Week9: 期中试卷评讲

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(1) Which of the following statements can guarantee that the following homogeneous system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

has a nonzero solution?

(A) $m \le n$.

(B) $m = n$.

(C) $m > n$.

(D) The rank of the coefficient matrix is less than n.

- 1(1)解析： D

要使原方程有非零解，即是说系数矩阵的列间线性相关。

A,B,C:只要n不大于m，则n个列向量之间就有可能线性无关，导致只有非零解。

D:系数矩阵的秩小于n，说明n个列向量间线性相关，有非零解。

(2) Which of the following matrices can be written as a product of elementary matrices? ( )

(A) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix}$.

(B) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(D) $\begin{bmatrix} -3 & 2 & 7 \\ -1 & 2 & 3 \\ 0 & -2 & -1 \end{bmatrix}$.

- 1(2)解析：**B**
- 能够写成初等矩阵的乘积表示，这个矩阵一定是方阵，并且可逆。则只需要判断哪个矩阵可逆即可.

(3) Let $\beta_1, \beta_2, \beta_3$ be a basis of the null space $N(A)$ of some matrix A. Another basis of $N(A)$ is (   )

(A) $\beta_1+\beta_2, \beta_2+\beta_3, \beta_3+\beta_1$.

(B) $\beta_1+\beta_2, \beta_2+\beta_3, \beta_3-\beta_1$.

(C) $\beta_1-\beta_2, \beta_2-\beta_3, \beta_3-\beta_1$.

(D) $\beta_1+2\beta_2, 2\beta_2+3\beta_3, 3\beta_3-\beta_1$.

- 1(3)解析：**A**

因为选项中的所有向量都已经在N(A)中了，所以判断这些向量能否构成一组基，则只需判断他们是否线性无关。

以A为例，设k_1,k_2,k_3使$k_1(\beta_1+\beta_2)+k_2(\beta_2+\beta_3)+k_3(\beta_3+\beta_1)=0$

$\therefore$原式等价于：$(\beta_1,\beta_2,\beta_3)\underbrace{\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}}_{A}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix}=0$ $\because(\beta_1,\beta_2,\beta_3)$已可逆，$\therefore$只需判断A是否可逆就可判出$k_1,k_2,k_3$是否全为0.

设 $a, b \in \mathbb{R}$. 集合

$$V=\{(x,y,z,w): x+2y+3z+4w=a+b+1,\ x-2y+4z-w=a-2b-5\}$$

在 (　) 成立时是 $\mathbb{R}^4$ 的子空间.

(A) $a=-1, b=1$.

(B) $a=-2, b=1$.

(C) $a=1, b=-2$.

(D) $a=1, b=-1$.

- 1(4)解析：**C**
- **V可以被看做两个平面的交线，这条交线是子空间当且仅当这条交线过原点。即是说当x=y=z=w=0的时候，a+b+1,a-2b-5也等于0.解方程组即可。**

(5) Let u and v be unit vectors in $\mathbb{R}^3$. If the vectors $u+2v$ and $5u-4v$ are orthogonal, then the angle α between u and v is ( )

(A) $\alpha = \frac{\pi}{6}$.

(B) $\alpha = \frac{\pi}{4}$.

(C) $\alpha = \frac{\pi}{3}$.

(D) $\alpha = \frac{3\pi}{4}$.

- 1(5)解析： C

$$\cos\langle u, v\rangle = \frac{u^T v}{|u||v|} = u^T v$$

(1) Let A be a 5×8 real matrix. If $\dim N(A) = 3$, then $\dim(N(A^T))=$ ___0___.

设 A 为 5×8 实矩阵. 若 $\dim N(A) = 3$, 则 $\dim(N(A^T))=$ ________.

$$\dim N(A^T) = m - \text{rank}(A) = m - (n - \dim N(A))$$
$$= m - n + \dim N(A)$$
$$= 5 - 8 + 3 = 0$$

(2) All the 2×2 matrices that commute with $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ can be written in the form $\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$.

所有和矩阵 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ 乘法可交换的 2 × 2 矩阵均可写成 ________ 的形式.

设 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\therefore$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}$

$\therefore$ $a = a+b \Rightarrow b=0$

$a+c=c+d \Rightarrow d=a$

$b+d=d \Rightarrow$ 无.

$\therefore \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$,

(3) Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & \lambda \\ 3 & 1 & -1 \end{bmatrix}$. If $AB = 0$ for some nonzero matrix B, then $\lambda =$ $\underline{\quad 1 \quad}$.

$\because AB = 0$ $\therefore$ A 的列线性相关. $\therefore$ A 不可逆.

$$\begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & \lambda \\ 3 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -2 \\ 0 & -5 & \lambda+4 \\ 0 & -5 & 5 \end{pmatrix}$$

$\therefore \lambda + 4 = 5 \Rightarrow \lambda = 1$

(4) An LU-factorization of $A = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$ is $L =$ $\begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}$, $U =$ $\begin{pmatrix} 2 & 1 \\ 0 & 11/2 \end{pmatrix}$.

