Lecture Notes 11&12: Page 25 Solution:

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$T: \mathbb{R}^3 \to \mathbb{R}^3$ linear transformation

$u_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad u_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}, \quad u_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}.$

$T(u_1) = v_1, \quad T(u_2) = v_2, \quad T(u_3) = v_3.$

新基底 = 原基底 · A （列组合）

1. 基 u_1, u_2, u_3 到 v_1, v_2, v_3 的过渡矩阵:

$[\ ,\ ,\ ]$

意思相同

$$(v_1, v_2, v_3) = (u_1, u_2, u_3)\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

等价于 $\Leftrightarrow$

$$\begin{cases} v_1 = a_{11}u_1 + a_{21}u_2 + a_{31}u_3\\ v_2 = a_{12}u_1 + a_{22}u_2 + a_{32}u_3\\ v_3 = a_{13}u_1 + a_{23}u_2 + a_{33}u_3\end{cases}$$

$$\overset{v_1}{\underset{\|}{}}\quad v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} = a_{11}\overset{u_1}{\begin{bmatrix}1\\0\\1\end{bmatrix}} + a_{21}\overset{u_2}{\begin{bmatrix}2\\1\\0\end{bmatrix}} + a_{31}\overset{u_3}{\begin{bmatrix}1\\1\\1\end{bmatrix}}$$

$$v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} = a_{12}\begin{bmatrix}1\\0\\1\end{bmatrix} + a_{22}\begin{bmatrix}2\\1\\0\end{bmatrix} + a_{32}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} = a_{13}\begin{bmatrix}1\\0\\1\end{bmatrix} + a_{23}\begin{bmatrix}2\\1\\0\end{bmatrix} + a_{33}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\Downarrow$

$$\Leftrightarrow \underbrace{\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}}_{C} = \underbrace{\begin{bmatrix}1&2&1\\0&1&1\\1&0&1\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}}_{A}$$

$\Rightarrow A = B^{-1}C.$

左乘为行变换

$$[B \mid C] = \left[\begin{array}{ccc|ccc}1&2&1&1&1&1\\0&1&1&0&1&1\\1&0&1&0&0&1\end{array}\right] \to \left[\begin{array}{ccc|ccc}1&2&1&1&1&1\\0&1&1&0&1&1\\0&-2&0&-1&-1&0\end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac12&-\frac12&0\\0&1&0&\frac12&\frac12&0\\0&0&1&-\frac12&\frac12&1\end{array}\right] \leftarrow \left[\begin{array}{ccc|ccc}1&2&0&\frac32&\frac12&0\\0&1&0&\frac12&\frac12&0\\0&0&1&-\frac12&\frac12&\frac22\end{array}\right] \leftarrow \left[\begin{array}{ccc|ccc}1&2&1&1&1&1\\0&1&1&0&1&1\\0&0&2&-1&1&2\end{array}\right]$$

A

$[\ ]$ 坐标

$$\begin{bmatrix}\frac12&-\frac12&0\\\frac12&\frac12&0\\-\frac12&\frac12&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\0\\2\end{bmatrix} \qquad \bigcirc = A\ \bigcirc$$

2. 求 T 在基 u_1, u_2, u_3 下的矩阵 D:

$A\begin{bmatrix}2\\0\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$$T(u_1, u_2, u_3) = (u_1, u_2, u_3)\,D \quad \Rightarrow D = A.$$

$$\overset{\|}{(Tu_1, Tu_2, Tu_3)} = (v_1, v_2, v_3) \xlongequal{\text{由第1问}} (u_1, u_2, u_3)\,A$$

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3. 求 T 在基 v_1, v_2, v_3 下的矩阵 F:

$$\underline{T(v_1,v_2,v_3) = (v_1,v_2,v_3)F}^{(=A)} = (v_1,v_2,v_3)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

||

$$T(\underline{v_1,v_2,v_3}) = T(\underline{(u_1,u_2,u_3)A}) \overset{?}{=} (T(u_1,u_2,u_3))A$$

由第1问

$$= (Tu_1, Tu_2, Tu_3)A$$

$$= \underline{(v_1, v_2, v_3)\ A}$$

$$\Rightarrow F = A.$$

?: (the proof)

$$T((u_1,u_2,u_3)A) = T\left((u_1,u_2,u_3)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}\right)$$

$$= T(a_{11}u_1 + a_{21}u_2 + a_{31}u_3,\ a_{12}u_1 + a_{22}u_2 + a_{23}u_3,$$

$$a_{13}u_1 + a_{23}u_2 + a_{33}u_3)$$

$$= (a_{11}Tu_1 + a_{21}Tu_2 + a_{31}Tu_3,\ a_{12}Tu_1 + a_{22}Tu_2 + a_{23}Tu_3,\ a_{13}Tu_1 + a_{23}Tu_2 + a_{33}Tu_3)$$

$$= \left((Tu_1,Tu_2,Tu_3)\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix},\ (Tu_1,Tu_2,Tu_3)\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix},\ (Tu_1,Tu_2,Tu_3)\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}\right)$$

$$= (Tu_1, Tu_2, Tu_3)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \underline{T(u_1,u_2,u_3)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}$$

$$= (Tu_1, Tu_2, Tu_3)\ A.$$

4. 求 $T^2(u_1) \overset{定义}{=} T(Tu_1) = T(v_1) = +\frac{1}{2}v_1 + \frac{1}{2}v_2 - \frac{1}{2}v_3$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}.$$

5. $$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}.$$