Assignment 2

12310520 蔺煜涵

T1、(1) let Event A: Fischer wins the match

$$P(A) = 0.4 + 0.3\times0.4 + 0.3^2\times0.4 + \cdots + 0.3^n\times0.4$$
$$= 0.4 + \frac{0.3-0.3^{n+1}}{1-0.3}\times0.4$$
$$= \frac{4}{7}$$

(2) let X be the during of the match

when $1\le i\le 9$ and i in an integer

$$P(X=i) = (1-0.3)\times0.3^{i-1}$$
$$= 0.7\times0.3^{i-1}$$

when $i=10$

$$P(X=i) = 1-\sum_{i=1}^{9}0.7\times0.3^{i-1} = 1-0.7\times(1+\cdots+0.3^8)$$
$$= 1-\cancel{0.7}\times\frac{1-0.3^9}{\cancel{1-0.3}} = 0.3^9$$

$$\Rightarrow P(X=i) = \begin{cases} 0.7\times0.3^{i-1} & i=1,2\cdots9 \\ 0.3^9 & i=10 \end{cases}$$

T2、(1) let X be the number of trials needed to open the door.

$$P(X=1) = \frac{1}{5} \quad P(X=2) = \frac{4}{5}\times\frac{1}{4} = \frac{1}{5}$$
$$P(X=3) = \frac{4}{5}\times\frac{3}{4}\times\frac{1}{3} = \frac{1}{5} \quad \cdots \quad P(X=4) = \frac{1}{5} \quad P(X=5) = $$
$$\Rightarrow P(X=i) = \frac{1}{5} \quad i=1,2,3,4,5$$

(2) $$P(X=i) = \frac{1}{5}\times\left(\frac{4}{5}\right)^{i-1} \quad i\in N^*$$

T3、$$P(X=0) = \frac{8}{10} = \frac{4}{5}$$
$$P(X=1) = \left(1-\frac{4}{5}\right)\times\frac{8}{9} = \frac{8}{45}$$
$$P(X=2) = \left(1-\frac{4}{5}\right)\times\frac{1}{9} = \frac{1}{45}$$
$$E(X) = 0\times\frac{4}{5} + 1\times\frac{8}{45} + 2\times\frac{1}{45} = \frac{2}{9}$$
$$E(X^2) = 0^2\times\frac{4}{5} + 1^2\times\frac{8}{45} + 2^2\times\frac{1}{45} = \frac{4}{15}$$
$$Var(X) = E(X^2) - (E(X))^2 = \frac{4}{15} - \left(\frac{2}{9}\right)^2$$
$$= \frac{88}{405}$$

T4、$$E(X) = \int_0^1 f(x)\cdot x\,dx$$
$$= \int_0^1 (ax^2+bx^3)\,dx$$
$$= \left(\frac{1}{3}ax^3 + \frac{1}{4}bx^4\right)\Big]_0^1$$
$$= \frac{1}{3}a + \frac{1}{4}b = \frac{2}{3}$$
$$\int_0^1 f(x)\,dx = \left(\frac{1}{2}ax^2 + \frac{1}{3}bx^3\right)\Big]_0^1$$
$$= \frac{1}{2}a + \frac{1}{3}b = 1$$
$$\Rightarrow \begin{cases} a=2 \\ b=0 \end{cases}$$
$$E(X^2) = \int_0^1 f(x)\cdot x^2\,dx$$
$$= \int_0^1 (ax^3+bx^4)\,dx$$
$$= \left(\frac{1}{4}ax^4 + \frac{1}{5}bx^5\right)\Big]_0^1$$
$$= \frac{1}{4}a + \frac{1}{5}b = \frac{1}{2}$$
$$Var(X) = E(X^2) - (E(X))^2$$
$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2$$
$$= \frac{1}{18}$$

T5. $$p_X(k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

when $1\le k\le\lambda$, $k\in N$

$$\frac{p_X(k)}{p_X(k-1)} = \frac{\lambda^k \cancel{e^{-\lambda}}(k-1)!}{k!\,\cancel{e^{-\lambda}}\lambda^{k-1}} = \frac{\lambda}{k} > 1$$

and $p_X(k) > 0$, $p_X(k-1) > 0$

$\Rightarrow$ increases monotonically with k up to the point where k reaches the largest integer not exceeding λ

when $k>\lambda$ $k\in N$

$$\frac{p_X(k+1)}{p_X(k)} = \frac{\lambda^{k+1}e^{-\lambda}k!}{\lambda^k e^{-\lambda}(k+1)!} = \frac{\lambda}{k+1} < 1$$

and $p_X(k) > 0$ $p_X(k+1) > 0$

$\Rightarrow$ after that point decreases monotonically with k

T6. let event A : the insurance company loses money in this life insurance.

$$\frac{12 \times 2500}{2000} = 15$$

more than 15 people die

$$P(X \geqslant 16) = 1 - P(X \leqslant 15) = 1 - \sum_{k=0}^{15} \binom{2500}{k} 0.002^{k} 0.998^{2500-k}$$

$n = 2500 > 100$, $p = 0.002 < 0.05$

use Possion $(2500 \times 0.002) =$ Possion (5)

to approximate Binomial $(2500, 0.002)$

$$P(X \leqslant 15) \approx \sum_{k=0}^{15} \frac{5^k}{k!} e^{-5} = 0.999931$$

$$P(X \geqslant 16) \approx 1 - \sum_{k=0}^{15} \frac{5^k}{k!} e^{-5}$$

$$= 0.000069$$

T7. (1) let X be the time between jobs.

$X \sim Exp(3)$

$E(X) = \frac{1}{\lambda} = \frac{1}{3}$

so expected time $\frac{1}{3}$ hour.

(2) $P(X \leqslant \frac{5}{60}) = 1 - e^{-3 \times \frac{5}{60}}$

$= 0.2212$

T8. let $X \sim Exp(\lambda)$

then let $Y = \lceil X \rceil$, $Y \sim$ Geometric $(1 - e^{-\lambda})$

$$P(Y=k) = P(\lceil X \rceil = k) = P(k-1 < X \leq k)$$

$$= \int_{k-1}^{k} \lambda e^{-\lambda x} dx = \left(-e^{-\lambda x}\right]_{k-1}^{k}$$

$$= e^{-\lambda(k-1)} - e^{-\lambda k} = e^{-\lambda(k-1)}(1 - e^{\lambda})$$

let $p = 1 - e^{-\lambda} \Rightarrow e^{-\lambda} = 1 - p$

$$P(Y=k) = p(1-p)^{k-1}$$

$\Rightarrow Y \sim G(1 - e^{-\lambda})$ proved.