Similarity Transformations (相似变换)

期末必考

Lecture 25 and 26

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Similarity Transformations

$S^{-1}AS = B$

A is similar to B

相似

$$S^{-1}AS = B \Leftrightarrow A = SBS^{-1}$$

$$P(\lambda) = \det(B - \lambda I)$$

$$= \det(S^{-1}AS - \lambda I)$$ 可与任意矩阵交换

$$= \det(S^{-1}AS - S^{-1}(\lambda I)S)$$

$$= \det\left(S^{-1}(A - \lambda I)S\right)$$

$$= \det(S^{-1}) \cdot \det(S) \det(A - \lambda I)$$

$$= \det(A - \lambda I)$$

不改变特征值 $\Leftarrow$ 特征多项式相同

Similar Matrices

Now we look at all combinations $M^{-1}AM$– formed with any invertible M on the right and its inverse on the left.

A whole family of matrices $M^{-1}AM$ is similar to A, there are two questions:

1. What do these similar matrices $M^{-1}AM$ have in common?
2. With a special choice of M, what special form can be achieved by $M^{-1}AM$?

Theorem

Suppose that $B = M^{-1}AM$. Then A and B have the same eigenvalues. Every eigenvector x of A corresponds to an eigenvector $M^{-1}x$ of B.

当 Λ 可对角化时
有 S 使得 $S^{-1}AS = \Lambda$

Example 1

Example 1 $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has eigenvalues 1 and 0. Each B is $M^{-1}AM$:

$M^{-1} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$

- If $M = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$: triangular with $\lambda = 1$ and 0.
- If $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: projection with $\lambda = 1$ and 0.
- If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (invertible), then $B =$ an arbitrary matrix with $\lambda = 1$ and 0.

Change of Basis = Similarity Transformation

Similar matrices represent the same transformation T with respect to different bases.

Theorem 同一线性变换在不同基下的表示

The matrices A and B that represent the same linear transformation T with respect to two different bases (the v's and the V's) are similar:

$$[T]_{V\to V} = [I]_{v\to V}[T]_{v\to v}[I]_{V\to v}$$
$$B = M^{-1}AM.$$

Proof: Sketch

在$\mathbb{R}$中取两组基向量
$\{V_1, V_2, \cdots V_n\}$ $\{v_1, v_2, \cdots v_n\}$

If

$$T(V_1, V_2, \cdots, V_n) = (V_1, V_2, \cdots, V_n)B$$

$$T(v_1, v_2, \cdots, v_n) = (v_1, v_2, \cdots, v_n)A$$

$$(V_1, V_2, \cdots, V_n) = (v_1, v_2, \cdots, v_n)M$$

$$(v_1, v_2, \cdots, v_n) = (V_1, V_2, \cdots, V_n)M^{-1}$$

$TV_1 = b_{11}V_1 + b_{21}V_2 + \cdots + b_{n1}V_n$
$TV_2 = b_{12}V_1 + b_{22}V_2 + \cdots + b_{n2}V_n$
$\vdots$
$\Leftrightarrow T(V_1, V_2, \cdots V_n) = (TV_1, TV_2, \cdots TV_n) = (V_1, V_2, \cdots V_n)\begin{bmatrix} b_{11} & b_{12} & \\ b_{21} & b_{22} & \ddots \\ \vdots & \vdots & \\ b_{n1} & & \end{bmatrix}$

线性变换在这组基下的变换矩阵 — B

then

$$\begin{aligned} T(V_1, V_2, \cdots, V_n) &= T((v_1, v_2, \cdots, v_n)M) \\ &= (T(v_1, v_2, \cdots, v_n))M \\ &= (v_1, v_2, \cdots, v_n)AM \\ &= (V_1, V_2, \cdots, V_n)M^{-1}AM. \end{aligned}$$

M是系数可以拿出去

同一基底下表示唯一确定

Therefore, $B = M^{-1}AM$.

