



考试科目: 高等数学(上)

开课单位: 数学系

考试时长: 120 分钟

命题教师:

题号	1	2	3	4	5	6	7	8
分值	20 分	20 分	8 分	10 分	7 分	10 分	16 分	9 分

本试卷共 8 道大题, 满分 100 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

注意: 本试卷里的中文为直译 (即完全按英文字面意思直接翻译), 所有数学词汇的定义请参考教材 (Thomas' Calculus, 13th Edition) 中的定义。如果其中有些数学词汇的定义不同于中文书籍 (比方说同济大学的高等数学教材) 里的定义, 以教材 (Thomas' Calculus, 13th Edition) 中的定义为准。

1. (20pts) **Multiple Choice Questions:** (only one correct answer for each of the following questions.)

- (1) If $f(x) = \frac{ax+b}{x^2-1}$ has a local extreme of 1 at $x = 3$, then
 - (A) $a = 3, b = -1$.
 - (B) $a = 4, b = -4$.
 - (C) $a = 5, b = -7$.
 - (D) $a = 6, b = -10$.
- (2) Let $f(x) = \frac{\tan x}{|x|(x - \frac{\pi}{2})^4}$. Which of the following statements must be **correct**?
 - (A) f is continuous at $x = 0$ and f has a jump discontinuity at $x = \frac{\pi}{2}$.
 - (B) f has a jump discontinuity at $x = 0$ and f is continuous at $x = \frac{\pi}{2}$.
 - (C) f has an infinite discontinuity at $x = 0$ and f has an oscillating discontinuity at $x = \frac{\pi}{2}$.
 - (D) f has a jump discontinuity at $x = 0$ and f has an infinite discontinuity at $x = \frac{\pi}{2}$.
- (3) The number of asymptotes of $y = e^{\frac{1}{x^2}} \arctan \frac{x^2 + x + 1}{(x - 1)(x + 2)}$ is
 - (A) 1.
 - (B) 2.
 - (C) 3.
 - (D) 4.
- (4) Which of the following improper integrals is divergent ?
 - (A) $\int_{-1}^1 \ln|x| dx$.
 - (B) $\int_0^1 \frac{e^x}{\sqrt{1-x}} dx$.
 - (C) $\int_0^{+\infty} \frac{1}{x^2} dx$.
 - (D) $\int_e^{+\infty} \frac{1}{x \ln^2 x} dx$.
- (5) Suppose that $a < 0 < b$, and $f(x)$ is continuous on (a, b) . Let

$$F(x) = \begin{cases} \frac{\int_0^x t f(t) dt}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Which of the following statements must be **correct**?

- (A) F is differentiable on (a, b) and F' is not continuous at $x = 0$.
- (B) F is differentiable on (a, b) and F' is continuous at $x = 0$.
- (C) F is not differentiable on (a, b) and F is continuous at $x = 0$.
- (D) None of the above statements is correct.

2. (20 pts) Fill in the blanks.

- (1) The number of the real roots for the equation $x^3 - 4x^2 + x + 1 = 0$ is _____.
- (2) If $f(x) = (1+x)(1+2x)\cdots(1+10x)$, then $f'(0) =$ _____.
- (3) Use Euler's method to find the approximation for the solution of

$$y' = 1 + xy, \quad y(0) = 1.$$

Take $dx = 0.5$, and start at $x_0 = 0, y_0 = 1$. Then $y_2 =$ _____.

- (4) If $f(x) = \arctan \frac{1+x}{1-x}$, then $f'(0) =$ _____.
- (5) If $\int_0^1 \frac{e^x}{x+1} dx = a$, then $\int_0^1 \frac{e^x}{(x+1)^2} dx =$ _____.

3. (8 pts) Solve the following first-order linear differential equation

$$xy' + 2y = x^2 + 1, \quad x > 0.$$

4. (10 pts)

- (1) Find the area of the region enclosed by the curves $y = x^2 - 2x$, $y = 0$, $x = 1$, and $x = 3$.
- (2) Find the volume of the solid generated by revolving the region in (1) about the y -axis.
- (5) (7 pts) Assume that f is differentiable at $x = 1$, and $f(1) = 1$, $f'(1) = 2$. Find the value of $\lim_{n \rightarrow \infty} \left(f\left(1 + \frac{1}{n}\right) \right)^n$.

6. (10 pts) Evaluate the following limits.

- (1) $\lim_{x \rightarrow 0^+} \frac{\ln \tan 7x}{\ln \tan 2x}$.
- (2) $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2} \sin^2 \left(1 + \frac{k}{n}\right)$.

7. (16 pts) Evaluate the integrals.

