Diagonalization of a Matrix (矩阵的对角化)

Lecture 22

Dept. of Math.

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Diagonalization of a Matrix

考试重点

找 $S^{-1}AS = \Lambda_{n\times n}$

$\Leftrightarrow AS = S\Lambda$

$$A[v_1\ v_2 \cdots v_n] = [v_1\ v_2 \cdots v_n]\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n\end{bmatrix}$$

$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$

$\vdots$

$Av_n = \lambda_n v_n$

$v_1, v_2, \cdots v_n$ 线性无关 (S 可逆)

eigenvector

$$\frac{du}{dt} = \begin{bmatrix} \frac{dv}{dt} \\ \frac{dw}{dt} \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$$

$$= Au = S\Lambda S^{-1}u$$

$$S^{-1}AS = \Lambda$$

$$\Leftrightarrow \quad A = S\Lambda S^{-1}$$

$$\frac{dS^{-1}u}{dt} = \Lambda S^{-1}u$$

$$\frac{du}{dt} = Au$$

eigenvalues of A : $-1, 2$

$$\begin{array}{cc} -1 & 2 \\ \updownarrow & \updownarrow \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 5 \\ 2 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \frac{dy_1}{dt} = -y_1 \qquad y_1 = c_1 e^{-t}$$

$$\frac{dy_2}{dt} = 2y_2 \qquad y_2 = c_2 e^{2t}$$

$$S = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$$

（S不唯一：特征向量可取数乘）

$$\Downarrow$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{bmatrix} = S^{-1}u$$

$$\Rightarrow u = Sy = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$S^{-1} = \frac{1}{-3} \begin{bmatrix} 2 & -5 \\ -1 & 1 \end{bmatrix}$$

$$S^{-1}AS = -\frac{1}{3} \begin{bmatrix} 2 & -5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$u = \begin{bmatrix} y_1 + 5y_2 \\ y_1 + 2y_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + 5c_2 e^{2t} \\ c_1 e^{-t} + 2c_2 e^{2t} \end{bmatrix}$$

$v(0) = 8 = c_1 + 5c_2$

$w(0) = 5 = c_1 + 2c_2$

$c_1 = 3, \; c_2 = 1$

对角化后入就不会有"挑剔的可能"

Introduction

We start right off with the one essential computation. It is perfectly simple and will be used in every section of this chapter. The eigenvectors diagonalize a matrix:

Theorem

Suppose the n by n matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ. The eigenvalues of A are on the diagonal of Λ:

特征值矩阵

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Diagonalization of a Matrix

We call S the "eigenvector matrix" and Λ the "eigenvalue matrix".

There are four remarks before giving any examples or applications.

1: Any matrix with distinct eigenvalues can be diagonalized. If the matrix A has no repeated eigenvalues–the numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct–then its n eigenvectors are automatically independent. Therefore any matrix with distinct eigenvalues can be diagonalized.

互不相同λ
一定可被对角化

2: The diagonalizing matrix is not unique. An eigenvector x can be multiplied by a constant, and remains an eigenvector.

S不唯一

反之不对：可以被对角化，不一定有n个
互不相同的λ

$A = I_n$

$\lambda = 1$

$|A - \lambda I| = (1-\lambda)^n$

$\dim N(A - \lambda I) = \dim N(0) = n$

geometric multiplicity 几何重数

对每个λ 得满足：

代数重数 = 几何重数时可以对角化

（代数重数一定 ≥ 几何重数）

$A_{m \times n}$

有 m 个不同特征值 $\lambda_1, \lambda_2, \cdots \lambda_m$

$$\begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \downarrow & \downarrow & & \downarrow \\ v_1 & v_2 & \cdots & v_m \end{array}$$ eigenvectors

linearly independent

$A v_1 = \lambda_1 v_1$

$A v_2 = \lambda_2 v_2$

$\vdots$

$A v_m = \lambda_m v_m$

证明特征向量线性无关

$m=1$ ✓

$m=2$ λ_1, λ_2 $\lambda_1 \neq \lambda_2$

$$\begin{array}{cc} \lambda_1 & \lambda_2 \\ \downarrow & \downarrow \\ v_1 & v_2 \end{array}$$

