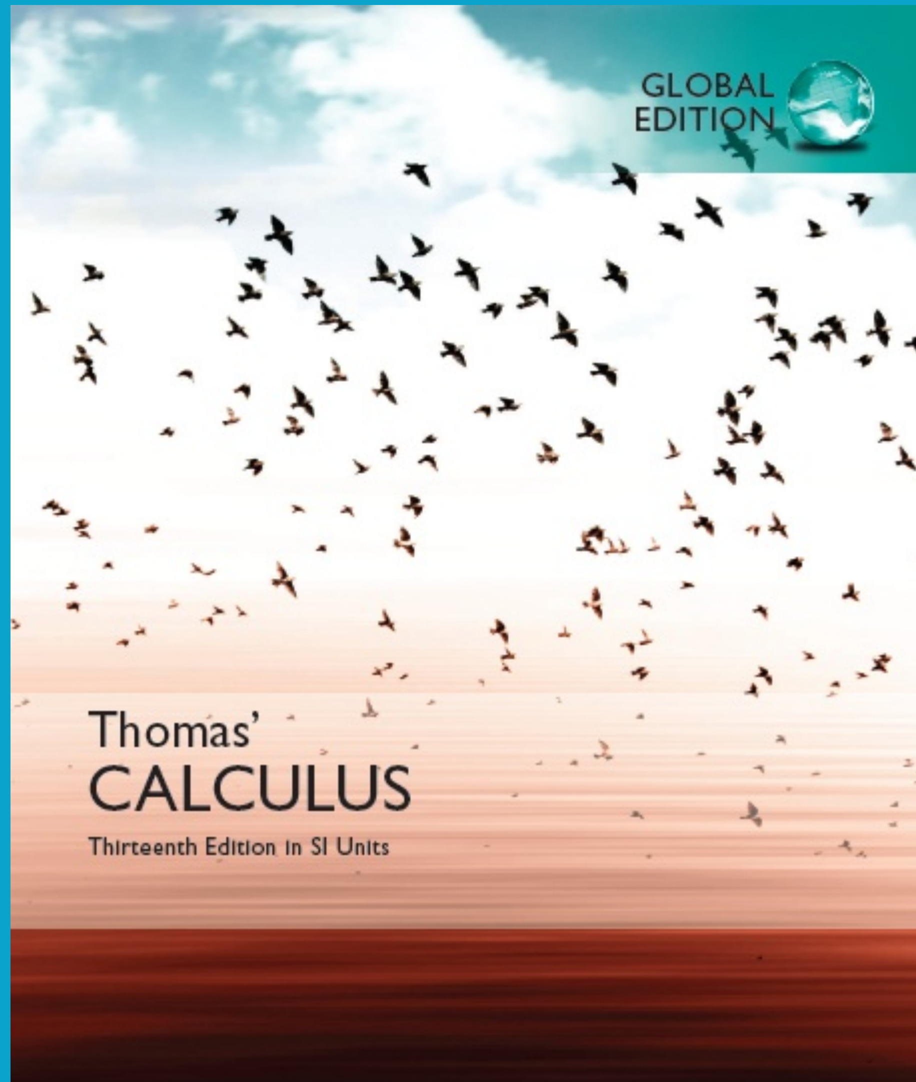

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Chapter 7

超越函数

Transcendental Functions

Transcendental Numbers and Transcendental Functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic**: -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers such as e and π that are not algebraic are called **transcendental**.

代数数

——整系数多项式的根

（移项作次方还原方程）

非代数数

We call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x + 1}$ is algebraic because it satisfies the equation $(x + 1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Functions that are not algebraic are called transcendental.

关于y的

可写成整系数多项式

$y = \frac{x^2+1}{x-1}$

$\Rightarrow (x-1)y - (x^2+1) = 0$

三角函数、

反三角

e^x、$\ln x$

（不考定义）

7.1

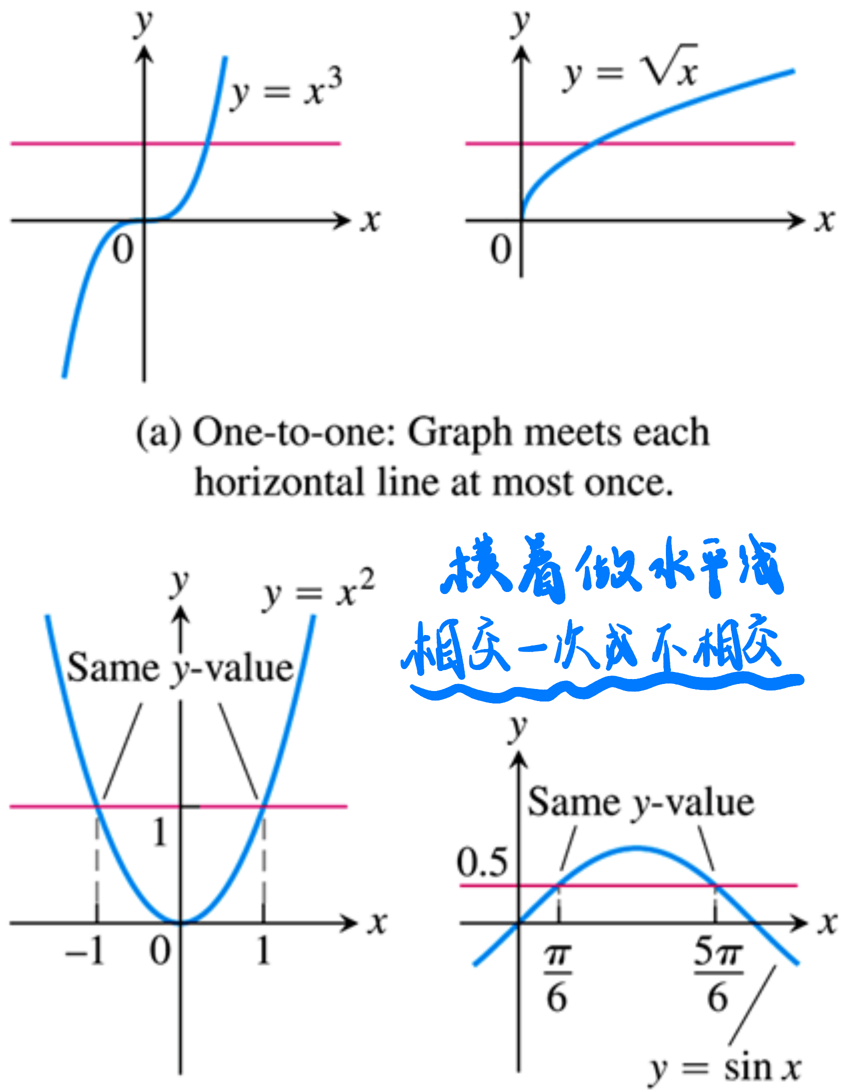
反函数

Inverse Functions and Their Derivatives

x不同 → y一定不同

一对一

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.



(a) One-to-one: Graph meets each horizontal line at most once.

横着做水平线
相交一次或不相交

(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

FIGURE 7.1 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

水平线测试

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

复合写法

外 内

定义域

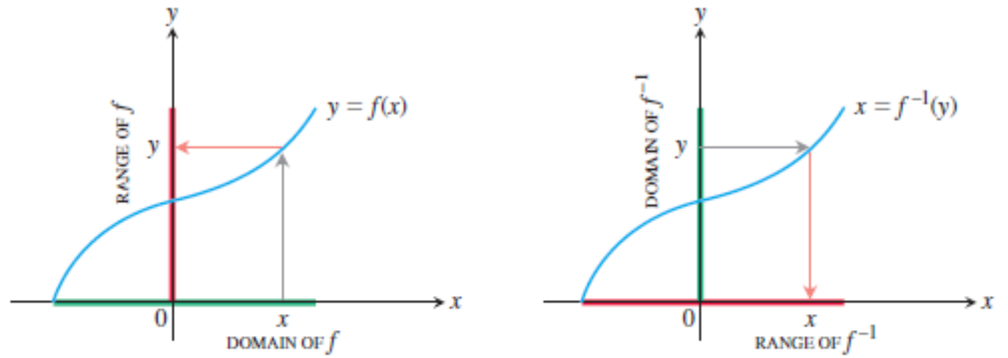
$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$

$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f\text{)}$$

先做哪个定义域是哪个

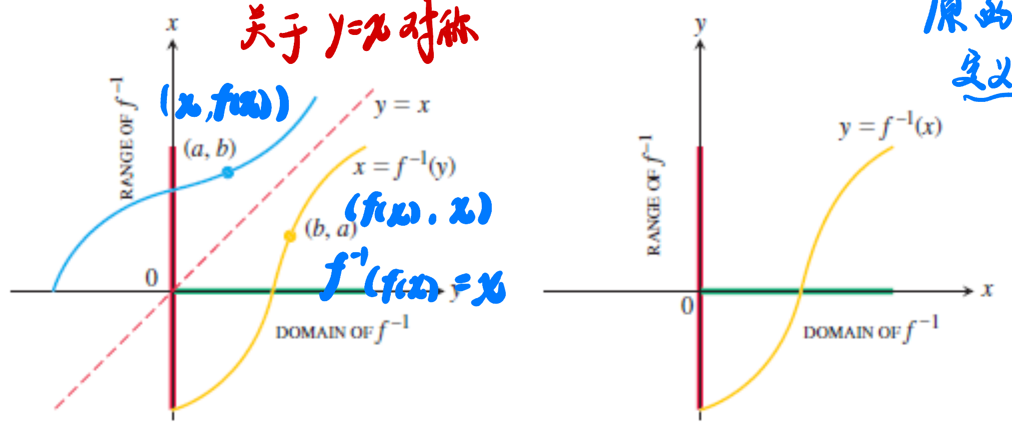
$y = f^{-1}(x)$

$f(y) = x$



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



关于 $y=x$ 对称

$(x, f(x))$

$(f(x), x)$

$f^{-1}(f(x))=x$

原函数与反函数定义域、值域互换

(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.

2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

1)交换 x, y 位置
2)整合

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

$x = \frac{1}{2}y + 1$

$y = 2x - 2$

EXAMPLE 4 Find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of x.

$x = y^2$

$y = \sqrt{x}$

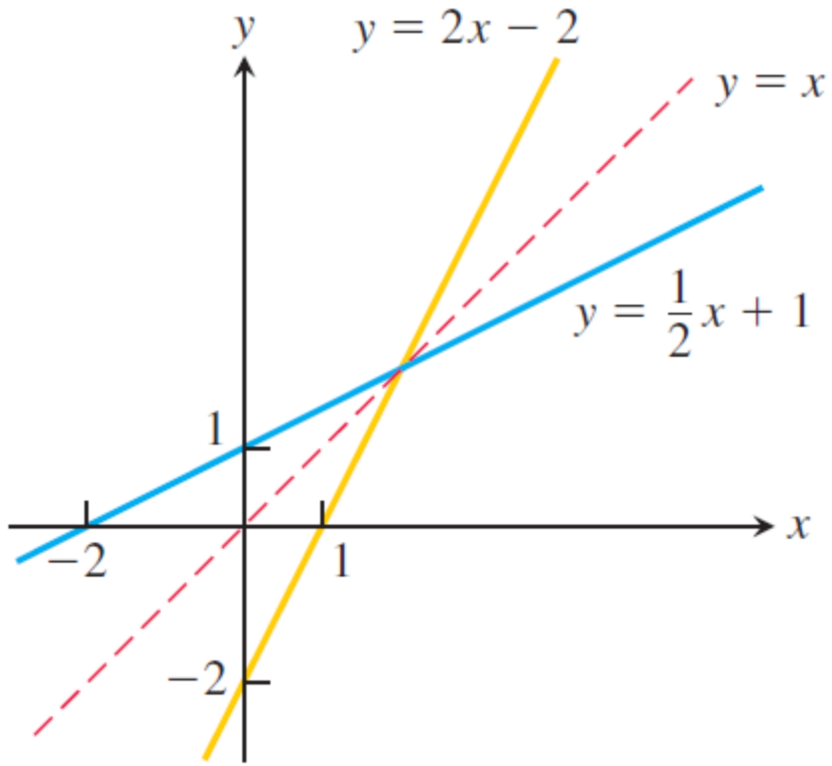


FIGURE 7.3 Graphing the functions $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

