Linear Transformations (线性变换)

Lecture 11 and 12

Dept. of Math., SUSTech

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Linear Transformations

$T: R^n \to R^m$

$x \to Ax$

* function

$T(v_1+v_2) = Tv_1 + Tv_2$

$T(\lambda v) = \lambda Tv \quad \lambda \in R$

验证 $T(x+x') = A(x+x') = Ax + Ax'$

$= Tx + Tx'$

$T(\lambda x) = A(\lambda x) = \lambda Ax = \lambda Tx$

Linear Transformations (线性变换)

- We know how a matrix moves subspaces around when we multiply by A. The nullspace goes into the zero vector. All vectors go into the column space, since Ax is always a combination of the columns.
- You will soon see something beautiful–that A takes its row space into its column space, and on those spaces of dimension r it is 100% invertible.
- That is the real action of A. It is partly hidden by nullspaces and left nullspaces, which lie at right angles and go their own way (toward zero).
- What matters now is what happens inside the space–which means inside n-dimensional space, if A is n by n. That demands a closer look.

$\underset{m\times n}{T(x)=A}\underset{n\times 1}{x}$ $T(u+v)=Tu+Tv$

$T(\lambda v)=\lambda Tv$

(该操作为左乘 A)

若取该操作为转置，也满足以上两个要求

例：是否有矩阵使 x 转置

$T(x)=Ax$ $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $x=\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

$T(x)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}=\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}$

不一定恒成立

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

取特殊情况 $= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\begin{cases} a=c=1 \\ d=0 \\ b=-1 \end{cases}$$

再取一个

矛盾

线性变换不一定是乘矩阵

Introduction

Suppose x is an n-dimensional vector. When A multiplies x, it transforms that vector into a new vector Ax. This happens at every point x of the n-dimensional space $\mathbb{R}^n$. The whole space is transformed, or "mapped into itself," by the matrix A. Here we consider four transformations that come from matrices:

1. A multiple of the identity matrix, $A = cI$, stretches every vector by the same number c.
2. A rotation matrix turns the whole space around the origin.
3. A reflection matrix transforms every vector into its image on the opposite side of a mirror.
4. A projection matrix takes the whole space onto a lower-dimensional subspace.

Four Typical Linear Transformations

See the figure:

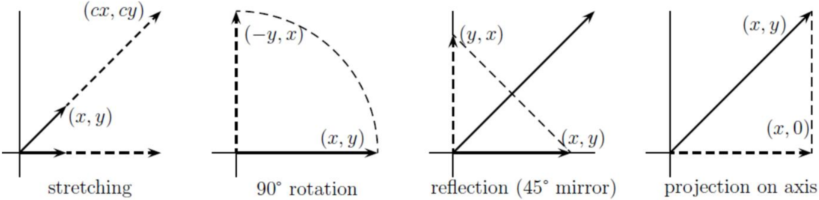
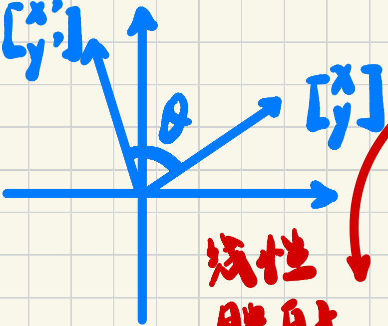


Figure 2.9: Transformations of the plane by four matrices.

The above figure illustrates four transformations that come from matrices:

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

线性映射

$$= T\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ y \end{bmatrix}\right)$$

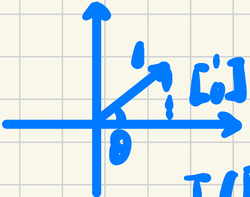
保加法

保数乘

变基向量

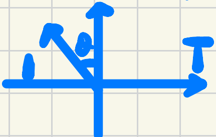
$$= xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$



$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ 旋轉矩陣

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}^{6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear Transformation

Those examples could be lifted into three dimensions. There are matrices to stretch the earth or spin it or reflect it across the plane of the equator. There is a matrix that projects everything onto that plane.

For all numbers c and d and all vectors x and y, matrix multiplication satisfies the rule of linearity:

$$A(cx+dy) = c(Ax) + d(Ay).$$

Every transformation $T(x)$ that meets this requirement is a linear transformation.