Assume: $c_1 v_1 + c_2 v_2 = 0$

$A(c_1 v_1 + c_2 v_2) = A \cdot 0$

$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0$

同乘 λ_2 $c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0$

相减: $c_1 \overset{\neq 0}{v_1} \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} = 0 \Rightarrow c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow$ 仅零解 v_1、v_2 线性无关

$m=k$ $\lambda_1, \lambda_2, \cdots \lambda_k$ distinct eigenvalues

$\downarrow \quad \downarrow \quad \cdots \quad \downarrow$

$v_1 \quad v_2 \quad \cdots \quad v_k$ 线性无关

Induction hypothesis 数归法

归纳假设 → 一般用于自然数

要证 $m=k+1$ $\lambda_1, \lambda_2, \cdots, \lambda_k, \lambda_{k+1}$ distinct eigenvalues

$v_1, v_2, \cdots v_k, v_{k+1}$ 线性无关

$c_1v_1+c_2v_2+\cdots+c_{k+1}v_{k+1}=0$ (使其都为 0)
c

$A(c_1v_1+c_2v_2+\cdots+c_{k+1}v_{k+1})=A\cdot 0$

$\lambda_1c_1v_1+\lambda_2c_2v_2+\cdots+\lambda_{k+1}c_{k+1}v_{k+1}=0$

$c_1\lambda_{k+1}v_1+c_2\lambda_{k+1}v_2+\cdots+c_{k+1}\lambda_{k+1}v_{k+1}=0$

作差 $(\lambda_{k+1}-\lambda_1)c_1v_1+(\lambda_{k+1}-\lambda_2)c_2v_2+\cdots+(\lambda_{k+1}-\lambda_k)c_kv_k=0$

$v_1 \sim v_k$ 线性无关 $(\lambda_{k+1}-\lambda_1)\sim(\lambda_{k+1}-\lambda_k)\neq 0$

$\Rightarrow c_1=c_2=\cdots=c_k=0$

$\Rightarrow c_{k+1}=0$ 得证.

Remarks

3: Other matrices S will not produce a diagonal Λ.

4: Not all matrices possess n linearly independent eigenvectors, so **not all matrices are diagonalizable**. The standard example of a "defective matrix" is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$\lambda = 0$ is a double eigenvalue–its algebraic multiplicity is 2. But the geometric multiplicity is 1–there is only independent eigenvector. We can not construct S.

Diagonalizability and Invertibility

There is **no** connection between diagonalizability and invertibility. Diagonalization can fail **only if** there are **repeated** eigenvalues. The problem is the shortage of eigenvectors–which are needed for S. That needs to be emphasized:

Proposition

Diagonalizability of A depends on enough eigenvectors. Invertibility of A depends on nonzero eigenvalues.

Algebraic multiplicity and geometric multiplicity.

Theorem

Theorem

If eigenvectors $x_1, x_2, \cdots, x_k$ *correspond to different eigenvalues* $\lambda_1, \cdots, \lambda_k$*, then those eigenvectors are linearly independent.*

Proof.

Suppose first that $k = 2$, and that some combination of x_1 and x_2 produces zero: $c_1x_1 + c_2x_2 = 0$. Multiplying by A, we find $c_1\lambda_1x_1 + c_2\lambda_2x_2 = 0$. Subtracting λ_2 times the previous equation, the vector x_2 disappears: $c_1(\lambda_1 - \lambda_2)x_1 = 0$. Since $\lambda_1 \neq \lambda_2$ and $x_1 \neq 0$, we are forced into $c_1 = 0$. Similarly $c_2 = 0$, and the two vectors are independent; only the trivial combination gives zero. The same argument extends to any number of vectors. $\square$

Examples of Diagonalization

The main point of this section is $S^{-1}AS = \Lambda$. The eigenvector matrix S converts A into its eigenvalue matrix Λ (diagonal). We see this for projections and rotations.

