8.1 Using Basic Integration Formulas

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \qquad \text{(any number } k)$
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
3. $\int \dfrac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$

18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

8.2 Integration By Parts

If f and g are differentiable functions of x,

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx,$$

integration by parts formula

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)}$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \tag{3}$$

分部积分法

由导数公式 $(uv)' = u'v + uv'$

积分得: $uv = \int u'v\mathrm{d}x + \int uv'\,\mathrm{d}x$

$$\Longrightarrow \left.\begin{aligned} &\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x \\ \text{或}\quad &\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \end{aligned}\right\} \text{分部积分公式}$$

选取 u 及 v' (或$\mathrm{d}v$) 的原则:

1) v 容易求得 ;

2) $\int u'v\,\mathrm{d}x$ 比 $\int u\,v'\,\mathrm{d}x$ 容易计算 .

解题技巧:

选取 u 及 v' 的一般方法 :

把被积函数视为两个函数之积 , 按 “ **反对幂指三** ” 的

顺序, 前者为 u 后者为 v'.

反: 反三角函数
对: 对数函数
幂: 幂函数
指: 指数函数
三: 三角函数

例1. 求 $\int x\cos x \, \mathrm{d}x$.

解： 令 $u = x$, $v' = \cos x$,

则 $u' = 1$, $v = \sin x$

$\therefore$ 原式 $= x\sin x - \int \sin x \, \mathrm{d}x$

$= x\sin x + \cos x + C$

思考： 如何求 $\int x^2 \sin x \, \mathrm{d}x$?

例2. 求 $\int x\arctan x\,\mathrm{d}x$.

解：令 $u=\arctan x$, $v'=x$

则 $u'=\dfrac{1}{1+x^2}$, $v=\dfrac{1}{2}x^2$

$$\therefore \text{原式} =\frac{1}{2}x^2\arctan x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,\mathrm{d}x$$

$$=\frac{1}{2}x^2\arctan x-\frac{1}{2}\int(1-\frac{1}{1+x^2})\,\mathrm{d}x$$

$$=\frac{1}{2}x^2\arctan x-\frac{1}{2}(x-\arctan x)+C$$

例3. 求 $\int e^x \sin x \,\mathrm{d}x$.

解: 令 $u=\sin x$, $v'=e^x$, 则

$u'=\cos x$, $v=e^x$

$\therefore$ 原式 $=e^x\sin x-\int e^x\cos x\,\mathrm{d}x$

再令 $u=\cos x$, $v'=e^x$, 则

$u'=-\sin x$, $v=e^x$

$=e^x\sin x-e^x\cos x-\int e^x\sin x\,\mathrm{d}x$

故 原式 $=\frac{1}{2}e^x(\sin x-\cos x)+C$

说明: 也可设 $u=e^x$, v' 为三角函数, 但两次所设类型必须一致.

例4. 求 $\int e^{\sqrt{x}}\,\mathrm{d}x$.

解： 令 $\sqrt{x}=t$，则 $x=t^2$, $\mathrm{d}x=2t\,\mathrm{d}t$

$$\text{原式}=2\int t\,e^t\,\mathrm{d}t$$

$$\Big\downarrow \text{令 } u=t\,,\ v'=e^t$$

$$=2(t\,e^t-e^t)+C$$

$$=2e^{\sqrt{x}}(\sqrt{x}-1)+C$$

例5. 证明递推公式

$$I_n = \int \tan^n x\,\mathrm{d}x = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \quad (n \geq 2)$$

证: $I_n = \int \tan^{n-2} x\,(\sec^2 x - 1)\,\mathrm{d}x$

$$= \int \tan^{n-2} x\,\mathrm{d}(\tan x) - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

注: $I_n \to \cdots \to I_0$ 或 I_1

$$I_0 = x + C\,, \qquad I_1 = -\ln|\cos x| + C$$

说明:

分部积分题目的类型:

1) 直接分部化简积分 ;

2) 分部产生循环式 , 由此解出积分式 ;

(注意: 两次分部选择的 u , v 函数类型不变 ,
解出积分后加 C)

3) 对含自然数 n 的积分, 通过分部积分建立递推公式 .

例6. 计算不定积分

$$1.\int \frac{1}{\cos^3 x} dx$$

$$2.\int \frac{x^2}{(a^2+x^2)^2} dx$$

$$3.\int \sqrt{a^2+x^2}\, dx$$

二、定积分的分部积分法

定理2. 设 $u(x), v(x) \in C^1[a, b]$, 则

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x$$

例计算 $\int_1^e x^2 lnx dx$.