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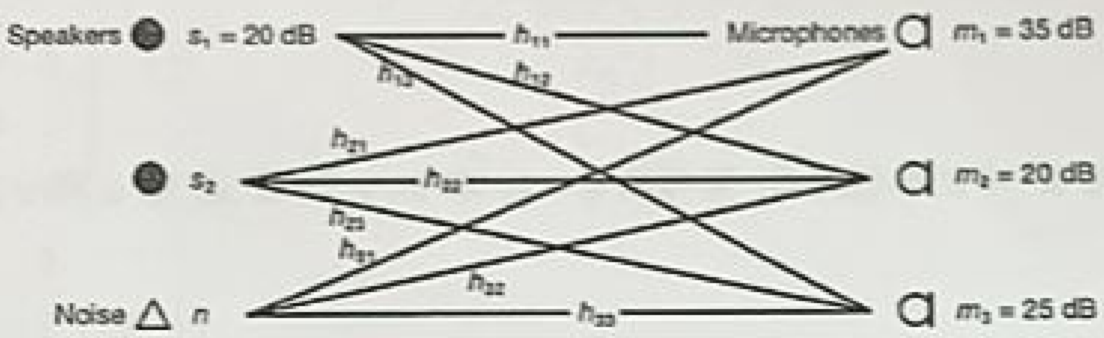
(5) Suppose b is a nonzero column vector. If η_1, η_2 are solutions to the system of linear equations $Ax = b$, and $\lambda_1\eta_1 + \lambda_2\eta_2$ is another solution to $Ax = b$, then λ_1, λ_2 must satisfy $\underline{\lambda_1 + \lambda_2 = 1}$

$$\begin{cases} A\eta_1 = b \\ A\eta_2 = b \\ A(\lambda_1\eta_1 + \lambda_2\eta_2) = b \end{cases} \Rightarrow \lambda_1 b + \lambda_2 b = b$$

$$\therefore \lambda_1 + \lambda_2 = 1$$

3. (10 points) An audio processing company develops technology for mobile devices and is proud of the capacity of their products to filter surrounding noise. Here is a simplified model (a *single-layer neural network*) showing how it works. Let s_1, s_2 be the volumes of a pair of speakers and n denote that of noise. Use 3 microphones to receive signals with recorded volumes m_1, m_2, and m_3. All values are in decibels and shown in the following diagram, where the linear factor h_{ij} indicates the rate of decay along each channel. (When a sound of 100 dB is transmitted along a channel with rate of decay h, the volume received is $100h$ dB.)



Suppose we are given the matrix $[h_{ij}] = \begin{bmatrix} 0.875 & 0.5 & 0.75 \\ 0.25 & 0.5 & 0.5 \\ 0.625 & 0.375 & 0.5 \end{bmatrix}$. Estimate the volume of the unknown source speaker by solving a linear system for s_2.

3. Solution.

$$\begin{cases} h_{11}S_1 + h_{21}S_2 + h_{31}n = m_1 \\ h_{12}S_1 + h_{22}S_2 + h_{32}n = m_2 \\ h_{13}S_1 + h_{23}S_2 + h_{33}n = m_3 \end{cases}$$

$$H = [h_{ij}] = \begin{bmatrix} \frac{7}{8} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}, \quad H^T \begin{bmatrix} S_1 \\ S_2 \\ n \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

$$\rightsquigarrow \Rightarrow S_2 = 20\ \text{dB}.$$

4. (20 points) Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T(\alpha_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ T(\alpha_2) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ T(\alpha_3) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \ \text{where } \alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \ \alpha_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(a) Show that $\alpha_1, \alpha_2, \alpha_3$ is a basis of $\mathbb{R}^3$.

(b) Find the representation matrix A of T (in the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$).

(c) Is the matrix A invertible? Why?

4. Solution. (a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = U$

Since U is invertible, so is $[\alpha_1, \alpha_2, \alpha_3]$; that is, $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

$\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of $\mathbb{R}^3$.

