题 号	1	2	3	4	5	6	7
分 值	15 分	25 分	15 分	15 分	10 分	12 分	8 分

本试卷共 (7) 大题, 满分 (100) 分. **请将所有答案写在答题本上**.

This exam includes **7** questions and the score is 100 in total. **Write all your answers on the examination book**.

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) Let A be an $m \times n$ matrix and suppose $Ax = 0$ has a nonzero solution. Which of the following must be true? ( )

   (A) The row vectors of A are linearly dependent.

   (B) The column vectors of A are linearly independent.

   (C) The rank of A is $< n$.

   (D) $m = n$ and $\det(A) = 0$.

   设 A 为 $m \times n$ 矩阵. 假设 $Ax = 0$ 有非零解. 下列哪一项一定是正确的? ( )

   (A) A 的行向量线性相关.

   (B) A 的列向量线性无关.

   (C) A 的秩 $< n$.

   (D) $m = n$ 且 $\det(A) = 0$.

   (2) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and let $\alpha_1, \alpha_2, \alpha_3$ be linearly independent column vectors in $\mathbb{R}^3$. Then the rank of the vector system $A\alpha_1, A\alpha_2, A\alpha_3$ ( )

   (A) must be 1.

   (B) must be 2.

   (C) must be 3.

   (D) can be 1 or 2.

   设 $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\alpha_1, \alpha_2, \alpha_3$ 为 $\mathbb{R}^3$ 中线性无关的向量组. 则向量组 $A\alpha_1, A\alpha_2, A\alpha_3$ 的秩 ( )

   (A) 一定是 1.

   (B) 一定是 2.

   (C) 一定是 3.

   (D) 可能是 1 也可能是 2.

(3) Let A and P be square matrices of order 3 with P invertible. Suppose $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. If $P = (\alpha_1, \alpha_2, \alpha_3)$ and $Q = (\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$, then $Q^{-1}AQ =$ ( )

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

设 A 和 P 为 3 阶方阵, P 可逆. 假设 $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. 若 $P = (\alpha_1, \alpha_2, \alpha_3)$, $Q = (\alpha_1 + \alpha_2, \alpha_2, \alpha_3)$, 则 $Q^{-1}AQ =$ ( )

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(4) Let A and B be real symmetric matrices of order n. Suppose A and B are congruent. Then ( )

(A) The null spaces $N(A)$ and $N(B)$ have the same dimension

(B) A and B have the same eigenvalues

(C) A and B have the same column space

(D) A and B have the same determinant

设 A 与 B 均为 n 阶实对称矩阵. 假设 A 与 B 合同 (也称相合). 则 ( )

(A) 零空间 $N(A)$ 与 $N(B)$ 有相同的维数

(B) A 与 B 有相同的特征值

(C) A 与 B 有相同的列空间

(D) A 与 B 有相同的行列式

(5) Let Q be a real orthogonal matrix of order 3. Which of the following is false？ ( )

(A) For every real symmetric matrix A of order 3, $Q^{-1}AQ$ is symmetric.

(B) For every column vector $v \in \mathbb{R}^3$, the vectors Qv and v have the same length.

(C) There is a nonzero column vector $v \in \mathbb{R}^3$ such that $Qv = v$ or $Qv = -v$.

(D) There is an invertible real matrix P of order 3 such that $P^{-1}QP$ is diagonal.

设 Q 为 3 阶实正交矩阵. 下列哪一项论断是错误的？ ( )

(A) 对任何 3 阶实对称阵 A, $Q^{-1}AQ$ 仍为对称阵.

(B) 对任何列向量 $v \in \mathbb{R}^3$, 向量 Qv 和 v 的长度相同.

(C) 存在非零列向量 $v \in \mathbb{R}^3$ 使得 $Qv = v$ 或 $Qv = -v$.

(D) 存在 3 阶可逆实矩阵 P 使得 $P^{-1}QP$ 为对角阵.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let A, B, C and D be square matrices of order n. Suppose A is invertible. Find two square matrices X, Y such that $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ X & I_n \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$. (We denote by I_n the identity matrix of order n.)

Answer: $X =$ __________, $Y =$ __________.

设 A, B, C, D 均为 n 阶方阵. 假设 A 可逆. 写出两个方阵 X, Y 使得 $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$. (我们用 I_n 表示 n 阶单位矩阵.)

答案: $X =$ __________, $Y =$ __________.

(2) Let A be a 3×3 matrix with determinant $|A| = 4$. Then $|2A^{-1}| =$ __________

设 A 为 3×3 矩阵, 行列式 $|A| = 4$. 则 $|2A^{-1}| =$ __________

(3) Let A be a 3×3 matrix. Suppose that the sum of the diagonal entries of A is -5, and $A^2 + 2A - 3I = 0$, then the three eigenvalues of A are __________.

设 A 是 3 阶矩阵. 假设 A 的主对角线元素之和为 -5, 且满足 $A^2 + 2A - 3I = 0$. 则矩阵 A 的三个特征值是 __________.

(4) Let $L \subseteq \mathbb{R}^3$ be the line through the vector $\beta = (1, -2, 2)^T$ (and the origin). Then the projection of the vector $\alpha = (1, 0, -1)^T$ onto the line L is __________

设 $L \subseteq \mathbb{R}^3$ 为经过 (原点和) 向量 $\beta = (1, -2, 2)^T$ 的直线. 则向量 $\alpha = (1, 0, -1)^T$ 在直线 L 上的投影是 __________

(5) Suppose that the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & a & b \\ 0 & 2 & 3 \end{bmatrix}$ is similar to the matrix $B = \begin{bmatrix} 3 & & \\ & 4 & \\ & & -1 \end{bmatrix}$.

Then $b =$ ___________.