解:

$$\int_1^e x^2 lnx dx = \frac{1}{3}x^3 lnx\Big|_1^e - \frac{1}{3}\int_1^e x^2 dx$$

$$= \frac{1}{3}(\mathrm{e}^3 - x^3|_1^e) = \frac{1}{9}(2\mathrm{e}^3 + 1)$$

定积分用分部积分法时要注意带上积分上下限

例 计算$\int_0^1 \sqrt{1-x^2}\,dx$.

*换元必换限

解 令$x = sint, t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,此时$cost \geq 0$, 规定范围

$$\text{则有}\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\, costdt$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t\, dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + cos2t)dt$$

$$= \frac{\pi}{4}$$

内容小结

$$\text{基本积分法}\begin{cases}\text{换元积分法}\\ \text{分部积分法}\end{cases}$$

换元必换限
配元不换限

若干补充

例计算下列不定积分

1. $\int \frac{1+x^2}{1+x^4}dx$ 2. $\int \frac{x^2-1}{1+x^4}dx$ 3. $\int \frac{1}{1+x^4}dx$

尝试分子与dx结合

解： 1. 原式 $=\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x=\int \frac{d(x-\frac{1}{x})}{(\sqrt{2})^2+(x-\frac{1}{x})^2}=\frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}+C$

2. 原式 $=\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x=\int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-(\sqrt{2})^2}=\frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+C$

3.提示： 原式 $=\frac{1}{2}(\int \frac{x^2+1}{1+x^4}dx-\int \frac{x^2-1}{1+x^4}dx)$

例计算不定积分

4. $\int \frac{xe^{x}}{(1+x)^{2}}dx$ 构造相同部分分考

解： 原式$=\int \frac{(x+1)e^{x}-e^{x}}{(x+1)^{2}}dx=\int \frac{e^{x}}{x+1}dx-\int \frac{e^{x}}{(x+1)^{2}}dx=\int \frac{e^{x}}{x+1}dx+\int e^{x}d(\frac{1}{x+1})$

$$=\int \frac{e^{x}}{x+1}dx+\frac{e^{x}}{x+1}-\int \frac{e^{x}}{x+1}dx=\frac{e^{x}}{x+1}+C$$

思考与练习

下列各题求积方法有何不同?

$$(1)\ \int\frac{\mathrm{d}x}{4+x}=\int\frac{\mathrm{d}(4+x)}{4+x}\qquad (2)\ \int\frac{\mathrm{d}x}{4+x^2}=\frac{1}{2}\int\frac{\mathrm{d}(\frac{x}{2})}{1+(\frac{x}{2})^2}$$

$$(3)\ \int\frac{x}{4+x^2}\mathrm{d}x=\frac{1}{2}\int\frac{\mathrm{d}(4+x^2)}{4+x^2}$$

$$(4)\ \int\frac{x^2}{4+x^2}\mathrm{d}x=\int[1-\frac{4}{4+x^2}]\mathrm{d}\ x$$

$$(5)\ \int\frac{\mathrm{d}x}{4-x^2}=\frac{1}{4}\int[\frac{1}{2-x}+\frac{1}{2+x}]\mathrm{d}x$$

$$(6)\ \int\frac{\mathrm{d}x}{\sqrt{4x-x^2}}=\int\frac{\mathrm{d}(x-2)}{\sqrt{4-(x-2)^2}}$$

8.3 Trigonometric Integrals

Products of Powers of Sines and Cosines

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero).

Case 1 If ***m* is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

$$\begin{aligned}
\sin mx \sin nx &= \frac{1}{2}\left[\cos (m-n)x - \cos (m+n)x\right], \\
\sin mx \cos nx &= \frac{1}{2}\left[\sin (m-n)x + \sin (m+n)x\right], \\
\cos mx \cos nx &= \frac{1}{2}\left[\cos (m-n)x + \cos (m+n)x\right].
\end{aligned}$$

8.4 Trigonometric Substitutions

8.5 Integration of Rational Functions by Partial Fractions

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. Example 3 of this section illustrates such a case.
- *We must know the factors of $g(x)$.* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

(1)分子次数比分母次数高

Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

(2)分母次数比分子高

Heaviside Method

1. *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$\begin{aligned}
A_1 &= \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}\\
A_2 &= \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}\\
&\vdots\\
A_n &= \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.
\end{aligned}$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

有理函数分解为多项式及部分分式之和以后,各个部分都能积出,且原函数都是初等函数.此外,由代数学知道,从理论上说,多项式**Q(x)**总可以在实数范围内分解成为一次因式及二次因式的乘积,从而把有理函数 $\frac{P(x)}{Q(x)}$ 分解为多项式与部分分式之和.因此,有理函数的原函数都是初等函数.

但是,用部分分式法求有理函数的积分,一般说来计算比较繁,只是在没有其它方法的情况下,才用此方法.

