Problem Set 9 —— Linear Algebra A (Spring 2024)

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1. If you know all 25 cofactors of a 5 by 5 invertible matrix A, how would you find A?

2. A function $\delta : \mathbb{R}^{n\times n} \to \mathbb{R}$ is called an n-**linear function** if it is a linear function of each row of an $n \times n$ matrix when the remaining $n-1$ rows are held fixed. And an n-linear function $\delta : \mathbb{R}^{n\times n} \to \mathbb{R}$ is called **alternating** if, for each $A \in \mathbb{R}^{n\times n}$, we have $\delta(A) = 0$ whenever two adjacent rows of A are identical. Suppose δ is an alternating n-function such that $\delta(I) = 1$. Show that:

   (a) If $A \in \mathbb{R}^{n\times n}$ and B is a matrix obtained from A by interchanging any two rows of A, then $\delta(B) = -\delta(A)$.

   (b) For any $A, B \in \mathbb{R}^{n\times n}$, we have $\delta(AB) = \delta(A) \cdot \delta(B)$.

   (c) $\delta(A) = \det(A)$ for every $A \in \mathbb{R}^{n\times n}$.

3. Find the determinant of

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix}.$$

   Where $a_1, a_2, \cdots, a_n$ are nonzero real numbers.

4. Find the following determinant of order n:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & \cdots & x_n^n \end{vmatrix}.$$

5. (**Lovy-Desplanques**) Let A be real matrix of order n, and $|a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|, i = 1, 2, \cdots, n$. Show that the determinant of A is nonzero.