Quadratic Forms（二次型）

Quadratic Forms

Dept. of Math.

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Quadratic Forms

Introduction

A quadratic equation in n variables $x_1, x_2, \cdots, x_n$ is one of the form

$$\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + \alpha = 0.$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, A is an $n \times n$ real symmetric matrix, B is a $1 \times n$ matrix, and α is a scalar. The vector function

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) x_i$$

is the quadratic form in n variables associated with the quadratic equation.

Three Unknowns

In the case of three unknowns, if

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}, B = \begin{bmatrix} g & h & i \end{bmatrix},$$

then the quadratic equation is

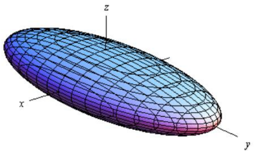
$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + \underline{gx + hy + iz} + \alpha = 0.$$

The graph of a quadratic equation in three variables is called a quadric surface（二次曲面）.

Ellipsoids (椭球面)

1. Ellipsoids

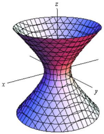
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ a>0,\ b>0,\ c>0.$$



Hyperboloid of One Sheet(单叶双曲面)

2. Hyperboloid of One Sheet:

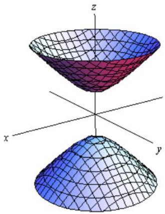
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\ (a>0,\ b>0,\ c>0).$$



Hyperboloid of Two Sheets(双叶双曲面)

3. Hyperboloid of Two Sheets:

$$\frac{z^2}{a^2} - \frac{x^2}{b^2} - \frac{y^2}{c^2} = 1,\ a,\ b,\ c > 0.$$

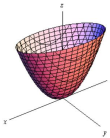


Elliptic Paraboloid(椭圆抛物面)

4. Elliptic Paraboloid:

$$\frac{x^2}{2p}+\frac{y^2}{2q}=z,$$

where $p,\ q$ have the same sign.



碗状

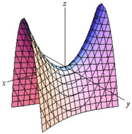
一定正定

Hyperbolic Paraboloid(双曲抛物面)

5. Hyperbolic Paraboloid:

$$\frac{x^2}{2p} - \frac{y^2}{2q} = z,$$

where $p,\ q$ have the same sign.

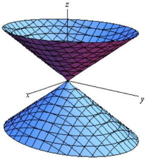


马鞍面

Cones (圆锥面)

6. Cones:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,\ a,\ b,\ c>0.$$



Example 1

$$x^T A x + Bx + \alpha = 0$$

Example

Given the quadratic equation

$$[x_1\ x_2\ x_3]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$2x_1x_2 + 2x_1x_3 - 6x_2x_3 = 1.$$

Find a change of coordinates so that the resulting equation represents a quadric surface in standard position.

The left hand side is of a quadratic form, and its matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} A \\ I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

行乘列操作

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -3 \\ -2 & -3 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

做了行操作再做匹配的列操作

C^TAC

E^TAE

$$E^TAE = \overset{E^T}{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \overset{E}{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

不改变对称性

每一步保持对称性

$$\rightarrow \begin{bmatrix} 2 & 1 & -2 \\ 0 & -\frac{1}{2} & -2 \\ -2 & -3 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -2 \\ 0 & -\frac{1}{2} & -2 \\ -2 & -2 & 0 \\ 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -2 \\ 0 & -\frac{1}{2} & -2 \\ 0 & -2 & -2 \\ 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & -2 \\ 0 & -2 & -2 \\ 1 & -\frac{1}{2} & 1 \\ 1 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & -2 \\ 0 & 0 & 6 \\ 1 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 6 \\ 1 & -\frac{1}{2} & 3 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

→ 对角矩阵 D

→ C

$$C^TAC = D$$

$$x^TAx = 1$$

$$(Cy)^TACy = 1$$

$$y^T\underbrace{C^TAC}_{D}y = 1$$

$2x_1x_2+2x_1x_3-6x_2x_3=1$

$\Rightarrow 2y_1^2-\frac{1}{2}y_2^2+6y_3^2=1$

两正一负　单叶双曲面

Approach 1: Completing the Squares (part 1)

Let

$$\begin{cases} x_1 = y_1 + y_2, \\ x_2 = y_1 - y_2, \\ x_3 = y_3, \end{cases}$$

引入平方项

then

$$f(x_1, x_2, x_3) = 2(y_1 - y_3)^2 - 2y_3^2 - 2y_2^2 + 8y_2y_3.$$

Let

$$\begin{cases} z_1 = y_1 - y_3, \\ z_2 = y_2, \\ z_3 = y_3, \end{cases} \Leftrightarrow \begin{cases} y_1 = z_1 + z_3, \\ y_2 = z_2, \\ y_3 = z_3, \end{cases}$$

