Tutorial 05 for Calculus I

Sect. 4.4-4.7

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Review Sect. 4.4-4.7

Section 4.4: The definition of concave up, concave down, inflection point, The Second Derivative Test for Concavity and Local Extrema, Procedure for graphing $y = f(x)$.

Section 4.5: Applied Optimization.

Section 4.6: Newton's Method

- Section 4.7 Antiderivatives

Review of Sect. 4.4

1. The Second Derivative Test for Concavity: Let $y = f(x)$ be twice-differentiable on an interval I.

(1)If $f'' > 0$ on I, the graph of f over I is concave up;

(2)If $f'' < 0$ on I, the graph of f over I is concave down.

2. A point of inflection: A point where the graph of a function has a tangent line (include the vertical tangent line) and where the concavity changes is a point of inflection.

3. At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Review of Sect. 4.4

Theorem (Second Derivative Test for Local Extrema)

Suppose $f0$ is continuous on an open interval that contains $x = c$.

1.If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

2.If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

3.If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Review of Sect. 4.4

Procedure for Graphing $y = f(x)$:

1.Identify the domain of f and any symmetries the curve may have.

2.Find the derivatives y' and y''.

3.Find the critical points of f, if any, and identify the function’ s behavior at each one.

4.Find where the curve is increasing and where it is decreasing.

5.Find the points of inflection, if any occur, and determine the concavity of the curve.

6.Identify any asymptotes that may exist.

7.Plot key points, such as the intercepts and the points found in Steps 3-5, and sketch the curve together with any asymptotes that exist.

Review of Sect. 4.6

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.

2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0$$

Newton's Method

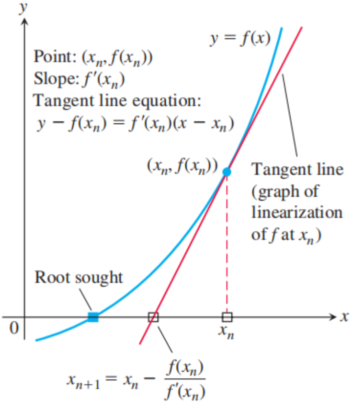


FIGURE 4.45 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

Review of Sect. 4.6

Newton's Method

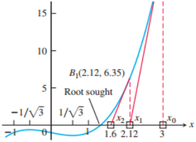


FIGURE 4.48 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root in Example 2.

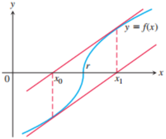


FIGURE 4.49 Newton's method fails to converge. You go from x_0 to x_1 and back to x_0, never getting any closer to r.

Convergence of the Approximations

In Chapter 10 we define precisely the idea of *convergence* for the approximations x_n in Newton's method. Intuitively, we mean that as the number n of approximations increases without bound, the values x_n get arbitrarily close to the desired root r. (This notion is similar to the idea of the limit of a function $g(t)$ as t approaches infinity, as defined in Section 2.6.)

In practice, Newton's method usually gives convergence with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for x_0. You can test that you are getting closer to a zero of the function by evaluating $|f(x_n)|$, and check that the approximations are converging by evaluating $|x_n - x_{n+1}|$.

Newton's method does not always converge. For instance, if

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \geq r, \end{cases}$$

the graph will be like the one in Figure 4.49. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

If Newton's method does converge, it converges to a root. Be careful, however. There are situations in which the method appears to converge but no root is there. Fortunately, such situations are rare.

When Newton's method converges to a root, it may not be the root you have in mind. Figure 4.50 shows two ways this can happen.

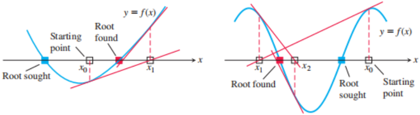


FIGURE 4.50 If you start too far away, Newton's method may miss the root you want.

专题一: 画图题.

例: Graphing equation $y = \dfrac{\sqrt{1-x^2}}{2x+1}$ $-\frac{1}{2}$ (书本32)

定义域

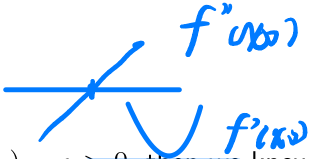
$y'=0$

$y''=0$

y''

y'

组合



专题二: 极值和拐点.

延伸: (1) If $f'(x_0) = f''(x_0) = 0, f'''(x_0) = a > 0$, then we know (   )

(A) f has local minimum at x_0.

(B) f has local maximum at x_0.

