Tutorial for chapter 13

Key words

Gravitational force
Universe
Attraction
Shell theorem
Uniform sphere
Superposition
Gravitational acceleration
Free fall acceleration
Gravitational potential energy
Mechanical energy
Escape speed
Kepler's law
Sweep
altitude

Equator
Pole
Infinity
Satellite
asteroid
Planet
Orbits
Elliptical
aphelion
perihelion
focus
Semi-major axis (a)
Eccentricity (e)
Period
Astronaut

Chapter 13 Gravitation

Newton's Law of Gravitation

$$F = G\frac{m_1 m_2}{r^2} \quad \text{(Newton's Law of Gravitation)}$$

$$\boxed{\vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{15} + \cdots + \vec{F}_{1n}}$$

The Shell Theorem

- A **uniform spherical** shell of matter attracts a particle that is **outside** the shell as if all the shell's mass **were concentrated at its center**.
- A **uniform shell** of matter exerts **no-net gravitational force** on a particle located **inside it**
- The gravitational force on a point with a distance from Earth's center comes entirely from the mass within the sphere of radius.

The force due to a uniform shell:

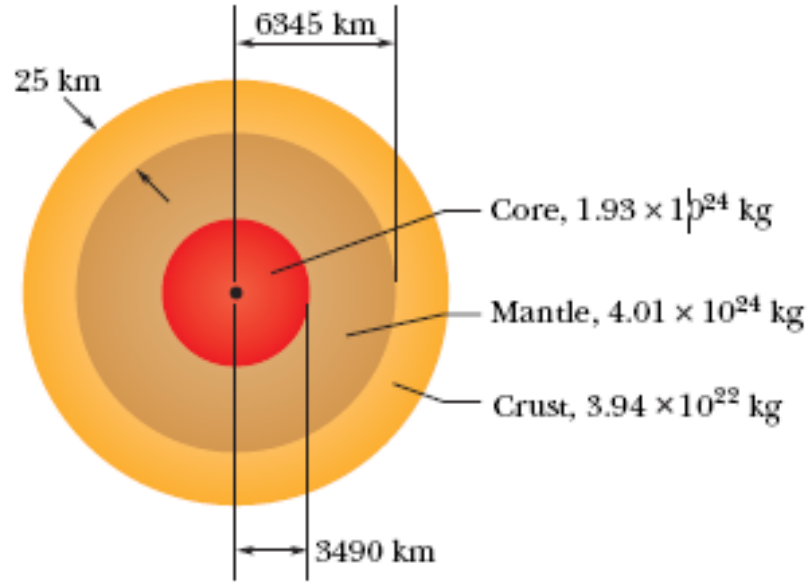
$$F(r) = -\frac{\partial U}{\partial r} = \begin{cases} -\dfrac{GM_1 M_s}{r^2}, & (r > R) \\ 0, & (r < R) \end{cases}$$

The force due to a uniform sphere:

$$F(r) = -\frac{\partial U}{\partial r} = \begin{cases} -\dfrac{GM_1 M_s}{r^2}, & (r > R) \\ -\dfrac{GM_1 M_s r}{R^3}, & (r < R) \end{cases}$$

Problem 1

Earth is divided into three zones: an outer crust, a mantle, and an inner core. The dimensions of these zones and the masses contained within them are shown in figure. Earth has a total mass of $\mathbf{5.98 \times 10^{24}}$ **kg** and a radius of **6370 km**. Ignore rotation and assume that Earth is spherical.



a) Calculate $\mathbf{a_g}$ at the surface.

b) Suppose that a bore hole is driven to the crust–mantle interface at a depth of 25.0 km; what would be the value of $\mathbf{a_g}$ at the bottom of the hole?

c) Suppose that Earth was a uniform sphere with the same total mass and size. What would be the value of $\mathbf{a_g}$ at a depth of 25.0 km?

