Chapter 2： limits and continuity

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

ATTENTION

$f(x)$有极限，$g(x)$无极限
则$h(x)$无极限

$f(x)=x, g(x)=\frac{1}{x}$ 则$h(x)=x+\frac{1}{x}$ (

$h(x)=f(x)$

（左某一点）

$f(x)$无极限，$g(x)$无极
则$h(x)$极限未知

$f(x)=-\frac{1}{x}, g(x)=\frac{1}{x}$ 则$h(x)=0$ $(x=$

$f(x)=\frac{1}{x}, g(x)=\frac{1}{x}$ 则$h(x)=\frac{2}{x}$ $(x=0$

$f(x)$有极限，$g(x)$无极限
则$h(x)$极限未知

$f(x)=x, g(x)=\frac{1}{x}$ 则$h(x)=1$ $(x=$

$f(x)=x, g(x)=\frac{1}{\ }$ 则$h(x)=\frac{1}{\ }(x=0$

$h(x)=f(x)g$

（左某一点）

$f(x)$无极限，$g(x)$无极
则$h(x)$极限未知

$f(x)=0, g(x)=\frac{1}{x}$ 则$h(x)=0$ $(x=$

$f(x)=\frac{1}{x}, g(x)=\frac{1}{x}$ 则$h(x)=\frac{1}{\ }$ $(x=$

分式函数极限

分子为多项式 $$\lim_{v\to 2}\frac{v^3-8}{v^4-16}$$

分子带根号 $$\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}.$$

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

C处可以不存在函数值（极限是否存在与该点是否有定义无关）

应用：

sin x　cos$x$　绝对值函数

Ex. 9　证明　$(a)\ \lim\limits_{x\to 0}\sin x=0$　　$(b)\ \lim\limits_{x\to 0}\cos x=1$

$(c)\ \lim\limits_{x\to c}|f(x)|=0 \Leftrightarrow \lim\limits_{x\to c}f(x)=0.$

$\lim\limits_{0^+}\left[\dfrac{1}{x}\right] sin$

证明：　$(a)\ -|x|\le \sin x\le |x|$

$(b)\ 0\le 1-\cos x=2\sin^2\dfrac{x}{2}\le\dfrac{x^2}{2}$

1　[1

$(x > sinx\ \ (x>0)\ \ tan\ x$

(c)“$\Rightarrow$”：$-|f(x)|\le f(x)\le |f(x)|.$

“$\Leftarrow$”：$|f(x)|=\sqrt{f^2(x)}.$

重要极限（之一)

THEOREM 7—Limit of the Ratio sin θ/θ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians})$$

趋近0而不是别的！！

ATTENTION

$$\lim_{0} \frac{tanx}{x} = 1$$

Find $\lim_{x \to \pi} \frac{\sin 2x}{5(\pi - x)}$.

$$\lim_{0} \frac{sinf(x)}{f(x)} = 1$$

$$原式 = \lim_{u \to 0} \frac{-\sin 2u}{5u} \qquad (u = \pi - x)$$

$$= -\frac{2}{5}$$

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

应用：分段函数

b. At what points c, if any, does $\lim_{x\to c} f(x)$ exist?

c. At what points does only the left-hand limit exist?

d. At what points does only the right-hand limit exist?

$$f(x) = \begin{cases} \sqrt{1-x^2}, & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & \text{x=2} \end{cases}$$

$$\lim_{x\to 1}\frac{x^{2020}-1}{x^{2019}-1}.$$

$$\lim_{x\to 0}\frac{\sin(x^2+x)}{x}$$

$$\lim_{x\to 0}\frac{\tan x}{x^2\cot 3x}.$$

$$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$$

$$\lim_{x\to 0}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)$$

$$\lim_{x\to 0^+} x\left[\frac{1}{x}\right];$$

$$\lim_{x\to -\infty}\frac{4-3x^3}{\sqrt{x^6+9}}.$$

Find a and b such that $\lim_{x\to 2}\frac{x-2}{x^2+ax+1}=b.$

Find k such that $\lim_{x\to 0}\frac{\sqrt{1+x\sin x}-\sqrt{\cos x}}{kx^2}=1.$

DEFINITIONS Let c be a real number on the x-axis.