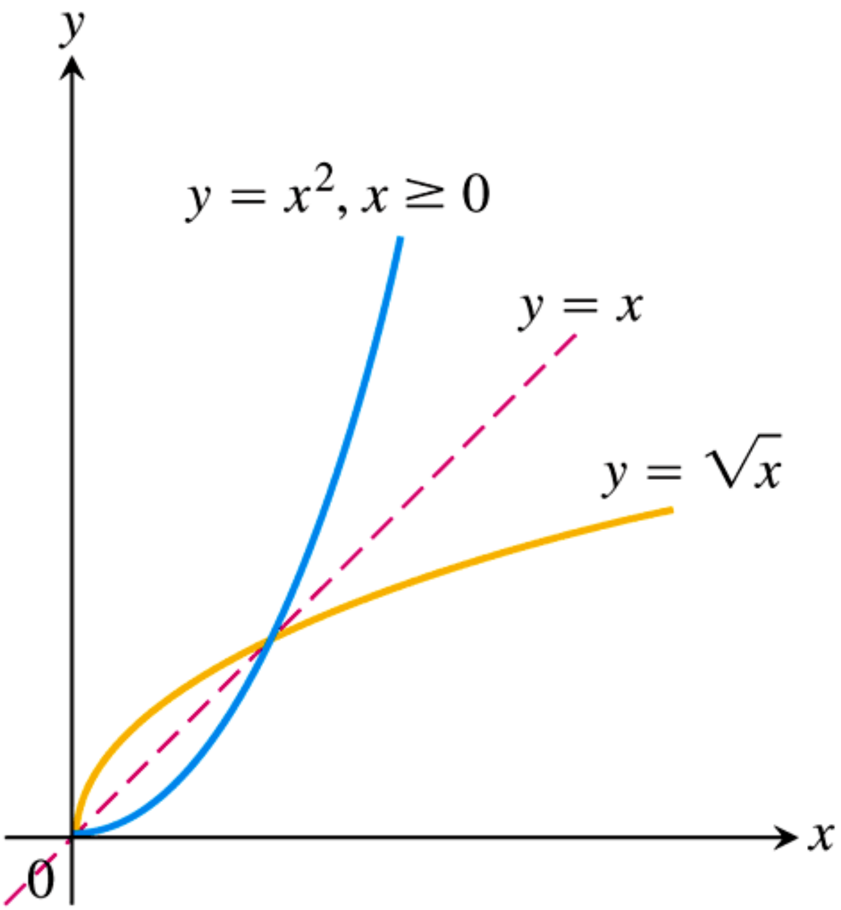
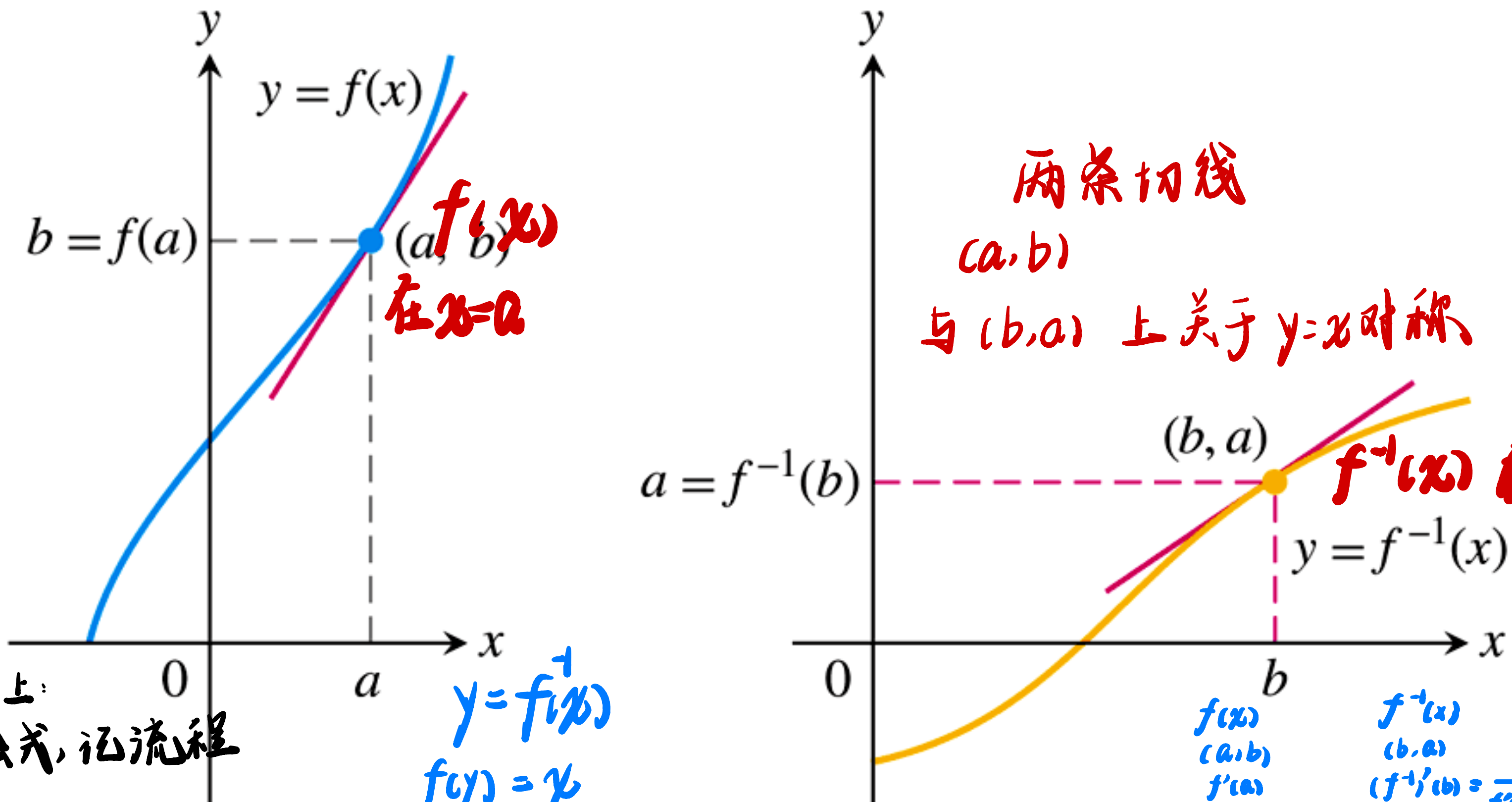


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 4).



f(x) 在x=a

两条切线
(a,b)
与(b,a)上关于y=x对称

$f^{-1}(x)$ 在 x=b

f(x)
(a,b)
f'(a)

$f^{-1}(x)$
(b,a)
$(f^{-1})'(b) = \frac{1}{f'(a)}$

* 实际上：
不记公式，记流程

$y = f^{-1}(x)$
$f(y) = x$
对x求导 $\frac{d}{dx} f(y) = 1$
$= \frac{df(y)}{dy} \frac{dy}{dx}$
$= f'(y) \frac{dy}{dx}$

$\underbrace{\frac{dy}{dx}}$ $= \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$ 何处

永远是切线斜率

The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

FIGURE 7.5 The graphs of inverse functions have reciprocal slopes at corresponding points.

$y = f^{-1}(x)$

$f^{-1\prime}(x)\ \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$

THEOREM 1—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df}{dx}\right|_{x=a} \cdot \left.\frac{df^{-1}}{dx}\right|_{x=f^{-1}(a)} = 1$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Theorem 1 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1} when it exists. While we omit the proof of the first assertion, the second one is proved in the following way:

$$f(f^{-1}(x)) = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx} f(f^{-1}(x)) = 1 \qquad \text{Differentiating both sides}$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1 \qquad \text{Chain Rule}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}. \qquad \text{Solving for the derivative}$$

EXAMPLE 5 The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/\left(2\sqrt{x}\right)$.

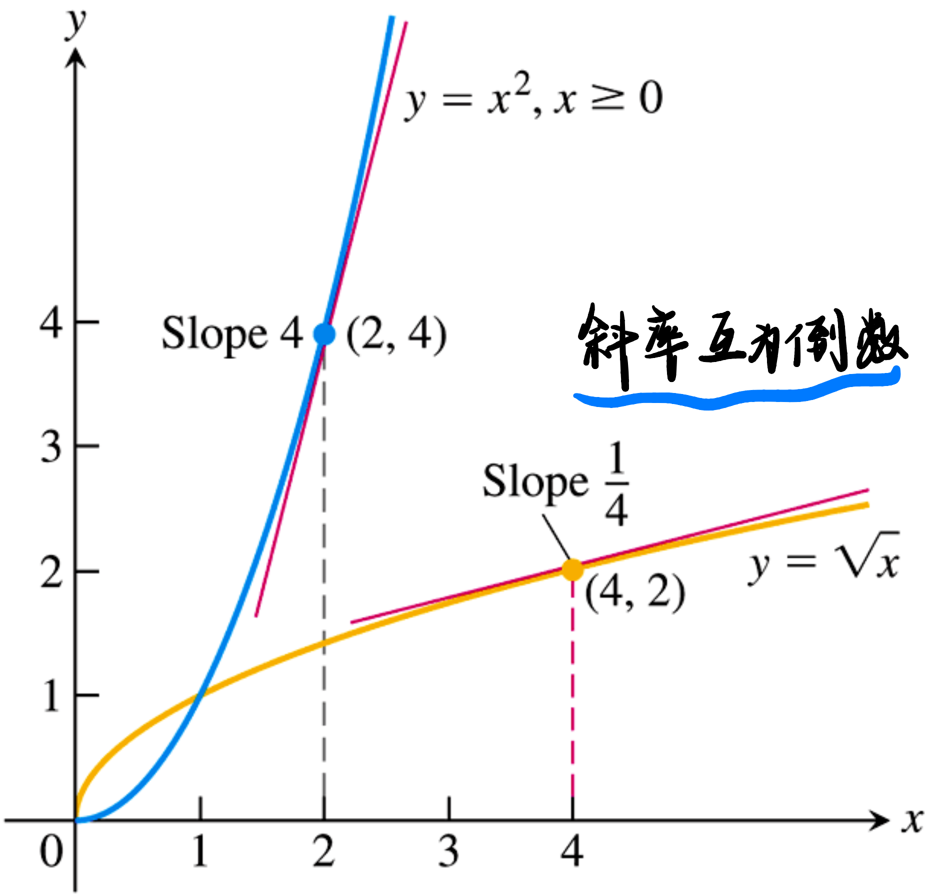


FIGURE 7.6 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 5).

7.2

自然对数

Natural Logarithms

Historically, logarithms played important roles in arithmetic computations, making possible the great seventeenth-century advances in offshore navigation and celestial mechanics. In this section we define the natural logarithm as an integral using the Fundamental Theorem of Calculus. While this indirect approach may at first seem strange, it provides an elegant and rigorous way to obtain the key characteristics of logarithmic and exponential functions.

$(x^n)' = n x^{n-1}$

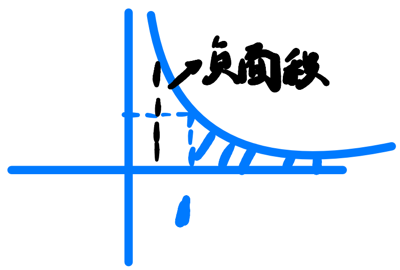
$f(x)$ $F(x)$

x^n $\frac{1}{n+1} x^{n+1}$

$n \neq -1$

那 x^{-1} 的原函数？

$\frac{1}{x}$



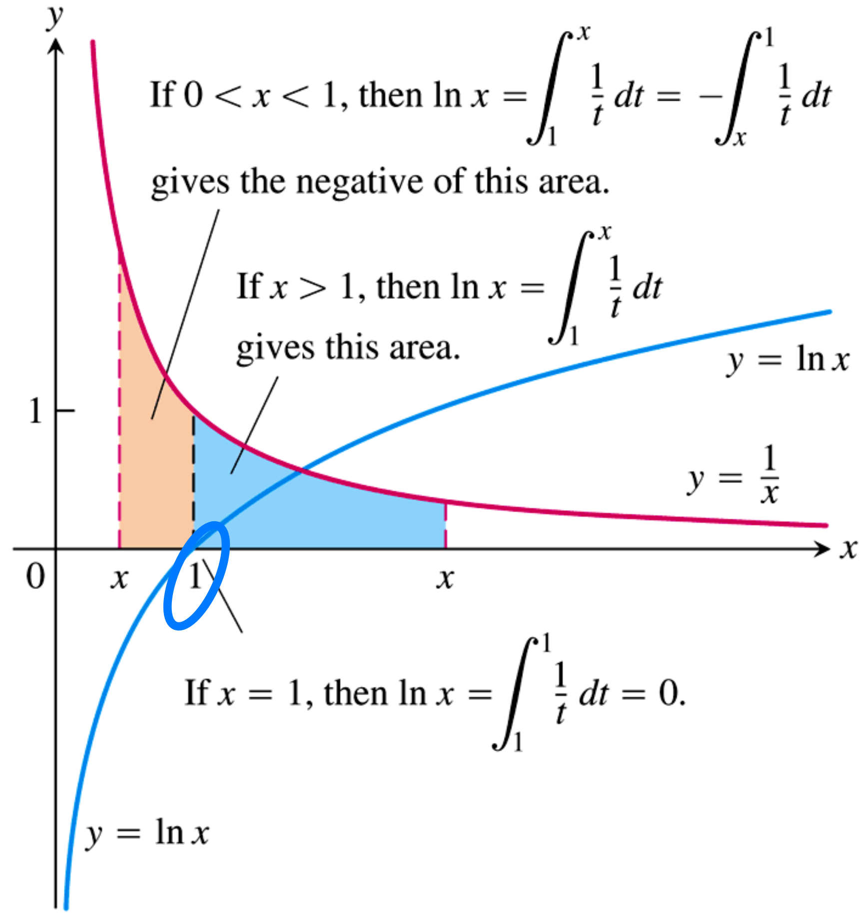
微积分基本定理：有连续函数即可有原函数

定义 $\ln x = \int_1^x \frac{1}{t}\,dt$ $x>0$ → 不能过 ∞

连续

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0. \tag{1}$$



ln x 单增变化

FIGURE 7.8 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the x-axis as x moves from 1 to the left.

TABLE 7.1 Typical 2-place values of ln x

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

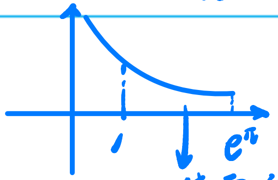
DEFINITION The **number *e*** is that number in the domain of the natural logarithm satisfying

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

面积为1

$\ln e^{\pi} = \pi$

$\ln x = \int_1^x \frac{1}{x}\,dx$

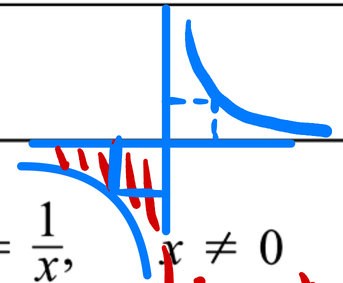


使面积正好为π

$2^{\pi} = e^{\pi \ln 2}$ 移到 2^{π}时面积为 $\pi \ln 2$

定义

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \tag{2}$$



* 一定加 ||

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad x \neq 0 \tag{4}$$

取出来是正的

统一

$x>0$ 同上

$x<0$ $\frac{d\ln|x|}{dx} = \frac{d\ln(-x)}{dx} = -\frac{1}{x}\frac{d(-x)}{dx} = \frac{1}{x}$

$\int \frac{1}{x}\,dx = \ln|x|$

还未定义指数函数

只能用求导+特殊点相同值证明

THEOREM 2—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

$(\ln bx)' = \frac{1}{bx} \cdot b = \frac{1}{x}$

$(\ln b + \ln x)' = \frac{1}{x}$

1. *Product Rule*: $\ln bx = \ln b + \ln x$ 取值 $x=1$ 发现同一个原函数

2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$

$(\ln \frac{1}{x})' = x\left(-\frac{1}{x^2}\right) = -\frac{1}{x}$

$(-\ln x)' = -\frac{1}{x}$ 取 $x=1$ 同原函数

3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$

4. *Power Rule*: $\ln x^r = r \ln x$ For r rational

不考严格证明

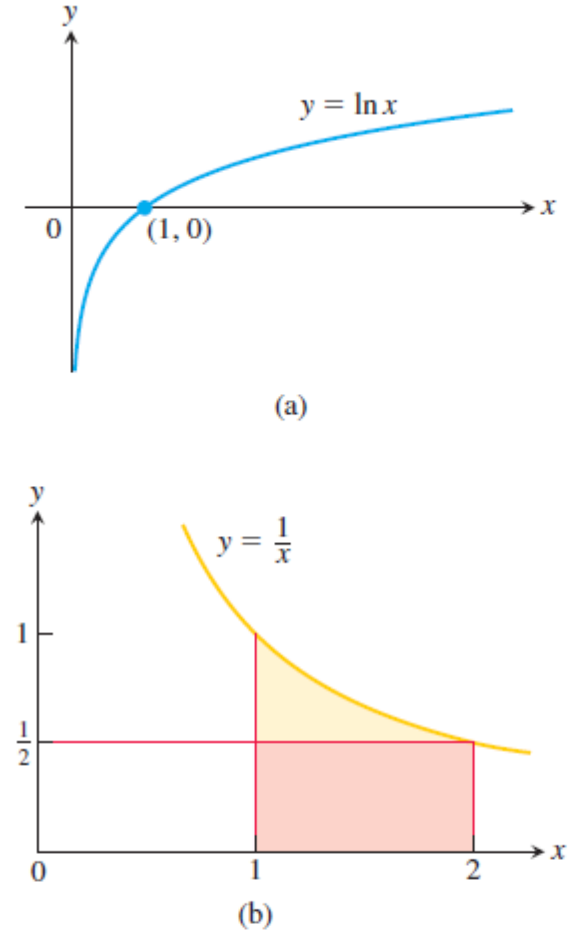
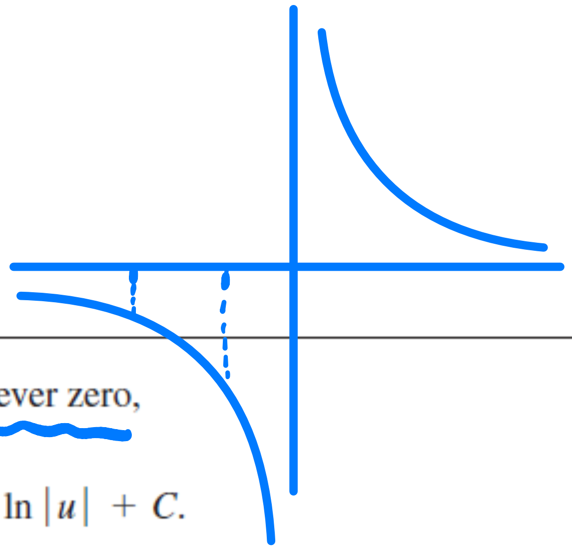


FIGURE 7.9 (a) The graph of the natural logarithm. (b) The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.



