Date. / /

复习专题一 . chapter 2 and 7.

1\. (1) ✓

(2) X. 洛必达中的前提是 $\lim\limits_{x\to0}\frac{f'(x)}{g'(x)}$ 存在，反过来不一定存在.

$$f(x)=\begin{cases}x^2\sin\frac{1}{x}, & x\neq0\\ 0, & x=0\end{cases}\qquad f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x}=\lim_{x\to0}x\sin\frac{1}{x}=0$$

$g(x)=x.\quad g(0)=0$

$$\Rightarrow f'(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\neq0\\ 0, & x=0\end{cases}\qquad f'(0)=0\quad g'(0)=1\neq0.$$

$$\Rightarrow \lim_{x\to0}\frac{f'(x)}{g'(x)}=\lim_{x\to0}\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)\text{ 不存在. but }\lim_{x\to0}\frac{f(x)}{g(x)}=\lim_{x\to0}x\sin\frac{1}{x}=0.$$

2\. $f(x)=\frac{\tan x}{|x|(x-\frac{\pi}{2})^4}$ 选(D)

$$\lim_{x\to0}f(x)=\lim_{x\to0}\frac{\tan x}{|x|(x-\frac{\pi}{2})^4}=\lim_{x\to0}\frac{\tan x}{|x|}\cdot\frac{1}{(x-\frac{\pi}{2})^4}=\begin{cases}\frac{1}{(\frac{\pi}{2})^4}, & x\to0^+\\ \frac{1}{-(\frac{\pi}{2})^4}, & x\to0^-\end{cases}\text{ jump.}$$

$$\lim_{x\to\frac{\pi}{2}}f(x)=\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{|x|(x-\frac{\pi}{2})^4}=\frac{1}{\frac{\pi}{2}}\cdot\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{(x-\frac{\pi}{2})^4}\overset{t=x-\frac{\pi}{2}}{=\!=}\begin{cases}-\infty, & x\to(\frac{\pi}{2})^+\\ \infty, & x\to(\frac{\pi}{2})^-\end{cases}\text{ infinite}$$

3\. 选(A).

$$f(x)=\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}\Rightarrow\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{2e^{-\frac{1}{x}}+1}{e^{-\frac{1}{x}}+1}=1$$

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}=2$$

$\Rightarrow$ jump.

Date. / /

4. 选(A).

$f(x)=\frac{\ln|x|}{|x-1|}\sin x$， $x=1$，$x=0$.

$\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\frac{\ln|x|}{|x-1|}\sin x=\lim\limits_{x\to 0}\underset{\downarrow -\infty}{\ln|x|}\cdot\underset{\downarrow 0}{\sin x}=\lim\limits_{x\to 0}\underset{\downarrow 1}{\frac{\sin x}{x}}\cdot\frac{\ln|x|}{\frac{1}{x}}=\lim\limits_{x\to 0}\frac{\ln|x|}{\frac{1}{x}}$

$=\begin{cases}\lim\limits_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=0\\ \lim\limits_{x\to 0^-}\frac{\ln(-x)}{\frac{1}{x}}=\lim\limits_{x\to 0^-}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=0\end{cases}$

$\Rightarrow \lim\limits_{x\to 0}f(x)=0$, but $f(0)$不存在

$\Rightarrow$ removable $x=0$.

$\lim\limits_{x\to 1}f(x)=\lim\limits_{x\to 1}\frac{\ln|x|}{|x-1|}\sin 1=\begin{cases}\lim\limits_{x\to 1^+}\frac{\ln x}{x-1}\cdot\sin 1=\lim\limits_{x\to 1^+}\frac{\frac{1}{x}}{1}\cdot\sin 1=\sin 1\\ \lim\limits_{x\to 1^-}\frac{\ln x}{1-x}\cdot\sin 1=\lim\limits_{x\to 1^-}\frac{\frac{1}{x}}{-1}\sin 1=-\sin 1\end{cases}$

$\Rightarrow$ jump $x=1$.

5. 选(D).

$x\to 0^-$，$x-\sin x\to 0^-$；$\lim\limits_{x\to 0^-}f(x-\sin x)=b$.

$x^2+x\to 0^-$ $\lim\limits_{x\to 0^-}f(x^2+x)=b$

$\Rightarrow b+2b=3b$.

6. 选(B).

