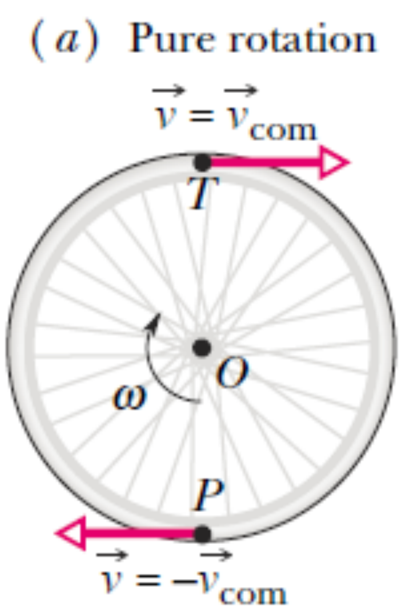
Lecture 11

Rolling, Torque, and Angular Momentum, Part I

Rolling as Translation and Rotation Combined



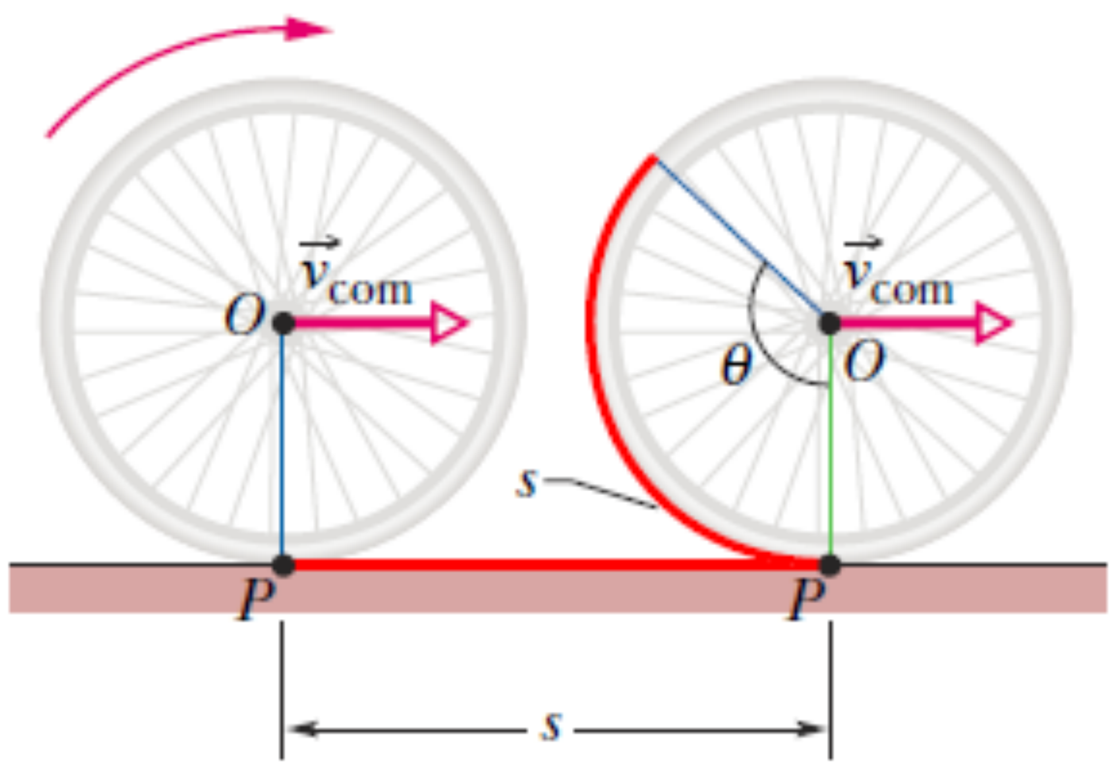
滚动是平动与转动的叠加

(b.) pure translation
$\vec{v} = \vec{v}_{com}$

A uniform round body (wheel, disk, ball, ring, cylinder, sphere, …) rolls on a ground surface. The rolling motion of the body is a combination of the translation of its center (center of mass) and the pure rotation of the body around an axis through its center of mass.

Smooth Rolling (Rolling Without Slipping)

In a smooth rolling, there is no relative motion of the contact point(s) ***P*** of the body with respect to the stationary surface.



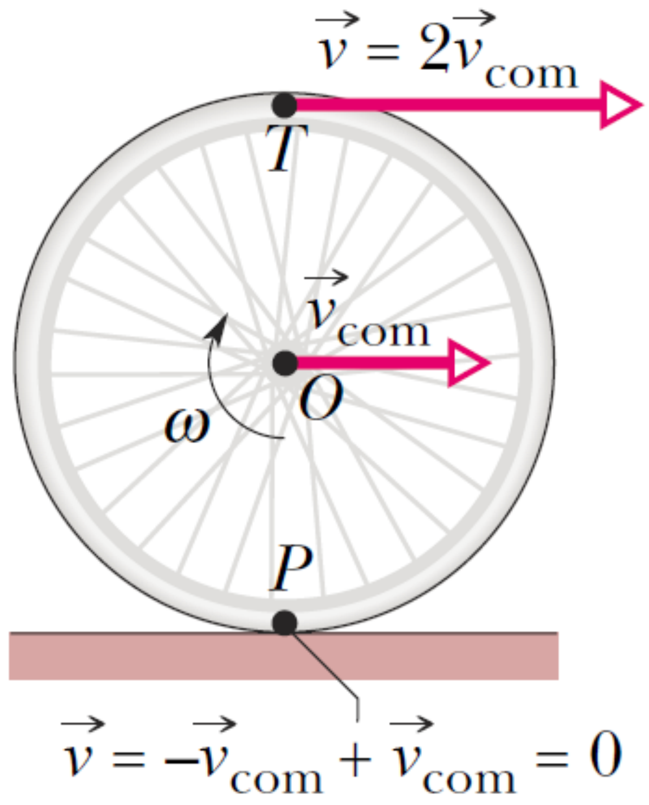
Distance:

$$\boxed{s = \theta R}$$

$V_{com} = \frac{ds}{dt} = \omega R$

Smooth Rolling (Rolling Without Slipping)

不打滑



The contact point P is stationary

$$\vec{v}_p = \vec{v}_{com} + \vec{v}_t = 0$$

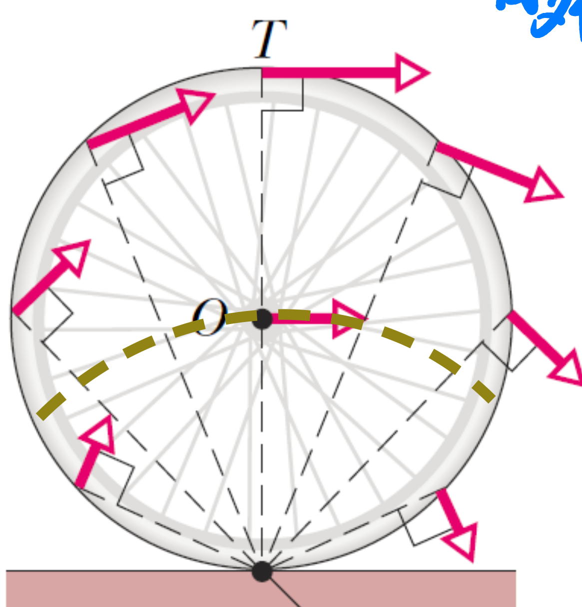
$$v_{com} - R\omega = 0$$

$$\boxed{v_{com} = \omega R \qquad \text{(smooth rolling motion)}}$$

Smooth Rolling as a Pure Rotation

- Smooth rolling of a round body at any moment may be viewed as a pure rotation about an axis passing through the contact point P.

一瞬间是绕P转动

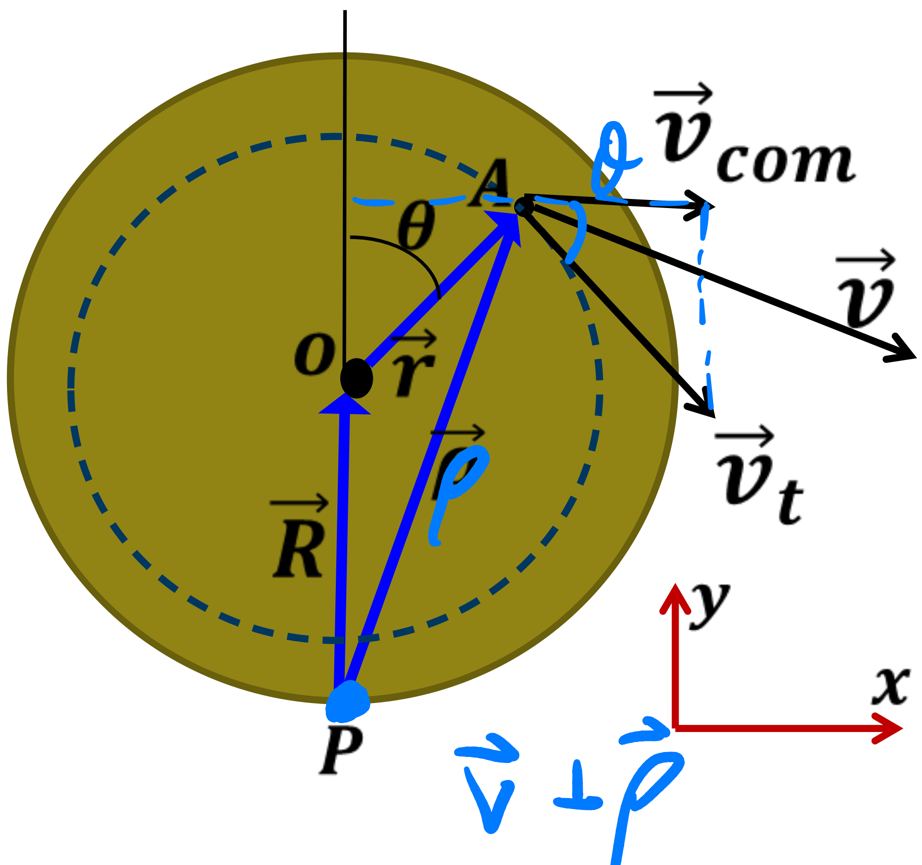


Rotation axis at P

Smooth Rolling as a Pure Rotation

The angular speed about the rotation axis at P is also ω.

Proof:



$$\vec{v} = \vec{v}_{com} + \vec{v}_t$$
$$= (v_{com} + v_t \cos\theta)\,\hat{i} + (-v_t \sin\theta)\,\hat{j}$$

$$v = \sqrt{(v_{com} + v_t\cos\theta)^2 + (-v_t\sin\theta)^2}$$
$$= \sqrt{{v_{com}}^2 + {v_t}^2 + 2v_t v_{com}\cos\theta}$$

$v_{com} = R\omega$, $v_t = r\omega$, and
$\rho^2 = R^2 + r^2 + 2rR\cos\theta$

$$\boxed{\boldsymbol{v = \rho\omega}}$$

One can check $\vec{v} \cdot \vec{\rho} = 0$,
i.e., $\vec{v} \perp \vec{\rho}$.