Example 1. The projection $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ has eigenvalue matrix $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Example 2

Example 2. The eigenvalues and eigenvectors of a rotation matrix. 90° rotation:

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$K - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$|K - \lambda I| = \lambda^2 + 1$

How can a vector be rotated and still have its direction unchanged?

Imaginary numbers? Complex numbers are needed even for real matrices! See section 5.5 for more!

实数域无解

$\lambda = \pm i$

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}$$

实数域中

一定会转

不共线

Powers and Products: A^k and AB

The eigenvalues of A^2 are exactly $\lambda_1^2, \cdots, \lambda_n^2$, and every eigenvector of A is also an eigenvector of A^2.

Theorem

The eigenvalues of A^k are $\lambda_1^k, \cdots, \lambda_n^k$, and each eigenvector of A is still an eigenvector of A^k. When S diagonalizes A, it also diagonalizes A^k:

$$\Lambda^k = S^{-1} A^k S$$

Each S^{-1} cancels an S, except for the first S^{-1} and the last S.

If A is invertible, this rule also applies to its inverse (the power $k = -1$). The eigenvalues of A^{-1} are $1/\lambda_i$. That can be seen without diagonalizing:

$$\text{if } Ax = \lambda x \text{ then } x = \lambda A^{-1} x \text{ and } \frac{1}{\lambda} x = A^{-1} x.$$

$A_{n\times n}$ real / complex

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = (\lambda-0)^{2}$ → 代数重数

$S^{-1}A\underline{\underline{S}} = \Lambda$ 相似变换

$\lambda = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$

$x = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 几何重数=1 无法对角化

$\dim N(A-\lambda I) = 1 < 2$

$B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$|B-\lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 1 = 0$

$\lambda = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3\pm\sqrt{5}}{2}$

Q: $A = A^T$ (A是实矩阵)

$\Rightarrow$ A 可对角化

$F_0=0$, $F_1=1$, $F_2=1$, $F_3=2$, $F_4=3$ 求F_{1000}

★ $\begin{cases} F_k + F_{k+1} = F_{k+2} \\ F_{k+1} = F_{k+1} \end{cases} \Rightarrow \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \underset{u_k}{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}} = \underset{u_{k+1}}{\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}}$ 数学建模

$\begin{bmatrix} F_{1001} \\ F_{1000} \end{bmatrix} = u_{1000}$

$Au_0 = u_1$

$A^2u_0 = u_2$

$\vdots$

$A^{1000}u_0 = u_{1000}$

$A^k u_0 = u_k$

$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$

$\lambda = \dfrac{-1\pm\sqrt{5}}{2}$

λ_1、λ_2 (较复杂不代入算)

$\begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} x = 0 \quad x = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$

(满足方程)

$\begin{bmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{bmatrix} x = 0 \quad x = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$ ($\dfrac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$ 代数余子式)

$S^{-1} = \dfrac{1}{\lambda_1-\lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$

求高次幂:对角化

$S^{-1}AS = \Lambda$

$A = S\Lambda S^{-1}$

$A^2 = S\Lambda \cancel{S^{-1}S}\Lambda S^{-1} = S\Lambda^2 S^{-1}$

$A^3 = S\Lambda^3 S^{-1}$

$\Rightarrow A^k = S\Lambda^k S^{-1}$

$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{bmatrix}$

$\Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots & \lambda_n^k \end{bmatrix}$

$A^k = \dfrac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$

$= \dfrac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1^{k+1} & \lambda_2^{k+1} \\ \lambda_1^k & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$