If u is a differentiable function that is never zero,

$$\int \frac{1}{u} du = \ln |u| + C. \tag{3}$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\,du = \underline{\ln|\sin u| + C} \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

$$\int \sec x\,dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{1}{u}\,du = \ln|\sec u + \tan u| + C$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\,dx$

$= \sec x(\tan x + \sec x)\,dx$

似乎提前已知结果

标准做法

$$\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx$$

弄出 $\sin^2 x$ 或 $\cos^2 x$

有时比一次有用

$u = \sin x \to du = \cos x\,dx$

负二次比负一次好处理

$$\text{so} \quad = \int \frac{1}{1-u^2}\,du = \int \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du$$

$\frac{1}{u}du$ 才好积

→ 一次

$$= \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

同乘 $1+\sin x$

$$= \frac{1}{2}\ln\frac{(1+\sin x)^2}{\cos^2 x} + C$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$= \ln|\sec x + \tan x| + C$$

EXAMPLE 5 Find dy/dx if *全是多项式乘除

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

化乘除为加减

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

取对

$$\begin{aligned}
\ln y &= \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1} \\
&= \ln((x^2+1)(x+3)^{1/2}) - \ln(x-1) && \text{Quotient Rule} \\
&= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) && \text{Product Rule} \\
&= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). && \text{Power Rule}
\end{aligned}$$

We then take derivatives of both sides with respect to x, using Equation (2) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

*实际隐函数求导

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y from the original equation:

复杂乘除不用简化

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

使用对数函数求导数

7.3

Exponential Functions

反函数定义指数函数

$a>0$

$\ln x \to e^x \to a^x \to \log_a x$

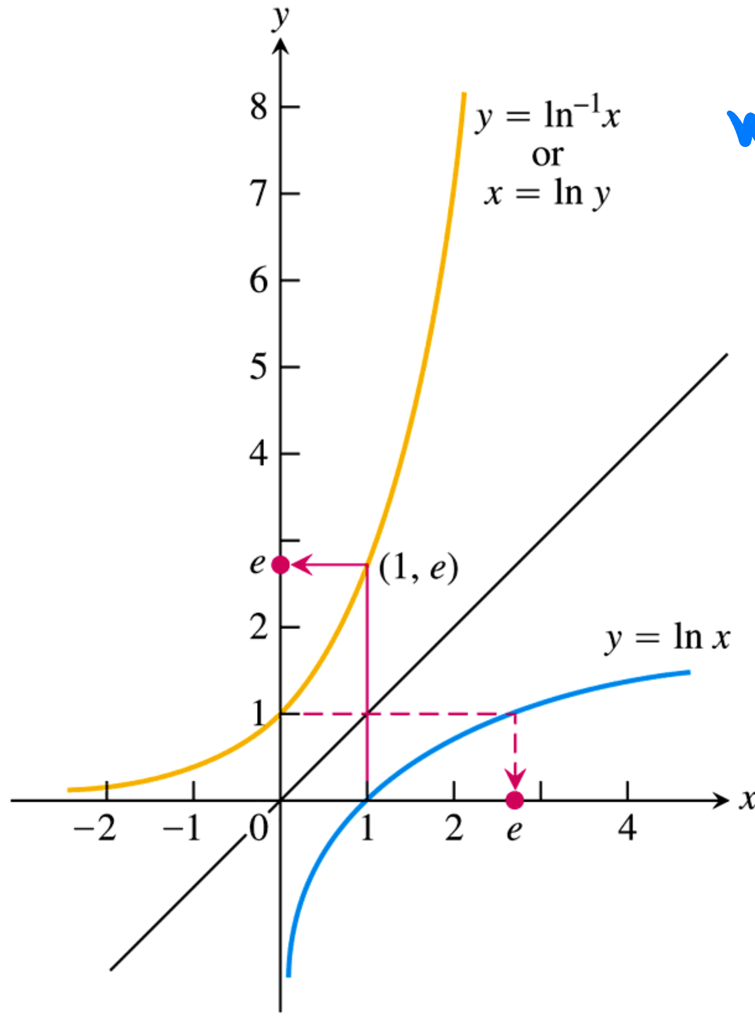
(n为任意实数)

$e^{n\ln x} = x^n$

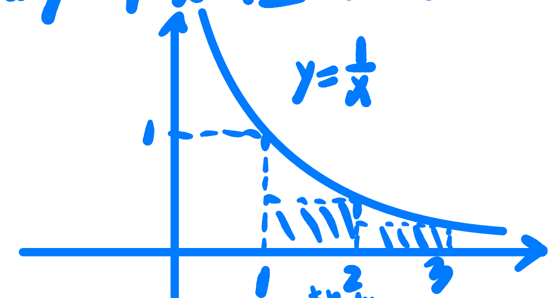
n为无理数只能这样定义

$a = (e^{\ln a})^x = e^{x\ln a}$

要求



why $\ln x$ 值域为 R



when n 为正整数 $\ln n = \int_1^n \frac{1}{x} dx$

取较小部分

$> \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

惯用放缩 $> \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) \cdots + (\frac{1}{2^m} + \cdots)$

$= \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$ 共 m 个

$= \frac{1}{2} m$

对倒数 $\underset{+\infty}{\ln x} + \underset{-\infty}{\ln \frac{1}{x}} = 0$

FIGURE 7.10 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$. 反函数而来

Typical values of e^x

x	e^x (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

DEFINITION For every real number x, we define the **natural exponential function** to be $e^x = \exp x$.

Inverse Equations for e^x and $\ln x$ 互为反函数

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$

$$\ln(e^x) = x \qquad (\text{all } x)$$

EXAMPLE 2 A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

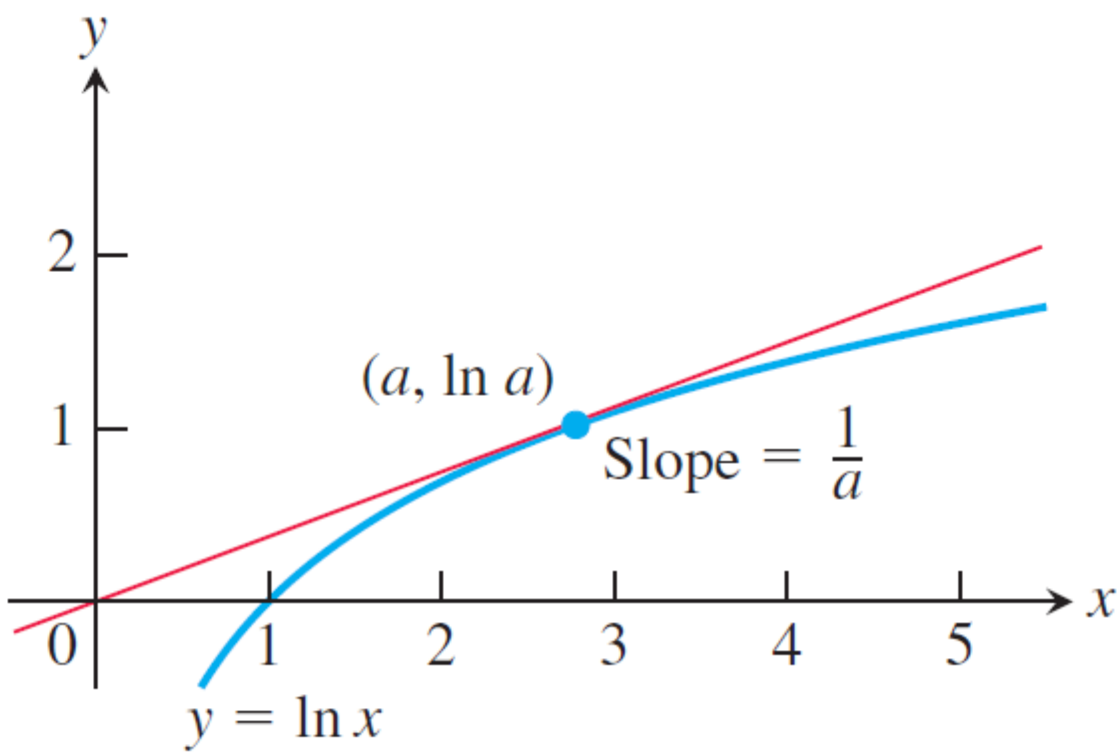


FIGURE 7.11 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$ (Example 2).

If u is any differentiable function of x, then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \tag{2}$$

求$\frac{dy}{dx}$

$y = \exp x$

$\ln y = x$

$\frac{1}{y}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = y = \exp x$

The general antiderivative of the exponential function

$$\int e^u \, du = e^u + C$$

1. exp a · exp b = exp(a+b)

取对

ln(exp a · exp b) = a+b ✓

THEOREM 3 For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^r = e^{rx_1}$, if r is rational

此时可以看成指数了

e^x

DEFINITION For any numbers $a > 0$ and x, the **exponential function with base a** is

导/积都化成 $\begin{cases} e^x \\ \ln x \end{cases}$

$$a^x = e^{x \ln a}.$$

$e^{\ln a} = a$

(反函数性质)

$$\begin{cases} x^a \cdot x^b = x^{a+b} \\ \dfrac{x^a}{x^b} = x^{a-b} \end{cases}$$

DEFINITION For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

$$(x^n)' = (e^{n \ln x})' = e^{n \ln x} \cdot \frac{n}{x}$$

$$= n x^{n-1}$$

General Power Rule for Derivatives

For $x > 0$ and any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

-1的无理数次方无定义

ch7

* 考点：
- 隐函数
- 对数隐函数求导
- 该类函数求导

EXAMPLE 5 Differentiate $f(x) = x^x$, $x > 0$.

* 常考!

$y = u^v$

u, v 至少一常数 才可求导法则

*底与指数得一个有常数才可用公式

取指数

① $(x^x)' = (e^{x\ln x})'$

$= e^{x\ln x}(\ln x + 1)$

$= x^x(\ln x + 1)$

② $\ln y = x\ln x$

$\frac{d(\ln y)}{dx} = 1 + \ln x$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = (1 + \ln x)x^x$

$y = a^x$

$\ln y = x\ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = \ln a \cdot a^x$

用取对验证

THEOREM 4—The Number e as a Limit The number e can be calculated as the limit

构型 $a^x = (e^{\ln a})^x$

$$e = \lim_{x\to 0} (1 + x)^{1/x}.$$

$\lim_{n\to\infty} (1+\frac{1}{n})^n$

$\ln(1+x)\frac{1}{x}$

$e = \lim_{x\to 0} e$ 自然底

$\Updownarrow$

$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

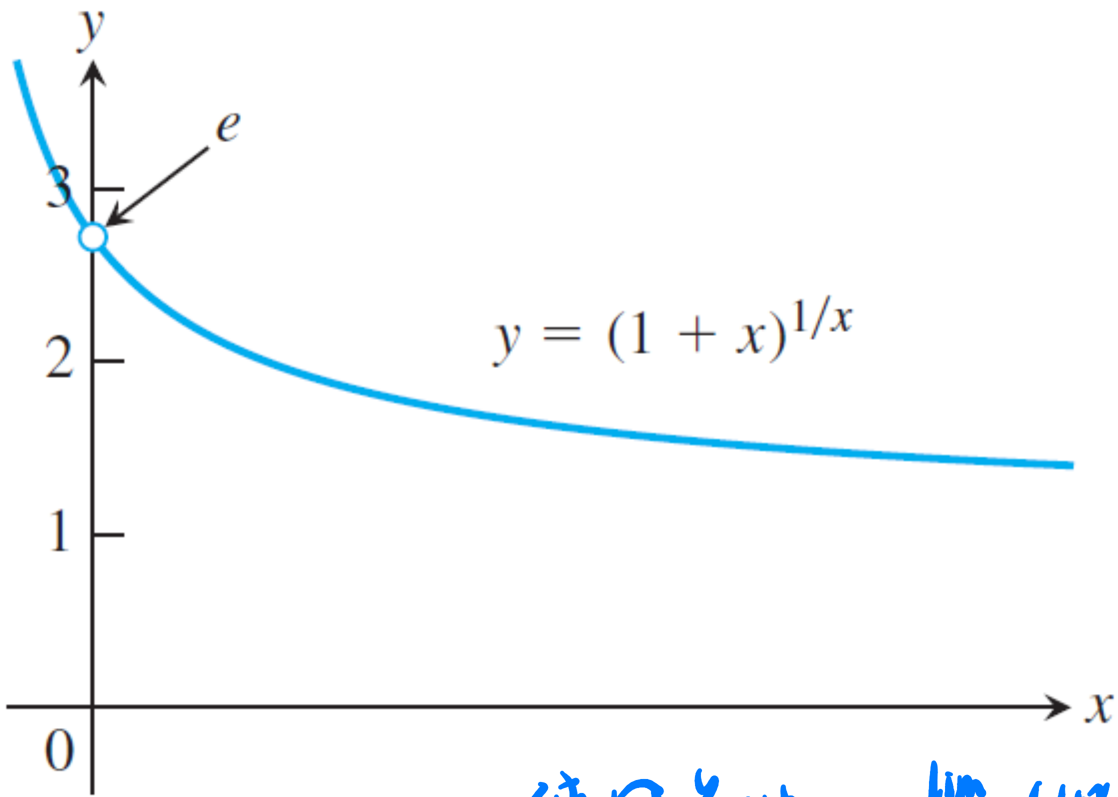
$\lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x} = 1$

*后续也用此技巧

仍是常用手段

$\ln x$ 在 $x=1$ 处导数!

构造 $\frac{f(x+\Delta x) - f(x)}{\Delta x}$ 定义



使用条件 $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

FIGURE 7.12 The number e is the limit of the function graphed here as $x \to 0$.

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$e^{\ln a \cdot u}$ $\quad (a^x)' = a^x \ln x$

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}. \tag{3}$$

$$\int a^u \, du = \frac{a^u}{\ln a} + C. \tag{4}$$

换元：慢但准

$$= \int (e^{\ln a})^u du = \int e^{\underbrace{u \ln a}_{v}} du$$

$$= \int e^v \frac{dv}{\ln a}$$

$$= \frac{1}{\ln a} e^v + C = \frac{1}{\ln a} a^u + C$$

EXAMPLE 6 We find derivatives and integrals using Equations (3) and (4).

