Applications of Determinants

Lecture 19 and 20

Dept. of Math., SUSTech

2024.04

Formulas for the Determinant; Applications of Determinants

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \det(A)$$

$$\underbrace{a_{j1}}\,\underbrace{C_{i1}} + a_{j2}C_{i2} + \cdots + a_{jn}C_{in} = 0$$

不匹配时 $i \neq j$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = A$$

$$1\,C_{31}\,(-1)\,C_{32} \underset{(1)}{+} C_{33} =$$

$$\parallel$$

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$ 只是改变该位置的元素

$$a_{j1}C_{i1} + a_{j2}C_{i2} + \cdots + a_{jn}C_{in}$$

对 ith row co-factor expansion ↓ 再把这行元素替换

$$= \begin{vmatrix} \cdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & & & \end{vmatrix} \begin{array}{l} \\ \text{— } i\text{th row} \\ \\ \text{— } j\text{th row} \\ \\ \end{array}$$

两行完全相同

$$= 0$$

Computation of A^{-1}

The 2 by 2 case shows how cofactors go into A^{-1}:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \underline{\frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}}.$$

Cofactor matrix C is transposed

$$A^{-1} = \frac{C^T}{\det A}$$

非对角元处为0

$$\overset{A}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}} \overset{C^T}{\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}} = \begin{bmatrix} \det A & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & \det A \end{bmatrix} = \det(A) I$$

(伴随矩阵) A^*

Note: The above C should be transposed.

The critical question is: Why do we get zeros off the diagonal?

Computation of A^{-1}

If we combine the entries a_{1j} from row 1 with cofactors C_{2j} for row 2, why is the result zero?

$$a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n} = 0.$$

- We are computing the determinant of a new matrix B with a new row 2.
- The first row of A is copied into the second row of B. Then B has two equal rows, and $\det\ B = 0$.

实际上该方法不如操作

A, A^* $\quad A^* = C^T$

$(A^*)^* = A$

$\text{rank}(A^*) = n$ (满秩)

$|A^*| = (\det(A))^{n-1}$

Example

Example

The inverse of a sum matrix is a difference matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{has } A^{-1} = \frac{C^T}{\det A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The minus signs enter because cofactors always include $(-1)^{i+j}$.

The Solution of $Ax = b$.

$A^{-1} = \frac{C^T}{|A|}$

知道，但考试不这么算)

Cramer's Rule: The j th component of $x = A^{-1}b$ is just $C^T b$ divided by $\det A$. There is a famous way in which to write the answer $(x_1, x_2, \cdots, x_n)$:

The jth component of $x = A^{-1}b$ is the ratio : $x_j = \dfrac{\det B_j}{\det A}$, where

$$x = A^{-1}b = \frac{C^T}{|A|}b = \frac{1}{|A|}C^T b$$

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}.$$

(替换掉第一列)

Can you prove this rule?

$$C^T b = \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \cdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

$$b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\Rightarrow x = \frac{1}{|A|} C^T b = \begin{bmatrix} |B_1|/|A| \\ |B_2|/|A| \\ \vdots \\ |B_n|/|A| \end{bmatrix}$$

$|B_i| \rightarrow$ 第i列用b
替换A

Example

Example

The solution of

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 2x_1 + 4x_2 &= 6 \end{aligned}$$

has 0 and 6 in the first column for x_1 and in the second column for x_2:

$$x_1 = \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{-18}{-2} = 9, \; x_2 = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{6}{-2} = -3.$$

- The denominators are always $\det A$.
- For 1000 equations Cramer's Rule would need 1001 determinants.

