

Answer to Midterm Exam

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Problem 1. Multiple Choice.

Solution. 1.A 2.C 3.D 4.D 5.C

Problem 2.

$$1. \begin{bmatrix} 0 & B & 1 \\ A & 1 & 0 \end{bmatrix} \quad 2. 1 \quad 3. \begin{bmatrix} 1 & & & \\ & a & & \\ ac & b & 1 & \\ & & c & 1 \end{bmatrix} \quad 4. 12 \quad 5. \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Problem 3. Find the LU decomposition of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

Solution. We use elementary matrix to do Gaussian elimination to A .

$$\begin{aligned} \bullet \quad A \rightarrow E_{12}A &= \begin{bmatrix} 1 & & \\ 1/3 & 1 & \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 1 & 1 & 3 \end{bmatrix} = A_2 \\ \bullet \quad A_2 \rightarrow E_{13}A_2 &= \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{bmatrix} = A_3 \\ \bullet \quad A_3 \rightarrow E_{23}A_3 &= \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{bmatrix} = U \end{aligned}$$

$$\text{So we have } E_{23}E_{13}E_{12}A = U \Rightarrow A = E_{12}^{-1}E_{13}^{-1}E_{23}^{-1}U = LU = \begin{bmatrix} 1 & & \\ 1/3 & 1 & \\ 1/3 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{bmatrix}.$$

Remark: 1. Do not directly write the answer! You should write the process like the answer above.

2. If you didn't know the definition or computation of LU decomposition in midterm, you're near the end. I hope you to take a series of self-help steps to save yourselves.

Problem 4. Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. Find the bases of four fundamental subspaces.

Solution. Notice that all basis vectors are **column** vectors, not **row** vectors.

$$\begin{aligned} \bullet \quad \text{Column space } C(A) \text{ has basis: } &\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}. \\ \bullet \quad \text{Row space } C(A^T) \text{ has basis: } &\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} \right\}. \\ \bullet \quad \text{Nullspace } N(A) \text{ has basis: } &\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}. \\ \bullet \quad \text{Left nullspace } N(A^T) \text{ has basis: } &\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

Remark: Some of you write $N(A^T)$ in \mathbb{R}^5 not in \mathbb{R}^3 . Incredible! I hope you to read the textbook from the beginning of chapter 2 to save yourselves.

Problem 5. $E = \{u_1 = (1, 0, -1)^T, u_2 = (1, 2, 1)^T, u_3 = (-1, 1, 1)^T\}$; $F = \{v_1 = (1, -1)^T, v_2 = (2, -1)^T\}$

Solution. Suppose

$$\begin{cases} T(u_1) = a_{11}v_1 + a_{21}v_2 \\ T(u_2) = a_{12}v_1 + a_{22}v_2 \\ T(u_3) = a_{13}v_1 + a_{23}v_2 \end{cases} \quad (1)$$

By definition, the matrix representation of T under bases $E = \{u_1, u_2, u_3\}$, $F = \{v_1, v_2\}$ is $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$.

So we only need to find the combination coefficients of (1). We can write the three equations of (1) as matrix form:

$$[T(u_1) \ T(u_2) \ T(u_3)] = [v_1 \ v_2]A \Leftrightarrow \begin{bmatrix} 0 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}A$$

We can do Gaussian elimination to find A :

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 4 & 2 \\ 0 & 1 & 1 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 3 & 3 \end{bmatrix}$$

$$\text{So } A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 3 \end{bmatrix}.$$

Problem 6. Let A, B be n by n matrices. Suppose A and B are both symmetric. Is AB necessarily symmetric?

Solution. No, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ is not symmetric. This is a counter-example.

Remark: When you want to give an example, you should firstly consider the 2 by 2 matrices, not higher orders. Because the computation is simplest for 2 by 2 matrices.

Problem 7.

a) A is the 2 by 2 rotation matrix of rotating $\frac{\pi}{3}$. Find A and A^{2020} .

b) Three planes $\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3$ in the space \mathbb{R}^3 are given by the equations:

$$\begin{cases} \mathbb{I}_1: x + y + z = 0 \\ \mathbb{I}_2: 2x - y + 4z = 0 \\ \mathbb{I}_3: x + 2y - z = 0 \end{cases}$$

Determine a matrix representation (in the standard basis of \mathbb{R}^3) of a linear transformation taking the xy plane to \mathbb{I}_1 , the yz plane to \mathbb{I}_2 , and the zx plane to \mathbb{I}_3 .

