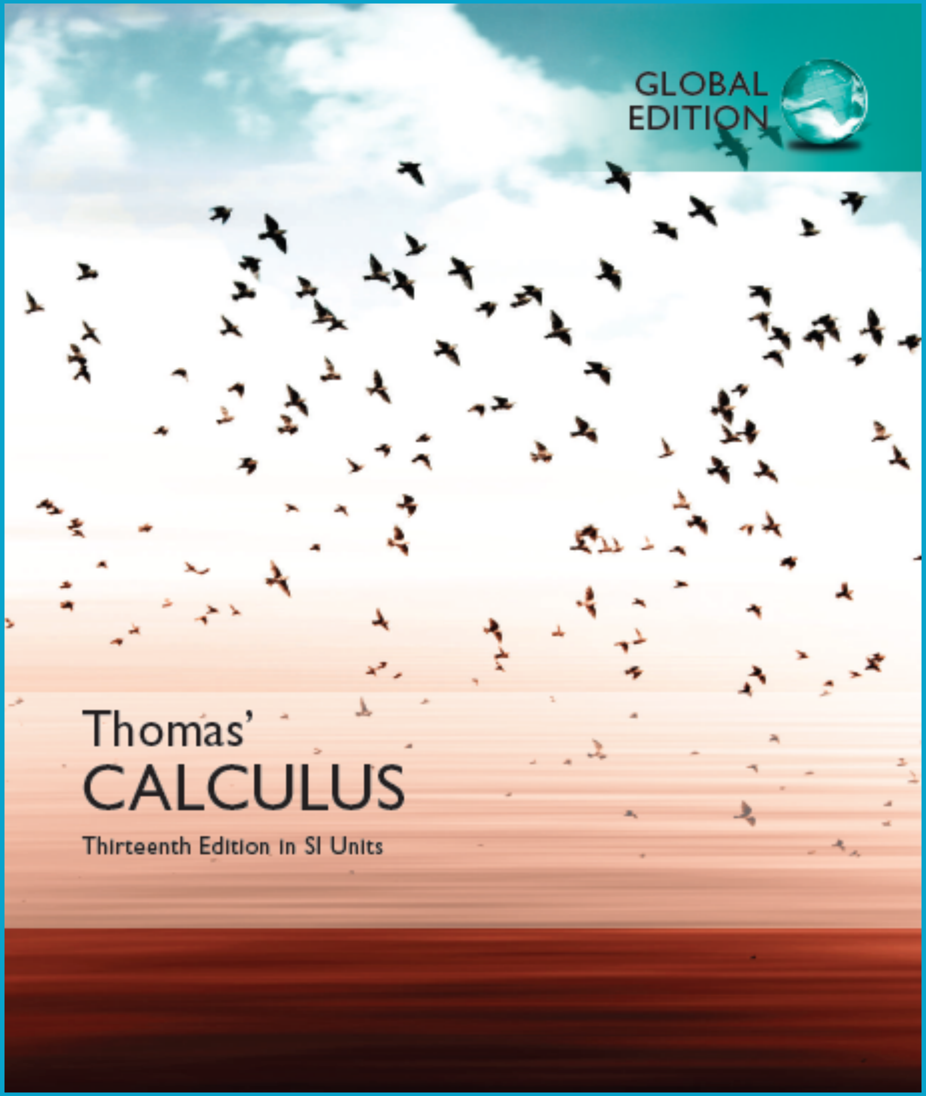

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 9

一阶线性常微分方程

First-Order Differential Equations

9.1

Solutions, Slope Fields, and Euler's Method

General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \qquad \text{and} \qquad \frac{d}{dx}y = f(x, y)$$

are equivalent to Equation (1) and all three forms will be used interchangeably in the text.

$y'' + \sin x \cdot y' + (x^2 + y^2) = C$

$f(y', y, x) = 0$

$((y'))^2 + e^{yy'} + \sin x = 0$

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into Equation (1), the resulting equation is true for all x over the interval I. The **general solution** to a first-order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant, but having this property doesn't mean a solution is the general solution. That is, a solution may contain an arbitrary constant without being the general solution. Establishing that a solution *is* the general solution may require deeper results from the theory of differential equations and is best studied in a more advanced course.

As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

EXAMPLE 2 Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

$y - y' = x$

$e^{-x}(y - y') = x e^{-x}$

$-(e^{-x}y)' = x e^{-x}$

$\Rightarrow -\int (e^{-x}y)'\,dx = \int x e^{-x}\,dx$

$-e^{-x}y = -(x+1)e^{-x} + C$

$y(x) = (x+1) + \bar{C}e^{x}$

怎么猜出来的

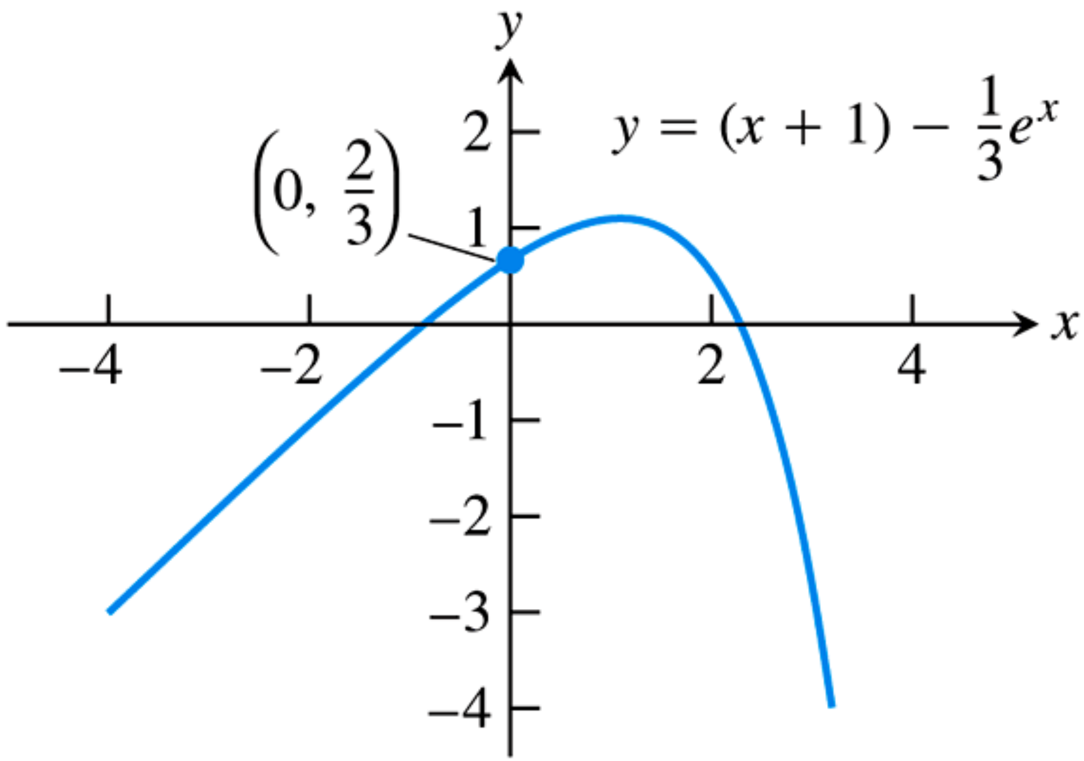
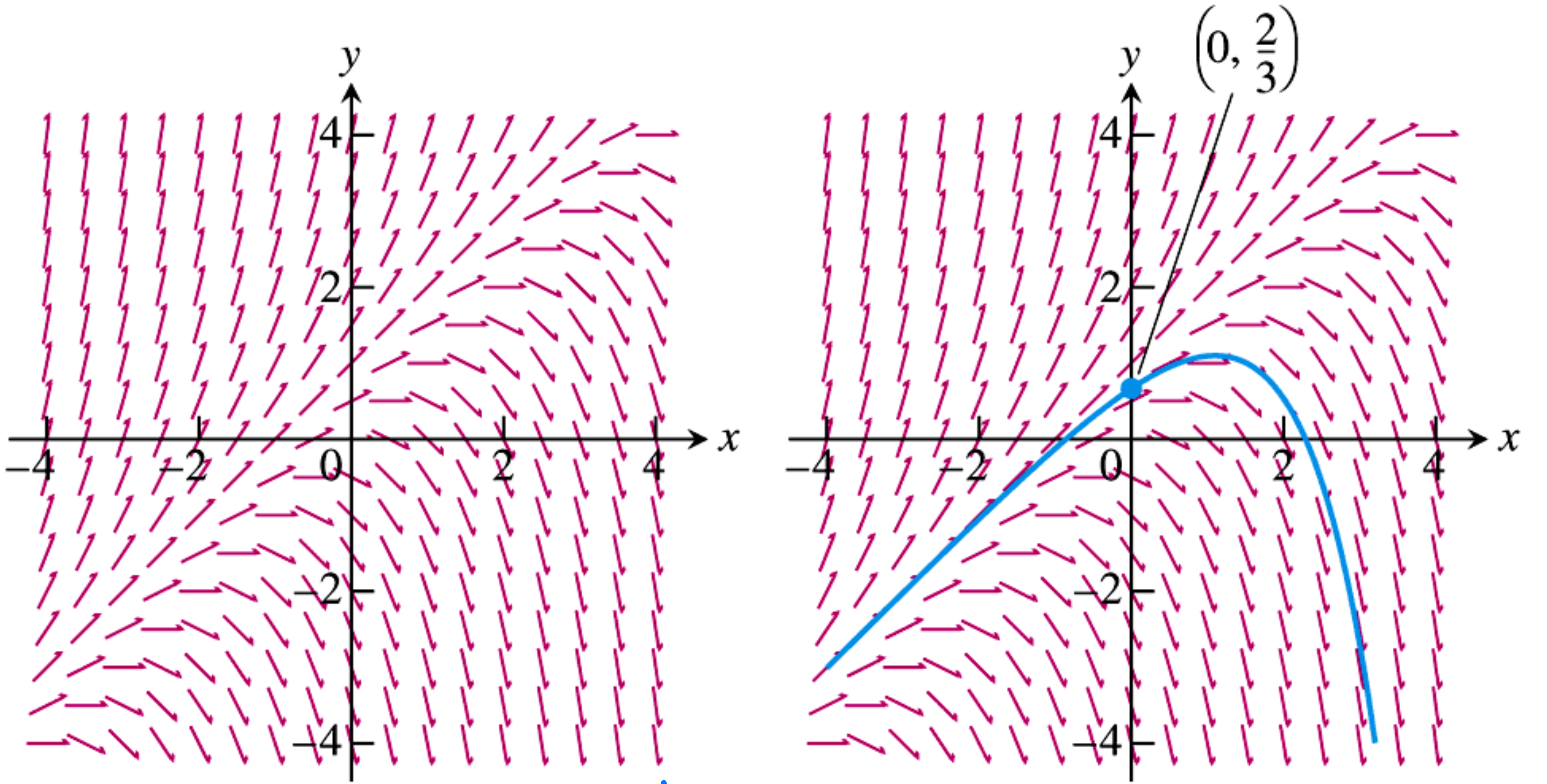


FIGURE 9.1 Graph of the solution to the initial value problem in Example 2.