直角

The Volume of a Box

The determinant equals the volume of a box. We first consider right-angled box, which has orthogonal rows:

$$AA^T = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } n \end{bmatrix} \begin{bmatrix} \text{column } 1 & \cdots & \text{column } n \end{bmatrix} = \begin{bmatrix} l_1^2 & & 0 \\ & \ddots & \\ 0 & & l_n^2 \end{bmatrix}.$$

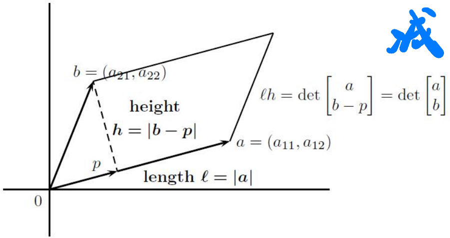
The l_i's are the lengths of the rows (the edges), and the zeros off the diagonal come because the rows are orthogonal. Using the product and transposing rules,

$$l_1^2 l_2^2 \cdots l_n^2 = \det(AA^T) = (\det A)(\det A^T) = (\det A)^2.$$

The square root of this equation says that the absolute value of the determinant of A equals the volume.

Not Right-Angle Case

It the angles are not 90°, the volume is **NOT** the product of the lengths. However, by rule 5, like the following figure shows that we can change the parallelogram to rectangle, where it is already that volume = |determinant|.

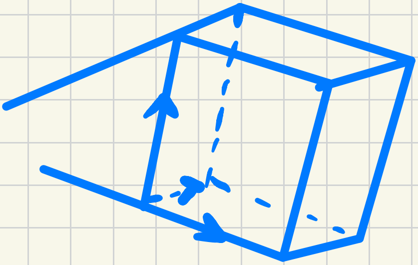


减去投影

⇒ 正交化

Figure 4.2: Volume (area) of the parallelogram = ℓ times $h = |\det A|$.

In n dimensions, the idea is the same. The Gram-Schmidt process produces orthogonal rows, with volume = |determinant|.



两异面直线 可构造
空间平行六面体

两条相异直线距离：$\dfrac{\text{体积}}{\text{底面}}$
(都用行列式算)

Volume = |Determinant|

We know that 作消元

$$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, \det\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = 1.$$

These determinants give volumes–or areas, since we are in two dimensions–drawn in Figure 4.3. The parallelogram has unit base and unit height; its area is also 1.

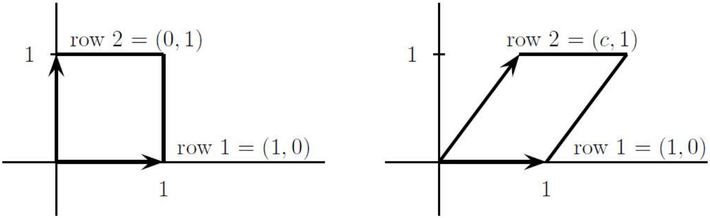


Figure 4.3: The areas of a unit square and a unit parallelogram are both 1.

A Formula for the Pivots

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{bmatrix}$$

Elimination on A includes elimination on A_2:

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A_2} = \begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix}}_{L_2}\underbrace{\begin{bmatrix} a & 0 \\ 0 & d-\frac{c}{a}b \end{bmatrix}}_{D_2}\underbrace{\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}}_{U_2}$$

$\det(A_1) = a$

$\det(A_2) = ad - bc$

$$A = \begin{bmatrix} a & b & e \\ c & d & f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} a & b & e \\ 0 & \frac{ad-bc}{a} & \frac{af-ec}{a} \\ g & h & i \end{bmatrix}$$

Actually it is not just the pivots, but the entire upper-left corners of L, D, and U, that are determined by the upper-left corner of A:

$$A = LDU = \begin{bmatrix} 1 & & \\ \frac{c}{a} & 1 & \\ * & * & 1 \end{bmatrix}\begin{bmatrix} a & & \\ & \frac{ad-bc}{a} & \\ & & * \end{bmatrix}\begin{bmatrix} 1 & \frac{b}{a} & * \\ & 1 & * \\ & & 1 \end{bmatrix}$$

$\det(A_k) = \det(L_k D_k U_k) = d_1 d_2 \cdots d_{k-1} d_k$

$\det(A_{k-1}) = \det(L_{k-1} D_{k-1} U_{k-1}) = d_1 d_2 \cdots d_{k-1}$

Formulas for pivots

What we see in the first two rows and columns is exactly the factorization of the corner submatrix A_2. This is a general rule if there are no row exchanges: 不需行交换

Proposition

If A is factored into LDU, the upper left corners satisfy $A_k = L_k D_k U_k$. For every k, the submatrix A_k is going through a Gaussian elimination of its own.