Solution 1

a) Calculate a_g at the surface:

Shell theorem: the magnitude of the force on a particle with mass *m* at the surface of Earth is given by $F = GMm/R^2$, where *M* is the total mass of Earth and *R* is Earth's radius.

$$a_g = \frac{F}{m} = \frac{GM}{R^2} = \frac{\left(6.67\times10^{-11}\ \mathrm{m^3/s^2\cdot kg}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)}{\left(6.37\times10^{6}\ \mathrm{m}\right)^2} = 9.83\ \mathrm{m/s^2}.$$

b) what would be the value of a_g at the bottom of the hole?

Shell theorem: Now $a_g = GM/R^2$, where $M=M_{Core}+M_{mantle}$ and $R=R_{Core}+R_{mantle}$

$$a_g = \frac{\left(6.67\times10^{-11}\ \mathrm{m^3/s^2\cdot kg}\right)\left(5.94\times10^{24}\ \mathrm{kg}\right)}{\left(6.345\times10^{6}\ \mathrm{m}\right)^2} = 9.84\ \mathrm{m/s^2}.$$

Solution 1

c) Suppose that Earth were a uniform sphere with the same total mass and size. What would be the value of a_g at a depth of 25.0 km?

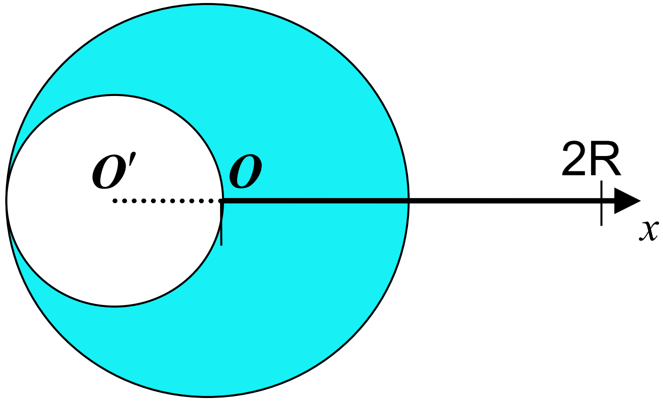
Since the mass is now assumed to be uniformly distributed, the mass within this sphere can be found by:

$$M' = \left(\frac{R'}{R}\right)^3 M$$

Then the acceleration due to gravity is

$$a_g = \frac{GM'}{R'^2} = GM\frac{R'}{R^3} = \frac{\left(6.67\times10^{-11}\ \mathrm{m^3/s^2\cdot kg}\right)\left(5.98\times10^{24}\ \mathrm{kg}\right)\left(6.345\times10^{6}\ \mathrm{m}\right)}{\left(6.370\times10^{6}\ \mathrm{m}\right)^2} = 9.79\ \mathrm{m/s^2}.$$

Problem 2



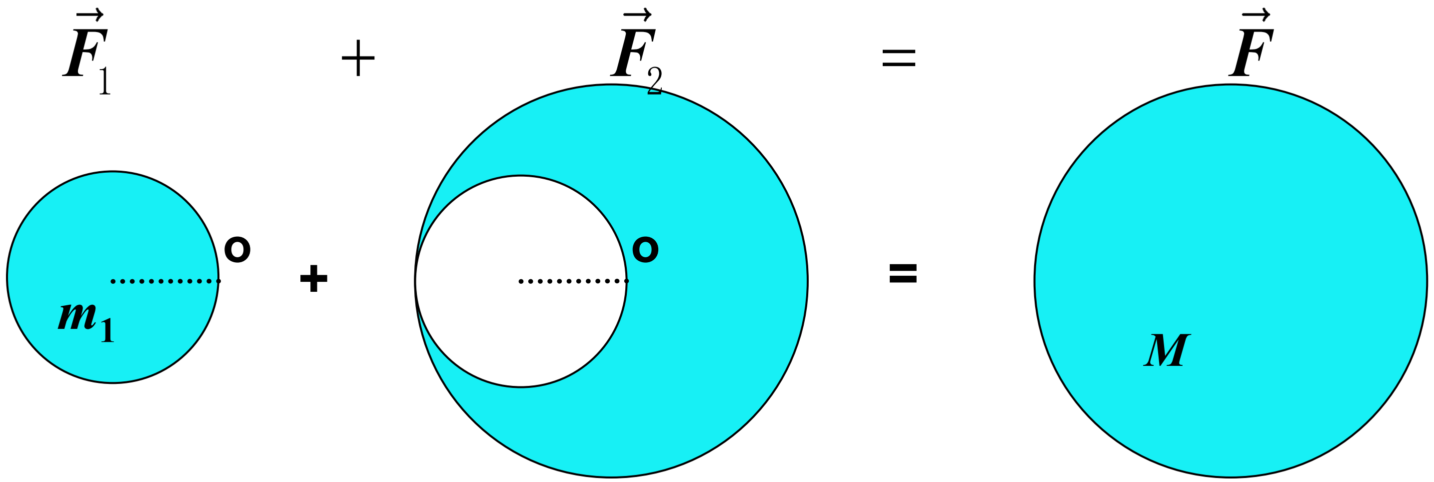
A **spherical** hollow inside a uniform lead sphere of radius R=4.00 cm; the surface of the hollow passes through the center of the sphere and "touches" the left side of the sphere. The mass of the sphere before hollowing was M=2.95 kg. What is the magnitude of the gravitational force due to the hollow sphere on a particle of mass m=0.5kg