$T: V \to W$ linear transformation
线性变换

$v_1, v_2, \cdots v_m$
basis for V

$T(v) = ?$

$v = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$

$T(v) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$
$= T(c_1 v_1) + T(c_2 v_2) + \cdots + T(c_n v_n)$

$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$

对 W ↘ w_1、$w_2 \cdots w_m$ basis for W

$Tv_1 = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$

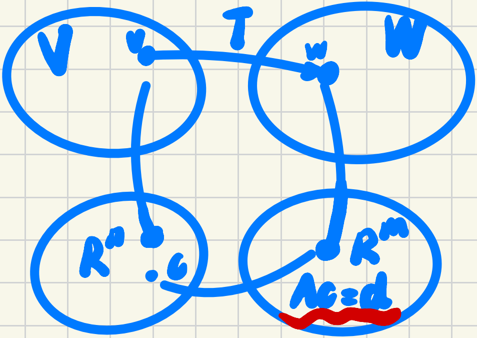
是 W 中的一个向量 → 人为写成列向量

$Tv_2 = a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m$

$\vdots$

$Tv_n = a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$



$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$T(v) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

$$= (Tv_1, Tv_2, \cdots, Tv_n) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= T(v_1, \cdots v_n) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$T(v_1, v_2, \cdots v_n) = (w_1, w_2, \cdots, w_m) \begin{bmatrix} a_{11} & a_{12} \cdots \\ a_{21} & \cdots \\ \vdots & \\ a_{m1} & \cdots a_{mn} \end{bmatrix}$$

$$\Downarrow$$

$$(w_1, w_2, \cdots w_m) AC$$

Definition

Definition

Let V, W be real vector spaces and T be a function from vector space V to W. If T satisfies the following two properties:

(a) Additivity: $T(v_1 + v_2) = T(v_1) + T(v_2)$;

(b) Homogeneity: $T(\lambda v) = \lambda Tv$;

then T is called a linear map from V to W. Where $v_1, v_2, v \in V$ and $\lambda \in \mathbb{R}$.

$Tx = Ax$ is a linear transformation.

It is also important to recognize that matrices can not do everything, and some transformations Tx are not possible with Ax.

A few remarks

- Any matrix leads immediately to a linear transformation. The more interesting question is in the opposite direction: Does every linear transformation lead to a matrix? The object of this section is to find the answer: **yes**.
- A transformation need not go from $\mathbb{R}^n$ to the same space $\mathbb{R}^n$. It is absolutely permitted to transform vectors in $\mathbb{R}^n$ to vectors in a different space $\mathbb{R}^m$.
- The operations in the linearity condition are addition and scalar multiplication, but x and y need not be column vectors in $\mathbb{R}^n$, and they may actually be polynomials or matrices or functions. As long as the transformation satisfies the rule of linearity, it is linear.

Linearity: Examples

- The operation of differentiation, $A = d/dt$, is linear:

$$Tp(t) = \frac{d}{dt}(a_0 + a_1 t + \cdots + a_n t^n) = a_1 + \cdots + n a_n t^{n-1}.$$

常值函数 ? 零空间维数

Nullspace? Column space? Nullity? Rank?

- Integration from 0 to t is also linear(it takes P_n to P_{n+1}):

$$Tp(t) = \int_0^t (a_0 + a_1 t + \cdots + a_n t^n) dt = a_0 t + \cdots + \frac{a_n}{n+1} t^{n+1}.$$

No nullspace?

- Multiplication by a fixed polynomial like $2 + 3t$ is linear:

$$Tp(t) = (2 + 3t)(a_0 + a_1 t + \cdots + a_n t^n) = 2a_0 + \cdots + 3a_n t^{n+1}.$$

No nullspace?

Transformations Represented by Matrices

Linearity has a crucial consequence:

Theorem

If we know Ax for each vector in a basis, then we know Ax for each vector in the entire space. * 抓基向量怎么变换

Example 4 What linear transformation takes x_1 and x_2 to Ax_1 and Ax_2?

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ goes to } Ax_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}; x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ goes to } Ax_2 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

Solution. $T(x) = Ax$, where $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$.

Transformation from one space to another

In short, the matrix carries all the essential information. If the basis is known, and the matrix is known, then the transformation of every vector is known.

The coding of the information is simple. To transform a space to itself, one basis is enough. A transformation from one space to another requires a basis for each.