$= \dfrac{1}{\lambda_1-\lambda_2}\begin{bmatrix} \lambda_1^{k+1}-\lambda_2^{k+1} & -\lambda_2\lambda_1^{k+1}+\lambda_1\lambda_2^{k+1} \\ \lambda_1^k-\lambda_2^k & -\lambda_2\lambda_1^k+\lambda_1\lambda_2^k \end{bmatrix}$

$$\begin{bmatrix}F_{k+1}\\F_k\end{bmatrix} = u_k = A^k\begin{bmatrix}F_1\\F_0\end{bmatrix} = A^k\overset{\text{取第一列}}{\begin{bmatrix}1\\0\end{bmatrix}} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1^{k+1}-\lambda_2^{k+1}\\ \lambda_1^{k}-\lambda_2^{k}\end{bmatrix}$$

$$F_k = \frac{1}{\lambda_1-\lambda_2}(\lambda_1^k-\lambda_2^k)$$

$$= \frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^k-\underbrace{\left(\frac{1-\sqrt5}{2}\right)^k}_{k\text{很大时}\to 0}\right)$$ (过程中始终未代入!)

但重要：将该项靠近整数

(5.3节只讲斐波那契数列)

$$\lim_{k\to\infty}\frac{F_{k+1}}{F_k} = \frac{1+\sqrt5}{2}$$

Q:?

A 特征值 $\lambda_1, \lambda_2, \cdots \lambda_n$

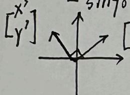
$f(A)$ 特征值 $f(\lambda_1), f(\lambda_2), \cdots f(\lambda_n)$

$f(\lambda) = \lambda^2-\lambda-1$ Page 306

$f(A) = A^2-A-1$

— — — —

$$K = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix}$$



$K\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x'\\y'\end{bmatrix} \neq \lambda\begin{bmatrix}x\\y\end{bmatrix}$ (正交)

$|K-\lambda I| = \begin{vmatrix}-\lambda & -1\\ 1 & -\lambda\end{vmatrix} = \lambda^2+1 = 0$

$\lambda = \pm i$ 复数域中才有解

$(K-iI)x=0$ $\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}x=0$ $x=\begin{bmatrix}i\\1\end{bmatrix}$ (消元易搞错)

$(K-(-i)I)x=0$, $\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}x=0$ $x=\begin{bmatrix}1\\i\end{bmatrix}$

$\Rightarrow S=\begin{bmatrix}i & 1\\ 1 & i\end{bmatrix}$ $S^{-1} = \frac{1}{i^2-1}\begin{bmatrix}i & -1\\ -1 & i\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}i & -1\\ -1 & i\end{bmatrix}$

$\Lambda = S^{-1}AS = \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$ $\Lambda^4 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\Rightarrow A^4 = S\Lambda^4S^{-1} = SS^{-1} = I$

(旋转90° 4次)

$A_{n\times n}$ real / complex

$p(\lambda) = \det(A-\lambda I)$ polynomid $\deg p = n$

$= (\lambda_1-\lambda)^{d_1}(\lambda_2-\lambda)^{d_2}\cdots(\lambda_m-\lambda)^{d_m}$

λ_1、λ_2 … λ_m distinct

d_i algebraic multiplicity of λ_i, $i=1,2,\cdots m$

$d_1+d_2+\cdots+d_m=n$

$S_i = \dim N(A-\lambda_i I)$ S_i geometric multiplicity

$A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $|A-\lambda I| = (\lambda-0)^2$

$\dim N(A-0\cdot I) = 1<2$

$\lambda=0$ $\lambda=0$ $\lambda=0,1$

$A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $B=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ $AB=\begin{bmatrix}1&0\\0&0\end{bmatrix}$

$\begin{cases} AB \text{ 的特征值不是 } A\text{的} \times B\text{的} \\ A+B \quad\quad A\text{的} + B\text{的}\end{cases}$