1) when the particle is located at a distance of 2R from the center of the hollow sphere (point $\boldsymbol{O}$)

2) when the particle is located at point $\boldsymbol{O'}$

3) when the particle is located at any point in the cavity of the sphere.

Solution 2



Superposition: $\vec{F}_1 + \vec{F}_2 = \vec{F} \Rightarrow \vec{F}_2 = \vec{F} - \vec{F}_1$

$$m_1 = \rho V_1 = \frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi(\frac{R}{2})^3 = \frac{M}{8}$$

Solution 2

1) the particle is located at a distance of 2R from the center of the hollow sphere:

Shell theorem:

$$\vec{F} = \frac{GMm}{r^2}\hat{r} = -\frac{GMm}{4R^2}\hat{x}$$

$$\vec{F}_1 = \frac{Gm_1 m}{r_1^{\,2}}\hat{r}_1 = -\frac{Gm_1 m}{(2R + R/2)^2}\hat{x}$$

Superposition:

$$\vec{F}_2 = \vec{F} - \vec{F}_1 = -\frac{GMm}{4R^2}\hat{x} + \frac{Gm_1 m}{(2R + R/2)^2}\hat{x} = \frac{GMm}{R^2}\left(\frac{1}{50} - \frac{1}{4}\right)\hat{x}$$

$$= -1.4\times 10^{-8}\,\hat{x}\ N$$

Solution 2

2) when the particle is located at point O'

Shell theorem:

$$\vec{F} = \frac{Gm'm}{r^2}\hat{r} = \frac{G\dfrac{M}{8}m}{(R/2)^2}\hat{x} = \frac{1}{2}\frac{GMm}{(R)^2}\hat{x}$$

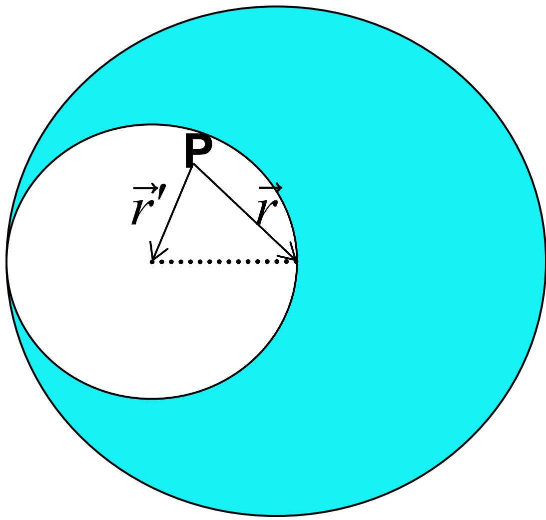
$$\vec{F}_1 = 0$$

Superposition:

$$\vec{F}_2 = \vec{F} - \vec{F}_1 = \frac{1}{2}\frac{GMm}{(R)^2}\hat{x} = 3.07\times10^{-8}\hat{x}\ N$$

