题　号	1	2	3	4	5	6	7
分　值	12 分	15 分	24 分	14 分	15 分	10 分	10分

本试卷共 （ 7 ）大题, 满分 （ 100 ）分.

This 2-hour long test includes 7 questions. Write ***all your answers*** on the examination book.

1. (12 points, 2 points each) Label the following statements as **True** or **False**. No need to justify. (12 分, 2 分一道) 判断正误, 不需要说明理由.

   (a) If A and B are invertible, then BA is invertible.

   如果 A 和 B是可逆矩阵，则 BA 也是可逆矩阵.

   (b) Let A be an $m \times n$ matrix with rank n, then $Ax = b$ is solvable for all $b \in \mathbb{R}^m$.

   设 A 为 $m \times n$ 矩阵且秩为 n, 则对于任意的 $b \in \mathbb{R}^m$, $Ax = b$ 都是可解的.

   (c) If x_p is a particular solution to $Ax = b$, then x_p is always in the row space of A.

   如果 x_p 是 $Ax = b$ 的一个特解, 那么 x_p 一定在矩阵 A 的行空间里.

   (d) Let the vectors v_1, v_2, v_3 be linearly independent. If $w_1 = v_1, w_2 = v_1 + v_2, w_3 = v_1 + v_2 + v_3$, then w_1, w_2, w_3 are linearly independent.

   假定向量 v_1, v_2, v_3 线性无关. 如果 $w_1 = v_1, w_2 = v_1 + v_2, w_3 = v_1 + v_2 + v_3$, 则 w_1, w_2, w_3 线性无关.

   (e) The transformation that takes x to $2x + 1$ is a linear transformation (from $\mathbb{R}^1$ to $\mathbb{R}^1$).

   变换把 x 变为 $2x + 1$ 是线性变换 (从 $\mathbb{R}^1$ 到 $\mathbb{R}^1$).

   (f) If the row space of A is the same as the column space of A, then the nullspace of A and the left nullspace of A must be the same.

   如果矩阵 A 的行空间和列空间相同, 则 A 的零空间和左零空间必定相同.

   Solution. (a)True (b) False (c) False (d) True (e) False (f) True.

2. (15 points, 5 points each) Fill in the blanks. (15 分, 5 分一道) 填空题.

(a) If $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & a+2 \\ 1 & a & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ has no solution, then $a = \underline{-1}$.

如果 $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & a+2 \\ 1 & a & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ 无解, 那么 $a = \underline{-1}$.

(b) Suppose A is a 4×3 matrix, and rank $A = 2$, and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$, then rank $(AB) = \underline{2}$.

如果 A 是一个 4×3 矩阵, 且 rank $A = 2$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$, 则 rank $(AB) = \underline{2}$.

(c) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 0 & -6 & 7 \end{bmatrix}$, and $B = (I+A)^{-1}(I-A)$, then $(I+B)^{-1} =$ $\underline{\frac{1}{2}(I+A)}$ (Here I is the 4×4 identity matrix).

设 $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 0 & -6 & 7 \end{bmatrix}$, $B = (I+A)^{-1}(I-A)$, 那么 $(I+B)^{-1} =$ $\underline{\frac{1}{2}(I+A)}$ (这里 I 是 4×4 单位矩阵).

3. (24 points) Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}.$$

(a) Find the complete solution to $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) Find the complete solution to $Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(c) Find the rank of A and dimensions of the four fundamental subspaces of A.

(d) Find bases of the four fundamental subspaces of A.

(24 分) 设

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}.$$

(a) 求 $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 的所有解.

(b) 求 $Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ 的所有解.

(c) 求 A 的秩和矩阵 A 的四个基本子空间的维数.

(d) 求矩阵 A 的四个基本子空间的基.

Solution. Let's put the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & b_1 \\ 1 & 2 & 2 & 2 & 3 & b_2 \\ -1 & -2 & 0 & 2 & 3 & b_3 \end{bmatrix}$$

into Reduced Row Echelon Form.

We get

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & 2 & 3 & b_2 - b_1 \\ 0 & 0 & 1 & 2 & 3 & b_3 + b_1 \end{bmatrix}$$

followed by

$$\begin{bmatrix} 1 & 2 & 0 & -2 & -3 & 2b_1 - b_2 \\ 0 & 0 & 1 & 2 & 3 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & 0 & 2b_1 - b_2 + b_3 \end{bmatrix}.$$

(a) We read the special solutions as follows:

$$x_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The complete solution to $Ax = 0$ is $x = c_1x_1 + c_2x_2 + c_3x_3$ for $c_1, c_2, c_3 \in \mathbb{R}$.

(b) Let

$$x_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Then $Ax_p = (1, 2, 0)^T$, so that the complete solution to $Ax = (1, 2, 0)^T$ is $x = x_p + c_1x_1 + c_2x_2 + c_3x_3$ for $c_1, c_2, c_3 \in \mathbb{R}$.

(c) We see there are two pivot columns in the Reduced Row Echelon Form of A and so the rank of A is 2. Thus

$$\dim C(A) = \dim C(A^T) = 2, \quad \dim N(A) = 5-2 = 3, \quad \dim N(A^T) = 3-2 = 1.$$

(d) To give a basis for A we read of the columns corresponding to pivot columns in the Reduced Row Echelon Form of A:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

We have already computed a basis for $N(A)$ in (a): $\{x_1, x_2, x_3\}$.

To give a basis for $C(A^T)$, we find two independent rows:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

To give a basis for $N(A^T)$, we read off the coefficients of the relation $2b_1 - b_2 + b_3 = 0$:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

4. (14 points) Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix}.$$

(a) Find the symmetric factorization of $A = LDL^T$.

(b) Use the Gauss-Jordan method to find A^{-1}.

