线性代数期末复习

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Determinant（行列式）

- properties

1. *The determinant of the identity matrix is* 1.

$$\det I = 1 \qquad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \text{and} \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad \text{and} \ldots$$

2. *The determinant changes sign when two rows are exchanged.*

$$\textbf{Row exchange} \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

3. *The determinant depends linearly on the first row.* Suppose A, B, C are the same from the second row down—and row 1 of A is a linear combination of the first rows of B and C. Then the rule says: $\det A$ *is the same combination of* $\det B$ *and* $\det C$.

Linear combinations involve two operations—adding vectors and multiplying by scalars. Therefore this rule can be split into two parts:

$$\textbf{Add vectors in row 1} \qquad \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$$

$$\textbf{Multiply by } t \textbf{ in row 1} \qquad \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

4. *If two rows of A are equal, then* $\det A = 0$.

$$\textbf{Equal rows} \qquad \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0.$$

5. *Subtracting a multiple of one row from another row leaves the same determinant.*

$$\textbf{Row operation} \qquad \begin{vmatrix} a - \ell c & b - \ell d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

6. *If A has a row of zeros, then* $\det A = 0$.

$$\textbf{Zero row} \qquad \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0.$$

7. *If A is triangular then* $\det A$ *is the product* $a_{11}a_{22}\cdots a_{nn}$ *of the diagonal entries. If the triangular A has 1s along the diagonal, then* $\det A = 1$.

Triangular matrix $$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \qquad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$$

8. *If A is singular, then* $\det A = 0$. *If A is invertible, then* $\det A \neq 0$.

9. *The determinant of AB is the product of* $\det A$ *times* $\det B$.

Product rule $|A||B| = |AB|$ $$\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{vmatrix}.$$

10. *The transpose of A has the same determinant as A itself:* $\det A^{\mathrm{T}} = \det A$.

Transpose rule $$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = |A^{\mathrm{T}}|.$$

Question

- 1.If a 4 by 4 matrix has det(A) = 2 ,   det(2A)=_____ ,
det(-A)=________, det(A2)=________, and det(A−1)=________.
- 2.If a 3 by 3 matrix has det(A) = 2 ,   det(2A)=_____ ,
- det(-A)=________, det(A2)=________, and det(A−1)=________.

How to calculate?

- essence of determinant

Big Formula $$\det A = \sum_{\text{all } P\text{'s}} (a_{1\alpha} a_{2\beta} \cdots a_{n\nu}) \det P.$$

$\det P = +1$ *or* -1 ***for an even*** *or* ***odd number of row exchanges***.

How to calculate?

- LU factorization

上/下三角行列式是
对角之积

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix} = LDU = L \begin{bmatrix} 2 & & & & \\ & 3/2 & & & \\ & & 4/3 & & \\ & & & \cdot & \\ & & & & (n+1)/n \end{bmatrix} U.$$

How to calculate?

- cofactors(代数余子式)

$$\textbf{Cofactors along row 1} \qquad \det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}. \qquad (8)$$

4B The determinant of A is a combination of any row i times its cofactors:

$$\det A \textbf{ by cofactors} \qquad \det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}. \qquad (10)$$

The cofactor C_{1j} is the determinant of M_{ij} with the correct sign:

$$\textbf{delete row } i \textbf{ and column } j \qquad C_{ij} = (-1)^{i+j} \det M_{ij}. \qquad (11)$$

Question

3. (20 points) Let A_n be the $n \times n$ matrix

$$A_n = \begin{bmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{bmatrix}$$

(a) Find constants b, c such that the sequence $\det(A_n)$ satisfies

$$\det(A_n) = b \cdot \det(A_{n-1}) + c \cdot \det(A_{n-2}) \quad \text{for all } n \geq 3.$$

(b) Find a matrix B such that $\mathbf{x}_n = B\mathbf{x}_{n-1}$ for $n \geq 3$, where $\mathbf{x}_n = \begin{bmatrix} \det(A_n) \\ \det(A_{n-1}) \end{bmatrix}$.

(c) For $a^2 = \frac{3}{16}$, find an expression for $\det(A_n)$ for all $n \geq 3$.