1. 有理式的不定积分

有理函数:

$$R(x)=\frac{P(x)}{Q(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_m}$$

$m\le n$ 时, $R(x)$ 为假分式; $m>n$ 时, $R(x)$ 为真分式

有理函数 $\underset{\text{相除}}{=\!=\!=}$ 多项式 + $\boxed{\text{真分式}}$

$\downarrow$ 分解

$\boxed{\text{若干部分分式之和}}$

其中部分分式的形式为

部分分式: $\dfrac{A}{x-a}$, $\dfrac{A}{\left(x-a\right)^{n}}$ $\left(n>1\right)$;

$$\frac{Bx+C}{x^{2}+px+q},\quad \frac{Bx+C}{\left(x^{2}+px+q\right)^{n}}\quad \left(n>1\right);$$

$(n\in \mathrm{N}^{+}, p^{2}-4q<0)$

有理函数积分法

(1) 假分式 $\xrightarrow{\text{多项式除法}}$ 多项式（+真分式）;

(2) 真分式 $\xrightarrow{\text{待定系数法}}$ 部分分式之和:

真分式 $\dfrac{P(x)}{Q(x)}$ $\underline{\underline{\text{分母因式分解}}}$

$$\frac{P(x)}{b_0(x-a_1)^{n_1}\cdots(x-a_k)^{n_k}(x^2+p_1x+q_1)^{m_1}\cdots(x^2+p_lx+q_l)^{m_l}}$$

（其中 $x^2+p_ix+q_i$ ， $i=1,\cdots,h$ 为不可约因式）

$$
\begin{aligned}
&= \frac{1}{b_0}\Big\{\frac{A_{11}}{x-a_1}+\cdots+\frac{A_{n_1 1}}{(x-a_1)^{n_1}} \\
&\quad +\cdots+\frac{A_{1k}}{x-a_k}+\cdots+\frac{A_{n_k k}}{(x-a_k)^{n_k}} \\
&\quad +\frac{B_{11}x+C_{1,1}}{x^2+p_1x+q_1}+\cdots+\frac{B_{m_1 1}x+C_{m_1 1}}{(x^2+p_1x+q_1)^{m_1}} \\
&\quad +\cdots+\frac{B_{1l}x+C_{1l}}{x^2+p_lx+q_l}+\cdots+\frac{B_{m_l l}x+C_{m_l l}}{(x^2+p_lx+q_l)^{m_l}}\Big\}
\end{aligned}
$$

（其中各系数待定）；

如果$Q(x)$有一个n 重实根a，则$P(x)/Q(x)$ 的部分分式中一定包含下列形式的n 项部分分式之和:

$$\frac{A_1}{x-a}+\cdots+\frac{A_n}{(x-a)^n}$$

如果 $Q(x)$中包含因子$\left(x^2+px+q\right)^m\left(q>p^2/4\right)$时，则 $P(x)/Q(x)$ 的部分分式中一定包含下列形式的m 项部分分式之和:

$$\frac{B_1x+C_1}{x^2+px+q}+\cdots+\frac{B_mx+C_m}{\left(x^2+px+q\right)^m}$$

例如 将真分式$\dfrac{x+1}{(x-1)(x-2)^2(x^2+1)^3(x^2+x+1)}$

分解成**部分分式.**

$$原式=\frac{\mathrm{A}_{11}}{(x-1)}+(\frac{A_{12}}{x-2}+\frac{A_{22}}{(x-2)^2})+(\frac{B_{11}x+C_{11}}{x^2+1}+$$

$$\frac{B_{21}x+C_{21}}{(x^2+1)^2}+\frac{B_{31}x+C_{31}}{(x^2+1)^3})\ +\frac{B_{12}x+C_{12}}{(x^2+x+1)}.$$

其中A_{ij},B_{ij}与C_{ij}均为常数，下面将用待定系数法求出.

四种典型部分分式的积分:

1. $\int \frac{A}{x-a}\mathrm{d}x = A\ln|x-a| + C$

2. $\int \frac{A}{(x-a)^n}\mathrm{d}x = \frac{A}{1-n}(x-a)^{1-n} + C \quad (n \neq 1)$

$$\left.\begin{array}{l} 3.\ \int \frac{Bx+C}{x^2+px+q}\mathrm{d}x \\[2ex] 4.\ \int \frac{Bx+C}{(x^2+px+q)^n}\mathrm{d}x \end{array}\right\}$$