$$T(v_1, v_2, \cdots v_n) = (v_1, v_2, \cdots v_n)\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & \\ a_{m1} & & \end{bmatrix}}_{A}$$

$$(V_1, V_2, \cdots V_n) = (v_1, v_2, \cdots v_n)\, M$$

$$= (v_1, v_2, \cdots v_n)\begin{bmatrix} m_{11} & m_{12} & \cdots \\ m_{21} & \ddots & \\ \vdots & & \\ m_{n1} & & \end{bmatrix}$$

$$V_1 = m_{11}v_1 + m_{21}v_2 + \cdots + m_{n1}v_n$$

对 $T((v_1, v_2, \cdots v_n)M)$

$= T(m_{11}v_1 + m_{21}v_2 + \cdots + m_{n1}v_n, \; m_{12}v_1 + m_{22}v_2 + \cdots + m_{n2}v_n, \cdots)$

$= (m_{11}Tv_1 + m_{21}Tv_2 + \cdots + m_{n1}Tv_n, \; \cdots)$

$= (Tv_1, Tv_2, \cdots Tv_n)\begin{bmatrix} m_{11} & m_{12} & \\ m_{21} & m_{22} & \\ \vdots & \vdots & \ddots \\ m_{n1} & & \end{bmatrix}$

$= T(v_1, v_2, \cdots v_n) \cdot M$

Figure 5.5 目标：找一个好的基向量组

Example Suppose T is projection onto the line L at angle $\theta (= 135^\circ)$. This linear transformation is completely determined without the help of a basis. But to represent T by a matrix, we do need a basis. Figure 5.5 offers two choices, the standard basis $v_1 = (1,0), v_2 = (0,1)$ and a basis V_1, V_2 chosen especially for T.

让矩阵尽量简单 $S^{-1}AS = \Lambda$ ✓

更多0

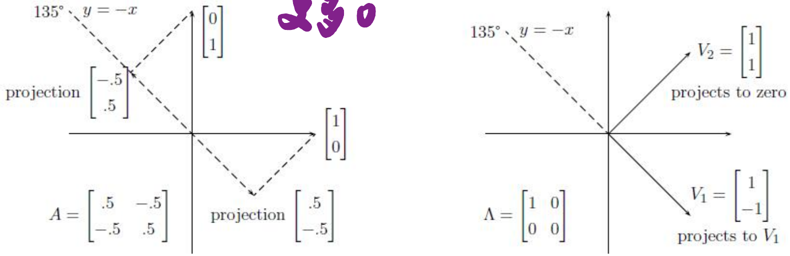

Figure 5.5: Change of basis to make the projection matrix diagonal.

Summary

- The way to simplify that matrix A–in fact to diagonalize it–is to find its eigenvectors. They go into the columns of M (or S) and $M^{-1}AM$ is diagonal. The algebraist says the same thing in the language of linear transformations: **Choose a basis consisting of eigenvectors**. The standard basis led to A, which was not simple. The right basis led to B, which was diagonal.

  关键是 λ 不变!

- $M^{-1}AM$ does not arise in solving $Ax = b$. There the basic operation was to multiply A (on the left side only!) by a matrix that subtracts a multiple of one row from another. Such a transformation preserved the nullspace and row space of A; it normally changes the eigenvalues.

$$T\underline{\underline{(v_1, v_2, \cdots v_n)}}_{\text{good}} = (v_1, v_2, \cdots v_n)\underline{A}$$

$$\begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_m \end{bmatrix}$$

准对角

$S^{-1}AS = \Lambda$(对角)

$U^{-1}AU = T$(上三角)　Jordan matrix

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ 不可对角化

eigenvalues : 1, 1

代数重数 > 几何重数
2 1

$(A-I)x=0$

$x = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$ $|A| = \lambda^3$

仅 $\lambda = 0$

$(A-\lambda I)x=0$ (缺2个)

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = 0$ $x = c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Triangular Forms with a Unitary M

Theorem

任意矩阵可用U

(Schur's lemma) *There is a unitary matrix $M = U$ such that $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of this similar matrix T.*

相似到上三角矩阵

Can you prove this theorem?

Remark:

- This lemma applies to all matrices, with no assumption that A is diagonalizable.
- We could use it to prove that the powers A^k approach zero when all $|\lambda_i| < 1$, and the exponentials e^{At} approach zero when all Re $\lambda_i < 0$–even without the full set of eigenvectors which was assumed in sections 5.3 and 5.4.

