1. True or false? No justification is necessary.

   判断对错. 不需要给出解释.

   (1) Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are column vectors of length n, and A is an $n \times n$ matrix. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent, then $A\alpha_1, A\alpha_2, \cdots, A\alpha_n$ are linearly independent.

   设 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 均为 n 维列向量, A 是 $n \times n$ 矩阵, 若 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性无关, 则 $A\alpha_1, A\alpha_2, \cdots, A\alpha_n$ 线性无关.

   (2) For the matrices $A = \begin{bmatrix} 2018 & & & \\ & 2019 & & \\ & & 2019 & \\ & & & 2020 \end{bmatrix}$ and $A' = \begin{bmatrix} 2020 & & & \\ & 2019 & & \\ & & 2018 & \\ & & & 2019 \end{bmatrix}$, there exists an invertible real matrix P of order 4 such that $P^{-1}AP = A'$.

   对于矩阵 $A = \begin{bmatrix} 2018 & & & \\ & 2019 & & \\ & & 2019 & \\ & & & 2020 \end{bmatrix}$ 和 $A' = \begin{bmatrix} 2020 & & & \\ & 2019 & & \\ & & 2018 & \\ & & & 2019 \end{bmatrix}$, 存在可逆的 4 阶实矩阵 P 使得 $P^{-1}AP = A'$.

   (3) Let A be a real square matrix. Then a real number λ is an eigenvalue of A if and only if it is an eigenvalue of the transpose A^T.

   设 A 为实方阵. 则一个实数 λ 是 A 的特征值当且仅当它是转置矩阵 A^T 的特征值.

   (4) If H is a Hermitian matrix, then $I + iH$ is an invertible matrix.

   若 H 为 Hermite 矩阵, 则 $I + iH$ 是可逆矩阵.

   (5) Suppose $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where a, b, c are positive real numbers. Then for all nonzero column vectors x in $\mathbb{R}^2$, $x^T A x \geq 0$.

   设 $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, 其中 a, b, c 均为正实数. 则对于 $\mathbb{R}^2$ 中所有的非零列向量 x, 均有 $x^T A x \geq 0$.

2. Write down your answers to the following questions. No further explanation is needed.

   请直接写出以下问题的答案. 不需要做进一步解释.

   (1) Let U be the subspace of $\mathbb{R}^4$ spanned by the two column vectors $u = (1, 0, 1, -1)^T$ and $v = (0, 1, 0, -1)^T$. Let $W = U^{\perp}$ be the orthogonal complement of U in $\mathbb{R}^4$, that is, W is the subspace of $\mathbb{R}^4$ consisting of vectors orthogonal to all vectors in U.

   Find a basis of W.

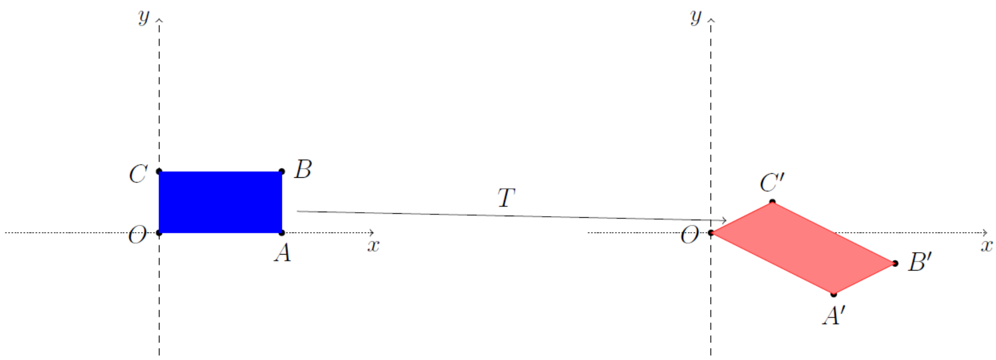
   设 U 为 $\mathbb{R}^4$ 中的两个列向量 $u = (1, 0, 1, -1)^T$ 和 $v = (0, 1, 0, -1)^T$ 张成 (生成) 的子空间. 设 $W = U^{\perp}$ 为 U 在 $\mathbb{R}^4$ 中的正交补, 即, W 由 $\mathbb{R}^4$ 中与 U 中向量全都正交的向量组成.

