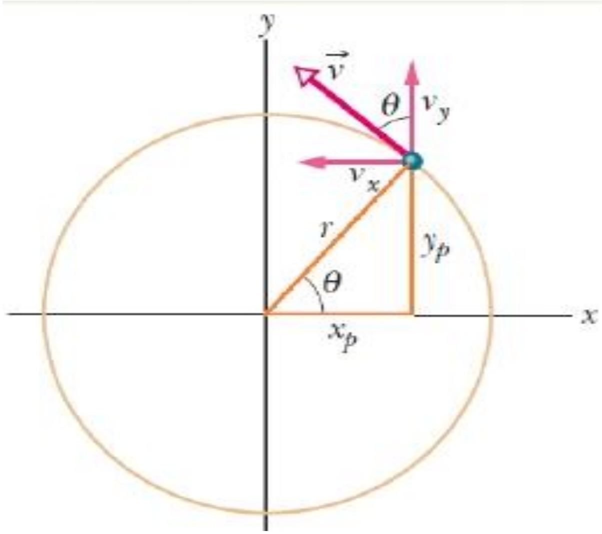
Keywords for chapter 10

- Rotation
- Revolution
- Torque
- Angular position
- Angular displacement
- Angular velocity
- Angular speed
- Angular acceleration
- Radian
- Reference line
- Rotation axis
- Rotational inertia
- Moment of inertia 转动惯量
- Parallel –axis theorem

Chapter 10 Rotation

Relating linear and angular variables:



$$\boxed{s = \theta r}$$

$$\boxed{\frac{ds}{dt} = \frac{d\theta}{dt} r \Rightarrow v = \omega r}$$

$$\boxed{T = \frac{2\pi r}{v} \Rightarrow \frac{2\pi r}{\omega r} = \frac{2\pi}{\omega}}$$

Just for constant angular velocity

$\vec{v} = v\hat{t}$ 乘积求导

$$\vec{v} = -v\sin\theta\,\hat{\imath} + v\cos\theta\,\hat{\jmath} = v\hat{t}$$

切向

$$\vec{a} = \frac{dv}{dt}\hat{t} + v\frac{d\hat{t}}{dt}$$

- Both the speed and the angle change with time.

$$\boxed{\frac{dv}{dt}\hat{t} = \frac{d\omega}{dt} r\hat{t} = \alpha r\hat{t} = a_t\hat{t}}$$

Magnitude change

$$\frac{d\hat{t}}{dt} = \frac{d(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})}{dt} = \frac{d\theta}{dt}(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}) = \omega\hat{r}_c$$

向心方向

$$\boxed{v\frac{d\hat{t}}{dt} = \omega r(\omega)\hat{r}_c = \omega^2 r\hat{r}_c = \frac{v^2}{r}\hat{r}_c = a_r\hat{r}_c}$$

Direction change

Eq.s of Motion with Constant Linear/Angular Acceleration:

1D pure translational motion	Pure rotation about a fixed axis
$v - v_0 = at$	$\boxed{\omega - \omega_0 = \alpha t}$
$x - x_0 = v_0 t + \frac{1}{2}at^2$	$\boxed{\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2}$
$v^2 - {v_0}^2 = 2a(x - x_0)$	$\boxed{\omega^2 - {\omega_0}^2 = 2\alpha(\theta - \theta_0)}$
1D pure translational motion with a constant linear acceleration.	Pure rotation about a fixed axis with a constant angular acceleration.

Chapter 10 Rotation

Kinetic energy of rotation and Rotational inertia:

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \tfrac{1}{2}m_3v_3^2 + \cdots$$
$$= \sum \underbrace{\tfrac{1}{2}m_iv_i^2}_{},$$

