Tutorial for chapter 9

Key words for chapter 9

Center of mass

Linear momentum

Closed system (constant mass)

Particle-like object

Impulse

守恒

Conservation of linear momentum

Component

Collision

Collide

Elastic collision

Inelastic collision

Projectile 抛射体

Target

Head on 迎面地

Stationary

Massive

Exhaust

Chapter 9 **Center of mass and linear momentum**

Center of mass of a system of particles :

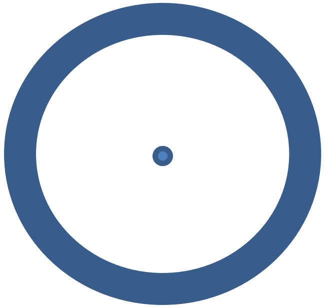
For a system of particles

$$\vec{r}_{\text{com}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots + m_n} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \qquad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \qquad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

For a solid body with uniform distribution of mass

$$\vec{r}_{\text{com}} = \frac{1}{V}\int \vec{r}\,dV$$



Chapter 9 **Center of mass and linear momentum**

Comparison:	One particle	A system of particles
Position	r	$r_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i r_i$
Velocity	v	$v_{com} = \frac{dr_{com}}{dt}$
Acceleration	a	$\vec{a}_{com} = \frac{d\vec{v}_{com}}{dt}$
Linear Momentum	$P = mv$	$P = \sum_i m_i v_i = M v_{com}$
Newton's 2nd Law	$F_{net} = ma$ $F_{net} = \frac{dP}{dt}$	$F_{external} = Ma_{com}$ ***M*** **must be constant, Closed system** $F_{external} = \frac{dP}{dt}$ ***F* :net force of all external forces**
Impulse	$\vec{F}(t) = \frac{d\vec{P}}{dt} \Rightarrow \int_i^f d\vec{P} = \int_{t_i}^{t_f} \vec{F}(t)dt = \vec{J} = \vec{F}_{avg}t$	

Chapter 9 **Center of mass and linear momentum**

Linear Momentum – Impulse theorem:

$$F(t) = \frac{dP}{dt} \Rightarrow \int_{p_i}^{p_f} dP = \int_{t_i}^{t_f} F(t)dt = J$$

$$\Rightarrow \Delta P = P_f - P_i = J$$

$$\Downarrow$$

$$F_{external} = 0 \Rightarrow \Delta P = 0 \Rightarrow P_i = P_f$$

$$p_{fx} - p_{ix} = \Delta p_x = J_x$$

$$p_{fy} - p_{iy} = \Delta p_y = J_y$$

$$p_{fz} - p_{iz} = \Delta p_z = J_z$$

$$Px_i = Px_f$$

$$Py_i = Py_f$$

$$Pz_i = Pz_f$$

If a system is isolated, so that no external force act on it, the linear momentum of the system must be constant.

And If the system is closed system, M is constant, then $\vec{v}_{com}$ is also constant

$$Mv_{comi} = Mv_{comf} \Rightarrow v_{comi} = v_{comf}$$

Chapter 9 **Center of mass and linear momentum**

Momentum and Kinetic Energy in Collisions:

Collisions due to a closed and isolated system:

	Elastic collision	Inelastic collision
Kinetic energy K	Conserved	Not Conserved
Linear momentum P	Conserved	Conserved
Velocity of center mass v_{com}	Unchanged	Unchanged

Momentum conserved: $m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$

Kinetic energy conserved : $\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$

For Elastic Collisions in 1D

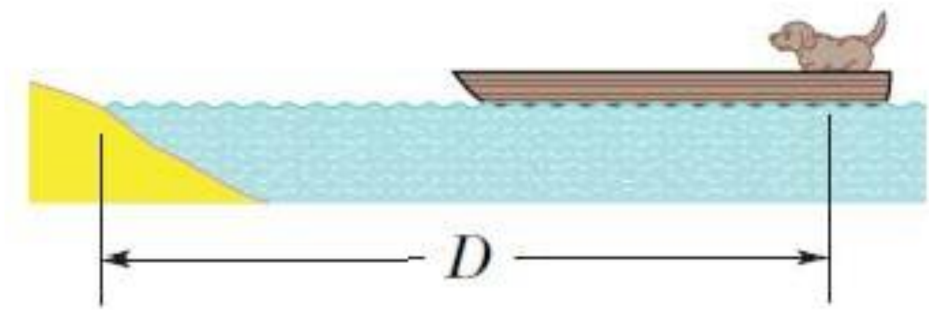
$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

