Complex Matrices(复矩阵)

Lecture 23 and 24

Dept. of Math.

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Complex Matrices

1. Complex Numbers and Their Conjugates
2. Lengths and Transposes in the Complex Case
3. Hermitian Matrices
4. Unitary Matrices
5. Homework Assignment 23 and 24

Complex Numbers

- It is no longer possible to work only with real vectors and real matrices. A real matrix has real coefficients in $\det(A-\lambda I)$, but the eigenvalues may be complex.
- We now introduce the space $\mathbb{C}^n$ of vectors with n complex components. Addition and matrix multiplication follow the same rules as before. Length is computed differently.
- We want to find out about symmetric matrices and Hermitian matrices: Where are their eigenvalues, and what is special about their eigenvectors?

$A = A^H$: hermitan matrix

例 $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ $A^H = \bar{A}^T = A$

Proposition

Every symmetric matrix(and Hermitian matrix) has real eigenvalues. Its eigenvectors can be chosen to be orthonormal.

实对称 / 厄米特 有实 λ，特征向量可正交

$$A=\begin{bmatrix}2 & 3-3i\\ 3+3i & 5\end{bmatrix}$$

$$z\cdot\bar{z}=|z|^2$$

$$|A-\lambda I|=\begin{vmatrix}2-\lambda & 3-3i\\ 3+3i & 5-\lambda\end{vmatrix}=\lambda^2-7\lambda+10-18$$

$$=\lambda^2-7\lambda-8$$

$$=(\lambda-8)(\lambda+1)$$

$\lambda=8\ /\ -1$ 实数

$(A+I)x=0$ 2x2 猜就行

$$\begin{bmatrix}3 & 3-3i\\ 3+3i & 6\end{bmatrix}x=0$$

$(3+3i)(1-i)$
$=3+3=6$

$$x=\begin{bmatrix}-1+i\\ 1\end{bmatrix}$$

$(A-8I)x=0$

$$\begin{bmatrix}-6 & 3-3i\\ 3+3i & -3\end{bmatrix}x=0$$

$$x=\begin{bmatrix}1\\ 1+i\end{bmatrix}$$

特征向量不仅无关，还正交

The Arithmetic of Complex Numbers

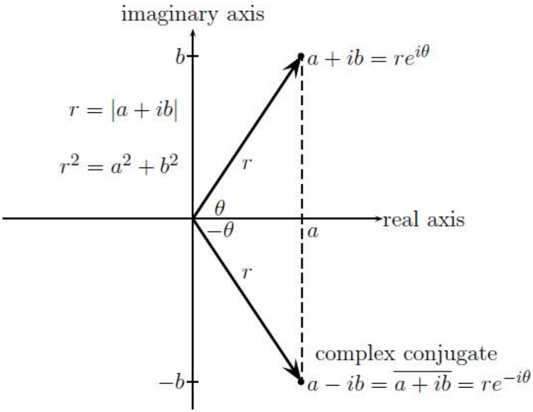


Figure 5.4: The complex plane, with $a + ib = re^{i\theta}$ and its conjugate $a - ib = re^{-i\theta}$.

The Arithmetic of Complex Numbers

Addition:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

Multiplication:

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)$$

Complex Conjugate:

$$\overline{a+ib}=a-ib$$

The sign of the imaginary part is reversed.

Complex Conjugate and Polar Form

The complex conjugate has the following three important properties:

(a) The conjugate of a product equals the product of the conjugates.

(b) The conjugate of a sum equals the sum of the conjugates.

(c) Multiplying any $a+bi$ by its conjugate $a-bi$ produces a real number a^2+b^2.

Polar form:

$$a+bi = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Lengths and Transposes in the Complex Case

- Complex vector. The complex vector space $\mathbb{C}^n$ contains all vectors x with n complex components:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  with components $x_j = a_j + ib_j$.
- Length Squared

$$||x||^2 = |x_1|^2 + \cdots + |x_n|^2.$$

- Inner Product

$$\bar{x}^T y = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n.$$

Lengths and Transposes in the Complex Case

- A Hermitian

$$A^H = \overline{A}^T$$

- Conjugate Transpose.

$$\begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix}^H = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}$$

Properties

Definition

The inner product of x *and* y *is* $x^H y$. *The squared length of* x *is* $||x||^2 = x^H x = |x_1|^2 + \cdots + |x_n|^2$.

共轭+转置

Orthogonal vectors have $x^H y = 0$.

Proposition

Conjugating $(AB)^T = B^T A^T$ *produces* $(AB)^H = B^H A^H$.

复数 → 转置与共轭绑定

Hermitian matrix

Definition

$A = A^H$

Matrices that equal their conjugate transpose are called Hermitian Matrices.

Our main goal is to establish three basic properties of Hermitian matrices. These properties apply equally well to symmetric matrices. A real symmetric matrix is certainly Hermitian.