Solution.

a) We know that the 2 by 2 rotation matrix is $R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$. So $A = R\left(\frac{\pi}{3}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. As for A^{2020} , we notice that $R(\theta) = R(\theta + 2k\pi)$, $k \in \mathbb{Z}^+$ (Positive integer). So

$$A^{2020} = R\left(\frac{\pi}{3} \times 2020\right) = R\left(\frac{4}{3}\pi\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b) **Method 1:** We want to find the matrix representation A under standard basis of \mathbb{R}^3 . So we only need to compute Ae_1, Ae_2, Ae_3 , where $\{e_1, e_2, e_3\}$ is the standard basis. The question is we don't know A , how to compute Ae_i ?

Consider the plane transformation. We notice that e_1, e_2 are in the xy plane, and A transform xy plane to \mathbb{I}_1 plane. This means that each vector in xy plane should in \mathbb{I}_1 plane after transformation. So Ae_1, Ae_2 should in \mathbb{I}_1 plane. Similarly, we know that Ae_2, Ae_3 are in \mathbb{I}_2 plane, and Ae_1, Ae_3 are in \mathbb{I}_3 plane. So we have

$$Ae_1 \in \mathbb{I}_1 \cap \mathbb{I}_3, Ae_2 \in \mathbb{I}_1 \cap \mathbb{I}_2, Ae_3 \in \mathbb{I}_2 \cap \mathbb{I}_3$$

Thus, we know that Ae_i is in the intersection of two planes. Find the 3 intersection lines of the 3 planes:

$$\mathbb{I}_1 \cap \mathbb{I}_3: \begin{cases} x+y+z=0 \\ x+2y-z=0 \end{cases}, \mathbb{I}_1 \cap \mathbb{I}_2: \begin{cases} x+y+z=0 \\ 2x-y+4z=0 \end{cases}, \mathbb{I}_2 \cap \mathbb{I}_3: \begin{cases} 2x-y+4z=0 \\ x+2y-z=0 \end{cases}$$

Notice that we only determine the directions of Ae_i , can not determine the norm(length) of Ae_i . So A is not unique. We choose

$$Ae_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, Ae_2 = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, Ae_3 = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$$

This implies that $A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix}$.

Remark: 1. Some of students do this question by finding normal vector of each plane. The normal vector of xy plane is $n_1 = (0, 0, 1)^T$, normal vector of \mathbb{I}_1 plane is $n_2 = (1, 1, 1)^T$, then they have $An_1 = n_2$. They find 3 equations between 6 normal vectors to find A . But this is incorrect, since A is not a rotation matrix in \mathbb{R}^3 , so the angle will be changed after transformation. Right angle will be not right after transformation. You can image a simple case to figure out this: T is a transformation in \mathbb{R}^2 .

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \end{bmatrix}$$

T is a transformation which transforms x axis to the line $x-y=0$ and leaves y axis unchanged. If you want to find the matrix representation A of T under standard basis by finding the normal vectors of each lines. We have

$$\begin{cases} \hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is the normal vector of } x \text{ axis, } v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is the normal vector of the line: } x-y=0 \\ \hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is the normal vector of } y \text{ axis, normal vector is unchanged since } y \text{ axis is unchanged} \end{cases}$$

But we notice that $T(\hat{y}) \neq v, T(\hat{x}) \neq \hat{x}$. This means that transformation T don't transform the normal vector of x axis to normal vector of the line: $x-y=0$, although T transform x axis to the line: $x-y=0$.

2. Actually, only when T is a conformal transformation(保角变换), the angle leaves unchanged after transformation. In \mathbb{R}^n , there are two special conformal transformation: rotation and reflection. These two kinds of matrices both are orthogonal matrices. You can verify that $(x, y) = (Ox, Oy)$ for any orthogonal matrix O and vectors x, y . Since the inner product of two vectors is unchanged after transformation, the angle between them is unchanged.

Method 2: We can directly compute A , not compute from Ae_1, Ae_2, Ae_3 .

