Problem Set 4 —— Linear Algebra (Spring 2024)

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1. Suppose R is m by n of rank r, with pivot columns first:

$$R = \begin{bmatrix} I & F \\ O & O \end{bmatrix}$$

   (a) What are the shapes of those four blocks?

   (b) Find a right-inverse B with $RB = I$ if $r = m$. The zero blocks are gone.

   (c) Find a left-inverse C with $CR = I$ if $r = n$. The F and O column is gone.

   (d) What is the reduced row echelon form of R^T (with shapes)?

   (e) What is the reduced row echelon form of $R^T R$ (with shapes)?

2. Suppose you know that the 3×4 matrix A has the vector $\mathbf{s} = (2, 3, 1, 0)$ as the only special solution to $A\mathbf{x} = \mathbf{0}$.

   (a) What is the rank of A and the complete solution to $A\mathbf{x} = \mathbf{0}$.

   (b) What is the exact row reduced echelon form R of A?

   (c) How do you know that $A\mathbf{x} = \mathbf{b}$ can be solved for all $\mathbf{b}$?

3. Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is is true that A equals C?

4. Which invertible matrices allow $A = LU$ (elimination without row exchanges)?

5. Suppose $Q^T = Q^{-1}$ (transpose equals inverse, so $Q^T Q = I$).

   (a) Show that the columns $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n$ are unit vectors: $||\mathbf{q}_i||^2 = 1$.

   (b) Find a 2×2 example with first entry $q_{11} = \cos\theta$.