Problem Set 7 —— Linear Algebra (Spring 2024)

Dr. Y. Chen

1. Let
$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}.$$

   (a) Find the projection matrix P_C onto the column space of the matrix A.

   (b) Find the 3×3 projection matrix P_R onto the row space of A.

   (c) Multiply $B = P_C A P_R$. Your answer B should be a bit surprising–can you explain it?

2. Three planes Π_1, Π_2, Π_3 in the space $\mathbb{R}^3$ are given by the equations
$$\begin{aligned} \Pi_1 &: \quad x + y + z = 0\,, \\ \Pi_2 &: \quad 2x - y + 4z = 0\,, \\ \Pi_3 &: \quad -x + 2y - z = 0\,. \end{aligned}$$
   Determine a matrix representative (in the standard basis of $\mathbb{R}^3$) of a linear transformation taking the xy plane to Π_1, the yz plane to Π_2 and the zx plane to Π_3.

3. Suppose P_1 and P_2 are projection matrices ($P_i^2 = P_i = P_i^T$). Prove: P_1P_2 is a projection if and only if $P_1P_2 = P_2P_1$.

4. The space M of 2 by 2 matrices has the basis
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$
   Suppose T multiplies each matrix by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, i.e., $T(X) = AX$. Find the matrix representing this linear transformation T with respect to the above mentioned basis.

5. Consider two bases of $\mathbb{R}^3$
$$u_1 = (1, 0, 1),\ u_2 = (2, 1, 0),\ u_3 = (1, 1, 1)$$
   and
$$v_1 = (1, 2, -1),\ v_2 = (2, 2, -1),\ v_3 = (2, -1, -1).$$
   Define a linear transformation as follows: $T(u_i) = v_i, i = 1, 2, 3$.

   (a) Find the transition matrix from u_1, u_2, u_3 to v_1, v_2, v_3 ;

   (b) Find the matrix representation, A, of T with respect to the basis u_1, u_2, u_3 ;

   (c) Find the matrix representation, B, of T with respect to the basis v_1, v_2, v_3 ;

   (d) Can you say something about the relation between A and B?