变分子为

$\frac{B}{2}(2x+p)+C-\frac{Bp}{2}$

再分项积分

$(p^2-4q<0\,,\,n\neq 1)$

3. $$\int \frac{Bx+C}{x^2+px+q}\mathrm{d}x = \frac{B}{2}\int \frac{2x+\frac{2C}{B}}{x^2+px+q}\mathrm{d}x$$

$$=\frac{B}{2}\int \frac{2x+p}{x^2+px+q}\mathrm{d}x + \int \frac{C-\frac{Bp}{2}}{x^2+px+q}\mathrm{d}x$$

$$=\frac{B}{2}\int \frac{2x+p}{x^2+px+q}\mathrm{d}x + (C-\frac{Bp}{2})\int \frac{d(x+\frac{p}{2})}{(x+\frac{p}{2})^2+\frac{4q-p^2}{4}}$$

$$=\frac{B}{2}\ln(x^2+px+q)+(C-\frac{Bp}{2})\cdot\frac{2}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}}+C.$$

$$=\frac{B}{2}\ln(x^2+px+q) + \frac{2C-Bp}{\sqrt{4q-p^2}}\cdot\arctan\frac{2x+p}{\sqrt{4q-p^2}}+C.$$

$$4.\ \int \frac{Bx+C}{(x^2+px+q)^n}\mathrm{d}x = \frac{B}{2}\int \frac{2x+\frac{2C}{B}}{[(x+\frac{p}{2})^2+(g-\frac{p^2}{4})]^n}dx$$

$$=\frac{B}{2}\int \frac{2x+p+\frac{2C}{B}-p}{(x^2+px+q)^n}\mathrm{d}x=\frac{B}{2}\int \frac{2x+p}{(x^2+px+q)^n}\mathrm{d}x+\int \frac{C-\frac{Bp}{2}}{(x^2+px+q)^n}\mathrm{d}x$$

$$=\frac{B}{2}\int \frac{d(x^2+px+q)}{(x^2+px+q)^n} +(C-\frac{Bp}{2})\int \frac{d(x+\frac{p}{2})}{[(x+\frac{p}{2})^2+\frac{4q-p^2}{4}]^n}$$

$$=\frac{B}{1-n}(x^2+px+q)^{1-n} +(C-\frac{Bp}{2})\int \frac{d(x+\frac{p}{2})}{[(x+\frac{p}{2})^2+\frac{4q-p^2}{4}]^n}$$

$$I_n = \int \frac{\mathrm{d}x}{(x^2+a^2)^n}$$

递推公式 $I_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n$

说明: 已知 $I_1 = \frac{1}{a}\arctan\frac{x}{a} + C$ 利用递推公式可求得 I_n.

例如,

$$I_3 = \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2} I_2$$

$$= \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{4a^2}\left(\frac{1}{2a^2}\frac{x}{x^2+a^2} + \frac{1}{2a^2} I_1\right)$$

$$= \frac{1}{4a^2}\frac{x}{(x^2+a^2)^2} + \frac{3}{8a^4}\frac{x}{x^2+a^2} + \frac{3}{8a^5}\arctan\frac{x}{a} + C$$

例1 求$\int \frac{x^3+1}{x(x-1)^3}dx$

解 $$\frac{x^3+1}{x(x-1)^3}=\frac{A_{11}}{x}+\frac{A_{12}}{x-1}+\frac{A_{22}}{(x-1)^2}+\frac{A_{32}}{(x-1)^3},$$

其中A_{ij}为常数，可以用如下的方法求出待定系数.

待定系数法，上式通分后得

$$\frac{x^3+1}{x(x-1)^3}=\frac{A_{11}(x-1)^3+A_{12}x(x-1)^2+A_{22}x(x-1)+A_{32}}{x(x-1)^3}$$

$$x^3+1\equiv A_{11}(x-1)^3+A_{12}x(x-1)^2+A_{22}x(x-1)+A_{32}x.$$

$$x^3+1\equiv(A_{11}+A_{12})x^3+(-3A_{11}-A_{12}+A_{22})x^2+$$
$$(3A_{11}+A_{12}-A_{22}+A_{32})x-A_{11}$$

比较恒等式两端同次幂的系数，得一方程组：

$$\begin{cases} A_{11}+A_{12}=1, \\ -3A_{11}-2A_{12}+A_{22}=0, \\ 3A_{11}+A_{12}-A_{22}+A_{32}=0, \\ -A_{11}=1. \end{cases}$$

从而解得 $A_{11}=-1,\quad A_{12}=2,\quad A_{22}=1,\quad A_{32}=2.$ 故有

$$\frac{x^3+1}{x(x-1)^3}=\frac{-1}{x}+\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}$$

于是

$$\int\frac{x^3+1}{x(x-1)^3}dx=-\ln|x|+2\ln|x-1|-\frac{1}{x-1}-\frac{1}{(x-1)^2}+C.$$

$$=\ln\frac{|x-1|^2}{|x|}-\frac{x}{(x-1)^2}+C.$$

例 2 求 $\int \frac{\mathrm{d}x}{(1+2x)(1+x^2)}$.