$a_n = a_{n-1} - \frac{3}{16} a_{n-2}$

$x^2 - x + \frac{3}{16} = 0$

$x = \dfrac{1 \pm \frac{1}{2}}{2}$

$x = \frac{1}{4} / \frac{3}{4}$

$a_n - \frac{1}{4} a_{n-1} = \frac{3}{4}(a_{n-1} - \frac{1}{4} a_{n-2})$

$a_n - \frac{3}{4} a_{n-1} = \frac{1}{4}(a_{n-1} - \frac{3}{4} a_{n-2})$

$a_1 = 1$

$a_2 = 1 - a^2$

$a_n - \frac{1}{4} a_{n-1} = (\frac{3}{4})^{n-2}(1 - a^2 - \frac{1}{4})$

$a_n - \frac{3}{4} a_{n-1} = (\frac{1}{4})^{n-1}(1 - a^2 - \frac{3}{4})$

$\begin{bmatrix} 1 & -\frac{1}{4} \\ 1 & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} (\frac{3}{4})^{n-2}(\frac{3}{4} - a^2) \\ (\frac{1}{4})^{n-1}(\frac{1}{4} - a^2) \end{bmatrix}$

$\ldots = \dfrac{1}{-\frac{3}{4} + \frac{1}{4}} \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} \\ -1 & 1 \end{bmatrix}$

$= -2 \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} \\ -1 & 1 \end{bmatrix}$

$\begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = -2 \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (\frac{3}{4})^{n-2}(\frac{3}{4} - a^2) \\ (\frac{1}{4})^{n-1}(\frac{1}{4} - a^2) \end{bmatrix}$

4. (10 points) Compute the determinant of an $n \times n$ matrix A:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

Extra contents

Wolfram MathWorld FROM THE MAKERS OF MATHEMATICA AND WOLFRAM|ALPHA

Algebra › Linear Algebra › Determinants ›

Vandermonde Determinant

$$\Delta(x_1, \ldots, x_n) \equiv \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$$

$$= \prod_{\substack{i,j \\ i>j}} (x_i - x_j)$$

Extra contents

In linear algebra, **Cramer's rule** is a specific formula used for solving a system of linear equations containing as many equations as unknowns, efficient whenever the system of equations has a unique solution. This rule is named after Gabriel Cramer (1704–1752), who published the rule for an arbitrary number of unknowns in 1750. This is the most commonly used formula for getting the solution for the given

Gram Schmidt orthogonalization

3P The vectors $q_1, \ldots, q_n$ are *orthonormal* if

$$q_i^{\mathrm{T}} q_j = \begin{cases} 0 & \text{whenever } i \neq j, \quad \text{giving the orthogonality;} \\ 1 & \text{whenever } i = j, \quad \text{giving the normalization.} \end{cases}$$

A matrix with orthonormal columns will be called Q.

乘正交矩阵长度不变

Lengths unchanged $\quad \|Qx\| = \|x\|$ for every vector x.

QR factorization

$q_1 = \frac{v_1}{\|v_1\|}$

$q_2 = \frac{v_2 - q_1^T v_2 q_1}{\| \quad \|}$

$q_3 = \frac{v_3 - q_1^T v_3 q_1 - q_2^T v_3 q_2}{\| \quad \|}$

$$\textit{QR}\ \textbf{factors} \qquad A = \begin{bmatrix} & & \\ a & b & c \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ q_1 & q_2 & q_3 \\ & & \end{bmatrix} \begin{bmatrix} q_1^{\mathrm{T}}a & q_1^{\mathrm{T}}b & q_1^{\mathrm{T}}c \\ & q_2^{\mathrm{T}}b & q_2^{\mathrm{T}}c \\ & & q_3^{\mathrm{T}}c \end{bmatrix} = QR \qquad (12)$$

v1=[1 0
1]T
v2=[1 0
0]T
v3=[2 1

v1=[1 0
0]T
v2=[1 0
1]T
v3=[2 1

有关前面的总结

- 前面的知识不是期中考后3.4开始，而是**从1.1开始!**
- 第二章的知识点在期末考试里面有不小的占比，一定要去重复看
- 行列式之后，算之前就要思考算法了，不建议（但可以）过度暴力计算

(4) Let A be a 6×7 real matrix of rank 2. Then $\dim N(A^T) =$ ( )

(A) 2. (B) 3.

(C) 4. (D) 5.

(1) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, then $\operatorname{rank}\left(A^2 - A\right) =$ __________.

Eigenvalues and Eigenvectors

- basic information
- algebraic multiplicities&geometric multiplicities
- complex numbers
- real symmetric & Hermition matrix
- Orthonormal matrix & Unitary matrix