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves. Figure 9.2a shows a slope field, with a particular solution sketched into it in Figure 9.2b. We see how these line segments indicate the direction the solution curve takes at each point it passes through.



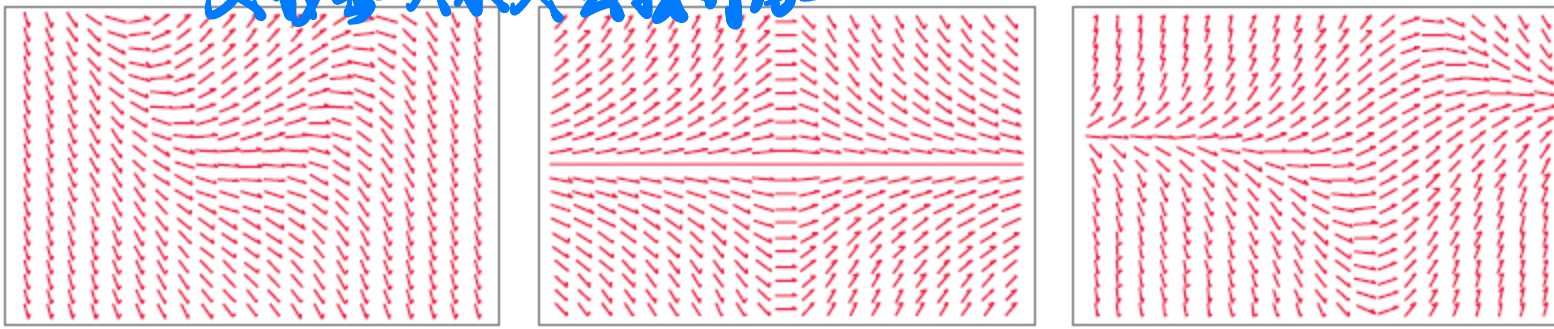
(a) 箭头放右边 (b)

FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

斜率场 — 辅助猜测性质

* 考过，但近几年没考

足够多 从案头会接到后上



(a) $y' = y - x^2$ (b) $y' = -\dfrac{2xy}{1 + x^2}$ (c) $y' = (1 - x)y + \dfrac{x}{2}$

找特殊规律

$x=0$ $y=0 \Rightarrow y'=0$

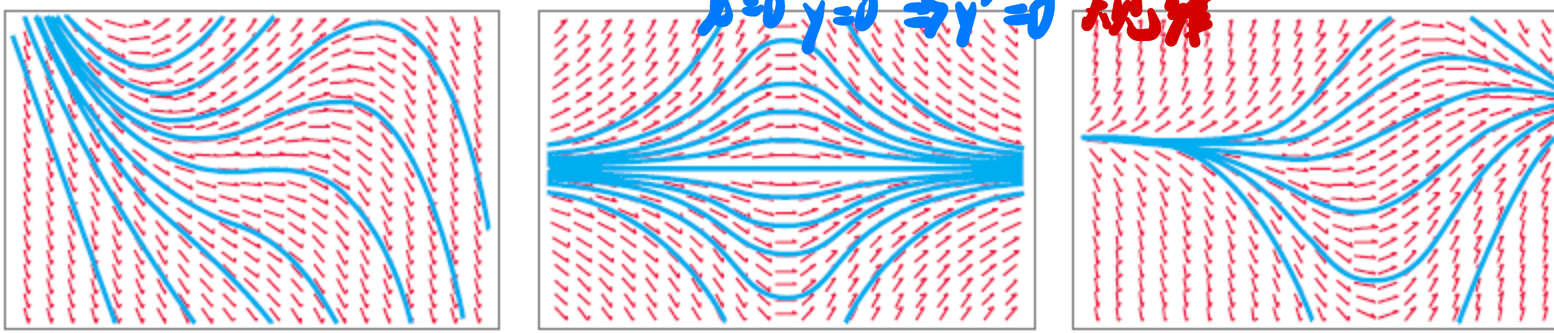
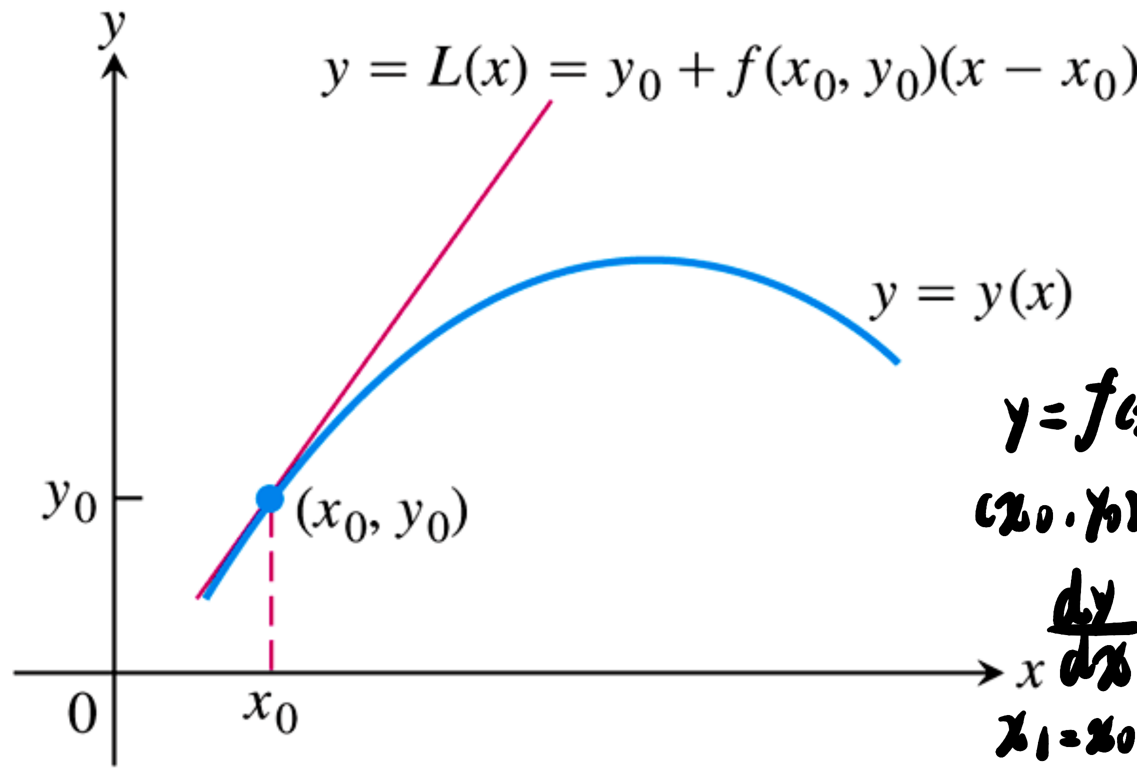


FIGURE 9.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here, but they should be considered as just tangent line segments.



$y = f(x_0, y_0)$

(x_0, y_0)

$\left.\frac{dy}{dx}\right|_{(x_0, y_0)}$

$x_1 = x_0 + \Delta x$

$y(x_0 + \Delta x)$

$\approx \underbrace{y_0 + f(x_0, y_0)\Delta x}_{y_1}$

FIGURE 9.4 The linearization $L(x)$ of $y = y(x)$ at $x = x_0$.

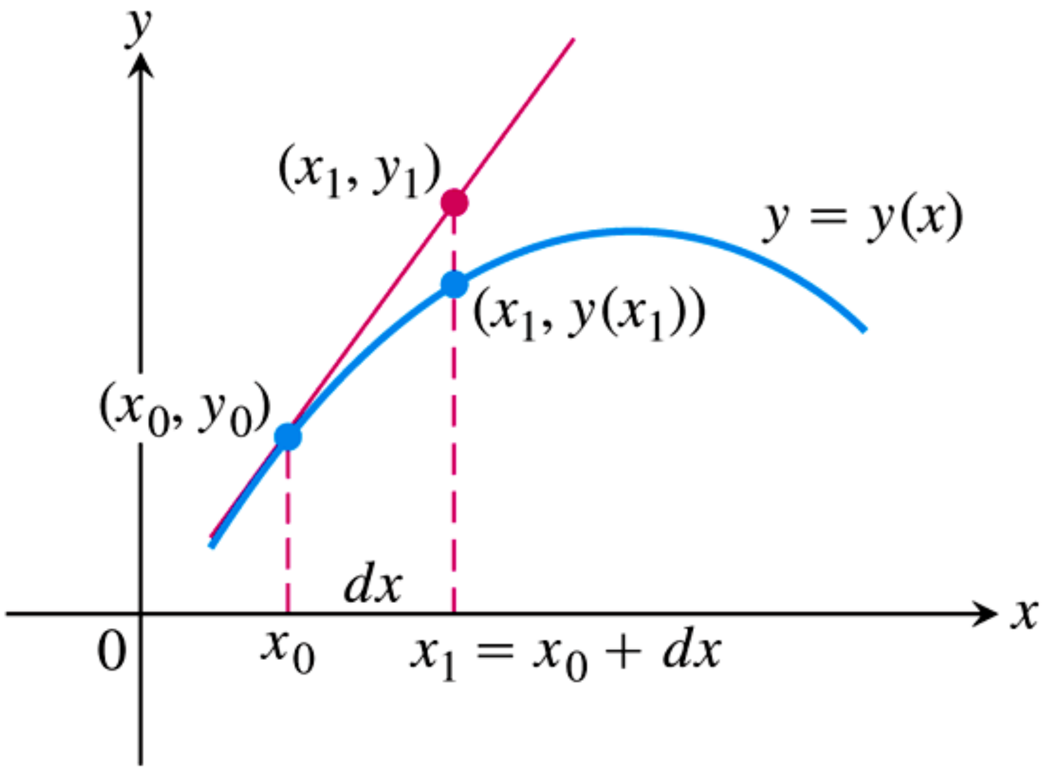


FIGURE 9.5 The first Euler step approximates $y(x_1)$ with $y_1 = L(x_1)$.

