We review the section 1 by doing the following exercise

Answer the following questions with the row operation on the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 1 & 1 & 3 \end{pmatrix}$$

1. Which of $(3, 2, 1)$, $(2, 1, 3)$ lie in the plane spanned by $(1, 1, 1)$ and $(1, 2, 3)$?
2. What (line, point, empty, etc.) is the intersection of the planes $x_1 + x_2 + 3x_3 = 2$, $x_1 + 2x_2 + 2x_3 = 3$, $x_1 + 3x_2 + x_3 = 1$?
3. What is the intersection of the planes $x_1 + x_2 + 2x_3 = 2$, $x_1 + 2x_2 + 3x_3 = 1$, $x_1 + 3x_2 + x_3 = 3$?
4. Find three columns of A to form an invertible 3×3 matrix, and find the inverse of this matrix.
5. Find the LU-decomposition of A.
6. Find the LU-decomposition of the matrix formed by the first, second, and fourth columns of A.

Answer

$$\begin{pmatrix} 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 1 & 1 & 3 \end{pmatrix} \xrightarrow[\text{Row}_3 - \text{Row}_1]{\text{Row}_2 - \text{Row}_1} \begin{pmatrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & -2 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{Row}_3 - \text{Row}_2} \begin{pmatrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & -3 & 3 \end{pmatrix}$$

1. $(1, 1, 1)$ and $(1, 2, 3)$ are the first and second columns of A. $(3, 2, 1)$, $(2, 1, 3)$ are the third and fifth columns of A.
By restricting the row operations to the first, second, third, the system has solution (the third column is not pivot). Therefore $(3, 2, 1)$ is in the plane.
By restricting the row operations to the first, second, fifth, the system has no solution (the fifth column is pivot). Therefore $(2, 1, 3)$ is not in the plane.

Answer

2. The augmented matrix is the first four columns. The right side (fourth column) is pivot. The system has no solution. The intersection is empty.

3. The augmented matrix is the first, second, fourth, fifth. The right side is not pivot, and other columns are pivot. The system has unique solution. The intersection is one point.

4. We get invertible matrix by picking pivot columns, i.e., first, second and fourth. Then we calculate the inverse

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 & -1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{7}{3} & -\frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

Then we get

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{7}{3} & -\frac{4}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}.$$

Answer

5. The following gives L

$$\begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{pmatrix} \xleftarrow[]{\substack{\text{Row}_2+\text{Row}_1 \\ \text{Row}_3+\text{Row}_1}} \begin{pmatrix} x_1 \\ x_2 \\ x_2 + x_3 \end{pmatrix} \xleftarrow[]{\text{Row}_3+\text{Row}_2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We have

$$\begin{pmatrix} 1 & 1 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 1 & 3 & 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & -3 & 3 \end{pmatrix}.$$

6. By picking the first, second, and fourth columns, we get

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{pmatrix}.$$

You should be familiar with all above.

Review of section 2

Key words:

- Subspace;
- Linear dependence and linear independence;
- Basis and dimension;
- Solve $Ax = b$;
- Fundamental subspaces $C(A)$, $N(A)$, $C(A^\top)$, $N(A^\top)$;
- Linear transformation.

Subspace

Prove the following subset is a subspace, and find its basis and dimension.

$$V = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbb{R}^{2\times 2} \mid x - 2y + z - w = 0 \right\}.$$

Let $v_1 = \begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix}$. Then we have

$$v_1 + v_2 = \begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix} + \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ z_1 + z_2 & w_1 + w_2 \end{bmatrix} \in \mathbb{R}^{2\times 2},$$

with $(x_1 + x_2) - 2(y_1 + y_2) + (z_1 + z_2) - (w_1 + w_2) = 0$. We also have

$$kv_1 = \begin{bmatrix} kx_1 & ky_1 \\ kz_1 & kw_1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \ \forall k \in \mathbb{R}, \ \text{with} \ \ kx_1 - 2ky_1 + kz_1 - kw_1 = 0.$$

Then $v_1 + v_2, kv_1 \in V$ with $\forall v_1, v_2 \in V$, $k \in \mathbb{R}$, which means V is a subspace.

Cont.

Through the relationship $x - 2y + z - w = 0$, we have

$$v = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 2y - z + w & y \\ z & w \end{bmatrix}$$

$$= y \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + x \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

Then the dimension of V is 3, and one basis is

$$\left\{ \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Solve $Ax = b$ and fundamental subspaces $C(A)$, $N(A)$, $C(A^\top)$, $N(A^\top)$

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 5 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

1. Find the dimension and a basis for the four fundamental subspaces of A.
2. Under what condition on $\mathbf{b}$ is the system $A\mathbf{x} = \mathbf{b}$ solvable?
3. Find all the solutions when $\mathbf{b} = (1, 1, 1)^\top$.

$$[A, \mathbf{b}] = \begin{bmatrix} 1 & 3 & 1 & 1 & b_1 \\ 0 & 1 & 1 & 0 & b_2 \\ 2 & 5 & 1 & 2 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 1 & b_1 \\ 0 & 1 & 1 & 0 & b_2 \\ 0 & -1 & -1 & 0 & b_3 - 2b_1 \end{bmatrix}$$

Cont.

$$\to \begin{bmatrix} 1 & 3 & 1 & 1 & b_1 \\ 0 & 1 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 + b_2 \end{bmatrix}.$$

1. We have

$$A \to \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Column space $C(A)$: dimension $= 2$, a basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \right\}$

Row space $C(A^\top)$: dimension $= 2$, a basis $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Cont.

