Assignment 3

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T1 (1) let income in some country be X

$$X \sim N(900, 200^2)$$

$$P(600 \le X \le 1200) = P\left(\frac{600-900}{200} \le \frac{X-\mu}{\sigma} \le \frac{1200-900}{200}\right)$$

$$= P(-1.5 \le Z \le 1.5)$$

$$= 2(\Phi(1.5) - 0.5) = 2\times(0.9332-0.5)$$

$$= 0.8664$$

the proportion of middle class is 0.8664

(2) $P(Z \le z_0) = 3\% = 0.03$

$$z_0 = -1.88$$

$$\Rightarrow \frac{X-\mu}{\sigma} \le -1.88$$

$$X \le -1.88 \times 200 + 900$$

$$X \le 524$$

below 524 coins

T2. $y^2+4y+X=0$

$\Delta = 16-4X<0$

when $X>4$, there is no real root

there is 0.5 probability that $X>4$ for $X\sim N(\mu,\sigma^2)$

then $\mu=4$

T3. let English score be X

$P(X>96)=2.3\%$

$\Rightarrow P\left(\frac{X-\mu}{\sigma}>\frac{96-72}{\sigma}\right)=2.3\%$

$P\left(Z>\frac{24}{\sigma}\right)=2.3\%=0.023$

$1-\Phi\left(\frac{24}{\sigma}\right)=0.023$

$\Phi\left(\frac{24}{\sigma}\right)=0.977$

$\frac{24}{\sigma}=2.00 \Rightarrow \sigma=12$

$P(60\le X\le 84)$

$=P\left(\frac{60-72}{12}\le\frac{X-\mu}{\sigma}\le\frac{84-72}{12}\right)$

$=P(-1\le Z\le 1)$

$=2(\Phi(1)-0.5)$

$=2\times(0.8413-0.5)$

$=0.6826$

So the probability is 0.6826.

T4. Let the diameter be X, the area be A

$X\sim U(a,b)$ $\quad A=\frac{\pi}{4}X^2$

$$E(X^2)=\int_a^b \frac{x^2}{b-a}\,dx=\frac{1}{b-a}\left(\frac{1}{3}x^3\right)\Big]_a^b$$

$$=\frac{1}{3}(a^2+ab+b^2)$$

$$E(A)=\frac{\pi}{4}E(X^2)=\frac{\pi}{12}(a^2+ab+b^2)$$

T5. $\Phi(Z)$ exists and $\Phi^{-1}(Z)$ exists

let $Y=\Phi(Z)$, then $Y \sim U(0,1)$

$E(Y)=\frac{0+1}{2}=\frac{1}{2}$ $\quad Var(Y)=\frac{(1-0)^2}{12}=\frac{1}{12}$

$E(\Phi(Z))=\frac{1}{2}$

$V(\Phi(Z))=\frac{1}{12}$

$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

T6. (1) $Y_1=g_1(X)=|X|$ increase on $(0,+\infty)$

when $y>0$

$F_{Y_1}(y)=P(Y_1\leq y)=P(|X|\leq y)=\Phi(y)-\Phi(-y)$

$f_{Y_1}(y)=F'_{Y_1}(y)=\phi(y)+\phi(-y)$

$=2\phi(y)=\frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}=\frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{y^2}{2}}$

when $y=0$

$f_{Y_1}(0)=\phi(0)=\frac{1}{\sqrt{2\pi}}$

$$\Rightarrow f_{Y_1}(y)=\begin{cases}\frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{y^2}{2}} & y>0\\ \frac{1}{\sqrt{2\pi}} & y=0\\ 0 & \text{otherwise}\end{cases}$$

(2) $Y_2=g_2(X)=2X^2+1$ increase on $(0,+\infty)$

when $y>1$

$F_{Y_2}(y)=P(Y_2\leq y)=P(2X^2+1\leq y)$

$=\Phi\left(\sqrt{\frac{y-1}{2}}\right)-\Phi\left(-\sqrt{\frac{y-1}{2}}\right)$

$f_{Y_2}(y)=F'_{Y_2}(y)=\phi\left(\sqrt{\frac{y-1}{2}}\right)\frac{\frac{1}{2}}{2\sqrt{\frac{y-1}{2}}}-\phi\left(-\sqrt{\frac{y-1}{2}}\right)\frac{\frac{1}{2}}{-2\sqrt{\frac{y-1}{2}}}$

