Problem Set 5 —— Linear Algebra (Spring 2024)

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可做行交换使得操作为[illegible]些

1. 设

$$\begin{cases} (2-\lambda)x_1+2x_2-2x_3 &= 1, \\ 2x_1+(5-\lambda)x_2-4x_3 &= 2, \\ -2x_1-4x_2+(5-\lambda)x_3 &= -\lambda-1, \end{cases}$$

问 λ 为何值时, 此方程组有唯一解、无解或有无穷多解? 并在有无穷多解时求其通解.

Let

$$\begin{cases} (2-\lambda)x_1+2x_2-2x_3 &= 1, \\ 2x_1+(5-\lambda)x_2-4x_3 &= 2, \\ -2x_1-4x_2+(5-\lambda)x_3 &= -\lambda-1, \end{cases}$$

For what values of λ, the above system has a unique solution, no solution, or an infinity of solutions. Find all the solutions to the above system if the system has infinitely many solutions.

2. 设 A 为 $m\times n$ 矩阵, B 为 $n\times p$ 矩阵. 证明:

$$\begin{bmatrix} AB & \\ & I_n \end{bmatrix} \to \begin{bmatrix} AB & A \\ & I_n \end{bmatrix}$$

$$\begin{bmatrix} 0 & A \\ -B & I_n \end{bmatrix}$$

$$\operatorname{rank} A+\operatorname{rank} B-n\le \operatorname{rank} AB.$$

同时探讨一下在什么时候上面的等号成立.

Let A be $m\times n$, B be $n\times p$ matrices. Show that:

$$\operatorname{rank} A+\operatorname{rank} B-n\le \operatorname{rank} AB.$$

And determine when the above inequality is actually an equality.

3. 假定:

$$W_1:=\left\{\begin{bmatrix} x\\ y\\ z\end{bmatrix}\in\mathbb{R}^3 : x+2y+3z=0\right\} \text{ 和 } W_2:=\left\{\begin{bmatrix} x\\ y\\ z\end{bmatrix}\in\mathbb{R}^3 : 3x+2y+z=0\right\}$$

(a) $W_1\cap W_2:=\{x\in\mathbb{R}^3 : x\in W_1 \text{ 且 } x\in W_2\}$ 是否为 $\mathbb{R}^3$ 的一个子空间? 阐明理由.

(b) 设 W_3 为 $\mathbb{R}^3$ 的另外一个子空间. 证明 $(W_1+W_2)\cap W_3$ 也是 $\mathbb{R}^3$ 的一个子空间, 其中

$$W_1+W_2:=\{w_1+w_2 : w_1\in W_1,\ w_2\in W_2\}.$$

Let

$$W_1:=\left\{\begin{bmatrix} x\\ y\\ z\end{bmatrix}\in\mathbb{R}^3 : x+2y+3z=0\right\} \text{ and } W_2:=\left\{\begin{bmatrix} x\\ y\\ z\end{bmatrix}\in\mathbb{R}^3 : 3x+2y+z=0\right\}$$

(a) Is $W_1\cap W_2:=\{x\in\mathbb{R}^3 : x\in W_1 \text{ and } x\in W_2\}$ a subspace of $\mathbb{R}^3$? Please explain.

(b) Let W_3 为 $\mathbb{R}^3$ be a third subspace of $\mathbb{R}^3$. Show that $(W_1+W_2)\cap W_3$ is a subspace of $\mathbb{R}^3$, where

$$W_1+W_2 := \{w_1+w_2 : w_1\in W_1,\ w_2\in W_2\}.$$

4. 设 A 为:

$$A=\begin{bmatrix} 1 & 3 & 3 & 2\\ 2 & 6 & 9 & 7\\ -1 & -3 & 3 & 4\end{bmatrix}.$$

(a) 如果 P 为一个 3×3 的可逆矩阵, Q 为一个 4×4 的可逆矩阵. 证明

$$\operatorname{rank} PAQ = 2.$$

(b) 求一个 3×3 的可逆矩阵 P 和一个 4×4 的可逆矩阵 Q 使得

$$PAQ=\begin{bmatrix} I_2 & O\\ O & O\end{bmatrix}.$$

这里的 O 都表示相应的零矩阵.

Let A be:

$$A=\begin{bmatrix} 1 & 3 & 3 & 2\\ 2 & 6 & 9 & 7\\ -1 & -3 & 3 & 4\end{bmatrix}.$$

(a) Suppose P is a 3×3 invertible matrix, Q is a 4×4 invertible matrix. Show that

$$\operatorname{rank} PAQ = 2.$$

(b) Find a 3×3 invertible matrix P and a 4×4 invertible matrix Q such that

$$PAQ=\begin{bmatrix} I_2 & O\\ O & O\end{bmatrix}.$$

where O denotes the zero matrix.

5. 考虑

$$v_1=\begin{bmatrix}1\\-1\\1\\5\end{bmatrix},\ v_2=\begin{bmatrix}2\\-2\\2\\10\end{bmatrix},\ v_3=\begin{bmatrix}1\\0\\2\\5\end{bmatrix},\ v_4=\begin{bmatrix}1\\3\\5\\5\end{bmatrix},\ v_5=\begin{bmatrix}2\\-3\\2\\13\end{bmatrix},\ v_6=\begin{bmatrix}0\\-1\\2\\9\end{bmatrix}.$$

(a) 找出 v_1,v_2,v_3,v_4,v_5,v_6 的一个极大线性无关组.

(b) 将其余向量表示为该极大线性无关组的线性组合. [ ] 肩宽不行列!

Consider

$$v_1=\begin{bmatrix}1\\-1\\1\\5\end{bmatrix},\ v_2=\begin{bmatrix}2\\-2\\2\\10\end{bmatrix},\ v_3=\begin{bmatrix}1\\0\\2\\5\end{bmatrix},\ v_4=\begin{bmatrix}1\\3\\5\\5\end{bmatrix},\ v_5=\begin{bmatrix}2\\-3\\2\\13\end{bmatrix},\ v_6=\begin{bmatrix}0\\-1\\2\\9\end{bmatrix}.$$

(a) Find a maximal linearly independent list of $v_1, v_2, v_3, v_4, v_5, v_6$.

(b) Express the remaining vectors in terms of the vectors in the maximal linearly independent list.