Kinetic Energy of Smooth Rolling

若桌子没有f则物体滑下来

滚得快 — 转动惯量小

$$K = \frac{1}{2} I_p \omega^2$$ (Rotation about axis P)

$$I_p = I_{com} + M R^2$$ (Parallel – axis Theorem)

每时刻 f静不做功

$$K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

两项分配比例不同

☆ $$\boxed{K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} M v_{com}^2}$$

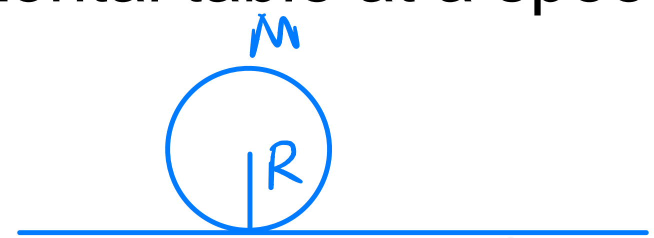
等效于绕O转动+平动

A rolling object has two types of kinetic energy: a rotational kinetic energy ($\frac{1}{2} I_{com} \omega^2$) due to its rotation about its center of mass and a translational kinetic energy ($\frac{1}{2} M v_{com}^2$) due to translation of its center of mass.

Sample Problem

A uniform solid cylindrical disk, of mass M = 1.4 kg and radius R = 8.5 cm, rolls smoothly across a horizontal table at a speed of $\boxed{\text{15 cm/s.}}$ **What is its kinetic energy *K*?**

$V=\omega R$



Solution:

Smooth rolling: $\omega = v_{com} / R$, and $v_{com} = 15\ cm/s$

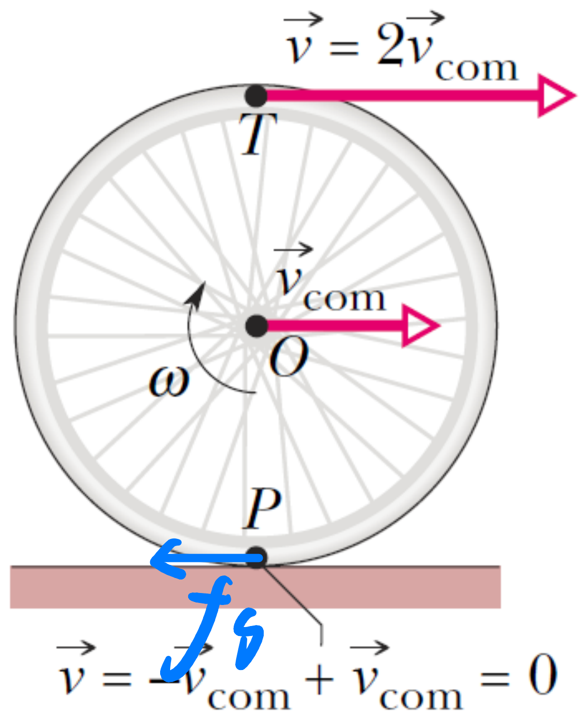
Given: $I_{com} = \frac{1}{2}MR^2$ for a disk,

$$K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2}M v_{com}^2$$

$$= (\tfrac{1}{2})(\tfrac{1}{2}MR^2)(v_{com}/R)^2 + \tfrac{1}{2}M v_{com}^2 = \tfrac{3}{4}M v_{com}^2$$

$$= \tfrac{3}{4}(1.4\ \text{kg})(0.15\ \text{m/s})^2 = 0.024\ \text{J} = 24\ \text{mJ}$$

Friction in Accelerating Smooth Rolling



If a body in smooth rolling accelerates:

$$v_{com} = \omega R$$

$$\frac{d\,v_{com}}{d\,t} = a_{com} = \frac{R\,d\omega}{d\,t}$$

$$\boxed{a_{com} = \alpha R \quad (\text{Smooth rolling motion})}$$

* 只存在 smooth rolling

前提：$s = \theta R$

$\boldsymbol{a_{com}}$ is the magnitude of linear acceleration of the center of mass,
$\boldsymbol{\alpha}$ is the magnitude of angular acceleration.

From the surface, a **static** friction is acted on the contact point *P* against **its tendency of motion**.

假设没有f

Friction in Accelerating Smooth Rolling

- From the surface, a **static** friction is acted on the contact point P of the rolling body against **its tendency of motion**.

- The value of the static frictional force f_s self-adjusts itself to keep the body roll without slipping.

- $f_s \leq f_{s,\max}$, where $f_{s,\max} = \mu_s N$.

f_s 不做功

- f_s does not transfer kinetic energy to thermal energy.

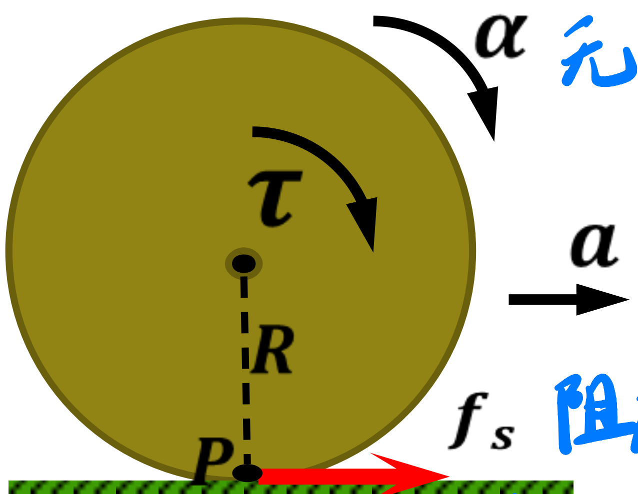
Accelerating a Rolling Wheel – Apply a Pure Torque

"Pure" means that the net force responsible for the torque is zero.

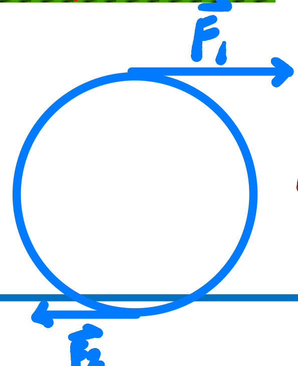
What is the direction of the static frictional force?

开始有V com

无 ω

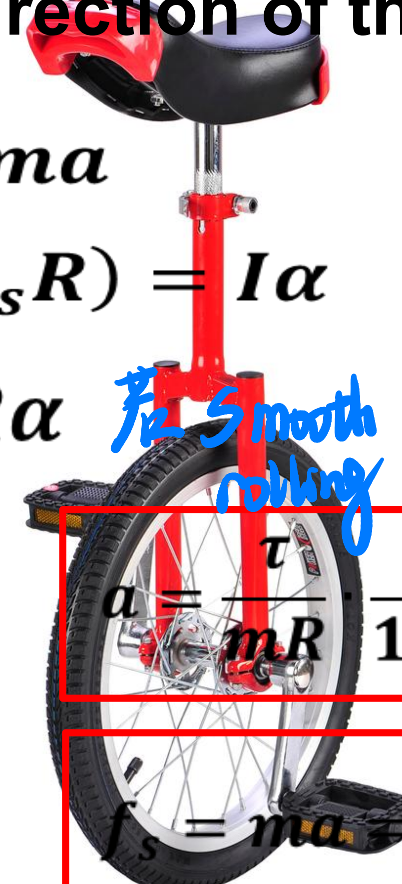


f_s 阻碍转动



$\vec{F}_{net} = 0$

$\vec{\tau}_{net} \neq 0$



$$f_s = ma \qquad \text{Translation}$$

$$(\tau - f_s R) = I\alpha \qquad \text{Rotation}$$

$$a = R\alpha \qquad \text{Rolling without slipping}$$

后 Smooth rolling

$\tau - maR = I\alpha$

$= \frac{Ia}{R}$

$\tau - maR$

$\tau R = a(I + mR^2)$

$a = \frac{\tau R}{I + mR^2}$

$$\boxed{a = \frac{\tau}{mR} \cdot \frac{1}{1+\gamma}}$$

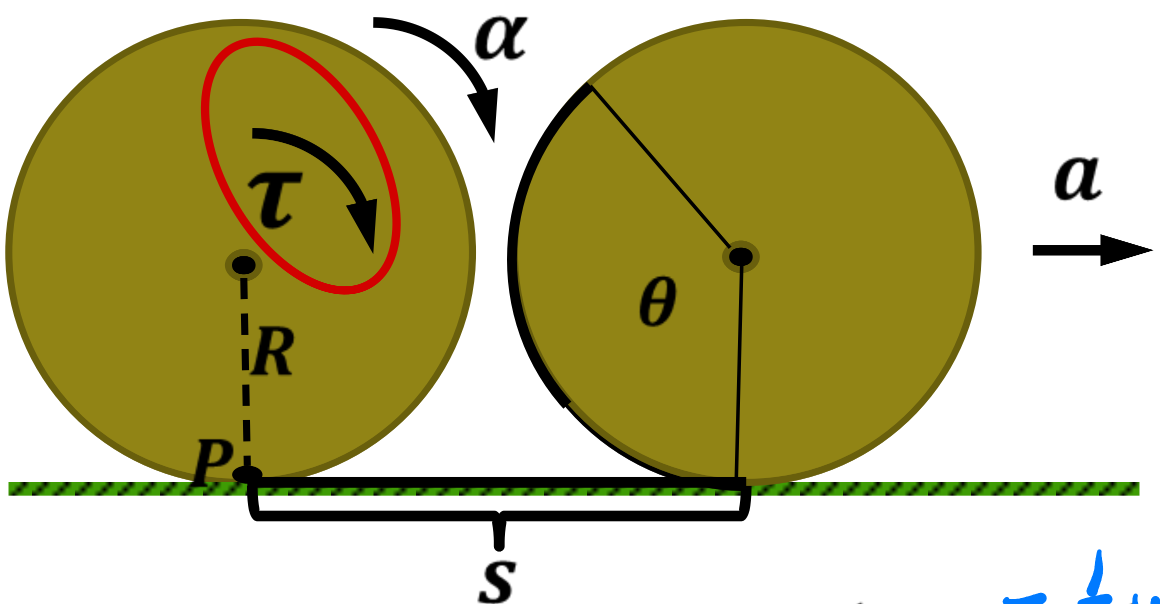
$$\boxed{f_s = ma = \frac{\tau}{R} \cdot \frac{1}{1+\gamma}}$$

Defining:

$$\boxed{\gamma = \frac{I}{mR^2}}$$

$\gamma = \frac{I}{mR^2}$

Accelerating a Rolling Wheel – Apply a Torque



Let: $v_0 = \omega_0 = 0$

Then: $v^2 = 2as$

$\omega^2 = 2\alpha\theta$

Smooth rolling:

$s = R\theta,\ a = R\alpha$

$= \frac{1}{2}\omega^2 mR^2(1+\gamma)$

力矩做功

K.E. of the body: $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mas(1+\gamma) = \tau\theta$

= Work done by applied torque.

