Test for Positive Definiteness (正定性的判定)

Lecture 28

Dept. of Math.

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Test for Positive Definiteness

讨论前提：

A real symmetric

(不然可能考虑 $\frac{1}{2}(A+A^T)$)

Positive Definiteness

Which symmetric matrices have the property that $x^TAx > 0$ for all nonzero vectors? There are four or five different ways to answer this question. Let's first consider a 2 by 2 matrix:

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

A is positive definite when $a > 0$ and $ac - b^2 > 0$. From those conditions, we can obtain that both eigenvalues are positive.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \frac{ac-b^2}{a}y^2.$$

We see that to make $x^TAx > 0$, we need all the pivots positive.

$$f(x_1, x_2, \cdots x_n) = x^T A x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(x_1, x_2) = a x_1^2 + 2b x_1 x_2 + c x_2^2$$

$$= a\left(x_1 + \frac{b}{a} x_2\right)^2 + \frac{ac - b^2}{a} x_2^2$$

$a > 0$ $\quad$ $ac - b^2 > 0$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix}$$ k阶子矩阵

Determinants

How about determinants? What can determinants tell about positive definiteness?

- The two parts of this book were linked by the chapter on determinants. Now we ask what part determinants play.
- It is not enough to require that the determinant of A is positive. If $a=c=-1$ and $b=0$, then $\det A=1$, but $A=-I$ is negative definite. The determinant test is applied not only to A itself, giving $ac-b^2>0$, but also to the 1 by 1 submatrix a in the upper left-hand corner.
- The natural generalization will involve all n of the upper left submatrices of A.

Test for Positive Definiteness

Here is the main theorem on positive definiteness: 充要

Theorem

Each of the following tests is a $\boxed{\text{necessary and sufficient condition}}$ for the real symmetric matrix A to be positive definite:

难判断

定义

(I) $x^TAx > 0$ *for all nonzero real vectors* x.

(II) *All the eigenvalues of* A *satisfy* $\lambda_i > 0$.

(III) *All the* ***upper left submatrices*** A_k *(顺序主子矩阵) have positive determinants.*

(IV) *All the pivots (without row exchanges) satisfy* $d_k > 0$.

Can you prove this theorem?

考试会考察证明+技巧

(III) $\Rightarrow$ (I)

$|A_1|>0$ $|A_2|>0$, $\cdots$ $|A_n|>0$

$\Rightarrow$ A 正定

数学归纳法

$A_{11}=[a_{11}]$

$x^TAx=a_{11}x^2>0$ for $x\neq 0$

Induction hypothesis:

$A_{(n-1)\times(n-1)}$ 结论成立

$A_{nn}=\begin{bmatrix} A_{n-1} & y \\ y^T & a_{nn} \end{bmatrix}$

$\forall\, z^T A_{n-1} z>0$

$$\begin{bmatrix} I_{n-1} & 0 \\ -y^TA_{n-1}^{-1} & 1 \end{bmatrix}\begin{bmatrix} A_{n-1} & y \\ y^T & a_{nn} \end{bmatrix}$$

$$=\begin{bmatrix} A_{n-1} & y \\ 0 & a_{nn}-y^TA^{-1}y \end{bmatrix}$$

$$\begin{bmatrix} A_{n-1} & y \\ 0 & a_{nn}-y^TA^{-1}y \end{bmatrix}\begin{bmatrix} I_{n-1} & -A_{n-1}^{-1}y \\ 0 & 1 \end{bmatrix}$$

