Assignment 5

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T1. X and Y are independent

$$P(Z=1)=\int_0^\infty\int_0^y \lambda e^{-\lambda x}\mu e^{-\mu y}dxdy$$
$$=\int_0^\infty (-e^{-\lambda x})\Big]_0^y \mu e^{-\mu y}dy$$
$$=\int_0^\infty (1-e^{-\lambda y})\mu e^{-\mu y}dy$$
$$=\int_0^\infty \mu e^{-\mu y}dy-\int_0^\infty \mu e^{-(\lambda+\mu)y}dy$$
$$=(-e^{-\mu y})\Big]_0^{+\infty}-\left(-\frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)y}\right)\Big]_0^{+\infty}$$
$$=1-\frac{\mu}{\lambda+\mu}=\frac{\lambda}{\lambda+\mu}$$

$$P(Z=0)=1-P(Z=1)=\frac{\mu}{\lambda+\mu}$$

$$\Rightarrow f(z)=\begin{cases}\frac{\lambda}{\lambda+\mu}, & z=1\\ \frac{\mu}{\lambda+\mu}, & z=0\end{cases}$$

T2、X、Y are independent, identically distributed r.v.s

$$P(Z=1)=P(X=1)\cdot P(Y=1)=p^2$$

$$P(Z=2)=P(X=1)P(Y=2)+P(X=2)P(Y=1)+P(X=2)(Y=2)$$
$$=(1-p)p^2\cdot 2+(1-p)^2p^2$$

$$P(Z=z)=2\sum_{i=1}^{z-1}P(X=i)P(Y=z)+P(X=z)P(Y=z)$$
$$=2(1-p)^{z-1}p\cdot p(1+(1-p)+\cdots+(1-p)^{z-2})+(1-p)^{2z-2}p^2$$
$$=2(1-p)^{z-1}p^2\frac{1-(1-p)^{z-1}}{p}+(1-p)^{2z-2}p^2$$

$$= 2p(1-p)^{z-1} - 2p(1-p)^{2z-2} + (1-p)^{2z-2}p^2$$

$$\Rightarrow P(Z=z) = 2p(1-p)^{z-1} - 2p(1-p)^{2z-2} + p^2(1-p)^{2z-2} \quad z=1,2,\cdots$$

T3. Let X、Y, Z be the lifespan of three components.

$$f_X(t) = f_Y(t) = f_Z(t) = \lambda e^{-\lambda t} \quad t>0$$

let $A = \min(X, Y)$, $T = \min(A, Z)$

$$F_A(t) = 1-(1-F_X(t))(1-F_Y(t)) = 1-e^{-2\lambda t} \quad t>0$$

$$F_T(t) = 1-(1-F_A(t))(1-F_Z(t)) = 1-e^{-2\lambda t}e^{-\lambda t}$$

$$= 1-e^{-3\lambda t} \quad t>0$$

$$\Rightarrow f_T(t) = F'_T(t) = 3\lambda e^{-3\lambda t} \quad t>0$$

T4. (1) $\binom{X}{Y} \sim N(\mu, \Sigma)$ $\mu = \binom{0}{0}$ $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

let $Z = X-Y$

$$Z \sim N(\mu_X - \mu_Y, \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2)$$

$$\Rightarrow Z \sim N(0, 2-2\rho)$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi(2-2\rho)}} e^{-\frac{z^2}{4-4\rho}} = \frac{1}{2\sqrt{\pi(1-\rho)}} e^{-\frac{z^2}{4-4\rho}}$$

(2) $\mathrm{Cov}(X-Y, XY) = \mathrm{Cov}(X, XY) - \mathrm{Cov}(Y, XY)$

$$= \left(E(X^2Y) - E(X)E(XY)\right) - \left(E(Y^2X) - E(Y)E(XY)\right)$$

$$= E(X^2Y) - E(Y^2X)$$

X、Y are independent, identically distributed r.v.s

$\Rightarrow E(X^2Y) = E(Y^2X)$

$\Rightarrow Cov(X-Y, XY) = 0$

$$\rho_{X-Y,XY} = \frac{Cov(X-Y, XY)}{\sqrt{Var(X-Y)Var(XY)}} = 0$$

T5. (1) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \frac{1}{2\pi 6_X 6_Y \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x}{6_X} - \rho\frac{y}{6_Y}\right)^2 + (1-\rho^2)\frac{y^2}{6_Y^2}\right] dx$$

$$= \frac{1}{2\pi 6_X 6_Y \sqrt{1-\rho^2}} \cdot \exp\left(-\frac{y^2}{2 6_Y^2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x}{6_X} - \rho\frac{y}{6_Y}\right)^2\right) dx$$

let $t = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x}{6_X} - \rho\frac{y}{6_Y}\right)$ $dx = dt\sqrt{1-\rho^2}\, 6_X$

$$= \frac{1}{2\pi 6_X 6_Y \sqrt{1-\rho^2}} \exp\left(-\frac{y^2}{2 6_Y^2}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}t^2\right) \sqrt{1-\rho^2}\, 6_X\, dt$$

$$= \frac{1}{2\pi 6_Y} \exp\left(-\frac{y^2}{2 6_Y^2}\right) \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} dt$$

$$= \frac{1}{2\pi 6_Y} e^{-\frac{y^2}{2 6_Y^2}} \sqrt{2\pi} = \frac{1}{\sqrt{2\pi}\, 6_Y} e^{-\frac{y^2}{2 6_Y^2}}$$