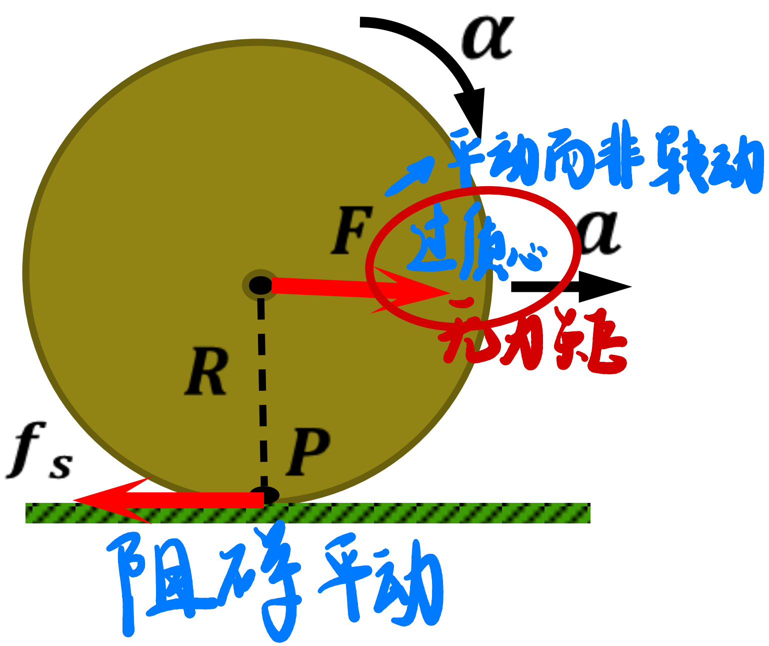
$E_k = m\frac{\tau}{mR}\frac{1}{1+\gamma}(1+\gamma)s$

$= \frac{\tau s}{R} = \tau\theta$

- Static friction f_s does not generate "heat". It converts part of rotational K.E. due to the torque to the translational K.E. If $f_s = 0$, the body only rotates without translation.

Accelerating a Rolling Wheel – Apply a Force

What is the direction of the static frictional force?



$$F - f_s = ma \qquad \text{Translation}$$

$$f_s R = I\alpha \qquad \text{Rotation}$$

$f_s = \frac{I\alpha}{R} = I\frac{a}{R^2}$

$F - I\frac{a}{R^2} = ma$

$a = \frac{F}{m + \frac{I}{R^2}}$

$= \frac{F}{m}\frac{1}{1+r}$

$$a = R\alpha \qquad \text{Rolling without slipping}$$

$f_s = \frac{Ia}{R^2} = \frac{I}{R^2}\frac{F}{m}\frac{1}{1+r} = F\frac{r}{1+r} < F$ 保证平运动

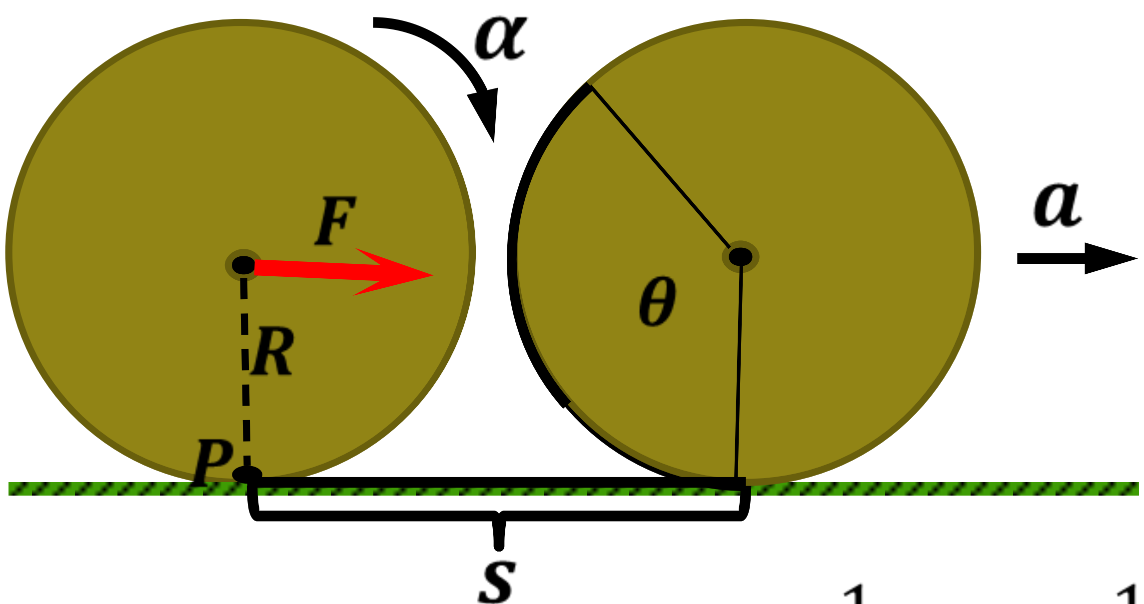
$$\boxed{a = \frac{F}{m} \cdot \frac{1}{1+\gamma}}$$

$$\boxed{f_s = \gamma ma = F\frac{\gamma}{1+\gamma}}$$

$F = ma + F\frac{r}{1+r}$

$F = ma(1+r)$

Accelerating a Rolling Wheel – Apply a Force



Let: $v_0 = \omega_0 = 0$

Then: $v^2 = 2as$

$\omega^2 = 2\alpha\theta$

Smooth rolling:

$\omega^2 = 2\alpha\theta$

$s = R\theta,\ a = R\alpha$

$F = ma(1+\gamma)$

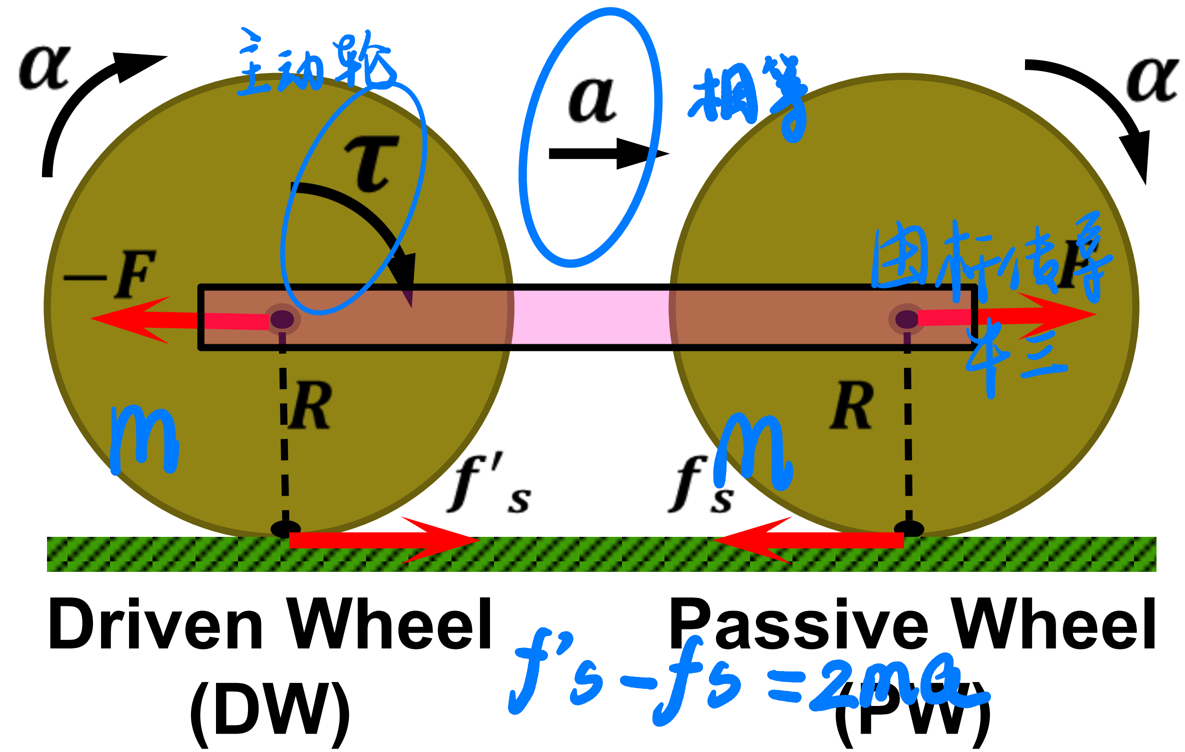
K.E. of the body: $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mas(1+\gamma) = FS$

mas $\quad I\alpha\theta = I\frac{a}{R}\frac{s}{R}$ $\quad FS = mas(1+\gamma)$

= Work done by applied force.

- Static friction f_s does not generate "heat". It converts part of translational K.E. due to the force to the rotational K.E. If $f_s = 0$, the body only slides without rotation.

Accelerating Connected Wheels – Apply a Torque



Are the magnitudes of the angular accelerations (linear accelerations at the centers of mass) of the two connected rolling wheels the same?

Yes! They are the same for two wheels.

Accelerating Connected Wheels – Apply a Torque

DW: $$\begin{cases} f'_s - F = ma \\ (\tau - f'_s R) = I\alpha \end{cases}$$

PW: $$\begin{cases} F - f_s = ma \\ f_s R = I\alpha \end{cases}$$

Rolling without slipping: $a = R\alpha$

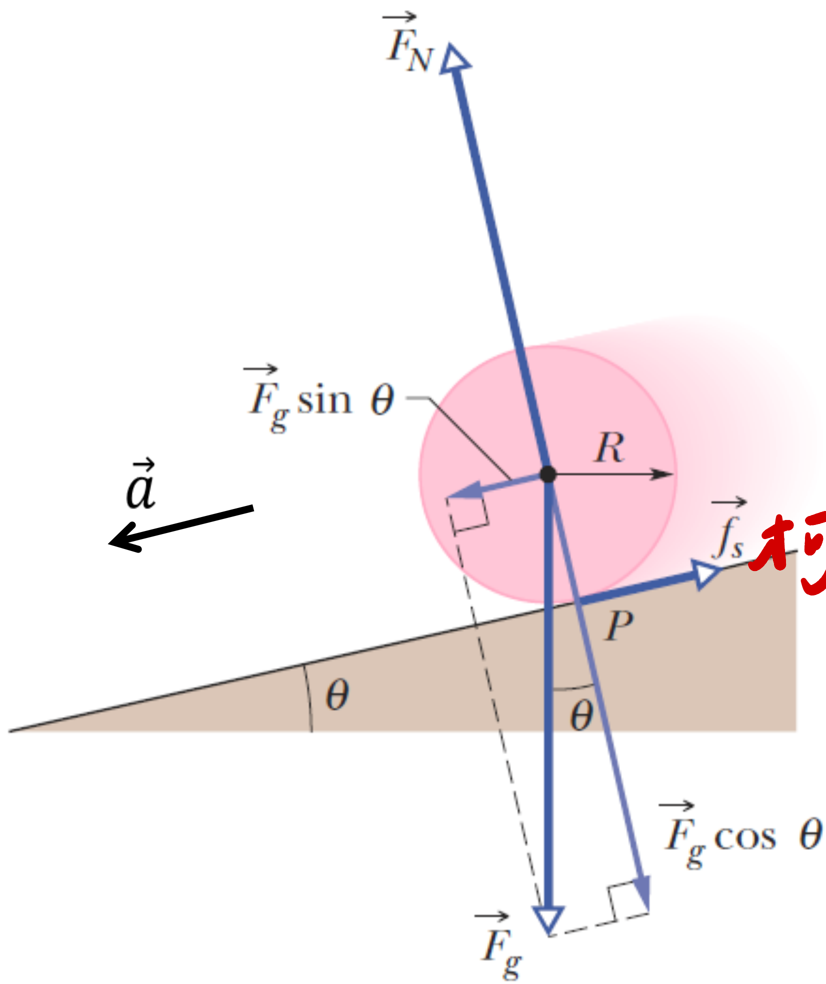
$$a = \frac{\tau}{2mR} \cdot \frac{1}{1+\gamma}$$

$$F = (1+\gamma)ma$$

$$f_s = \gamma ma$$

$$f'_s = (2+\gamma)ma$$

Smooth Rolling along a Ramp



才可提供 τ

- Static frictional force is up along ramp.
- Angular acceleration is counterclockwise.