分成 $\begin{bmatrix} A_{n-1} & \\ & x \end{bmatrix}$ 才能算 x^TAx

$$x^T A x = x^T \underbrace{\begin{bmatrix} I_{n-1} & 0 \\ y^T A_{n-1}^{-1} & 1 \end{bmatrix}}_{\substack{\| \\ [z^T \ z_n]}}^{\det A>0 \quad \det=1} \begin{bmatrix} A_{n-1} & \\ & \underbrace{a_{nn} - y^T A_{n-1}^{-1} y}_{\substack{z_n \text{为数} \\ z\in\mathbb{R}^{n-1}}} \end{bmatrix}^{\det A_{n-1}>0 \,\cdot\, \text{该数}} \underbrace{\begin{bmatrix} I_{n-1} & A_{n-1}^{-1} y \\ 0 & 1 \end{bmatrix}}_{\substack{\| \\ \begin{bmatrix} z \\ z_n \end{bmatrix}}}^{\det=1} x$$

$$x^T A x = \underbrace{z^T A_{n-1} z}_{>0} + \underbrace{(a_{nn} - y^T A_{n-1}^{-1} y)}_{\text{说明} >0} z_n^2$$

（假设前提）

$\left(x \text{ 与 } \begin{bmatrix} z \\ z_n \end{bmatrix} \text{ 同时为 } 0 / \text{不为 } 0 \right)$

$\begin{bmatrix} I_{n-1} & A_{n-1}^{-1} y \\ 0 & 1 \end{bmatrix}$ 可逆！

(I) ⇒ (II) $x^TAx>0$ for all $x\neq 0$

$Ax=\lambda x \Rightarrow x^TAx=\lambda x^Tx$

$\Rightarrow \lambda = \dfrac{x^TAx^{>0}}{x^Tx^{>0}}$ $\lambda>0$

(II ⇒ III) 所有 $\lambda>0$

问题 A 的特征值不是小就特征值

先证 (II) ⇒ (I)

$\lambda_i>0$，$i=1,2,\cdots n$

(谱定理) $v_1, v_2, \cdots v_n$ 特征向量 ⇒ (线性表示) $x=c_1v_1+\cdots+c_nv_n$

线性无关，成为 $\mathbb{R}^n$ basis

$Av_i=\lambda_iv_i$

$i=1,2,\cdots n$

取成标准正交 (即可对角化)

$x^T A x$

$= (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)^T A (c_1 v_1 + \cdots + c_n v_n)$

$= c_1^2 \lambda_1 + c_2^2 \lambda_2 + \cdots + c_n^2 \lambda_n > 0$

因特殊向量表示一切向量

(II → III) 即可由 (II → I → III) 给出

Proof.

Condition I defines a positive definite matrix. Our first step shows that each eigenvalue will be positive:

$$\text{If } Ax = \lambda x, \text{then} x^T Ax = x^T \lambda x = \lambda ||x||^2.$$

A positive definite matrix has positive eigenvalues, since $x^T Ax > 0$.

Now we go in the other direction. If all $\lambda_i > 0$, we have to prove $x^T Ax > 0$ for every vector x (not just the eigenvectors). Since symmetric matrices have a full set of orthonormal eigenvectors, any x is a combination $c_1x_1 + c_2x_2 + \cdots + c_nx_n$. Then

$$Ax = c_1Ax_1 + \cdots + c_nAx_n = c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n.$$

Because of the orthogonality $x_i^T x_j = 0$, and the normalization $x_i^T x_i = 1$,

Proof.

$$x^T A x = (c_1 x_1^T + \cdots + c_n x_n^T)(c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n)$$
$$= c_1^2 \lambda_1 + \cdots + c_n^2 \lambda_n.$$

If every $\lambda_i > 0$, then the above equation shows that $x^T A x > 0$. Thus condition II implies condition I.