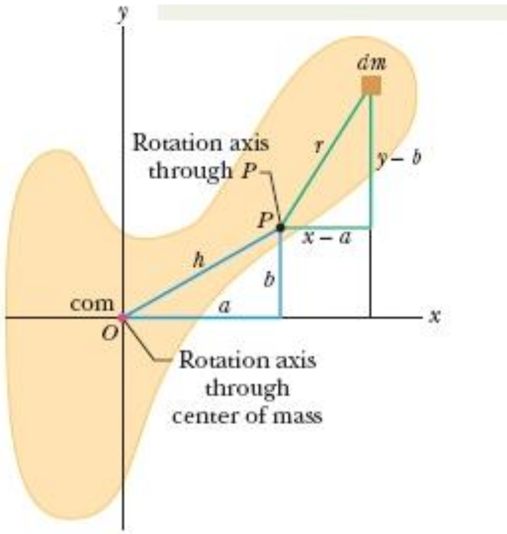
$$K = \sum \tfrac{1}{2}m_i(\omega r_i)^2 = \tfrac{1}{2}\left(\sum m_ir_i^2\right)\omega^2 = \tfrac{1}{2}I\omega^2 \qquad \text{(radian measure)}$$

Rotational inertia (or moment of inertia) *I of* *the body* *with respect to* the axis of rotation.

$$I = \sum m_ir_i^2 \qquad \text{(rotational inertia)}$$

$$\boxed{I = \int \underline{r^2\,dm}}$$

Chapter 10 Rotation



Set origin at center of mass. Then:

$$I = \int r^2\, dm = \int [(x-a)^2 + (y-b)^2]\, dm,$$

which we can rearrange as

$$I = \int (x^2 + y^2)\, dm - 2a\boxed{\int x\, dm} - 2b\boxed{\int y\, dm} + \int (a^2 + b^2)\, dm. \quad (10\text{-}37)$$

$$\boxed{x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i = 0}$$

$$\boxed{y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i = 0}$$

$$\boxed{I = I_{com} + Mh^2 \qquad \text{(parallel-axis theorem).}}$$

Chapter 10 Rotation

Torque for body rotating about a Fixed Axis:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The SI unit of torque is $\mathrm{N \cdot m}$

$\vec{\tau} = \vec{r} \times \vec{F}$

- **Magnitude**: $\tau = rF\sin\phi = rF_t$

  zero torque if extended line of force passes through axis

- **Direction**: **Right-hand**

  Positive if driven rotation is counterclockwise
  Negative if driven rotation is clockwise

$$\boxed{\tau_{\text{net,ext}} = I\alpha}$$

Newton's 2nd law for rotation about a fixed axis

$$\vec{\tau}_i = \vec{r}_i \times \vec{F}_i = \vec{r}_i \times m_i \frac{d\vec{v}_i}{dt} = (\vec{r}_i \times m_i \frac{dv_i}{dt}\vec{t} + \vec{r}_i \times m_i a_{r_i}\vec{r}_a)$$

$$\Rightarrow \tau_i = r_i m_i r_i \alpha \Rightarrow \sum \tau_i = \sum r_i m_i r_i \alpha = \alpha \sum m_i r^2{}_i = I\alpha$$

Chapter 10 Rotation

Work for Rotation about a Fixed Axis:

Work done on a particle of the body for the angular position from θ_i to θ_f:

$$dW_{i(\text{net})} = F_{i(\text{net}),t}ds_i = F_{i(\text{net}),t}r_i d\theta = \tau_{i(\text{net})}d\theta \Rightarrow W_{i(\text{net})} = \int_{\theta_i}^{\theta_f} \tau_{i(\text{net})}d\theta$$

$$\Rightarrow W_{\text{net}} = \sum_i W_{i(\text{net})} = \int_{\theta_i}^{\theta_f}\left(\sum_i \tau_{i(\text{net})}\right)d\theta = \int_{\theta_i}^{\theta_f}\tau_{\text{net}}d\theta$$

$dW = \tau d\theta$

Sum of work and sum of torque from paired internal forces are zero.

$P = \tau\omega$

$$\boxed{W_{\text{net}} = \int_{\theta_i}^{\theta_f}\tau_{\text{net,ext}}d\theta}$$

For a constant torque:

$$\boldsymbol{W = \tau(\theta_f - \theta_i)}$$

Power (instantaneous): $\boldsymbol{P = \dfrac{dW}{dt} = \tau\omega}$

Chapter 10 Rotation

Work – K.E. Theorem for Rotation about a Fixed Axis:

Work – K.E. theorem for a particle of body between final (f) and initial (i) states:

$$W_{j(\mathrm{net})} = \Delta K_j = \frac{1}{2}m_j {v_{j,f}}^2 - \frac{1}{2}m_j {v_{j,i}}^2$$

