考试科目: 线性代数 期中考试 样卷

题	号	1	2	3	4	5	6	7	8
分	值	15 分	25 分	10 分	16 分	10 分	6 分	16 分	12 分

本试卷共 (8) 大题, 满分 (110) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 110 in total. Write all your answers on the examination book.

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共15分,每小题3分)选择题,只有一个选项是正确的.

- (1) Let A be an  $m \times n$  real matrix and b be a column vector in  $\mathbb{R}^m$ . Which of the following statements is correct? ( )
  - (A) If Ax = b has infinite many solutions, then Ax = 0 has a nonzero solution.
  - (B) If the system Ax = 0 has only zero solution, then Ax = b has one and only one solution.
  - (C) If the rank of A is n, then the system Ax = b must have a solution.
  - (D) If A is a square matrix (i.e., m = n), then the system Ax = b is consistent if and only if A is invertible.

设 A 为  $m \times n$  实矩阵, b 是  $\mathbb{R}^m$  中的列向量. 下列陈述中哪个是正确的?

- (A) 如果 Ax = b 有无穷多个解,则 Ax = 0 有非零解.
- (B) 如果方程组 Ax = 0 只有零解, 则 Ax = b 有且仅有一个解.
- (C) 如果 A 的秩为 n, 则方程组  $Ax = \sqrt{2}$  必有解
- (D) 如果 A 是方阵 (即 m=n), 则方程组 Ax=b 是相容的当上仅当 A 可逆
- (2) Suppose A is an  $m \times n$  matrix, B is an  $n \times m$  matrix, and I is the  $m \times m$  identity matrix. If AB = I, then (
  - (A) the column vectors of A are linearly independent, and the row vectors of B are linearly independent.
  - (B) the column vectors of A are linearly independent, and the column vectors of B are linearly independent.
  - (C) the row vectors of A are linearly independent, and the column vectors of B are linearly independent.
  - (D) the row vectors of A are linearly independent, and the row vectors of B are linearly independent.

设 A 为  $m \times n$  型矩阵, B 为  $n \times m$  型矩阵 I 为 m 阶单位矩阵. 若 AB = I, 则 (

- (A) 4 的列向量组线性无关, B 的行向量组线性无关.
- (B) A 的列向量组线性无关, B 的列向量组线性无关.
- (C) A 的行向量组线性无关, B 的列向量组线性无关.

- (D) A 的行向量组线性无关, B 的行向量组线性无关.
- (3) Let A be a  $3 \times 3$  matrix, and let B be the matrix formed by adding the second column of A to its first column. Suppose that after exchanging the second and third rows of B, the

resulting matrix is the  $3 \times 3$  identity matrix. Let  $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$ 

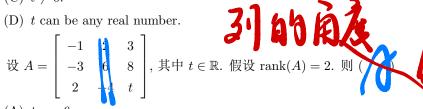
Then A = (

- (A)  $P_1P_2$ .
- (B)  $P_1^{-1}P_2$ .
- (C)  $P_2P_1$ .
- (D)  $P_2P_1^{-1}$ .

设 
$$A$$
 为 3 阶方阵,将  $A$  的第二列加到第一列得矩阵  $B$ . 假设交换  $B$  的第二行与第三行可以得到 3 阶单位矩阵. 它  $P_1=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$ ,  $P_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ .则  $A=($ 

(A)  $P_1P_2$ .

- (B)  $P_1^{-1}P_2$ .
- (C)  $P_2P_1$ .
- (D)  $P_2 P_2^{-1}$
- (4) Let  $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$ , where  $t \in \mathbb{R}$ . Suppose rank(A) = 2. Then (
  - (A) t = -6.
  - (B) t = 6.
  - (C)  $t \neq 0$ .
  - (D) t can be any real number.



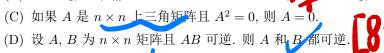
- (A) t = -6.
- (B) t = 6.
- (C)  $t \neq 0$ .
- (D) t 可以是住意买数.
- (5) Which of the following statements is incorrect? ( )
  - (A) For any matrix A, rank $(A) = \dim C(A)$ .
  - (B) If  $v_1, \dots, v_m$  are pairwise orthogonal nonzero vectors, then the vectors  $v_1, \dots, v_m$ are linear independent.

- (C) If A is an upper triangular  $n \times n$  matrix such that  $A^2 = 0$ , then A = 0.
- (D) Let A, B be  $n \times n$  matrices such that AB is invertible. Then both A and B are invertible.

下列哪个论断是错误的?(



- (B) 如果  $v_1, \dots, v_m$  是一组两两正交的非零  $\mathbb{Z}$  量,





- 2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.
  - (1) Let A, B be invertible  $n \times n$  matrices. Then the inverse of the block matrix  $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$  is 设 A, B 均为  $n \times n$  可逆矩阵. 则分块矩阵  $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$  的逆为 \_
  - (2) Suppose A is a  $3 \times 4$  matrix and dim N(A) = 2. Then dim  $N(A^T) =$ 设 A 为  $3 \times 4$  矩阵且  $\dim N(A) = 2$ . 则  $\dim N(A^T) =$

  - Then  $||2u + 3v|| = _____$

设 u, v 为  $\mathbb{R}^n$  中的向量, 满足 ||u|| = 3, ||v|| = 4 以及  $u^T v = -3$ . 则  $||2u + 3v|| = ___$ 

(5) Let  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ .

Then the least squares solution to Ax = b is  $\hat{x} =$ 

设 
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$
则  $Ax = b$  的最小二乘解是  $\hat{x} =$ 

3. (10 points) Find the LU factorization of the matrix  $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ .

b=3

4. (16 points) Let 
$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$
.