$\lim\limits_{x\to 1^+}y=\frac{\pi}{2}e$，$\lim\limits_{x\to -2^+}y=e^{\frac{1}{4}}\cdot\arctan\frac{3}{(-3)\cdot 1}=e^{\frac{1}{4}}\cdot(-\frac{\pi}{2})$.

$\Rightarrow x=1$ 不是 $x=-2$ 不是

$\lim\limits_{x\to 0^+}y=-\infty$ $\Rightarrow x=0$ 是；$\lim\limits_{x\to\infty}y=1\cdot\frac{\pi}{4}=\frac{\pi}{4}\Rightarrow y=\frac{\pi}{4}$ 是

7. 选(B)

$$F(x)=\begin{cases}\dfrac{\int_0^x tf(t)dt}{x}, & x\neq 0\\ 0, & x=0.\end{cases} \Rightarrow F'(0)=\lim_{x\to 0}\frac{F(x)-F(0)}{x}=\lim_{x\to 0}\frac{\int_0^x tf(t)dt}{x^2}$$

$$=\lim_{x\to 0}\frac{xf(x)}{2x}=\frac{1}{2}\lim_{x\to 0}f(x)=\frac{1}{2}f(0)$$

$$\lim_{x\to 0}F(x)=\lim_{x\to 0}\frac{\int_0^x tf(t)dt}{x}=\lim_{x\to 0}\frac{xf(x)}{1}=\lim_{x\to 0}x\cdot f(x)=0=F(0)$$

$\Rightarrow F(x)$ is continuous at $x=0$.

$$F'(x)=\begin{cases}\dfrac{xf(x)-\int_0^x tf(t)dt}{x^2}, & x\neq 0\\ \frac{1}{2}f(0), & x=0\end{cases}$$

$$\lim_{x\to 0}F'(x)=\lim_{x\to 0}\frac{x^2f(x)-\int_0^x tf(t)dt}{x^2}=$$

$$=\lim_{x\to 0}f(x)-\lim_{x\to 0}\frac{\int_0^x tf(t)dt}{x^2}=f(0)-\lim_{x\to 0}\frac{xf(x)}{2x}=f(0)-\frac{1}{2}f(0)=\frac{1}{2}f(0)$$

$$=F'(0)$$

$\Rightarrow F'(x)$ is continuous at $x=0$.

8. $$\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x=\lim_{x\to\infty}\left(1+\frac{2a}{x-a}\right)^x=\lim_{x\to\infty}\left[\underbrace{\left(1+\frac{2a}{x-a}\right)^{\frac{x-a}{2a}}}_{\to e}\right]^{\frac{2ax}{x-a}\to 2a}=e^{2a}=8$$

$\Rightarrow 2a=\ln 8 \Rightarrow a=\frac{1}{2}\ln 8$

或 $$e^{x\ln\left(\frac{x+a}{x-a}\right)}=e^{\frac{\ln(x+a)-\ln(x-a)}{\frac{1}{x}}}\to e^{\frac{\frac{1}{x+a}-\frac{1}{x-a}}{-\frac{1}{x^2}}}\to e^{\frac{2ax^2}{(x+a)(x-a)}}\to e^{2a}=8.$$

Date: / /

9. $\lim_{x\to0^+}\frac{\ln\tan7x}{\ln\tan2x}=\lim_{x\to0^+}\frac{\frac{1}{\tan7x}\cdot7\sec^27x}{\frac{1}{\tan2x}\cdot2\sec^22x}=\lim_{x\to0^+}\frac{7}{2}\cdot\frac{\tan2x}{\tan7x}=\frac{7}{2}\cdot\frac{2}{7}=1$

10. (1) $\lim_{x\to0}\frac{(1+x)^{\frac{1}{x}}-e}{x}\to(1+x)^{\frac{1}{x}}\cdot\frac{x-(1+x)\ln(1+x)}{\underbrace{(1+x)}_{1}x^2}\to e\cdot\frac{x-(1+x)\ln(1+x)}{x^2}\to e\cdot\frac{\cancel{x}-\ln(1+x)-\cancel{x}}{2x}$

let $y=(1+x)^{\frac{1}{x}}\Rightarrow\ln y=\frac{\ln(1+x)}{x}$ $\to e\cdot\frac{-\frac{1}{1+x}}{2}\to\frac{e}{-2}$.