Particle j moves in a circular path with radius r_j, $v_{j,i} = r_j\omega_i$ and $v_{j,f} = r_j\omega_f$

$$W_{j(\mathrm{net})} = \frac{1}{2}m_j\left(r_j\omega_f\right)^2 - \frac{1}{2}m_j\left(r_j\omega_i\right)^2 = \frac{1}{2}\left(m_j {r_j}^2\right)\left({\omega_f}^2 - {\omega_i}^2\right)$$

$$\Rightarrow \quad W_{net} = \sum_j W_{j(\mathrm{net})} = \frac{1}{2}\left(\sum_j m_j {r_j}^2\right)\left({\omega_f}^2 - {\omega_i}^2\right) = \frac{1}{2}I{\omega_f}^2 - \frac{1}{2}I{\omega_i}^2$$

Net work only takes from the applied external forces.

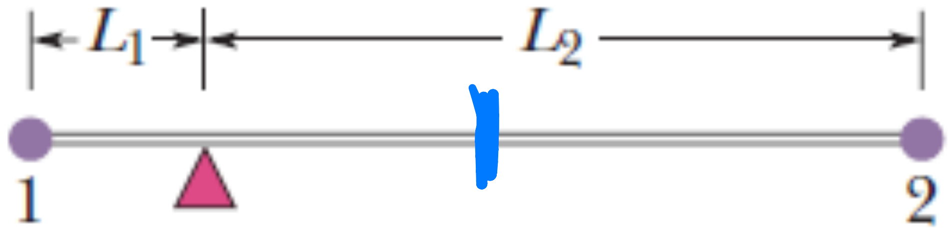
$$\boxed{W_{\mathrm{net}} = \Delta K_{\mathrm{rotation}} = \frac{1}{2}I{\omega_f}^2 - \frac{1}{2}I{\omega_i}^2}$$

Chapter 10 Rotation

Some Corresponding Relations for Translational and Rotational Motion

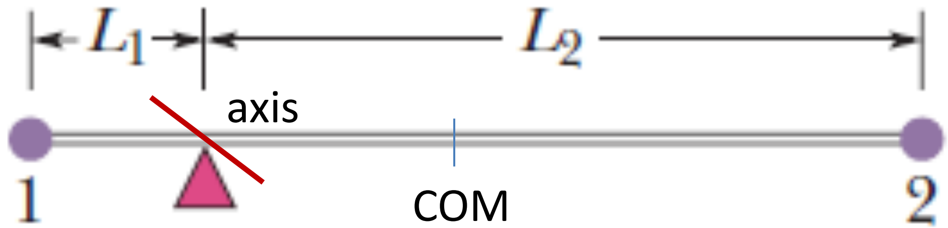
Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{\text{net}} = ma$	Newton's second law	$\tau_{\text{net}} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

Problem 1:



As the figure shows, particles 1 and 2, each of mass m, attached to the ends of a uniform rigid rod of mass $2m$ and length $L_1 + L_2$, with $L_1 = L$ and $L_2 = 3L$. The rod is held horizontally on the fulcrum (支点) and then released. What are the magnitudes of the initial accelerations of particle 1 and particle 2?

Problem 1:



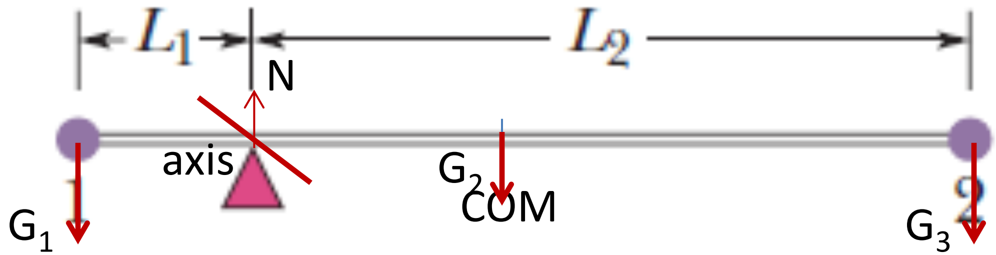
First, let's calculate the moment of inertial for the particles-rod system relative to the horizontal axis through the fulcrum. (We apply parallel-axis theorem to calculate the rotational inertial of the rod about that axis)