(C) there exists $\delta > 0$, such that f is concave up on the interval $(x_0 - \delta, x_0)$, and concave down on the interval $(x_0, x_0 + \delta)$.

(D) there exists $\delta > 0$, such that f is concave down on the interval $(x_0 - \delta, x_0)$, and concave up on the interval $(x_0, x_0 + \delta)$.

勤画图 不要猜

$f''(0)=0$ $f'''(0)=1$

专题二: 极值和拐点.

f''

延伸: (2) Let $f(x)$ ~~has continuous~~ three derivative,
$f''(x) + [f'(x)]^2 = x$, $f'(0) = 0$, then ( )

$f'''(x)+2f'(x)f''(x)=1$

(A) f has local maximum at $x = 0$.

(B) f has local minimum at $x = 0$.

(C) f has inflection point $(0, f(0))$. ✓

(D) f does not have local extreme at $x = 0$, and $(0, f(0))$ is not the inflection point.

专题二: 极值和拐点.

只设对应好

延伸: (3) $f(x)$ has continuous second derivative, and $f'(0) = 0$, $\lim\limits_{x \to 0} \frac{f''(x)}{|x|} = 1$, then (B)

x^4

$4x^3$

$12x^2$

(A) f has local maximum at $x = 0$.

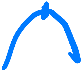
(B) f has local minimum at $x = 0$.

(C) f has inflection point $(0, f(0))$.

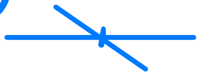
(D) f does not have local extreme at $x = 0$, and $(0, f(0))$ is not the inflection point.

Homework of Section 4.4

f'



$f'(a)=0$



专题二: 极值和拐点.

延伸: (4) Let $f'(x)$ is continuous at $x=a$, and $\lim\limits_{x\to a}\frac{f'(x)}{x-a}=-1$, then (B)

(A) f has local minimum at $x=a$.

(B) f has local maximum at $x=a$.

(C) f has inflection point $(a, f(a))$.

(D) f does not have local extreme at $x=a$, and $(a, f(a))$ is not the inflection point.

Homework of Section 4.4

专题二: 极值和拐点.

延伸: (5) $f(x) = |x(1-x)|$, then (   )

A. $f(x)$ has local extreme at $x = 0$, but $(0,0)$ is not the inflection point.

B. $f(x)$ does not have local extreme at $x = 0$, but $(0,0)$ is the inflection point.

C. $f(x)$ has local extreme at $x = 0$, and $(0,0)$ is the inflection point.

D. $f(x)$ does not have local extreme at $x = 0$, and $(0,0)$ is not the inflection point.

专题二: 极值和拐点.

$g(x)$ $f''(x) = 2(x-1)g(x) + g'(x)(x-1)^2$

延伸: (6) Given $f'(x) = (x-1)^2(x-2)^3$, which of the following statements is wrong ? (   )

$f'' = 2g$

A. $(1, f(1))$ is an inflection point.

$f'' = 0$

B. $(2, f(2))$ is an inflection point.

$f''' \neq 0$

C. $(3, f(3))$ is not the inflection point.

D. There exists other inflection points.

$\frac{dy}{dx} \ln y + 2\frac{dy}{dx} - 1 = 0$

(7) $y = y(x)$ is defined by the equation $y \ln y - x + y = 0$, identify the concavity of $y = y(x)$ at the neighbourhood of the point $(1, 1)$.

$\frac{dy}{dx}$

专题三: 极值最值的应用题.

延伸: (1) Designing a tank: Your iron works has contracted to design and build a 4 m^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible. (书本9)

a. What dimensions do you tell the shop to see ?

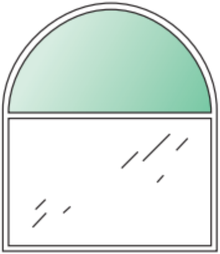
b. Briefly describe how you took weight into account.

(2) Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3. (书本12)

Homework of Section 4.5

专题三: 极值最值的应用题.

延伸: A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame. (书本22)



Homework of Section 4.7

专题四: 不定积分.

例: Find the most general antiderivative or indefinite integral.

(1) $\int (\frac{1}{x^2} - x^2 - \frac{1}{3})\ dx$ (书本23)

(2) $\int (2 + \tan^2 x)\ dx$ (书本52)

(3) $\int \cot^2 x\ dx$ (书本53)

延伸:(1) $\int \sin^2 \frac{x}{4}\ dx$

(2) $\int \cos^2 \frac{x}{2}\ dx$