Matrix representation of Linear Transformation

Definition

Suppose the vectors

$$v_1, v_2, \cdots, v_n$$

are a basis for the space V, and vectors

$$w_1, w_2, \cdots, w_m$$

are a basis for W. 用具体的矩阵表示抽象变换

Each linear transformation T from V to W is represented by a matrix A. The j th column is found by applying T to the j th basis vector v_j, and writing $T(v_j)$ as a combination of the y's: 写成列向量形式

$$\text{Column } j \text{ of } A: \ T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m.$$

$T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$

$X \longmapsto X^T$

$T(X) = X^T$

採作basis

$M_{2\times2}(\mathbb{R})$ basis 对矩阵空间 : $\underset{v_1}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}$ $\underset{v_2}{\begin{bmatrix}0&1\\0&0\end{bmatrix}}$ $\underset{v_3}{\begin{bmatrix}0&0\\1&0\end{bmatrix}}$ $\underset{v_4}{\begin{bmatrix}0&0\\0&1\end{bmatrix}}$

$$Tv_1 = 1\begin{bmatrix}1&0\\0&0\end{bmatrix} + 0\begin{bmatrix}0&1\\0&0\end{bmatrix} + 0\begin{bmatrix}0&0\\1&0\end{bmatrix} + 0\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$Tv_2 = 0\begin{bmatrix}1&0\\0&0\end{bmatrix} + 0\begin{bmatrix}0&1\\0&0\end{bmatrix} + 1\begin{bmatrix}0&0\\1&0\end{bmatrix} + 0\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

…… Tv_3 , Tv_4

系数写成列向量

$$A=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}_{4\times4}$$

矩阵大小与基向量有关　操作几个基向量

$Ac=d$

得和坐标相乘　坐标变换

$$X=a\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+d\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$=(v_1,v_2,v_3,v_4)\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\vec{c}$$

$$A\vec{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix}$$ 新的坐标

$$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Example

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T\right) = \begin{bmatrix} x_2 & x_1 + x_2 & x_1 - x_2 \end{bmatrix}^T$$

Find the matrix representation of T with respect to the ordered bases $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$, where

$$u_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T, u_2 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$$

and

$$v_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T,\ v_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T,\ v_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T.$$

写成一列一列的坐标变换

$$Tu_1 = \underline{a_{11}}\, v_1 + \underline{a_{21}}\, v_2 + \underline{a_{31}}\, v_3$$

坐标分量 × 基底

$$Tu_2 = a_{12}\, v_1 + a_{22}\, v_2 + a_{32}\, v_3$$

$$u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

坐标 $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{变换}} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$

$$Tu_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{21} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_{31} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$T_{u2}=\begin{bmatrix}1\\4\\2\end{bmatrix}=a_{12}\begin{bmatrix}1\\0\\0\end{bmatrix}+a_{22}\begin{bmatrix}1\\1\\0\end{bmatrix}+a_{32}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix}$$

新基底　　基底变换后

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}=\begin{bmatrix}2&1\\3&4\\-1&2\end{bmatrix}$$

基向量组模表示-矣证

$$\underset{B}{\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}}\ \underset{C}{\begin{bmatrix}2&1\\3&4\\-1&2\end{bmatrix}}\rightarrow\begin{bmatrix}1&0&0&-1&-3\\0&1&0&4&2\\0&0&1&-1&2\end{bmatrix}$$

$B^{-1}B=I$　　$B^{-1}C$

作同样行变换

高斯约旦方法

Solution.

求: 新基底 · A = 像的坐标 (原基础变换后)

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 2 \end{bmatrix}.$$

Then the matrix representation is given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = A^{-1}B = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}.$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1+x_2 \\ x_1-x_2 \end{bmatrix}$$

两基底

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

这才是坐标

$$c = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(原基底下)坐标

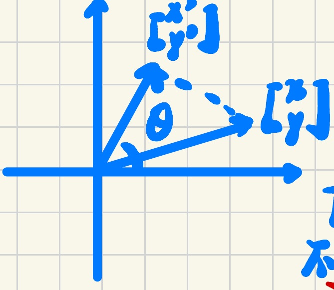
$$A \cdot c = d$$

新坐标(新基底下)

$$Ac = \begin{bmatrix} -1 & 3 \\ 4 & 2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} = d \rightarrow \text{3维坐标}$$

$$2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$



（线性变换）

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ y \end{bmatrix}\right)$$

$T(x) = ax+b \ (b \neq 0)$ → 平移不是线性变换

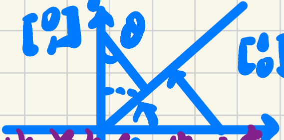
不是线性变换

一定得把0映成0

$$T(0) = T(0) + T(0) \Rightarrow T(0) = 0$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x\begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix} + y\begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