(a) $\dfrac{d}{dx}3^x = 3^x \ln 3$ — Eq. (3) with $a = 3$, $u = x$

$e^{\ln 3 \cdot y}$

(b) $\dfrac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\dfrac{d}{dx}(-x) = -3^{-x}\ln 3$ — Eq. (3) with $a = 3$, $u = -x$

(c) $\dfrac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\dfrac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$ — $\ldots, u = \sin x$

(d) $\displaystyle\int 2^x\,dx = \frac{2^x}{\ln 2} + C$ — Eq. (4) with $a = 2$, $u = x$

(e) $\displaystyle\int 2^{\sin x}\cos x\,dx = \int 2^u\,du = \frac{2^u}{\ln 2} + C$ — $u = \sin x$, $du = \cos x\,dx$, and Eq. (4)

$\displaystyle = \frac{2^{\sin x}}{\ln 2} + C$ — u replaced by $\sin x$ ■

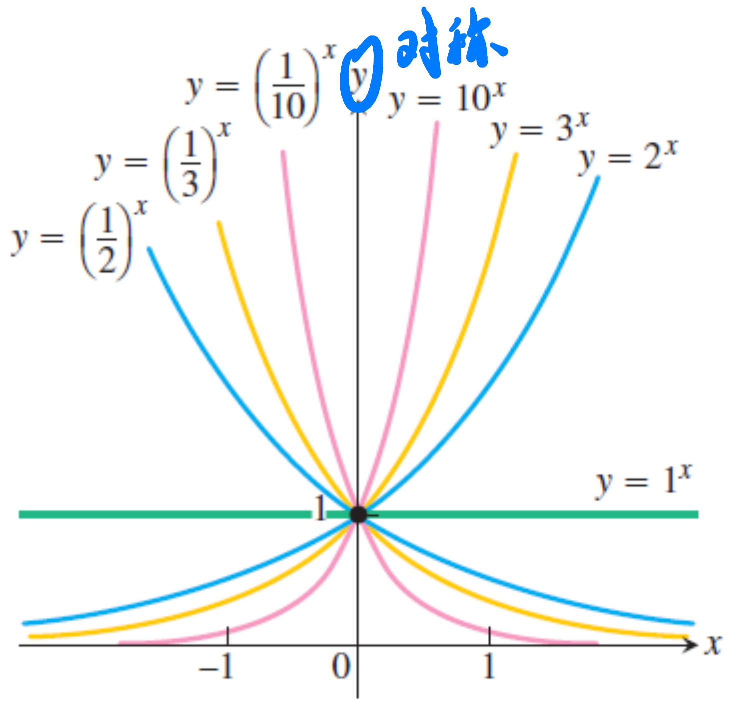


FIGURE 7.13 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

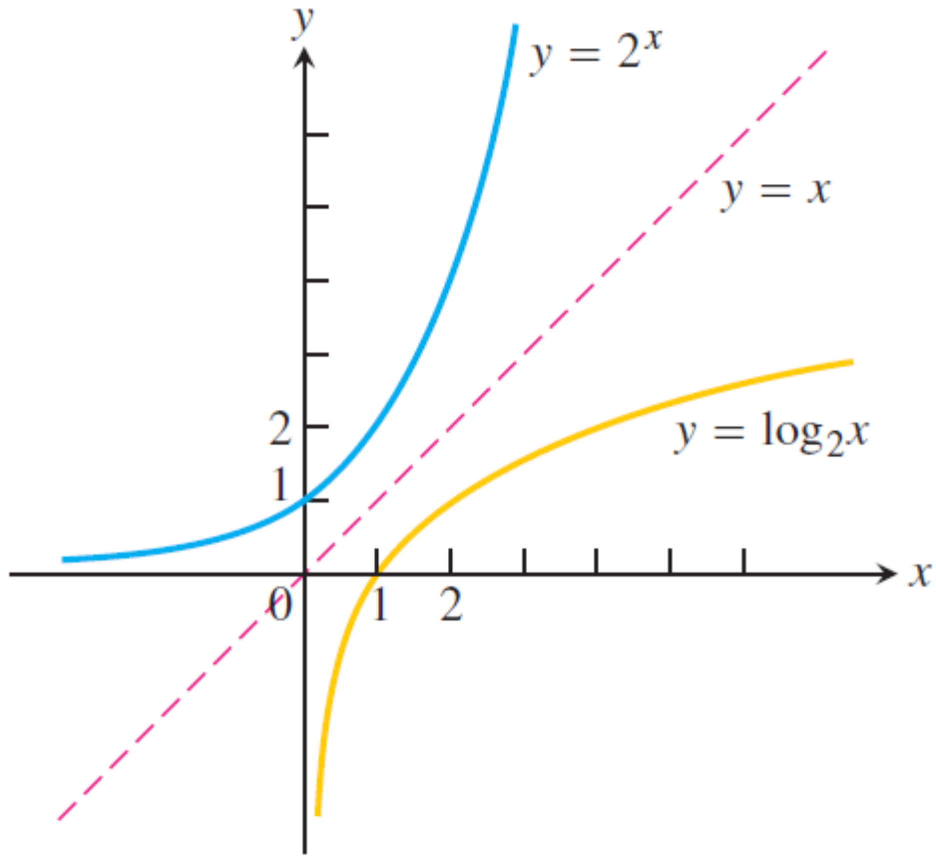
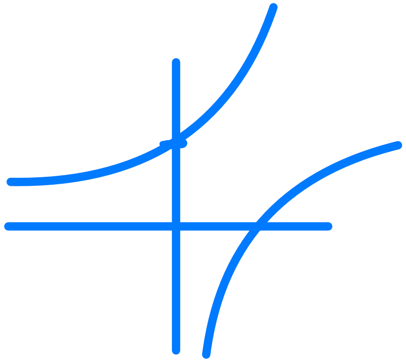


FIGURE 7.14 The graph of 2^x and its inverse, $\log_2 x$.

$y = \log_a x$

与 $y = a^x$



DEFINITION For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

$a=1$ 无反函数

$\log_a x$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a (a^x) = x \qquad (\text{all } x)$$

$y = \log_a x$

$a^y = x$

$\frac{dy}{dx}\ a^y \ln a$

$\frac{dy}{dx} = \frac{1}{\ln a \cdot a^y} = \frac{1}{x \ln a}$

反函数 写回去

$$\log_a x = \frac{\ln x}{\ln a}. \qquad (5)$$

$\left(\frac{\ln x}{\ln a}\right)' = \frac{1}{x \ln a}$

取 $x=1$ 相等 $c=0$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}$$

$\frac{\ln u}{\ln a}$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*

$$\log_a xy = \log_a x + \log_a y$$

2. *Quotient Rule:*

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

3. *Reciprocal Rule:*

$$\log_a \frac{1}{y} = -\log_a y$$

4. *Power Rule:*

$$\log_a x^y = y \log_a x$$

7.4

指数

Exponential Change and Separable Differential Equations

2侧都积分 开积

The solution of the initial value problem

$$\frac{dy}{y} = k\,dt$$

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

积分

is

$$\int \frac{dy}{y} = k\int dt$$

$$y = y_0 e^{kt}. \tag{2}$$

$$\ln|y| = kt + C \qquad \bar{c} = e^{c} > 0$$

$$|y| = \bar{c}e^{kt}$$

Quantities changing in this way are said to undergo **exponential growth** if $k > 0$ and **exponential decay** if $k < 0$. The number k is called the **rate constant** of the change. (See Figure 7.15.)

一般解 $y = \pm\bar{c}e^{kt}$

$= \underbrace{\tilde{c}}_{\neq 0} e^{kt}$ 表示通解

$y(0) = \tilde{c}$

ALWAYS LEARNING

PEARSON

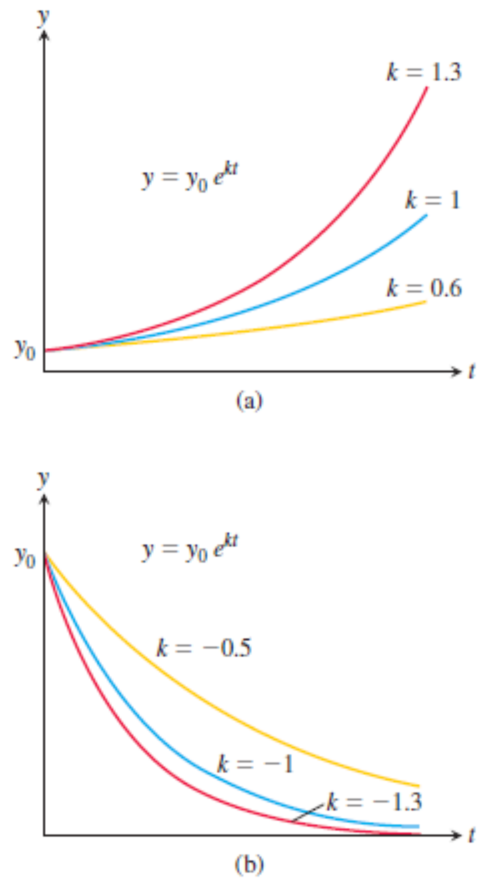


FIGURE 7.15 Graphs of (a) exponential growth and (b) exponential decay. As $|k|$ increases, the growth ($k > 0$) or decay ($k < 0$) intensifies.

Equation (3) is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y).$$

g is a function of x;
H is a function of y.

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)},$$

$H(y) = \dfrac{1}{h(y)}$

追求 $dx\, f(x) = dy\, h(y)$ 式子

d 各放一边，含x与dx，含y与dy

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

可分解为
一侧只有x，一侧只有y

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx. \tag{4}$$

After completing the integrations we obtain the solution y defined implicitly as a function of x.

EXAMPLE 1 Solve the differential equation

运气好，简单原函数

$$\frac{dy}{dx} = (1 + y)e^x, \quad y > -1.$$

$1+y>0$ 一般到这步

有时太麻 没必要

$$\boxed{\ln(1+y) = e^x + C}$$

$$\int \frac{dy}{1+y} = \int e^x dx$$

$$\ln|1+y| = e^x + C$$

$$y = e^{e^x + C} - 1$$

$$= \tilde{C} e^{e^x} - 1$$

EXAMPLE 2 Solve the equation $y(x + 1)\dfrac{dy}{dx} = x(y^2 + 1)$.

$$\int \frac{y}{y^2+1} dy = \int \frac{x}{x+1} dx$$

$$= \int \left(1 - \frac{1}{x+1}\right) dx$$

$$= x - \ln|x+1| + C_2$$

$u = \underbrace{y^2+1}_{>0}$ $du = 2y\,dy$

$$\int \frac{\frac{1}{2}}{u} du$$

$$= \frac{1}{2}\ln|u| + C_1$$

$$= \frac{1}{2}\ln u + C_1$$

so $\frac{1}{2}\ln(y^2+1) = x - \ln|x+1| + C$

要回代

酵母

EXAMPLE 3 The biomass of a yeast culture in an experiment is initially 29 grams. After 30 minutes the mass is 37 grams. Assuming that the equation for unlimited population growth gives a good model for the growth of the yeast when the mass is below 100 grams, how long will its take for the mass to double from its initial value?

指数型

$$y(t) = y_0 e^{kt}$$

$$37 = 29e^{30k} \Rightarrow e^{30k} = \frac{37}{29}$$

$$58 = 29e^{kt} \Rightarrow e^{kt} = 2 = \left(e^{30k}\right)^{\frac{t}{30}} = \left(\frac{37}{29}\right)^{\frac{t}{30}}$$

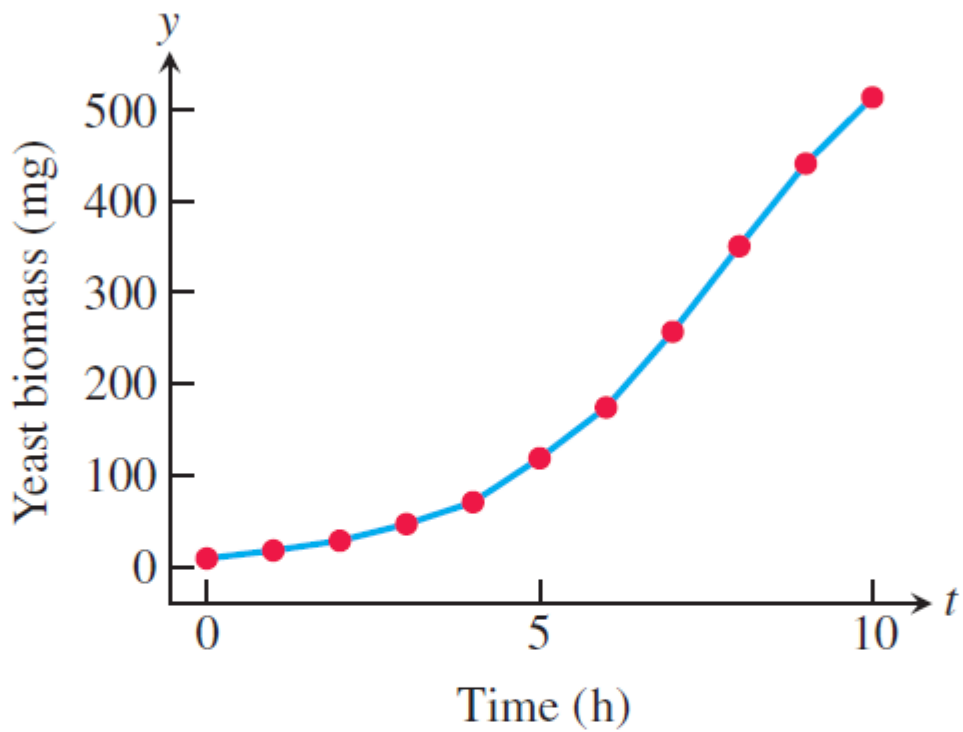


FIGURE 7.16 Graph of the growth of a yeast population over a 10-hour period, based on the data in Example 3.

Time (h)	Yeast biomass (mg)
0	9.6
1	18.3
2	29.0
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441.0
10	513.3

Some atoms are unstable and can spontaneously emit mass or radiation. This process is called **radioactive decay**, and an element whose atoms go spontaneously through this process is called **radioactive**. Sometimes when an atom emits some of its mass through this process of radioactivity, the remainder of the atom re-forms to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium, through a number of intermediate radioactive steps, decays into lead.

半衰期 —指数形式

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is an interesting fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

$$y = e^{kt} \cdot y_0$$

$$\frac{dy}{dx} = kdt$$

$$y(t) = y_0 e^{-kt}$$

$$\frac{1}{2} y_0 = y(t) = y_0 e^{-kT}$$

$$2 = e^{kT}$$

$$T = \frac{\ln 2}{k}$$

$$\text{Half-life} = \frac{\ln 2}{k} \tag{7}$$

If H is the temperature of the object at time t and H_S is the constant surrounding temperature, then the differential equation is

当前温度 环境温度

要求周围环境不变

$$\frac{dH}{dt} = -k(H - H_S). \tag{8}$$

越接近环境变化越慢

$W(t) = H(t) - H_S$

$\frac{dW}{dt} = -kW$

$\frac{dH}{dt} = -k(H - H_S)$

$H(t) = H_S + (H_0 - H_S)e^{-kt}$

If we substitute y for $(H - H_S)$, then

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}(H - H_S) = \frac{dH}{dt} - \frac{d}{dt}(H_S) \\
&= \frac{dH}{dt} - 0 && H_S \text{ is a constant.} \\
&= \frac{dH}{dt} \\
&= -k(H - H_S) && \text{Eq. (8)} \\
&= -ky. && H - H_S = y
\end{aligned}$$

当前Δ 原Δ

$H - H_S = e^{-kt}(H_0 - H_S)$

$$H - H_S = (H_0 - H_S)e^{-kt}, \tag{9}$$

where H_0 is the temperature at $t = 0$.