($I \Rightarrow III$) The determinant of A is the product of the eigenvalues. And if condition I holds, we already know that these eigenvalues are positive. But we also have to deal with every **upper left submatrix** A_k (顺序主子矩阵). The trick is to look at all nonzero vectors whose last $n-k$ components are zero:

当x取特殊值来(后面全为0)

$$x^T A x = \begin{bmatrix} x_k^T & 0 \end{bmatrix} \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} \begin{bmatrix} x_k \\ 0 \end{bmatrix}.$$

$= x_k^T A_k x_k > 0$ (条件>0) $\Rightarrow A_k$ 正定

A 正交

取第i个分量为1

$$[0\ 0\ \cdots\ 0\ \underset{i^{th}}{1}\ 0\ \cdots\ 0]\, A \begin{array}{c} \begin{bmatrix}0\\0\\\vdots\\1\\0\\\vdots\\0\end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \to i^{th} \\ \\ \\ \\ \end{array}$$

$$= a_{ii} > 0$$

$\Rightarrow$ 正交矩阵对角元一定为正

Proof.

Thus A_k is positive definite. Its eigenvalues (not the same λ_i!) must be positive. Its determinant is their product, so all upper determinants are positive.

($III \Rightarrow IV$) According to Section 4.4, the kth pivot d_k is the ratio of $\det A_k$ to $\det A_{k-1}$. If the determinants are all positive, so are the pivots.

($IV \Rightarrow I$) We are given positive pivots, and must deduce that $x^T Ax > 0$. This is what we did in the 2×2 case, by completing the square. The pivots were the numbers outside the squares. To see how that happens for symmetric matrices of any size, we go back to elimination of a symmetric matrix:

$$A = LDL^T.$$

Example 1

Example 1 Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

第3个(算行列式)比较好
算 λ 可能慢一些

(a) Each test is enough by itself.

(b) It is beautiful that elimination and completing the square are actually the same.

(c) Every diagonal entry a_{ii} must be positive.

Example 1

The pivots are d_i are not to be confused with the eigenvalues. For a typical positive definite matrix, they are two completely different sets of positive numbers. In our 3 by 3 example, probably the determinant test is the easiest:

$$\det(A_1) = 2, \det(A_2) = 3, \det(A_3) = 4.$$

The pivots are the ratios $d_1 = 2, d_2 = \frac{3}{2}, d_3 = \frac{4}{3}$. Ordinarily the eigenvalue test is the longest computation. For this A we know the λ's are all positive:

$$\lambda_1 = 2 - \sqrt{2}, \lambda_2 = 2, \lambda_3 = 2 + \sqrt{2}.$$

Even though it is the hardest to apply to a single matrix, eigenvalues can be the most useful test for theoretical purposes.

Positive Definite Matrices and Least Squares

So far, we connected positive definite matrices to pivots(Chapter 1), determinants (Chapter 4), and eigenvalues (Chapter 5). Now we see them in the least-squares problems of Chapter 3, coming from the rectangular matrices of Chapter 2.

A正定 $\Leftrightarrow A = R^T R$ independent columns

Theorem 第一个判定条件：

The symmetric matrix A is positive definite if and only if (V) There is a matrix R with independent columns such that $A = R^T R$.

The key is to recognize $x^T A x$ as $x^T R^T R x = (Rx)^T (Rx)$. Thus $x^T R^T R x > 0$ and $R^T R$ is positive definite. It remains to find an R for which $A = R^T R$.

$\Leftarrow$ $A = R^TR$

$x^TAx = \underbrace{x^TR^T}_{y^T}\underbrace{Rx}_{y} = y^Ty = y_1^2 + y_2^2 + \cdots + y_n^2$

>0

$y_1 \sim y_n$ 不同时为 0

$\left(\begin{array}{c} Rx \neq 0 \\ x \neq 0 \Rightarrow y \neq 0 \\ y \neq 0 \Rightarrow x \neq 0 \end{array} \right)$

$\Rightarrow$ A 正定

$Q^TAQ = \Lambda \Rightarrow A = Q\Lambda Q^T$ λ全正

诸实性

$= Q\sqrt{\Lambda}\sqrt{\Lambda}Q^T$

$= \underbrace{Q\sqrt{\Lambda}Q^T}_{R^T}\underbrace{Q\sqrt{\Lambda}Q^T}_{R}$

(构造出 rank = n、2个 对称阵)

Choices for R

- We almost done this twice already: (消元容易实现)

$$A = LDL^T = (L\sqrt{D})(\sqrt{D}L^T)$$

This Cholesky decomposition has the pivots split evenly between L and L^T.

