Spring 2022 Linear Algebra A Final Exam Solution Set.

 June 14, 2022.

Question 1. (15 points)

(1) C (2) D (3) B (4) C (5) A.

Question 2. (15 points)

(1) 3

(2) $\begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$.

(4) $\begin{bmatrix} 1 & 2022 \\ 0 & 1 \end{bmatrix}$

(5) $2, \sqrt{2}$.

Question 3. (15 points)

(a) $\det A = -2$

(b) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

(c) $L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Question 4. (15 points)

$AX - XA = B$ implies that

$$\begin{cases} -x_2 + ax_3 = 0 \\ -ax_1 + x_2 + ax_4 = 1 \\ x_1 - x_3 - x_4 = 1 \\ x_2 - ax_3 = b \end{cases}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & a & 0 & 0 \\ -a & 1 & 0 & a & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -a & 0 & b \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & a+1 \\ 0 & 0 & 0 & 0 & b \end{bmatrix} \xRightarrow{\text{consistent}}$$

augmented matrix

$$\left.\begin{matrix} a+1=0 \\ b=0 \end{matrix}\right\} \Rightarrow \begin{cases} a=-1, \\ b=0. \end{cases}$$

augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$x_1 = 1 + k_1 + k_2, \quad x_2 = -k_1, \quad x_3 = k_1, \quad x_4 = k_2$

$$X = \begin{bmatrix} 1+k_1+k_2 & -k_1 \\ k_1 & k_2 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}.$$

Question 5. (20 points)

(a) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

(b) A has an eigenvalue 3.

$$\det(A - 3I) = 8(2-k) = 0 \quad \Rightarrow \quad k = 2.$$

$$Q^{-1}AQ = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 3 & \\ & & & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(c) $f(x_1, x_2, x_3, x_4)$ indefinite.

Question 6. (8 points)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1.01 \\ 1.35 \\ 1.68 \\ 2.07 \\ 2.53 \\ 2.77 \end{bmatrix}$$

Best Fitting Line:

$$y = kx + t$$

$$A^T A = \begin{bmatrix} 55 & 15 \\ 15 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 34.89 \\ 11.41 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b \quad , \quad \hat{x} = \begin{bmatrix} k \\ t \end{bmatrix}$$

$$\begin{bmatrix} k \\ t \end{bmatrix} = \frac{1}{105} \begin{bmatrix} 6 & -15 \\ -15 & 55 \end{bmatrix} \begin{bmatrix} 34.89 \\ 11.41 \end{bmatrix} = \frac{1}{105} \begin{bmatrix} 38.19 \\ 104.2 \end{bmatrix}$$

$$y = kx + t = \frac{6}{105} 38.14 + \frac{104.2}{105} \approx 3.17.$$

Question 7. (12 points)

(a) $\sum_{i=1}^{r} \sum_{j=1}^{s} c_{ij} \alpha_i \beta_j^T = 0$

$$\Leftrightarrow \underbrace{\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \end{bmatrix}}_{m \times r} \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s} \\ c_{21} & c_{22} & \cdots & c_{2s} \\ \vdots & \vdots & & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rs} \end{bmatrix}}_{r \times s} \underbrace{\begin{bmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_s^T \end{bmatrix}}_{s \times n} = O_{m \times n}$$

$$B \qquad C \qquad D = O_{m \times n}$$

$$\text{rank } B = r \Rightarrow CD = O_{r \times n}$$

$$\text{rank } D = s \Rightarrow C = O_{r \times s}. \qquad \#.$$

(b) If $AB = BA$, then AB is symmetric. It then suffices to show that the eigenvalues of AB are all positive.

Since A is positive definite, there exists a positive definite matrix P such that $A = P^2$. Thus $AB = P^2 B$, which is similar to

$$P^{-1} P^2 B P = PBP = P^T B P.$$

Since $P^T B P$ is congruent to B, and B is a positive definite matrix, we know that $P^T B P$ has positive ~~definite~~ eigenvalues.

Therefore, the eigenvalues of AB are positive.

It follows that AB is positive definite.