假设矩阵 $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & a & b \\ 0 & 2 & 3 \end{bmatrix}$ 和 $B = \begin{bmatrix} 3 & & \\ & 4 & \\ & & -1 \end{bmatrix}$ 相似. 则 $b =$ ___________.

3. (15 points) Let $V = \mathbf{M}_2(\mathbb{R})$ be the space of real square matrices of order 2. Let T be the linear transformation

$$T : V \longrightarrow V ; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 0 & c \\ b & a \end{bmatrix} .$$

(a) Find the matrix A of T in the ordered basis v_1, v_2, v_3, v_4, where

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad v_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} , \quad v_3 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} , \quad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

(b) Is T invertible？Why？

(c) Investigate whether the matrix A is diagonalizable.

(15 分) 设 $V = \mathbf{M}_2(\mathbb{R})$ 为 2 阶实方阵构成的向量空间. 令 T 表示如下线性变换

$$T : V \longrightarrow V ; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} 0 & c \\ b & a \end{bmatrix} .$$

(a) 求 T 在有序基 v_1, v_2, v_3, v_4 下的矩阵 A, 其中

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad v_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} , \quad v_3 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} , \quad v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

(b) T 是否是可逆的？为什么？

(c) 判定矩阵 A 是否可对角化.

4. (15 points) Let $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$.

(a) Decide whether A is positive (or negative) definite, or positive (or negative) semidefinite.

(b) Find an orthogonal matrix Q such that $Q^{-1}AQ$ is a diagonal matrix.

(c) Let S be the surface in $\mathbb{R}^3$ defined by the equation $2x^2 - 4xy + y^2 - 4yz + 1 = 0$.

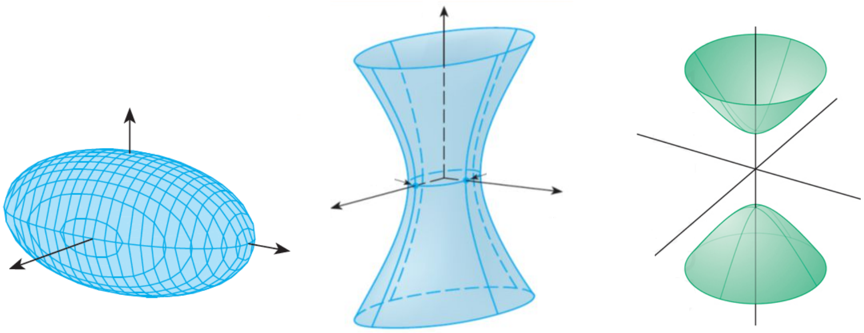
Which of the following graphs best illustrates the shape of the surface S (when the coordinate axes are suitably chosen)？(A), (B) or (C)？

(15 分) 设 $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}$.

(a) 判定 A 是否正定或负定、是否半正定或半负定.

(b) 找出一个正交矩阵 Q 使 $Q^{-1}AQ$ 为对角阵.

(c) 设 S 为 $\mathbb{R}^3$ 中由方程 $2x^2 - 4xy + y^2 - 4yz + 1 = 0$ 定义的曲面.
(当坐标轴适当选取时) 以下那个图最适合描述曲面 S 的形状? (A), (B) 还是 (C)?



(A) An ellipsoid 椭球面　　(B) A hyperboloid of one sheet 单叶双曲面　　(C) A hyperboloid of two sheets 双叶双曲面

5. (10 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) Find all the singular values of A.

(b) Find the singular value decomposition of A. That is, find two orthogonal matrices U and V (of suitable size) such that $A = U\Sigma V^T$.

(10 分) 令 $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) 求 A 的所有奇异值.

(b) 求 A 的奇异值分解. 即, 找出两个 (适当大小的) 正交矩阵 U 和 V 使得 $A = U\Sigma V^T$.

6. (12 points) Let A be an $m \times n$ complex matrix and set $B = A^H A$ (where $A^H = \overline{A}^T$ denotes the conjugate transpose of A).

(a) Prove that the eigenvalues of B in $\mathbb{C}$ are all real numbers.

(b) Suppose $m < n$. Show that 0 is an eigenvalue of B.

(c) Suppose $m = n > 1$. Is it possible that -1 is an eigenvalue of B? If yes, write down explicitly a matrix A with this property and justify your answer. Otherwise explain why such a phenomenon is impossible.

(12 分) 设 A 为 $m \times n$ 复矩阵, $B = A^H A$ (其中 $A^H = \overline{A}^T$ 表示 A 的共轭转置).

(a) 证明 B 在 $\mathbb{C}$ 中的特征值都是实数.

(b) 假设 $m < n$. 证明 0 是 B 的一个特征值.

(c) 假设 $m = n > 1$. 是否有可能 -1 是 B 的一个特征值? 若是, 请具体写出一个满足此条件的矩阵 A 并且解释你给的答案为何满足要求. 若否, 请解释为何此现象不可能出现.

7. (8 points) Let A be a real (symmetric) positive definite matrix of order n and let $\alpha_1, \cdots, \alpha_n$ be column nonzero vectors in $\mathbb{R}^n$ such that for all distinct indices $i, j \in \{1, 2, \ldots, n\}$, $\alpha_i^T A \alpha_j = 0$.

Prove that the vectors $\alpha_1, \cdots, \alpha_n$ are linearly independent.

(8 分) 设 A 为 n 阶实 (对称) 正定矩阵. 设 $\alpha_1, \cdots, \alpha_n$ 为 $\mathbb{R}^n$ 中的非零列向量. 假设对任意不同的指标 $i, j \in \{1, 2, \ldots, n\}$ 均有 $\alpha_i^T A \alpha_j = 0$.

证明向量组 $\alpha_1, \cdots, \alpha_n$ 是线性无关的.