$$LDU = \begin{bmatrix} L_k & 0 \\ B & C \end{bmatrix} \begin{bmatrix} D_k & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} U_k & F \\ 0 & G \end{bmatrix} = \begin{bmatrix} L_k D_k U_k & L_k D_k F \\ B D_k U_k & B D_k F + CEG \end{bmatrix}.$$

Formula for pivots

Comparing the last matrix with A, the corner $L_k D_k U_k$ coincides with A_k. Then:

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$$\det A_k = \det L_k \det D_k \det U_k = \det D_k = d_1 d_2 \cdots d_k.$$

The product of the first k pivots is the determinant of A_k. This is the same rule we know already for the whole matrix. Since the determinant of A_{k-1} will be given by $d_1 d_2 \cdots d_{k-1}$, we can isolate each pivot d_k as a ratio of determinants:

$$\frac{\det A_k}{\det A_{k-1}} = \frac{d_1 d_2 \cdots d_k}{d_1 d_2 \cdots d_{k-1}} = d_k$$

Formulas for pivots

Multiplying together all the individual pivots, we recover

$$d_1 d_2 \cdots d_n = \frac{\det A_1}{\det A_0} \frac{\det A_2}{\det A_1} \cdots \frac{\det A_n}{\det A_{n-1}} = \frac{\det A_n}{\det A_0} = \det A.$$

The pivot entries are all nonzero whenever the numbers $\det A_k$ are all nonzero:

Proposition

Elimination can be completed without row exchanges (so $P = I$ and $A = LU$), if and only if the leading submatrices $A_1, A_2, \cdots, A_n$ are all nonsingular.

The determinant of a permutation matrix

The determinant of a permutation matrix P was the only questionable point in the big formula. Independent of the particular row exchanges linking P to I, is the number of exchanges always even or always odd?

If so, its determinant is well defined by rule 2 as either 1 or -1.

- An even number of exchanges can never produce the natural order, beginning with $(3,2,1)$.
- Let N count the pairs in which the lager number comes first. Every exchange alters N by an odd number.

$$\det(A) = \sum_{\text{all } P\text{'s}} a_{1\alpha} a_{2\beta} \cdots a_{n\gamma} \det(P)$$

$$\det(P) = (-1)^{\sigma(\alpha\beta\cdots\gamma)}$$

$(3,2,1) \rightarrow$ 逆序数 $=3$

$\downarrow$

$(2,3,1)$

$\downarrow$

$(2,1,3)$

$\downarrow$

$(1,2,3) \rightarrow$ 逆序数 $=0$

Determinant of a permutation matrix P

- Certainly $N = 0$ for the natural order $(1,2,3)$. The order $(3,2,1)$ has $N = 3$ since all pairs $(3,2),(3,1)$, and $(2,1)$ are wrong.
- We will show that every exchange alters N by an odd number. Then to arrive at $N = 0$ (the natural order) takes a number of exchanges having the same evenness and oddness as N.

每次邻居交换

- When neighbors are exchanged, N changes by $+1$ or -1.

$+1/-1$

- Any exchange can be achieved by an odd number of exchanges of neighbors. This will complete the proof; an odd number of odd numbers is odd.

考虑排序奇偶性

逆序数 $n \to k$

每次交换改变奇偶性

$(-1)^{n-k}$

One Example

- To exchange the first and fourth entries below, which happen to be 2 and 3, we use five exchanges (an odd number) of neighbors:

$$(\mathbf{2},1,4,\mathbf{3}) \to (1,\mathbf{2},4,\mathbf{3}) \to (1,4,\mathbf{2},\mathbf{3}) \to (1,4,\mathbf{3},\mathbf{2})$$
$$\to (1,\mathbf{3},4,\mathbf{2}) \to (\mathbf{3},1,4,\mathbf{2}).$$

- We need $l-k$ exchanges of neighbors to move the entry in place k to place l.
- Then $l-k-1$ exchanges move the originally in place l (and now found in place $l-1$) back down to place k.
- Since $(l-k)+(l-k-1)$ is odd, the proof is complete.