(2) $$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{2\pi 6_X 6_Y \sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{6_X^2} - 2\rho\frac{xy}{6_X 6_Y} + \frac{y^2}{6_Y^2}\right]\right)}{\frac{1}{\sqrt{2\pi}\, 6_Y} e^{-\frac{y^2}{2 6_Y^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\, 6_X} \exp\left(-\frac{1}{2(1-\rho^2)6_X^2}\left[x - \frac{\rho 6_X y}{6_Y}\right]^2\right)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\,\sigma_X} \exp\left(-\frac{1}{2(1-\rho^2)\sigma_X^2}\left[x-\left(\frac{\rho\sigma_X}{\sigma_Y}y\right)^2\right]\right)$$

$$\mu = \frac{\rho\sigma_X}{\sigma_Y}y \qquad \sigma^2 = (1-\rho^2)\sigma_X^2$$

T6. $\binom{X}{Y} \sim N(\mu, \Sigma) \qquad \mu = \binom{4500}{5500}$

$$\rho\sigma_X\sigma_Y = 0.65\times1500\times2000 = 1950000 \quad \Sigma = \begin{pmatrix} 1500^2 & 1950000 \\ 1950000 & 2000^2 \end{pmatrix}$$

$$X|Y=y \sim N\left(4500 + \frac{0.65\times1500}{2000}(6800-5500),\ (1-0.65^2)\times1500^2\right)$$

$$\sim N(5133.75,\ 1299375)$$

$$P(X<6800|Y=6800) = P\left(\frac{X-5133.75}{\sqrt{1299375}} < \frac{6800-5133.75}{\sqrt{1299375}}\right)$$

$$= \Phi(1.46) = 0.9279$$

(2)

$$X-Y \,|\, X+Y=12000 \qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$A\binom{X}{Y} \sim N(A\mu,\ A\Sigma A^T)$$

$$\sim N\left(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\binom{4500}{5500},\ \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1500^2 & 1950000 \\ 1950000 & 2000^2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right)$$

$$\sim N\left(\begin{pmatrix} -1000 \\ 10000 \end{pmatrix},\ \begin{pmatrix} 235\times10^4 & -175\times10^4 \\ -175\times10^4 & 1015\times10^4 \end{pmatrix}\right)$$

Let $M = X-Y,\ N = X+Y \quad \mu_M = -1000,\ \mu_N = 10000$

$$\sigma_M = \sqrt{235}\times100 \quad \sigma_N = \sqrt{1015}\times100 \quad \rho = \frac{-175}{\sqrt{235\times1015}}$$

$$M|N=12000 \sim N\left(-1000 + \frac{\rho\sigma_M}{\sigma_N}\times(12000-10000),\ (1-\rho^2)\sigma_M^2\right)$$

$$\sim N(-1344.83,\ 2048275)$$

$$P(M>0 \mid N=12000) = 1 - P(M \le 0) = 1 - P\left(\frac{M+1344.83}{\sqrt{2048275}} \le \frac{1344.83}{\sqrt{2048275}}\right)$$

$$= 1 - \Phi(0.94) = 1 - 0.8264 = 0.1736$$

T7. $X \sim \text{Poisson}(\lambda)$ then $P(X=k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$

$\mu = \lambda \quad Var(x) = \lambda$

$\Rightarrow$ when $\lambda \to +\infty \quad x \overset{\text{approx}}{\sim} N(\lambda, \lambda)$

$$P(X=k) = \frac{1}{\sqrt{2\pi\lambda}}\, e^{-\frac{(k-\lambda)^2}{2\lambda}}$$

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import poisson, norm

# 定义不同的 λ 值
lambda_values = [5, 10, 20, 50]

# 创建图形
plt.figure(figsize=(12, 10))

for i, lam in enumerate(lambda_values):
    # 泊松分布
    x_poisson = np.arange(0, lam + 3 * np.sqrt(lam))  # 范围从 0 到 λ + 3σ
    y_poisson = poisson.pmf(x_poisson, lam)

    # 正态分布近似
    x_normal = np.linspace(0, lam + 3 * np.sqrt(lam), 1000)
    y_normal = norm.pdf(x_normal, lam, np.sqrt(lam))

    # 创建子图
    plt.subplot(2, 2, i + 1)
    plt.bar(x_poisson, y_poisson, width=0.5, color='yellow', alpha=0.6, label=f'Poisson λ={lam}')
    plt.plot(x_normal, y_normal, 'b-', label=f'Normal Approx. λ={lam}')
    plt.title(f'Poisson and Normal Approximation (λ={lam})')
    plt.xlabel('x')
    plt.ylabel('Probability')
    plt.legend()

# 显示图形
plt.tight_layout()
plt.show()
```