Thomas Calculus | Chapter 5 Integrals

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自我介绍

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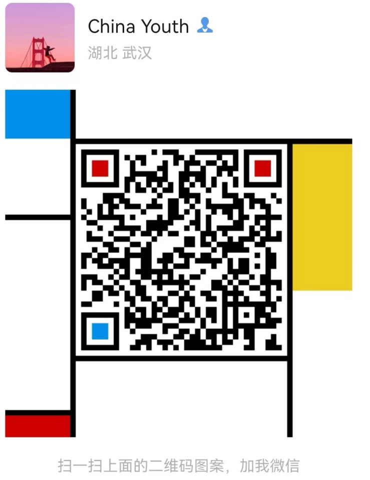


目录

- 基础知识回顾
- 提升训练（习题课，补充题，《同济》高数）

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i, j, and k are customary.

The index k ends at $k = n$.

The summation symbol (Greek letter sigma) $\qquad \displaystyle\sum_{k=1}^{n} a_k \qquad$ a_k is a formula for the kth term.

The index k starts at $k = 1$.

The first n squares: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

The first n cubes: $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Algebra Rules for Finite Sums

1. *Sum Rule:* $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. *Difference Rule:* $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. *Constant Multiple Rule:* $\displaystyle\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)

4. *Constant Value Rule:* $\displaystyle\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

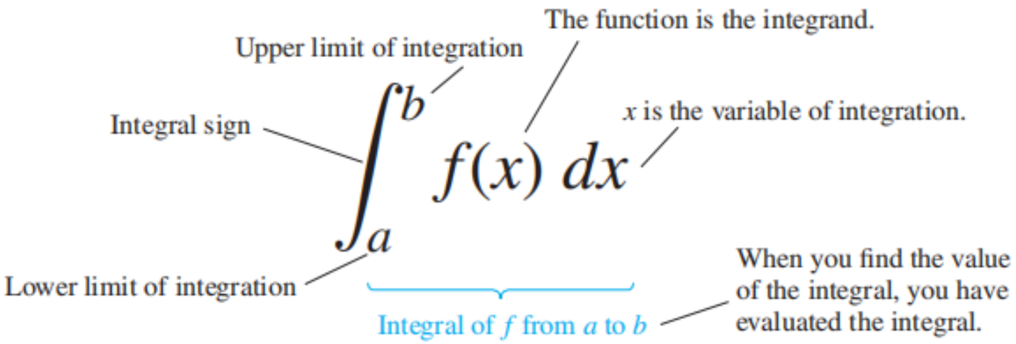
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon.$$

The symbol for the number J in the definition of the **definite integral** is

$$\int_a^b f(x)\,dx,$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



注意:

（1）积分值仅与被积函数及积分区间有关，

而与积分变量的字母无关. **dummy variable**

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

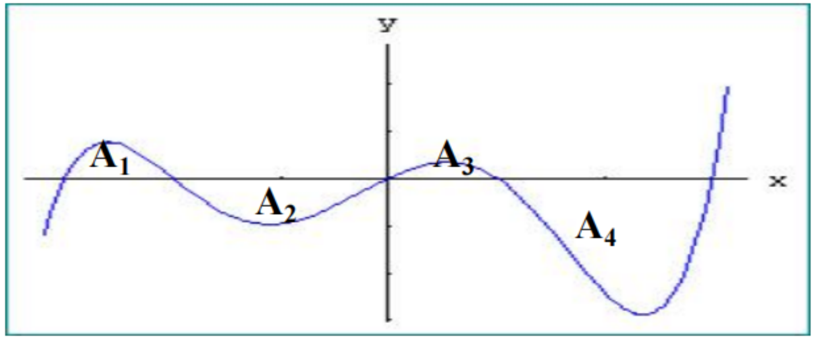
（2）定义中区间的分法和c_i的取法是任意的.

