Midterm Copy 1

Suggested Solutions.

1. D D C B B

2. (1) $\begin{bmatrix} a & b \\ 2-a & 3-b \end{bmatrix}$, $a, b \in \mathbb{R}$

(2) $t = 5$

(3) $K = 10$

(4) $\dim N(A^TA) = 1$

(5) $\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. (1) let ξ_1, ξ_2, ξ_3 be three linearly independent solutions.

$\xi_1-\xi_2, \xi_1-\xi_2$ linearly independent

Solutions to $Ax = 0$

$\Rightarrow 4 - \text{rank}(A) \geq 2 \quad \Rightarrow \quad \text{rank}(A) \leq 2.$

Also, $\text{rank}(A) \geq 2. \quad \Rightarrow \quad \text{rank}(A) = 2.$

(the first two ~~columns~~ rows of A are linearly independent)

(2) $\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 4 & 3 & 5 & -1 & -1 \\ a & 1 & 3 & b & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -4 & 2 \\ 0 & 1 & -1 & 5 & -3 \\ 0 & 0 & 4-2a & 4a+b-5 & 4-2a \end{bmatrix}$

$\text{rank}(A) = 2. \quad \Rightarrow \quad 4-2a = 0 \quad 4a+b-5 = 0 \quad \Rightarrow \quad a = 2, \ b = -3.$

Complete solution :

$$x = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 4 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}.$$

4. (1) $A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & a^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(2) $A^{-1} = \begin{bmatrix} 1 & -\frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \\ 0 & -a & 1 & 0 \\ 0 & -a^2 & 0 & 1 \end{bmatrix}$

(3) $x = \begin{bmatrix} 0 \\ \frac{1}{a} \\ 0 \\ 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) $N(A)$'s basis: $\left\{ \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) $C(A^T)$'s basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right\}$

(c) $C(A)$'s basis: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \right\}$

(d) $\begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = 9 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

6. (a) $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ (b) L's basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) Projection : $\begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix}$.

7. (a) $\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ 0 & -4 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow x_n = \begin{bmatrix} +\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \quad x_p = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \Rightarrow \xi_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + c \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}, \quad c \in \mathbb{R}.$

$A^2 = \begin{bmatrix} 2 & 2 & 0 \\ -2 & -2 & 0 \\ 4 & 4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A^2 & b \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & -1 \\ -2 & -2 & 0 & 1 \\ 4 & 4 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \xi_3 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}.$

(b) $c_1 \xi_1 + c_2 \xi_2 + c_3 \xi_3 = 0$

$A\xi_1 = 0 \qquad A^2(c_1\xi_1 + c_2\xi_2 + c_3\xi_3) = A^2 0$

$A\xi_2 = \xi_1 \qquad \Rightarrow c_3 = 0 \qquad A(c_1\xi_1 + c_2\xi_2) = 0$

$\Rightarrow c_2 = 0 \Rightarrow$

$A\xi_3 = \xi_2 \qquad c_1 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

ξ_1, ξ_2, ξ_3 are linearly independent.

8. (a) $A^{-1} = I_n + \cancel{\frac{uv^T}{1 - v^T u}}$

(b) $(I_n - UV^T)^{-1} = I_n + U\underline{\underline{(I_m - V^T U)^{-1}}}V^T.$

Assume $I_m - V^T U$ is invertible.

Midterm Copy 2.

Suggested Solutions.

1. A C D D C

2. (1) $\begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix}$ (2) 1 (3) $\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$ (4) 12 (5) $\begin{bmatrix} 4 \\ -5 \end{bmatrix}$

3. $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 0 & 0 & 5/2 \end{bmatrix}$

4. $C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$N(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$

$C(A^T) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

$N(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

5. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

7. (a) $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix}$, $A^{2020} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & -5 & -7 \\ 0 & 2 & -2 \\ 1 & 3 & 3 \end{bmatrix}$ 每一列可替换为它的任意常数倍.

8. (a) $A^3 = 0 \Rightarrow \left.\begin{matrix} C(A^2) \subseteq N(A) \\ \dim N(A) = 1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \dim C(A^2) = \text{rank}(A^2) \leq 1 \\ \text{If } A^2 = 0 \text{ then } C(A) \subseteq N(A) \\ \Rightarrow \text{rank}(A) \leq 1 \end{matrix}\right\} A^2 \neq 0$

$\Rightarrow \text{rank}(A^2) = 1$.

(b) $C(A^2) \subset N(A)$ $\quad N(A) = \text{span}(\alpha_1)$ $\quad A^2\alpha_3 = A(\underbrace{A\alpha_3}_{\alpha_2}) = \alpha_1$

$\underline{A^2\alpha_3 = \alpha_1}$. existence. $\quad C(A^2) = \text{span}(\alpha_1)$ $\quad A\alpha_2 = \alpha_1$.

(c) $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$ 用A^2乘 $\Rightarrow c_1 = c_2 = c_3 = 0$

Midterm Copy 3:

Suggested Solutions:

1. (a) True (b) True (c) ~~True~~ False (d) False (f) True

2. (a) m (b) $3, 1$ (c) $\begin{bmatrix} 9/7 \\ 6/7 \\ 3/7 \end{bmatrix}$

3. (i) $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $A^{-1} = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

4. (a) Row space : $\left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$ dimension : 2

Column space : $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$ dimension : 2

Null space : $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ dimension: 2

left nullspace : $\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$ dimension: 1

(b) $b_3 - 2b_2 + 5b_1 = 0$

(c) $x = x_p + x_n = \begin{bmatrix} -1 \\ 0 \\ 1/3 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$

5. $Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{66}} & \frac{-3}{\sqrt{11}} \\ \frac{1}{\sqrt{6}} & \frac{7}{\sqrt{66}} & \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{6}} & \frac{-4}{\sqrt{66}} & \frac{1}{\sqrt{11}} \end{bmatrix}$

6. (i) $q_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad q_2 = \begin{bmatrix} -6/\sqrt{90} \\ -2/\sqrt{90} \\ 1/\sqrt{90} \\ 7/\sqrt{90} \end{bmatrix}$

(ii) $A = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix}$

$= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}.$

(iii) $\hat{x} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}.$

Midterm Copy 3.

7. (i) By definition.

(ii) $\begin{bmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$

8. (i) $\dim W = 3$.

(ii) Three independent vectors.

\# of independent vectors $= \dim W$.

(iii) One possible choice: u_1, u_3, u_5.

9. (i) Check the definitions: Columnspace, rank.

(ii) $C(B) \subseteq N(A) \Rightarrow \underset{\text{rank}(B)}{\dim C(B)} \leq \underset{n - \text{rank}\, A}{\dim N(A)}$ ✓

$$\text{rank}(B) \leq n - \text{rank}\, A$$

(iii) $N(A^TA) = N(A)$ (prove this!)

$\Rightarrow \text{rank}(A^TA) = \underset{\sim}{\text{rank}(A)}$

$A_{m\times n}$

If $\text{rank}(A) = n$, A^TA is invertible $\Rightarrow$ projection matrix is invertible

If $m > n$, $\text{rank}(A) < m$, Projection Matrix is NOT invertible.