解 $$\frac{1}{(1+2x)(1+x^2)}=\frac{A}{1+2x}+\frac{Bx+C}{1+x^2},$$

两端去分母,得 $$1=A(1+x^2)+(Bx+C)(1+2x),$$

或 $$1=(A+2B)x^2+(B+2C)x+C+A.$$

比较两端的各同次幂的系数及常数项，有

$$\begin{cases} A+2B=0, \\ B+2C=0, \\ A+C=1. \end{cases}$$ 解之得 $$A=\frac{4}{5}, B=-\frac{2}{5}, C=\frac{1}{5}.$$

$$\therefore \quad \frac{1}{(1+2x)(1+x^2)}=\frac{\frac{4}{5}}{1+2x}+\frac{-\frac{2}{5}x+\frac{1}{5}}{1+x^2}.$$

$$\therefore \quad \frac{1}{(1+2x)(1+x^2)} = \frac{\frac{4}{5}}{1+2x} + \frac{-\frac{2}{5}x+\frac{1}{5}}{1+x^2}.$$

$$\int \frac{\mathrm{d}x}{(1+2x)(1+x^2)} = \frac{2}{5}\int \frac{\mathrm{d}(1+2x)}{1+2x} - \frac{1}{5}\int \frac{\mathrm{d}(1+x^2)}{1+x^2} + \frac{1}{5}\int \frac{\mathrm{d}x}{1+x^2}$$

$$= \frac{2}{5}\ln\left|1+2x\right| - \frac{1}{5}\ln(1+x^2) + \frac{1}{5}\arctan x + C$$

例3 求 $\int \frac{x^2}{x^3+1}\mathrm{d}x$.

解 $\int \frac{x^2}{x^3+1}\mathrm{d}x = \frac{1}{3}\int \frac{d(x^3+1)}{x^3+1} = \frac{1}{3}\ln|x^3+1| + C.$

例4. 求 $\displaystyle\int \frac{x-2}{x^2+2x+3}\,\mathrm{d}x$.

解: 原式 $\displaystyle = \int \frac{\frac{1}{2}(2x+2)-3}{x^2+2x+3}\,\mathrm{d}x$

$$= \frac{1}{2}\int \frac{\mathrm{d}(x^2+2x+3)}{x^2+2x+3} - 3\int \frac{\mathrm{d}(x+1)}{(x+1)^2+(\sqrt{2})^2}$$

$$= \frac{1}{2}\ln\left|x^2+2x+3\right| - \frac{3}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$$

例5 求 $\int \frac{dx}{x^3+1}$. **解** $\because \frac{1}{x^3+1} = \frac{1}{3}(\frac{1}{x+1} - \frac{x-2}{x^2-x+1})$.

$$\int \frac{x-2}{x^2-x+1}dx = \frac{1}{2}\int \frac{2x-4}{x^2-x+1}dx = \frac{1}{2}\int \frac{2x-1-3}{x^2-x+1}dx$$

$$= \frac{1}{2}\int \frac{(2x-1)dx}{x^2-x+1} - \frac{3}{2}\int \frac{dx}{x^2-x+1}$$

$$= \frac{1}{2}\int \frac{d(x^2-x+1)}{x^2-x+1} - \frac{3}{2}\int \frac{d(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{1}{2}\ln(x^2-x+1) - \sqrt{3}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

$$\int \frac{dx}{x^3+1} = \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

例 6 求 $\int \frac{x^3+x^2+2}{x(x^2+2)^2}dx.$

解 $\because \frac{x^3+x^2+2}{x(x^2+2)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+2}+\frac{B'x+C'}{(x^2+2)^2},$

即有

$$x^3+x^2+2\equiv A(x^2+2)^2+(Bx+C)x(x^2+2)+B'x^2+C'x$$

令$x=0$,得$A=\frac{1}{2}$,则

$$x^3+x^2+2-\frac{1}{2}(x^2+2)^2=(Bx+C)x(x^2+2)+B'x^2+C'x$$

即 $$x^2-x-\frac{1}{2}x^3=Bx^3+Cx^2+2Bx+B'x+2C+C'.$$

$$\therefore\ B=-\frac{1}{2},\quad C=1,\quad B'=0,\quad 2C+C'=0\Rightarrow C'=-2.$$

$$\int \frac{x^3+x^2+2}{x(x^2+2)^2}dx = \frac{1}{2}\int \frac{1}{x}dx \quad -\frac{1}{4}\int \frac{2x}{x^2+2}dx \quad -\int \frac{2}{(x^2+2)^2}dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+2) \quad + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} - \int \frac{2}{(x^2+2)^2}dx$$

$$\int \frac{1}{(x^2+2)^2}dx \ = -\int \frac{1}{2x}d\left(\frac{1}{(x^2+2)}\right)$$

$$\int \frac{Ax+B}{(x^2+px+q)^r}dx$$

选择分部积分降次

$$= -\frac{1}{2\mathrm{x}(\mathrm{x}^2+2)} - \frac{1}{2}\int \frac{dx}{x^2(x^2+2)}$$

$$= -\frac{1}{2\mathrm{x}(\mathrm{x}^2+2)} - \frac{1}{4}\int \left(\frac{1}{x^2} - \frac{1}{x^2+2}\right)dx$$