投影后 坐标有:

$y = \tan\theta\, x$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

投影矩阵 P (projection matrix)

性质：$P^T = P$

$P^2 = P$ (再投就不变了)

Examples

线性变换都可以写成$Ac=d$（变成坐标）

Example 5 Differentiation matrix.

$$p(x)=2\cdot 1+1\cdot x-x^2-x^3 = (1,x,x^2,x^3)\underbrace{\begin{bmatrix}2\\1\\-1\\-1\end{bmatrix}}_{=c}$$

$$A_{diff} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T(p)= 1-2x-3x^2 = (1,x,x^2,x^3)\underbrace{\begin{bmatrix}1\\-2\\-3\\0\end{bmatrix}}_{=d}$$

$$Ac=d$$

This matrix can differentiate $p(t)$, because matrices built in linearity!

$$\frac{dp}{dt} = Tp \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \end{bmatrix} \rightarrow 1-2t-3t^2.$$

Example

Example 6 Integration matrix.

$$\int_0^1 1dt = t \text{ or } Ax_1 = y_2, \cdots, \int_0^t t^3 dt = \frac{1}{4}t^4 \text{ or } Ax_4 = \frac{1}{4}y_5.$$

Integration matrix

$$P(t) = 1 - 2t + t^2 - \tfrac{1}{2}t^3 = (1, t, t^2, t^3)\begin{bmatrix} 1 \\ -2 \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

$$A_{int} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ \frac{1}{3} \\ -\frac{1}{8} \end{bmatrix}$$

It can be verified that differentiation is a left-inverse of integration.

Inverse Operations

Differentiation and integration are inverse operations.

$$A_{diff}A_{int} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Rotations Q

This section began with 90° rotations, projections onto the x-axis, and reflections through the 45° line. But rotations through other angles, projections onto other lines, and reflections in other mirrors are almost as easy to visualize. Figure 2.10:

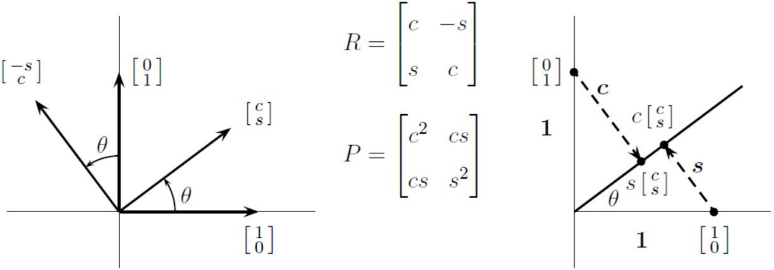


Figure 2.10: Rotation through θ (left). Projection onto the θ-line (right).

Rotations Q

Rotation through θ:

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Q_θ^{-1} and $Q_\theta Q_\varphi$:

Does the inverse of Q_θ equal $Q_{-\theta}$ (rotation backward through θ)? **Yes.**

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Does the square of Q_θ equal $Q_{2\theta}$? **Yes.**

Does the product of Q_θ and Q_φ equal $Q_{\theta+\varphi}$ (rotation through θ and φ)? **Yes.**

$A^2 = -I$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^2$$

$2\theta = \pi$

The product of the transformations

Matrix multiplication is defined exactly so that **the product of the matrices corresponds to the product of the transformations.**

Theorem

Suppose S and T are linear transformations from V to W and from U to V.

(1) *Their product ST starts with a vector u in U, goes to Tu in V, and finishes with STu in W.*

(2) *This "composition" ST is again a linear transformation (from U to W). Its matrix is the product of the individual matrices representing S and T.*

Projections P

Projection onto the θ-line

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

Remark:

- **A projection matrix equals its own square:** $P^2 = P$. 两行成比例
- The matrix P has no inverse. $P = \begin{bmatrix} c \\ s \end{bmatrix} \begin{bmatrix} c & s \end{bmatrix}$
- Points on the perpendicular line are projected onto the origin; that line is the nullspace of P.