何时 $A+B$ 特征值 $= A$的 $+B$的

Q:? $AB=BA$

A、B 都是 diagonalizable

$S^{-1}AS=\Lambda_1$ $S^{-1}BS=\Lambda_2$ 共享特征向量矩阵

$\Leftrightarrow$ $AB=BA$

证明: $\Rightarrow$ $A=S\Lambda_1S^{-1}$ $B=S\Lambda_2S^{-1}$

$AB = S\Lambda_1\cancel{S^{-1}S}\Lambda_2S^{-1}$

$= S\Lambda_2\Lambda_1S^{-1}$ (对角矩阵相乘可换)

$= S\Lambda_2S^{-1}S\Lambda_1S^{-1}$

$=BA$

$\Leftarrow$ Q:? 较长

Complex Matrix

$z = a+bi$, $a,b \in R$ $i^2=-1$

$e^{ix} = \cos x + i\sin x$

$\to e^{i\pi} = -1 + i\cdot 0$

$e^{i\pi}+1 = 0$

$V = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ $z_i \in \mathbb{C}$ 加粗（R）

$x = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\|x\|^2 = \bar{x}^T x$ 取共轭 $\Rightarrow [1 \ -i]\begin{bmatrix} 1 \\ i \end{bmatrix} = 2$

$\|x\| = \sqrt{2}$

$x^T x = [1 \ i]\begin{bmatrix} 1 \\ i \end{bmatrix} = 1-1 = 0$ ✗

做内积同样

$V_1 = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$ $V_2 = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$

$\bar{V_1}^T V_2 = [\bar{z_1} \ \bar{z_2} \cdots \bar{z_n}]\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \bar{z_1}w_1 + \bar{z_2}w_2 + \cdots + \bar{z_n}w_n$

$\bar{V_2}^T V_1 = \bar{w_1}z_1 + \bar{w_2}z_2 + \cdots + \bar{w_n}z_n$ 不一样！

取共轭才一样 $\overline{(\bar{V_1}^T V_2)} = \bar{V_2}^T V_1$

$\bar{x}^T x = x^H x$ "厄米特"

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$|A - \lambda I| = (\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$\cos\theta - \lambda = \pm \sin\theta\, i$

$\lambda = \cos\theta \pm i\sin\theta$

$= e^{\pm i\theta}$

旋转矩阵特征值是一对共轭复数

矩阵^H：对每个元素取共轭，整体转置

$A^H = \bar{A}^T$ (A Hermitian)

$A = A^H$ (A is Hermitian) A为厄米特矩阵

$A = A^T$ real

$Q^{-1}AQ = \Lambda$ Q为正交矩阵（置换矩阵就是正交矩阵）

$Q^T = Q$

$A = A^H$

$U^H A U = \Lambda$

$U^H = U = UU^H = I$

Example 3

Example 3. If K is rotation through 90°, then K^2 is rotation through 180°(which means $-I$) and K^{-1} is rotation through -90°:

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{and } K^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of K are i and $-i$; their squares are -1 and -1; their reciprocals are $1/i = -i$ and $1/(-i) = i$. Then K^4 is a complete rotation through 360°.

$$K^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and also } \Lambda^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Notice also that complex numbers are needed even for real matrices.

Product

- For a product of two matrices, we can ask about the eigenvalues of AB–but we won't get a good answer.
- In general, if A has an eigenvalue λ and B has an eigenvalue μ, AB does not have $\lambda\mu$ as its eigenvalue. For instance, we have two matrices with zero eigenvalues, while AB has $\lambda = 1$:

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

  The eigenvectors of this A and B are completely different, which is typical.
- Similarly, the eigenvalues of $A+B$ generally have nothing to do with $\lambda + \mu$.

Theorem

If the eigenvector is the same for A and B, then the eigenvalues multiply and AB has the eigenvalue $\mu\lambda$. But there is something more important.