Solution 2

3) At any point P in the cavity of the sphere



$$\vec{F} = \frac{Gm'm}{r^2}\hat{r} = \frac{Gm}{r^2}(\rho V')\hat{r} = \frac{Gm}{r^2}\left(\frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3\right)\hat{r} = \frac{GMmr}{R^3}\hat{r}$$

$$\vec{F}_1 = \frac{Gm''m}{r'^2}\hat{r}' = \frac{Gm}{r'^2}(\rho V'')\hat{r}' = \frac{Gm}{r'^2}\left(\frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r'^3\right)\hat{r}' = \frac{GMmr'}{R^3}\hat{r}'$$

$$\vec{F}_2 = \vec{F} - \vec{F}_1 = \frac{GMmr}{R^3}\hat{r} - \frac{GMmr'}{R^3}\hat{r}' = \frac{GMm}{R^3}(\vec{r} - \vec{r}')$$

$$= \frac{GMm}{R^3}\left(\frac{R}{2}\right)\hat{x} = 3.07\times10^{-8}\,\hat{x}\ N$$

is a constant in the cavity

Gravitational Potential Energy:

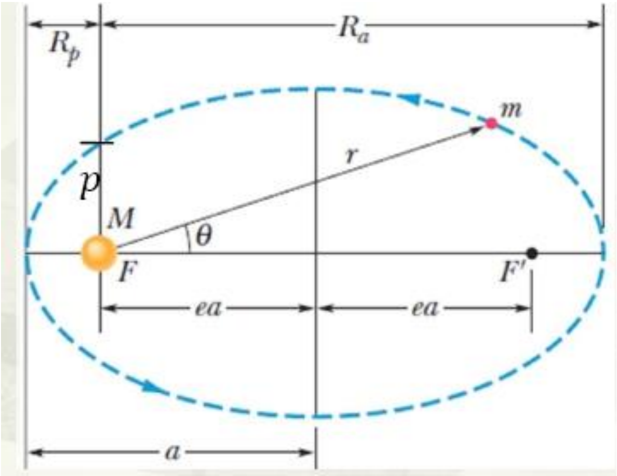
for two particles system $$U = -\frac{GMm}{r}$$

for three particles system $$U = -\frac{Gm_1 m_2}{r_{12}} - \frac{Gm_1 m_3}{r_{13}} - \frac{Gm_2 m_3}{r_{23}}$$

Escape Speed $$E = K + U = 0 = \frac{1}{2} m v_e^2 - \frac{GMm}{r} = 0 \Rightarrow v_e = \sqrt{\frac{2GM}{r}}$$

Kepler's Laws

- **The Law of Orbits: all planets move in elliptical orbits** with the Sun at one focus.



$$r = \frac{p}{1 + e\cos\theta}$$

$0 < e < 1$ **Elliptical orbits**

e=0 **Circular orbits**

- **The Law of Areas: a line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals,**

Angular momentum conserved:

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} \boxed{= \text{Constant}}$$

Kepler's Laws

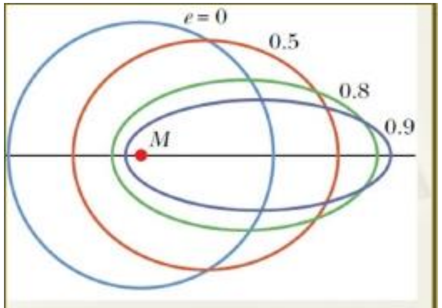
- **The Law of Periods: The square of the period of any** planet is proportional to the cube of semimajor axis of its orbit