THEOREM 1—Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

定积分的几何意义

$f(x) > 0,\quad \int_a^b f(x)dx = A$ 曲边梯形的面积

$f(x) < 0,\quad \int_a^b f(x)dx = -A$ 曲边梯形的面积的负值



$$\int_a^b f(x)dx = A_1 - A_2 + A_3 - A_4$$

EXAMPLE 4 Compute $\int_0^b x\,dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

$$\begin{aligned}
\sum_{k=1}^{n} f(c_k)\,\Delta x &= \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n} && f(c_k) = c_k \\
&= \sum_{k=1}^{n} \frac{kb^2}{n^2} \\
&= \frac{b^2}{n^2}\sum_{k=1}^{n} k && \text{Constant Multiple Rule} \\
&= \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2} && \text{Sum of First } n \text{ Integers} \\
&= \frac{b^2}{2}\left(1 + \frac{1}{n}\right).
\end{aligned}$$

As $n \to \infty$ and $\|P\| \to 0$, this last expression on the right has the limit $b^2/2$. Therefore,

$$\int_0^b x\,dx = \frac{b^2}{2}.$$

Ex. 3 **Show that the integral on the interval [0,1] does not exist for the function**

Solution

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

$$\int_0^1 f(x)dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$= \begin{cases} \lim\limits_{\|P\|\to 0} \sum\limits_{i=1}^{n} 1\Delta x_i = 1, & c_i \text{ chosen rational}, \\ \lim\limits_{\|P\|\to 0} \sum\limits_{i=1}^{n} 0\Delta x_i = 0, & c_i \text{ chosen irrational}, \end{cases}$$

Since the limit depends on tchoices of c_k, the function f is not integrable.

THEOREM 2 When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules in Table 5.6.

1. *Order of Integration:* $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\displaystyle\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7. *Domination:* $\displaystyle f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$$f(x) \ge 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0 \quad \text{(Special case)}$$

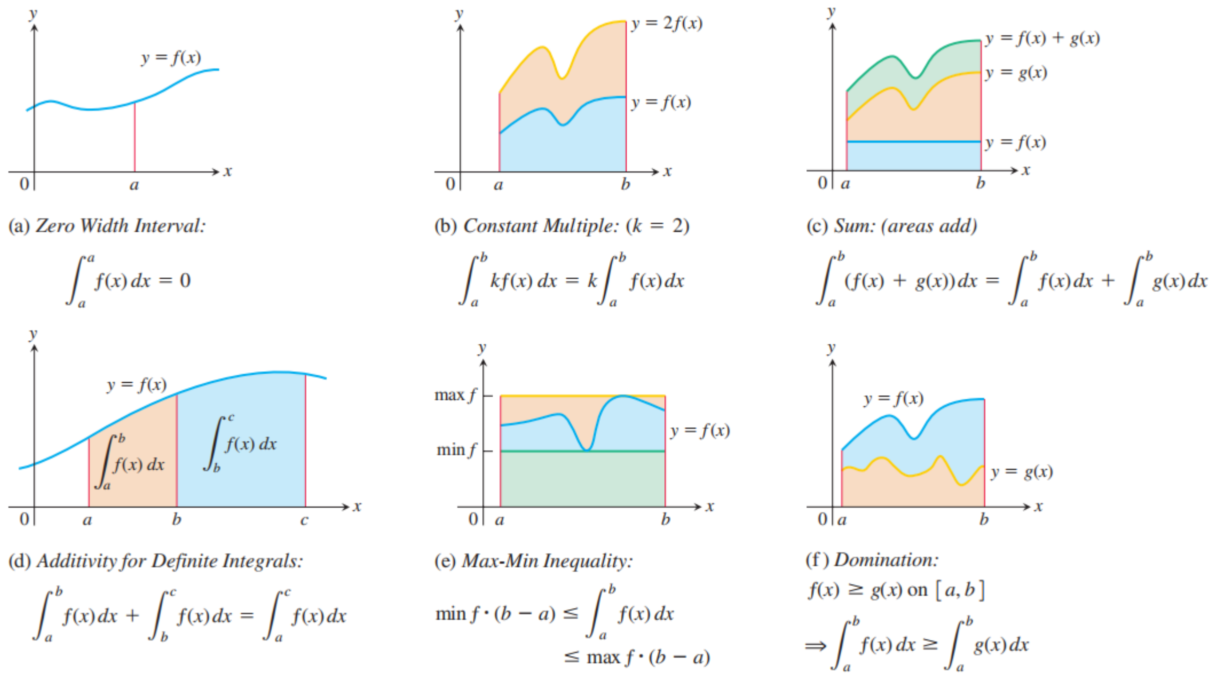


FIGURE 5.11 Geometric interpretations of Rules 2–7 in Table 5.6.

