考试科目:    线 性 代 数 A       开课单位:      数 学 系

考试时长:    120 分钟

| 题 号 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 分 值 | 15 分 | 25 分 | 10 分 | 16 分 | 10 分 | 6 分 | 16 分 | 12 分 |

本试卷共（8）大题, 满分（110）分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 110 in total. Write all your answers on the examination book.

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1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

   (共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

   (1) Let $A$ be an $m \times n$ real matrix and $b$ be a column vector in $\mathbb{R}^m$. Which of the following statements is correct？（　　）

   (A) If $Ax = b$ has infinite many solutions, then $Ax = 0$ has a nonzero solution.

   (B) If the system $Ax = 0$ has only zero solution, then $Ax = b$ has one and only one solution.

   (C) If the rank of $A$ is $n$, then the system $Ax = b$ must have a solution.

   (D) If $A$ is a square matrix (i.e., $m = n$), then the system $Ax = b$ is consistent if and only if $A$ is invertible.

   设 $A$ 为 $m \times n$ 实矩阵, $b$ 是 $\mathbb{R}^m$ 中的列向量. 下列陈述中哪个是正确的？（　　）

   (A) 如果 $Ax = b$ 有无穷多个解, 则 $Ax = 0$ 有非零解.

   (B) 如果方程组 $Ax = 0$ 只有零解, 则 $Ax = b$ 有且仅有一个解.

   (C) 如果 $A$ 的秩为 $n$, 则方程组 $Ax = b$ 必有解.

   (D) 如果 $A$ 是方阵 (即 $m = n$), 则方程组 $Ax = b$ 是相容的当且仅当 $A$ 可逆.

   (2) Suppose $A$ is an $m \times n$ matrix, $B$ is an $n \times m$ matrix, and $I$ is the $m \times m$ identity matrix. If $AB = I$, then（　　）

   (A) the column vectors of $A$ are linearly independent, and the row vectors of $B$ are linearly independent.

   (B) the column vectors of $A$ are linearly independent, and the column vectors of $B$ are linearly independent.

   (C) the row vectors of $A$ are linearly independent, and the column vectors of $B$ are linearly independent.

   (D) the row vectors of $A$ are linearly independent, and the row vectors of $B$ are linearly independent.

   设 $A$ 为 $m \times n$ 型矩阵, $B$ 为 $n \times m$ 型矩阵, $I$ 为 $m$ 阶单位矩阵. 若 $AB = I$, 则（　　）

   (A) $A$ 的列向量组线性无关, $B$ 的行向量组线性无关.

    (B) $A$ 的列向量组线性无关, $B$ 的列向量组线性无关.

    (C) $A$ 的行向量组线性无关, $B$ 的列向量组线性无关.

    (D) $A$ 的行向量组线性无关, $B$ 的行向量组线性无关.

(3) Let $A$ be a $3 \times 3$ matrix, and let $B$ be the matrix formed by adding the second column of $A$ to its first column. Suppose that after exchanging the second and third rows of $B$, the resulting matrix is the $3 \times 3$ identity matrix. Let $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$ Then $A = ($     $)$

    (A) $P_1 P_2$.

    (B) $P_1^{-1} P_2$.

    (C) $P_2 P_1$.

    (D) $P_2 P_1^{-1}$.

设 $A$ 为 3 阶方阵, 将 $A$ 的第二列加到第一列得矩阵 $B$. 假设交换 $B$ 的第二行与第三行可以得到 3 阶单位矩阵. 记 $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$ 则 $A = ($     $)$

    (A) $P_1 P_2$.

    (B) $P_1^{-1} P_2$.

    (C) $P_2 P_1$.

    (D) $P_2 P_1^{-1}$.

(4) Let $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$, where $t \in \mathbb{R}$. Suppose $\mathrm{rank}(A) = 2$. Then (     )

    (A) $t = -6$.

    (B) $t = 6$.

    (C) $t \neq 0$.

    (D) $t$ can be any real number.

设 $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$, 其中 $t \in \mathbb{R}$. 假设 $\mathrm{rank}(A) = 2$. 则 (     )

    (A) $t = -6$.

    (B) $t = 6$.

    (C) $t \neq 0$.

    (D) $t$ 可以是任意实数.

(5) Which of the following statements is incorrect? (     )

    (A) For any matrix $A$, $\mathrm{rank}(A) = \dim C(A)$.

(B) If $v_1, \cdots, v_m$ are pairwise orthogonal nonzero vectors, then the vectors $v_1, \cdots, v_m$ are linear independent.

(C) If $A$ is an upper triangular $n \times n$ matrix such that $A^2 = 0$, then $A = 0$.

(D) Let $A$, $B$ be $n \times n$ matrices such that $AB$ is invertible. Then both $A$ and $B$ are invertible.

下列哪个论断是错误的? (      )

(A) 对于任意矩阵 $A$, $\mathrm{rank}(A) = \dim C(A)$.

(B) 如果 $v_1, \cdots, v_m$ 是一组两两正交的非零向量, 则向量组 $v_1, \cdots, v_m$ 线性无关.

(C) 如果 $A$ 是 $n \times n$ 上三角矩阵且 $A^2 = 0$, 则 $A = 0$.

