

# Solving $Ax = 0$ and $Ax = b$ (part 2)

非齐次

## Lecture 8

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# Complete Solution

- 1 Solving  $Ax=b$ ,  $Ux=c$ , and  $Rx=d$
- 2 Homework Assignment 8

## Solving $Ax = b$ , $Ux = c$ , and $Rx = d$

The case  $b \neq 0$  is quite different from  $b = 0$ . The row operations on  $A$  must act also on the right-hand side (on  $b$ ). We begin with letters  $(b_1, b_2, b_3)$  to find the solvability condition— for  $b$  lie in the column space.

For the original example  $Ax = b = (b_1, b_2, b_3)$ , apply to both sides the operations that led from  $A$  to  $U$ . The result is an upper triangular system

$Ux = c$ :

$$Ux = c \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}.$$

做消元时考虑  
增广矩阵

非齐次时可能有条件  
有解  
第三次变换  
row 2 变换

## Remarks

有解的条件

- (a) The equations are inconsistent unless  $b_3 - 2b_2 + 5b_1 = 0$ .
- (b) The dependent columns, the second and the fourth, are exactly the ones without pivots.

[5, -2, 1]  
是  $C(A)$   
法向量

# Solving $Ax = b$ , $Ux = c$ , and $Rx = d$ : Continue

For a specific example with  $b_3 - 2b_2 + 5b_1 = 0$ , choose  $b = (1, 5, 5)$ :

$$Ax = b \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$

$\begin{bmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The complete solution is as follows

$$x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(等出组)

$\downarrow$

free variable  
 $= 0$

1.  $x=0$  时的解

## Remark

Remark:

Every solution to  $Ax = b$  is the sum of one particular solution and a solution to  $Ax = 0$ :

$$x_{\text{complete}} = x_{\text{particular}} + x_{\text{nullspace}}$$

加了一个向量就  
不一定是一个 subspace 了

## \* Rank 重要!

Elimination reveals the pivot variables and free variables. If there are  $r$  pivots, there are  $r$  pivot variables and  $n - r$  free variables. That important number  $r$  will be given a name—it is the rank of the matrix.

### Definition

The rank of  $A$  is the number of pivots. This number is  $r$ .

立元數量的個數

$$\text{rank}(A) = \# \text{ of pivots} = r = \text{dimension of CLA}$$

$$\# \text{ free variables} = n - r = \# \text{ of special solutions}$$

特殊解個數 = dimension of NCA

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

↓  
invertible    invertible

经过初等行 + 列变换

相抵标准形  
(equivalent)

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

有 r 个 1 (主元)

$A^T = A$       矩阵对称：主对角线能大  
 $\Rightarrow \text{tr}(A) = \text{rank}(A)$   
 ↓ 主对角元素之和

$$A = A^T \quad 0/1$$

# Theorem

## Theorem

Suppose elimination reduces  $Ax = b$  to  $Ux = c$  and  $Rx = d$ , with  $r$  pivot rows and  $r$  pivot columns. **The rank of those matrices is  $r$ .** The last  $m - r$  rows of  $U$  and  $R$  are zero, so there is a solution only if the last  $m - r$  entries of  $c$  and  $d$  are also zero.

The complete solution is  $x = x_p + x_n$ . One particular solution  $x_p$  has all free variables zero. Its pivot variables are the first  $r$  entries of  $d$ , so  $Rx_p = d$ .

The nullspace solution  $x_n$  are combinations of  $n - r$  **special solutions**, with one free variable equal to 1. The pivot variables in that special solution can be found in the corresponding column of  $R$  (with sign reversed).

有多少行为 0:  $m - r$

(只有有 pivot 与 free 个数相  
同时有冗余)

维度不一定  
分量个数

## Another Worked Example

There are several remarks regarding the previous theorem:

1. You see how the rank  $r$  is crucial. It counts the pivot rows in the “row space” and the pivot columns in the column space.
2. There are  $n - r$  special solutions in the nullspace.
3. There are  $m - r$  solvability conditions on  $b$  or  $c$  or  $d$ .

The full picture uses elimination and pivot columns to find the column space, nullspace, and rank. The 3 by 4 matrix  $A$  has rank 2:

$$Ax = b \text{ is } \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

## Another Worked Example: Continue

$$Ax = b \text{ is } \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We follow the following steps to solve this system:

1. Reduce  $[A \quad b]$  to  $[U \quad c]$ , to reach a triangular system  $Ux = c$ .
2. Find the condition on  $b_1, b_2, b_3$  to have a solution.
3. Describe the column space of  $A$ : Which plane in  $\mathbb{R}^3$ ?
4. Describe the nullspace of  $A$ : Which special solutions in  $\mathbb{R}^4$ ?

solvability condition:  $b_3 - 2b_2 + 5b_1 = 0$

$$\Leftrightarrow [5 - 2 \ 1] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$Ax = b$$

consistent  $\Leftrightarrow b \in C(A)$

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ 1 & -3 & 3 & 4 \end{bmatrix}$$

$[5 - 2 \ 1]$  为  $C(A)$  平面的法向量

与  $[5 - 2 \ 1]$  垂直才能在  $C(A)$  中

$$C(A) = \{c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$
 线性组合

$$[5 - 2 \ 1] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0$$

$$[5 - 2 \ 1] \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} = 0$$

与构成平面的基向量垂直

线性独立  
列向量

## Remarks

5. Find a particular solution to  $Ax = (0, 6, -6)$  and the complete  $x_p + x_n$ .
6. Reduce  $[U \quad c]$  to  $[R \quad d]$ : Special solutions from  $R$  and  $x_p$  from  $d$ .

Now let us work out the details on blackboard together!

$$x_{\text{complete}} = x_p + x_n$$

$$Ax = b$$

$$Ax_p = b$$

$$A(x - x_p) = 0$$

$$x_n = x - x_p$$

解的结构性定理

## Two More Examples

### Example

Describe the set of attainable right-hand sides  $b$  (in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on  $b$  that turn the third equation into  $0 = 0$  (after elimination). What is the rank, and a particular solution?

### Example

Suppose  $A$  and  $B$  are  $n$  by  $n$  matrices, and  $AB = I$ . Prove from  $\text{rank}(AB) \leq \text{rank}(A)$  that the rank of  $A$  is  $n$ . So  $A$  is invertible and  $B$  must be its two-sided inverse. Therefore,  $BA = I$ .

# Homework Assignment 8

2.2: 2, 5, 9, 15, 24, 30, 45, 48, 58.