$\Rightarrow\frac{y'}{y}=\frac{\frac{x}{1+x}-\ln(1+x)}{x^2}\Rightarrow y'=(1+x)^{\frac{1}{x}}\cdot\frac{x-(1+x)\ln(1+x)}{(1+x)x^2}$

(2) $\lim_{x\to0,}\frac{3\sin x+x^2\cos\frac{1}{x}}{\underbrace{(1+\cos x)}_{2}\ln(1+x)}$ $\to\frac{3\sin x+x^2\cos\frac{1}{x}}{2.\frac{\cancel{\ln(1+x)}}{\cancel{x}}\cdot x}\to\frac{1}{2}\left(\underbrace{\frac{\sin x}{x}}_{1}+\underbrace{x\cos\frac{1}{x}}_{0}\right)\to\frac{1}{2}$

($\cancel{\times}$ $\frac{1}{2}\cdot\frac{3\sin x+x^2\cos\frac{1}{x}}{\ln(1+x)}\to\frac{1}{2}\cdot\frac{\overset{\to1}{3\cos x}+\overset{\to0}{2x\cos\frac{1}{x}}-\sin\frac{1}{x}}{\frac{1}{1+x}\to1}$ $\lim\frac{f'(x)}{g'(x)}$不存在. 不存在,條件不满足. $\Rightarrow$ 方法错)

11. $y=\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{e^x-1}}$ $\Rightarrow\ln y=\frac{\ln\left(\frac{\ln(1+x)}{x}\right)}{e^x-1}=\frac{\ln(\ln(1+x))-\ln x}{e^x-1}\to\frac{\frac{1}{\ln(1+x)}\cdot\frac{1}{1+x}-\frac{1}{x}}{e^x}$

$x\to0$

$\to\frac{1}{(1+x)\ln(1+x)}-\frac{1}{x}=\frac{x-(1+x)\ln(1+x)}{x\underbrace{(1+x)}_{1}\ln(1+x)}$

$\to\frac{x-(1+x)\ln(1+x)}{x\ln(1+x)}\to\frac{\cancel{1}-\ln(1+x)-\cancel{1}}{\ln(1+x)+\frac{x}{1+x}}\to\frac{-\ln(1+x)}{(1+x)\ln(1+x)+x}$

$\to\frac{-\frac{1}{1+x}}{\ln(1+x)+1+1}\to-\frac{1}{2}$

$\Rightarrow y\to e^{-\frac{1}{2}}$

方法二: $\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{e^x-1}}=\left(\underbrace{\left[1+\left(\frac{\ln(1+x)}{x}-1\right)\right]^{\frac{x}{\ln(1+x)-x}}}_{\to e}\right)^{\frac{\ln(1+x)-x}{x(e^x-1)}}\to\frac{\ln(1+x)-x}{x^2}\to\frac{\frac{1}{1+x}-1}{2x}=\frac{-1}{2(1+x)}\to-\frac{1}{2}$

$\to e^{-\frac{1}{2}}$

Date. / /

12. (1) $\frac{\tan^{-1}x - x}{x\tan^2 x} \rightarrow \frac{\tan^{-1}x - x}{x^3\cdot\left(\underbrace{\frac{\tan x}{x}}_{\downarrow 1}\right)^2} \rightarrow \frac{\tan^{-1}x - x}{x^3} \rightarrow \frac{\frac{1}{1+x^2} - 1}{3x^2} \rightarrow \frac{-x^2}{\underbrace{(1+x^2)}_{\downarrow 1}\cdot 3x^2} \rightarrow \frac{1}{-3}$

$x \to 0.$

(2) $\cancel{x\to\infty,\ y=(x+100)^{100x} \rightarrow \ln y = 100x\ln(x+100) \Rightarrow \frac{y'}{y} = 100\ln(x+100) + \frac{100x}{x+100}}$

$\cancel{\Rightarrow y' = (x+100)^{100x}}$

$\frac{(x+100)^{100x}}{x^{100x}} = \left(\frac{x+100}{x}\right)^{100x} = \left(1+\frac{100}{x}\right)^{100x} = \left[\underbrace{\left(1+\frac{100}{x}\right)^{\frac{x}{100}}}_{\downarrow e}\right]^{\frac{100x\cdot 100}{x}} \rightarrow e^{10^4}$