This is the equation for Newton's Law of Cooling.

EXAMPLE 6 A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

$$38 - 18 = (98 - 18) e^{-k \cdot 5}$$

$$20 - 18 = (98 - 18) e^{-kt}$$

7.5

Indeterminate Forms and L'Hôpital's Rule

一定要验证条件

THEOREM 5— L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

① 含a开区间可导

✓右侧不存在，左侧不一定

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

EXAMPLE 1 The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

适用条件

(a) $$\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{3-\cos x}{1}=\frac{3-\cos x}{1}\bigg|_{x=0}=2$$

$\frac{d\sqrt{x}}{dx}\Big|_{x=1}$ 导数

(b) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$$

有理化

(c) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$$ $(\sqrt{1+x}+1+x/2)$ $\frac{0}{0}$; apply l'Hôpital's Rule.

每步让式子简单

$$=\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=-\frac{1}{8}$$ Not $\frac{0}{0}$; limit is found.

(d) $$\lim_{x\to 0}\frac{x-\sin x}{x^3}$$ 暂时只能洛 $\frac{0}{0}$; apply l'Hôpital's Rule.

一定要放同一个分式才能洛

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to 0}\frac{\sin x}{6x}$$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$$ Not $\frac{0}{0}$; limit is found. ■

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE 2 Be careful to apply l'Hôpital's Rule correctly:

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} \qquad \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sin x}{1+2x} \qquad \text{Not } \frac{0}{0}$$

It is tempting to try to apply l'Hôpital's Rule again, which would result in

$$\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2},$$

另：$= \lim_{x\to\frac{\pi}{2}} \dfrac{\frac{1}{\cos x}}{1+\frac{\sin x}{\cos x}} = \lim_{x\to\frac{\pi}{2}} \dfrac{1}{\sin x+\cos x} = 1$

$\lim_{x\to\infty} \dfrac{\ln x}{x^p}$ $p>0$

$= \lim_{x\to\infty} \dfrac{\frac{1}{x}}{p x^{p-1}} = \lim_{x\to\infty} \dfrac{1}{p x^p} = 0$

说明 $\ln x < x^p$ $p>0$

EXAMPLE 4 Find the limits of these ∞/∞ forms:

(a) $\lim_{x\to\pi/2} \dfrac{\sec x}{1+\tan x}$ $= \lim_{x\to\frac{\pi}{2}^+} \dfrac{\sec x \tan x}{\sec^2 x}$ $= \lim_{x\to\frac{\pi}{2}^+} \sin x = 1$

单边极限

(b) $\lim_{x\to\infty} \dfrac{\ln x}{2\sqrt{x}}$ $= \lim_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}$ $= \lim_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$

(c) $\lim_{x\to\infty} \dfrac{e^x}{x^2}$. $= \lim_{x\to\infty} \dfrac{e^x}{2} = \infty$

$e^x > x^p$

EXAMPLE 5 Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x\to\infty} \left(x \sin \dfrac{1}{x}\right)$ $u = \frac{1}{x}$ $\frac{1}{u}\sin u = 1$

(b) $\lim_{x\to 0^+} \sqrt{x}\ln x$ $\lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{2}\frac{1}{x^{\frac{3}{2}}}} = \lim_{x\to 0^+} 2\sqrt{x} = 0$

构造除法

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

$= \lim_{x\to 0} \dfrac{x - \sin x}{x \sin x}$

使用时转化为 [illegible]

$= \lim_{x\to 0} \dfrac{x - \sin x}{x^2}$ $\dfrac{x}{\sin x} = 1$

加减替换可能有问题

乘除可换 → 用 $\lim_{x\to 0} \dfrac{x}{\sin x} = 1$

If $\lim_{x \to a} \ln f(x) = L$, then *定理!

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^{L}.$$

Here a may be either finite or infinite.

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x\to 0^+}(1 + x)^{1/x} = e$.

$\lim_{x\to 0^+} e^{\ln(1+x)^{\frac{1}{x}}}$

$\bigstar\ \lim_{x\to 0^+} \frac{\ln(1+x)}{x}$

$= \lim_{x\to 0^+} \frac{1}{x+1} = 1$

EXAMPLE 8 Find $\lim_{x\to\infty} x^{1/x}$.

$x^{\frac{1}{x}}$

$(e^{\ln x})^{\frac{1}{x}}$

$\lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^0 = 1$

$1^\infty \quad f_1^{g\to\infty} = e^{g\ln f}$

$\infty^0 \quad f_{\downarrow\infty}^{g\to 0} = e^{g\ln f \searrow \infty}$

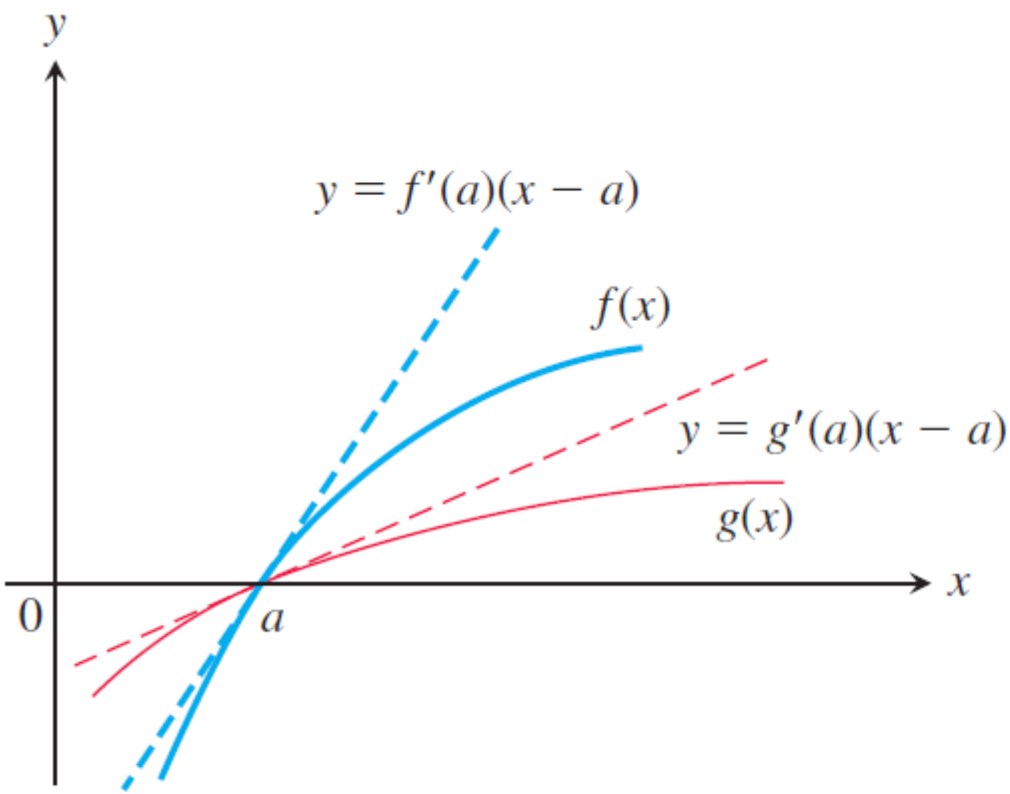


FIGURE 7.19 The two functions in l'Hôpital's Rule, graphed with their linear approximations at $x = a$.

Before we prove l'Hôpital's Rule, we consider a special case to provide some geometric insight for its reasonableness. Consider the two functions $f(x)$ and $g(x)$ having *continuous* derivatives and satisfying $f(a) = g(a) = 0$, $g'(a) \neq 0$. The graphs of $f(x)$ and $g(x)$, together with their linearizations $y = f'(a)(x - a)$ and $y = g'(a)(x - a)$, are shown in Figure 7.19. We know that near $x = a$, the linearizations provide good approximations to the functions. In fact,

$$f(x) = f'(a)(x - a) + \epsilon_1(x - a) \quad \text{and} \quad g(x) = g'(a)(x - a) + \epsilon_2(x - a)$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $x \to a$. So, as Figure 7.19 suggests,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f'(a)(x - a) + \epsilon_1(x - a)}{g'(a)(x - a) + \epsilon_2(x - a)} \\ &= \lim_{x\to a}\frac{f'(a) + \epsilon_1}{g'(a) + \epsilon_2} = \frac{f'(a)}{g'(a)} && g'(a) \neq 0 \\ &= \lim_{x\to a}\frac{f'(x)}{g'(x)}, && \text{Continuous derivatives}\end{aligned}$$

THEOREM 6—Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

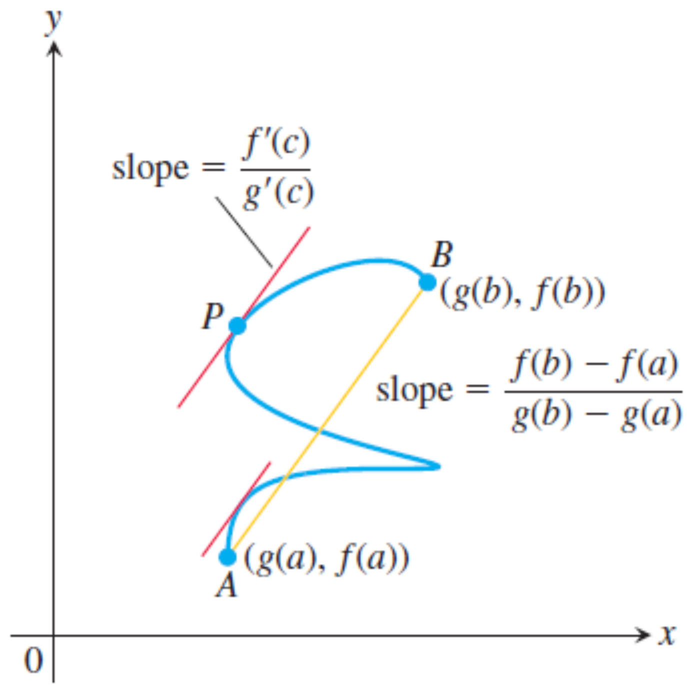


FIGURE 7.20 There is at least one point P on the curve C for which the slope of the tangent to the curve at P is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

求导分子分母都变复杂

上下同除 4^x

$$\lim_{x\to\infty}\frac{2^x-3^x}{3^x+4^x}=\lim_{x\to\infty}\frac{(\frac{1}{2})^x-(\frac{3}{4})^x}{1+(\frac{3}{4})^x}$$

$=0$

预处理

$$\lim_{x\to\infty}\frac{e^{x^2}}{xe^x}=\lim_{x\to\infty}\frac{e^{x^2-x}}{x}$$

$=\lim_{x\to\infty} e^{x^2-x}(2x-1)$

$=\infty$

For what values of a and b is

$$\lim_{x\to 0}\left(\frac{\tan 2x}{x^3}+\frac{a}{x^2}+\frac{\sin bx}{x}\right)=0?$$

$=b$

$\lim_{x\to 0}\frac{\tan 2x+ax}{x^3}=-b$

$=\lim_{x\to 0}\frac{2\sec^2 2x+a}{3x^2}$ → 分子$=0$ $\boxed{a=-2}$

$=\lim_{x\to 0}\frac{2(\sec^2 2x-1)}{3x^2}$

$=\lim_{x\to 0}\frac{2}{3}\frac{\tan^2 2x}{x^2}$

$=\frac{2}{3}\times 4=\frac{8}{3}$ $b=-\frac{8}{3}$

Find $f'(0)$ for $f(x)=\begin{cases} e^{-1/x^2}, & x\neq 0\\ 0, & x=0.\end{cases}$

分段不可直接求导 单边 ×2

左导数=右导数

so $f'(0)=0$

$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{e^{-\frac{1}{h^2}}}{h}$

直接硬求做不出来

$u=\frac{1}{h}$

$=\lim_{u\to\infty^+}\frac{u}{e^{u^2}}=\lim_{u\to\infty^+}\frac{1}{e^{u^2}\cdot 2u}=0$

$u\to-\infty$

$=\frac{1}{e^{u^2}\cdot 2u}=0$

$x=0$处任意阶导数都为0

分段找导数

讨论 $x\to c^+$, $x\to c^-$

易忽略

$*\ \lim_{x\to0}\frac{e^x-1}{x}=1$

$f\to0$ $\quad ((1+f)^{\frac{1}{f}})^{fg}$

$$\lim_{x\to\infty}\left(\frac{x^2}{(x-a)(x+b)}\right)^x$$

$\lim_{t\to0}(1+t)^{\frac{1}{t}}=e$

$\lim_{x\to0}(1+f(x))^{g(x)\to\infty}$, $f(x)\to0$

$\lim_{x\to\infty}\left(1+\frac{x^2}{(x-a)(x+b)}-1\right)^x$ $\quad f(x)=\frac{(a-b)x+ab}{(x-a)(x+b)}$

$=\lim_{x\to\infty}\left(1+\frac{(a-b)x+ab}{(x-a)(x+b)}\right)^x=\lim_{x\to\infty}(1+f(x))^{\frac{1}{f(x)}f(x)x}=\lim_{x\to0}((1+f(x))^{\frac{1}{f(x)}})^{f(x)g(x)}$

先构出来后调整

$=\lim_{x\to\infty}e^{\frac{(a-b)x^2+abx}{(x-a)(x+b)}}=e^{(a-b)}$

$$\lim_{x\to0}\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{e^x-1}}$$

$e^{-\frac{1}{2}}$

$\lim_{x\to0}\left(1+\underbrace{\frac{\ln(1+x)}{x}-1}_{f(x)}\right)^{\underline{\frac{1}{e^x-1}}\,g(x)}$

$f(x)\cdot g(x)$ $\quad \lim_{x\to0}\frac{\ln(1+x)-x}{(e^x-1)x}$

乘除可等价替换

$=\lim_{x\to0}\frac{\ln(1+x)-x}{x\cdot x}\cdot\frac{x}{e^x-1}$

再洛

$=\lim_{x\to0}\frac{\frac{1}{x+1}-1}{2x}=\lim_{x\to0}\frac{-\frac{x}{x+1}}{2x}=-\frac{1}{2}$

$\lim_{x\to0}\frac{\ln(1+x\sin(x))}{x^2}\to\frac{\ln(1+\underline{x\sin(x)}_u)}{\underline{x\sin x}_u}$

0·0 积分为0

$$\lim_{x\to0}\frac{\int_0^x t\ln(1+t\sin t)\,dt}{1-\cos(x^2)}$$

微积分基本定理

$=\lim_{x\to0}\frac{\cancel{x}\ln(1+x\sin x)}{\sin x^2\cdot 2\cancel{x}}$ 变得简单一点 $=\lim_{x\to0}\frac{\ln(1+x\sin x)}{2x\sin x}$ 相同构型 $=\frac{1}{2}$