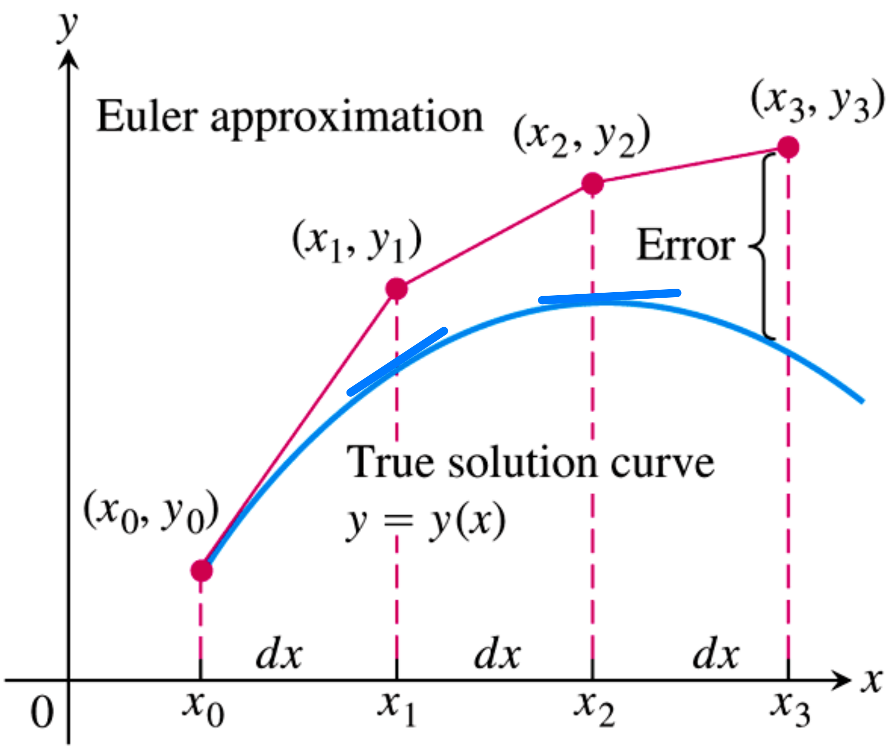


FIGURE 9.6 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$. As we take more steps, the errors involved usually accumulate, but not in the exaggerated way shown here.

$$
\begin{aligned}
x_1 &= x_0 + dx \\
x_2 &= x_1 + dx \\
&\vdots \\
x_n &= x_{n-1} + dx.
\end{aligned}
$$

（递推）

后一个点由前一个点推出

$$
\begin{aligned}
y_1 &= y_0 + f(x_0, y_0)\, dx \\
y_2 &= y_1 + f(x_1, y_1)\, dx \\
&\vdots \\
y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\, dx.
\end{aligned}
$$

不一定准确

$$\frac{y_{n+1} - y_n}{x_{n+1} - x_n} = f(x_n, y_n)$$

割线斜率逼近切线斜率

EXAMPLE 4 Use Euler's method to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking **(a)** $dx = 0.1$ and **(b)** $dx = 0.05$. Compare the approximations with the values of the exact solution $y = 2e^x - 1$.

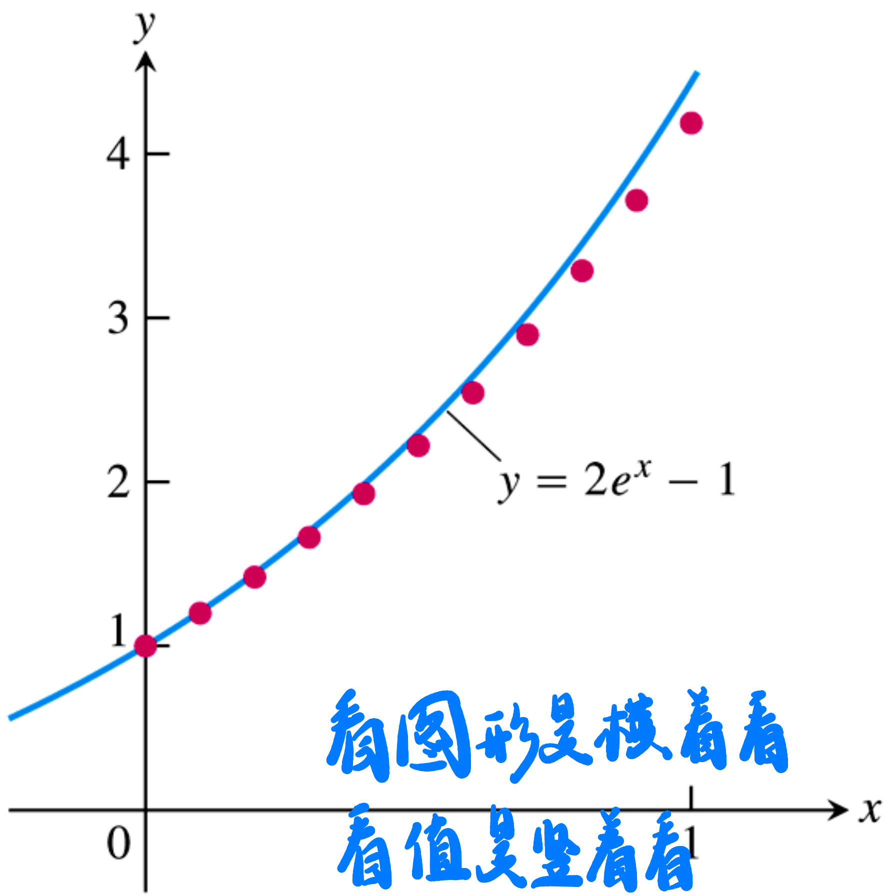


FIGURE 9.7 The graph of $y = 2e^x - 1$ superimposed on a scatterplot of the Euler approximations shown in Table 9.1 (Example 4).

TABLE 9.1 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

TABLE 9.2 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.05$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.05	1.1	1.1025	0.0025
0.10	1.205	1.2103	0.0053
0.15	1.3153	1.3237	0.0084
0.20	1.4310	1.4428	0.0118
0.25	1.5526	1.5681	0.0155
0.30	1.6802	1.6997	0.0195
0.35	1.8142	1.8381	0.0239
0.40	1.9549	1.9836	0.0287
0.45	2.1027	2.1366	0.0340
0.50	2.2578	2.2974	0.0397
0.55	2.4207	2.4665	0.0458
0.60	2.5917	2.6442	0.0525
0.65	2.7713	2.8311	0.0598
0.70	2.9599	3.0275	0.0676
0.75	3.1579	3.2340	0.0761
0.80	3.3657	3.4511	0.0853
0.85	3.5840	3.6793	0.0953
0.90	3.8132	3.9192	0.1060
0.95	4.0539	4.1714	0.1175
1.00	4.3066	4.4366	0.1300

一阶方法
dx（步长）变成$\frac{1}{2}$
误差变成$\frac{1}{2}$

要求：计算2步
（要记得改变）

9.2

First-Order Linear Equations

$y' + \underbrace{P(x)}\, y$

A first-order **linear** differential equation is one that can be written in the form

关于y而言是
线性方程（视x为常数）

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**. Since the exponential growth/decay equation $dy/dx = ky$ (Section 7.2) can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y) because y and its derivative dy/dx occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function $\left(\text{such as } \sin y,\ e^{y}, \text{ or } \sqrt{dy/dx}\right)$.

To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\underbrace{\int P(x)\,dx}_{\text{原函数}}}$ and integrate both sides.

只找一个特殊原函数即可

$y' - y = -x$

$P(x) = -1$

$\int p(x)\,dx = -x$

$e^{-x}(y' - y) = e^{-x}(-x)$

$$y' + P(x)y = Q(x)$$

左右同乘 $v(x)$ 希望左侧是 $(vy)'$

$$v(x)y' + P(x)y\,v(x) = v(x)Q(x)$$

构造出 $(vy)' = v(x)Q(x)$

$v(x)$ $\int (v(x)y)'\,dx = \int v(x)Q(x)\,dx$

$$y = \frac{\int v(x)Q(x)\,dx}{v(x)}$$

$$v'(x)y + v(x)y' = v(x)y' + P(x)v(x)y$$

so $v'(x) = v(x)P(x)$

$$\frac{dv}{dx} = vP$$

$$\int \frac{dv}{v} = \int P\,dx$$

$$\ln|v| = \int P\,dx$$

$v = e^{\int P(x)\,dx}$ 只要用一个特殊的 v

标准做法：
写成 $y' + P(x)y = Q(x)$

EXAMPLE 2 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

$y' = x + \frac{3}{x}y$

$y' - \frac{3}{x}y = x$

$P(x) = -\frac{3}{x}$

$\int P(x)\,dx = -3\ln x + C$

$v(x) = e^{-3\ln x} = x^{-3}$

$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = \frac{1}{x^2}$

$\left(\frac{1}{x^3}y\right)' = \frac{1}{x^2}$

$\frac{1}{x^3}y = \int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$

$y(x) = -x^2 + Cx^3$

EXAMPLE 3 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$,

$y' - \frac{y}{3x} = \frac{\ln x + 1}{3x}$

$P(x) = -\frac{1}{3x}$ $\quad v(x) = e^{\left(-\frac{1}{3}\ln x\right)} = x^{-\frac{1}{3}}$

$x^{-\frac{1}{3}}y' - x^{-\frac{1}{3}}\frac{y}{3x} = \frac{\ln x + 1}{3x}x^{-\frac{1}{3}}$

每次验一下

$\left(x^{-\frac{1}{3}}y\right)' = \frac{\ln x + 1}{3x^{\frac{4}{3}}}$

$\int \frac{1}{3x^{\frac{4}{3}}}(\ln x + 1)\,dx = -\int (\ln x + 1)\,d\left(\frac{1}{x^{\frac{1}{3}}}\right)$

$= -\frac{1}{x^{\frac{1}{3}}}(\ln x + 1) + \int \frac{1}{x^{\frac{1}{3}}}\frac{1}{x}dx$