   找出 W 的一组基.

(2) Consider the following points in the plane $\mathbb{R}^2$:

$$O(0,\,0)\,;\quad A(2,\,0)\,,\;B(2,\,1)\,,\;C(0,\,1)\,;\quad A'(2,\,-1)\,,\;B'(3,\,-0.5)\,,\;C'(1,\,0.5)\,.$$

(Here we write $M(a,\,b)$ to mean that the point M has coordinates $(a,\,b)$ in the plane.) Find a linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ that transforms the rectangular $OABC$ (with sides and interior) onto the parallelogram $OA'B'C'$. (*Please write down the expression $T(x,\,y)$ in terms of the coordinate $(x,\,y)$ of a general point in $\mathbb{R}^2$.*)



考虑平面 $\mathbb{R}^2$ 内的下列各点:

$$O(0,\,0)\,;\quad A(2,\,0)\,,\;B(2,\,1)\,,\;C(0,\,1)\,;\quad A'(2,\,-1)\,,\;B'(3,\,-0.5)\,,\;C'(1,\,0.5)\,.$$

(这里我们用 $M(a,\,b)$ 表示点 M 的坐标为 $(a,\,b)$.)
找出一个线性变换 $T:\mathbb{R}^2\to\mathbb{R}^2$, 它将长方形 $OABC$ (包含四条边和内部) 变换成 (映射为) 平行四边形 $OA'B'C'$. (请通过 $\mathbb{R}^2$ 中一般点的坐标 $(x,\,y)$ 写出 $T(x,\,y)$ 的表达式.)

(3) Let $A=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&-1\end{bmatrix}$. Compute A^{2020}.

设 $A=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&-1\end{bmatrix}$. 求 A^{2020}.

(4) Let $A=\begin{bmatrix}2&0&2\\0&1&0\\0&0&0\end{bmatrix}$. Write down all the singular values of A.

设 $A=\begin{bmatrix}2&0&2\\0&1&0\\0&0&0\end{bmatrix}$. 写出 A 的所有奇异值.

3. For each natural number $n \geq 3$, consider the $n \times n$ matrix $A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$ and define $a_n = \det(A_n)$.

   (a) Compute a_3 and a_4.

   (b) For each $n \geq 5$, find a recursive formula relating a_n to a_{n-1} and a_{n-2}.

   (c) For general $n \geq 3$, find an explicit expression of a_n (in terms of n).

   对每个自然数 $n \geq 3$, 考虑 $n \times n$ 矩阵 $A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$ 并设 $a_n = \det(A_n)$.

   (a) 计算 a_3 和 a_4.

   (b) 对每个 $n \geq 5$, 找出一个递推公式将 a_n 和 a_{n-1}, a_{n-2} 联系起来.

   (c) 对一般的 $n \geq 3$, 找出 a_n (关于 n) 的一个显式表达式.

4. Suppose that a data set consists of points $(-6,-1)$, $(-2,2)$, $(1,1)$ and $(7,6)$ on the xy-plane. Find an equation for the line that best models the relation between the x and y coordinates of these sample values in the sense of least-squares.

   假设一组数据由 xy-平面内的点 $(-6,-1)$, $(-2,2)$, $(1,1)$ 和 $(7,6)$ 给出.
   求能够在最小二乘法意义下最好地拟合这些样本点 x 和 y 坐标关系的直线方程.

5. Consider the quadratic form $f(x_1,x_2,x_3) = -x_1^2 - 5x_2^2 - 9x_3^2 - 4x_1x_2 - 6x_1x_3 - 8x_2x_3$.

   (a) Find the matrix A of the quadratic form $f(x_1,x_2,x_3)$.

   (b) Decide for or against the positive definiteness of A.

   (c) Is there a real solution to the quadratic equation $f(x_1,x_2,x_3) = 1$ (in the unknowns x_1, x_2, x_3)? Explain why.

   考虑二次型 $f(x_1,x_2,x_3) = -x_1^2 - 5x_2^2 - 9x_3^2 - 4x_1x_2 - 6x_1x_3 - 8x_2x_3$.

   (a) 求二次型 $f(x_1,x_2,x_3)$ 的矩阵 A.

   (b) 判定矩阵 A 的正定性.