There is an easy way to recognize when A and B share a full set of eigenvectors, and that is a key question in quantum mechanics:

Theorem

Diagonalizable matrices share the same eigenvector matrix S if and only if $AB = BA$.

Proof

Proof. If the same S diagonalizes both $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$, we can multiply in either order:

$$AB = S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1\Lambda_2 S^{-1} \text{ and } BA = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = S\Lambda_2\Lambda_1 S^{-1}.$$

Since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ (diagonal matrices always commute) we have $AB = BA$.

Proof

In the opposite direction, suppose $AB = BA$. Starting from $Ax = \lambda x$, we have

$$ABx = BAx = B\lambda x = B(\lambda x) = \lambda Bx.$$

Thus x and Bx are both eigenvectors of A, sharing the same λ. If we assume for convenience that the eigenvalues of A are distinct–the eigenspaces are all one-dimensional–then Bx must be a multiple of x. In other words, x is an eigenvector of B as well as A. The proof with repeated eigenvalues is a little longer(left as an exercise).

Heisenberg's uncertainty principle

Heisenberg's uncertainty principle comes from non-commuting matrices, like position P and momentum Q. Position is symmetric, momentum is skew-symmetric, and together they satisfy:

$$QP - PQ = I.$$

The uncertainty principle follows directly from the Schwarz inequality $(Qx)^T(Px) \leq ||Qx||||Px||$ of Section 3.2:

$$||x||^2 = x^T x = x^T(QP - PQ)x \leq 2||Qx||||Px||$$

The product of $||Qx||/||x||$ and $||Px||/||x||$– momentum and position errors, when the wave function is x–is at least $\frac{1}{2}$. It is impossible to get both errors small, because when you try to measure the position of a particle you change its momentum.

Final Note

At the end we come back to $A = S\Lambda S^{-1}$.

- That factorization is particularly suited to take powers of A, and the simplest case A^2 makes the point.
- The LU factorization is hopeless when squared, but $S\Lambda S^{-1}$ is perfect. The square is $S\Lambda^2 S^{-1}$, and the eigenvectors are unchanged.
- By following those eigenvectors we will solve difference equations and differential equations.

例：

数列：$\{x_n\}, \{y_n\}, \{z_n\}$

满足 $x_0=-1, y_0=0, z_0=2$

且 $x_n = -2x_{n-1} + 2z_{n-1}$

$y_n = -2y_{n-1} - 2z_{n-1}$

$z_n = -6x_{n-1} - 3y_{n-1} + 3z_{n-1}$

记 $\alpha_n = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$ 写出满足 $\alpha_n = A\alpha_{n-1}$ 的矩阵 A，并求 A^n 及 x_n, y_n, z_n

$$\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & -2 \\ -6 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{bmatrix}$$

$\alpha_n = A\alpha_{n-1}$

$S^{-1}AS = \Lambda$

$(A-\lambda I)x = 0$

$$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 0 & 2 \\ 0 & -2-\lambda & -2 \\ -6 & -3 & 3-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left[(-2-\lambda)(3-\lambda)-6\right] - 2(-6)(-2-\lambda)$$

$$= (-2-\lambda)\lambda(\lambda-1)$$

$\lambda = -2、0、1$ → 互不相同 可对角化

3个独立特征向量

$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -6 & -3 & 5 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & -2 \\ -6 & -3 & 3 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -3 & 0 & 2 \\ 0 & -3 & -2 \\ -6 & -3 & 2 \end{bmatrix} x = 0 \qquad x = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

$\Lambda = \begin{bmatrix} -2 & & \\ & 0 & \\ & & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \qquad S^{-1} = \ldots$

$$A^n = S\Lambda^n S^{-1} = S\begin{bmatrix} (-2)^n & 0 \\ & 1 \end{bmatrix} S^{-1}$$

$$\alpha_n = A^n \alpha_0$$

Homework Assignment 22

5.2: 7, 8, 12, 13, 19, 23, 32, 40.