$A_{4\times 4}$ complex （实矩阵不一定能做到）

$Ax_1 = \lambda x_1 \quad \|x_1\| = 1$

$[x_1 \ x_2 \ x_3 \ x_4] = U_1$ 找特征向量 x_1 将其补3个变成正交矩阵

（施密特正交化）

$x_1 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \quad x_1^T x = 0 \Leftrightarrow [z_1 \ z_2 \ z_3 \ z_4]\, x_{4\times 1} = 0$

$$A U_1 = A[x_1 \ \underbrace{x_2 \ x_3 \ x_4}_{B}]$$

$$= [A x_1 \ \ AB]$$

$\lambda x_1 = U_1\begin{bmatrix} \lambda_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [x_1 x_2 x_3 x_4]\begin{bmatrix} \lambda_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$= [\lambda_1 x_1 \ \ AB]$$

$$= U_1 \begin{bmatrix} \lambda_1 & \\ 0 & \\ 0 & U_1^{-1}AB \\ 0 & \end{bmatrix} = U_1 \begin{bmatrix} \lambda_1 & \\ 0 & \\ 0 & C \\ 0 & \end{bmatrix}$$

$\Rightarrow U_1^{-1} A U_1 = \begin{bmatrix} \lambda_1 & \\ 0 & C \\ 0 & \end{bmatrix}$

先将第一列变成上三角

之后的做法完全雷同

$U_1^{-1}AB$

结果: $U_1^{-1}AU_1 = \begin{bmatrix}\lambda_1 & \times & \times & \times\\ 0 & \times & \times & \times\\ 0 & \times & \times & \times\\ 0 & \times & \times & \times\end{bmatrix}$ A_2

相似变换 而非消元 ⇒ 特征值不改变

同理 $M_2^{-1}A_2M_2 = \begin{bmatrix}\lambda_2 & \times & \times\\ 0 & \times & \times\\ 0 & \times & \times\end{bmatrix}$ A_3

$M_3^{-1}A_3M_3 = \begin{bmatrix}\lambda_3 & \times\\ 0 & \times\end{bmatrix}$

取 $U_2 = \begin{bmatrix}1 & 0\ 0\ 0\\ 0 & \\ 0 & M_2\\ 0 & \end{bmatrix}$ $U_3 = \begin{bmatrix}1 & 0 & 0\ 0\\ 0 & 1 & 0\ 0\\ 0 & 0 & \\ 0 & 0 & M_3\end{bmatrix}$

$$\underbrace{U_3^{-1}U_2^{-1}U_1^{-1}}_{U^{-1}}A\underbrace{U_1U_2U_3}_{U} = \begin{bmatrix}\lambda_1 & & & *\\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4\end{bmatrix}$$

Example

Example 2. $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalues $\lambda = 1$ (twice).

1. The only line of eigenvectors goes through $(1, 1)$.
2. After dividing by $\sqrt{2}$, this is the first column of U, and the triangle $U^{-1}AU = T$ has the eigenvalues on its diagonal.
3. The triangular T is given as follows:

$$
\begin{aligned}
U^{-1}AU &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T.
\end{aligned}
$$

This triangular form will show that any symmetric or Hermitian matrix–whether its eigenvalues are distinct or not–has a **complete set of orthonormal eigenvectors.**

Spectral Theorem

We need a unitary matrix such that $U^{-1}AU$ is diagonal. **Schur's lemma** has just found it. This triangular T must be diagonal, because it is also Hermitian when $A = A^H$:

$$T = T^H$$

$$(U^{-1}AU)^H = U^H A^H (U^{-1})^H = U^{-1}AU.$$

Spectral Theorem

The diagonal matrix $U^{-1}AU$ represents a key theorem in linear algebra:

Theorem

Every real symmetric A can be diagonalized by an orthogonal matrix Q.
Every Hermitian matrix can be diagonalized by a unitary U:
(Real)

$$Q^{-1}AQ = \Lambda \text{ or } A = Q\Lambda Q^T.$$

(Complex)

$$U^{-1}AU = \Lambda \text{ or } A = U\Lambda U^H.$$

The columns of Q (or U) contain orthonormal eigenvectors of A.

