Tutorial 14 for Calculus I

Sect. 9.1-9.4

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通过上一点算出 y'

dx、y' 算 Δy

13. Use Euler's method to calculate the first three approximations to the given initial value problem

$y' = 2xy + 2y, \quad y(0) = 3, \quad dx = 0.2$

for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximation.

Review of Sect. 9.2

无系数

y'

1. First-Order Linear Equations: $\frac{dy}{dx} + P(x)y = Q(x)$, where P and Q are continuous of x.

一次

$v(x) = \int P(x)dx$

multiply both sides by the integrating factor $v(x) = e^{\int P(x)\ dx}$

$\Rightarrow v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x)$

$\Rightarrow \frac{d}{dx}(v(x) \cdot y) = v(x)Q(x)$

$\Rightarrow v(x) \cdot y = \int v(x)Q(x)\ dx$ 有 c

再待定系数

$\Rightarrow y = \frac{1}{v(x)} \int v(x)Q(x)\ dx, \quad v'(x) = P(x)v(x)$

$\Rightarrow v(x) = e^{\int P(x)dx}$

$y' \to u'$

2. A Bernoulli differential equation: $\frac{dy}{dx} + P(x)y = Q(x)y^n$.

* $u = y^{1-n}$ 核心代换

Observe that, if n=0 or 1, the Bernoulli equation is linear.

$\frac{du}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx}$

To solve the equation, let $u = y^{1-n}$,

$\frac{du}{dx} \frac{1}{1-n} + P(x)u = Q(x)$

$\Rightarrow \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx}$

$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$

$\Rightarrow \frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx} \Rightarrow \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$

$P(x) = \frac{3}{x}$ $v(x) = \int P(x)\,dx = 3\ln x$

Solve the differential equations:

$e^{v(x)} = x^3$

3. $xy' + 3y = \frac{\sin x}{x^2}, \quad x > 0$

$y' + \frac{3y}{x} = \frac{\sin x}{x^3}$

$(x^3 y)' = \sin x$

$x^3 y = -\cos x + C$

$y = \frac{-\cos x + C}{x^3}$

① $xy' + y(\ln x - \ln y) = 0, \quad y(1) = e^3$

Q: ?

31. $xy' + y = y^{-2}$

$u = y^3$

$\frac{du}{dx} = 3y^2 \frac{dy}{dx}$

$\frac{xu'}{3y^2} + y = y^{-2}$

$xu' + u = 3$

$(xu)' = 3$

$xu = 3x + C$

$u = 3 + \frac{C}{x} = y^3$

$y = \sqrt[3]{3 + \frac{C}{x}}$

V y

Mixture Problems: let $y(t)$=the amount of chemical in the container at time t,y_0=the amount of chemical in the container at time t=0, and $V(t)$=the total volume of liquid in the container at time t, then

$\text{rate out} = \frac{y}{V(t)} \cdot v_{out}$

$\frac{dy}{dt}$=Rate in -Rate out, or dy=amount in-amount out.

$y' = (\ )t \cdot \frac{y}{V(t)} V_{out}$

$c_1(t)$=the concentration of the chemical flowing in at time t, $v_1(t)$=the velocity at which chemical arrives, so Rate in$=c_1(t) \cdot v_1(t)$.

V_0=the total volume of liquid in the container at time t=0, $v_2(t)$=the velocity at which chemical departs, so $V(t) = V_0 + (v_1 - v_2)t$.

$c_2(t)$=the concentration of the chemical running out of container at time t, Rate out$=c_2(t) \cdot v_2(t) = \frac{y(t)}{V(t)} v_2(t)$

13. Salt mixture: A tank initially contains 400 L of brine in which 20 kg/L of salt are dissolved. A brine containing 0.2 kg/L of the salt runs into the tank at rate of 20 L/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 16 L/min.

a. At what rate (kilograms per minute) does salt enter the tank at time t? b. What is the volume of the brine in the tank at time t?

c. At what rate (kilograms per minute) does salt leave the tank at time t?

d. Write down and solve the initial value problem describing the mixing process.

e. Find the concentration of the salt in the tank 25 min after the process starts.