Approach 1: Completing the Squares (part 2)

It follows that

$$f(x_1,x_2,x_3)=2z_1^2-2(z_2-2z_3)^2+6z_3^2.$$

Let

$$\begin{cases} w_1=z_1, \\ w_2=z_2-2z_3, \\ w_3=z_3, \end{cases} \Leftrightarrow \begin{cases} z_1=w_1, \\ z_2=w_2+2w_3, \\ z_3=w_3, \end{cases}$$

then

$$f(x_1,x_2,x_3)=2w_1^2-2w_2^2+6w_3^2.$$

标准型不唯一
（规范型唯一）
只要对角矩阵就好

Approach 1: Completing the Squares (part 3)

The change of coordinates is give by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

The quadratic equation becomes $f(x_1, x_2, x_3) = 2w_1^2 - 2w_2^2 + 6w_3^2 = 1$. Therefore it represents a hyperboloid of one sheet in standard position.

Approach 2: Using Matrix Multiplication (part 1)

Let

$$C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$A_1 = C_1^T A C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & 4 \\ -2 & 4 & 0 \end{bmatrix}.$$

Approach 2: Using Matrix Multiplication (part 2)

Let

$$C_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$A_2 = C_2^T A_1 C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & 4 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & -2 \end{bmatrix}.$$

Approach 2: Using Matrix Multiplication (part 3)

Let

$$C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$A_3 = C_3^T A_2 C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Approach 2: Using Matrix Multiplication (part 4)

A_3 is already diagonal, therefore if we let

$$C = C_1C_2C_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$C^TAC = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Approach 2: Using Matrix Multiplication (part 5)

If we choose

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

then the quadratic equation in this example becomes

$$2y_1^2 - 2y_2^2 + 6y_3^2 = 1.$$

Therefore the equation represents a hyperboloid of one sheet in standard position.

Approach 3: Spectral Theorem

Since A is a symmetric matrix, therefore by the spectral theorem, we can find an orthogonal matrix Q such that $Q^{-1}AQ = \Lambda$. In particular

$$A = Q\Lambda Q^T = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ q_3^T \end{bmatrix}.$$

If we let $y = Q^T x$, then

特征值2正1负

$$x^T A x = x^T Q\Lambda Q^T x = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

The eigenvalues of A are $3, \frac{-3+\sqrt{17}}{2}, \frac{-3-\sqrt{17}}{2}$. Therefore the equation represents a hyperboloid of one sheet in standard position.

Exercises

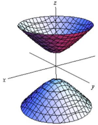
1. 设二次型 $f(x_1,x_2,x_3)=x_1^2+x_2^2+x_3^2+4x_1x_2+4x_1x_3+4x_2x_3$, 则 $f(x_1,x_2,x_3)=2$ 在空间直角坐标系下表示的二次曲面为
   (A) 单叶双曲面
   (B) 双叶双曲面
   (C) 椭球面
   (D) 柱面
2. 已知二次曲面方程 $x^2+ay^2+z^2+2bxy+2xz+2yz=4$ 可以经过正交变换
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$
   化为椭圆柱面方程 $\eta^2+4\xi^2=4$, 求 a, b 的值和正交矩阵 P.

Exercises

3. 已知二次型 $f(x_1,x_2,x_3)=x^TAx$ 在正交变换 $x=Qy$ 下的标准形为 $y_1^2+y_2^2$, 且 Q 的第三列为 $(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2})^T$.
   (I) 求矩阵 A.
   (II) 证明 $A+E$ 为正定矩阵，其中 E 为三阶单位矩阵.
4. 设 A 为三阶实对称矩阵，如果二次曲面方程

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

在正交变换下的标准方程的图形如图所示，



则 A 的正特征值的个数为 ________.

Exercises

5. 已知二次型

$$f(x_1,x_2,x_3)=(1-a)x_1^2+(1-a)x_2^2+2x_3^2+2(1+a)x_1x_2$$

的秩为 2.

   (I) 求 a 的值;
   (II) 求正交变换 $x=Qy$, 把 $f(x_1,x_2,x_3)$ 化为标准形;
   (III) 求方程 $f(x_1,x_2,x_3)=0$ 的解.

6. 设二次型

$$f(x_1,x_2,x_3)=ax_1^2+ax_2^2+(a-1)x_3^2+2x_1x_3-2x_2x_3.$$

   (I) 求二次型 f 的矩阵的所有特征值;
   (II) 若二次型 f 的规范形为 $y_1^2+y_2^2$, 求 a 的值.