含正交特征向量

$U^H U = U U^H = I \Rightarrow U^H = U^{-1}$

$(U^H A U)^H = T^H$ → 上三角

$U^H A^H (U^H)^H = U^H A U$

$\Rightarrow T^H = T$ T 还是对角矩阵

Remark

In the real symmetric case, the eigenvalues and eigenvectors are real at every step. That produces a real unitary U–an orthogonal matrix.

例 $A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$ $A^T=A$ 实对称

找 $Q^TAQ=\Lambda$ 特征向量并正交

明确些

$$|A-\lambda I| = \begin{vmatrix}-\lambda&1&0\\1&-\lambda&0\\0&0&1-\lambda\end{vmatrix} = (1-\lambda)(-1)^{3+3}(\lambda^2-1)$$

$$= -(\lambda-1)^2(\lambda+1)$$

$\lambda=1$ 谱定理保证可对角化 $(s_1=n_1)$

$$\begin{bmatrix}-1&1&0\\1&-1&0\\0&0&0\end{bmatrix}x=0 \qquad x=\begin{bmatrix}0\\0\\1\end{bmatrix}、\begin{bmatrix}1\\1\\0\end{bmatrix} \qquad q_1=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} \quad q_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\lambda=-1$

$$\begin{bmatrix}1&1&0\\1&1&0\\0&0&2\end{bmatrix}x=0 \qquad x=\begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad q_3=\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix}$$

三向量正交

$x_1=\begin{bmatrix}1\\1\\0\end{bmatrix}$ $x_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$

恰好正交

同一特征值对应v

如果不交 要进行施密特正交化

不同λ，v一定正交

如 $\begin{bmatrix}1\\-1\\0\end{bmatrix}$

$$Q=[q_1\ q_2\ q_3]$$

$$Q^TAQ=\begin{bmatrix}1&&\\&1&\\&&-1\end{bmatrix}\rightarrow\Lambda\text{由}\lambda\text{组成}$$

$$A=Q\Lambda Q^T=[q_1\ q_2\cdots q_n]\begin{bmatrix}\lambda_1&&0\\&\ddots&\\0&&\lambda_n\end{bmatrix}\begin{bmatrix}q_1^T\\q_2^T\\\vdots\\q_n^T\end{bmatrix}$$

在q_1方向上的投影

$$=\lambda_1\underbrace{q_1q_1^T}+\lambda_2q_2q_2^T\cdots+\lambda_nq_nq_n^T$$

(矩阵相加)

投影矩阵

所有分量相加

$$A=1(q_1q_1^T+q_2q_2^T)-1(q_3q_3^T)=\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\\0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0&0&1\end{bmatrix}$$

$$-\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\\0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}&0\end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$$

$$= \lambda_1 P_1 + \lambda_2 P_2$$

$P_1 P_2 = 0$ 投到正交空间

$$QQ^T = q_1 q_1^T + q_2 q_2^T + q_3 q_3^T$$

$$= I$$

Example

Example 3 The spectral theorem says that this $A = A^T$ can be diagonalized:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) A has eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$.

(b) Every Hermitian matrix with k different eigenvalues has a spectral decomposition into $A = \lambda_1 P_1 + \cdots + \lambda_k P_k$, where P_i is the projection onto the eigenspace for λ_i. 可分解为投影相加

(c) Since there is a full set of eigenvectors, the projections add up to the identity. And since the eigenspaces are orthogonal, two projections produce zero: $P_j P_i = 0$. 交叉投影乘成0

Normal Matrices

We are very close to answering an important question, so we keep going: For which matrices is $T = \Lambda$?

(最大的一类) 可交换

Theorem

The matrix N is normal if it commutes with N^H: $NN^H = N^H N$. For such matrices, and no others, the triangular $T = U^{-1}NU$ is the diagonal Λ. Normal matrices are exactly those that have a complete set of orthonormal eigenvectors.

可用 U 对角化

Remarks:

- Symmetric, skew-Symmetric, and Orthogonal are normal.
- Hermitian, skew-Hermitian, and Unitary are normal.

