Practice Problems Set 2
Linear Algebra A

(1) True or false. No need to justify.

(a) The diagonal entries of a positive definite matrix are positive. ( )

(b) If A is similar to B, then A^2 is similar to B^2. ( )

(c) If A and B are diagonalizable, so is AB. ( )

(d) If A is a 3×3 skew-symmetric ($A^T = -A$), then $|A| = 0$. ( )

(e) If A is negative definite, then all the upper left submatrices A_k of A have negative determinants. ( )

(f) Let A be an $n \times n$ matrix, then the number of nonzero eigenvalues of A (counting the multiplicities) is equal to the rank of A. ( )

(2) Fill in the blanks.

(a) Let A be a 3×3 real matrix whose column vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. If $A\alpha_1 = \alpha_1 + \alpha_2, A\alpha_2 = \alpha_2 + \alpha_3, A\alpha_3 = \alpha_3 + \alpha_1$, then $|A| =$ __________.

(b) If $A \in \mathbb{R}^{3\times 3}$ has eigenvalues $0, 1, 2$, then the eigenvalues of $A(A-I)(A-2I)$ are __________.

(c) A box has edges from $(0,0,0)$ to $(3,1,1), (1,3,1), (1,1,3)$, then its volume is __________.

(3) (10 points) Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find all the eigenvalues of A and their associated eigenvectors.

(ii) Is A diagonalizable? Explain why.

(4) Let

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}.$$

(i) Verify that A is Hermitian.

(ii) Find a unitary matrix U that diagonalizes A.

(5) Let

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(i) Find all the singular values of A.

(ii) Find the singular value decomposition of A, in other words, find orthogonal matrices U and V, such that $A = U\Sigma V^T$.

(6) Let A be

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

(i) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

(ii) Find A^k, where k is a positive integer.

(7) Consider the quadratic form

$$f(x_1, x_2, x_3, x_4) = t(x_1^2 + x_2^2 + x_3^2) + x_4^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3.$$

(i) Find A, such that $f(x_1, x_2, x_3, x_4) = x^T Ax$.

(ii) For which t is $f(x_1, x_2, x_3, x_4)$ positive definite?

(8) Let N be a normal matrix ($N^H N = NN^H$).

(i) Show that $||Nx|| = ||N^H x||$ for every vector x.

(ii) Deduce that the ith row of N has the same length as the ith column.

(iii) If N is upper triangular, then N must be diagonal.

(9) Prove the following statements:

(i) Suppose A is an $n \times n$ real symmetric positive definite matrix, show that $|A + I_n| > 1$.

(ii) Let A be an $n \times n$ matrix, show that $A^T A$ is similar to AA^T.

(10) (6 points) If $A^k = O$ for some positive integer k, then A is called a "nilpotent" matrix. O is the $n \times n$ zero matrix.

(i) Show that all the eigenvalues of a nilpotent matrix must be zero.

(ii) Prove that a nonzero nilpotent matrix can not be symmetric.

(11) Let A be an $n \times n$ real symmetric positive definite matrix, and $\beta \in \mathbb{R}^n$ be a nonzero vector. Consider

$$D = \begin{bmatrix} A & \beta \\ \beta^T & c \end{bmatrix}$$

(i) Find a condition on c to guarantee that D is positive definite.

(ii) In the case that D is positive semi-definite, find a basis for the nullspace of D, $N(D)$.