For $v_2 = 0$:

- If $m_1 = m_2$, $v_{1f} = 0$ and $v_{2f} = v_{1i}$. Two bodies switch their speed.
- If $m_1 \ll m_2$, $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx \left(\frac{2m_1}{m_2}\right) v_{1i}$. Projectile body bounces back. Target body moves forward very slowly.
- If $m_1 \gg m_2$, $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$. Projectile body almost feels no effect while target body moves forward with a speed of $2v_{1i}$.

Chapter 9 Tutorial Problem 1



In the figure, a dog of mass $m = 4.5$ **kg** stands on a flatboat of mass $M = 18$ **kg**. Initially both the dog and the flatboat are stationary and the dog is at a distance $D = 6.1$ **m** from the shore. The dog walks $d =$ **2.4 m** along the boat (i.e., with respect to the boat) toward shore and then stops. Assuming no friction between the boat and the water, **what is the distance S of the dog then from the shore**.

Solution

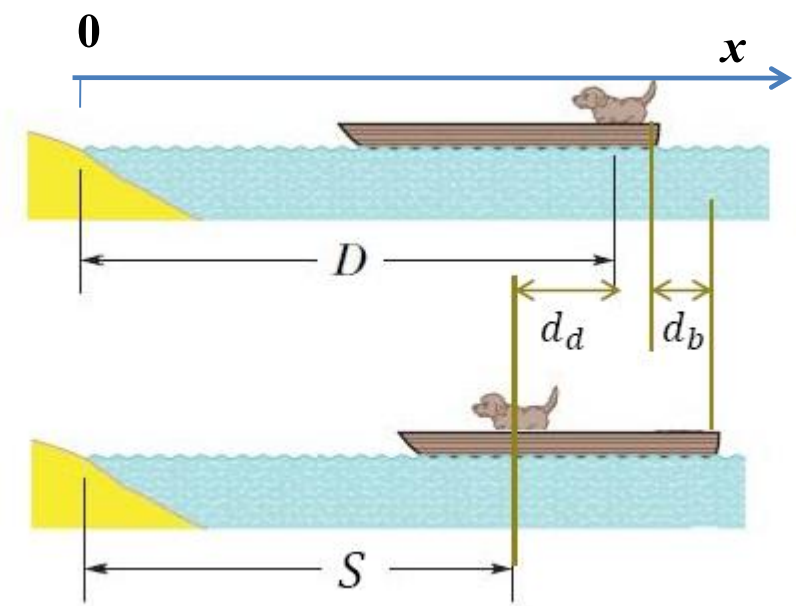
Conserved of linear momentum:

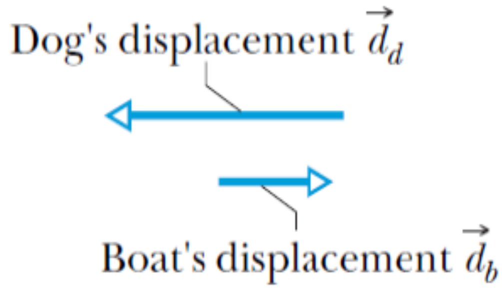
For the system of dog + flatboat, the $\boxed{\text{net external force is zero.}}$ Initially the system is at rest. Thus:

- the linear momentum of the system remains as zero. This means when dog walks toward shore the boat must moves oppositely, in a direction away off shore. When the dog stops, the boat also stops. The *displacement (w.r.t. shore)* of the dog must be in *opposite* direction to that of the flatboat.
- the acceleration of the center of mass (COM) of the system is zero. Therefore, the velocity of the COM of the system is a constant. The system will remain zero velocity of COM. Thus, the ***COM*** of the system ***does not change***. $\Delta x_{com} = 0$

Solution

We can choose the x axis along the boat **with the positive direction toward the boat and the origin is at the shore** . The displacements (w.r.t. (with respect to) shore) of the dog and the boat are d_d and d_b respectively.





$$d = d_d - d_b$$

$$displacement : d = -2.4\hat{i} \quad (m)$$

$$position : x_i = D\hat{i} = 6.1\hat{i} \quad (m)$$

Solution

Relative motion: $d = d_d - d_b$.............(1)

