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Orthogonality :
- Orthogonal vectors & Subspaces
- Projections
- Orthnormal Basis
- Gram-Schmidt

$$\underset{\text{linearly independent}}{v_1, v_2, \cdots, v_n} \xrightarrow[\text{Gram-Schmidt}]{} \underset{\text{Orthonormal}}{e_1, e_2, \cdots, e_n}$$ (Examples)

$$A_{m\times n} = Q_{m\times n} R_{n\times n}$$

R upper-triangular + invertible
diagonal entries > 0.

The columns of Q are orthnormal

$\text{rank}(A) = r$.

Determinants :

$A_{n\times n}$

$|A|$ or $\det A$

- Definition and Examples
- Properties 1-10.
- Formulas :
  (a) LDU(?)
  (b) Cofactor expansion :
  $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$
  (c) Techniques : Recursion Relation, Vandermonde Matrix.
  (d) Applications : Inverse / Volume / Cramer's Rule / Pivots.

Important :

(a) $\det(AB) = \det(A)\det(B)$;

(b) $\det(A^T) = \det(A)$;

(c) $\det A = \sum_{\text{All } P\text{'s}} a_{1\alpha} a_{2\beta} \cdots a_{n\gamma} \det P$.

Eigenvalues & Eigenvectors

$A_{n \times n}$ Square matrix

$Ax = \lambda x$ $\qquad \lambda$: eigenvalue

$\qquad v\,(\neq 0)$: eigenvector

$\Leftrightarrow (A - \lambda I)x = 0$

$\Leftrightarrow x \in N(A - \lambda I)$ $\qquad P(\lambda) = \det(A - \lambda I)$: Characteristic polynomial

$$\deg P(\lambda) = n$$

$$P(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

$$= (-\lambda)^n + \underbrace{(\lambda_1 + \cdots + \lambda_n)}_{\substack{= a_{11} + \cdots + a_{nn} \\ = \operatorname{Trace}(A)}}(-\lambda)^{n-1} + \cdots + \underbrace{\lambda_1 \cdots \lambda_n}_{\det A}$$

Computation:

$\lambda = \lambda_1$: λ_1 eigenvalue

Solve $Ax = \lambda_1 x$ for x

$x \to$ eigenvector.

Diagonalizability : $A_{n \times n}$ diagonalizable

$\Leftrightarrow$ There exist n linearly independent eigenvectors :

$v_1, v_2, \cdots, v_n$

$$S = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \longrightarrow \text{eigenvector matrix}$$

$$\begin{array}{ccc} \downarrow & \downarrow & & \downarrow \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{array} \longrightarrow \text{eigenvalues}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \longrightarrow \text{eigenvalue matrix}$$

$$S^{-1} A S = \Lambda \quad \text{(putting together)}$$

$$\Leftrightarrow A = S \Lambda S^{-1} \qquad A^k = S \Lambda^k S^{-1}, \; k = 1, 2, \cdots.$$

Table on Page 306. Properties of Eigenvalues and Eigenvectors.

Complex number / vector / matrix.

- $z = a + bi$: $a, b \in \mathbb{R}$

- $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$, $\|x\|^2 = x^H x$ Hermitian = Conjugate Transpose.

- $A^H = (\bar{A})^T$.

  $A^H = A$ Hermitian matrix : Properties.

- Unitary Matrix : $U^H U = U U^H = I_n$

  Properties.

- Similarity Transformations:

  $A \longrightarrow M^{-1} A M = B$
  - Same eigenvalues
  - x eigenvector of A $\Rightarrow$ $M^{-1}x$ eigenvector of B.
  - Change of Basis

- Spectral Theorem (Schur's Lemma)
  - Real : $Q^{-1} A Q = \Lambda$ given that $A = A^T$
  - Complex : $U^{-1} A U = \Lambda$ given that $A^H = A$.

- Normal Matrix

- Jordan Forms $(n = 2, 3)$

Positive Definite Matrices + Quadratic Forms

- Definitions: Positive Definiteness + Quadratic Forms
  $x^T A x = f(x_1, \cdots, x_n)$
- Tests: 6 B + 6 C $\longleftrightarrow$ Positive Definiteness.
  6.D $\longleftrightarrow$ Positive Semidefiniteness.
- Qurdric Surfaces.
- Law of Inertia.
- Quadratic Forms.
- Singular Value Decomposition.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = U \Sigma V^T.$$

- Applications.

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Additional Page.

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$Ax = b$ Nonhomogeneous $Ax = 0$ Homogeneous

$A_{m\times n}$: $m = n$, $m < n$, $m > n$.

Special Solution(s)

Complete Solutions

Matrix :
- Invertible matrix
- rank
- LU Factorization / QR / SVD.
- Four Fundamental Subspaces
- Eigenvalues and Eigenvectors
- Determinants.

Special Matrices :
- Projection matrix.
- Rotation matrix.
- Orthogonal matrix.
- Positive definite matrix.
- Hermitian Matrix.
- Unitary Matrix.

Example: $A_{n\times m}$, $B_{m\times n}$

(a) $\begin{vmatrix} I_m & B \\ A & I_n \end{vmatrix} = |I_n - AB| = |I_m - BA|$

$$\begin{bmatrix} I_m & 0 \\ -A & I_n \end{bmatrix}\begin{bmatrix} I_m & B \\ A & I_n \end{bmatrix} = \begin{bmatrix} I_m & B \\ 0 & I_n - AB \end{bmatrix}$$

$\Rightarrow |I_n - AB| = \begin{vmatrix} I_m & B \\ A & I_n \end{vmatrix} \cdots (1)$ Meanwhile,

$$\begin{bmatrix} I_m & B \\ A & I_n \end{bmatrix}\begin{bmatrix} I_m & 0 \\ -A & I_n \end{bmatrix} = \begin{bmatrix} I_m - BA & B \\ 0 & I_n \end{bmatrix}$$

$\Rightarrow |I_m - BA| = \begin{vmatrix} I_m & B \\ A & I_n \end{vmatrix} \cdots (2)$ (1) + (2) $\Rightarrow$ Done! #.

$\lambda \neq 0$: $|\lambda I_n - AB| = \lambda^{n-m} |\lambda I_m - BA|$

$\left| I_n - A\left(\frac{B}{\lambda}\right)\right| = \left| I_m - \left(\frac{B}{\lambda}\right) A \right| \Rightarrow \frac{1}{\lambda^n}|\lambda E_n - AB| = \frac{1}{\lambda^m}|\lambda I_m - BA|$

$\Rightarrow |\lambda I_n - AB| = \lambda^{n-m} |\lambda I_m - BA|$.