$$\tau_{net} = Rf_s = I_{com}\alpha \qquad \text{(rotation)}$$

$$mg\sin\theta - f_s = ma \qquad \text{(translation)}$$

*Smooth rolling: $a = R\alpha$

$$\boxed{a_{com} = \frac{g\sin\theta}{1 + I_{com}/MR^2}}$$ down ramp

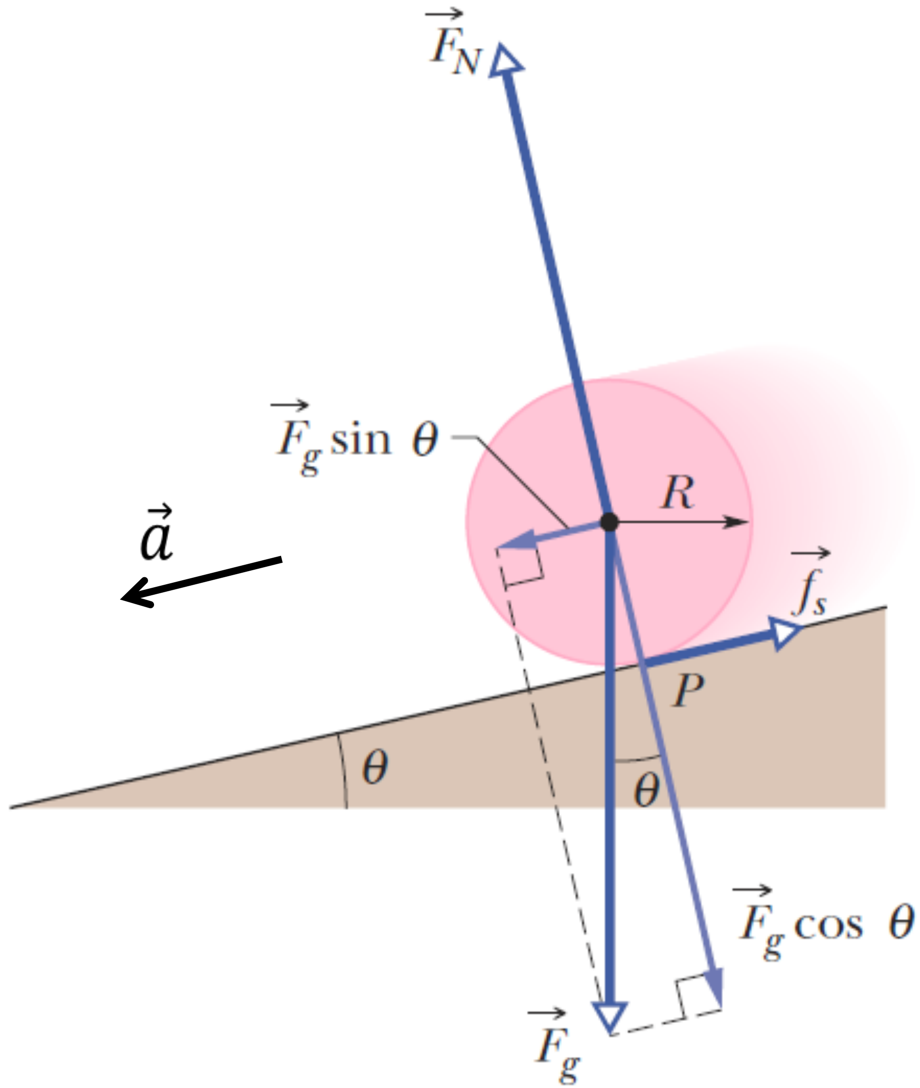
$$\frac{g\sin\theta}{1+\gamma}$$

$$\boxed{f_s = I_{com}\frac{a_{com}}{R^2} = \frac{\gamma}{1+\gamma}mg\sin\theta}$$

$$\gamma = \frac{I}{MR^2}$$

$$mg\sin\theta\,\frac{\gamma}{1+\gamma}$$

Sample Problem



A uniform ball, of mass M = 6.00 kg and radius R, rolls smoothly from rest down a ramp at angle $\boldsymbol{\theta}$= 30.0°. The rotation inertia of the ball around its center axis is in form of $I_{com} = \frac{2}{5}MR^2$.

(a) What are the magnitude and direction of the friction force on the ball as it rolls down the ramp?

Solution

$$a_{com} = \frac{g \sin\theta}{1 + I_{com} / MR^2} = \frac{g \sin\theta}{1 + \frac{2}{5} MR^2 / MR^2}$$

down along ramp

$$= \frac{(9.8\ m/s^2)\ \sin 30.0^\circ}{1 + \frac{2}{5}} = 3.50\ m/s^2$$

$$f_s = I_{com} \frac{a_{com}}{R^2} = \frac{2}{5} MR^2 \frac{a_{com}}{R^2} = \frac{2}{5}\ M a_{com}$$

up along ramp

$$= \frac{2}{5}\ (6.00\ kg)(3.50\ m/s^2) = 8.40\ N$$

Solution

(b) The ball descends a vertical height *h* =1.20 m to reach the bottom of the ramp. What is its speed at the bottom?

Using kinematics in translational motion:

$$S = h/\sin\theta, \quad 2a_{com}S = v_{com}{}^2 \Rightarrow v_{com} = \sqrt{\frac{10}{7}gh}$$

Using energy:

机械能守恒

分配

$$mgh = \frac{1}{2}I_{com}\omega^2 + \frac{1}{2}Mv_{com}^2$$

- System of ball + Earth.
- No work done by normal force. Static friction does not generate "heat". So system is isolated. Only gravitational force transfers energy inside system.
- So mechanical energy of the system is conserved.

Solution

$$K_f + U_f = K_i + U_i$$

$$\left(\tfrac{1}{2}I_{com}\omega^2 + \tfrac{1}{2}Mv_{com}^2\right) + 0 = 0 + Mgh$$

For smooth rolling: $v_{com} = R\omega$

$$v_{com} = \sqrt{\left(\tfrac{10}{7}\right)gh} = \sqrt{\left(\tfrac{10}{7}\right)(9.8\ m/s^2)(1.2\ m)}$$

$$= 4.1\ m/s$$

Non-smooth Rolling with Slipping

- When a round body rolls with slipping on a surface, there exists relative motion at contact. Therefore, the friction on the body is a **kinetic** frictional force, always against the relative motion of the contacting point on the body with respect to the surface.

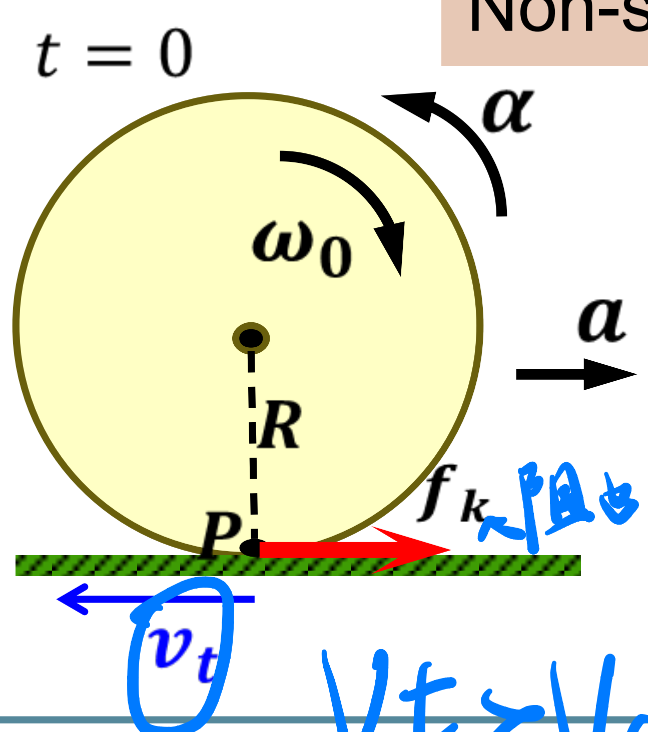
- The nonslip conditions do not hold for non-smooth rolling.

$$\cancel{S = R\theta}$$
$$\cancel{v_{com} = R\omega}$$
$$\cancel{a_{com} = I\alpha}$$

Sample Problem

A yo-yo, which has angular velocity ω_0 but zero speed of center of mass, at $t_0 = 0$ sec comes into contact with a flat surface with friction.

Non-smooth rolling



平动↗
转动↓

$V_t > V_{com}$ 但V合方向不变

Sample Problem

接触点 vp ≠0

- At contact, there is slipping motion ($\vec{v}_p$ is not zero). Then a kinetic frictional force f_k appears, pointing to right, which decrease angular speed but increase translational speed of center of mass.

- At point P, the tangential speed due to the rotation v_t (toward left) is larger than the speed of center of mass v_{com} (toward right) until a smooth rolling starts.

- when smooth rolling starts, at point P, $\boxed{v_t = v_{com}}$, i.e. $\vec{v}_p = 0$. There is no slipping, $f_k = 0, a = 0, \alpha = 0$. The yo-yo rolls smoothly with a constant angular velocity and a constant translational velocity if there is no external force or torque applied on it.

Solution

(a) When the yo-yo starts a smooth rolling motion, What is the speed of its center of mass and the angular speed?