$y(x) = -(\ln x + 4) + Cx^{\frac{1}{3}}$

$y(1) = -4 + C = -2 \qquad C = 2$

$$L\frac{di}{dt} + Ri = V,$$

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{V}{L}$$

$$e^{\frac{R}{L}t} i' + \frac{R}{L} e^{\frac{R}{L}t} i = \frac{V}{L} e^{\frac{R}{L}t}$$

$$\int (e^{\frac{R}{L}t} i)' dt = \int \frac{V}{L} e^{\frac{R}{L}t} dt$$

$$i' + P(t) i = \frac{V}{L}$$

$$e^{\frac{R}{L}t} i = \frac{V}{R} e^{\frac{R}{L}t} + C$$

$$P(t) = \frac{R}{L}$$

EXAMPLE 4 The switch in the *RL* circuit in Figure 9.8 is closed at time $t = 0$. How will the current flow as a function of time?

i

$$\int P(t)\,dt = \frac{R}{L} t + C$$

$$i(0) = 0 \qquad C = -\frac{V}{R}$$

$$v(t) = e^{\frac{R}{L}t}$$

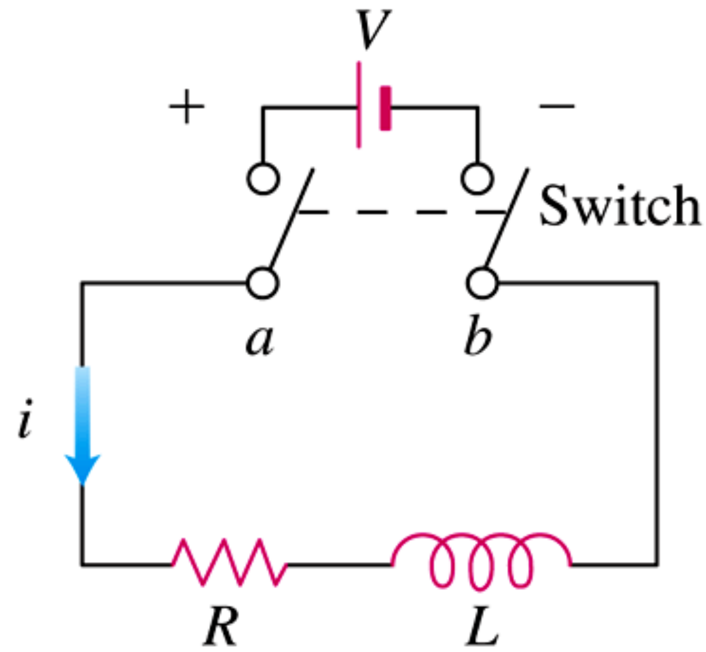


FIGURE 9.8 The *RL* circuit in Example 4.

HISTORICAL BIOGRAPHY

James Bernoulli

(1654–1705)

A **Bernoulli differential equation** is of the form

格式 $y' + P(x)\overset{一次}{y} = Q(x)\overset{n次}{y^n}$

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^{n-1}} = Q(x)$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

转化为 $u' + P(x)u$

$$du = (1-n)\frac{1}{y^n}\,dy$$

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

$$\frac{1}{1-n}\frac{du}{dx} + uP(x) = Q(x)$$

$$\Rightarrow \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

$$y' + 2xy = 3\sqrt{y}$$

$$u = \frac{1}{y^{n-1}}$$

$n = \frac{1}{2}$ 此处n可取任意实数

$$\frac{y'}{\sqrt{y}} + 2x\sqrt{y} = 3 \qquad u = \sqrt{y}$$

$$2u' + 2xu = 3 \qquad u' = \frac{1}{2\sqrt{y}} y'$$

中值定理

★ Suppose $f(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$ and $f(2) = 3f(0)$. Prove that there exists at least one $c \in (0, 2)$ such that $(1 + c)f'(c) = f(c)$.

$$f'(c) - \frac{1}{1+c} f(c) = 0 \text{ 有解}$$

变成一个函数的导数

用积分因子构造函数

$$f'(x) - \frac{1}{1+x} f(x)$$

$$p(x) = -\frac{1}{1+x}$$

$$\int p(x)\,dx = -\ln(1+x) + C$$

$$v(x) = e^{-\ln(1+x)} = \frac{1}{1+x}$$

$$\frac{f'(x)}{1+x} - \frac{1}{(1+x)^2} f(x) = \left(\frac{f(x)}{x+1}\right)'$$

要证该函数在 $(0,2)$ 存在导数 $=0$

$$g(x) = \frac{f(x)}{x+1}$$

$$g(0) = f(0)$$

$$g(2) = f(0)$$

闭区间连续

开区间可导

罗尔定理

$$g'(c) = 0$$

要证明的结论	可考虑的辅助函数
$\xi f'(\xi) + f(\xi) = 0$	$xf(x)$
$\xi f'(\xi) + n\,f(\xi) = 0$	$x^n f(x)$
$\xi f'(\xi) - f(\xi) = 0$	$\frac{f(x)}{x}$
$\xi f'(\xi) - n\,f(\xi) = 0$	$\frac{f(x)}{x^n}$
$f'(\xi) + \lambda\,f(\xi) = 0$	$e^{\lambda x} f(x)$
$f'(\xi) + f(\xi) = 0$	$e^{x} f(x)$
$f'(\xi) - f(\xi) = 0$	$e^{-x} f(x)$

9.3

Applications

In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes. Picture the object as a mass m moving along a coordinate line with position function s and velocity v at time t. From Newton's second law of motion, the resisting force opposing the motion is

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

If the resisting force is proportional to velocity, we have

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

$v' + \frac{k}{m}v = 0$

This is a separable differential equation representing exponential change. The solution to the equation with initial condition $v = v_0$ at $t = 0$ is (Section 7.4)

$ma = -kv$

$mv' = -kv$

$$v = v_0 e^{-(k/m)t}. \qquad (1)$$

指数衰减

$v = v_0 e^{-\frac{k}{m}t}$

Suppose that an object is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Equation (1) and solve the initial value problem

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right). \tag{2}$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned}\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)\\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.\end{aligned}$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{3}$$

EXAMPLE 1 For a 90-kg ice skater, the k in Equation (1) is about 5 kg/s. How long will it take the skater to coast from 3.3 m/s (11.88 km/h) to 0.3 m/s? How far will the skater coast before coming to a complete stop?

$V_0 = 3.3$

$0.3 = 3.3\, e^{-\frac{1}{18}t}$

Suppose a chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process, it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula

$$\begin{array}{c}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{array} = \left(\begin{array}{c}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{array}\right) - \left(\begin{array}{c}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{array}\right). \tag{6}$$

$$\frac{dy}{dx} = \text{rate in} - \text{rate out}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)} \cdot (\text{outflow rate}) \\ &= \begin{pmatrix}\text{concentration in} \\ \text{container at time } t\end{pmatrix} \cdot (\text{outflow rate}).\end{aligned} \tag{7}$$

Accordingly, Equation (6) becomes

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}). \tag{8}$$

EXAMPLE 3 In an oil refinery, a storage tank contains 10,000 L of gasoline that initially has 50 kg of an additive dissolved in it. In preparation for winter weather, gasoline containing 0.2 kg of additive per liter is pumped into the tank at a rate of 200 L/min. The well-mixed solution is pumped out at a rate of 220 L/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 9.14)?

Ans = 675

$$\frac{dy}{dt}$$

$$\text{rate in} = 200 \times 0.2 = 40 \text{ kg/min}$$

$$\text{rate out} = 220 \text{ L/min} \cdot \frac{y}{10000 - 20t} \text{ kg/L}$$

$$= \frac{220y}{1000 - 20t} \text{ kg/min}$$

$y \to$ 盐总量

$1000 - 20t \to$ 水总体积

$$y' + \frac{11}{500 - t} y = 40$$

$$P(t) = \frac{11}{500 - t}$$

$$\int P(t)\,dt = -11 \ln(500 - t) + C$$

$$v(t) = (500 - t)^{-11}$$

$$[(500 - t)^{-11} y]' = 40(500 - t)^{-11}$$

$$(500 - t)^{-11} y = -4(500 - t)^{-10} + C$$

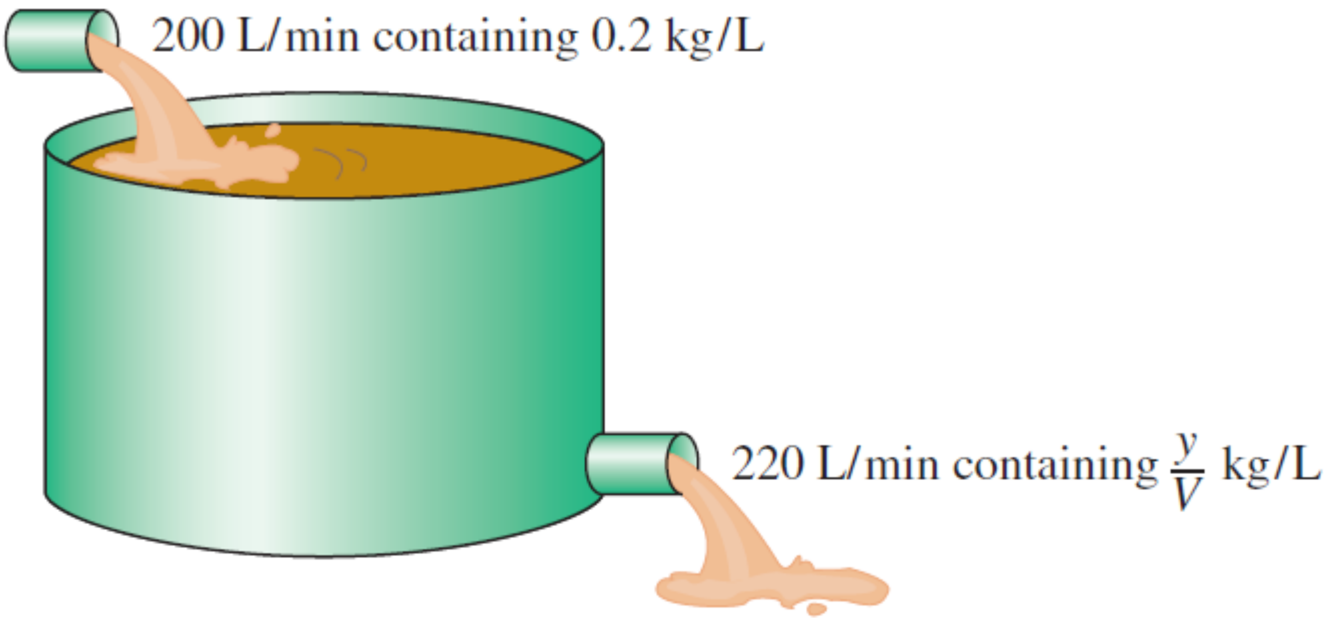


FIGURE 9.14 The storage tank in Example 3 mixes input liquid with stored liquid to produce an output liquid.