$*\ \lim_{x\to0}\frac{x^2}{\sin x^2}=1$

$*\ \lim_{t\to0}\frac{\ln(1+t)}{t}=1$

几个基本极限化简

$$\lim_{x\to0}\left(\frac{1}{e^x-1}-\frac{1}{\ln(1+x)}\right)$$

$\lim_{x\to0}\frac{\ln(1+x)-(e^x-1)}{\cancel{(e^x-1)}_x\cancel{\ln(1+x)}_x}=\lim_{x\to0}\frac{\ln(1+x)-(e^x-1)}{x^2}$

$=\lim_{x\to0}\frac{\frac{1}{x+1}-e^x}{2x}=\lim_{x\to0}\frac{-\frac{1}{(x+1)^2}-e^x}{2}=-1$

7.6

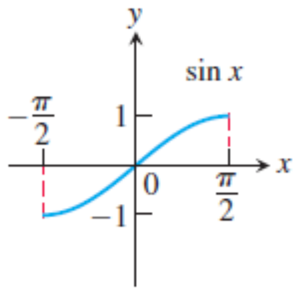
反三角函数

Inverse Trigonometric Functions

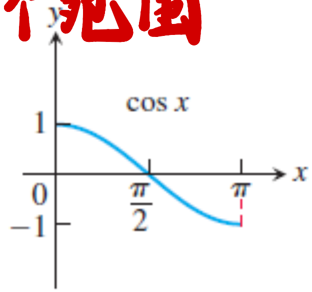
周期函数一定非一对一

→ 无反函数

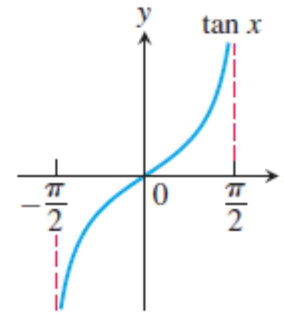
取一个范围



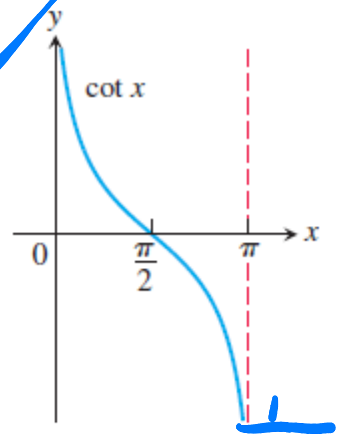
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$

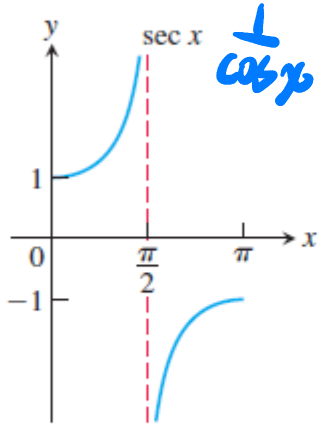


$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



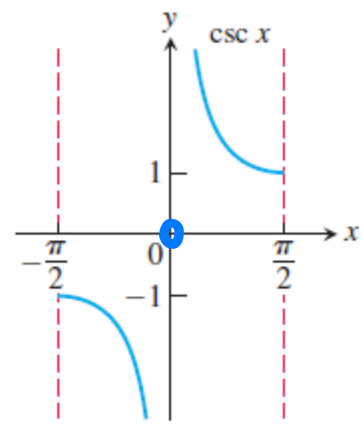
$\frac{1}{\tan x}$

$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$\frac{1}{\cos x}$

$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

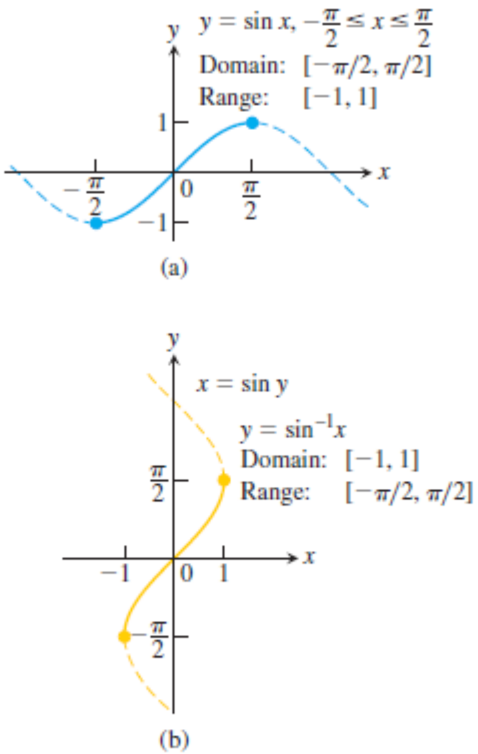


FIGURE 7.22 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

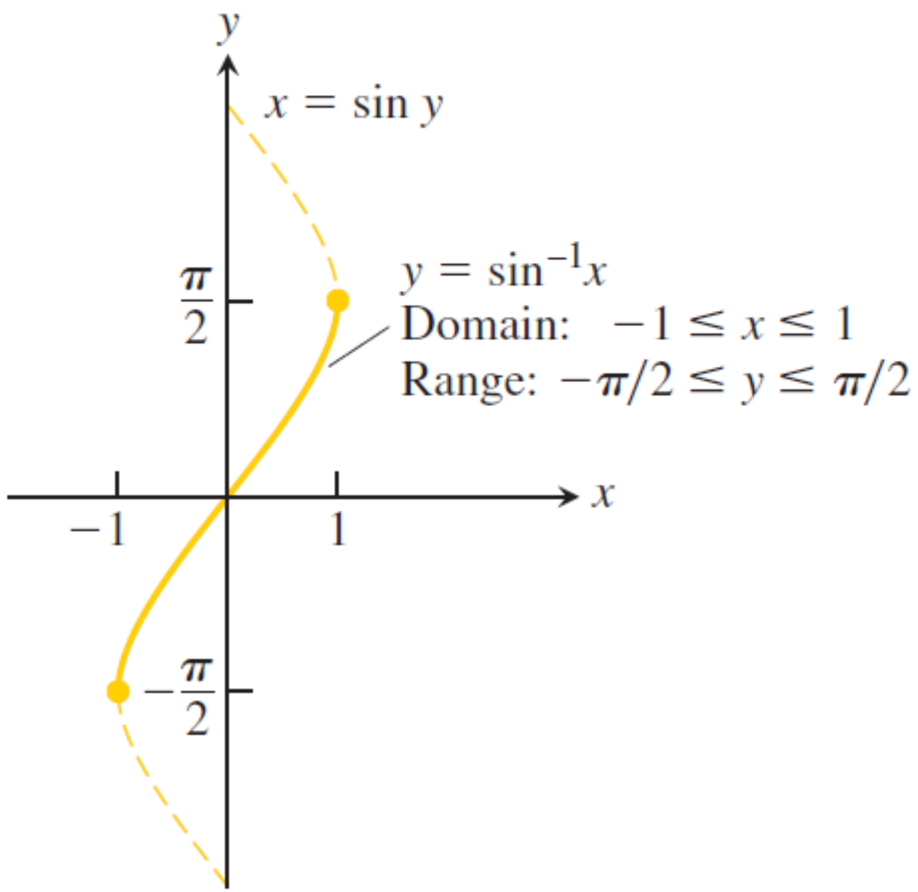


FIGURE 7.21 The graph of $y = \sin^{-1} x$.

定义域与值域

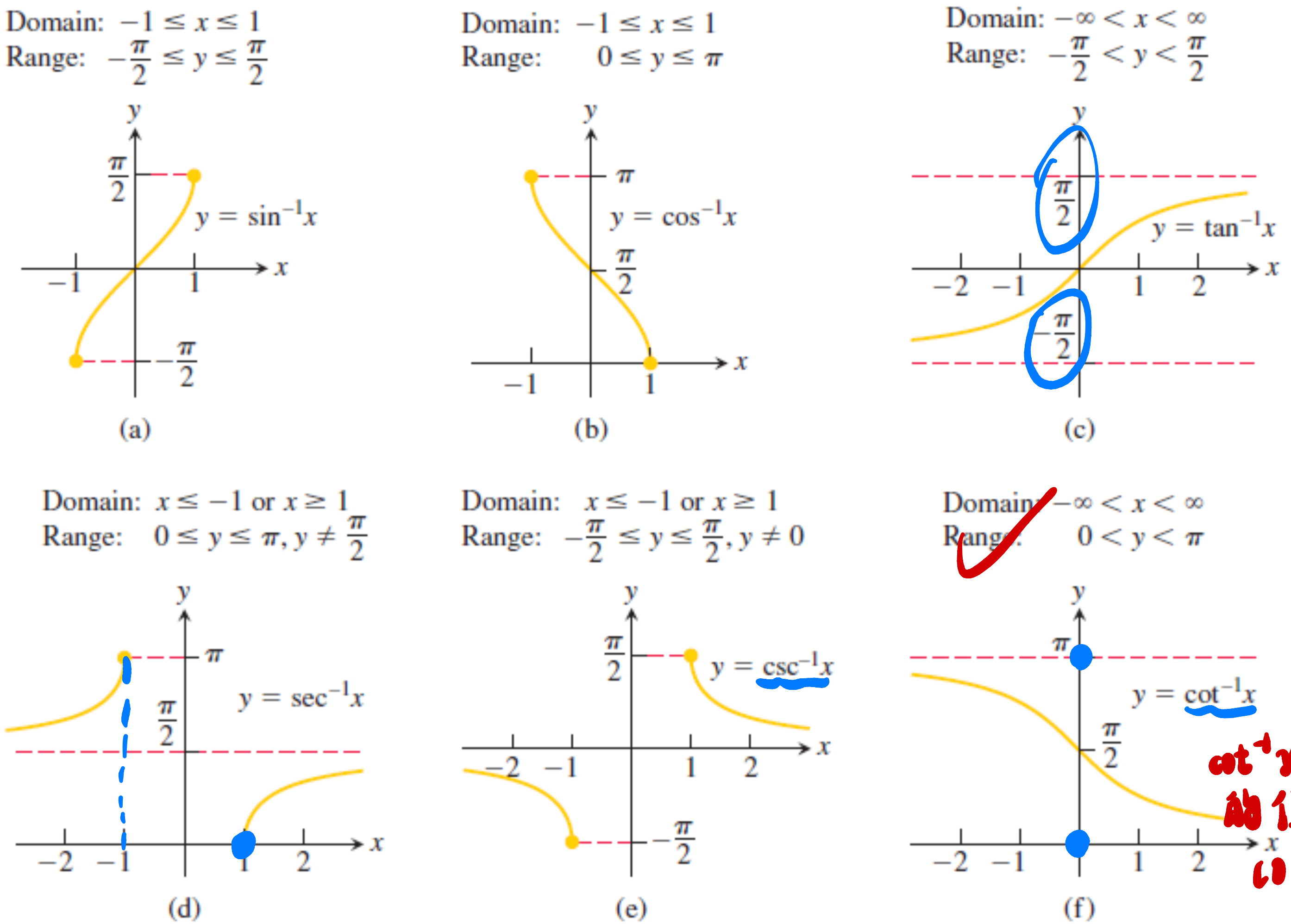
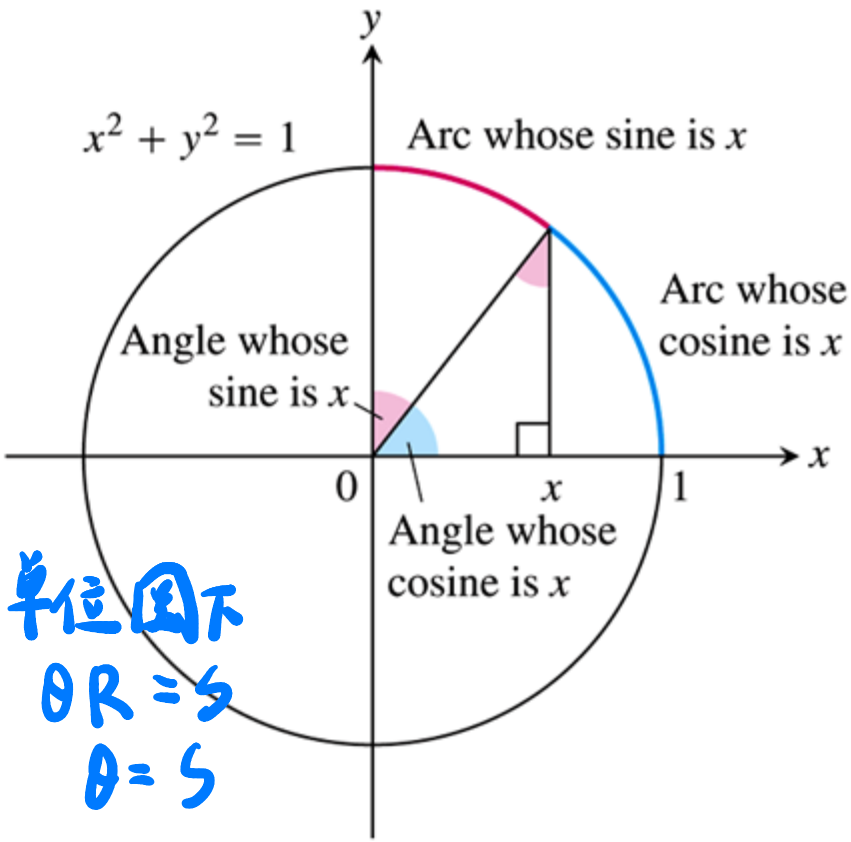


FIGURE 7.23 Graphs of the six basic inverse trigonometric functions.

弧

The “Arc” in Arcsine and Arccosine

For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y “the arc whose sine is x.”