$$I_{p1} + I_{p2} = mL_1^2 + mL_2^2$$

$$I_{rod} = \frac{1}{12}(2m)(L_1 + L_2)^2 + (2m)(\frac{L_1 + L_2}{2} - L_1)^2 = \frac{14}{3}mL^2$$

$$I = I_{p1} + I_{p2} + I_{rod} = mL_1^2 + mL_2^2 + \frac{14}{3}mL = mL^2 + m(3L)^2 + \frac{14}{3}mL = \frac{44}{3}mL^2$$

Problem 1:



After the rod is released from rest, the particles-rod system will then rotate about that axis due to the gravities. And we can apply newton's second law to find the angular acceleration for this rigid combined body.
Let's draw the free-body-diagrams for particles 1 and 2 and the rod and apply newton's second law for the body:

$$\tau_{net} = m_1 g L_1 - m_2 g L_2 - m_r g\left(\frac{L_1 + L_2}{2} - L_1\right) = mgL - mg(3L) - (2m)gL = I\alpha$$

$$\Rightarrow -4mgL = \frac{44}{3} m L^2 \alpha \Rightarrow \alpha = -\frac{3}{11}\frac{g}{L}$$

minus means α is clockwise, the direction is into the paper

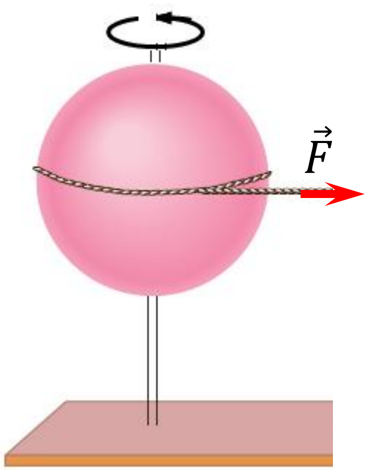
Problem 1:

Just after releasing, the angular speed of the system is 0, so the linear speeds of the particles 1 and 2 are also 0. As a result, the centripetal acceleration of the two particle are also 0. so initially, Only tangential acceleration is involved:

$$a_1 = \alpha L_1 = \frac{3}{11} g \quad \text{upward}$$

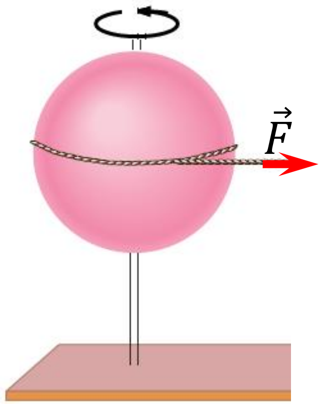
$$a_2 = \alpha L_2 = \frac{9}{11} g \quad \text{downward}$$

Problem 2:



A uniform sphere of radius $\boldsymbol{R}$**=8 cm** can rotate about a vertical axis (through it's center) on **frictionless** bearings. A **massless cord** passes around the equator of the sphere **without slipping**. When we applied a constant horizontal force with the magnitude $\boldsymbol{F = 3}$ **N** on the free end of the cord, the sphere starts to **rotate from rest.** After **5 revolutions**, the angular speed of the sphere arrives $\boldsymbol{\omega =}$ **2 rad/s**. Find the rotational inertial $\boldsymbol{I_s}$ of the sphere about the vertical axis .

Problem 2:



Solution: The sphere rotates at a constant angular acceleration α under the applied force.