(14 分) 假设

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix}.$$

(a) 求 A 的一个 LDL^T 分解.

(b) 用高斯约旦方法求 A 的逆矩阵, A^{-1}.

Solution. (a) The symmetric factorization of A is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) The inverse of A is

$$\begin{bmatrix} \frac{25}{3} & -\frac{11}{3} & \frac{4}{3} \\ -\frac{11}{3} & \frac{11}{6} & -\frac{2}{3} \\ \frac{4}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

5. (15 points) Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}.$$

(a) Explain why $Ax = b$ is inconsistent.

(b) Find the lease squares solution to $Ax = b$.

(c) Split b into a column space component x_c and a left nullspace component x_l, i.e., $b = x_c + x_l$.

(15 分) 设

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}.$$

(a) 说明为什么线性方程组 $Ax = b$ 没有解.

(b) 求 $Ax = b$ 的最小二乘解.

(c) 把 b 分解成一个列空间分量 x_c 和一个左零空间分量 x_l, 换言之, $b = x_c + x_l$.

Solution. (a) Gaussian elimination shows that $Ax = b$ is inconsistent.

(b)We consider the normal equations:

$$A^T A\hat{x} = A^T b.$$

$$\begin{bmatrix} 4 & -5 & 1 \\ -5 & 7 & -2 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -10 \end{bmatrix}.$$

$$\hat{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(c)

$$b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix} = A\hat{x} + (b - A\hat{x}) = \begin{bmatrix} 0 \\ -2 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = x_c + x_l.$$

6. (10 points) The space of all 2×2 real matrices, denoted $\mathbb{R}^{2\times 2}$, has the four basis "vectors"

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define the transformation of transposing from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ as follows:

$$T(X) = X^T.$$

(a) Show that T is a linear transformation.

(b) Find the matrix A representing T with respect to the above basis for $\mathbb{R}^{2\times 2}$.

(c) Explain why $A^2 = I$.

(10 points) 包含所有 2×2 实矩阵的向量空间 $\mathbb{R}^{2\times 2}$ 有以下四个基向量

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

从 $\mathbb{R}^{2\times 2}$ 到 $\mathbb{R}^{2\times 2}$ 的转置变换定义如下:

$$T(X) = X^T.$$

(a) 证明 T 是一个线性变换.

(b) 找出线性变换 T 在上述基向量组下的矩阵表示, A.

(c) 为什么有 $A^2 = I$? 说明理由.

Solution. (a) By definition. (b) The matrix representation is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) A is a permutation matrix, which is formed by exchanging the second row and third row of the 4 by 4 identity matrix, therefore $A^2 = I$.

7. (10 points)

(a) Let $v_1, v_2, \cdots, v_m$ be linearly independent vectors in $\mathbb{R}^n$ $(n > m)$, and

$$A = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}.$$

It follows that A is an $m \times n$ matrix with rank m. Let

$$w_1, w_2, \cdots, w_{n-m}$$

be a sequence of linearly independent vectors in $\mathbb{R}^n$ satisfying

$$Aw_j = 0,\ j = 1, 2, \cdots, n-m.$$

Show that

$$v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_{n-m}$$

are linearly independent.

(b) Let A be an $n \times n$ real matrix and A^T be its transpose. Show that the column spaces of A^TA and A^T are the same, i.e., $C(A^TA) = C(A^T)$.

(10 分)

(a) 如果 $v_1, v_2, \cdots, v_m$ 是 $\mathbb{R}^n$ 中的线性无关向量$(n > m)$. 假定

$$A = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}.$$

由此可见, A 是一个 $m \times n$ 行满秩矩阵. 如果 $\mathbb{R}^n$ 中线性无关向量组

$$w_1, w_2, \cdots, w_{n-m}$$

满足 $Aw_j = 0,\ j = 1, 2, \cdots, n-m$. 证明:

$$v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_{n-m}$$

线性无关.

(b) 设 A 为一个 $n \times n$ 实矩阵, A^T 为它的转置. 证明: $A^T A$ 和 A^T 的列空间相同, 换言之, $C(A^T A) = C(A^T)$.

Solution. (a) Suppose

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m + b_1 w_1 + b_2 w_2 + \cdots + b_{n-m} w_{n-m} = 0.$$

If we let $a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = v$ and $b_1 w_1 + b_2 w_2 + \cdots + b_{n-m} w_{n-m} = w$, the above equality becomes

$$v + w = 0.$$

Note that $v \in C(A^T)$ and $w \in N(A)$. Taking inner product of v with respect to both sides of the equation above to obtain

$$v^T(v + w) = v^T 0 = 0.$$

Since $C(A^T)$ and $N(A)$ are a pair of orthogonal complements, v is orthogonal to w, the above equation becomes $v^T v = 0$, this only happens when $v = 0$. It follows immediately that $w = 0$. In other words,

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = 0, \; b_1 w_1 + b_2 w_2 + \cdots + b_{n-m} w_{n-m} = 0.$$

Since $v_1, v_2, \cdots, v_m$ and $w_1, w_2, \cdots, w_{n-m}$ are linearly independent, therefore all a's and b's are zero. Thus

$$v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_{n-m}$$

are linearly independent.

(b) Since the column space of $A^T A$ is contained in the column space of A^T, it is sufficient to prove that $\dim C(A^T A) = \dim C(A^T)$. We know that $\dim(C(A^T)) = \operatorname{rank}(A)$ and that

$$\dim C(A^T) + \dim N(A) = n, \quad \dim C(A^T A) + \dim N(A^T A) = n.$$

Since the nullspaces of $A^T A$ and A are the same, and therefore $\dim N(A^T A) = \dim N(A)$. The above equalities imply that

$$\dim C(A^T A) = \dim C(A^T).$$

This completes the proof.