Example

Example

Find the following determinant:

$$|A| = \begin{vmatrix} 1 & 4 & -1 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 2 & 3 & 11 \\ 3 & 0 & 9 & 2 \end{vmatrix}.$$

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Further Examples

Example

Suppose the entries of A satisfy

$$a_{ij} = -a_{ji} \quad (i, j = 1, 2, \cdots, n).$$

Find det (A) if n is odd.

$$|A| = \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & & \\ \vdots & & \ddots & \\ -a_{1n} & & & 0 \end{vmatrix} = (-1)^n \det \begin{vmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ a_{12} & 0 & & \\ \vdots & & \ddots & \\ a_{1n} & & & 0 \end{vmatrix}$$

$= (-1)^n |A^T|$

$= (-1)^n |A|$

n为奇 $2|A| = 0$ $|A| = 0$

Solution. 0.

Example

Find

$$|A| = \begin{vmatrix} a_1 + b_1 & b_1 + c_1 & c_1 + a_1 \\ a_2 + b_2 & b_2 + c_2 & c_2 + a_2 \\ a_3 + b_3 & b_3 + c_3 & c_3 + a_3 \end{vmatrix}.$$

Example

Example

Find the determinant:

加起来一样

↑ 可提公因式

全加到左column

$$|A| = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix}.$$

$$= \begin{vmatrix} a+(n-1)b & b & b & \cdots & b \\ a+(n-1)b & a & b & \cdots & b \\ \vdots & & & \ddots & \\ a+(n-1)b & b & b & \cdots & a \end{vmatrix}$$

$$= (a+(n-1)b) \begin{vmatrix} 1 & b & \cdots & b \\ 1 & a & \cdots & b \\ \vdots & & \ddots & \\ 1 & b & & a \end{vmatrix}$$

$$= (a+(n-1)b) \begin{vmatrix} 1 & b & \cdots & b \\ 0 & a-b & 0 \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & a-b \end{vmatrix}$$

$$= (a+(n-1)b) \begin{vmatrix} a-b & & \\ & \ddots & \\ & & a-b \end{vmatrix}$$

$$= (a+(n-1)b)(a-b)^{n-1}$$

Solution. Add all the columns to the first column, and factor out an common factor in the first column. Then add -1 times the first row to the rows beneath it to obtain a determinant of an upper triangular matrix, where its determinant can be found by taking the product of the diagonal entries. The determinant of A is $[a+(n-1)b](a-b)^{n-1}$.

Example

$$|A| = a(-1)^{1+1}|A_{n-1}| + c\,(-1)^{1+2}\, b\,(-1)^{1+1}|A_{n-2}|$$
$$= a|A_{n-1}| - bc|A_{n-2}|$$
$$= (\alpha+\beta)|A_{n-1}| - \alpha\beta|A_{n-2}|$$

Example

Find the determinant:

$$|A| = \begin{vmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{vmatrix}.$$

故

$$D_n - \alpha D_{n-1} = \beta(D_{n-1} - \alpha D_{n-2})$$

$$\because D_n = \underline{(\alpha+\beta)} D_{n-1} \underline{- \alpha\beta} D_{n-2} \Rightarrow D_n - \alpha D_{n-1} = \beta^{n-2}(D_2 - \alpha D_1)$$

对应起来

$$= \beta^{n-2}(a^2 - bc - \alpha a)$$
$$= \beta^{n-2}((\alpha+\beta)^2 - \alpha\beta - \alpha(\alpha+\beta))$$
$$= \beta^n$$

Solution.

$$D_n = \begin{cases} \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, & \alpha \neq \beta; \\ \frac{(n+1)a^n}{2^n}, & \alpha = \beta. \end{cases}$$

Where $\alpha+\beta = a, \alpha\beta = bc$.