   (c) (以 x_1, x_2, x_3 为未知数的) 二次方程 $f(x_1,x_2,x_3) = 1$ 是否有实数解? 请阐述理由.

6. For any real matrix M, let $C(M)$ be its column space, $N(M)$ be its null space and $\operatorname{rank}(M)$ be its rank.

(a) Write down a 2×3 real matrix A such that $C(A)$ has dimension 2.

(b) For the matrix A you give in the previous question, find $\dim N(A)$ and $\operatorname{rank}(A^T A)$.

(c) Is there a real matrix M such that $\operatorname{rank}(M^T M) < \operatorname{rank}(M)$? If yes, provide such an example; otherwise, explain why such a matrix cannot exist.

对任意实矩阵 M, 以 $C(M)$ 表示它的列空间, $N(M)$ 表示它的零空间 (也称零化空间), $\operatorname{rank}(M)$ 表示它的秩.

(a) 写出一个 2×3 实矩阵 A 使 $C(A)$ 的维数是 2.

(b) 对于你在上一个问题中写出的矩阵 A, 求 $\dim N(A)$ 和 $\operatorname{rank}(A^T A)$.

(c) 是否存在实矩阵 M 满足 $\operatorname{rank}(M^T M) < \operatorname{rank}(M)$? 若是, 请给出一个这样的例子; 若否, 请解释为什么这样的矩阵不存在.

7. Let A be a square matrix of order n and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be column vectors in $\mathbb{R}^n$. Suppose that

$$\alpha_i^T A \alpha_j = 0 \quad \text{whenever } i \neq j\,, \quad \text{and } \ \alpha_i^T A \alpha_i = 1 \text{ for all } i = 1, 2, \cdots, n\,.$$

Show that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent.

设 A 是一个 n 阶方阵, $\alpha_1, \alpha_2, \cdots, \alpha_n$ 是 $\mathbb{R}^n$ 中的列向量. 假设

$$\text{当 } i \neq j \text{ 时, 总有 } \ \alpha_i^T A \alpha_j = 0 \quad \text{且 } \alpha_i^T A \alpha_i = 1 \text{ 对所有 } i = 1, 2, \cdots, n \text{ 成立}\,.$$

证明 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性无关.

8. Let A be a square matrix of order n such that $A^2 = A$.

(a) Show that if λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

(b) Suppose $A \neq I$ (where I is the identity matrix of order n). Show that $\det(A) = 0$.

(c) Suppose B is a square matrix of order n, and the only eigenvalues of B are 0 and 1. Is it necessarily true that $B^2 = B$? If yes, provide a proof. Otherwise give a counterexample.

设 A 为 n 阶方阵, $A^2 = A$.

(a) 证明: 若 λ 是 A 的特征值, 则 $\lambda = 0$ 或 $\lambda = 1$.

(b) 假设 $A \neq I$ (这里 I 表示 n 阶单位矩阵). 证明 $\det(A) = 0$.

(c) 假设 B 是 n 阶方阵, 且 B 的特征值只有 0 和 1. 等式 $B^2 = B$ 是否一定成立？若是, 请给出证明. 若否, 请举出反例.

9. Let

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \ f(x) = x^3 - 2x + 5\,, \quad B = f(A)\,.$$

(a) Prove that every eigenvector of A is an eigenvector of B.

(b) Show that B is diagonalizable.

设

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \; f(x) = x^3 - 2x + 5\,, \quad B = f(A)\,.$$

(a) 证明 A 的特征向量都是 B 的特征向量.

(b) 证明 B 可以对角化.

10. Let v be a nonzero column vector in $\mathbb{R}^n$ with $n \geq 2$.

(a) Find all the eigenvalues of the $n \times n$ matrix $v\,v^T$.

(b) Let I_n be the identity matrix of order n and

$$H = I_n - 2\frac{vv^T}{v^T v}\ .$$

Find the rank of the matrix $I_n + H$.

设 v 为 $\mathbb{R}^n$ 中的非零列向量, 其中 $n \geq 2$.

(a) 求出 $n \times n$ 矩阵 $v\,v^T$ 的所有特征值.

(b) 令 I_n 为 n 阶单位矩阵,

$$H = I_n - 2\frac{vv^T}{v^T v}\ .$$

求矩阵 $I_n + H$ 的秩.