9.4

Graphical Solutions of Autonomous Equations

DEFINITION If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

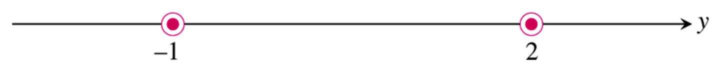
EXAMPLE 1 Draw a phase line for the equation

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

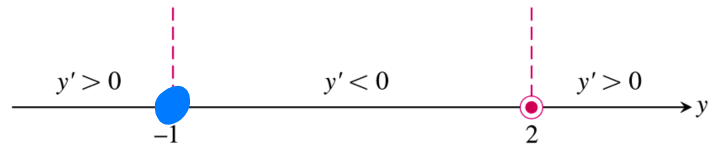
and use it to sketch solutions to the equation.

Solution

1. *Draw a number line for y and mark the equilibrium values* $y = -1$ *and* $y = 2$, *where* $dy/dx = 0$.



2. *Identify and label the intervals where* $y' > 0$ *and* $y' < 0$. This step resembles what we did in Section 4.3, only now we are marking the y-axis instead of the x-axis.



由y 一阶导可知符号

画图：y轴

We can encapsulate the information about the sign of y' on the phase line itself. Since $y' > 0$ on the interval to the left of $y = -1$, a solution of the differential equation with a y-value less than -1 will increase from there toward $y = -1$. We display this information by drawing an arrow on the interval pointing to -1.

Similarly, $y' < 0$ between $y = -1$ and $y = 2$, so any solution with a value in this interval will decrease toward $y = -1$.

For $y > 2$, we have $y' > 0$, so a solution with a y-value greater than 2 will increase from there without bound.

In short, solution curves below the horizontal line $y = -1$ in the xy-plane rise toward $y = -1$. Solution curves between the lines $y = -1$ and $y = 2$ fall away from $y = 2$ toward $y = -1$. Solution curves above $y = 2$ rise away from $y = 2$ and keep going up.

3. *Calculate y'' and mark the intervals where $y'' > 0$ and $y'' < 0$.* To find y'', we differentiate y' *with respect to* x, using implicit differentiation.

$$y' = (y + 1)(y - 2) = y^2 - y - 2 \qquad \text{Formula for } y' \ldots$$

$$\begin{aligned} y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - y - 2) \\ &= 2yy' - y' \qquad \text{differentiated implicitly with respect to } x \\ &= (2y - 1)y' \\ &= (2y - 1)(y + 1)(y - 2). \end{aligned}$$

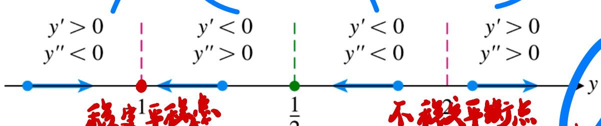
复合函数求导

对x

*之前考过

$\frac{1}{2}$ -1 2

From this formula, we see that y'' changes sign at $y = -1$, $y = 1/2$, and $y = 2$. We add the sign information to the phase line. *



稳定平衡点

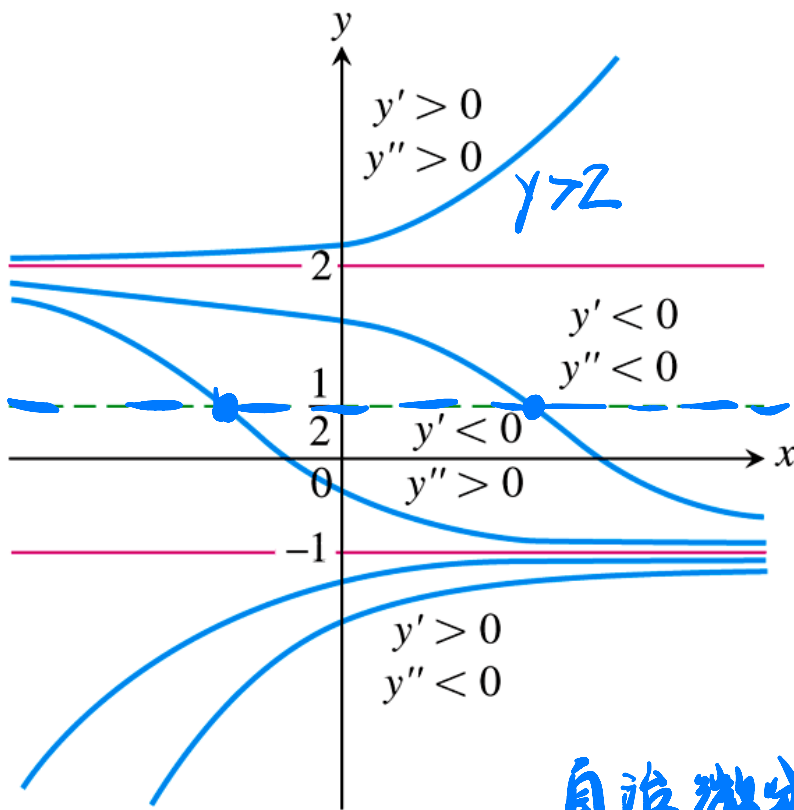
不稳定平衡点

箭头要标——由y'决定

4. *Sketch an assortment of solution curves in the xy-plane.* The horizontal lines $y = -1$, $y = 1/2$, and $y = 2$ partition the plane into horizontal bands in which we know the signs of y' and y''. In each band, this information tells us whether the solution curves rise or fall and how they bend as x increases (Figure 9.15).

The "equilibrium lines" $y = -1$ and $y = 2$ are also solution curves. (The constant functions $y = -1$ and $y = 2$ satisfy the differential equation.) Solution curves that cross the line $y = 1/2$ have an inflection point there. The concavity changes from concave down (above the line) to concave up (below the line).

As predicted in Step 2, solutions in the middle and lower bands approach the equilibrium value $y = -1$ as x increases. Solutions in the upper band rise steadily away from the value $y = 2$.



每个区域
都有一条线就够了

自治微分方程画图

FIGURE 9.15 Graphical solutions from Example 1 include the horizontal lines $y = -1$ and $y = 2$ through the equilibrium values. No two solution curves can ever cross or touch each other.

Stable and Unstable Equilibria

Look at Figure 9.15 once more, in particular at the behavior of the solution curves near the equilibrium values. Once a solution curve has a value near $y = -1$, it tends steadily toward that value; $y = -1$ is a **stable equilibrium**. The behavior near $y = 2$ is just the opposite: All solutions except the equilibrium solution $y = 2$ itself move *away* from it as x increases. We call $y = 2$ an **unstable equilibrium**. If the solution is *at* that value, it stays, but if it is off by any amount, no matter how small, it moves away. (Sometimes an equilibrium value is unstable because a solution moves away from it only on one side of the point.)

In free fall, the constant acceleration due to gravity is denoted by g and the one force propelling the body downward is

$$F_p = mg,$$

the force due to gravity. If, however, we think of a real body falling through the air—say, a penny from a great height or a parachutist from an even greater height—we know that at some point air resistance is a factor in the speed of the fall. A more realistic model of free fall would include air resistance, shown as a force F_r in the schematic diagram in Figure 9.19.

For low speeds well below the speed of sound, physical experiments have shown that F_r is approximately proportional to the body's velocity. The net force on the falling body is therefore

$$F = F_p - F_r,$$

giving

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = g - \frac{k}{m}v. \qquad (4)$$

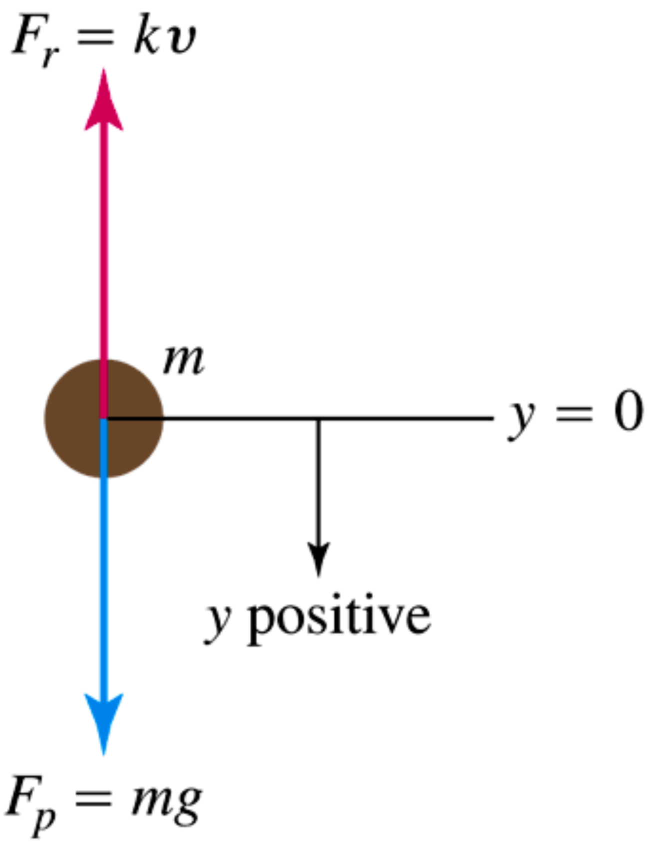
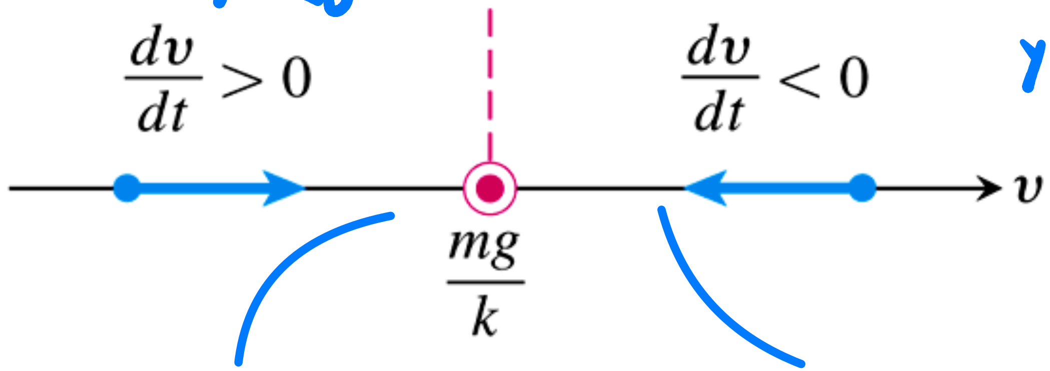


FIGURE 9.19 An object falling under the influence of gravity with a resistive force assumed to be proportional to the velocity.

$$y' = g - \frac{k}{m} y$$

没有t → 自治微分方程

$y'' < 0$ $\quad$ $y'' > 0$



$$y'' = -\frac{k}{m} y'$$
$$= -\frac{k}{m}\left(g - \frac{k}{m} y\right)$$
$$= -\frac{k}{m} g + \frac{k^2}{m^2} y$$

FIGURE 9.20 Initial phase line for the falling body encountering resistance.