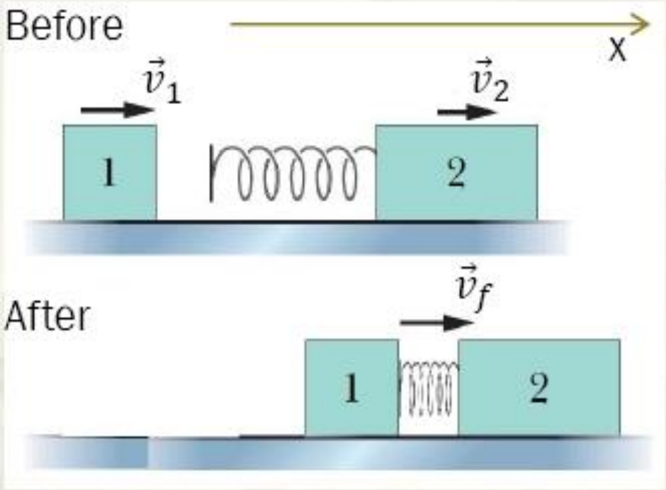
$\Delta x_{com} = 0 : md_d + Md_b = 0$..................(2)

$$\Rightarrow (m+M)d_d = Md \Rightarrow d_d = \frac{M}{m+M}d$$

$$d_d = x_f - x_i$$

$$\Rightarrow x_f = d_d + x_i = \frac{M}{m+M}d + x_i = (-1.92 + 6.1)\hat{i} = 4.18\hat{i} \quad (m)$$

Chapter 9 Tutorial Problem 2



In the figure, block 1 of mass $m_1 = 2.0$ kg is moving rightward at speed $v_1 = 10$ m/s and block 2 of mass $m_2 = 5.0$ kg is moving rightward at speed $v_2 = 3.0$ m/s. The surface is frictionless, and a spring with a spring constant $k = 1120$ N/m is fixed to block 2.

Find the maximum compression d.

Solution

Solution:

For the system of two blocks + spring, the net external force is zero.

Conservation of linear momentum of the system:

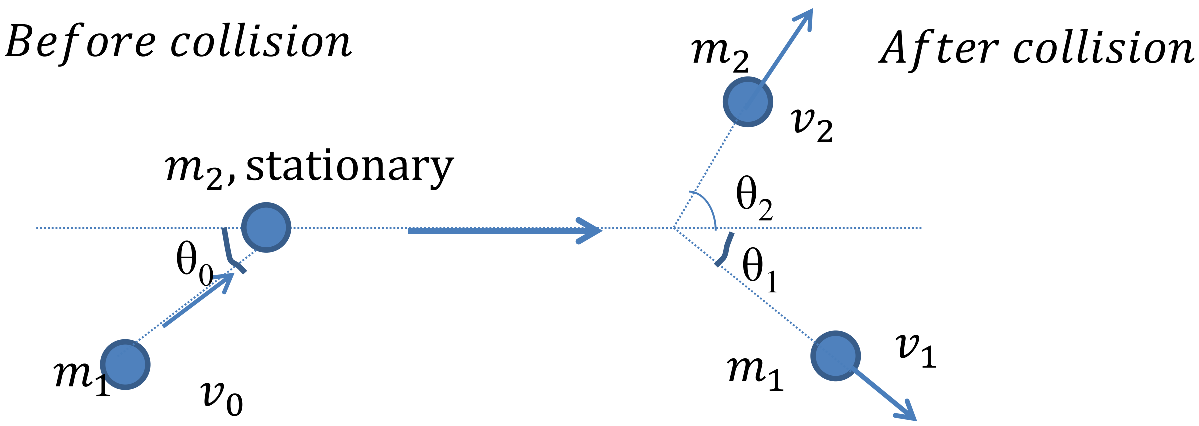
$$m_1v_1 + m_2v_2 = (m_1 + m_2)v_f \quad \rightarrow \quad v_f = 5 \text{ m/s}$$

Conservation of energy of the isolated system:

$$\Delta K + \Delta U = 0 \quad \rightarrow \quad \frac{1}{2}(m_1 + m_2){v_f}^2 - \left(\frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2\right) + \frac{1}{2}kd^2 = 0$$

