



南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线性代数 A

开课单位: 数学系

考试时长: 120 分钟

命题教师: 线性代数教学团队

题号	1	2	3	4	5	6	7	8
分值	15 分	25 分	10 分	10 分	10 分	10 分	10 分	10 分

本试卷共 (8) 大题, 满分 (100) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 100 in total. **Write all your answers on the examination book.**

本套试卷为 A 卷 Version A

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Which one of the following statements must be true? ()

- (A) If A and B are $m \times n$ matrices and $Ax = 0$ has the same solution set as $Bx = 0$, then A and B have the same column space.
- (B) If A is an $n \times n$ real symmetric positive definite matrix, then all the square submatrices of A have positive determinants.
- (C) If real symmetric matrices A and B are congruent, then they are similar.
- (D) If $AB = 0$ and A and B are not zero matrices, then A has linearly dependent columns and B has linearly dependent rows.

下列陈述一定正确的是? ()

- (A) 若 A 和 B 为 $m \times n$ 矩阵, 且 $Ax = 0$ 与 $Bx = 0$ 同解, 则 A 和 B 具有相同的列空间.
- (B) 若实对称矩阵 A 正定, 则它的所有正方形子矩阵的行列式都为正.
- (C) 若实对称矩阵 A 和 B 相合 (也称合同), 则它们相似.
- (D) 设 A, B 为满足 $AB = 0$ 的两个非零矩阵, 则 A 的列向量组线性相关, B 的行向量组线性相关.

(2) The plane curve defined by the equation $2x^2 - 8xy + 2y^2 = 1$ is ()

- (A) an ellipse.
- (B) a hyperbola.
- (C) a parabola.
- (D) a union of two intersecting lines.

由方程 $2x^2 - 8xy + 2y^2 = 1$ 定义的平面曲线是 ()

- (A) 椭圆.
- (B) 双曲线.
- (C) 抛物线.
- (D) 一对相交直线.

(3) Let A be an $n \times n$ real matrix. Suppose that for all column vectors $x \in \mathbb{R}^n$ we have $x^T Ax = 0$. Then ()

- (A) The determinant $|A|$ of A is 0.
- (B) $A = 0$.
- (C) The trace, $\text{trace}(A)$, of A is 0.
- (D) The only eigenvalue of A in \mathbb{C} is 0.

设 A 为 $n \times n$ 实矩阵. 假设对任意列向量 $x \in \mathbb{R}^n$ 都有 $x^T Ax = 0$. 则 ()

- (A) A 的行列式 $|A|$ 为 0.
- (B) $A = 0$.
- (C) A 的迹, $\text{trace}(A)$, 为 0.
- (D) A 在 \mathbb{C} 中唯一的特征值是 0.

(4) Let $n \geq 2$. Let A be an $n \times n$ real matrix of rank 1. Then ()

- (A) A is necessarily diagonalizable.
- (B) A has only one nonzero column.
- (C) The trace, $\text{trace}(A)$, of A is nonzero.
- (D) A has at least $n - 1$ linearly independent eigenvectors.

设 $n \geq 2$, A 是秩为 1 的 $n \times n$ 实矩阵. 则 ()

- (A) A 一定可以对角化.
- (B) A 只有一列是非零列.
- (C) A 的迹, $\text{trace}(A)$, 不为零.
- (D) A 有至少 $n - 1$ 个线性无关的特征向量.

(5) Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The matrices A and B are ()

- (A) congruent and similar.
- (B) congruent but not similar.
- (C) similar but not congruent.
- (D) neither similar nor congruent.

设 $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. 则 A 与 B ()

- (A) 合同且相似. (这里的“合同”也称“相合”.)