$=\phi\left(\sqrt{\frac{y-1}{2}}\right)\frac{1}{\sqrt{2(y-1)}}=\frac{1}{2\sqrt{\pi(y-1)}}e^{\frac{1-y}{4}}$

when $y=1$ $f_{Y_2}(y)=\phi(0)=\frac{1}{\sqrt{2\pi}}$

$$\Rightarrow f_{Y_2}(y)=\begin{cases}\frac{1}{2\sqrt{\pi(y-1)}}e^{\frac{1-y}{4}} & y>1\\ \frac{1}{\sqrt{2\pi}} & y=1\\ 0 & \text{otherwise}\end{cases}$$

T7. $X\sim Exp(2)$ 知X $Y=f(X)$ → 可以用 $|h'(y)|\cdot f_X(h(y))$

(1) $g_1(x)=e^{-2x}$ $h_1(y)=g_1^{-1}(y)=\frac{-\ln y}{2}$

$$f_{Y_1}(y)=|h_1'(y)|\cdot f_X(h_1(y))$$
$$=\frac{1}{2y}\,2e^{+\ln y/2\times 2}$$
$$=1$$

$Y_1\sim U(0,1)$

(2) $g_2(x)=1-e^{-2x}$ $h_2(y)=\frac{\ln(1-x)}{-2}$

$$f_{Y_2}(y)=|h_2'(y)|\,f_X(h_2(y))$$
$$=\left|\frac{1}{2}\frac{1}{1-x}\right|\cdot 2e^{+\frac{\ln(1-x)}{2}\cdot 2}$$
$$=1$$

$Y_2\sim U(0,1)$

T8. (1) $\int_0^{+\infty}\int_0^{+\infty}ke^{-(3x+4y)}dxdy$

$$=k\int_0^{+\infty}e^{-3x}dx\int_0^{+\infty}e^{-4y}dy$$
$$=k(e^{-3x})\Big]_0^{+\infty}(e^{-4y})\Big]_0^{+\infty}(-\tfrac{1}{3})(-\tfrac{1}{4})$$
$$=\tfrac{1}{12}k\,(-1)\times(-1)=\tfrac{1}{12}k=1\quad\Rightarrow k=12$$

(2) $F(x,y)=\int_0^y\int_0^x 12e^{-3u}e^{-4v}\,du\,dv$

$$= e^{-3u}\Big]_0^x\, e^{-4v}\Big]_0^y$$

$$=(e^{-3x}-1)(e^{-4y}-1)$$

$\Rightarrow F(x,y)=(e^{-3x}-1)(e^{-4y}-1)$

(3) $P(X+Y)\leq 1=\iint\limits_{x+y\leq 1,\,x,y>0} f(x,y)\,dx\,dy$

$$=\int_0^1\int_0^{1-x} 12e^{-(3x+4y)}\,dy\,dx$$

$$=12\int_0^1 \left(-\tfrac{1}{4}\right)e^{-(3x+4y)}\Big]_0^{1-x}\,dx$$

$$=(-3)\int_0^1\left(e^{-(3x+4-4x)}-e^{-3x}\right)dx$$

$$=(-3)\int_0^1\left(e^{x-4}-e^{-3x}\right)dx$$

$$=(-3)\left(e^{x-4}+\tfrac{1}{3}e^{-3x}\right)\Big]_0^1=(-3)\left(e^{-3}+\tfrac{1}{3}e^{-3}-e^{-4}-\tfrac{1}{3}\right)$$

$$=0.856$$

when $x,y>0$

T9. $f_X(x)=\int_x^{+\infty} f(x,y)\,dy$

$$=\int_x^{+\infty} e^{-y}\,dy=\left(-e^{-y}\right)\Big]_x^{+\infty}=e^{-x}$$

$f_Y(y)=\int_0^y f(x,y)\,dx=\int_0^y e^{-y}\,dx=ye^{-y}$

$$\Rightarrow f_X(x)=\begin{cases} e^{-x} & 0<x<y<\infty \\ 0 & \text{otherwise}\end{cases}$$

$$f_Y(y)=\begin{cases} ye^{-y} & 0<x<y<\infty \\ 0 & \text{otherwise}\end{cases}$$