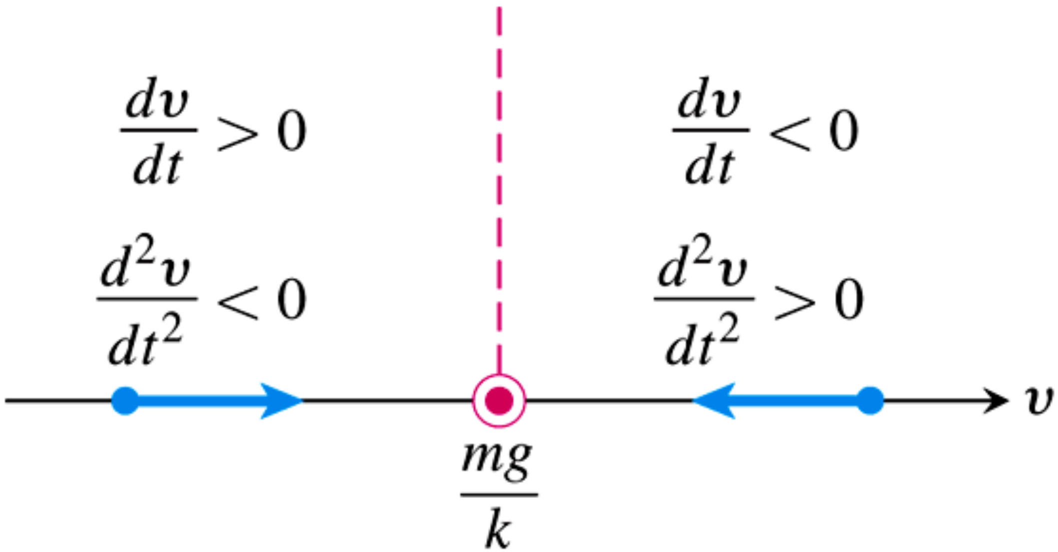


FIGURE 9.21 The completed phase line for the falling body.

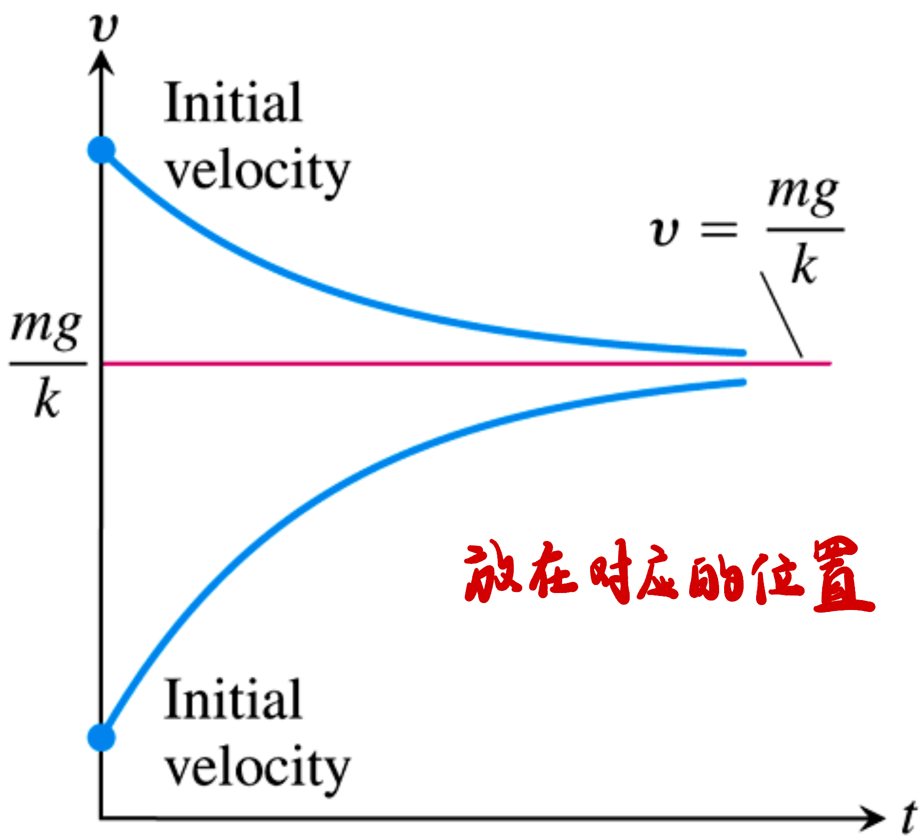


FIGURE 9.22 Typical velocity curves for a falling body encountering resistance. The value $v = mg/k$ is the terminal velocity.

Logistic Population Growth

In Section 9.3 we examined population growth using the model of exponential change. That is, if P represents the number of individuals and we neglect departures and arrivals, then

$$\frac{dP}{dt} = kP, \tag{5}$$

where $k > 0$ is the birth rate minus the death rate per individual per unit time.

Because the natural environment has only a limited number of resources to sustain life, it is reasonable to assume that only a maximum population M can be accommodated.

理想状态平衡数

As the population approaches this **limiting population** or **carrying capacity**, resources become less abundant and the growth rate k decreases. A simple relationship exhibiting this behavior is

$$k = r(M - P),$$

where $r > 0$ is a constant. Notice that k decreases as P increases toward M and that k is negative if P is greater than M. Substituting $r(M - P)$ for k in Equation (5) gives the differential equation

$$\frac{dP}{dt} = r(M - P)P = rMP - rP^2. \tag{6}$$

The model given by Equation (6) is referred to as **logistic growth**.

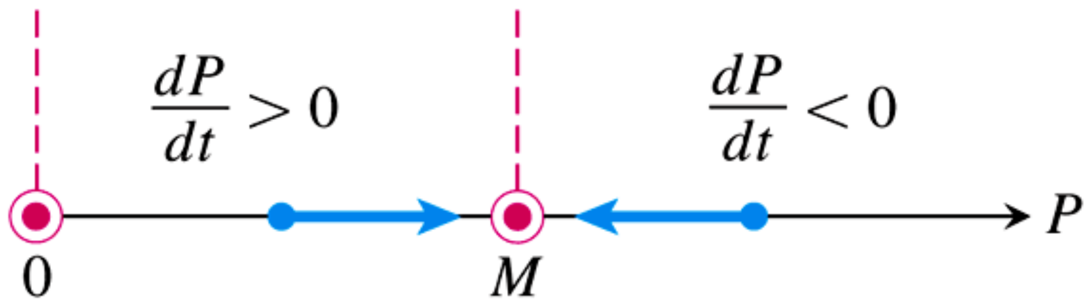


FIGURE 9.23 The initial phase line for logistic growth (Equation 6).

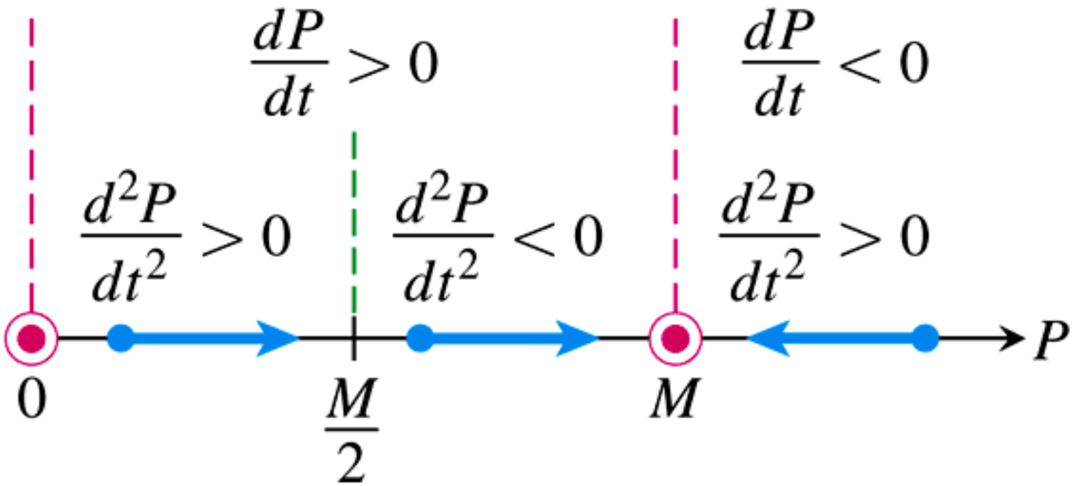


FIGURE 9.24 The completed phase line for logistic growth (Equation 6).