- Eigenvalues:

再乘个Q^T也行

$$A = Q\Lambda Q^T = (Q\sqrt{\Lambda})(\sqrt{\Lambda}Q^T)$$

So take $R = \sqrt{\Lambda}Q^T$. 不是QR分解!

- We also can take QR.

Semidefinite Matrices

The tests for semidefiniteness will relax $x^T Ax > 0, \lambda > 0, d > 0$, and $\det > 0$, to allow zeros to appear.

Theorem

Each of the following tests is a necessary and sufficient condition for the real symmetric matrix A to be positive semidefinite:

(I') $x^T Ax \geq 0$ *for all nonzero real vectors* x*(this defines positive semidefinite).*

(II') *All the eigenvalues of A satisfy $\lambda_i \geq 0$.*

(III') *No* ***principal submatrices*** (主子矩阵) *have negative determinants.*

不同于顺序主子矩阵

(IV') *No pivots are negative.* ≥ 0

(V') *There is a matrix R, possibly with dependent columns, such that $A = R^T R$.*

允许线性相关

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

所以不选该矩阵

判定半正定不太合适

算的太多了

principle submatrices

1×1: $[a_{11}]$ $[a_{22}]$ $[a_{33}]$

2×2: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$ $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

3×3: A

顺序主子: 3个

主子: 7个

对于负反：

表示为 正反 取反

但第三条关于交错正负的

Example 2

Example 2 Decide whether the following matrix is positive definite, negative definite, semidefinite, or indefinite:

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

A is positive semidefinite, by all five tests.

Ellipsoids in n Dimensions

Throughout this book, geometry has helped the matrix algebra. A linear equation produced a plane. The system $Ax = b$ gives an intersection of planes.

- Ellipse in two dimensions, and an ellipsoid in n dimensions.
- The equation to consider is $x^T Ax = 1$.
- A is identity, diagonal, general matrices.

Example 3

$\lambda_1 = 1 \quad x_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$\lambda_2 = 9 \quad x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Example 3 $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and $x^T A x = 5u^2 + 8uv + 5v^2 = 1$.

- The axes of the ellipse point toward the eigenvectors of A. Because $A = A^T$, those eigenvectors and axes are orthogonal.
- The way to see the ellipse properly is to rewrite $x^T A x = 1$:

$$5u^2 + 8uv + 5v^2 = \left(\frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}}\right)^2 + 9\left(\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}\right)^2 = 1$$

- This is different from completing the square to $\underline{5(u + \frac{4}{5}v)^2 + \frac{9}{5}v^2}$, with the pivots outside.

是二次型

去掉交叉项并有几何意义

实对称矩阵重要结论: 谱定理

$[x, y] \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\vec{x}^T A \vec{x} = \vec{x}^T Q \Lambda Q^T \vec{x}$

$\vec{y} = Q^T \vec{x}$

$= \vec{y}^T \Lambda \vec{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots$

椭圆方程 (正定时)

Example 3 轴方向是特征向量方向

$A = Q\Lambda Q^T$
$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T \cdots + \lambda_n q_n q_n^T$

$Q = [q_1\ q_2\ \cdots\ q_n]$ 正交矩阵

$y = Q^T x = \begin{bmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt2}u - \frac{1}{\sqrt2}v \\ \frac{1}{\sqrt2}u + \frac{1}{\sqrt2}v \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

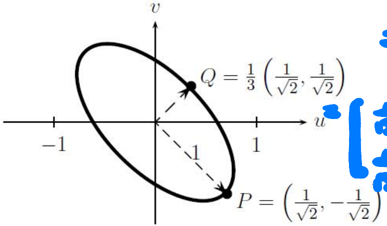


Figure 6.2: The ellipse $x^TAx = 5u^2 + 8uv + 5v^2 = 1$ and its principal axes.