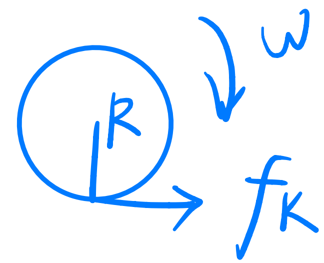
During non-smooth rolling

$f_k = ma$ translation

$\tau = f_k R = I\alpha$ rotation

$v_{com} = at$ 仅f_k提供 constant acceleration

$\omega = \omega_0 - \alpha t$



$\Rightarrow$ $\boxed{a = \gamma R\alpha}$

Defining:

$$\gamma = \frac{I}{mR^2}$$

When smooth rolling starts

$v_t = R\omega = v_{com}$ $\Rightarrow$ $\boxed{\omega = \frac{\gamma}{1+\gamma}\omega_0 \qquad v_{com} = \frac{\gamma}{1+\gamma}\omega_0 R}$

$$\left.\begin{aligned} f_k &= ma \\ f_k R &= I\alpha \end{aligned}\right\} \Rightarrow maR = I\alpha$$

$$a = \frac{I\alpha}{mR} = \frac{I}{mR^2} R\alpha = \gamma R\alpha$$

末速度 $V_{com} = at = \left\{\begin{aligned} &\gamma \alpha R t = \omega R \\ \omega = &\gamma \alpha t \end{aligned}\right.$

$$\omega = \omega_0 - \alpha t$$

$$\omega_0 = \omega + \alpha t = (\gamma + 1)\alpha t$$

$$\omega = \frac{\gamma}{\gamma + 1}\omega_0 \quad , \quad V_{com} = \frac{\gamma}{\gamma + 1}\omega_0 R$$

Solution

情况1：全转 → SR

(b) How much kinetic energy is left when the yo-yo starts the smooth rolling?

Initially: $K_i = \frac{1}{2} I \omega_0{}^2$

Smooth rolling:

$$K_f = \frac{1}{2} I \omega^2 + \frac{1}{2} m v_{com}{}^2 = \frac{1}{2} I \omega_0{}^2 \frac{\gamma}{1+\gamma}$$

$$\boxed{\frac{K_f}{K_i} = \frac{\gamma}{1+\gamma}}$$

$$\frac{W_f}{W_i} = \frac{r}{1+r}$$

- For a disk, $\gamma = \frac{1}{2}$. Only 1/3 K.E. is left when the smooth rolling starts. The “lost” K.E. is transferred to thermal energy due to the kinetic friction during sliding.

Lecture 12

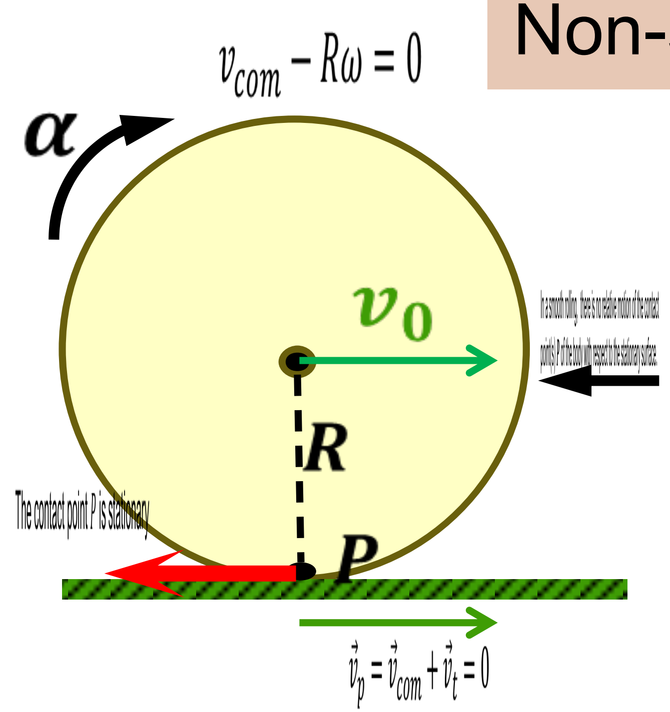
Rolling, Torque, and Angular Momentum, Part II

Sample Problem

情况2：全平 →SR

A soccer ball is kicked with $v_{com} = v_o$ and $\omega_0 = 0$. It slides over a flat surface with friction. The sliding creates a kinetic friction f_k , which decreases v_{com} and introduces a rotational motion.

Non-smooth rolling



Sample Problem

y

Solution

(a) When the ball starts a smooth rolling motion, What is the speed of its center of mass and the angular speed?

During non-smooth rolling

$f_k = ma$ (translational motion)
$\tau = f_k R = I\alpha$ (rotational motion)
$v_{com} = v_0 - at$
$\omega = \alpha t$

$\Rightarrow$ $\boxed{a = \gamma R\alpha}$

Defining:

$\boldsymbol{v = \rho\omega}$

When smooth rolling starts

$v_t = R\omega = v_{com}$ $\Rightarrow$ $\boxed{\omega = \dfrac{v_0/R}{1+\gamma} \qquad v_{com} = \dfrac{v_0}{1+\gamma}}$

Solution

(b) How much kinetic energy is left when the ball starts the smooth rolling?

Initially: $K_i = \frac{1}{2} m {v_0}^2$

Smooth rolling: $K_f = \frac{1}{2} I \omega^2 + \frac{1}{2} m {v_{com}}^2 = \frac{1}{2} m {v_0}^2 \frac{1}{1+\gamma}$

(Rotation about axis P)

- For a sphere, $\gamma = \frac{2}{5}$. Only 5/7 K.E. is left when the smooth rolling starts. The "lost" K.E. is transferred to thermal energy due to the kinetic friction during sliding.

Smooth vs. Non-smooth Rolling

- For a smooth rolling motion without external force or torque, there is no friction, $s = R\theta$, $a = \alpha = 0$, and at point P, $v_{com} = v_t = R\omega$.

- For a smooth rolling motion with external force or torque, $s = R\theta$ and at point P, $v_{com} = v_t = R\omega$. There exists a static friction f_s and $a \neq 0, \alpha \neq 0$ with $a = R\alpha$. The f_s does not generate “heat”.

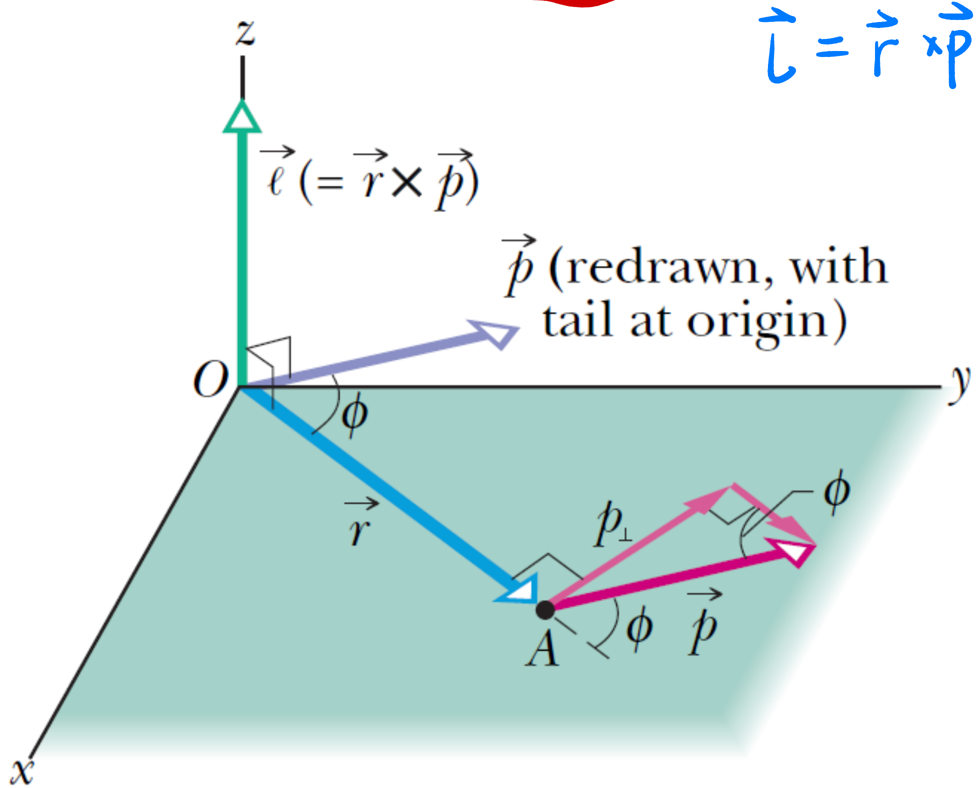
- For a non-smooth rolling motion with sliding, there exists a kinetic friction f_k and $a \neq 0, \alpha \neq 0$ with $a = \gamma R\alpha$. The f_k generates “heat”, reducing body’s kinetic energy.

Angular Momentum

Angular momentum of a particle about a point (origin):

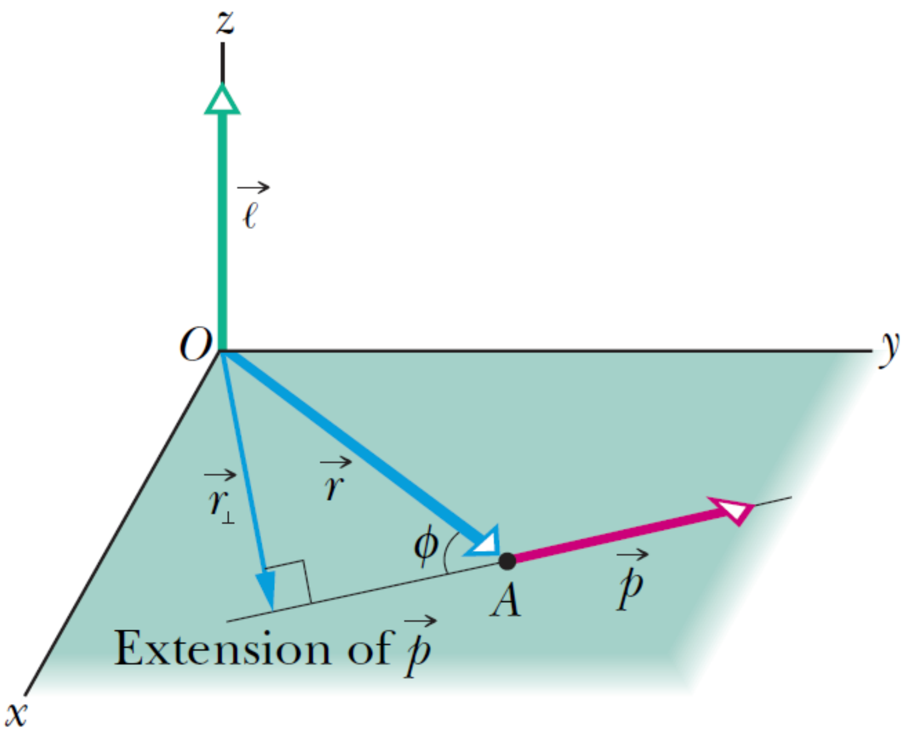
$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

$\vec{L} = \vec{r} \times \vec{p}$ 角动量



- Angular momentum is defined with respect to a point (origin).
- Angular momentum is a vector with SI units $\mathrm{kg \cdot m^2/s}$ or $\mathrm{J \cdot s}$.
- Direction of angular momentum is determined by right-hand rule (RHR).