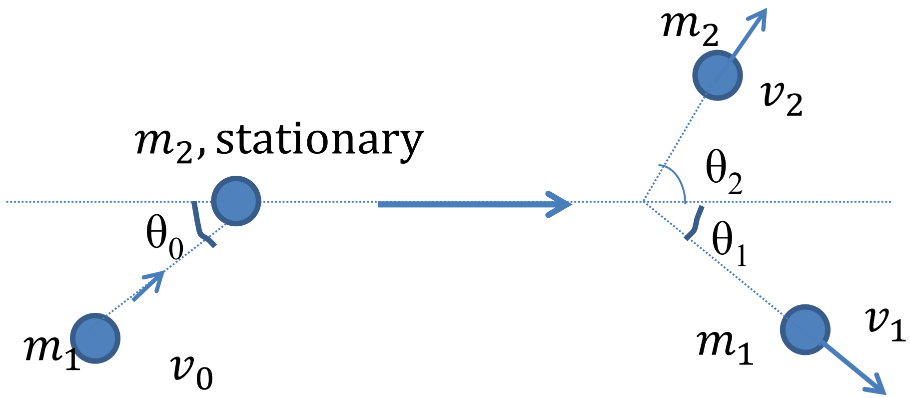
$$\rightarrow \quad \boxed{d = 0.25 \text{ m}}$$

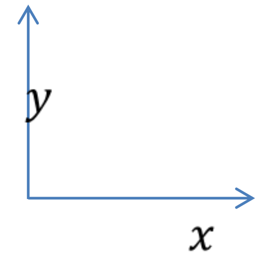
Chapter 9 Tutorial Problem 3



In a two-dimensional collision as shown in the figure, the projectile particle of **mass $\boldsymbol{m_1 = m}$**, speed v_0 hits the initially stationary target particle of mass $\boldsymbol{m_2 = 2m}$ at an angle $\boldsymbol{\theta_0 = 60^o}$. After collision, the projectile particle is scattered at an angle $\boldsymbol{\theta_1 = 30^o}$. If the **collision is elastic**, find v_1, v_2 and θ_2 in terms of v_0.

Solution





The total linear momentum must be conserved in *x* component and *y* component:

$$m_1 v_0 \cos\theta_0 = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \text{.....(1)}$$

$$m_1 v_0 \sin\theta_0 = -m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2 \text{....(2)}$$

The collision is elastic , then the total kinetic energy is also conserved

$$\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \text{.......................(3)}$$

Solution

$$mv_0\cos 60° = mv_1\cos 30° + 2mv_2\cos\theta_2.....(1)$$

$$\Rightarrow m^2v_0^2\cos^2 60° - 2m^2v_0v_1\cos 60°\cos 30° + m^2v_1^2\cos^2 30° = 4m^2v_2^2\cos^2\theta_2...(4)$$

$$mv_0\sin 60° = -mv_1\sin 30° + 2mv_2\sin\theta_2....(2)$$

$$\Rightarrow m^2v_0^2\sin^2 60° + 2m^2v_0v_1\sin 60°\sin 30° + m^2v_1^2\sin^2 30° = 4m^2v_2^2\sin^2\theta_2....(5)$$

$$(4)+(5) \Rightarrow m^2v_0^2 + m^2v_1^2 - 2m^2v_0v_1\cos(60° + 30°) = 4m^2v_2^2$$

$$\Rightarrow m\ v_2^2 = \frac{1}{4m}(m^2v_0^2 + m^2v_1^2) = \frac{1}{4}(mv_0^2 + mv_1^2)....(6)$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}\times 2mv_2^2..................................(3)$$