$z = \bar{z} \leftrightarrow A = A^H$

$z = -\bar{z} \leftrightarrow A = -A^H$

$|z| = 1 \leftrightarrow U^H U = U U^H = I$

$z \leftrightarrow A^H A = A A^H$

$\bar{z} z = z \bar{z}$

(所有复数与对应 Normal matrix)

Proof: Sketch

Step 1: If N is normal, then so is the triangular $T = U^{-1}NU$:

$$TT^H = U^{-1}NUU^HN^HU = U^{-1}NN^HU$$
$$= U^{-1}N^HNU = U^HN^HUU^{-1}NU = T^HT.$$

Step 2: A triangular matrix T that is normal must be diagonal. (See Problems 19-20 at the end of this section).

Thus, if N is normal, the triangular $T = U^{-1}NU$ must be diagonal. Since T has the same eigenvalues as N, it must be Λ. The eigenvectors of N are the columns of U, and they are orthonormal. That is the good case. We turn now from the best possible matrices (normal) to the worst possible (defective). See:

$$\texttt{\textbf{Normal}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } \texttt{\textbf{Defective}} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

The Jordan Form

- Our next goal is to make $M^{-1}AM$ as nearly diagonal as possible.
- The result of this supreme effort at diagonalization is the **Jordan form** J.
- If A has a full set of eigenvectors, we take $M = S$ and we arrive at $J = S^{-1}AS = \Lambda$. Then the Jordan form coincides with the diagonal Λ.
- This is impossible for a nondiagonalizable matrix. For every missing eigenvector, the Jordan form will have a 1 just above its main diagonal.

The Jordan Block

Theorem

If A has s independent eigenvectors, it is similar to a matrix with s blocks:

$$J = M^{-1}AM = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}.$$

Each Jordan block J_i is a triangular matrix that has only a single eigenvalue λ_i and only one eigenvector.

Jordan Block

The Jordan Block:

$$\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

- The same λ_i will appear in several blocks, if it has several independent eigenvectors.

Jordan Block

The Jordan Block:

$$\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

缺少几个特征向量，出现几个1

- The same λ_i will appear in several blocks, if it has several independent eigenvectors.
- Two matrices are similar if and only if they share the same Jordan form J.

$M^{-1}AM = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$

J " Jordan block

求 A^k

$A = MJM^{-1}$ 有规律

$A^k = M\underbrace{J^k}M^{-1}$

$AM = MJ$

$A[v_1\ v_2\ v_3] = [v_1\ v_2\ v_3]\begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$

$Av_1 = \lambda v_1 \Leftrightarrow (A-\lambda I)v_1 = 0$ v_1是λ对应特征向量

v_2、v_3 不是特征向量

$Av_2 = v_1 + \lambda v_2$

$(A-\lambda I)v_2 = v_1$

$Av_3 = v_2 + \lambda v_3$

$(A-\lambda I)v_3 = v_2$

$v_2 = (A-\lambda I)v_3$

$v_1 = (A-\lambda I)v_2 = (A-\lambda I)^2 v_3$

v_1 易知

但要求 v_3

$M = [v_1 \ \underline{v_2 \ v_3}]$

↓ 广义上特征向量

Example

Example 4 $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ all lead to $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(T)

$$M^{-1}TM = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

(B)

$$P^{-1}BP = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

(A)

$$U^{-1}AU = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T \quad \text{and then} \quad M^{-1}TM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

$T=\begin{bmatrix}1&2\\0&1\end{bmatrix} \to J=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ 模取特征值，补上1

$M^{-1}TM=J$

$\begin{bmatrix}1&2\\0&1\end{bmatrix}(v_1\ v_2)=(v_1\ v_2)\begin{bmatrix}1&1\\0&1\end{bmatrix}$

$Tv_1=1\cdot v_1 \quad \begin{bmatrix}0&2\\0&0\end{bmatrix}v_1=0 \quad \Rightarrow v_1=\begin{bmatrix}1\\0\end{bmatrix}$

$Tv_2=v_1+v_2 \Leftrightarrow (T-1I)v_2=v_1$

$\begin{bmatrix}0&2\\0&0\end{bmatrix}v_2=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow v_2=\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}$

$M=\begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}$

Example

Example 5 $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

- Zero is a triple eigenvalue for A and B, so it will appear in all their Jordan blocks.
- There can be a single 3 by 3 block, or a 2 by 2 and a 1by 1 block, or three 1 by 1 blocks.
- A count of the eigenvectors will determine J when there is nothing more complicated than a triple eigenvalue.