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof If we divide both sides of the Max-Min Inequality (Table 5.6, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \max f.$$

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2 If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

THEOREM 5—The Net Change Theorem The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx. \tag{6}$$

$Ex.$ 设$y = y(x)$是由方程$x - \int_1^{x+y} e^{-t^2}\,dt = 0$所确定的隐函数，求 $\left.\frac{dy}{dx}\right|_{x=0}$.

$$1 - e^{-(x+y)^2}(1+y') = 0,$$

在方程$x - \int_1^{x+y} e^{-t^2}\,dt = 0$中令$x = 0$, 得$\int_1^y e^{-t^2}\,dt = 0$, 显然$y(0) = 1$,

将$x = 0$带入方程 $1 - e^{-(x+y)^2}(1+y') = 0$, 得$y'(0) = e - 1$.

$$Ex.\text{求}\lim_{x\to 0}\frac{\int_0^x e^{-t^2}\,dt}{x} = \lim_{x\to 0}\frac{e^{-c^2}x}{x} = \lim_{x\to 0} e^{-c^2} = 1.$$

Ex.8 Evaluate $\int_0^{\frac{\pi}{2}} |\frac{1}{2} - \sin x|\, dx$

Solute $\int_0^{\frac{\pi}{2}} |\frac{1}{2} - \sin x|\, dx = \int_0^{\frac{\pi}{6}} (\frac{1}{2} - \sin x)dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin x - \frac{1}{2})dx$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \quad - \frac{\pi}{6} + \frac{\sqrt{3}}{2} \qquad = \sqrt{3} - 1 - \frac{\pi}{12}.$$

Ex.9 Evaluate $\lim\limits_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} (p \neq -1)$

Solute $\lim\limits_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} = \lim\limits_{n\to\infty} ((\frac{1}{n})^p + (\frac{2}{n})^p + \cdots + (\frac{n}{n})^p) \frac{1}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{i}{n})^p \frac{1}{n} \quad = \int_0^1 x^p dx = \frac{1}{p+1}$$

Express the limit as a definite integral, and evaluate the resulting definite integral.

(1) $\lim_{n\to\infty} \frac{1^5+2^5+3^5+\cdots+n^5}{n^6}$

(2) $\lim_{n\to\infty} \frac{1}{n^2}(2+4+6+\cdots+2n)$

(3) $\lim_{n\to\infty} \frac{1}{n}(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{n\pi}{n})$

(4) $\lim_{n\to\infty} \frac{1^2+3^2+\cdots+(2n-1)^2}{n^3}$

(5) $\lim_{n\to\infty} \frac{1}{n}(\sqrt{1+\cos\frac{\pi}{n}}+\sqrt{1+\cos\frac{2\pi}{n}}+\cdots+\sqrt{1+\cos\frac{n\pi}{n}})$

(1) If $f(x) > 0, f'(x) < 0, f''(x) > 0, x \in [a,b]$, let $S_1 = \int_a^b f(x)\,dx$, $S_2 = f(b)(b-a), S_3 = \frac{1}{2}[f(a)+f(b)](b-a)$, then ( )

(A) $S_1 < S_2 < S_3$ (B) $S_2 < S_1 < S_3$

(C) $S_3 < S_1 < S_2$ (D) $S_2 < S_3 < S_1$

(2) Let $I_1 = \int_0^{\frac{\pi}{4}} \frac{\tan x}{x}\,dx, I_2 = \int_0^{\frac{\pi}{4}} \frac{x}{\tan x}\,dx$, then ( )

(A) $I_1 > I_2 > 1$ (B) $1 > I_1 > I_2$

(C) $I_2 > I_1 > 1$ (D) $1 > I_2 > I_1$

(3) Let $I_k = \int_0^{k\pi} e^{x^2} \sin x\,dx, (k = 1,2,3)$, then ( )

(A) $I_1 < I_2 < I_3$ (B) $I_3 < I_2 < I_1$

(C) $I_2 < I_3 < I_1$ (D) $I_2 < I_1 < I_3$

(1) Find $\frac{dy}{dx}$.