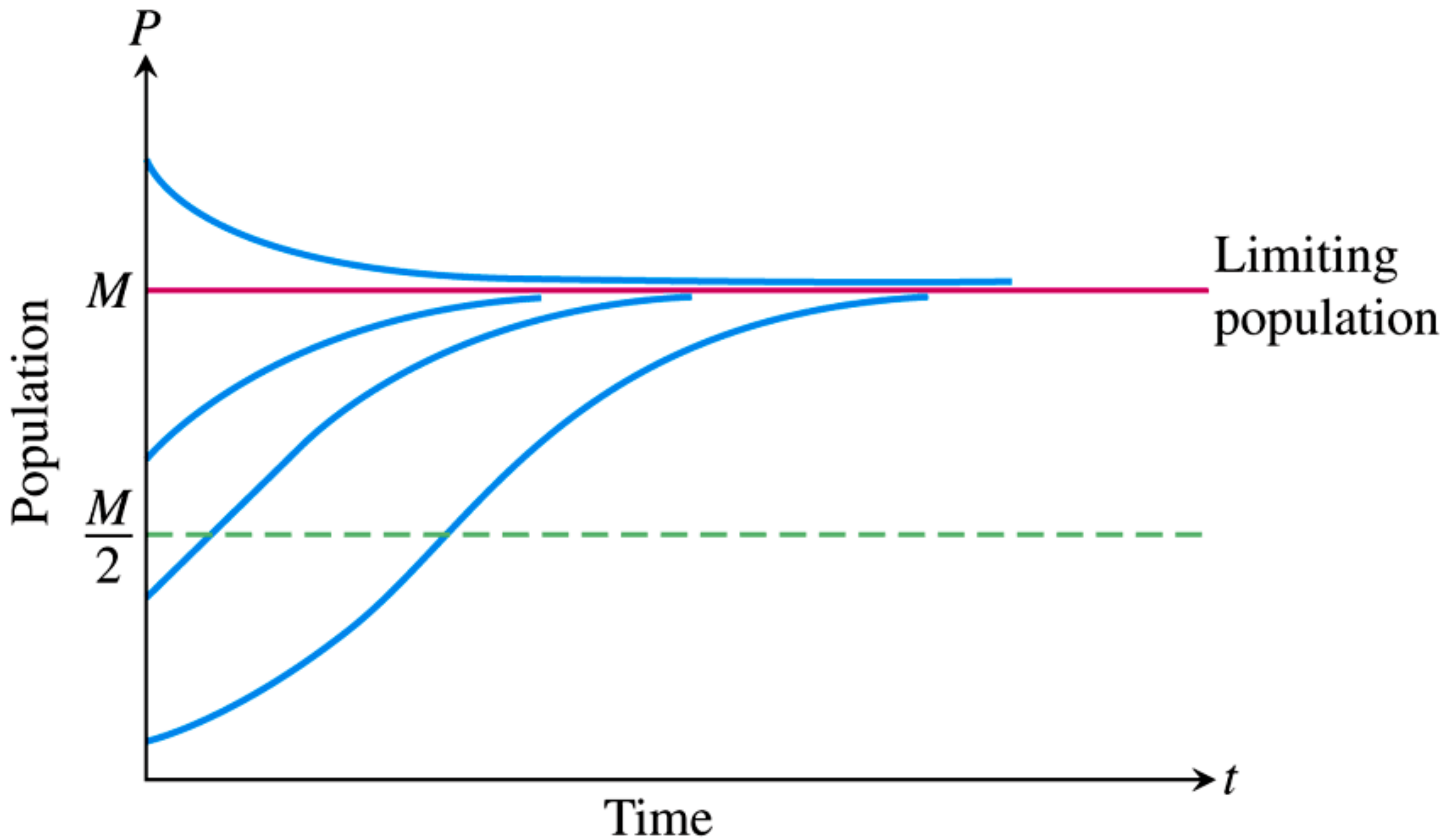


FIGURE 9.25 Population curves for logistic growth.

9.5

Systems of Equations and Phase Planes

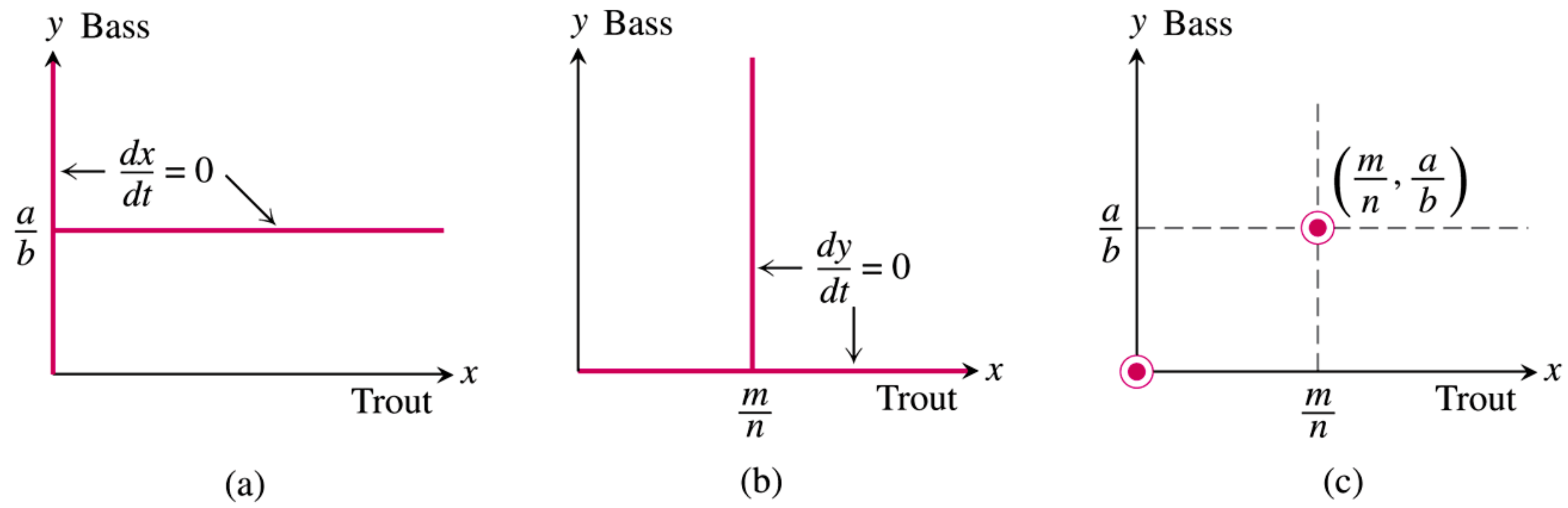


FIGURE 9.26 Rest points in the competitive-hunter model given by Equations (1a) and (1b).

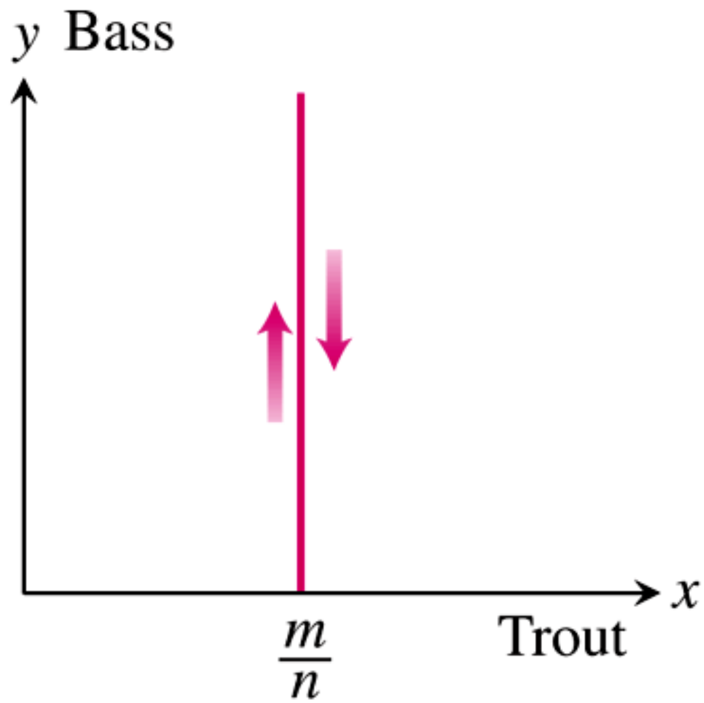


FIGURE 9.27 To the left of the line $x = m/n$ the trajectories move upward, and to the right they move downward.

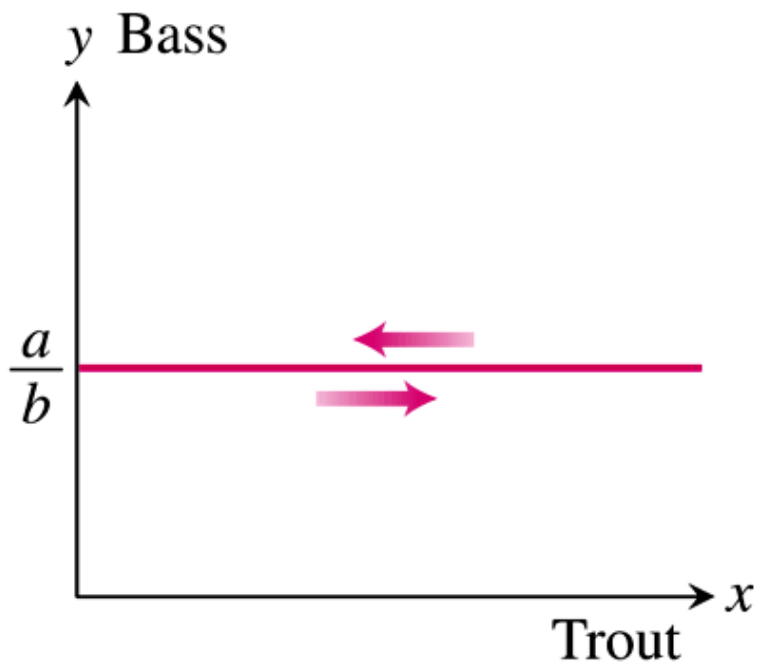


FIGURE 9.28 Above the line $y = a/b$ the trajectories move to the left, and below it they move to the right.

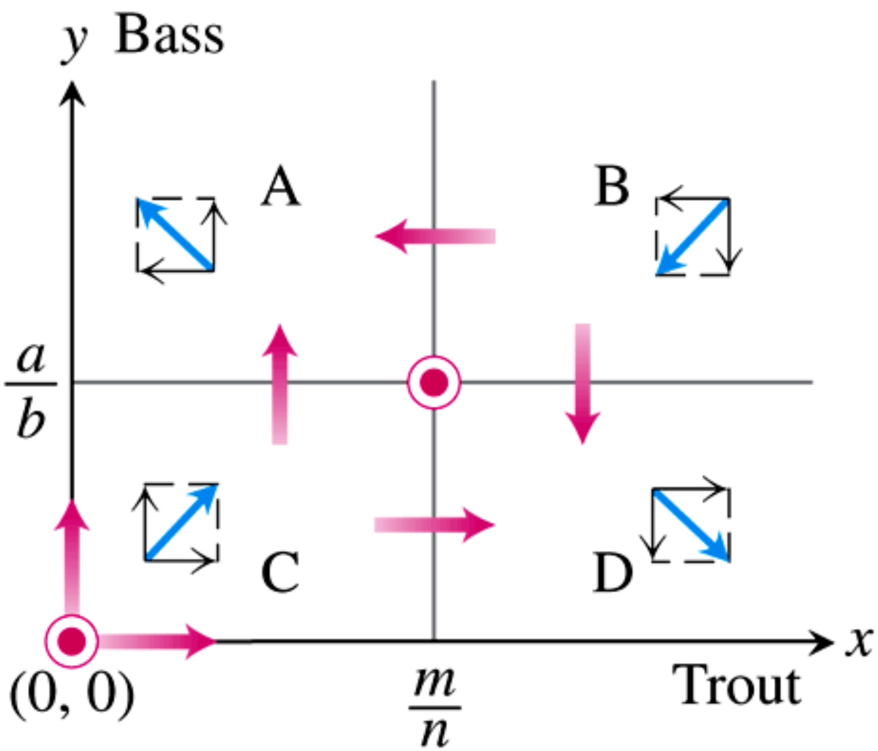


FIGURE 9.29 Composite graphical analysis of the trajectory directions in the four regions determined by $x = m/n$ and $y = a/b$.

