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Coordinates and Basis

In previous slides, we always represent vectors by coordinates such as $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$. That is natural because we use the standard basis! For example:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

But, a space can have infinite bases, if we change the basis, the coordinates will also change. Suppose we choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a basis, then the coordinates becomes $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ because:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A summary: coordinates come from the choice of basis.

The Transformation Matrix with Other Basis

Our transformation matrix A do the easy thing: input the coordinates before transformation, and output the coordinates after transformation.

Let's now consider the new basis for $\mathbb{R}^2$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose an input coordinates $\begin{bmatrix} x \\ y \end{bmatrix} \left(x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$, the coordinates after transformation is $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Things get easier now. Every column in transformation matrix A is the output coordinates of an input basis vector.

Example

Example

Let $L\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(\mathbf{x}) = \begin{bmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

输入是以{u1,u2}为基底坐标

Find the matrix representations of L with respect to the ordered bases $\{\mathbf{u_1}, \mathbf{u_2}\}$ and $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$, where

输出是以{b1,b2,b3}为基底坐标 $\begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$阵

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$\mathbf{b_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example

Solution:

Input coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in natural basis. $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The output in natural basis will be

$$L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1+2 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Transform to the coordinates under new basis **b**.

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = a_{11}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{21}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_{31}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Output coordinates $\begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$, which is $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ in natural basis.

Solution:

Input coordinates $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in natural basis.

The output in natural basis will be

$$L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3+1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

Transform to the coordinates under new basis **b**.

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = a_{12}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_{22}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_{32}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Output coordinates $\begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$, which is $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ in natural basis.