(i) $y = \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}}$ (ii)y=$\int_{\sin x}^{\cos x} [xt - \cos(\pi t^2)]\ dt$

(2) Find $f''(x)$ if $f(x) = \int_0^x [\int_1^{\sin t} \sqrt{1+u^4}]\ du\ dt$.

(3) $f(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $f'(x) \le 0$, $F(x) = \frac{1}{x-a} \int_a^x f(t)\ dt$. Show that $F'(x) \le 0, x \in (a,b)$.

(4) $f(x)$ is continuous on $[0,1]$, and $0 < f(x) < \frac{1}{2}, \forall x \in [-1,1]$, Let $F(x) = \int_{-1}^x f(t)\ dt$. Show that $\exists c \in (0,1)$ such that $F(c) = c$.

(5) Let $F(x) = \int_x^{2x} \frac{1}{\sqrt{1+t^2}}\ dt$. (a) Show that $F(x)$ is increasing. (b) Find $\lim_{x\to\infty} \frac{F(x)}{x}$.

1. 求极限 $\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\cos(t^2)dt}{x^2}$

（2）设 $f(x)$ 连续，且 $\int_0^{x^3-1}f(t)dt=x$，求 $f(7)$

（3）设 $f(x)$ 连续，且 $f(x)=\sqrt{2x-x^2}+x\int_0^1 f(t)dt,$，求 $f(x)$.

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

基本方法：找微分，凑微分（求不定积分别忘了加常数！）

Ex.1 **Find the integral** $\int (x^3+x)^5(3x^2+1)dx$.

solution $\int (x^3+x)^5(3x^2+1)dx$

$$= \int (x^3+x)^5 d(x^3+x)$$

$$= \int u^5 du (u = x^3+x) = \frac{u^6}{6} + C = \frac{(x^3+x)^6}{6} + C.$$

Ex. 2 **Find** $\int \sin 2x dx$.

Solution 1. $\int \sin 2x dx = \frac{1}{2}\int \sin 2x d(2x) = -\frac{1}{2}\cos 2x + C;$

2. $\int \sin 2x dx = 2\int \sin x \cos x dx = 2\int \sin x d(\sin x)$

$$= (\sin x)^2 + C;$$

3. $\int \sin 2x dx = 2\int \sin x \cos x dx = -2\int \cos x d(\cos x)$

$$= -(\cos x)^2 + C.$$

Ex. 3 求 $\int \sec^2(5x+1)dx$.

Solution $\int \sec^2(5x+1)dx = \frac{1}{5}\int \sec^2(5x+1)d(5x+1)$

$$= \frac{1}{5}\int \sec^2 u du = \frac{1}{5}\tan u + C = \frac{1}{5}\tan(5x+1) + C$$

Ex. 5 Find $(a)\int \cos^2 x dx.\ (b)\int \sin^2 x dx.\ (c)\int (1-2\sin^2 x)\sin 2x dx.$