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

For circular orbit

$$\boxed{T^2 = \left(\frac{4\pi^2}{GM}\right) a^3}$$

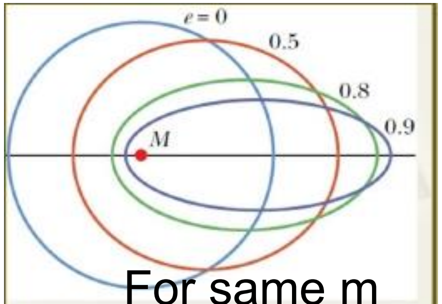
For ellipse **orbit**



Same a, same **T**

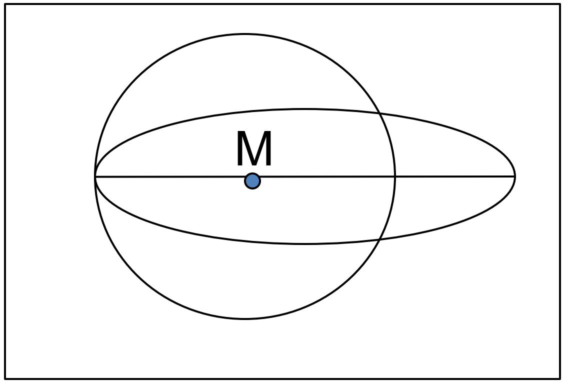
For the system of Earth-satellite, mechanical energy is conserved

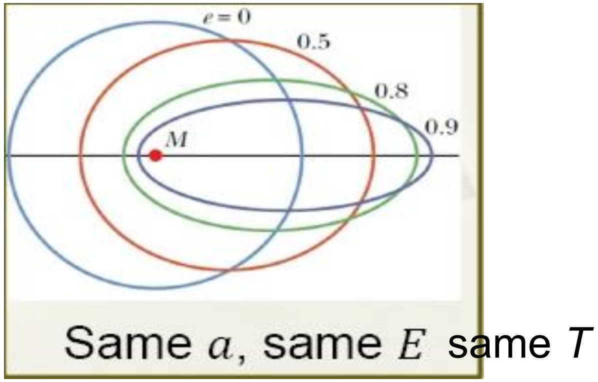
$$\boxed{E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}}$$



For same m
Same a, same E

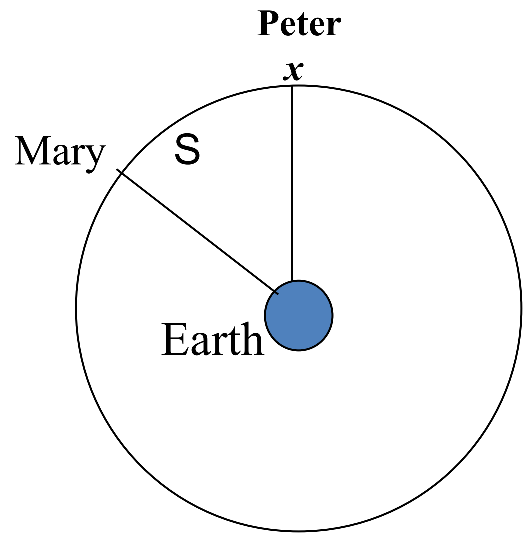
Chapter 13 Circular orbits and elliptical orbits





Variable	Circular orbits	Elliptical orbits
r	constant	change
v	constant	change
T	constant $T=\sqrt{\frac{4\pi^2 r^3}{GM}}$	constant $T=\sqrt{\frac{4\pi^2 a^3}{GM}}$
K	constant $K=\frac{1}{2}mv^2=\frac{GMm}{2r}$	change $K=\frac{1}{2}mv^2=-\frac{GMm}{2a}-(-\frac{GMm}{r})$
U	constant $U=-\frac{GMm}{r}$	change $U=-\frac{GMm}{r}$
E	constant $E=-\frac{GMm}{2r}$	constant $E=-\frac{GMm}{2a}$
L	Constant to Point M $\frac{dA}{dt}=\frac{L}{2m}=\text{constant}$	Constant to Point M $\frac{dA}{dt}=\frac{L}{2m}=\text{constant}$