出现 $\frac{1}{x^r(x^r+A)}$ 拆分

例7 求 $\int \frac{\mathrm{d}x}{x^4+1}$

解 原式 $= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}\mathrm{d}x$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x$$

注意本题技巧

按常规方法较繁

$$= \frac{1}{2}\int \frac{\mathrm{d}(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} - \frac{1}{2}\int \frac{\mathrm{d}(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}$$

$$= \frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x} - \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C \quad (x \neq 0)$$

2. 三角函数有理式的不定积分

（1）三角有理式：

——由三角函数和常数经过有限次四则运算构成的函数．三角函数有理式可记为 $R(\sin x, \cos x)$

（2）三角有理式的积分法：

$$\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},$$

万能代换

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},$$

万能替换公式： 令 $t = \tan\frac{x}{2}$ 则 $x = 2\arctan t$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2}dt$$

则 $\int R(\sin x, \cos x)\,\mathrm{d}x$

令 $t = \tan\frac{x}{2}$

t 的有理函数的积分

$$= \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt.$$

例8. 求 $\displaystyle\int \frac{1+\sin x}{\sin x(1+\cos x)}\mathrm{d}x$.

解: 令 $t=\tan\frac{x}{2}$, 则 $\mathrm{d}x=\dfrac{2}{1+t^2}\mathrm{d}t$

$$\int \frac{1+\sin x}{\sin x(1+\cos x)}\mathrm{d}x=\int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}(1+\frac{1-t^2}{1+t^2})}\cdot\frac{2}{1+t^2}\mathrm{d}t=\frac{1}{2}\int\left(t+2+\frac{1}{t}\right)\mathrm{d}t$$

$$=\frac{1}{2}\left(\frac{1}{2}t^2+2t+\ln|t|\right)+C$$

绝对值

$$=\frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+C$$

例 9 求 $\int \frac{\cot x dx}{\sin x+\cos x-1}$. **解** 令 $t=\tan\frac{x}{2}$ ，则

$$dx=\frac{2}{1+t^2}dt \quad \sin x=\frac{2t}{1+t^2}, \ \cos x=\frac{1-t^2}{1+t^2}, \ \cot x=\frac{1-t^2}{2t},$$

$$\int \frac{\cot x dx}{\sin x+\cos x-1}=\int \frac{\frac{1-t^2}{2t}\cdot\frac{2}{1+t^2}dt}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-1}=\int \frac{1+t}{2t^2}dt$$

$$=\frac{1}{2}\int \frac{1}{t^2}dt+\frac{1}{2}\int \frac{1}{t}dt=-\frac{1}{2t}+\frac{1}{2}\ln|t|+C$$

$$=-\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}}+\frac{1}{2}\ln|\tan\frac{x}{2}|+C.$$

注（1）用万能代换一定能将三角函数有理式的积分化为有理函数的积分；

（2）万能代换不一定是最好的；

（3）常用的将三角函数有理式的积分化为有理函数的积分的代换方法（非“万能的”）：

1）若 $R(-\sin x, \cos x) = -R(\sin x, \cos x)$，可取 $u=\cos x$ 为积分变量；

2）若 $R(\sin x, -\cos x) = -R(\sin x, \cos x)$，可取 $u=\sin x$ 为积分变量；

3）若 $R(-\sin x, -\cos x) = R(\sin x, \cos x)$，可取 $u=\tan x$ 为积分变量.