$\vec{x}^T A \vec{x} = x^T(\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T)$

$x^T A x = \lambda_1 y_1^2 + \lambda_2 y_2^2$
$= 1\left(\frac{1}{\sqrt2}u - \frac{1}{\sqrt2}v\right)^2 + 9\left(\frac{1}{\sqrt2}u + \frac{1}{\sqrt2}v\right)^2$

- The major axis of the ellipse corresponds to the smallest eigenvalue of A.
- The first square equals 1 at $(1/\sqrt2, -1/\sqrt2)$ at the end of the major axis. The minor axis is one-third as long since we need $\left(\frac{1}{3}\right)^2$ to cancel the 9.

$\frac{y_1^2}{1^2} + \frac{y_2^2}{(\frac{1}{3})^2} = 1$

算 λ_1、λ_2 ⋯
x_1 x_2 单位化

轴 示 x

Ellipsoids in n Dimensions

The equation to consider is $x^TAx = 1$. Any ellipsoid $x^TAx = 1$ can be simplified in the same way. The key step is to diagonalize $A = Q\Lambda Q^T$, We straightened the picture by rotating the axes. Algebraically, the change to $y = Q^Tx$ produces a sum of squares:

$$x^TAx = (x^TQ)\Lambda(Q^Tx) = y^T\Lambda y = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = 1$$

长轴

The major axis has $y_1 = \frac{1}{\sqrt{\lambda_1}}$ along the eigenvector with the smallest eigenvalue. The other axes are along the other eigenvectors. Their lengths are $1/\sqrt{\lambda_2}, \cdots, 1/\sqrt{\lambda_n}$. Notice that the λ's must be positive–the matrix must be positive definite–or these square roots are in trouble.

Simplifying an ellipsoid in n dimensions

The change from x to $y = Q^T x$ rotates the axes of the space, to match the axes of the ellipsoid. In the y variables we can see that it is an ellipsoid, because the equation becomes so manageable:

Theorem

Suppose $A = Q\Lambda Q^T$ *with* $\lambda_i > 0$. *Rotating* $y = Q^T x$ *simplifies* $x^T A x = 1$*:*

$$x^T Q\Lambda Q^T x = 1, \quad y^T \Lambda y = 1, \quad \textit{and} \quad \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = 1.$$

This is the equation of an ellipsoid. Its axes have lengths $1/\sqrt{\lambda_1}, \cdots, 1/\sqrt{\lambda_n}$ *from the center. In the original* x*-space they point along the eigenvectors of* A*.*

The Law of Inertia 惯性定理

What are the elementary operations and their invariants for $x^T Ax$? The basic operation on a quadratic form is to change variables. A new vector y is related to x by some nonsingular matrix, $x = Cy$. The quadratic form becomes $y^T C^T ACy$. This shows the fundamental operation on A: 代换

$$A \to C^T AC$$ 保持对称

for some nonsingular C. The symmetry of A is preserved, since $C^T AC$ remains symmetric. The real question is, what other properties are shared by A and $C^T AC$? The answer is given by Sylvester's Law of Inertia:

Congruence Transformation

相合变换 / 合同变换

Theorem

(***Sylvester's Law of Inertia***) C^TAC *has the same number of positive eigenvalues, negative eigenvalues, and zero eigenvalues as* A*.*

C是可逆的

The signs of the eigenvalues are preserved by a congruence transformation. Can you prove the above theorem?

Remarks: 改变特征值，但正负号不改变

1. The number of positive eigenvalues of the real symmetric matrix A p, is called the positive index of inertia (正惯性指数) of x^TAx.
2. The number of negative eigenvalues of the real symmetric matrix A q, is called the negative index of inertia (负惯性指数) of x^TAx.
3. $p-q$ is the signature (符号差) of x^TAx. $p+q=r$, where r is the rank of A.