Let v is a vector in xy plane, we know that Av must be a vector in plane \mathbb{I}_1 . This means that

$$\forall v \text{ with } \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} v = 0 \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} v = 0 \quad \text{s.t.} \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} Av = 0$$

The first equation $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} v = 0$ means that v is in xy plane, the second equation $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} Av = 0$ means that Av is in the plane \mathbb{I}_1 . Then we have the following result

Claim: $k \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A, k \in \mathbb{R}$

Proof. Any vector v in xy plane satisfies $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} v = 0$, this means that $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is orthogonal to xy plane. And we know that Any vector v in xy plane also satisfies $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} Av = 0$, this means that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A$ is orthogonal to xy plane. In other words, $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A$ are both in orthogonal complement of xy plane, we know that the orthogonal complement has dimension 1. So they two vectors are linearly dependent. \square

Let $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A$ (we choose the simplest case $k=1$, they are same)

Similarly, we have $[1 \ 0 \ 0] = [2 \ 1 \ 4]A$ and $[0 \ 1 \ 0] = [1 \ 2 \ 1]A$. Write the three equations together, we have

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} A$$

We can compute A by Gaussian elimination:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 4 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 2 & 1 & 0 & 2 \\ 0 & 3 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{5}{6} & \frac{7}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ So } A = \begin{bmatrix} \frac{1}{2} & \frac{5}{6} & \frac{7}{6} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Since A is not unique, the columns of A can multiple any constants. We multiple -2 to column 1, -6 to column 2, and 6 to column 3. We have

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

This is the same result as we get in method 1.

Problem 8. Let $A \in \mathbb{R}^{3 \times 3}$ with $r(A) = 2$, and $A^3 = 0$.

- Prove that $r(A^2) = 1$.
- Let $\alpha_1 \in \mathbb{R}^3$ is a vector s.t. $A\alpha_1 = 0$. Prove that there exist vectors α_2 and α_3 s.t. $A\alpha_2 = \alpha_1, A^2\alpha_3 = \alpha_2$.
- Prove that $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

Proof.

- From rank theorem: $AB = 0 \Rightarrow r(A) + r(B) \leq n$, we know that

$$r(A) + r(A^2) \leq 3 \text{ since } A(A^2) = 0$$

So we have $r(A^2) \leq 1$. Next, we need to prove $r(A^2) \neq 0$.

Argue by contradiction, suppose $r(A^2) = 0$, we have $A^2 = 0$. Using rank theorem again, then $r(A) + r(A) \leq 3$. But we already know that $r(A) = 2$. Contradiction!

So $r(A^2) = 1$.

- We only need to prove the two equations $A\alpha_2 = \alpha_1$ and $A^2\alpha_3 = \alpha_2$ has solution, namely, $\alpha_1 \in C(A)$ and $\alpha_1 \in C(A^2)$

- $A^3 = A^2(A) = 0 \Rightarrow C(A) \subset N(A^2)$. And we know that

$$\dim C(A) = r(A) = 2, \dim N(A^2) = 3 \quad \dim C(A^2) = 3 \quad r(A^2) = 2$$

So $C(A) = N(A^2)$, then $\alpha_1 \in N(A) \subset N(A^2) = C(A)$.

- $A^3 = A(A^2) = 0 \Rightarrow C(A^2) \subset N(A)$. And we know that

$$\dim C(A^2) = r(A^2) = 1, \dim N(A) = 3 \quad \dim C(A) = 3 \quad r(A) = 1$$

So $C(A^2) = N(A)$, then $\alpha_1 \in N(A) = C(A^2)$.

- Suppose $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$, then

$$A^2(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3) = c_1A^2\alpha_1 + c_2A^2\alpha_2 + c_3A^2\alpha_3 = c_3\alpha_1 = 0 \Rightarrow c_3 = 0 \text{ since } \alpha_1 \neq 0$$

Now we have $c_1\alpha_1 + c_2\alpha_2 = 0$, then

$$A(c_1\alpha_1 + c_2\alpha_2) = c_1A\alpha_1 + c_2A\alpha_2 = c_2\alpha_1 = 0 \Rightarrow c_2 = 0 \text{ since } \alpha_1 \neq 0$$

Now we have $c_1\alpha_1 = 0 \Rightarrow c_1 = 0$ since $\alpha_1 \neq 0$. All c_i 's are zero, so $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. \square