Final Exam Solution Set Fall 2023

Jan. 12, 2024.

15'

1. (1) D (2) C (3) B (4) B (5) A

25'

2. (1) u^Tu+1, 1 ; 1, $n-1$.

(2) $\sqrt{5}$, $\sqrt{7}$

(3) $$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2024 & 1012\times 2023 \\ 0 & 0 & 0 & 1 & 2024 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(4) $$\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{100}-(1-\sqrt{2})^{100}\right]$$

(5) 1, 0 .

2′

Question 3 : 12′

(a) There are two eigenvalues of A : $\lambda_1 = 0$ & $\lambda_2 = 1$

A basis for the eigenspace $N(A)$ corresponding to $\lambda_1 = 0$ is:

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the eigenspace, $N(A-I)$, corresponding to $\lambda_2 = 1$ is:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$S = \begin{bmatrix} -1 & 3 & 7 & 1 & 3 \\ -1 & -2 & -3 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(b) Since $S^{-1}AS = \Lambda = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$,

$$A = S\Lambda S^{-1} \Rightarrow A^k = S\Lambda^k S^{-1}$$

$$= S\Lambda S^{-1} = A.$$

8'

Question 4: 8'

Orthonormalize the columns of A by Gram-Schmidt to obtain

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

20'

Question 5: 20'

(a) $A^H = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}^H = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix} = A \Rightarrow A$ is Hermitian.

(b) $$|A - \lambda I| = \begin{vmatrix} 0-\lambda & -i & 0 \\ i & 1-\lambda & i \\ 0 & -i & 0-\lambda \end{vmatrix} = (-\lambda)(-1)^{1+1}\left[(1-\lambda)(-\lambda)+i^2\right] + (-i)(-1)^{1+2}\left[(-\lambda)i\right]$$

$$= -\lambda(\lambda^2-\lambda-1) + \lambda = -\lambda(\lambda^2-\lambda-2) = 0$$

$\Rightarrow \lambda = 0, -1, 2.$

$\lambda = 0$: $(A-\lambda I)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda = -1$: $(A-\lambda I)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$

$\lambda = 2$: $(A-\lambda I)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix}$

(c)

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{i}{\sqrt{3}} & 2i/\sqrt{6} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

10'

Question 6 : 10'

(a) $A = \begin{bmatrix} \lambda-3 & 2 & 0 \\ 2 & \lambda & 0 \\ 0 & 0 & \lambda-1 \end{bmatrix}$.

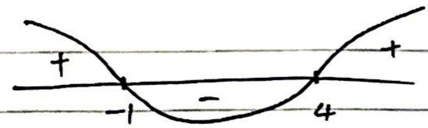
$(\lambda-4)(\lambda+1) > 0$

$(\lambda-1)(\lambda-4)(\lambda+1) < 0$

(b) A is negative definite if and only if

1) $\det A_1 = \lambda - 3 < 0$

2) $\det A_2 = \lambda^2 - 3\lambda - 4 > 0 \longrightarrow$



3) $\det A_3 = (\lambda-1)(\lambda^2-3\lambda-4) < 0$

$$\left.\begin{array}{l} \lambda-4>0 \text{ or } \lambda<-1 \\ \lambda-3<0 \\ \lambda-1<0 \end{array}\right\} \Rightarrow \lambda < -1.$$

Question 7. (a) Since A has n distinct eigenvalues, A is diagonalizable, i.e., there exists an invertible P, such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$AB = BA \Rightarrow P^{-1}APP^{-1}BP = P^{-1}ABP = P^{-1}BAP = P^{-1}BPP^{-1}AP.$$

Therefore, we can assume A is a diagonal matrix, that is,

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

$$AB = \begin{bmatrix} \lambda_1 b_{11} & \lambda_1 b_{12} & \cdots & \lambda_1 b_{1n} \\ \lambda_2 b_{12} & \lambda_2 b_{22} & \cdots & \lambda_2 b_{2n} \\ \cdots & \cdots & & \\ \lambda_n b_{n1} & \lambda_n b_{n2} & \cdots & \lambda_n b_{nn} \end{bmatrix} = \begin{bmatrix} \lambda_1 b_{11} & \lambda_2 b_{12} & \cdots & \lambda_n b_{1n} \\ \lambda_1 b_{21} & \lambda_2 b_{22} & \cdots & \lambda_n b_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_1 b_{n1} & \lambda_2 b_{n2} & \cdots & \lambda_n b_{nn} \end{bmatrix}$$

Comparing the entries, we see that $\lambda_i b_{ij} = \lambda_j b_{ij}$, $\lambda_i \neq \lambda_j$, $(i \neq j)$ hence $b_{ij} = 0$ $(i \neq j)$. It follows that B is a diagonal matrix.

(b) & (c) A and B can be diagonalized by some invertible matrix P, i.e.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad P^{-1}BP = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{bmatrix}$$

Where λ_i, μ_i are the eigenvalues of A and B correspondingly.

Choose n polynomials as follows,

$$f_i(x) = \frac{(x-\lambda_1)\cdots(x-\lambda_{i-1})(x-\lambda_{i+1})\cdots(x-\lambda_n)}{(\lambda_i-\lambda_1)\cdots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\cdots(\lambda_i-\lambda_n)}, \quad i = 1, 2, \cdots, n$$

Let $f(x) = \mu_1 f_1(x) + \mu_2 f_2(x) + \cdots + \mu_n f_n(x)$. Then it can be verified that $f(\lambda_i) = \mu_i$, $i = 1, 2, \cdots, n$.

$$P^{-1}BP = \begin{bmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n) \end{bmatrix} = f(P^{-1}AP) = P^{-1}f(A)P,$$

thus $B = f(A)$.