Solution

$$(a)\int \cos^2 x dx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{x}{2}+\frac{1}{2}\cdot\frac{\sin 2x}{2}+C.$$

$$(b)\int \sin^2 x dx = \frac{1}{2}\int (1-\cos 2x)dx = \frac{x}{2}-\frac{\sin 2x}{4}+C.$$

$$(c)\int (1-2\sin^2 x)\sin 2x dx = \int \cos 2x \sin 2x dx$$

$$= \frac{1}{2}\int \sin 4x dx = -\frac{\cos 4x}{8}+C.$$

偶次幂的三角函数可以利用二倍角公式化简

Ex. 7 Find $\int x\sqrt{2x+1}dx.$

Solution $u=\sqrt{2x+1}, du=\dfrac{1}{\sqrt{2x+1}}dx=\dfrac{dx}{u}, x=\dfrac{u^2-1}{2},$

$$dx = udu$$

$$\int x\sqrt{2x+1}dx = \int \frac{(u^2-1)u^2}{2}du = \frac{1}{2}\int (u^4-u^2)du$$

$$= \frac{u^5}{10}-\frac{u^3}{6}+C = \frac{\sqrt{2x+1}^5}{10}-\frac{\sqrt{2x+1}^3}{6}+C.$$

若积分式带根号可以考虑使用换元法

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Ex.4 **Evaluate** $\int_{-1}^{1} \frac{2x^2 + x\cos x}{1+\sqrt{1-x^2}}\,dx.$

Solution $\int_{-1}^{1} \frac{2x^2 + x\cos x}{1+\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{2x^2}{1+\sqrt{1-x^2}}\,dx + \int_{-1}^{1} \frac{x\cos x}{1+\sqrt{1-x^2}}\,dx$

$$= 4\int_0^1 \frac{x^2}{1+\sqrt{1-x^2}}\,dx = 4\int_0^1 \frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}\,dx$$

$$= 4\int_0^1 (1-\sqrt{1-x^2})\,dx = 4 - 4\int_0^1 \sqrt{1-x^2}\,dx = 4-\pi.$$

Ex. Proof $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx.$

Proof Let $x = \frac{\pi}{2} - u$, then

$$\int_0^{\pi/2} \sin^n x dx = -\int_{\pi/2}^{0} \cos^n u du = \int_0^{\pi/2} \cos^n x dx$$

计算$\int_0^{\frac{\pi}{2}} \frac{\cos^{10} x}{\sin^{10} x + \cos^{10} x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{10} x}{\sin^{10} x + \cos^{10} x} dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^{10} x}{\sin^{10} x + \cos^{10} x} dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^{10} x}{\sin^{10} x + \cos^{10} x} dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 dx = \pi / 4.$$

例　若$f(x)$是在$(-\infty,+\infty)$内以 T 为周期的连续函数，证明：对于任何的实数 a，有

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx.$$

证：$\int_0^T f(x)dx = \int_0^a f(x)dx + \int_a^{a+T} f(x)dx + \int_{a+T}^{T} f(x)dx$

令$u = x - T$，$\int_{a+T}^{T} f(x)dx = \int_a^0 f(u+T)du = -\int_0^a f(u)du$

所以 $\int_0^T f(x)dx = \int_a^{a+T} f(x)dx.$

计算$\int_2^{2+100\pi} |\sin x| dx = \int_0^{100\pi} |\sin x| dx$

$$= \int_0^{\pi} |\sin x| dx + \int_{\pi}^{2\pi} |\sin x| dx + \cdots + \int_{99\pi}^{100\pi} |\sin x| dx$$

$$= 100\int_0^{\pi} |\sin x| dx = 100\int_0^{\pi} \sin x dx = 200.$$

例 计算 $\dfrac{d(\int_0^1 f(x-t)dt)}{dx}$，其中$f(x)$连续。

解. 令$x-t=u$，$\int_0^1 f(x-t)dt=-\int_x^{x-1} f(u)du$

$$\therefore \frac{d(\int_0^1 f(x-t)dt)}{dx}=\frac{d\int_{x-1}^{x} f(u)du}{dx}=f(x)-f(x-1)$$

设$F'(x)=f(x)$,

$$\int_0^1 f(x-t)dt=-\int_0^1 f(x-t)d(x-t)$$

$$=-\int_x^{x-1} f(u)du=-F(u)\Big|_x^{x-1}=F(x)-F(x-1)$$

$$\therefore \frac{d(\int_0^1 f(x-t)dt)}{dx}=f(x)-f(x-1)$$

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Ex.8 Find the area of the region bounded below by $y = 2 - x$, and above by $y = \sqrt{2x - x^2}$.