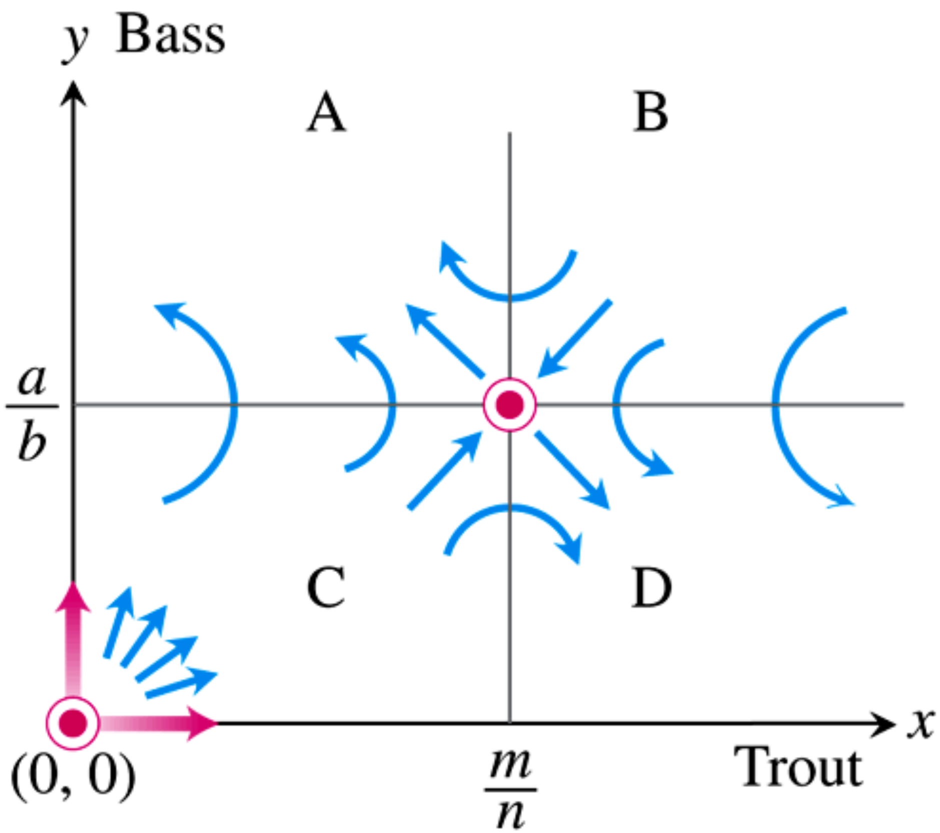


FIGURE 9.30 Motion along the trajectories near the rest points (0, 0) and $(m/n, a/b)$.

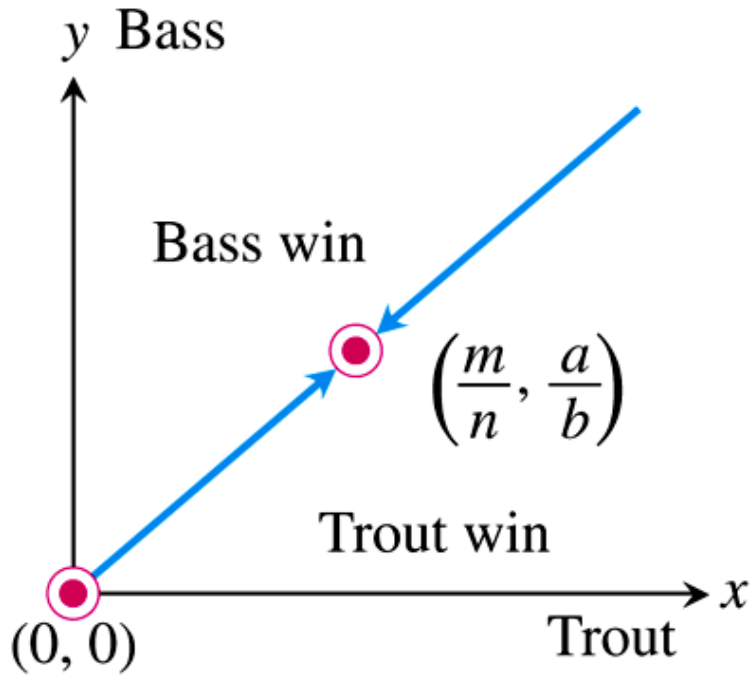


FIGURE 9.31 Qualitative results of analyzing the competitive-hunter model. There are exactly two trajectories approaching the point $(m/n, a/b)$.

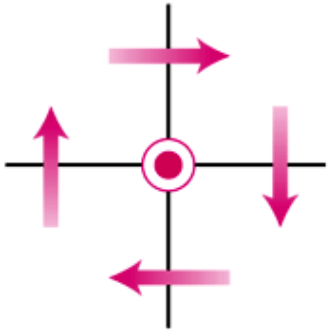


FIGURE 9.32 Trajectory direction near the rest point (0, 0).

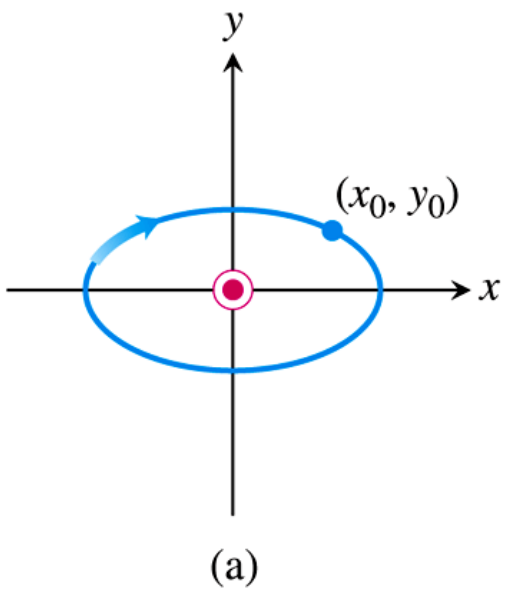


FIGURE 9.33 Three possible trajectory motions: (a) periodic motion, (b) motion toward an asymptotically stable rest point, and (c) motion near an unstable rest point.

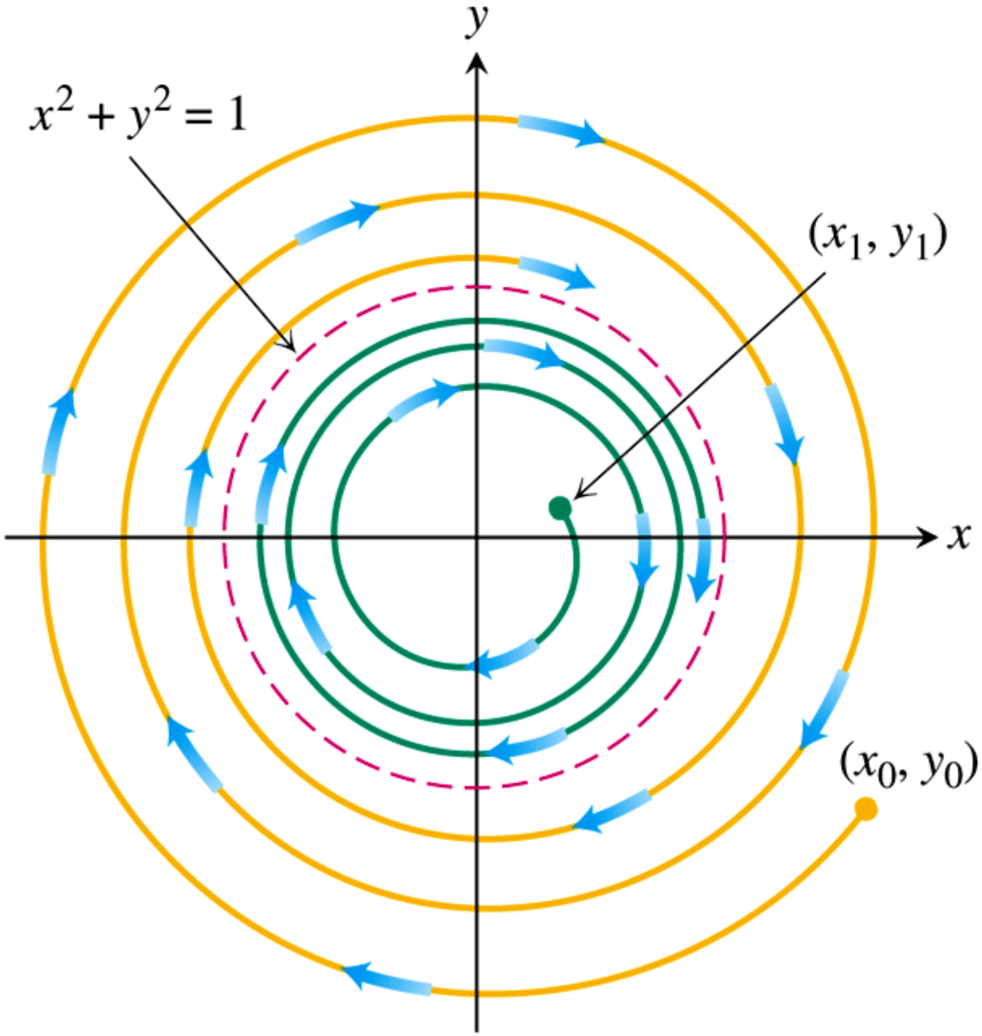


FIGURE 9.34 The solution $x^2 + y^2 = 1$ is a limit cycle.