例 10 求 $\int \frac{\sin x \cos x}{1+\sin^2 x} dx.$

写成 $f(\sin x) \cdot d(\sin x)$ 形式

解 此题中被积函数可写成 $f(\sin x)\cos x$的形式. 这时很自然想到"凑微分法".

$$\int \frac{\sin x \cos x}{1+\sin^2 x} dx = \int \frac{\sin x}{1+\sin^2 x} d\sin x \xlongequal{令 t=\sin x}$$

$$= \int \frac{tdt}{1+t^2} = \frac{1}{2}\ln(1+t^2)+C$$

$$= \frac{1}{2}\ln(1+\sin x^2)+C.$$

$$= \frac{1}{2}\int \frac{1}{1+\sin^2 x} d(1+\sin^2 x)$$

例 11 求 $\int \frac{\cos x}{\sin x+\cos x}dx.$

解 $\int \frac{\cos x}{\sin x+\cos x}dx \xlongequal{\text{同}\div\cos x} \int \frac{1}{1+\tan x}dx$

$$\xlongequal{\text{令}t=\tan x} \int \frac{1}{1+t}\cdot\frac{dt}{1+t^2}$$

$$=\frac{1}{2}\int\left(\frac{1}{1+t}-\frac{t-1}{1+t^2}\right)dt$$

$$=\frac{1}{2}[\ln|1+t|-\frac{1}{2}\ln(1+t^2)+\arctan t]+C$$

$$=\frac{1}{2}[\ln|1+\tan x|-\ln|\sec x|+x]+C$$

$$=\frac{1}{2}[\ln|\cos x+\sin x|+x]+C$$

例 12 求$\int \sin^4 x\cos^2 x dx.$

解 $$\int \sin^4 x\cos^2 x dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \cdot \frac{1+\cos 2x}{2} dx$$

$$= \frac{1}{8}\int (\cos^3 2x - \cos^2 2x - \cos 2x + 1) dx$$

$$= \frac{1}{16}\int (1-\sin^2 2x) d\sin 2x - \frac{1}{8}\int \frac{1+\cos 4x}{2} dx$$

$$+ \frac{1}{8}\int (1-\cos 2x) dx$$

$$= \frac{1}{16}\left(x - \frac{1}{3}\sin^3 2x - \frac{1}{4}\sin 4x\right) + C.$$

注 被积函数都是$\sin x$及$\cos x$的偶次方幂,可以利用倍角公式降低其方幂.

3. 某些根式的不定积分

被积函数为简单根式的有理式，可通过根式代换化为有理函数的积分. 例如:

$$\int R(x, \sqrt[n]{ax+b})\,\mathrm{d}x, \quad 令\ t=\sqrt[n]{ax+b}$$

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)\mathrm{d}x, \quad 令\ t=\sqrt[n]{\frac{ax+b}{cx+d}}$$

$$\int R(x, \sqrt[n]{ax+b}, \sqrt[m]{ax+b})\,\mathrm{d}x,$$

令 $t=\sqrt[p]{ax+b}$, p 为 m,n 的最小公倍数.

例 13 求 $\int \frac{\mathrm{d}x}{1+\sqrt[3]{x+2}}$.

解 令 $u=\sqrt[3]{x+2}$，则 $x=u^3-2$, $\mathrm{d}x=3u^2\,\mathrm{d}u$

$$原式 =\int \frac{3u^2}{1+u}\,\mathrm{d}u \quad =3\int \frac{(u^2-1)+1}{1+u}\mathrm{d}u$$

$$=3\int (u-1+\frac{1}{1+u})\,\mathrm{d}u$$

$$=3[\ \tfrac{1}{2}u^2-u+\ln\left|1+u\right|\ ]+C$$

$$=\frac{3}{2}\sqrt[3]{(x+2)^2}-3\sqrt[3]{x+2}$$

$$+3\ln\left|1+\sqrt[3]{x+2}\right|+C$$

例 14 求 $\int \frac{1}{x}\sqrt{\frac{1+x}{x}}\mathrm{d}x$.

解 令 $t=\sqrt{\frac{1+x}{x}}$,则 $x=\frac{1}{t^2-1}$, $\mathrm{d}x=\frac{-2t\,\mathrm{d}t}{(t^2-1)^2}$

原式 $=\int (t^2-1)t\cdot\frac{-2t}{(t^2-1)^2}\mathrm{d}t$

$$=-2\int\frac{t^2}{t^2-1}\,\mathrm{d}t=-2t-\ln\left|\frac{t-1}{t+1}\right|+C$$

$$=-2\sqrt{\frac{1+x}{x}}+\ln\left|2x+2x\sqrt{x+1}+1\right|+C$$

例15 求 $\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}}$.