Newton's 2nd Law on sphere gives: $\tau_{net} = I\alpha$

$$\tau = FR = I_s\alpha \Rightarrow \alpha = \frac{FR}{I_s} \cdots\cdots (1)$$

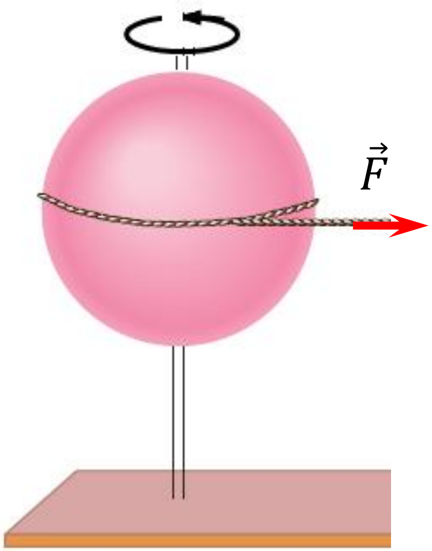
For the rotation about a fixed axis with a constant angular acceleration

$$2\alpha(\theta - \theta_0) = \omega_f^2 - \omega_0^2 \cdots\cdots (2)$$

$$\left.\begin{array}{l} (1)\,\&\,(2) \Rightarrow I_s = 2\dfrac{FR(\theta-\theta_0)}{\omega_f^2 - \omega_0^2} \\ \theta - \theta_0 = 5\text{ rev} = 10\pi\text{ rad} \\ F = 3\,\text{N}, R = 0.08\,\text{m} \\ \omega_0 = 0, \omega_1 = 2\text{ rad/s} \end{array}\right\} \Rightarrow I_s = 3.77\text{ kg}\cdot\text{m}^2$$

Problem 2:

Alternatively, using Work-Kinetic energy theorem $W_{\text{net}} = \Delta K$

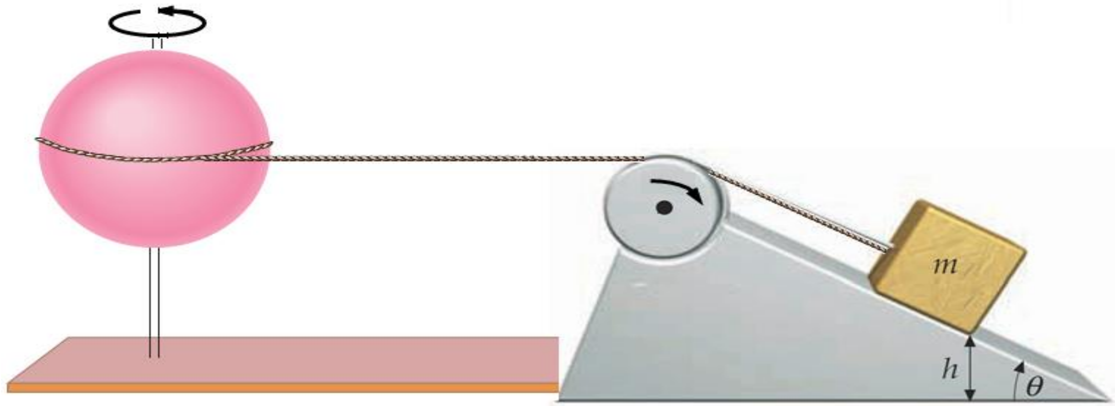


$$W = \tau(\theta - \theta_0) = FR(\theta - \theta_0) \qquad (1)$$

$$\Delta K = \frac{1}{2} I_s \omega_f^2 - \frac{1}{2} I_s \omega_0^2 \qquad (2)$$

$$(1) \,\&\, (2) \Rightarrow I_s = 2\frac{FR(\theta - \theta_0)}{\omega_f^2 - \omega_0^2} = 3.77 \text{ kg} \cdot \text{m}^2$$

Problem 3:

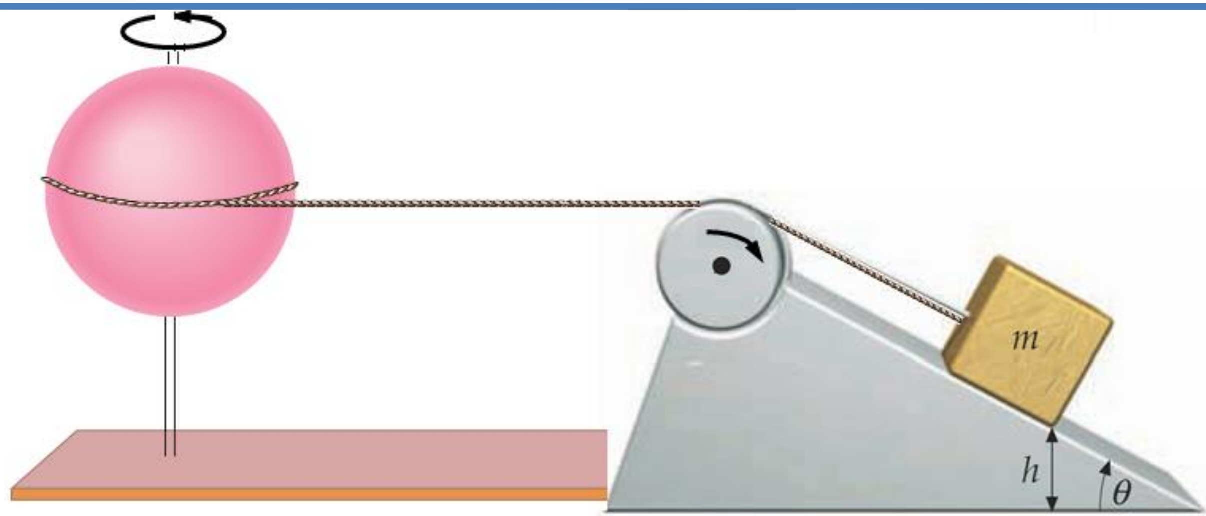


Given the same sphere, now the free end of the cord is connected to a block of mass $\boldsymbol{m = 0.6}$ **kg** over a pulley of rotational inertia $\boldsymbol{I_p = 3.0\ kg \cdot m^2}$ and radius $\boldsymbol{r = 5.0}$ **cm**. There is **no friction** on the pulley's axis. **The cord does not slip on the pulley and the sphere**. The block is on a **frictionless** ramp with inclined angle $\boldsymbol{\theta = 40°}$.

(a) What is the speed $\boldsymbol{v}$ of the block when it has dropped a vertical distance $\boldsymbol{h = 0.8}$ **m** after being released **from rest**?

(b) What is the magnitudes of the acceleration $\boldsymbol{a}$ of block, the angular acceleration $\boldsymbol{\alpha_s}$ of the sphere , and the angular acceleration $\boldsymbol{\alpha_p}$ of the pulley? What is the magnitudes of the tension $\boldsymbol{T_s}$ on the sphere and the tension $\boldsymbol{T_b}$ on the block?

Problem 3:



Solution:

(a) System of **Earth + sphere + pulley + block + cord**.

External forces: force on sphere from its axis, no work
force on pulley from its axis, no work
normal force on block, no work

$\Longrightarrow$ Isolated system $\Delta K + \Delta U = 0$

Only conservative internal force involves.

Therefore, **mechanical energy of the system is conserved.**

Problem 3:

$$\left.\begin{aligned} &\Delta K+\Delta U=0\\ &\Delta K=\frac{1}{2}I_s\omega_{s,f}^2+\frac{1}{2}I_p\omega_{p,f}^2+\frac{1}{2}v_f^2-0\\ &\Delta U=-mgh \end{aligned}\right\}\Rightarrow \frac{1}{2}I_s\omega_{s,f}^2+\frac{1}{2}I_p\omega_{p,f}^2+\frac{1}{2}mv_f^2=mgh\cdots\cdots(1)$$