- (B) 合同但不相似.
 (C) 相似但不合同.
 (D) 既不相似也不合同.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Evaluate the determinant:
$$\begin{vmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & n-1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 2 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} = \underline{\hspace{2cm}}.$$

计算行列式:
$$\begin{vmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & n-1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 2 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} = \underline{\hspace{2cm}}.$$

(2) Let $\mathbb{R}^{2 \times 2}$ be the real vector space of 2×2 real matrices. Let V be the subspace of $\mathbb{R}^{2 \times 2}$ spanned by the 4 matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}, A_3 = \begin{bmatrix} 5 & 1 \\ 1 & -5 \end{bmatrix}, A_4 = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}.$$

Then $\dim V = \underline{\hspace{2cm}}.$

设 $\mathbb{R}^{2 \times 2}$ 为所有 2×2 实矩阵构成的实向量空间. 令 V 为以下 4 个矩阵在 $\mathbb{R}^{2 \times 2}$ 中张成 (也称“生成”) 的子空间:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}, A_3 = \begin{bmatrix} 5 & 1 \\ 1 & -5 \end{bmatrix}, A_4 = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}.$$

则 $\dim V = \underline{\hspace{2cm}}.$

(3) Let A be a 2×2 matrix and suppose α_1, α_2 are 2-dimensional linearly independent column vectors such that $A\alpha_1 = 0, A\alpha_2 = 4\alpha_1 + 2\alpha_2$. Then the eigenvalues of A are $\underline{\hspace{2cm}}.$

设 A 是 2 阶方阵, α_1, α_2 是线性无关的二维列向量, 满足 $A\alpha_1 = 0, A\alpha_2 = 4\alpha_1 + 2\alpha_2$. 则 A 的所有特征值为 0, 2.

(4) Suppose that the quadratic form $f(x_1, x_2, x_3) = x_1^2 + ax_2^2 + x_3^2 + 2bx_1x_2 + 2x_1x_3 + 2x_2x_3$ can be transformed by an orthogonal transformation $(x_1, x_2, x_3)^T = Q(y_1, y_2, y_3)^T$ to $y_2^2 + 4y_3^2$. Then $a = \underline{\hspace{2cm}}, b = \underline{\hspace{2cm}}.$

假定二次型 $f(x_1, x_2, x_3) = x_1^2 + ax_2^2 + x_3^2 + 2bx_1x_2 + 2x_1x_3 + 2x_2x_3$ 可由正交变换 $(x_1, x_2, x_3)^T = Q(y_1, y_2, y_3)^T$ 化为 $y_2^2 + 4y_3^2$. 则 $a = \underline{\hspace{2cm}}, b = \underline{\hspace{2cm}}.$

(5) Let A be a 3×3 matrix with eigenvalues $-1, 0, 1$. Suppose that

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors belonging to the eigenvalues $-1, 0, 1$ respectively. Then $A^{2021} =$ _____.

假设 A 为 3×3 矩阵, 其特征值为 $-1, 0, 1$. 假设

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

分别为属于特征值 $-1, 0, 1$ 的特征向量. 则 $A^{2021} =$ _____.

3. (10 points 本题共 10 分) Suppose $A = \begin{bmatrix} a & 1 & 0 \\ 1 & a & -1 \\ 0 & 1 & a \end{bmatrix}$ and $A^3 = 0$.

- (a) Find $|A|$.
- (b) Find the value of a .
- (c) Show that $I - A$ is invertible. (Here I denotes the 3×3 identity matrix.)
- (d) Find an invertible matrix X of order 3 such that $(I - A)^{-1}X = (X^{-1} - X^{-1}A)^{-1}A^2 + I$.
(Hint: $X^{-1} - X^{-1}A = X^{-1}(I - A)$.)

设 $A = \begin{bmatrix} a & 1 & 0 \\ 1 & a & -1 \\ 0 & 1 & a \end{bmatrix}$, 且 $A^3 = 0$.

- (a) 求 A 的行列式 $|A|$.
- (b) 求 a 的值.
- (c) 证明 $I - A$ 可逆. (这里 I 表示 3 阶单位矩阵.)
- (d) 求一个 3 阶可逆矩阵 X 使得 $(I - A)^{-1}X = (X^{-1} - X^{-1}A)^{-1}A^2 + I$.
(提示: $X^{-1} - X^{-1}A = X^{-1}(I - A)$.)