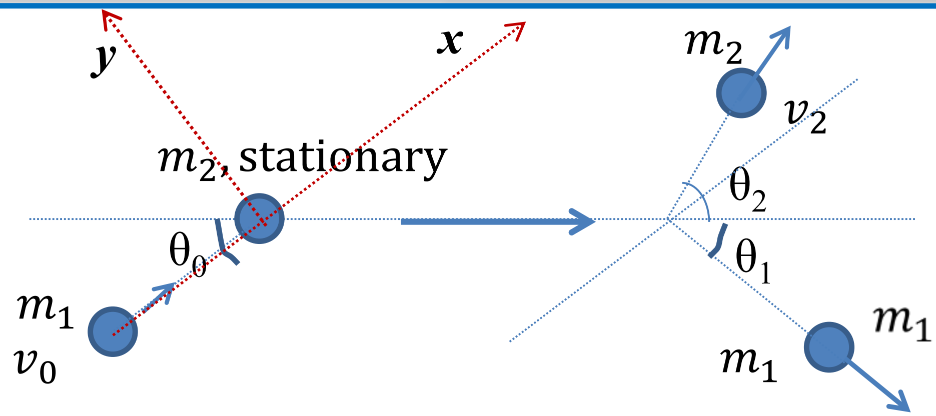
$$(6)+(3) \Rightarrow \frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + \frac{1}{4}(mv_0^2 + mv_1^2) \Rightarrow 3v_1^2 = v_0^2 \Rightarrow v_1 = \frac{\sqrt{3}}{3}v_0$$

Solution

$$v_2^2 = \frac{1}{4}(v_0^2 + v_1^2) = \frac{1}{4}(3v_1^2 + v_1^2) = v_1^2 \Rightarrow v_2 = \frac{\sqrt{3}}{3} v_0$$

$$\operatorname{ctan}\theta_2 = \frac{v_0 \cos\theta_0 - v_1 \cos\theta_1}{v_0 \sin\theta_0 - v_1 \sin\theta_1} = \frac{v_0 \cos 60° - \sqrt{3} v_0 / 3 \cos 30°}{v_0 \sin 60° - \sqrt{3} v_0 / 3 \sin 30°} = 0 \Rightarrow \theta_2 = 90°$$

Solution



Here, why $\theta_2 > \theta_0$?

For the system of m_1 and m_2:
initial: $p_{yi}=0$, so final: $p_{yf}=0$
$p_{yf}=p_{1,yf}+p_{2,yf}=0$
$p_{1,yf}<0 \rightarrow p_{2,yf}>0$, that meas $\theta_2 > \theta_0$

The total linear momentum must be conserved in *x* component and *y* component:

$$m_1 v_0 = m_1 \mathbf{v_1} \cos(\theta_1 + \theta_0) + m_2 \mathbf{v_2} \cos(\boldsymbol{\theta_2} - \theta_0) .....(1)$$

$$m_1 \mathbf{v_1} \sin(\theta_1 + \theta_0)_1 = m_2 \mathbf{v_2} \sin(\boldsymbol{\theta_2} - \theta_0) ....(2)$$

The collision is also elastic (a special case), then the total kinetic energy is also conserved

$$\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 \mathbf{v_1^2} + \frac{1}{2} m_2 \mathbf{v_2^2} ....(3)$$

Solution

$$m_1 v_0 = m_1 v_1 \cos(60° + 30°) + m_2 v_2 \cos(\theta_2 - \theta_0) \ldots\ldots(1)$$

$$\Rightarrow m^2 v_0^2 - 2m^2 v_0 v_1 \cos(90°) + m^2 v_1^2 \cos^2(90°) = 4m^2 v_2^2 \cos^2(\theta_2 - \theta_0)$$

$$\Rightarrow v_0^2 = 4v_2^2 \cos^2(\theta_2 - \theta_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$m_1 v_1 \sin(\theta_1 + \theta_0) = m_2 v_2 \sin(\theta_2 - \theta_0) \ldots\ldots(2)$$

$$\Rightarrow m^2 v_1^2 \sin^2(90°) = 4m^2 v_2^2 \sin^2(\theta_2 - \theta_0)$$

$$\Rightarrow v_1^2 = 4v_2^2 \sin^2(\theta_2 - \theta_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$(4) + (5) \Rightarrow v_0^2 + v_1^2 = 4v_2^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$\frac{1}{2} m\, v_0^2 = \frac{1}{2} m\, v_1^2 + \frac{1}{2} \times 2m\, v_2^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$(6) + (3) \Rightarrow v_1 = v_2 = \frac{\sqrt{3}}{3} v_0$$

$$c\tan(\theta_2 - \theta_0) = \frac{v_0 - v_1 \cos(\theta_1 + \theta_0)}{v_1 \sin(\theta_1 + \theta_0)} = \sqrt{3} \Rightarrow \theta_2 - \theta_0 = 30° \Rightarrow \theta_2 = 90°$$