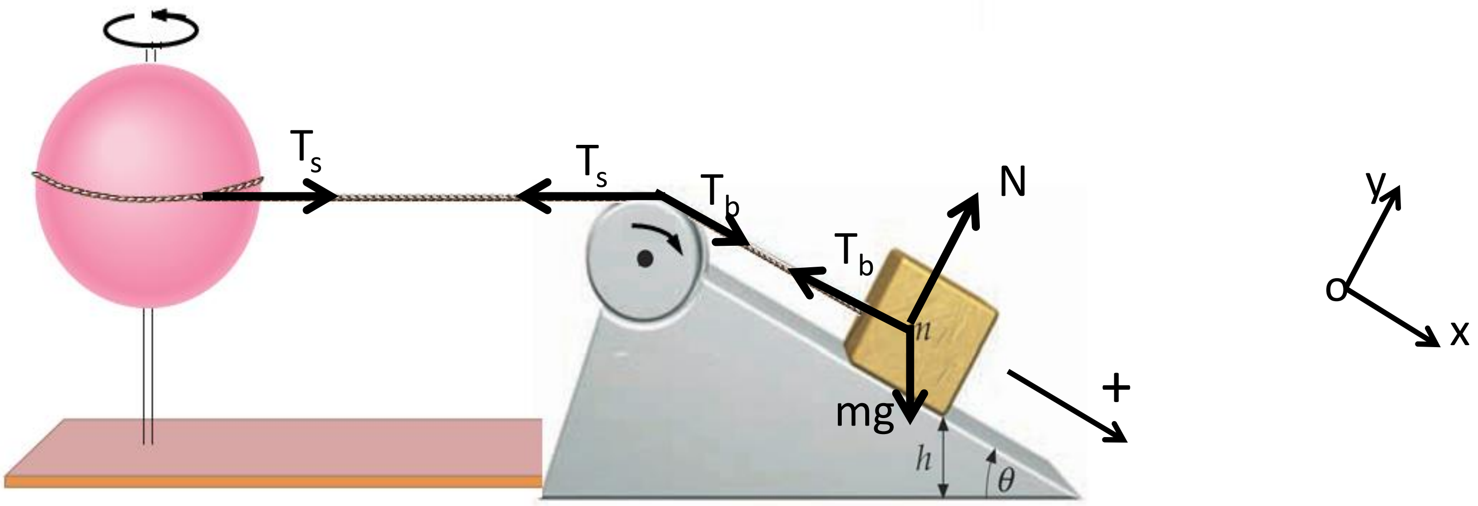
Nonslip cord:

- **Tangential velocity** of the sphere's equator is the **same** as the **linear velocity** of cord around it.
- **Tangential velocity** of the pulley's rim is the **same** as the **linear velocity** of cord around it.
- **The speed** of any piece of the cord is the same as the speed of the block , v_f.

$$v_f=R\omega_{s,f}=r\omega_{p,f}\cdots\cdots(2)\,\&\,(3)$$

$$(1),(2)\,\&\,(3)\Rightarrow v_f=\sqrt{\frac{m}{m+\dfrac{I_s}{R^2}+\dfrac{I_p}{r^2}}\cdot 2gh}=0.1\text{m/s}$$

Problem 3:



(b) The block slide down from rest, so with the acceleration downward along the ramp. Consequently, the pulley rotates clockwise and the sphere rotates in shown direction.

Newton's 2nd law for block: $mg\sin\theta - T_b = ma$ (1)

Newton's 2nd law for pulley: $\tau_{net} = T_b r - T_s r = I_p \alpha_p$ (2)

Newton's 2nd law for sphere: $\tau_{net} = T_s R = I_s \alpha_s$ (3)

Nonslip condition: $a = \alpha_p r = \alpha_S R$ (4 & 5)

Problem 3:

According to (1)~(5)

$$a=\frac{mg\sin\theta}{m+\frac{I_s}{R^2}+\frac{I_p}{r^2}}=0.002\ \mathrm{m/s^2}$$

$$\alpha_s=\frac{mg\sin\theta}{(m+\frac{I_s}{R^2}+\frac{I_p}{r^2})\cdot R}=0.03\ \mathrm{rad/s^2}$$

$$\alpha_r=\frac{mg\sin\theta}{(m+\frac{I_s}{R^2}+\frac{I_p}{r^2})\cdot r}=0.04\ \mathrm{rad/s^2}$$

$$T_s=\frac{I_s\frac{a}{R}}{R}=\frac{I_s}{R^2}\times\frac{mg\sin\theta}{m+\frac{I_s}{R^2}+\frac{I_p}{r^2}}=1.2\ \mathrm{N}$$

$$T_b=\frac{I_p\frac{a}{r}}{r}+T_s=(\frac{I_p}{r^2}+\frac{I_s}{R^2})\times\frac{mg\sin\theta}{m+\frac{I_s}{R^2}+\frac{I_p}{r^2}}=3.8\ \mathrm{N}$$

Discussion:

1） $I_p=0$： $T_1=T_2$

2） $I_s=I_p=0$： $T_1=T_2=0,\ a=g\sin\theta$

3） $\theta=90^\circ$ $I_s=I_p=0$： $a=g$