Tutorial 06 for Calculus I

Sect. 5.1-5.4

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- Section 5.1 Area and Estimating with Finite Sums
- Section 5.2 Sigma Notation and Limits of Finite Sums
- Section 5.3 Definition of the Definite Integral
- Section 5.4 The Mean Value Theorem for Definite Integrals, The Fundamental Theorem of Calculus

有限个一类点

Theorem (Integrability of Continuous Functions)

If a function f is continuous over the interval $[a, b]$, of if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\, dx$ exists and f is integrable over $[a, b]$.

专题一: 定积分的定义.

The definition of the Definite Integral of f over $[a, b]$:

If $\forall \epsilon > 0, \exists\ \delta > 0$, s.t for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $||P|| < \delta$ and $\forall\ c_k \in [x_{k-1}, x_k]$, we have $|\sum\limits_{k=1}^{n} f(c_k)\Delta x_k - J| < \epsilon$.

We say that a number J is the definite integral of f over $[a, b]$ and
$J = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(c_k)\Delta x_k = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(a + k\frac{(b-a)}{n})\underline{(\frac{b-a}{n})}$
$= \lim\limits_{||P||\to 0} \sum\limits_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\ dx$.

We define the norm of a partition P, written $||P||$, bo be the largest of all interval widths, i. e $||P|| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$.

专题一: 定积分的定义.

例: (1) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}(1-\frac{k^2}{n^2})$ (2) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{\sqrt{4n^2-k^2}}{n^2}$

$\frac{k}{n}\to x$

延伸: Express the limit as a definite integral, and evaluate the resulting definite integral.

$\sqrt{\frac{2k+1}{2n}}$ $2k+1$

(1) $\lim\limits_{n\to\infty}\frac{1}{n}(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{n\pi}{n})$

(2) $\lim\limits_{n\to\infty}\frac{1}{n}(\sqrt{1+\cos\frac{\pi}{n}}+\sqrt{1+\cos\frac{2\pi}{n}}+\cdots+\sqrt{1+\cos\frac{n\pi}{n}})$

(3) $\lim\limits_{n\to\infty}\frac{1^5+2^5+3^5+\cdots+n^5}{n^6}$

$\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}\cdots+\frac{1}{1+\frac{n}{n}}\right)$

(4) $\lim\limits_{n\to\infty}\frac{1}{n^2}(2+4+6+\cdots+2n)$

x^2

(5) $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n})$

$\frac{x^3}{3}$ $\int_0^1\frac{1}{1+x}$ $\ln(1+x)+C.$

(6) $\lim\limits_{n\to\infty}\frac{1^2+3^2+\cdots+(2n-1)^2}{n^3}$ $\frac{(2k-1)^2}{2n}$ (7) $\lim\limits_{n\to\infty}\frac{1}{n\sqrt{2n}}(\sqrt{1}+\sqrt{3}+\cdots+\sqrt{2n-1})$

Homework of Section 5.3

专题一: 定积分的定义.

乘一项

例: Use the formula 除一项构造和差化积

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos\frac{h}{2} - \cos((m+\frac{1}{2})h)}{2\sin\frac{h}{2}}$$

$\sin\frac{h}{2}$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$ in two steps. (书本85) $-\cos$和 $+\cos$差

a. Partition the interval $[0, \frac{\pi}{2}]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b. Find the limit of U as $n \to \infty$ and $\Delta x = \frac{b-a}{n} \to 0$.

Homework of Section 5.3

专题二：定积分的性质.

Theorem (Max-Min Inequality)

If f is has maximum value $\max f$ *and maximum value* $\min f$ *on* $[a, b]$,
then $\min f \cdot (b-a) \leq \int_a^b f(x)\, dx \leq \max f \cdot (b-a)$

Theorem (Domination)

$f(x) \geq g(x)$ *on* $[a, b] \Rightarrow \int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$
$f(x) \geq 0$ *on* $[a, b] \Rightarrow \int_a^b f(x)\, dx \geq 0$

Theorem (The Mean Value Theorem for Definite Integrals)

If f is continuous on $[a, b]$, *then at some point c in* $[a, b]$,
$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$

Homework of Section 5.3

专题二: 定积分的性质.

$f(x)$ is continuous,

(1) If $f(x)$ is a even function, then $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$.

(2) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\ dx = 0$.

(3) If $f(x)$ is a periodic with period T, then $\int_{a}^{a+T} f(x)\ dx = \int_{0}^{T} f(x)\ dx$.