可以换一种递推方式:

$$D_n-\beta D_{n-1}=\alpha(D_{n-1}-\beta D_{n-2})$$

$$\begin{cases} D_n-\beta D_{n-1}=\alpha^n \\ D_n-\alpha D_{n-1}=\beta^n \end{cases}$$

$\alpha\neq\beta$ $\quad (\alpha-\beta)D_{n-1}=\alpha^n-\beta^n$

$$D_{n-1}=\frac{\alpha^n-\beta^n}{\alpha-\beta}$$

$$D_n=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

$\alpha=\beta$ $\quad D_n-\frac{a}{2}D_{n-1}=\left(\frac{a}{2}\right)^{n-2}(D_2-D_1)=\left(\frac{a}{2}\right)^n$

$$\frac{D_n}{\left(\frac{a}{2}\right)^n}-\frac{D_{n-1}}{\left(\frac{a}{2}\right)^{n-1}}=1 \Rightarrow \frac{D_n}{\left(\frac{a}{2}\right)^n}=\frac{D_1}{\left(\frac{a}{2}\right)^1}+(n-1)$$

$$=n+1 \Rightarrow D_n=(n+1)\frac{a^n}{2^n}$$

一些题目

(1) 设 $A=(a_{ij})$ 是三阶非零矩阵，$|A|$ 为 A 的行列式，A_{ij} 为 a_{ij} 的代数余子式，若 $a_{ij}+A_{ij}=0(i,j=1,2,3)$, 则 $|A|=$ ____.

$A^2=\begin{bmatrix}a^2&0\\0&a^2\end{bmatrix}$ $B^2=\begin{bmatrix}2&-2\\-2&2\end{bmatrix}$

(2) $$\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = \underline{a^4-4a^2}$$

可交换

$\begin{vmatrix}A&B\\B&A\end{vmatrix}=|A^2-B^2|$

$\begin{vmatrix}A&B\\C&D\end{vmatrix}=|AD-BC|$ (若 $AC=CA$)

$=\begin{vmatrix}a^2-2&2\\2&a^2-2\end{vmatrix}=(a^2-2)^2-4=a^4-4a^2$

(3) 设 $A=(a_{ij})$ 为三阶矩阵, A_{ij} 为元素 a_{ij} 的代数余子式, 若 A 的每行元素的和均为 2, 且 $|A|=3$, 则 $A_{11}+A_{21}+A_{31}=$ ____.

(4) n 阶行列式 $$\begin{vmatrix} 2 & 0 & \cdots & 0 & 2 \\ -1 & 2 & \cdots & 0 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & \cdots & -1 & 2 \end{vmatrix} = \_\_\_\_.$$

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & -CA^{-1}B+D \end{bmatrix}$$ 准上三角

在这里 A不可逆 最终结论也正确

取 det

1. $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \left| -ACA^{-1}B + AD \right|$

额外条件 $AC=CA$

$\Rightarrow = \left| AD - CB \right|$

加一个扰动

$A+cI$ 易变成 invertible

$u(t)$, $v(t)$

$u(0)=8$ $v(0)=5$

求 $u(t)$, $v(t)$

$$\begin{cases} \dfrac{du}{dt} = 4u-5v \\ \dfrac{dv}{dt} = 2u-3v \end{cases}$$

依线性方程组

$$\Leftrightarrow \frac{d}{dt}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \quad \frac{d\vec{x}}{dt} = A\vec{x}$$

猜测形式：e^{At} 假定同样指数才能消

$$u(t) = y\,e^{\lambda t}, \quad v(t) = z\,e^{\lambda t}$$

$$\frac{du}{dt} = \lambda y e^{\lambda t} = 4ye^{\lambda t} - 5ze^{\lambda t}$$

$$\frac{dv}{dt} = \lambda z e^{\lambda t} = 2ye^{\lambda t} - 3ze^{\lambda t}$$

$$\lambda\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

$x = \begin{bmatrix} y \\ z \end{bmatrix}$ $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$Ax = \lambda x$

$\vec{x}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} y e^{\lambda t} \\ z e^{\lambda t} \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix} e^{\lambda t}$

$(A - \lambda I)x = 0$

$\det(A - \lambda I) = 0$ 才有非零解

$= \begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = (\lambda-4)(\lambda+3)+10$

$= \lambda^2 - \lambda - 2 = 0$

$\lambda = 2 / -1$ 特征值 eigenvalue

(有限个)

Homework Assignment 19 and 20

4.4: 1, 5, 14, 16, 22, 23, 29, 40.