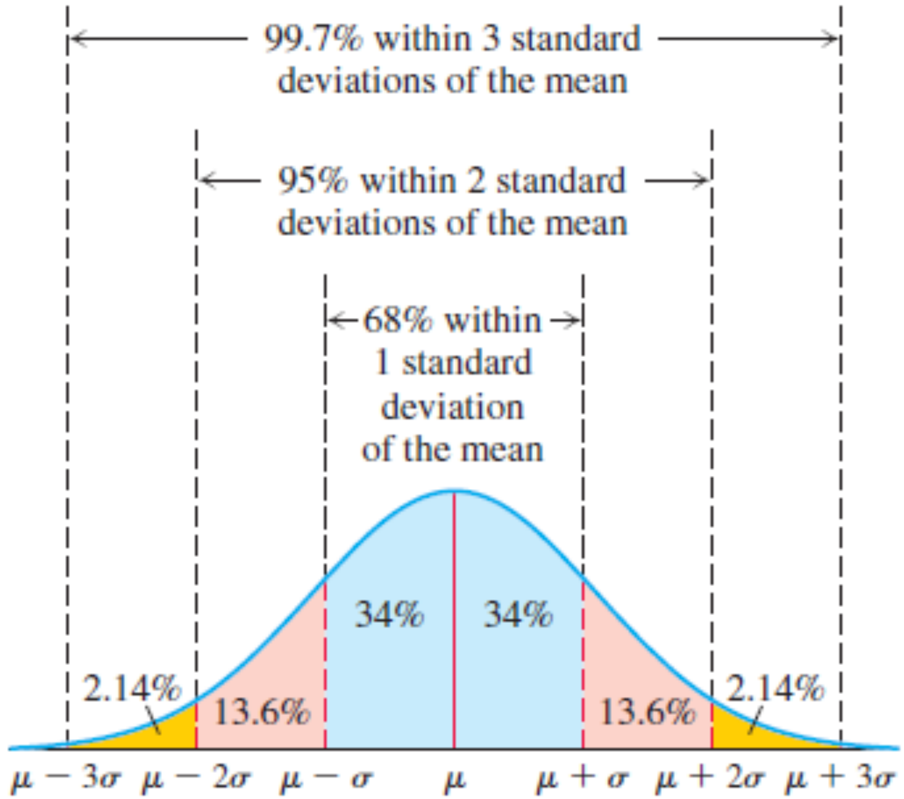


FIGURE 8.27 Probabilities of the normal distribution within its standard deviation bands.