几何意义 (圆面)

专题二: 积分的性质.

例: (1) Evaluate $\int_{-1}^{1}(1+\sqrt{1-x^2})\ dx$ (书本22)

(2) What values of a and b maximize the value of $\int_a^b (x-x^2)\ dx$? (书本71)

延伸: (1) If $f(x)>0, f'(x)<0, f''(x)>0, x\in[a,b]$, let $S_1=\int_a^b f(x)\ dx$, $S_2=f(b)(b-a), S_3=\frac{1}{2}[f(a)+f(b)](b-a)$, then ( )

(A) $S_1<S_2<S_3$ (B) $S_2<S_1<S_3$

(C) $S_3<S_1<S_2$ (D) $S_2<S_3<S_1$

专题二: 积分的性质.

延伸: (2) $I_1 = \int_0^{\frac{\pi}{4}} \frac{\tan x}{x}\, dx, I_2 = \int_0^{\frac{\pi}{4}} \frac{x}{\tan x}\, dx$, then ( )

(A) $I_1 > I_2 > 1$

(B) $1 > I_1 > I_2$

~~(C) $I_2 > I_1 > 1$~~

~~(D) $1 > I_2 > I_1$~~

(3) Let $I_k = \int_0^{k\pi} e^{x^2} \sin x\, dx, (k = 1, 2, 3)$, then ( )

(A) $I_1 < I_2 < I_3$

(B) $I_3 < I_2 < I_1$

(C) $I_2 < I_3 < I_1$

(D) $I_2 < I_1 < I_3$

$P > N > 0$

(4) If $M = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100} \sin x\, dx, N = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100} \sin^{100} x\, dx, P = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^{100} \sin^{90} x\, dx$. Compare them.

0

(5) Evaluate $\int_0^1 \sqrt{2x - x^2}\, dx$.

$y = \sqrt{2x - x^2}$

$x^2 - 2x + y^2 = 1$

$(x-1)^2 + y^2 = 1$

$x = 1$

专题二: 积分的性质.

延伸: (6) Assume $f(x)$ is a periodic with period T, Prove that $\int_x^{x+T} f(t)\ dt$ is a constant.
Furthermore, if $\int_0^T f(t)\ dt = 0$, then $\int_0^x f(t)\ dt$ is also a periodic function with period T.

(7) $f(x)$ is continuous, $F(x) = \int_0^x f(t)\ dt$ prove that
a. if $f(x)$ is an odd function, then $F(x)$ is a even function.
b. if $f(x)$ is a even function, then $F(x)$ is an odd function.

专题二: 积分的性质.

延伸: (8) $f(x)$ is continuous, $F(x)$ is the anti-derivative of $f(x)$, so (　　)

(A) $F(x)$ is a even function $\Longleftrightarrow$ $f(x)$ is an odd function.

(B) $F(x)$ is an odd function$\Longleftrightarrow$ $f(x)$ is a even function.

(C) $F(x)$ is a periodic function $\Longleftrightarrow$ $f(x)$ is a periodic function,

(D) $F(x)$ is a monotonic function $\Longleftrightarrow$ $f(x)$ is a monotonic function.

可导的偶函数的导函数为奇函数, 而可导的基函数的导函数为偶函数.
奇函数的原函数都是偶函数, 而偶函数的原函数之一为奇函数.
可导的周期函数其导函数仍为周期函数(<u>但周期函数的原函数并不一定</u>是周期函数).

专题三: 求原函数和变限积分.

Theorem (The Fundamental Theorem of Calculus, Part 1)

If f is continuous on $[a,b]$, then $F(x)=\int_a^x f(t)\ dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x)=\frac{d}{dx}\int_a^x f(t)\ dt=f(x).$$

Furthermore, $(\int_{u(x)}^{v(x)} f(t)\ dt)'=f(v(x))v'(x)-f(u(x))u'(x)$

Homework of Section 4.7， 5.4

专题三: 求原函数和变限积分.

延伸: (1) Given $f(x) = \begin{cases} 2x, & x \le 0 \\ \sin x, & x > 0 \end{cases}$, $F(x)$ is the anti-derivative of f with $F(0) = 1$, then find $F(x)$.

x^2

$-\cos x$

(2) If $f'(\sin x) = \cos(2x)$, then find $f(x)$.

t

$f'(t) = 1 - 2t^2$

$= 1 - 2\sin^2 x$

(3) Find an antiderivative for the function $f'(\ln x) = \begin{cases} 1, & 0 < x \le 1 \\ x, & x > 1 \end{cases}$.