Example

Example 6 Application to difference and differential equations(powers and exponentials). If $A = MJM^{-1}$, we have

$$A^k = MJM^{-1}MJM^{-1}\cdots MJM^{-1} = MJ^kM^{-1}$$

J is block diagonal, and the powers of each block can be taken separately:

$$(J_i)^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

This block J_i will enter when λ is a triple eigenvalue with a single eigenvector.

对于 $A_{3\times 3}$

λ 有 λ_1、λ_2、λ_3

① λ_1、λ_2、λ_3 互不相同

$$J=\Lambda=\begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{bmatrix}$$

② $\lambda_1=\lambda_2\neq\lambda_3$

i $\begin{bmatrix}\lambda & & \\ & \lambda & \\ & & \lambda_3\end{bmatrix}$ (仍可对角化

ii $\begin{bmatrix}\lambda & 1 & \\ & \lambda & \\ & & \lambda_3\end{bmatrix}$

③ $\lambda_1=\lambda_2=\lambda_3$

$$\begin{bmatrix}\lambda & & \\ & \lambda & \\ & & \lambda\end{bmatrix}$$

$$\begin{bmatrix}\lambda & 1 & \\ & \lambda & \\ & & \lambda\end{bmatrix}$$ 一般从大到小排布

$$\begin{bmatrix}\lambda & 1 & \\ & \lambda & 1 \\ & & \lambda\end{bmatrix}$$

Exponential

Its exponential is in the solution to the corresponding differential equation:

$$e^{J_i t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

Here

$$I + J_i t + \frac{(J_i t)^2}{2!} + \cdots$$

produces

$$1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots = e^{\lambda t}$$

on the diagonal.

Similarity Transformations

1. A is diagonalizable: The columns of S are eigenvectors and $S^{-1}AS = \Lambda$.
2. A is arbitrary: The columns of M include "generalized eigenvectors " of A, and the Jordan form $M^{-1}AM$ is block diagonal.
3. A is arbitrary: The unitary U can be chosen so that $U^{-1}AU = T$ is triangular.
4. A is normal, $AA^H = A^HA$: then U can be chosen so that $U^{-1}AU = \Lambda$.

Special Cases of Normal Matrices, all with orthonormal eigenvectors

(a) If $A = A^H$ is Hermitian, then all λ_i are real.

(b) If $A = A^T$ is real symmetric, then Λ is real and $U = Q$ is orthogonal.

(c) If $A = -A^H$ is skew-Hermitian, then all λ_i are purely imaginary.

(d) If A is orthogonal or unitary, then all $|\lambda_i| = 1$ are on the unit circle.

Exercise $A^T=A$

$\det(A)=0$ $\lambda=0$

设 A 是三阶实对称矩阵，A 的秩为 2，即 $r(A)=2$，且

线性无关

且正交

$$A\begin{pmatrix}1&1\\0&0\\-1&1\end{pmatrix}=\begin{pmatrix}-1&1\\0&0\\1&1\end{pmatrix}.$$

$\lambda=-1$

x_1

(1) $A\begin{bmatrix}1\\0\\-1\end{bmatrix}=(-1)\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$A\begin{bmatrix}1\\0\\1\end{bmatrix}=1\begin{bmatrix}1\\0\\1\end{bmatrix}$ x_2 $\lambda=1$

(I) 求 A 的所有特征值和特征向量;

(II) 求矩阵 A.

$\begin{bmatrix}1&0&-1\\1&0&1\end{bmatrix}x=0$

$x=\begin{bmatrix}0\\1\\0\end{bmatrix}$

(不用加c)

$A=S\begin{bmatrix}1&&\\&-1&\\&&0\end{bmatrix}S^{-1}$

Exercise

通过可逆矩阵 ~~相~~ 相似

已知矩阵 $A = \begin{pmatrix} -2 & -2 & 1 \\ 2 & x & -2 \\ 0 & 0 & -2 \end{pmatrix}$ 与 $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & y \end{pmatrix}$ 相似.