Problem 3

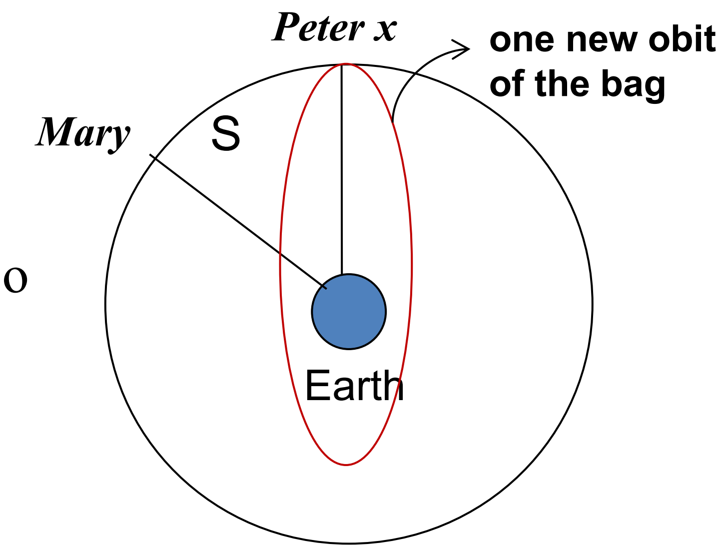


Peter and Mary are two astronauts and they are both in the same counterclockwise circular orbit with a radius of R = 7000 km around the earth. Mary is ahead of Peter at a distance $S = 2\pi RF$ along the circular orbit with F = 0.05. At one moment, Peter at the position of point x as shown, and he throws a bag along the tangential direction to Mary at that point. **In order for Mary catching the bag within one period**, what is the optimal velocity of the bag should Peter **throw relative to his satellite**.

Solution 3

The motion of the bag after being thrown:

The potential energy of the bag at position x would not change, but the kinetic energy of the bag will change, and as well as its total energy. So the orbit will also change, but the point of x will always be on the new orbit.



At what case that Mary can get the bag:

If Mary and bag can arrive at point x simultaneously, then Mary have chance to get the bag, that means:

$$(1-F)T_M = T_B \Rightarrow \frac{T_M}{T_B} = \frac{1}{1\text{-F}} \quad (or\ (n-F)T_M = mT_B)$$

"In order for Mary catching the bag within one period" means n=1

The law of period:

$$\frac{T_M^2}{T_B^2} = \frac{(4\pi)^2/GM}{(4\pi)^2/GM}\frac{R^3}{a^3} = \frac{R^3}{a^3} = \left(\frac{1}{1-F}\right)^2 \Rightarrow a = R(1-F)^{2/3} = 6765km$$

Solution 3

Potential energy of bag at position x:

$$U_B = -\frac{GMm}{r}$$

Total energy of bag after being thrown:

$$E_B = -\frac{GMm}{2a}$$

Kinetic energy of bag at position x after being thrown:

$$K_{B,x} = E_B - U_B = -\frac{GMm}{2a} - (-\frac{GMm}{R}) = GMm(\frac{1}{R} - \frac{1}{2a}) = \frac{1}{2} m v_{B,x}^2$$

$$\Rightarrow v_{B,x} = \sqrt{2GM(\frac{1}{R} - \frac{1}{2a})} = 7.42 km/s$$

Solution 3

Kinetic energy of the Peter and the bag at the circular orbit:

$$K_{P,x} = E_P - U_P = -\frac{GMm}{2R} - (-\frac{GMm}{R}) = GMm\frac{1}{2R} = \frac{1}{2}mv_P^2$$

$$\Rightarrow v_P = \sqrt{GM\frac{1}{R}} = 7.55km/s$$

$$v_{B,x,p} = v_{B,x} - v_P = \sqrt{2GM(\frac{1}{R} - \frac{1}{2a})} - \sqrt{GM\frac{1}{R}}$$

$$= \sqrt{\frac{GM}{R}}(\sqrt{(2 - \frac{1}{(1-F)^{2/3}})} - 1)$$

$$= -0.13km/s$$