(b) $T[e_1, e_2, e_3] = [e_1, e_2, e_3]A = I_3 A = A.$

$[\alpha_1, \alpha_2, \alpha_3]Q = [e_1, e_2, e_3] = I_3$

$\therefore A = T[e_1, e_2, e_3] = T([\alpha_1, \alpha_2, \alpha_3]Q) \underset{\text{linear map.}}{\uparrow=} (T[\alpha_1, \alpha_2, \alpha_3])Q$

$A = [T\alpha_1, T\alpha_2, T\alpha_3]Q, \quad Q = [\alpha_1, \alpha_2, \alpha_3]^{-1}$

Using Gauss-Jordan Elimination, we get

$$Q = \begin{bmatrix} 5 & 2 & -4 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & -4 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 1 \\ -3 & -1 & 3 \end{bmatrix}$$

(c) Yes. Since Q is invertible,

A is invertible $\Leftrightarrow P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ is invertible

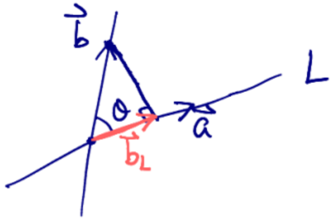
$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ $\therefore$ rank$(P) = 3$, P is invertible $\square$

5. (10 points) Let L be the line of intersection of $x_1+x_2+x_3=0$ and $2x_1-x_2-2x_3=0$ in $\mathbb{R}^3$.

Find the orthogonal projection of $b=\begin{bmatrix}1\\0\\1\end{bmatrix}$ onto L.

5. Solution.

$$\begin{cases} x_1+x_2+x_3=0 \\ 2x_1-x_2-2x_3=0 \end{cases} \Longrightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = t\begin{bmatrix}1\\-4\\3\end{bmatrix} = t\vec{a}, \quad a=\begin{bmatrix}1\\-4\\3\end{bmatrix}$$



$$\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$$\vec{b}_L = |\vec{b}|\cos\theta \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}^T\vec{b}}{|\vec{a}|^2}\vec{a}$$

$$= \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\vec{a} = \frac{2}{13}\begin{bmatrix}1\\-4\\3\end{bmatrix},$$

6. (10 points) Let u, v be nonzero column vectors in $\mathbb{R}^n$ and $A = uv^T$.

(a) Prove that the rank of A is 1.

(b) What are the possible values of the rank of the matrix $\begin{bmatrix} u^T v & 0 \\ 0 & vu^T \end{bmatrix}$? Justify your answer.

(a) A的所有列由u张成 行由V张成

(b) $u^T v$ 为一个数.

$$\text{rank}\begin{pmatrix} u^T v & 0 \\ 0 & vu^T \end{pmatrix} = \text{rank}(u^T v) + \text{rank}(vu^T) = 1 + 1 = 2$$

或 $0 + 1 = 1$

$\uparrow$

$u \perp v$

7. (10 points) Let $\alpha_1, \cdots, \alpha_n$ be column vectors in $\mathbb{R}^n$. Suppose that the system $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ is linearly dependent, and that the system $\alpha_2, \alpha_3, \cdots, \alpha_n$ is linearly independent.

Let $A = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

(a) Show that α_1 can be written as a linear combination of $\alpha_2, \alpha_3, \cdots, \alpha_n$, i.e., there exist constants $k_2, k_3, \cdots, k_n$ so that $\alpha_1 = k_2\alpha_2 + k_3\alpha_3 + \cdots + k_n\alpha_n$.

(b) Show that the linear system $Ax = \beta$ has infinitely many solutions.

(c) Prove that if $n > 2$, then $A^2 \neq O$. Here O denotes the zero matrix of order n.

7. proof. (a) Since $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are linearly independent,

there exist numbers $x_1, x_2, \cdots, x_n$, in which at least one of them is non-zero, such that

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} = 0.$$

If $x_1 \neq 0$, we are done:

$$\alpha_1 = -\frac{x_2}{x_1}\alpha_2 - \cdots - \frac{x_{n-1}}{x_1}\alpha_{n-1} + 0 \cdot \alpha_n$$

Assume $x_1 = 0$, then $x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} + 0 \cdot \alpha_n = 0$

Since $\alpha_2, \cdots, \alpha_n$ are linearly independent,

we have $x_2 = \cdots = x_{n-1} = 0$, which contradicts.

Thus $x_1 \neq 0$, and the proof of (a) is completed.

(b) $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_n = [\alpha_1, \alpha_2, \cdots, \alpha_n] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$\therefore A x = \beta$ has a solution $x_p = [1, 1, \cdots, 1]^T$.

Since $x = x_p + x_{null}$, $x_{null} \in N(A)$

it suffices to show $N(A) \neq 0$, or equivalently $\dim N(A) \geq 1$.

By (a) we get $\operatorname{rank}(A) = n-1$, hence

$$\dim N(A) = n - \operatorname{rank}(A) = 1 \geq 1.$$

Thus $N(A)$ is of dimension 1, $Ax = \beta$ has infinitely many solutions.

(c) "$n>2 \Rightarrow A^2 \neq 0$" $\Leftrightarrow$ "$A^2=0 \Rightarrow n \leq 2$".

Suppose $A^2=0$, then $C(A) \subseteq N(A)$, $\underset{\substack{\| \\ n-1}}{\text{rank}(A)} \leq \dim N(A) = 1$

Thus $n \leq 2$. $\square$