$\Uparrow$ $\lambda = -2, 2, -1$ $\begin{cases} y = -2 \\ x = 3 \end{cases}$

(1) (I) 求 x, y;

(II) 求可逆矩阵 P, 使得 $P^{-1}AP = B$.

$|A - \lambda I| = (-2-\lambda)\left((-2-\lambda)(x-\lambda)+4\right) = 0$

$\lambda = -2, \ldots$

$|B - \lambda I| = (y-\lambda)(2-\lambda)(-1-\lambda)$

$\lambda = 2、-1、y$

相似 λ 相等

(2) $A = \begin{bmatrix} -2 & -2 & 1 \\ 2 & 3 & -2 \\ 0 & 0 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 & \\ & -1 & \\ & & -2 \end{bmatrix}$

一定可找到 $P_1^{-1} B P_1 = \begin{bmatrix} 2 & & \\ & -1 & \\ & & -2 \end{bmatrix}$

$P_2^{-1} A P_2 = \begin{bmatrix} 2 & & \\ & -1 & \\ & & -2 \end{bmatrix}$

*直接构造困难
找共同标准型连接

$$P_1^{-1}BP_1 = P_2^{-1}AP_2$$

$$\underbrace{P_1P_2^{-1}}_{P^{-1}}A\underbrace{P_2P_1^{-1}}_{P} = B$$

cP 随一定可以找到)

Properties of Eigenvalues and Eigenvectors

1. Symmetric Matrices: $A = A^T$;
   real λ's; orthogonal eigenvectors: $x_i^T x_j = 0$.
2. Orthogonal: $Q^T = Q^{-1}$;
   all $|\lambda| = 1$; orthogonal $\overline{x_i}^T x_j = 0$.
3. Skew-symmetric: $A^T = -A$
   imaginary λ's; orthogonal $\overline{x_i}^T x_j = 0$.
4. Complex Hermitian: $\overline{A}^T = A$
   real λ's; orthogonal eigenvectors: $\overline{x_i}^T x_j = 0$.
5. Positive definite: $x^T A x > 0$, A is symmetric
   all $\lambda > 0$; eigenvectors can be chosen to be orthogonal

Properties of Eigenvalues and Eigenvectors

6. Similar Matrices:$B = M^{-1}AM$;
   $\lambda(A) = \lambda(B)$; $\quad x(B) = M^{-1}x(A)$.
7. Projection: $P = P^2 = P^T$;
   $\lambda = 1; 0$; $\quad$ column space; nullspace.
8. Reflection: $I - 2uu^T$
   $\lambda = -1; 1, 1, \cdots, 1$; $\quad u; u^{\perp}$.
9. Rank-1 matrix: uv^T
   $\lambda = v^T u; 0, \cdots, 0 \quad u; v^{\perp}$.
10. Inverse: A^{-1}
    $\frac{1}{\lambda(A)}$; $\quad$ eigenvectors of A.

Properties of Eigenvalues and Eigenvectors

11. Shift: $A + cI$;
    $\lambda(A) + c$; eigenvectors of A.
12. Cyclic permutation: $P^n = I$;
    $\lambda_k = e^{\frac{2\pi ik}{n}}$; $x_k = (1, \lambda_k, \cdots, \lambda_k^{n-1})$.
13. Diagonalizable: $S\Lambda S^{-1}$
    diagonal of Λ; columns of S are independent.
14. Symmetric: $Q\Lambda Q^T$
    diagonal of Λ(real); columns of Q are orthonormal.
15. Jordan: $J = M^{-1}AM$
    diagonal of J; each block gives 1 eigenvector
16. Every matrix: $A = U\Sigma V^T$
    rank(A)=rank(Σ); eigenvectors of A^TA, AA^T in V, U.

Homework Assignment 25 and 26

5.6: 4, 7, 13, 16, 17, 22, 24, 30, 34.