- (1) $\int x \tan^2 x dx$.
- (2) $\int_0^{+\infty} \frac{\tan^{-1} e^x}{e^{2x}} dx$.
- (3) $\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)^2} dx$.

$$(4) \int \frac{\ln(1-x^2)}{x^2\sqrt{1-x^2}} dx.$$

8. (9 pts) Assume $f(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. If $f(0) = f(1) = 0$, $f(\frac{1}{2}) = 1$, prove that:

- (1) there exists $c \in (\frac{1}{2}, 1)$, such that $f(c) = c$.
- (2) For any real number k , there always exists $\xi \in (0, c)$, such that $f'(\xi) - k[f(\xi) - \xi] = 1$.

一、(20分) 单项选择题:

$$f'(x) = \frac{a(x^2-1)-(ax+b)2x}{(x^2-1)^2} = 0$$

(1) 若函数 $f(x) = \frac{ax+b}{x^2-1}$ 在 $x=3$ 处有一个局部极值, 其值为 $f(3)=1$. 则

$$(A) a=3, b=-1. \quad (B) a=4, b=-4.$$

$$(C) a=5, b=-7. \quad (D) a=6, b=-10.$$

(2) 设函数 $f(x) = \frac{\tan x}{|x|(x-\frac{\pi}{2})^4}$. 下列叙述中哪一个是正确的?

(A) f 在 $x=0$ 处连续, 且在 $x=\frac{\pi}{2}$ 处有一个跳跃间断.

(B) f 在 $x=0$ 处有一个跳跃间断, 且在 $x=\frac{\pi}{2}$ 处连续.

(C) f 在 $x=0$ 处有一个无穷间断, 且在 $x=\frac{\pi}{2}$ 处有一个震荡间断.

(D) f 在 $x=0$ 处有一个跳跃间断, 且在 $x=\frac{\pi}{2}$ 处有一个无穷间断.

(3) 曲线 $y = e^{\frac{1}{x^2}} \arctan \frac{x^2+x+1}{(x-1)(x+2)}$ 的渐近线有 $y=0$

(A) 1 条. $x \rightarrow 0^+$

(B) 2 条.

(C) 3 条. $x \rightarrow 0^-$

(D) 4 条.

(4) 下列反常积分中发散的是

$$(A) \int_{-1}^1 \ln|x| dx. \quad x \ln x - x \Big|_0^1$$

$$(C) \int_0^{+\infty} \frac{1}{x^2} dx. \quad -\frac{1}{x} \Big|_0^{+\infty}$$

(5) 若 $a < 0 < b$, 且函数 $f(x)$ 在 (a, b) 上连续, 设

$$F(x) = \begin{cases} \frac{\int_a^x t f(t) dt}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

参数是否连续

$$\Rightarrow \lim_{x \rightarrow 0} F'(x) = f(0)$$

下列叙述中哪一个是正确的?

(A) F 在 (a, b) 可导, 且 F' 在 $x=0$ 处不连续,

(B) F 在 (a, b) 可导, 且 F' 在 $x=0$ 处连续.

(C) F 不在 (a, b) 中的所有点处可导, 且 F 在 $x=0$ 处连续.

(D) 上述结论都不对.

二、(20分) 填空题:

(1) 方程 $x^3 - 4x^2 + x + 1 = 0$ 的实根个数是 3.

(2) 设函数 $f(x) = (1+x)(1+2x) \cdots (1+10x)$, 则 $f'(0) = 55$.

(3) 使用欧拉法求微分方程

$$\frac{1}{(1-x)^2} \cdot \frac{y'}{1-y^2} = 1 + xy, \quad y(0) = 1$$

的近似解. 若步长 $dx = 0.5$, 且初始值选取为 $x_0 = 0, y_0 = 1$, 则 $y_2 = \frac{19}{8}$.

(4) 设 $f(x) = \arctan \frac{1+x}{1-x}$, 则 $f'(0) = 1$.

(5) 若 $\int_0^1 \frac{e^x}{x+1} dx = a$, 则 $\int_0^1 \frac{e^x}{(x+1)^2} dx = \frac{a+1}{2}$.

三、(8分) 求解一阶线性微分方程

$$= \frac{e^x}{x+1} \int_0^1 + \int_0^1 \frac{e^x}{(x+1)^2} dx$$

$$y' + \frac{2}{x+1} y = x + 1$$

$$P(x) = \frac{2}{x+1}$$

$$\text{第4页/共5页} \quad \text{总页数} = 2 \ln(11+1) + C$$

$$y = \frac{1}{x+1} x^2 + \frac{1}{x+1} C$$

$$= \frac{1}{x+1} x^2 + \frac{1}{2} + \frac{C}{x+1}$$

$$= \frac{1}{x+1} x^2 + \frac{1}{2} + \frac{C}{x$$