单位圆下
$\theta R = s$
$\theta = s$

DEFINITION

角度

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

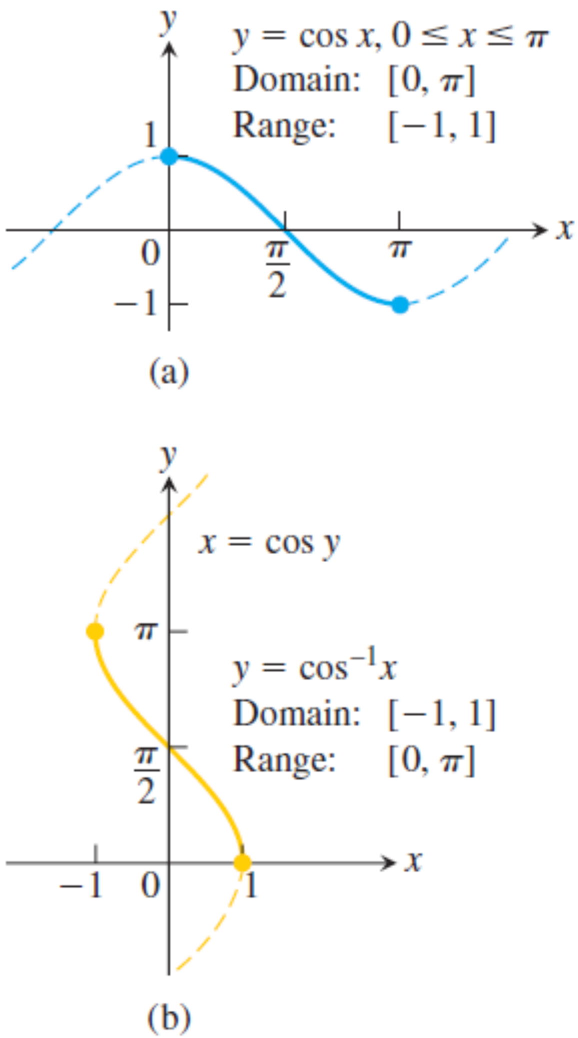


FIGURE 7.24 The graphs of (a) $y = \cos x$, $0 \leq x \leq \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

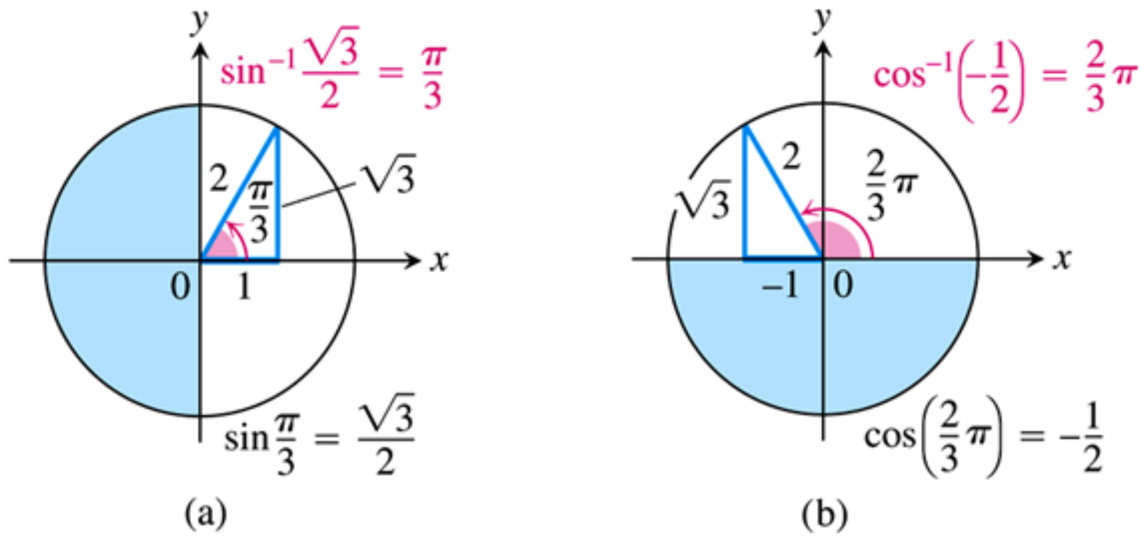


FIGURE 7.25 Values of the arcsine and arccosine functions (Example 1).

$$\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$$

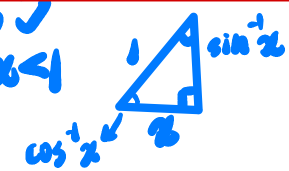
x	$\sin^{-1}x$	$\cos^{-1}x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

$\sin^{-1}x + \sin^{-1}(-x) = 0$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

(0) $x=0$ ✓

(1) $0<x<1$



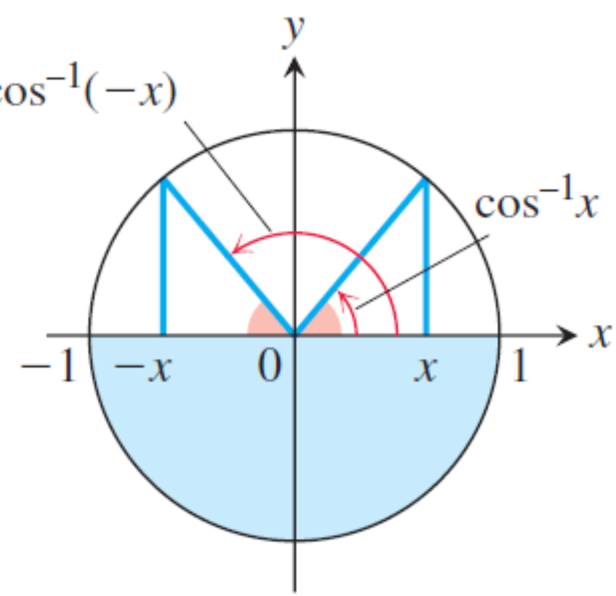
(2) $-1<x<0$

换元 $t=-x$ $0<t<1$

$\sin^{-1}x = -\sin^{-1}t$

$\cos^{-1}x = \pi - \cos^{-1}t$

So $= \pi - (\sin^{-1}t + \cos^{-1}t) = \frac{\pi}{2}$



与单位圆
找对应的角

FIGURE 7.27 $\cos^{-1}x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

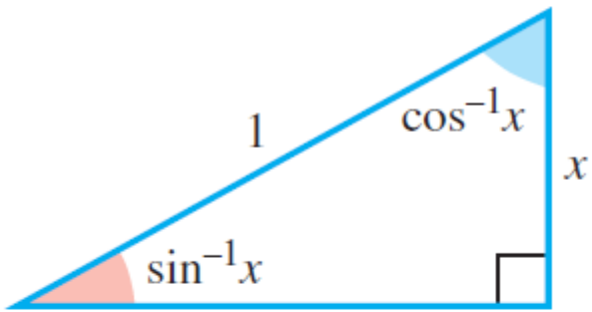


FIGURE 7.28 $\sin^{-1}x$ and $\cos^{-1}x$ are complementary angles (so their sum is $\pi/2$).

Identities Involving Arcsine and Arccosine

As we can see from Figure 7.27, the arccosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{2}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{3}$$

Also, we can see from the triangle in Figure 7.28 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{4}$$

DEFINITIONS

$\mathbf{y = \tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$\mathbf{y = \cot^{-1} x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

$\mathbf{y = \sec^{-1} x}$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

$\mathbf{y = \csc^{-1} x}$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

Domain: $|x| \geq 1$
Range: $0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}$

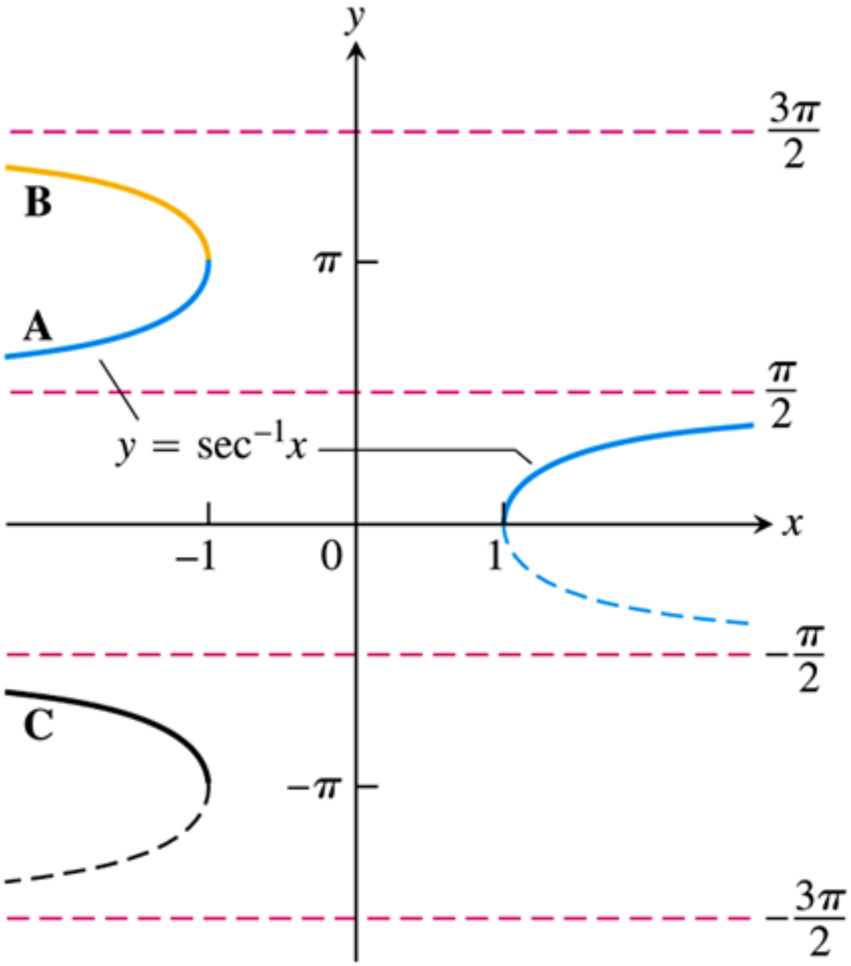


FIGURE 7.29 There are several logical choices for the left-hand branch of $y = \sec^{-1}x$. With choice **A**, $\sec^{-1}x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

Caution There is no general agreement about how to define $\sec^{-1} x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some texts choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 7.29). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$. From this, we can derive the identity

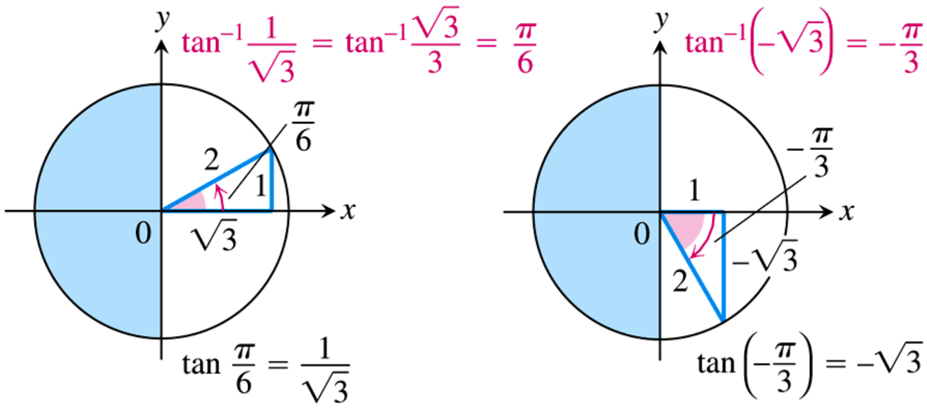
$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \tag{5}$$

$\sec^{-1} x = \cos^{-1}(\frac{1}{x})$

$\csc^{-1} x = \sin^{-1}(\frac{1}{x})$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

EXAMPLE 3 The accompanying figures show two values of $\tan^{-1} x$.



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

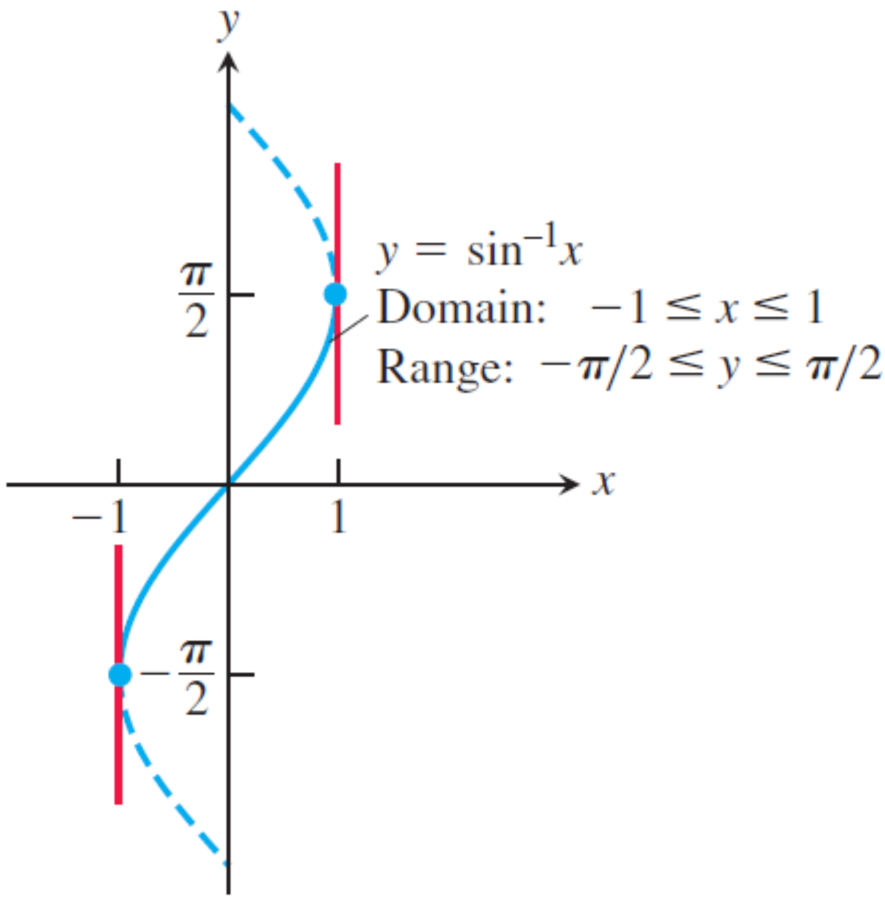


FIGURE 7.30 The graph of $y = \sin^{-1}x$ has vertical tangents at $x = -1$ and $x = 1$.