4. (10 points 本题共 10 分) Let

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

- (a) Find the determinant and the trace of H .
- (b) Find all the singular values of H .
- (c) Find a real number α such that $\text{rank}(\alpha I_4 - H)$ is the smallest possible.

令

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

- (a) 求 H 的行列式和迹.
- (b) 求 H 的所有奇异值.
- (c) 求一个实数 α 使得 $\text{rank}(\alpha I_4 - H)$ 达到最小可能的值.

5. (10 points 本题共 10 分) Suppose that the complex matrix $A = \begin{bmatrix} 1 & 1+i \\ \alpha & 2 \end{bmatrix}$ is a Hermitian matrix.

- (a) Find the value of α .
- (b) Find all the complex eigenvalues of A .
- (c) Find a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix.
- (d) Let $B = A + A^T$, where A^T denotes the transpose of A . Show that B is a real symmetric matrix, and decide whether B is positive definite.

假设复矩阵 $A = \begin{bmatrix} 1 & 1+i \\ \alpha & 2 \end{bmatrix}$ 是个埃尔米特矩阵.

- (a) 求 α 的值.
- (b) 求 A 的所有复特征值.
- (c) 求一个酉矩阵 U 使得 $U^{-1}AU$ 为对角阵.
- (d) 令 $B = A + A^T$, 其中 A^T 表示 A 的转置. 证明 B 是实对称阵, 并确定 B 是否正定.

6. (10 points 本题共 10 分) Let A be a 3×3 real matrix whose second and third columns are $(1, 0, 0)^T$ and $(2, 1, 0)^T$ respectively. Suppose that the QR factorization of A takes the form $A = QR$ with

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & x \\ 0 & 0 & y \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & z \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & \frac{1}{\sqrt{2}} & c \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) Find the values of x, y, z and a, b, c .
- (b) Find the determinant $|A|$.

设 A 为 3 阶实方阵, 其第二列和第三列分别为 $(1, 0, 0)^T$ 和 $(2, 1, 0)^T$. 假设 A 的 QR 分解 $A = QR$ 满足

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & x \\ 0 & 0 & y \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & z \end{bmatrix} \quad \text{及} \quad R = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & \frac{1}{\sqrt{2}} & c \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) 求 x, y, z 和 a, b, c 的值.
 (b) 求行列式 $|A|$.

7. (10 points 本题共 10 分) Let A be an $n \times n$ real symmetric positive definite matrix.

- (a) Show that there exists an $n \times n$ invertible matrix R such that $A = R^T R$.
 (b) Show that for all column vectors $x, y \in \mathbb{R}^n$,

$$(x^T A y)^2 \leq (x^T A x)(y^T A y).$$

设 A 为 n 阶正定实对称矩阵.

- (a) 证明: 存在 n 阶可逆阵 R 使得 $A = R^T R$.
 (b) 证明: 对任意列向量 $x, y \in \mathbb{R}^n$ 都有

$$(x^T A y)^2 \leq (x^T A x)(y^T A y).$$

8. (10 points 本题共 10 分) Let A, B be two $n \times n$ real symmetric matrices.

- (a) Suppose A is positive definite. Show that there exists an invertible $n \times n$ matrix C such that $C^T A C = I_n$ and $C^T B C$ is diagonal. (Here I_n denotes the $n \times n$ identity matrix).
 (b) Suppose $B - A$ and A are positive semidefinite matrices. Show that: $\det B \geq \det A$.

设 A, B 都为 n 阶实对称矩阵.

- (a) 设 A 为正定实对称阵. 证明: 存在 n 阶可逆实矩阵 C 使得 $C^T A C = I_n$ 且 $C^T B C$ 是对角阵. (这里 I_n 为 n 阶单位阵).
 (b) 设 $B - A$ 和 A 都是半正定矩阵. 证明: $\det B \geq \det A$.