解 为去掉被积函数分母中的根式，取根指数 2，3 的最小公倍数 6，令 $x=t^6$，则有

$$\text{原式} = \int \frac{6t^5\,\mathrm{d}t}{t^3+t^2}$$

$$= 6\int (t^2 - t + 1 - \frac{1}{1+t})\,\mathrm{d}t$$

$$= 6\left[\tfrac{1}{3}t^3 - \tfrac{1}{2}t^2 + t - \ln|1+t|\right] + C$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(1+\sqrt[6]{x}) + C$$

(5) $\int \sqrt{\frac{a-x}{x-b}}\,dx$

2. Find the integral.

$$\int_0^{\pi^2} \sqrt{x}\sin\sqrt{x}dx$$

3. Find the integral.

$$\int_0^1 \frac{f(x)}{\sqrt{x}}dx,$$

where $f(x) = \int_1^x \frac{\ln(t+1)}{t}dt.$

7. Use integration by parts to obtain the formula

$$\int \sqrt{1-x^2}dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx.$$

(1) $\int \frac{\ln(1+x^2)}{x^3}\,dx$

(5) $\int x\sqrt{\frac{x}{x+1}}\,dx$

(3) $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$

(5) $\int_0^{\pi} x\sin^3 x\,dx$

(6) $\int_0^{2\pi} \sqrt{1-\cos x}\,dx$

(5) $\int \cos\sqrt{x}\,dx$

内容小结

1. 可积函数的特殊类型

$$
\begin{tikzcd}
\text{有理函数} \arrow[d, "\text{分解}"] & & \text{三角函数有理式} \arrow[ll, "\text{万能代换}"'] \\
\text{多项式及部分分式之和} & & \text{简单无理函数} \arrow[u, "\text{三角代换}"'] \arrow[ull, "\text{根式代换}"]
\end{tikzcd}
$$

2. 特殊类型的积分按上述方法虽然可以积出, 但不一定简便 , 要注意综合使用基本积分法 , 简便计算 .

1. 求积分

(1) $\int \sin^{-1} x dx$　　　　(2) $\int x^n \ln x dx$

(3) $\int \cos\sqrt{x} dx$　　　　(4) $\int \sin(\ln x) dx$

猜: xe^{x^3}

(5) $\int (1+3x^3)e^{x^3} dx$　　　　(6) $\int_0^{2\pi} \sqrt{1-\cos x} dx$

(7) $\int_0^1 f(x)dx$,其中，$f(x)=\int_1^x e^{t^2} dt$

(8) 证明$\int_0^a \frac{1}{x+\sqrt{a^2-x^2}} dx = \frac{\pi}{4}$.

2. 设 $f(x)=\int_1^x \frac{\ln u}{1+u} du, x\in(0,\infty)$，求 $f(x)+f(\frac{1}{x})$.

3. 设 $f(x)=\begin{cases}x^2, & 0\le x<1,\\ x, & 1\le x\le 2,\end{cases}$ 问 $g(x)=\int_0^x f(u)du$ 在 $x=1$ 处连续吗？可导吗？.

4. 求 $f(x)=\int_0^x e^{-t}\cos t dt$ 在 $[0,\pi]$ 上的最大值最小值.

5. 设 $I_n=\int_0^{\frac{\pi}{2}}\sin^n x dx$，证明递推公式 $I_n=\dfrac{n-1}{n}I_{n-2}$，并计算 $\int_0^{\pi}\sin^{10}xdx$。

6. 设 $f(x)$ 在 $[a,b]$ 上有二阶连续导数，在 $x=a$ 处切线的倾角为 $\dfrac{\pi}{6}$，在 $x=b$ 处切线的倾角为 $\dfrac{\pi}{4}$，求 $\int_a^b f'(x)f''(x)dx$.

7. 设 $b>a>0$，证明 $1-\dfrac{a}{b}<\ln\dfrac{b}{a}<\dfrac{b}{a}-1$.（提示：用柯西中值定理证明）

1. 求积分

（1） $\int \frac{\tan^{-1} x}{x^2(1+x^2)} dx$　　（2） $\int_0^1 \frac{\ln(1+x)}{(2-x)^2} dx$

（3） $\int_1^2 \frac{1}{x(1+x^2)^2} dx$　　（4） $\int_0^1 \arcsin\sqrt{\frac{x}{1+x}}\, dx$

（5） $\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{(1+x^2)\sqrt{1+x^2}} dx$　　（6） $\int_{2\sqrt{2}}^4 \frac{1}{x^2\sqrt{x^2-4}} dx$

（7） $\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}} dx$　　（8） $\int_0^{\ln 2} \sqrt{1-e^{-2x}}\, dx$

2. 用万能替换求积分 $\int \frac{1}{\sin x+2}dx$.

3. 设可导函数 $f(x)$ 满足 $f(0)=0$, $f'(0)=a$ 且 $\lim\limits_{x\to 0}\frac{\int_0^x (x-t)f(t)dt}{x^k}=2$，求常数 a,k.

4. 设可导函数 $f(x)$ 满足 $\int_0^1 f(tx)dt=2f(x)\ (x>0)$，求 $f(x)$.

5. 设 $I_n=\int_0^{\infty} x^n e^{-x}dx$，推导递推公式 $I_n=nI_{n-1}$，并计算 $\int_0^{\infty} x^{100}e^{-x}dx$ 。