Angular Momentum



- Magnitude of angular momentum:

$$\ell = rmv \sin\phi$$

$$\ell = rp_\perp = rmv_\perp$$

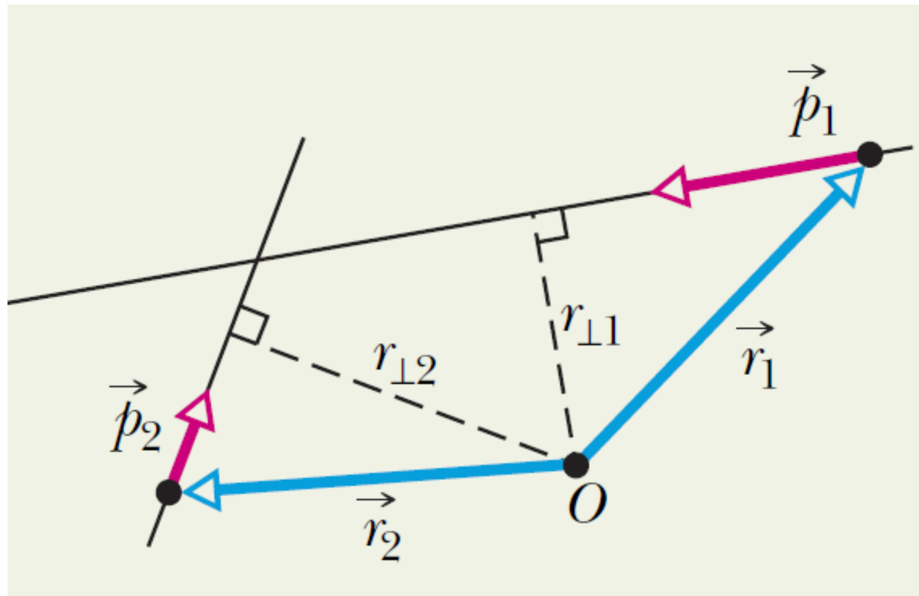
$$\ell = r_\perp p = r_\perp mv$$

对同一个参考点

Angular momentum of a system of particles about a point (origin):

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i$$

Sample Problem



$$(\tau - f_s R) = I\alpha$$

Solution

What is the angular momentum $\vec{L}$ about point O of the two-particle system?

Taking the "out of screen" direction as positive direction,

$$l_1 = r_{\perp 1}\, p_1 = (2.0\, m)(5.0\, kg \cdot m / s) = 10\ \mathrm{kg \cdot m^2 / s}$$

The RHR indicates that l_1 is positive.

$$l_1 = +10\, kg \cdot m^2 / s$$

$$l_2 = r_{\perp 2}\, p_2 = (4.0\, m)(2.0\, kg \cdot m / s) = 8.0\ \mathrm{kg \cdot m^2 / s}$$

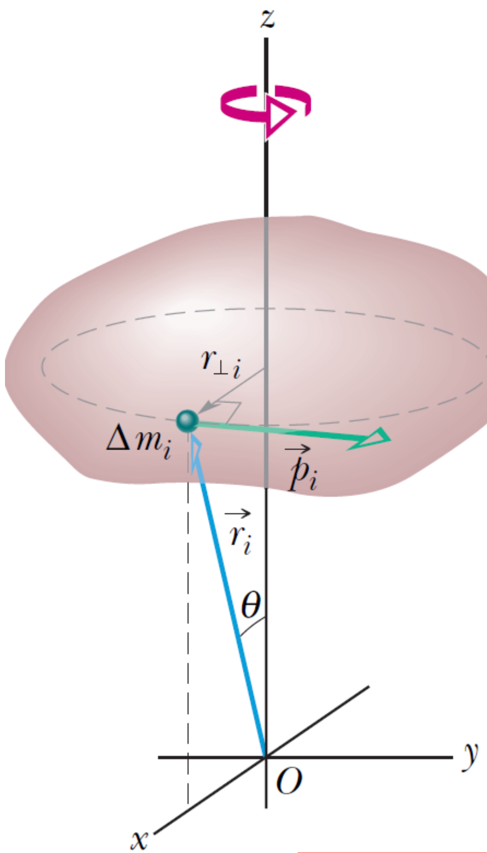
The RHR indicates that l_2 is negative.

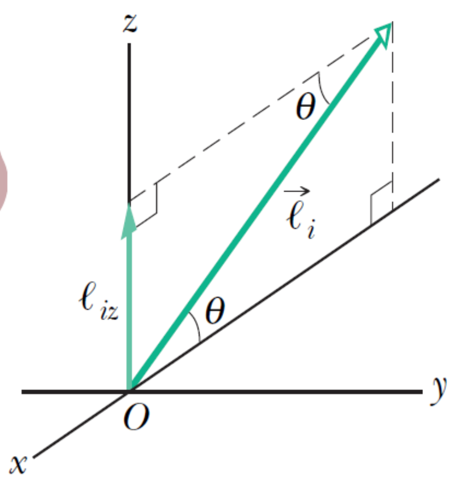
$$l_2 = -\,8.0\, kg \cdot m^2 / s$$

$$L = l_1 + l_2 = +10\, kg \cdot m^2 / s + (-8.0\, kg \cdot m^2 / s) = +2.0\ \mathrm{kg \cdot m^2 / s}$$

Angular Momentum of a Rigid Body about a Fixed Axis

Choosing origin at rotation axis, for a mass element Δm_i with momentum $\vec{p}_i$





$$l_i = (r_i)(p_i)(\sin 90^\circ) = (r_i)(\Delta m_i v_i)$$

$$l_{iz} = l_i \sin\theta = (r_i \sin\theta)(\Delta m_i v_i) = r_{\perp i} \Delta m_i v_i$$

Because $v = \omega r_\perp$, we get

$$L_z = \sum_{i=1}^{n} l_{iz} = \sum_{i=1}^{n} \Delta m_i v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp i}) r_{\perp i}$$

$$= \omega \left(\sum_{i=1}^{n} \Delta m_i r_{\perp i}^{2} \right) = I\omega$$

$\tau = I\alpha \quad L = I\omega$

$$\boxed{L = I\omega \qquad \text{(rigid body, fixed axis)}}$$

Newton's 2nd Law in Angular Form

For a single particle:

*叉乘求导

$$\frac{d\vec{\ell}}{dt} = \frac{d(\vec{r}\times\vec{p})}{dt} = m\overbrace{\frac{d\vec{r}}{dt}}^{\vec{v}}\times\vec{v} + \vec{r}\times\frac{d\vec{p}}{dt} = \cancel{m\vec{v}\times\vec{v}} + \overbrace{\vec{r}\times\vec{F}_{net}}^{\vec{\tau}_{net}}$$

$$\boxed{\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt} \qquad \text{(single particle)}}$$

力矩是改变角动量的原因

The torque $\vec{\tau}$ and angular momentum $\vec{l}$ must be defined with respect to the same origin.

Newton's 2nd Law in Angular Form

For a system of particles:

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{\ell}_i}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\text{net},i}$$

Sum of Internal torques are zero due to the newton's 3rd law paired forces.

$$\Rightarrow \boxed{\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \qquad \text{(system of particles)}}$$

$$\omega^2 = 2\alpha\theta$$

More corresponding Variables and Relations for Translational and Rotational Motion

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{\text{com}}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{\text{net}} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{\text{net}} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

[a]See also Table 10-3.
[b]For systems of particles, including rigid bodies.
[c]For a rigid body about a fixed axis, with L being the component along that axis.
[d]For a closed, isolated system.

Conservation of Angular Momentum

- If **no net external torque** acting on a system, the angular momentum $\vec{L}$ of the system remains constant, no matter what changes take place within the system.

$$\vec{L} = \text{a constant} \qquad \text{(isolated system)}$$

全是对同一点考虑

or

$$\boxed{\vec{L}_i = \vec{L}_f \qquad \text{(isolated system)}}$$

Conservation of Angular Momentum

$\tau \to L$

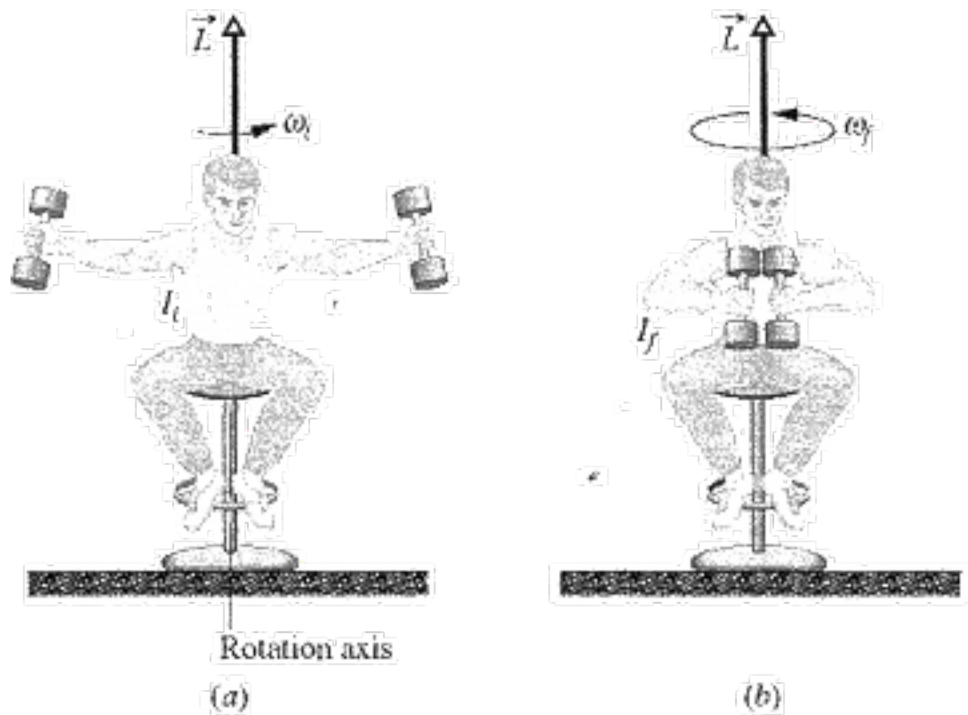
$F \to p$

- If the component of the net *external* torque on a system along a certain axis is zero, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

For a rigid body rotating about a fixed axis, the conservation law gives:

$$I_i \omega_i = I_f \omega_f$$

Change of Angular Speed



Without external torque on a system of chair + students + dumb bells, the system's angular momentum is constant. (a) has a bigger I than (b), therefore its angular speed is smaller than that of (b). Same skills used by figure skaters.