Nullspace $N(A)$: dimension $= 2$, a basis $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$A^\top = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 5 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Left nullspace $N(A^\top)$: dimension $= 1$, a basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Cont.

2. When $b_3 - 2b_1 + b_2 = 0$ the system $A\mathbf{x} = \mathbf{b}$ is solvable.

3. When $\mathbf{b} = (1, 1, 1)^\top$, then

$$[A, \mathbf{b}] \to \begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solutions to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

Linear transformation

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

We define a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ as follow:

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix}.$$

1. Explain why $\mathbf{b}_1, \mathbf{b}_2$ is a basis for $\mathbb{R}^2$.
2. Find the matrix representation of L with respect to bases $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and $\mathbf{b}_1, \mathbf{b}_2$.

Cont.

1. We have

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Then $\mathbf{b}_1, \mathbf{b}_2$ are linearly independent. Because the dimension of $\mathbb{R}^2$ is 2, then $\mathbf{b}_1, \mathbf{b}_2$ is a basis for $\mathbb{R}^2$.

2. The key point is to represent $L(\mathbf{u}_1)$, $L(\mathbf{u}_2)$, $L(\mathbf{u}_3)$ with the basis $\mathbf{b}_1, \mathbf{b}_2$ as

$$L(\mathbf{u}_1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_{21} \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$L(\mathbf{u}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a_{12} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_{22} \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$L(\mathbf{u}_3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = a_{13} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_{23} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is the required matrix.

Cont.

We have

$$\begin{bmatrix} L(\mathbf{u}_1) & L(\mathbf{u}_2) & L(\mathbf{u}_3) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Then the matrix A can be solved by

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Or we can solve the matrix A by

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & -1 & -3 & 1 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix},$$

which means

$$A = \begin{bmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Calculate A^n

Calculate A^n with the following matrices:

1. $A = \begin{bmatrix} -2 & -1 & -1 & -2 \\ 4 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 2 \end{bmatrix}$.
2. $A = \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$.
3. $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$.

1. Notice that $\operatorname{rank}(A) = 1$, then

$$A = \alpha\beta^\top, \quad \text{with} \quad \alpha = (-1, 2, 0, 1)^\top, \quad \beta = (2, 1, 1, 2)^\top,$$

which means $A^n = (\alpha\beta^\top)^n = \alpha(\beta^\top\alpha)^{n-1}\beta^\top = 2^{n-1}\alpha\beta^\top = 2^{n-1}A$.

Cont.

2. We have $A = 2B$ with

$$B = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta = 2\pi/3.$$

Here B represents the $2\pi/3$ counterclockwise rotation, which means

$$A^n = 2^n B^n = 2^n \begin{bmatrix} \cos\frac{2n\pi}{3} & -\sin\frac{2n\pi}{3} \\ \sin\frac{2n\pi}{3} & \cos\frac{2n\pi}{3} \end{bmatrix}.$$

3. We have $A = \lambda I + B$ with

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that $IB = BI$ and $B^3 = B^4 = \cdots = B^n = 0$, then

$$A^n = (\lambda I + B)^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} B^k = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}.$$

Review of section 3.1-3.3

Key words:

- Orthogonality;
- Projection and least Squares.

Orthogonality

1. If x_p is a particular solution to $Ax = b \neq 0$, then x_p is always in the row space of A. (False)
2. $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has a solution and $A^{\mathrm{T}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (False)
3. The subspace spanned by $(1,1,0,0,0)^\top$ and $(0,0,0,1,1)^\top$ is the orthogonal complement of the subspace spanned by $(1,-1,0,0,0)^\top$ and $(2,-2,3,4,-4)^\top$. (False)

1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then a particular solution $x_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin C(A^\top)$.

2. The first condition means $(1,1)^\top \in C(A)$ and the second condition means $(1,0)^\top \in N(A^\top)$. $C(A) \perp N(A^\top)$, but $(1,1) \cdot (1,0)^\top = 1 \neq 0$.

3. This tow subspace are orthogonal, but the dimension of the $\mathbb{R}^5$ is 5 rather than 4.

Projection and least Squares

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$$

1. Find the lease squares solution to $Ax = b$.
2. Split b into a column space component x_c and a left nullspace component x_l, i.e., $b = x_c + x_l$.

Cont.

1. We have

$$A^\top A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & -1 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -5 & 1 \\ -5 & 7 & -2 \\ 1 & -2 & 9 \end{bmatrix},$$

$$A^\top b = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & -1 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -10 \end{bmatrix}.$$

Then the lease squares solution is $\hat{x} = (1, 1, -1)^\top$ by $A^\top A\hat{x} = A^\top b$.

2. We have $x_c = A\hat{x} = (0, -2, -2, 2)^\top$, and $x_l = b - x_c = (2, 0, 1, 1)^\top$ with $A^\top x_l = 0$.