Please give a basis for each of the four fundamental subspaces C(A), N(A),  $C(A^T)$  and  $N(A^T)$ , respectively.

(16 分) 设 
$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$
. C(A) 3

5. (10 points) Let  $E = \{u_1, u_2, u_3\}$  and  $F = \{v_1, v_2\}$ , where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ and } v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Define the linear transformation  $T: \mathbb{R}^3 \to \mathbb{R}^2$  by

$$T\left(\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right]\right) = \left[\begin{array}{c} 2x_2 \\ -x_1 \end{array}\right].$$

Find the matrix A representing T with respect to the ordered bases E and F. 0+3b=2

$$(10 分)$$
 设  $E = \{u_1, u_2, u_3\}, F = \{v_1, v_2\},$  其中

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$
 and  $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ . 线性变换  $T: \mathbb{R}^3 \to \mathbb{R}^2$  如下

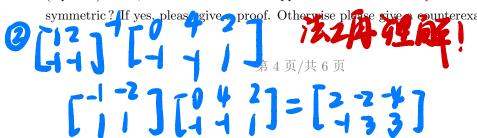
定义线性变换  $T: \mathbb{R}^3 \to \mathbb{R}^2$  如下

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

$$TU_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

求 T 在 E 和 F 这两组有序基下的矩阵表示 A.

6. (6 points) Let A, B be  $n \times n$  natrices. Suppose A and B are both symmetric. Is AB necessary



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 $(6 \ \%)$  设 A, B 均为  $n \times n$  矩阵. 假设 A 和 B 都是对称矩阵. AB 是否一定是对称矩阵? 若 是,请给出证明. 否则请给出一个反例.

- 7. (16 points) The following two questions are independent:
  - (a) Let A be the  $2 \times 2$  matrix such that the linear transformation  $\mathbb{R}^2 \to \mathbb{R}^2$ ,  $v \mapsto Av$  rotates every vector in  $\mathbb{R}^2$  through 60° counter-clockwise (all out the origin). Find A and  $A^{2020}$ .
  - (b) Three planes  $\Pi_1,\,\Pi_2,\,\Pi_3$  in the space  $\mathbb{R}^3$  are given by the equations

$$\Pi_1: \quad x+y+z=0\,,$$

$$\Pi_2: 2x - y + 4z = 0$$
,

$$\Pi_3 : -x + 2y - z = 0.$$

Determine a matrix representative (in the standard basis of  $\mathbb{R}^3$ ) of a linear transformation taking the xy plane to  $\Pi_1$ , the yz plane to  $\Pi_2$  and the zx plane to  $\Pi_3$ .

- (16 分) 以下两个小题是相互独立的:
- (a) 设 A 是  $2 \times 2$  矩阵使得线性变换  $\mathbb{R}^2 \to \mathbb{R}^2$ ,  $v \mapsto Av$  把  $\mathbb{R}^2$  中每个向量 (绕原点) 逆时针 转动 60°.

求 A 和 A<sup>2020</sup>.

(b) 在空间  $\mathbb{R}^3$  中由以下方程给出三个平面  $\Pi_1$ ,  $\Pi_2$ ,  $\Pi_3$ :

$$\Pi_1: \quad x+y+z=0\,,$$

$$\Pi_2$$
:  $2x - y + 4z = 0$ .

$$\Pi_2$$
:  $2x - y + 4z = 0$ ,  
 $\Pi_3$ :  $-x + 2y - z = 0$ .

求一个矩阵, 使它 (在  $\mathbb{R}^3$  的标准基下) 表示的线性变换将 xy 平面映射成  $\Pi_1$ , 将 yz 平面 映射成  $\Pi_2$  并将 zx 平面映射成  $\Pi_3$ .

- 8. (12 points) Let A be a  $3 \times 3$  matrix such that  $\operatorname{rank}(A) = 2$  and  $A^3 = 0$ .
  - (a) Prove that  $rank(A^2) = 1$ .
  - (b) Let  $\alpha_1 \in \mathbb{R}^3$  be a nonzero vector such that  $A\alpha_1 = 0$ . Prove that there exist vectors  $\alpha_2$ ,  $\alpha_3$ such that  $A\alpha_2 = \alpha_1$ ,  $A^2\alpha_3 = \alpha_1$ .
  - (c) For any vectors  $\alpha_2, \alpha_3$  described as above, show that  $\alpha_1, \alpha_2, \alpha_3$  are linearly independent.

(In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.)

(12 分) 设  $A \neq 3 \times 3$  矩阵, 它满足 rank(A) = 2 及  $A^3 = 0$ 



N(A)

- (a) 证明  $rank(A^2) = 1$ .
- (b) 设  $\alpha_1 \in \mathbb{R}^3$  是满足  $A\alpha_1 = 0$  的非零向量. 证明: 存在向量  $\alpha_2$ ,  $\alpha_2$  使得  $A\alpha_2 = \alpha_1$ ,  $A^2\alpha_3 = \alpha_1$ .
- (c) 证明: 对于任意满足上述条件的向量  $\alpha_2,\alpha_3$ , 向量组  $\alpha_1,\alpha_2,\alpha_3$  线

(本题中,允许承认前面小题的结果来是于后续问题 7解答.)

EN(A)
Span foll

di= A2d3=(8d2)

M (Ad3) EC(A)
ENLAIR

C(A) C(A)C(A)

(#A') > C3=0

乘力

Cz=1 => C1=0