2021年高数上期末考试答案.

1、选择题 A A C B B

2、填空题 (1) 7200 (2) $\frac{3}{8}$ (3) $\frac{11}{10}$ (4) $\frac{3}{2}\ln 2$. (5) $\frac{2\sqrt{2}}{\pi}$.

3、 $\int_1^3 2\pi\, 2\sqrt{y}\sqrt{1+\frac{1}{y}}\, dy = 4\pi \int_1^3 \sqrt{1+y}\, dy = \frac{16\pi}{3}(4-\sqrt{2})$

4、 $\frac{1}{x}(y' - \frac{1}{x}y) = \frac{2}{x}\ln x$

$\frac{1}{x}y = (\ln x)^2 + C$

$y = x(\ln x)^2 + Cx$

5、 Assume the tangent point is (b, b).

$$\begin{cases} b = \frac{\ln b}{\ln a} \\ \frac{1}{b\ln a} = 1 \end{cases}$$

$\Rightarrow b = e, \quad a = e^{\frac{1}{e}}$

、let ΔABC be a triangle inscribed in a circle of radius
and $AB = BC$. Put $\angle BAC = \theta$, then

$AB = BC = 2R\sin\theta$.

$AC = 4R\sin\theta\cos\theta = 2R\sin 2\theta$

Hence, the perimeter of triangle ΔABC

$L(\theta) = 2R(2\sin\theta + \sin 2\theta)$, $(0 < \theta <$

$L'(\theta) = 4R(2\cos\theta - 1)(\cos\theta + 1)$

Set $L'(\theta)=0$. We have $\theta_0=\frac{\pi}{3}$, And

$$L''(\theta_0)=-4R(\sin\theta_0+2\sin 2\theta_0)<0$$

The perimeter has a maximum $L(\frac{\pi}{3})=3\sqrt{3}R$

7. $$\int_0^{+\infty}\frac{e^{-x}}{x^p}dx=\int_0^1\frac{e^{-x}}{x^p}dx+\int_1^{+\infty}\frac{e^{-x}}{x^p}dx.$$

由于

$$\lim_{x\to 0^+}\frac{\frac{e^{-x}}{x^p}}{\frac{1}{x^p}}=1$$

when $p<1$, $\int_0^1\frac{e^{-x}}{x^p}dx$ converges.

当 x 充分大时，有 $\frac{e^{-x}}{x^p}<e^{-\frac{x}{2}}$

由于 $\int_1^{+\infty}\frac{1}{e^{\frac{x}{2}}}$ converges, $\int_1^{+\infty}\frac{e^{-x}}{x^p}dx$ converges no matter what value of p is.

综上，当 $p<1$ 时，$\int_0^{+\infty}\frac{e^{-x}}{x^p}dx$ converges.

8. (1) $$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}\cdot\frac{\frac{x}{1+x}-\ln(1+x)}{x^2}}{1}$$

$$=e\lim_{x\to 0}\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}$$

$$=e\lim_{x\to 0}\frac{x-(1+x)\ln(1+x)}{x^2}$$

$$= e^{\lim\limits_{x\to 0} \frac{-\ln(1+x)}{2x}}$$

$$= -\frac{e}{2}$$

(2). $\lim\limits_{x\to 0} \frac{3\sin x + x^2\cos(\frac{1}{x})}{(1+\cos x)\ln(1+x)}$

$$= \lim_{x\to 0} \frac{3\sin x + x^2\cos\frac{1}{x}}{2x}$$

$$= \lim_{x\to 0} \frac{3}{2}\frac{\sin x}{x} + \frac{1}{2}x\cos\frac{1}{x}$$

$$= \frac{3}{2}$$

(1) let $u = \ln x$, $\int_{\frac{1}{e}}^{e} \frac{\ln^2 x}{x} dx = \int_{-1}^{1} u^2 du = \frac{2}{3}$

2) let $\theta = \sec^{-1} x$, $\int_{1}^{\sqrt{2}} \frac{1}{x^3\sqrt{x^2-1}} dx = \int_{0}^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \frac{\pi}{8} + \frac{1}{4}$

) $\int_{1}^{+\infty} \frac{1}{x^6(x^5+1)} dx = \int_{1}^{+\infty} \frac{x^4}{x^{10}(x^5+4)} dx$

$$\xlongequal{\text{let } u = x^5} \frac{1}{5}\int_{1}^{+\infty} \frac{1}{u^2(4+u)} du.$$

$$= \frac{1}{5}\int_{1}^{+\infty} \left(\frac{-\frac{1}{16}}{u} + \frac{\frac{1}{4}}{u^2} + \frac{\frac{1}{16}}{u+4}\right) du$$

$$= \frac{1}{20} - \frac{1}{80}\ln 5$$

(4) let $u = x + \frac{1}{2}$, $\int \frac{1}{(1+x+x^2)^2} dx$

$$= \int \frac{1}{(u^2+\frac{3}{4})^2} du.$$

$$\xrightarrow{v=\frac{2}{\sqrt{3}}u} \frac{8}{3\sqrt{3}} \int \frac{1}{(1+v^2)^2} dv$$

$$= \frac{8}{3\sqrt{3}} \left(\frac{1}{2} \left(\tan^{-1}(v) + \frac{v}{v^2+1} \right) \right) + C$$

$$= \frac{4}{3\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + \frac{2x+1}{3(1+x+x^2)} + C$$