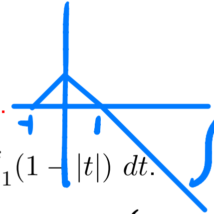
t

e^t $t \le 0$

e^t e^t $t > 0$

Homework of Section 4.7， 5.4

专题三：求原函数和变限积分.



延伸：(4) Let $x \geq -1$, find $\int_{-1}^{x} (1 - |t|)\, dt$.

$\int_{-1}^{0} + \int_{0}^{x}$

(5) Compute $\int_{-1}^{x} f(t)\, dt$, where $f(t) = \begin{cases} t, & t \in [0, 1] \\ 0, & t < 0 \text{ or } t > 1 \end{cases}$.

u du

(6) Compute $\int_{0}^{3} f(x-1)\, dx$ where $f(x) = \begin{cases} 1 + x^2, & x \leq 0 \\ e^{-x}, & x > 0 \end{cases}$.

$\int_{-1}^{2}$

专题四: 微积分基本定理.

$\int_2^{x^2}\sin(t^3)\,dt + 2x\cdot x^2\sin(t^6)$ $\qquad -\frac{1}{1+\tan^2 x}\sec^2 x$

例: (1) Find $\frac{d}{dx}[x\int_2^{x^2}\sin(t^3)\ dt]$. (2) Find $\frac{d}{dx}\int_{\tan x}^{0}\frac{dt}{1+t^2}$. (书本42,46)

$=-1$

延伸: (1) Find $\frac{dy}{dx}$. $\int_0^{x^3} - \int_0^{x^2}$

(1) $y=\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^4}}$ (2)y=$\int_{\sin x}^{\cos x}[\sin t-\cos(\pi t^2)]\ dt$

$f(x)=\int_1 \sqrt{1+u^4}\,du$

(2) Assume $f(x)$ is continuous and $\int_0^{x^2-1} f(t)\ dt = x$. Compute $f(7)$.

(3) Find $f''(x)$ if $f(x)=\int_0^x[\int_1^{\sin t}\sqrt{1+u^4}]\ du\ dt$.

(4) $f(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $f'(x)\le 0$, $F(x)=\frac{1}{x-a}\int_a^x f(t)\ dt$. Show that $F'(x)\le 0, x\in(a,b)$.

$\frac{1}{x-a}f(x)$

(5) Assume $f(x)$ is continuous on $(-\infty,\infty)$, and $f(x)>0$. Show that $F(x)=\frac{\int_0^x tf(t)\ dt}{\int_0^x f(t)\ dt}$ is increasing on the interval $(0,\infty)$.

$xf(x)\int_0^x f(t)dt - f(x)\int_0^x tf(t)dt$

专题四: 微积分基本定理.

延伸:

(6) Assume $f(x)$ is continuous on $[a,b]$, and $f(x) > 0$. Prove that the equation has only one real root between(a,b):

简单构造

$$\int_a^x f(t)\ dt = 2\int_x^b f(t)\ dt$$

.

$u = F(x) - x$

(7) $f(x)$ is continuous on $[0,1]$, and $0 < f(x) < \frac{1}{2}, \forall x \in [-1,1]$, let $F(x) = \int_{-1}^{x} f(t)\ dt$. $-x$ Show that $\exists c \in (0,1)$ such that $F(c) = c$.

$u' = f(x) - 1 < 0$

专题四. 微积分基本定理.

由…

延伸:

(8) $f(x)$ is continuous, $g(x)=\int_0^x xf(t)\ dt$, $g(1)=1, g'(1)=5$, find $f(1)$.

$x\int_0^x f(t) - \int_0^x f(t)\,t\,dt$

(9) Assume f is continuous on $[a,b]$. Define $F(x)=\int_a^x f(t)(x-t)\ dt$. Prove that F is second order differentiable and $F''(x)=f(x)$ in (a,b).

tt

$\int_a^x f(t)dt + xf(x)$

$-f(x)x$

(10)$f(x)=\int_0^x \cos(x-t)^2\ dt$, find $f'(x)$.

(11) Let $F(x)=\int_0^x tf(x^2-t^2)\ dt$. Compute $F'(x)$.

$x^2-t^2=u$

$du=-2tdt$