Reflections H

反射

Figure 2.11:

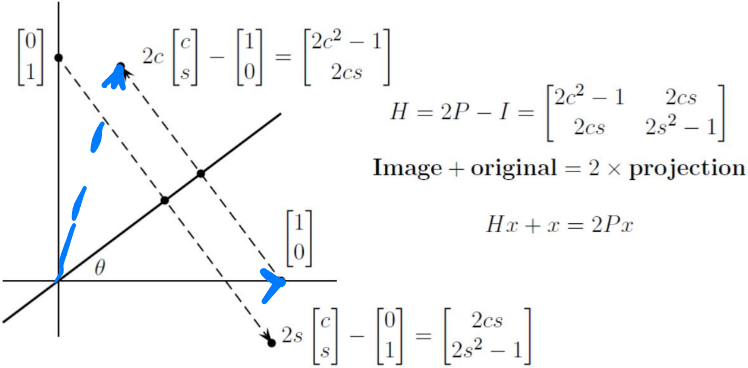
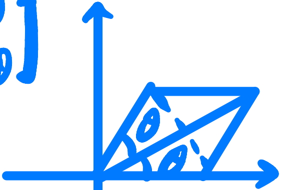


Figure 2.11: Reflection through the θ-line: the geometry and the matrix.

Remarks

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$



Reflection Matrix:

$$H = \begin{bmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to 2\begin{bmatrix} c^2 \\ cs \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2c^2-1 \\ 2cs \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 2cs \\ 2s^2-1 \end{bmatrix}$$

Remarks:

- Two reflections bring back the original: $H^2 = I$.
- $H = 2P - I$.

Remarks

- The matrix depends on the choice of basis.
- How to choose the best basis? 最好的基向量
- Change of basis.
- A single transformation is represented by different matrices.
- The theory of eigenvalues will lead to this formula $S^{-1}AS$, and to the best basis.

练习题

在 $\mathbb{R}^3$ 中考虑以下两组向量:

$[v_1\ v_2\ v_3] =$

$$u_1 = (1,0,1),\ u_2 = (2,1,0),\ u_3 = (1,1,1)$$

和

$$v_1 = (1,0,0),\ v_2 = (1,1,0),\ v_3 = (1,1,1).$$

可以验证这两组向量都是 $\mathbb{R}^3$ 的基. 假定线性变换 T 把基 u_1, u_2, u_3 映到基 v_1, v_2, v_3.

$A = (u_1, u_2, u_3)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$Tu_1 = v_1 = a_{11}u_1 + a_{21}u_2 + a_{31}u_3$

$Tu_2 = v_2 = a_{12}u_1 + a_{22}u_2 + a_{32}u_3$

$Tu_3 = v_3 = a_{13}u_1 + a_{23}u_2 + a_{33}u_3$

线性组合

(唯一性)

1. 求基 u_1, u_2, u_3 到基 v_1, v_2, v_3 的过渡矩阵;
2. 求 T 在基 u_1, u_2, u_3 下的矩阵;
3. 求 T 在基 v_1, v_2, v_3 下的矩阵;
4. 求 $T^2(u_1) = T(T(u_1))$;
5. 求 $(1,2,3)$ 在基 u_1, u_2, u_3 下的坐标.

T 在基 u_1, u_2, u_3 下的矩阵 B　　$T(u_1, u_2, u_3) = (u_1, u_2, u_3)\underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{B}$

$\Longleftrightarrow$ $Tu_1 = b_{11}u_1 + b_{21}u_2 + b_{31}u_3$

同一个空间，可取同一个基向量

$Tu_2 = b_{12}u_1 + b_{22}u_2 + b_{32}u_3$

$Tu_3 = b_{13}u_1 + b_{23}u_2 + b_{33}u_3$

练习题

设

$$v_1 = (1, -1, 5, 2),\ v_2 = (-2, 3, 1, 0),\ v_3 = (4, -5, 9, 4),$$
$$v_4 = (0, 4, 2, -3),\ v_5 = (-7, 18, 2, -8).$$

求向量组 v_1, v_2, v_3, v_4, v_5 的一个极大线性无关组, 并用极大线性无关组线性表出向量组中的其它的向量.

Homework Assignment 12

2.6: 1, 5, 6, 13, 15, 19, 33, 44, 47.