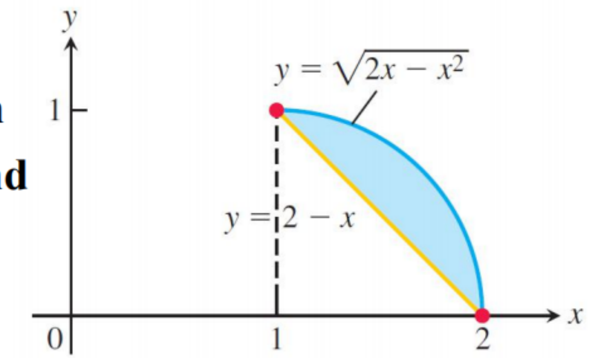


FIGURE 5.31 The region described by the curves in Example 7.

Solution $A = \int_1^2 [\sqrt{2x - x^2} - (2 - x)]dx$

$$= \int_0^1 [(1 + \sqrt{1 - y^2}) - (2 - y)]dy = \int_0^1 [\sqrt{1 - y^2} + y - 1]dy$$

$$= \frac{\pi}{4} + \frac{1}{2} - 1 = \frac{\pi}{4} - \frac{1}{2}.$$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\ dx$

31. $\int \frac{\sin(2x+1)}{\cos^2(2x+1)}\ dx$

51. $\int \frac{18\tan^2 x \sec^2 x}{(2+\tan^3 x)^2}\ dx$

54. $\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}}\ dx$

35. $\int \frac{1}{x^2}\sin\frac{1}{x}\cos\frac{1}{x}\ dx$

22. $\int \sqrt{\sin x}\cos^3 x\ dx$

40. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\ dx$

48. $\int 3x^5\sqrt{x^3+1}\ dx$

11. $\int_0^1 t\sqrt{4+5t}\ dt$

18. $\int_{\pi}^{\frac{3\pi}{2}} \cot^5(\frac{t}{6})\sec^2(\frac{t}{6})\ dt$

23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{t}\cos^2(t^{\frac{3}{2}})\ dt$

83. Suppose that $F(x)$ is an antiderivative of $f(x)=\frac{\sin x}{x}, x>0$. Express $\int_1^3 \frac{\sin 2x}{x}dx$ in terms of F.

84. Show that if f is continuous, then $\int_0^1 f(x)dx = \int_0^1 f(1-x)dx$.

85. Suppose that $\int_0^1 f(x)dx = 3$, find $\int_{-1}^0 f(x)dx$ if
(1)f is odd; (b)f is even.

87. If f is a continuous function, find the value of the integral $I=\int_0^a \frac{f(x)dx}{f(x)+f(a-x)}$ by making the substitution $u=a-x$ and adding the resulting integral to I.

88. By using a substitution, prove that all positive numbers x and y, $\int_x^{xy}\frac{1}{t}dt = \int_1^y \frac{1}{t}dt$.

(1) $f(x)$ is continuous on $[a, b]$, prove that
$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx.$

(2) Show that $\int_x^1 \frac{dt}{1+t^2} = \int_1^{\frac{1}{x}} \frac{dt}{1+t^2}.$

(3) $f(x)$ is differentiable on (a, ∞), and $\int_0^1 f(tx)\, dt = 2f(x) + 1$,
$f(1) = 1$. Find $f(x)$.

3. 计算

（1）$\int_{2}^{2+100\pi} |\sin x|\, dx$.

（2）$\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x} dx$.

（3）$\int_{0}^{1} x^2(1-x)^{10} dx$.

4. 设 $f(x)$ 在 $[a,b]$ 上连续且恒正，证明方程 $\int_a^x f(t)dt = 2\int_x^b f(t)dt$ 在 (a,b) 内有唯一实根.

5. 设 $f(x), g(x)$ 在 $[a,b]$ 上连续，且 $g(x)$ 恒正. 证明在 $[a,b]$ 至少存在一点 c，使得

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

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谢谢大家！