As the World Turns

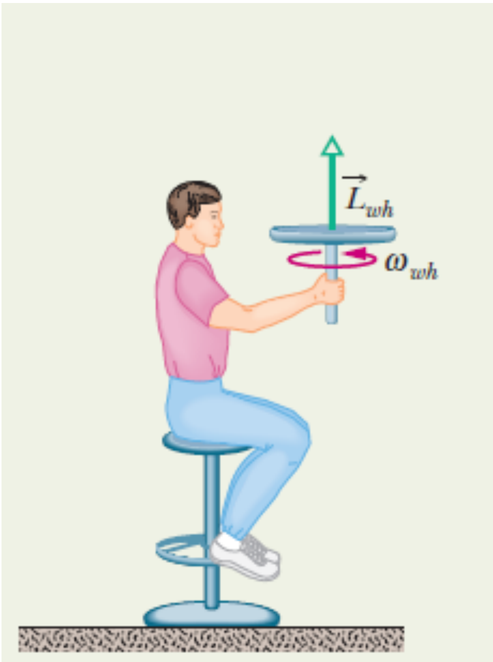
- The angular momentum of the entire Earth-atmosphere system is conserved. Both Earth's spin angular momentum and the total atmospheric angular momentum (AAM) are in the same direction – from west to east. They vary from time to time. When AAM speeds up, the angular speed of Earth itself slows down, and the length of day (LOD) is increased.

In 1984, measurements at Goddard Space Flight Center showed that the day had lengthened by over a millisecond during El Nino. In 1997, the day grew by four-tenths of a millisecond during the El Nino event.

A Shrinking Star

When the nuclear fire in the core of a star burns low, the star may eventually begin to collapse, building up pressure in its interior. The collapse may go so far as to reduce the radius of the star from something like that of the Sun to the incredibly small value of a few kilometers. The star then becomes a *neutron star* --- its material has been compressed to an incredibly dense gas of neutrons. During this shrinking process, the star is an isolated system and its angular momentum cannot change. Because its rotational inertia is greatly reduced, its angular speed is correspondingly greatly increased, to as much as 600 to 800 revolutions per *second.* For comparison, the Sun, a typical star, rotates at about one revolution per month.

Sample Problem

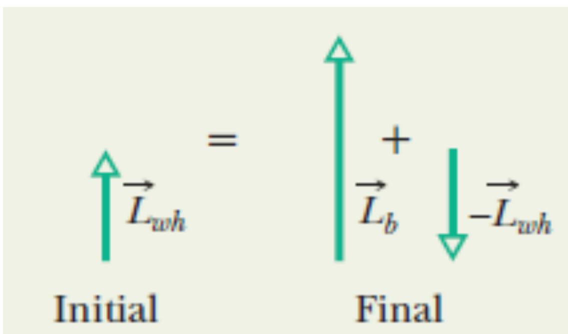


A student sits on a stool that can rotate freely (without friction) about a vertical axis. The student, initially at rest, is holding a bicycle wheel with rotational inertia about its central axis $I_{wh} = 1.2\ \mathrm{km \cdot m^2}$.The wheel is rotating at an angular speed ω_{wh} =3.9 rev/s; and the angular momentum $\vec{L}_{wh}$ of the wheel points vertically upward.

The student now inverts the wheel so that its angular momentum is then $-\vec{L}_{wh}$. The inversion results in the student, the stool, and the wheel's center rotating together as a composite rigid body with rotational inertial $I_b = 6.8\ \mathrm{km \cdot m^2}$ about the stool's rotation axis. What's the angular velocity $\vec{\omega}_b$ of the composite body right after inversion?

Solution

For the system of **student + stool + wheel**, no external torque on it during inverting wheel.



=0 due to initial rest

$$L_{b,f} + L_{wh,f} = L_{b,i} + L_{wh,i}$$

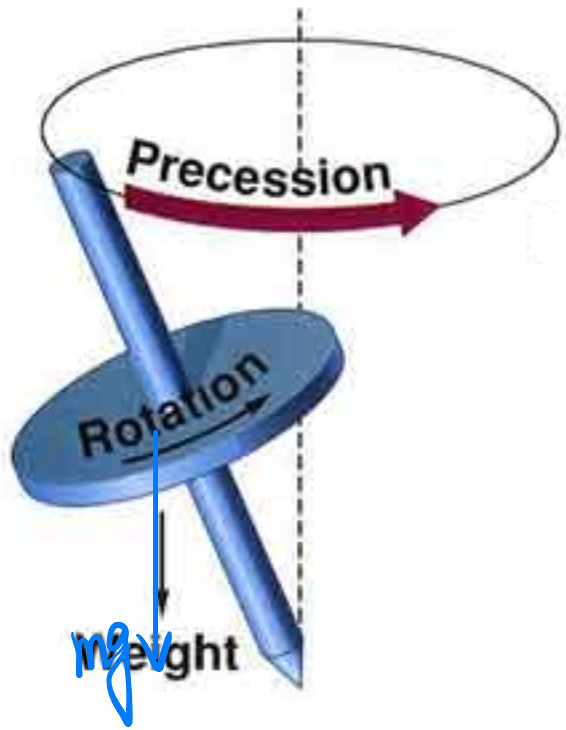
$$\alpha$$

$$v^2 = 2as$$

$$\omega_b = \frac{2 I_{wh}}{I_b} \omega_{wh} = \frac{(2)(1.2\ \mathrm{kg \cdot m^2})(3.9\ \mathrm{rev/s})}{6.8\ \mathrm{kg \cdot m^2}} = 1.4\ \mathrm{rev/s}$$

Counterclockwise seen from overhead.

Precession of a Gyroscope



A gyroscope is a round rigid body with a fixed shaft. When a gyroscope rapidly spins about its shaft, it begins rotate horizontally about a vertical axis through the support point of the shaft in a motion called **precession**.

进动

Precession of a Gyroscope

$\tau = mgr \qquad \frac{d\vec{L}}{dt} = \tau$

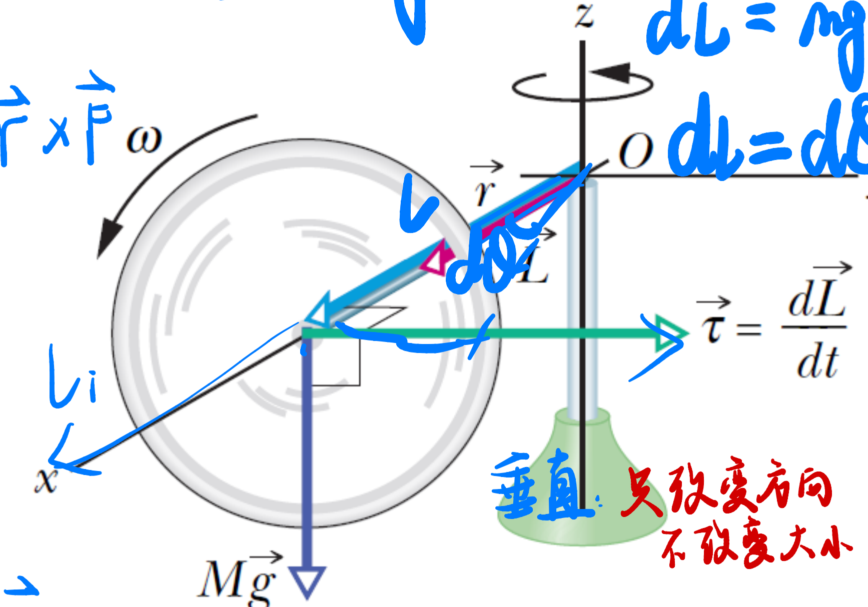
$dL = mgr\,dt$

$dL = d\theta\, L$, $\quad L = I\omega$

R

Magnitude $L = I\omega$ no change for rapid spin. Direction along shaft.

$\vec{\tau} = \vec{r} \times \vec{F}$



垂直：只改变方向
不改变大小

$\vec{\tau} = \frac{dL}{dt} \Rightarrow \vec{\tau}\,dt = dL$

α

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow d\vec{L} = \vec{\tau}\,dt$$

$\frac{d\vec{L}}{dt} = \tau = mgr$

F

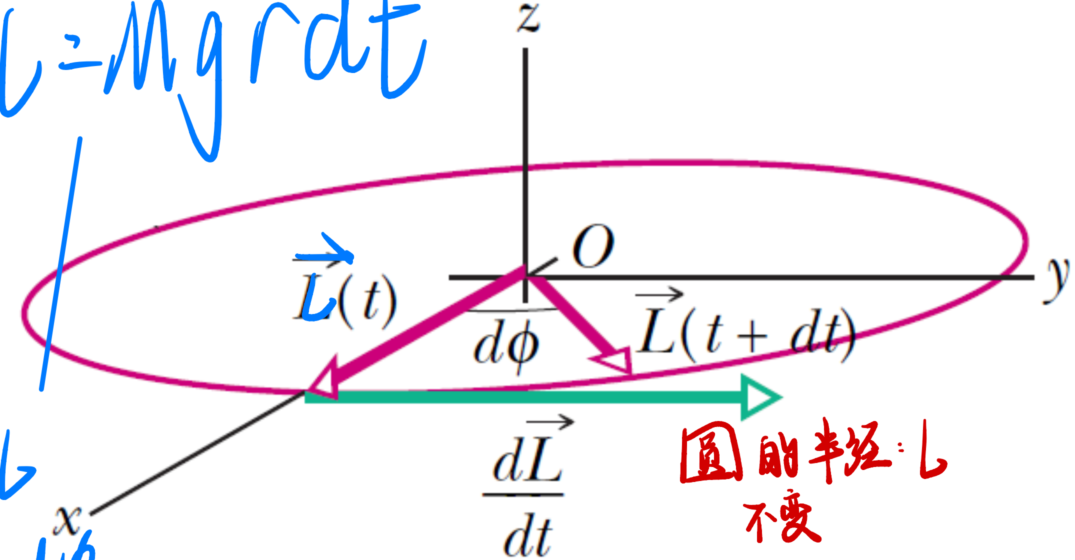
$$dL = \tau\,dt = Mgr\,dt$$

Precession Rate of a Gyroscope

角动量定理

$dL = Mgr\,dt$

$$\boldsymbol{F - f_s = ma} \qquad d\phi = \frac{dL}{L}$$



图

$d\vec{L} = d\phi \cdot L$

$= I\omega\, d\phi$

圆的半径: L 不变

$$\Rightarrow \quad \boldsymbol{f_s R = I\alpha}$$

$$\boxed{\boldsymbol{f'_s} \quad \frac{Mgr}{I\omega} \quad \text{(precession rate)}}$$