$y = \sin^{-1} x$ $\quad \sin^2 y + \cos^2 y = 1$

$\sin y = x$ $\quad \cos y = \sqrt{1-x^2}$

$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$，即只可能正

对x求导 $\quad \cos y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y}$ $\quad$ so $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ → 2个端点无导数

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}.$$

$\tan y = x$

$\sec^2 y \frac{dy}{dx} = 1$ $\quad \sec^2 y = 1 + \tan^2 y$

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$

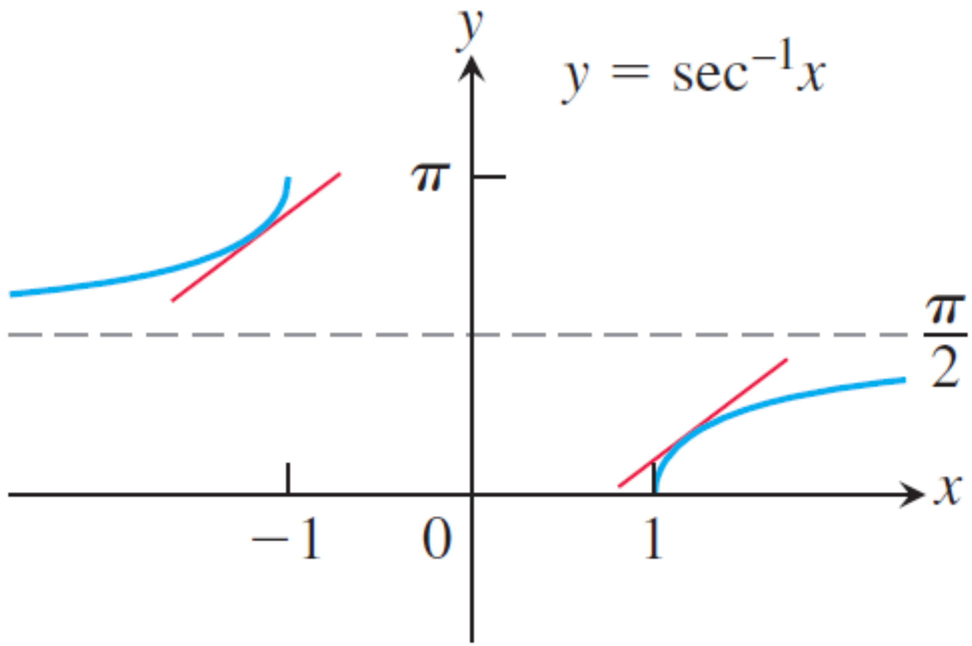


FIGURE 7.31 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

$y = \sec^{-1} x \quad y \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

$y = 0 / \pi$ 时垂直切线 $\quad x \geq 1 \quad x \leq -1$

$\sec y = x$

$\sec y \tan y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$

① $x \geq 1 \quad \tan y > 0$

$\sec^2 y = 1 + \tan^2 y$

$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$

② $x \leq -1$

$\tan y < 0$

$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}}$

综上 $\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

“统一”

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \boxed{\pi/2} - \sin^{-1} x$$
$$\cot^{-1} x = \boxed{\pi/2} - \tan^{-1} x$$
$$\csc^{-1} x = \boxed{\pi/2} - \sec^{-1} x$$

知一导数
⇒另一导数 + 负号即可

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1 + u^2} \frac{du}{dx}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$x = \frac{u}{a}$ $du = dx \cdot a$ 换元 ← 最基本的

$$\Rightarrow \sin^{-1}\frac{u}{a} + C = \int \frac{1}{\sqrt{1-\frac{u^2}{a^2}}}\frac{du}{a} = \int \frac{1}{\sqrt{a^2-u^2}}\,du$$

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx$$

$$\sec^{-1}(-x) + \sec^{-1}x = \pi$$

$$\sec^{-1}x = \pi - \sec^{-1}(-x)$$

when $x > 1$ $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$

统一

$$= \sec^{-1}|x| + C$$

$x < -1$ $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = -\sec^{-1}x + C$

$$= \sec^{-1}(-x) + C$$

ALWAYS LEARNING

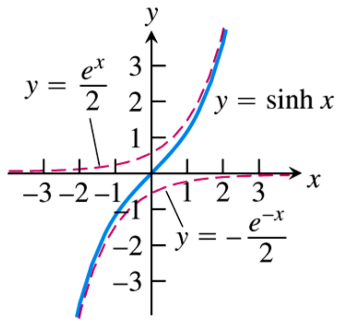
PEARSON

7.7

Hyperbolic Functions

双曲

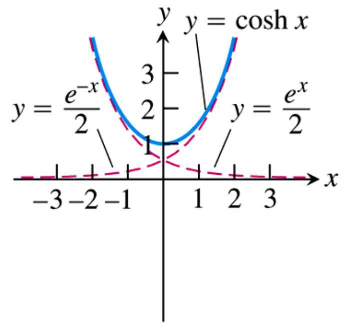
TABLE 7.5 The six basic hyperbolic functions



(a)

Hyperbolic sine:

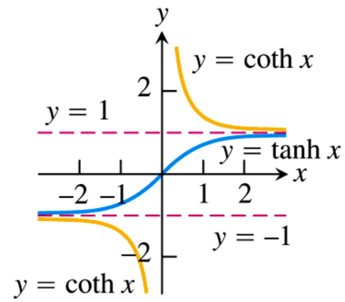
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



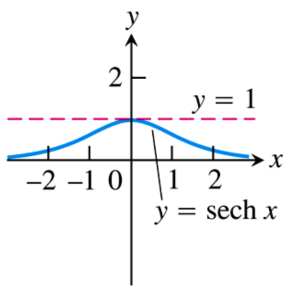
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

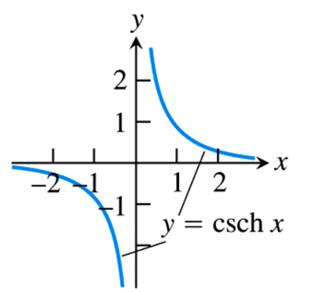
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$ 平方差

$$\sinh 2x = 2 \sinh x \cosh x$$ 倍角公式

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

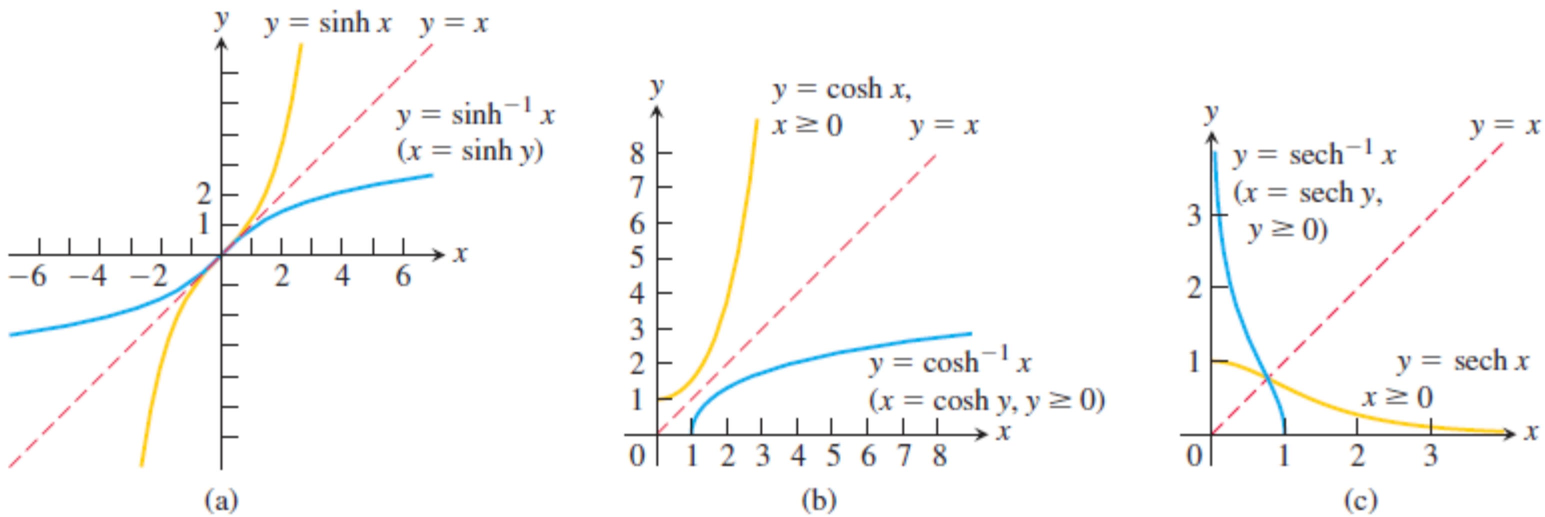


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

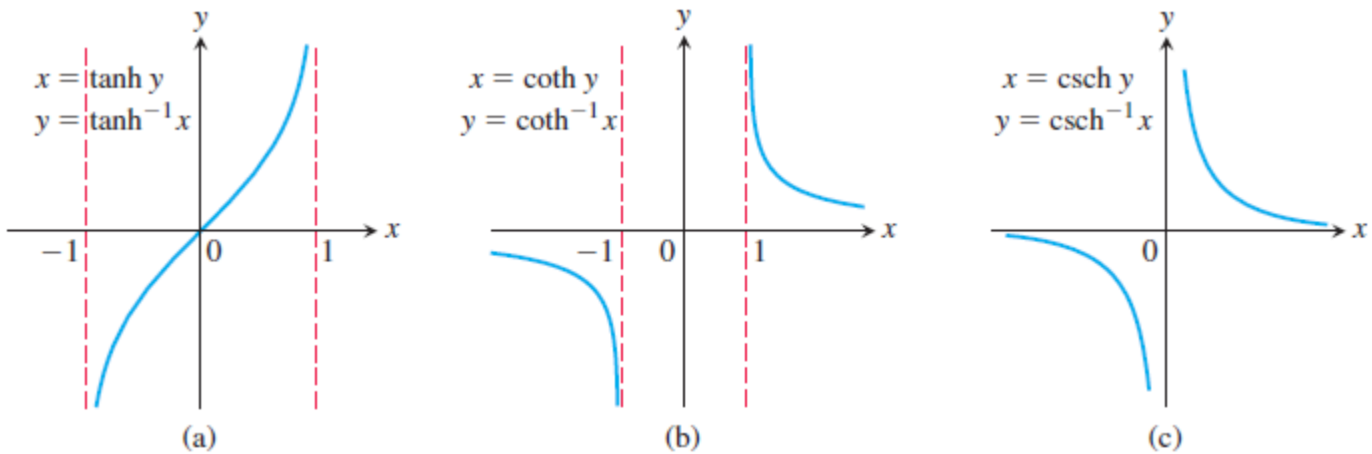


FIGURE 7.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1 - u^2}} \frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1 + u^2}} \frac{du}{dx}, \qquad u \neq 0$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1.	$\displaystyle\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,$	$a > 0$
2.	$\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,$	$u > a > 0$
3.	$\displaystyle\int \frac{du}{a^2 - u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & u^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$	
4.	$\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C,$	$0 < u < a$
5.	$\displaystyle\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left\lvert\frac{u}{a}\right\rvert + C,$	$u \neq 0$ and $a > 0$

7.8

（可不讲）

Relative Rates of Growth

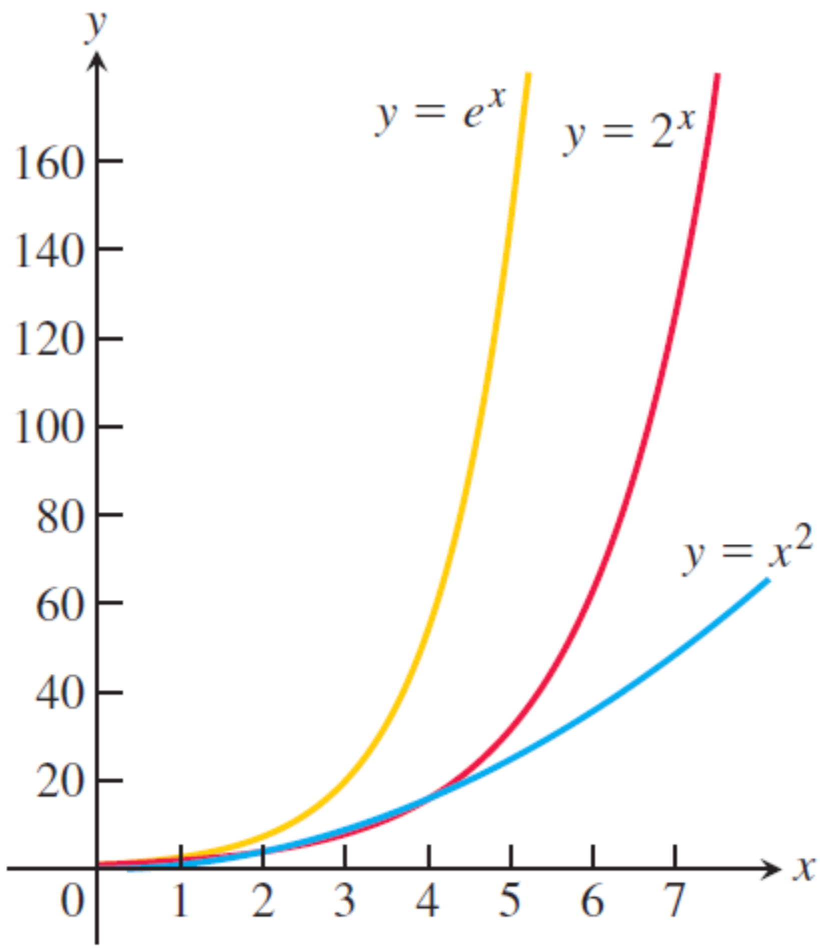


FIGURE 7.34 The graphs of e^x, 2^x, and x^2.

$\ln x < x^p < e^x$
$p > 0$

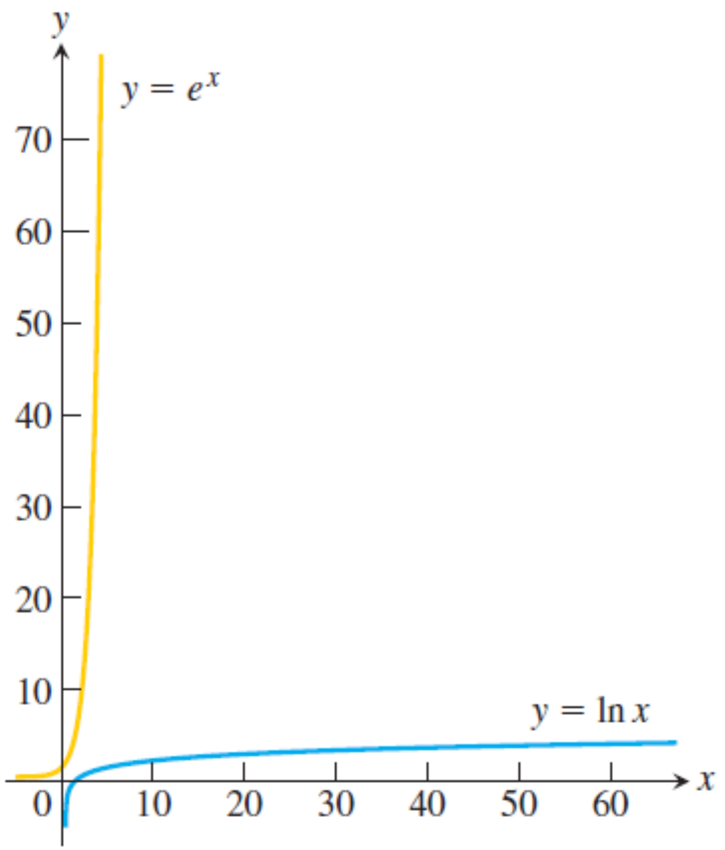


FIGURE 7.35 Scale drawings of the graphs of e^x and $\ln x$.

无穷大分级

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

一定是∞判断

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

低阶

DEFINITION A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f} = \boldsymbol{o}(\boldsymbol{g})$ ("f is little-oh of g").

无穷时判断

低阶

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \le M,$$

若 $f(x) = \frac{1}{2} g(x)$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

$o(g) \to O(g) \to$ 可以同阶

但 $O(g)$ 不一定 $o(g)$

EXAMPLE 3 Here we use little-oh notation. 记号

(a) $\ln x = o(x)$ as $x \to \infty$ because $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = 0$

(b) $x^2 = o(x^3 + 1)$ as $x \to \infty$ because $\displaystyle\lim_{x\to\infty} \frac{x^2}{x^3 + 1} = 0$ ■

EXAMPLE 4 Here we use big-oh notation.

O(g) 可以同阶

(a) $x + \sin x = O(x)$ as $x \to \infty$ because $\dfrac{x + \sin x}{x} \le 2$ for x sufficiently large.

(b) $e^x + x^2 = O(e^x)$ as $x \to \infty$ because $\dfrac{e^x + x^2}{e^x} \to 1$ as $x \to \infty$.

(c) $x = O(e^x)$ as $x \to \infty$ because $\dfrac{x}{e^x} \to 0$ as $x \to \infty$. ■