Midterm Linear Algebra Fall 2023

Suggested Solutions. Nov. 12, 2023

Question 1 :

(1) A (2) B (3) D (4) B (5) D

Question 2 :

(1) 21.

(2) $-A^{-1}CB^{-1}$.

(3) 1 or -3.

(4) $\begin{bmatrix} \frac{19}{11} \\ -\frac{5}{11} \end{bmatrix}$

(5) $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ or $-\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Question 3.

(a) A basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$.

A basis for $C(A^T)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

or $\begin{pmatrix} 1 \\ 2 \\ 5 \\ 1 \\ 6 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}$

A basis for $N(A)$: $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

A basis for $N(A^T)$: $\left\{ \begin{bmatrix} -5 \\ 13 \\ -3 \\ 1 \end{bmatrix} \right\}$.

(b) $\begin{bmatrix} 5 \\ 2 \\ 1 \\ 2 \end{bmatrix}$.

Question 4:

Gaussian Eliminations give:

$$\left[\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 2 & a & 1 & 1 & a \\ -1 & 1 & a & -a-1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|cc} 1 & -1 & -1 & 2 & 2 \\ 0 & a+2 & 3 & -3 & a-4 \\ 0 & 0 & a-1 & 0 & 0 \end{array}\right]$$

- If $a=-2$, then $\operatorname{rank} A = 2 \neq 3 = \operatorname{rank}(A \vdots B)$,

  $AX=B$ has no solution.

- If $a \neq 1$ and $a \neq -1$, $AX=B$ has a unique solution.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & a+2 & 3 & -3 \\ 0 & 0 & a-1 & 1-a \end{array}\right] \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & a+2 & 3 & a-4 \\ 0 & 0 & a-1 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} \frac{3a}{a+2} \\ \frac{a-4}{a+2} \\ 0 \end{bmatrix}.$$

$$X = \begin{bmatrix} 1 & \frac{3a}{a+2} \\ 0 & \frac{a-4}{a+2} \\ -1 & 0 \end{bmatrix}.$$

- If $a=1$, $AX=B$ has infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 & 1 \\ -k_1-1 & -k_2-1 \\ k_1 & k_2 \end{bmatrix},$$ k_1, k_2 arbitrary constants.

Question 5:

(a) Let $X, Y \in M_{2\times 2}(\mathbb{R})$ and $c \in \mathbb{R}$, then we have

$$T(cX+Y) = \begin{bmatrix} tr\, A^T(cX+Y) \\ tr\, B^T(cX+Y) \\ tr\, C^T(cX+Y) \end{bmatrix}$$

$$= c\begin{bmatrix} tr(A^T X) \\ tr(B^T X) \\ tr(C^T X) \end{bmatrix} + \begin{bmatrix} tr(A^T Y) \\ tr(B^T Y) \\ tr(C^T Y) \end{bmatrix}$$

$$= c\,T(X) + T(Y).$$

(b)
$$T(v_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{=w_1} + 0\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{=w_2} + 0\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=w_3}$$

$$T(v_2) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(v_3) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(v_4) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + -1\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the matrix representation of T with respect to v_1, v_2, v_3, v_4 and w_1, w_2, w_3 is : $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

(c) Since $T(A) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $T(B) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $T(C) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

we can take X to be

$$\frac{1}{2}A - 2B + C$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -2 \\ 1 & -\frac{1}{2} \end{bmatrix}.$$

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Question 6.

Apply Elementary Row and Column Operations to A and $\cancel{B}$ C to obtain $D_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ for A and $D_2 = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}$ for $\cancel{B}$ C,

Where $r = \text{rank } A$, $s = \text{rank } \cancel{B}$ C.

Let $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$. Then M can be converted to $M_1 = \begin{bmatrix} D_1 & C_1 \\ 0 & D_2 \end{bmatrix}$ via elementary row and column operations.

Furthermore, the pivots in D_1 and D_2 can be used to eliminate the nonzero entries in C_1, to obtain

$$M_2 = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In conclusion,

$$\text{rank } M = \text{rank } M_1 = \text{rank } M_2 = r+s+\text{rank } C_2 \geq r+s$$
$$= \text{rank } A + \text{rank } C.$$

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