The function f is **continuous at c** if

$$\lim_{x \to c} f(x) = f(c).$$

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c).$$

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

间断点

(1) 在 $x=x_0$ 没有定义；

(2) 虽在 $x=x_0$ 有定义，但 $\lim\limits_{x \to x_0} f(x)$ 不存在；

(3) 虽在 $x=x_0$ 有定义，且 $\lim\limits_{x \to x_0} f(x)$ 存在，但 $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$，

第一间断点：左右极限都存在

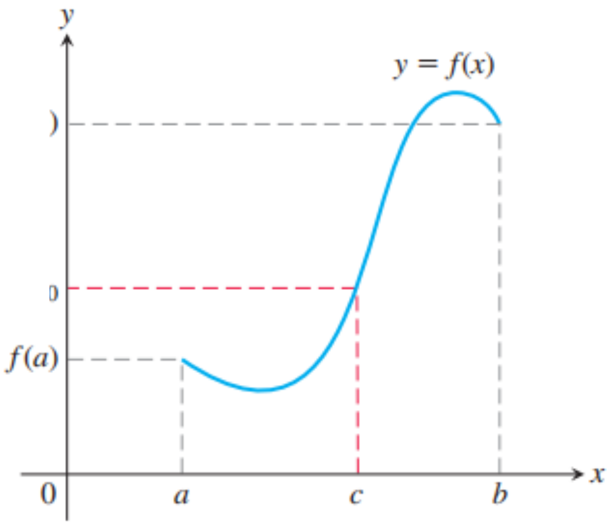
可去间断点：左右极限相等但函数在该点

第二间断点

无穷间断点　　$f(x)=$

零点存在定理和介质定理

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



定理 2(零点定理) 设函数 $f(x)$ 在闭区间 $[a,b]$ 上连续，且 $f(a)$ 与 $f(b)$ 异号(即 $f(a)\cdot f(b)<0$)，那么在开区间 (a,b) 内至少有一点 ξ，使

$$f(\xi)=0.$$

（有界性）在闭区间上连续的函数一定在该区间上有界且在能取得它的最大值和最小值.

e.g.

13. 证明方程 $\sin x+x+1=0$ 在开区间 $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ 内至少有一个根.

渐近线

水平渐近线

1. Horizontal asymptotes:A line $y = b$ is a horizontal asymptote of the function $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

垂直渐近线

3. Vertical asymptotes: if a line $x = a$ is a vertical asymptote of the function $y = f(x)$ if either $\lim \quad = \pm\infty$ or $\lim \quad = \pm\infty$.

斜渐近线

A line $y = kx + b$ is a oblique asymptote if

either $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = k \ (k \neq 0, \infty)$, $\lim\limits_{x\to\infty} [f(x) - kx] = b$

or $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = k \ (k \neq 0, \infty)$, $\lim\limits_{x\to-\infty} [f(x) - kx] = b$.

e.g.

求斜渐近线 $f(x) = \dfrac{x^2 - 3}{2x - 4}$

（长除法或传统方法）

If $\lim_{x \to c} f(x) = A$ and $\lim_{y \to A} g(y) = B$, imply that $\lim_{x \to c} g(f(x)) = B$.

Let $f(x)$ and $g(x)$ are continuous at x_0, prove $\varphi(x) = \max\{f(x), g(x)\}$ and $\psi(x) = \min\{f(x), g(x)\}$ are also continuous at x_0.

2. Determine whether the following statements are true or false?

g. If $f^2(x)$ is continuous, then $f(x)$ is continuous.

h. If $f^3(x)$ is continuous, then $f(x)$ is continuous.

then: (1) If $f(x)$ is continuous, then $|f(x)|$ is continuous.

(2) If $|f(x)|$ is continuous, then $f(x)$ is continuous.

At which points the functions fail to be continuous? At which points, if any, are the discontinuities removable? Not removable?

$$f(x) = \begin{cases} \frac{x^3-8}{x^2-4}, & x \neq 2, x \neq -2 \\ 3, & x=2 \\ 4, & x=-2 \end{cases}$$

Suppose that a function f is continuous on the closed interval $[0,1]$ and that $0 \leq f(x) \leq 1$ for every x in $[0,1]$. Show that there must exist a number c in $[0,1]$ such that $f(c) = c$.

Show that the function $F(x) = (x-a)^2 \cdot (x-b)^2 + x$ takes on the value $\frac{a+b}{2}$ for some value of x.

Graph the rational functions in Exercises 101, 104. Include the graphs and equations of the asymptotes.

$y = \frac{x^2-4}{x-1}$.