MID-SEMESTER TEST

Linear Algebra I A

This two-hour long test has 9 problems in total. Write ***all your answers*** on the examination book.

(1) (12 points, 2 points each) True or false. No need to justify.

(a) Every subspace of $\mathbb{R}^4$ is a nullspace of some matrix.

(b) If the rows of a square matrix are orthonormal, then its columns are also orthonormal.

(c) If a square matrix A has independent columns, so does A^2.

(d) If A and B are symmetric, then AB is symmetric.

(e) If the columns of A are linearly independent, then $Ax = b$ has exactly one solution for every b.

(f) Suppose that $A = A_{m\times n}, B = B_{s\times t}, C = C_{s\times n}$ are matrices, then

$$\operatorname{rank}\begin{bmatrix} A & \mathbf{0} \\ C & B \end{bmatrix} \geq \operatorname{rank}(A) + \operatorname{rank}(B).$$

(2) (9 points, 3 points each) Fill in the blanks.

(a) Suppose that A is an $m \times n$ matrix. If for any $m \times 1$ column vector b, the system of linear equations $Ax = b$ always has a solution, then $\operatorname{rank}(A) =$ __________.

(b) Suppose that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \\ 2 & 1 & t \end{bmatrix},$$

$B_{3\times 3} \neq \mathbf{0}$. If $AB = \mathbf{0}$, then $t =$ __________ and $\operatorname{rank}(B) =$ __________.

(c) The projection of a vector $b = (1,1,1)^T$ onto the line through $a = (3,2,1)^T$ is __________.

(3) (12 points) Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 1 & 3 & 5 \end{bmatrix}.$$

(i) Find an LU factorization of A.

(ii) Find the inverse A^{-1} of A.

(4) (12 points) Let

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}.$$

(i) Find a basis and the dimension for each of the four fundamental subspaces, i.e., row space, column space, nullspace and left nullspace, for the matrix A.

(ii) Let

$$x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Under what condition(s) on b_1, b_2, b_3 does $Ax = b$ have a solution?

(iii) If

$$b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

find the complete solution to $Ax = b$.

(5) (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 5 & 4 \end{bmatrix}.$$

Give a 3 by 3 orthogonal matrix $Q = [q_1\ q_2\ q_3]$, such that $q_1 \in C(A^T)$ and $q_3 \in N(A)$.

(6) (12 points)

(i) Find an orthonormal basis for the column space of

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}.$$

(ii) Write A as QR, where Q has orthonormal columns and R is upper triangular.

(iii) Find the least squares solution to $Ax = b$, if

$$b = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}.$$

(7) (9 points) Let

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \text{ and } \gamma = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

We define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ as follows:

$$T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} a_1 - a_2 \\ a_1 \\ 2a_1 + a_2 \end{bmatrix}.$$

(i) Explain why α is a basis for $\mathbb{R}^2$ and γ is a basis for $\mathbb{R}^3$.
(ii) Find the matrix representation of T with respect to α and γ.

(8) (12 points) Let W denote the subspace of $\mathbb{R}^4$ consisting of all the vectors whose components add to zero.
(i) Find the dimension of W.
(ii) Show that the vectors

$$u_1 = \begin{bmatrix} 2 \\ -3 \\ 4 \\ -3 \end{bmatrix}, u_2 = \begin{bmatrix} -6 \\ 9 \\ -12 \\ 9 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ -2 \\ 7 \\ -8 \end{bmatrix}, u_4 = \begin{bmatrix} 2 \\ -8 \\ 2 \\ 4 \end{bmatrix}, u_5 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$$

span W.
(iii) Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for W.

(9) (12 points)
(i) Let $Ax = b$ be a system of linear equations. Prove that the system is consistent if and only if $\text{rank}(A) = \text{rank}(A|b)$. The matrix $(A|b)$ is called the augmented matrix of the system $Ax = b$.

(ii) Suppose A is m by n, B is n by p, and $AB = 0$. Prove that $\text{rank}(A) + \text{rank}(B) \leq n$.

(iii) If A is an m by n matrix and rank (A) $= n$, show that $A^T A$ is invertible. Is $P = A(A^T A)^{-1} A^T$ invertible? Explain why.