$I\omega\, d\phi = Mgr\,dt$

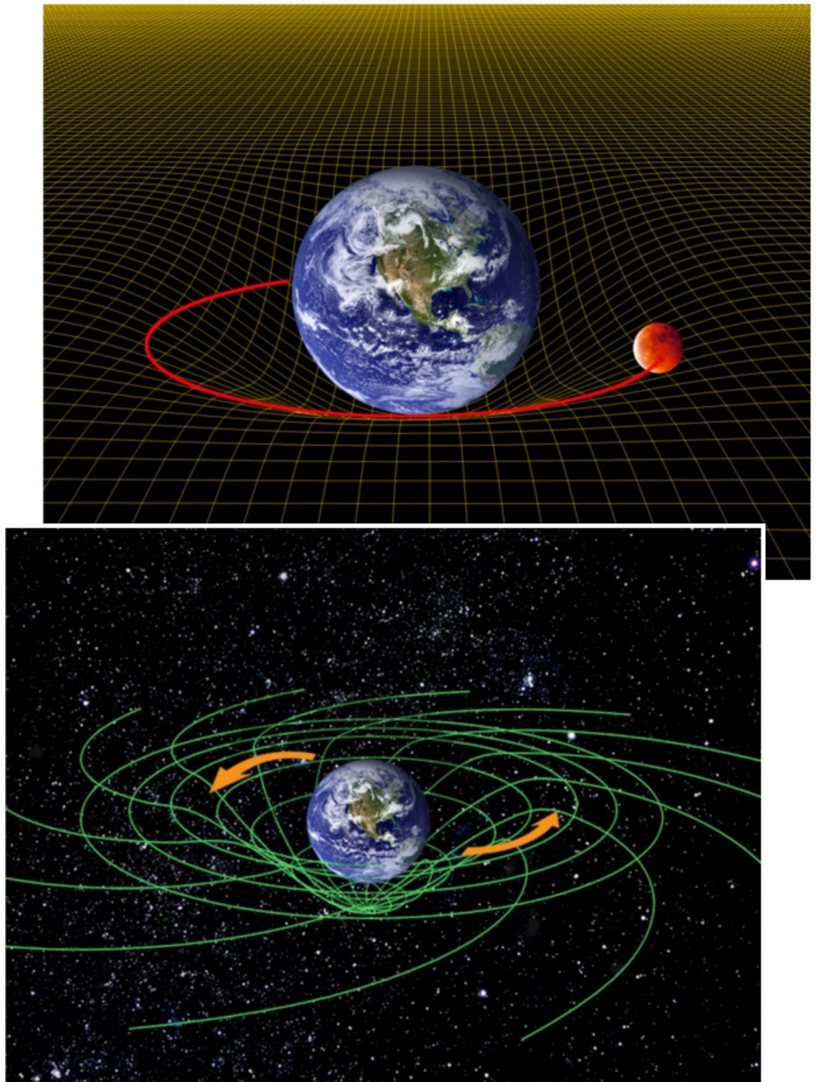
$\omega = \frac{Mgr}{I\omega}$ 进动角速度

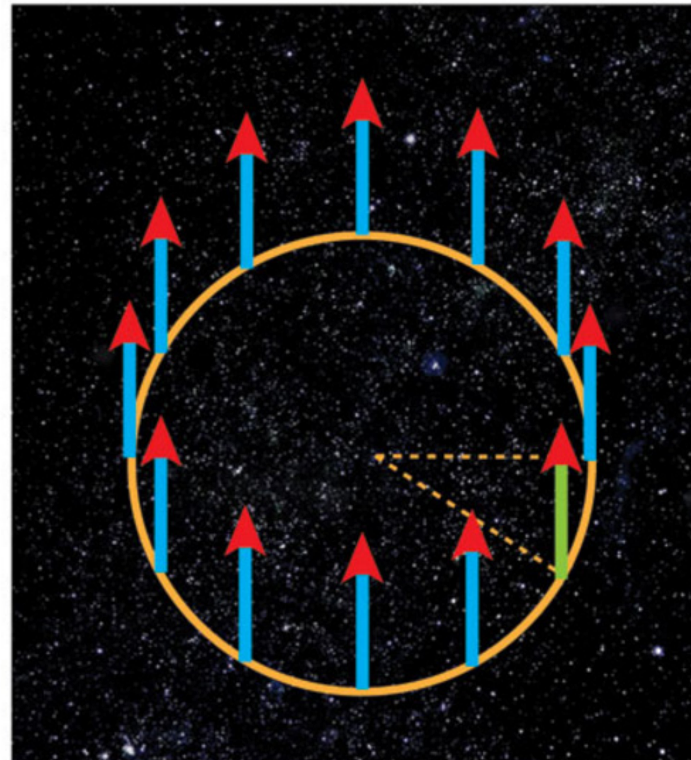
$$\boldsymbol{a = R\alpha}$$

对陀螺也成立

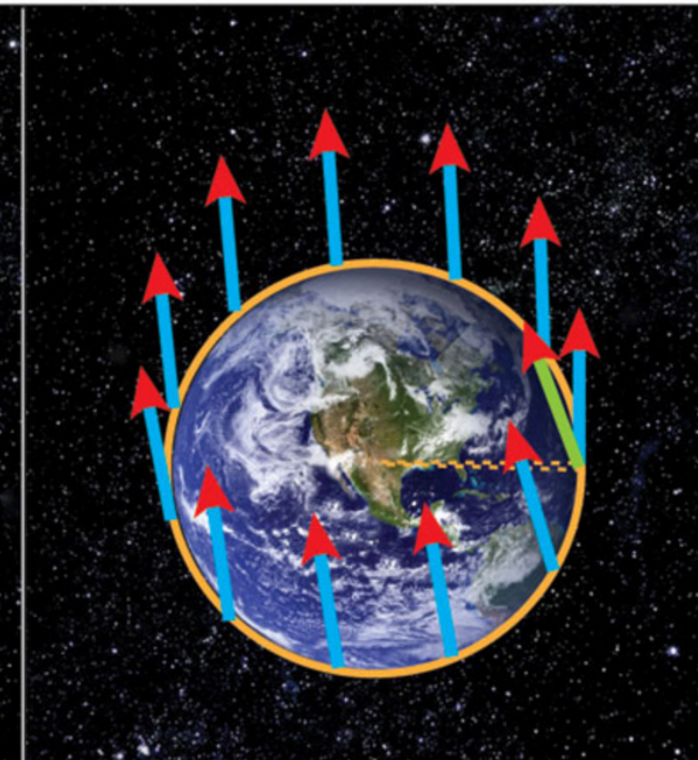
与θ → 无关

Gravity Probe B – Testing Einstein's Universe



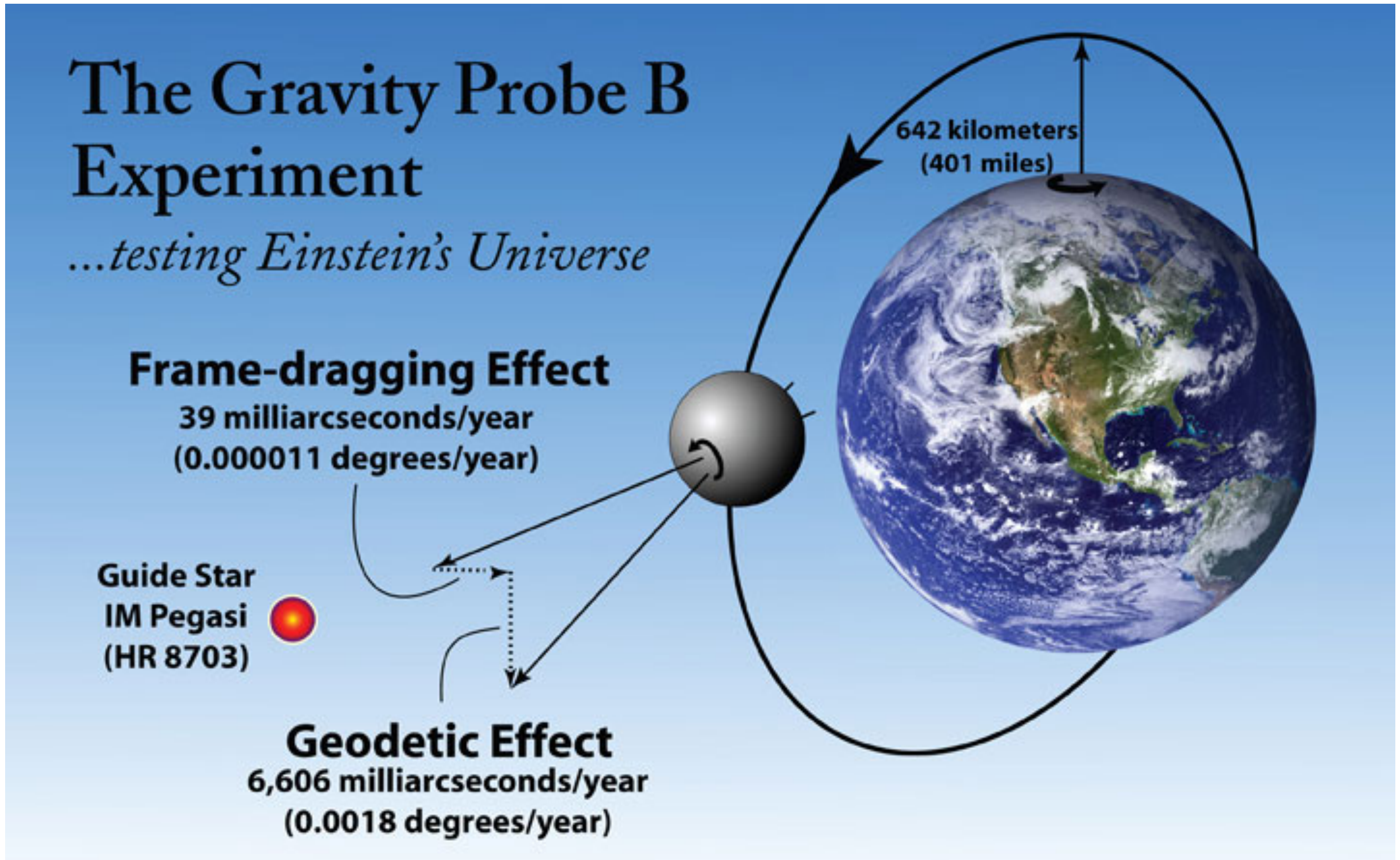


A circle with Earth's equatorial diameter (~7,926 mi) in empty space has a circumference of πD (~24,901 mi). A gyroscope following this circular path in empty space will always point in the same direction, as indicated by the arrows above.



Earth's mass warps spacetime inside the circle into a cone, formed by removing a pie-shaped wedge (dotted lines). This reduces the circle's circumference by 1.1 inches. A gyroscope will now change its orientation while tracing the conical path, as shown in the drawing above.

Gravity Probe B – Testing Einstein's Universe



http://einstein.stanford.edu/index.html

Summary of Chapter 11

- Rolling
  smooth rolling, non-smooth rolling,
  K.E. and forces of rolling

- Angular momentum
  Newton's 2nd law in angular momentum form;
  Angular momentum of system of particle and of a rigid body rotating about a fixed axis;
  Conservation of angular momentum;
  Procession of a gyroscope