共轭等于
自身 实数对称
不成立

1. If $A = A^H$, then for all complex vectors x, the number $x^H A x$ is real.

(1)可证(2)

2. If $A = A^H$, every eigenvalue is real. λ 实数 与实对称 H

3. Two eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.

不同λ的
特征向量 →正交

$B=\begin{bmatrix} 3 & 3+3i \\ 3+3i & 5 \end{bmatrix}$ $9-9+18i$

$|B-\lambda I| = \lambda^2-8\lambda+15-(3+3i)^2$
$= \lambda^2-8\lambda+15-18i$ X

A^H 的性质

$A = A^H$

$x^H\lambda x = x^H A x$

$\lambda = \dfrac{x^H \cdot Ax}{x^H x} \in R$ 只要证性质1

"实数"

$(x^H A x)^H = x^H A^H (x^H)^H$
1×1 数

共轭等于本身 $\to \mathbb{R}$ $= x^H A x$

$\lambda \in \mathbb{R}$

$A=A^H$

$\lambda_1, \lambda_2 \in R$
$\downarrow \quad \downarrow$
$\alpha \quad \beta$ $(\lambda_1 \neq \lambda_2)$

$\alpha^H \beta = 0$

特征值关联

$A\alpha = \lambda_1 \alpha$

$A\beta = \lambda_2 \beta$ real

$\lambda_1 \alpha^H \beta = (A\alpha)^H \beta$ 分别结合

$= \alpha^H A^H \beta$ $A^H = A$

$= \alpha^H A \beta = \alpha^H \lambda_2 \beta = \lambda_2 \alpha^H \beta$

$\Rightarrow \alpha^H \beta = 0$

Real Symmetric Matrix

Now we state one of the great theorems of linear algebra:

Theorem 实对称一定可以对角化

A real symmetric matrix can be factored into $A = Q\Lambda Q^T$*. Its orthonormal eigenvectors are in the orthogonal matrix* Q *and its eigenvalues are in* Λ*.*

Example 3 A is a combination of two projections.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

All symmetric matrices are combinations of one-dimensional projections–which are symmetric matrices of rank 1.

Unitary matrices

酉矩阵

Definition

$U^H U = I$

A complex matrix with orthonormal columns is called a unitary matrix.

A Hermitian matrix can be compared to a real number. A unitary matrix can be compared to a number on the unit circle.

Three properties of U:

保留长度

1' : $(Ux)^H(Uy) = x^H U^H U y = x^H y$ and lengths are preserved by U.

2' : Every eigenvalue of U has absolute value $|\lambda| = 1$. 单位圆上

3' : Eigenvectors corresponding to different eigenvalues are orthogonal.

Unitary matrix U矩阵

$Q^TQ = QQ^T = I$ (正交矩阵) 一定是U矩阵

$U^HU = UU^H = I$

Examples

Example 4 $U = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ has eigenvalues e^{it} and e^{-it}.

Example 5

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdot & 1 \\ 1 & \omega & \cdot & \omega^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \omega^{n-1} & \cdot & \omega^{(n-1)^2} \end{bmatrix} = \frac{\textit{Fourier matrix}}{\sqrt{n}}.$$

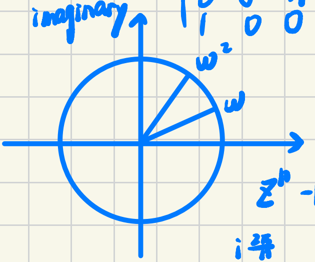
Example 6 $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$$|P-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = \lambda^4 - 1 = 0$$

特征值

$1 \quad i \quad -1 \quad -i$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} \begin{bmatrix} 1 \\ i^2 \\ i^4 \\ i^6 \end{bmatrix} \begin{bmatrix} 1 \\ i^3 \\ i^6 \\ i^9 \end{bmatrix}$$



$z^n - 1 = 0$

$$e^{i\frac{2\pi}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$1, w, w^2, \dots w^{n-1}$

$$1 + w + w^2 + \dots + w^{n-1} = \frac{w^n - 1}{w - 1} = 0$$

$$\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix}$$

Final Note

Definition

Skew-Hermitian matrices satisfy $K^H = -K$.

Just as skew-symmetric matrices satisfy $K^T = -K$. Their properties follow immediately from their close link to Hermitian matrices:

Proposition

If A is Hermitian then $K = iA$ is skew-Hermitian.

$A = A^H$

$iA = iA^H$

$= -(iA)^H$

- The eigenvalues of K are purely imaginary instead of purely real.
- The eigenvectors are still orthogonal, and we still have $K = U\Lambda U^H$–with a unitary U instead of a real orthogonal Q.

Real versus Complex

Real versus Complex

$\mathbf{R}^n$ (n real components)	$\leftrightarrow$	$\mathbf{C}^n$ (n complex components)
length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\|x\|^2 = \lvert x_1\rvert^2 + \cdots + \lvert x_n\rvert^2$
transpose: $A_{ij}^{\mathrm{T}} = A_{ji}$	$\leftrightarrow$	Hermitian transpose: $A_{ij}^{\mathrm{H}} = \overline{A_{ji}}$
$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	$(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
inner product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	inner product: $x^{\mathrm{H}}y = \overline{x}_1y_1 + \cdots + \overline{x}_ny_n$
$(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	$\leftrightarrow$	$(Ax)^{\mathrm{H}}y = x^{\mathrm{H}}(A^{\mathrm{H}}y)$
orthogonality: $x^{\mathrm{T}}y = 0$	$\leftrightarrow$	orthogonality: $x^{\mathrm{H}}y = 0$
symmetric matrices: $A^{\mathrm{T}} = A$	$\leftrightarrow$	Hermitian matrices: $A^{\mathrm{H}} = A$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian $K^{\mathrm{H}} = -K$
orthogonal $Q^{\mathrm{T}}Q = I$ or $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary $U^{\mathrm{H}}U = I$ or $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|Qx\| = \|x\|$	$\leftrightarrow$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|Ux\| = \|x\|$

The columns, rows, and eigenvectors of Q and U are orthonormal, and every $\lvert\lambda\rvert = 1$

Homework Assignment 23 and 24

5.5: 7, 8, 11, 13, 36, 46, 47.