(D) 设 $A$, $B$ 为 $n \times n$ 矩阵且 $AB$ 可逆. 则 $A$ 和 $B$ 都可逆.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let $A$, $B$ be invertible $n \times n$ matrices. Then the inverse of the block matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is _____

设 $A$, $B$ 均为 $n \times n$ 可逆矩阵. 则分块矩阵 $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ 的逆为 _____

(2) Suppose $A$ is a $3 \times 4$ matrix and $\dim N(A) = 2$. Then $\dim N(A^T) =$ _____

设 $A$ 为 $3 \times 4$ 矩阵且 $\dim N(A) = 2$. 则 $\dim N(A^T) =$ _____

(3) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. Then $A^{-1} =$ _____

设 $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. 则 $A^{-1} =$ _____

(4) Let $u$, $v$ be vectors in $\mathbb{R}^n$ such that $\|u\| = 3$, $\|v\| = 4$ and $u^T v = -3$.
Then $\|2u + 3v\| =$ _____

设 $u$, $v$ 为 $\mathbb{R}^n$ 中的向量, 满足 $\|u\| = 3$, $\|v\| = 4$ 以及 $u^T v = -3$. 则 $\|2u+3v\| =$ _____

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.
Then the least squares solution to $Ax = b$ is $\hat{x} =$ _____

设 $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.
则 $Ax = b$ 的最小二乘解是 $\hat{x} =$ _____

3. (10 points) Find the LU factorization of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

求矩阵 $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ 的 LU 分解.

4. (16 points) Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Please give a basis for each of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$ and $N(A^T)$, respectively.

(16 分) 设 $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

请对四个基本子空间 $C(A)$, $N(A)$, $C(A^T)$ 和 $N(A^T)$ 分别给出各自的一组基.

5. (10 points) Let $E = \{u_1, u_2, u_3\}$ and $F = \{v_1, v_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

Find the matrix $A$ representing $T$ with respect to the ordered bases $E$ and $F$.

(10 分) 设 $E = \{u_1, u_2, u_3\}$, $F = \{v_1, v_2\}$, 其中

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

定义线性变换 $T : \mathbb{R}^3 \to \mathbb{R}^2$ 如下

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

求 $T$ 在 $E$ 和 $F$ 这两组有序基下的矩阵表示 $A$.

6. (6 points) Let $A$, $B$ be $n \times n$ matrices. Suppose $A$ and $B$ are both symmetric. Is $AB$ necessarily symmetric ? If yes, please give a proof. Otherwise please give a counterexample.

   (6 分) 设 $A$, $B$ 均为 $n \times n$ 矩阵. 假设 $A$ 和 $B$ 都是对称矩阵. $AB$ 是否一定是对称矩阵？若是, 请给出证明. 否则请给出一个反例.

7. (16 points) The following two questions are independent:

   (a) Let $A$ be the $2 \times 2$ matrix such that the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto Av$ rotates every vector in $\mathbb{R}^2$ through $60°$ counter-clockwise (about the origin).
   Find $A$ and $A^{2020}$.

   (b) Three planes $\Pi_1$, $\Pi_2$, $\Pi_3$ in the space $\mathbb{R}^3$ are given by the equations

   $$\begin{aligned} \Pi_1 &: \quad x + y + z = 0\,, \\ \Pi_2 &: \quad 2x - y + 4z = 0\,, \\ \Pi_3 &: \quad -x + 2y - z = 0\,. \end{aligned}$$

   Determine a matrix representative (in the standard basis of $\mathbb{R}^3$) of a linear transformation taking the $xy$ plane to $\Pi_1$, the $yz$ plane to $\Pi_2$ and the $zx$ plane to $\Pi_3$.

   (16 分) 以下两个小题是相互独立的:

   (a) 设 $A$ 是 $2 \times 2$ 矩阵使得线性变换 $\mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto Av$ 把 $\mathbb{R}^2$ 中每个向量 (绕原点) 逆时针转动 $60°$.
   求 $A$ 和 $A^{2020}$.

   (b) 在空间 $\mathbb{R}^3$ 中由以下方程给出三个平面 $\Pi_1$, $\Pi_2$, $\Pi_3$:

   $$\begin{aligned} \Pi_1 &: \quad x + y + z = 0\,, \\ \Pi_2 &: \quad 2x - y + 4z = 0\,, \\ \Pi_3 &: \quad -x + 2y - z = 0\,. \end{aligned}$$

   求一个矩阵, 使它 (在 $\mathbb{R}^3$ 的标准基下) 表示的线性变换将 $xy$ 平面映射成 $\Pi_1$, 将 $yz$ 平面映射成 $\Pi_2$ 并将 $zx$ 平面映射成 $\Pi_3$.

8. (12 points) Let $A$ be a $3 \times 3$ matrix such that $\operatorname{rank}(A) = 2$ and $A^3 = 0$.

   (a) Prove that $\operatorname{rank}(A^2) = 1$.

   (b) Let $\alpha_1 \in \mathbb{R}^3$ be a nonzero vector such that $A\alpha_1 = 0$. Prove that there exist vectors $\alpha_2$, $\alpha_3$ such that $A\alpha_2 = \alpha_1$, $A^2\alpha_3 = \alpha_1$.

   (c) For any vectors $\alpha_2, \alpha_3$ described as above, show that $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

   (*In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.*)

(12 分) 设 $A$ 是 $3 \times 3$ 矩阵, 它满足 $\mathrm{rank}(A) = 2$ 及 $A^3 = 0$.

(a) 证明 $\mathrm{rank}(A^2) = 1$.

(b) 设 $\alpha_1 \in \mathbb{R}^3$ 是满足 $A\alpha_1 = 0$ 的非零向量. 证明: 存在向量 $\alpha_2$, $\alpha_3$ 使得 $A\alpha_2 = \alpha_1$, $A^2\alpha_3 = \alpha_1$.

(c) 证明: 对于任意满足上述条件的向量 $\alpha_2, \alpha_3$, 向量组 $\alpha_1, \alpha_2, \alpha_3$ 线性无关.

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)