$C(t) = tQ + (1-t)QR$

$C = QR$

$C(0) = QR = C$

$C(1) = Q$

$C(t)$ 一定可逆　对角元>0的上三角

$C(t) = Q(tI_n + (1-t)R)$

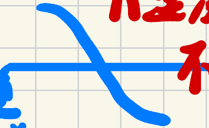
可逆 · 可逆

$C(t)$　$C \to Q$

$C(t)^T A C(t)$

$t=0$　$C(t)=C$　可逆

$t=1$　$C(t)=Q$　可逆



λ连续变化

不可能在中间突变为0

$A + \varepsilon I$　可逆　小扰动

(A不可逆 ⇒ 有 $\lambda=0$，$+\varepsilon$ 以移动)

$$x^T A x$$

$$x^T \underset{\substack{\shortparallel \\ C^T}}{L} D \underset{\substack{\shortparallel \\ C}}{L^T} x$$

$$x^T \underset{\substack{\shortparallel \\ C^T}}{Q} \Lambda \underset{\substack{\shortparallel \\ C}}{Q^T} x$$

- $PAQ=B$ 相抵变换 不改变 rank

  P、Q 可逆

- $M^{-1}AM=B$ 相似变换 不改变 λ

- $C^TAC=B$ 相合变换 不改变 λ 的正负号

$$\lim_{t\to\infty} e^{\lambda t}=0 \quad (\lambda<0)$$

$$\to\infty \quad (\lambda>0)$$

Examples

$R^T R$

- Example 4 Suppose $A = I$. Then $C^TAC = C^TC$ is positive definite. Both I and C^TC have n positive eigenvalues, confirming the law of inertia.
- Example 5 If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then C^TAC has a negative determinant:

$$\det C^TAC = -(\det C)^2 < 0.$$

Then C^TAC must have one positive and one negative eigenvalue, like A.

惯性定理的中文表述

Theorem

设 $f(x_1,x_2,\cdots,x_n)$ 是秩为 r 的 n 元二次型, 则一定存在可逆线性替换 $X=CY$, 把 $f(x_1,x_2,\cdots,x_n)$ 变为

$$g(y_1,y_2,\cdots,y_n)=y_1^2+y_2^2+\cdots+y_p^2-y_{p+1}^2-y_{p+2}^2-\cdots-y_r^2.$$

这个标准形称为实二次型 $f(x_1,x_2,\cdots,x_n)$ 的规范形. $f(x_1,x_2,\cdots,x_n)$ 的规范形由 $f(x_1,x_2,\cdots,x_n)$ 唯一决定.

Remark: 该定理中的 “惯性” 是指在变换下保持不变的东西.

Example 6

The following is an important application:

Theorem A实对称、主元符号与λ符号一致

For any symmetric matrix A, the signs of the pivots agree with the signs of the eigenvalues. The eigenvalue matrix Λ and the pivot matrix D have the same number of positive entries, negative entries, and zero entries.

高斯消元

- This is both beautiful and practical. It is beautiful because it brings together(for symmetric matrices) two parts of this book that were previously separate: pivots and eigenvalues.
- It is also practical, because the pivots can **locate** the eigenvalues.