14. Mixture problem: An 800 L tank is half full of distilled water. At time t=0, a solution containing 50 grams/L of the concentrate enters the tank at the rate of 20 L/min, and the well-stirred mixture is withdrawn at the rate of 12 L/min. a. At what time will the tank be full?

b. At the time the tank is full, how many kilograms of the concentrate will it contain?

Notices: chemical: concentrate, liquid: distilled water$\Rightarrow y_0 = 0$, half full$\Rightarrow$ V_0=800/2=400 L, $c_1(t)$=50 grams/L, $v_1(t)$=20 L/min, $v_2(t)$=12 L/min.

Homework of Sect. 9.3

15. Fertilizer mixture: A tank contains 400 L of fresh water. A solution containing 0.1 kg/L of soluble lawn fertilizer runs into the tank at the rate of 4 L/min, and the mixture is pumped out of the tank, at the rate of 12 L/min. Find the maximum amount of the fertilizer in the tank and the time required to reach the maximum.

    Notices: chemical: fertilizer, liquid: fresh water $\Rightarrow y_0 = 0$, V_0=400 L, $c_1(t)$=0.1 kg/L, $v_1(t)$=4 L/min, $v_2(t)$=12 L/min.

Review of Sect. 9.4

1. Equilibrium values: If $\frac{dy}{dx} = g(y)$ is an autonomous differential equation, then the values of y for which $\frac{dy}{dx} = 0$ are called equilibrium values or rest points.

2. Phase line.

3. Stable equilibrium and Unstable equilibrium.

7. $y' = (y-1)(y-2)(y-3)$.

a. Identify the equilibrium values. Which are stable and which are unstable?

b. Construct a phase line. Identify the signs of y' and y''.

c. Sketch several solution curves.

补充作业

12.(2019年期末) Let g be a function that is differentiable throughout an open interval containing the origin. Suppose g has the following properties:

i) $g(x+y) = \frac{g(x)+g(y)}{1-g(x)g(y)}$ for all real numbers x, y, and $x+y$ in the domain of g.

ii) $\lim\limits_{h\to 0} g(h) = 0$.

iii) $\lim\limits_{h\to 0} \frac{g(h)}{h} = 1$.

Find $g(x)$.

5.(2020年期末) Evaluate the following limits.

(1) $\lim\limits_{n\to+\infty}(\frac{n}{2n^2+3n+1^2} + \frac{n}{2n^2+6n+2^2} + \cdots + \frac{n}{2n^2+3nk+k^2} + \cdots + \frac{n}{2n^2+3n^2+n^2})$.

(2) $\lim\limits_{x\to 0}(\frac{\ln(1+x)}{x})^{\frac{1}{e^x-1}}$.

只能一侧有问题

7.(2021年期末) Find all values for p such that the improper integral $\int_0^\infty \frac{e^{-x}}{x^p}\, dx$ converges.

$\int_0^1 \frac{e^{-x}}{x^p} + \int_1^\infty \frac{e^{-x}}{x^p}$

$= \frac{\frac{e^{-x}}{x^p}}{\frac{1}{x^p}} = e^{-x}$

$p>1$ ×

$p<1$

9. (2021年期末) Evaluate the integrals.

(3) $\int_1^\infty \frac{1}{x^6(x^5+4)}\, dx$.

(4) $\int \frac{1}{(1+x+x^2)^2}\, dx$.

8. (2022年期末) Assume $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$. If $f(0)=f(1)=0$, $f(\frac{1}{2})=1$, prove that:

(1) there exists $c \in (\frac{1}{2}, 1)$, , such that $f(c)=c$.

(2) For any real number k, there always exists $\xi \in (0,c)$, such that $f'(\xi) - k[f(\xi)-\xi] = 1$.