(不考虑行交换时 $A = L\underline{\underline{D}}L^T$

$A = Q\underline{\underline{\Lambda}}Q^T$

$$A=\begin{bmatrix}3&3&0\\3&10&7\\0&7&8\end{bmatrix}$$

$$|A| = 3\times(80-49) - 3\times 24$$
$$= 21 > 0 \quad \text{正定}$$

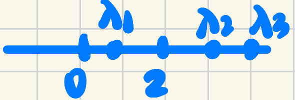
$\lambda_1, \lambda_2, \lambda_3 > 0$

平移λ

$$A-2I=\begin{bmatrix}1&3&0\\3&8&7\\0&7&6\end{bmatrix}\to\begin{bmatrix}1&3&0\\0&-1&7\\0&7&6\end{bmatrix}\to\begin{bmatrix}1&3&0\\0&-1&7\\0&0&55\end{bmatrix}$$

λ 2正1负

$A-2I$的特征值: $\lambda_1-2, \lambda_2-2, \lambda_3-2$



可以帮助定位

One Application of the Law of Inertia

- This was almost the first practical method of computing eigenvalues. It was dominant about 1960, after one important improvement–to make A tridiagonal first.
- The the pivots are computed in $2n$ steps instead of $\frac{1}{6}n^3$. Elimination becomes fast, and the search for eigenvalues becomes simple. The current favorite is the QR method in Chapter 7.

The Generalized Eigenvalue Problem

广义特征值问题

Sometimes $Ax = \lambda x$ is replaced by $Ax = \lambda Mx$. There are two matrices rather than one. An example is the motion of two unequal masses in a line of springs:

M是正定矩阵

$M = R^T R$ (R可逆)

$$m_1 \frac{d^2v}{dt^2} + 2v - w = 0$$

$$m_2 \frac{d^2w}{dt^2} - v + 2w = 0$$

$Ax = \lambda R^T R x$

令 $y = Rx$

$x = R^{-1} y$

This can be reduced to an eigenvalue problem:

$\Rightarrow A R^{-1} y = \lambda R^T y$

$$Ax = \lambda Mx.$$

$\underbrace{(R^T)^{-1}}_{C^T} A \underbrace{R^{-1}}_{C} y = \lambda y$

As long as M is positive definite, the generalized eigenvalue problem $Ax = \lambda Mx$ behaves exactly like $Ax = \lambda x$.

$C^T A C y = \lambda y$

λ是广义特征值

Equivalent problem and "M-orthogonality"

In the following discussion, M is assumed to be positive definite. As a consequence, M can be split into $R^T R$.

- Equivalent problem:
$$C^T A C y = \lambda y.$$
- The properties of $C^T AC$ lead directly to the properties of $Ax = \lambda Mx$, when $A = A^T$ and M is positive definite.
- A and M are being simultaneously diagonalized.
- As long as M is positive definite, the generalized eigenvalue problem $Ax = \lambda Mx$ behaves exactly like $Ax = \lambda x$.

一些习题

1. 设二次型 $f(x_1,x_2,x_3)=x_1^2-x_2^2+2ax_1x_3+4x_2x_1$ 的负惯性指数是 1, 则 a 的取值范围是 ____.
2. 二次型 $f(x_1,x_2,x_3)=(x_1+x_2)^2+(x_2+x_3)^2-(x_3-x_1)^2$ 的正惯性指数和负惯性指数分别为 ____.
3. 设 A 是三阶实对称矩阵，E 为三阶单位矩阵，若 $A^2+A=2E$, 且 $|A|=4$, 则二次型 x^TAx 的规范形为 ____.
4. 设二次型 $f(x_1,x_2,x_3)$ 在正交变换为 $x=Py$ 下的标准形为 $2y_1^2+y_2^2-y_3^2$, 其中 $P=(e_1,e_2,e_3)$, 若 $Q=(e_1,-e_3,e_2)$, 则 $f(x_1,x_2,x_3)$ 在正交变换 $x=Qy$ 下的标准形为 ____.
5. 设二次型 $f(x_1,x_2,x_3)=x_1^2+x_2^2+x_3^2+4x_1x_2+4x_2x_3+4x_1x_3$, 则 $f(x_1,x_2,x_3)=2$ 在空间直角坐标系下表示的二次曲面为 ____.

Homework Assignment 28

6.2: 1, 6, 9, 11, 13, 17, 24, 30, 36.