工程概率统计

Probability and Statistics for Engineering

第三章 联合分布

Chapter 3 Joint Distributions

Chapter 3 Joint Distributions

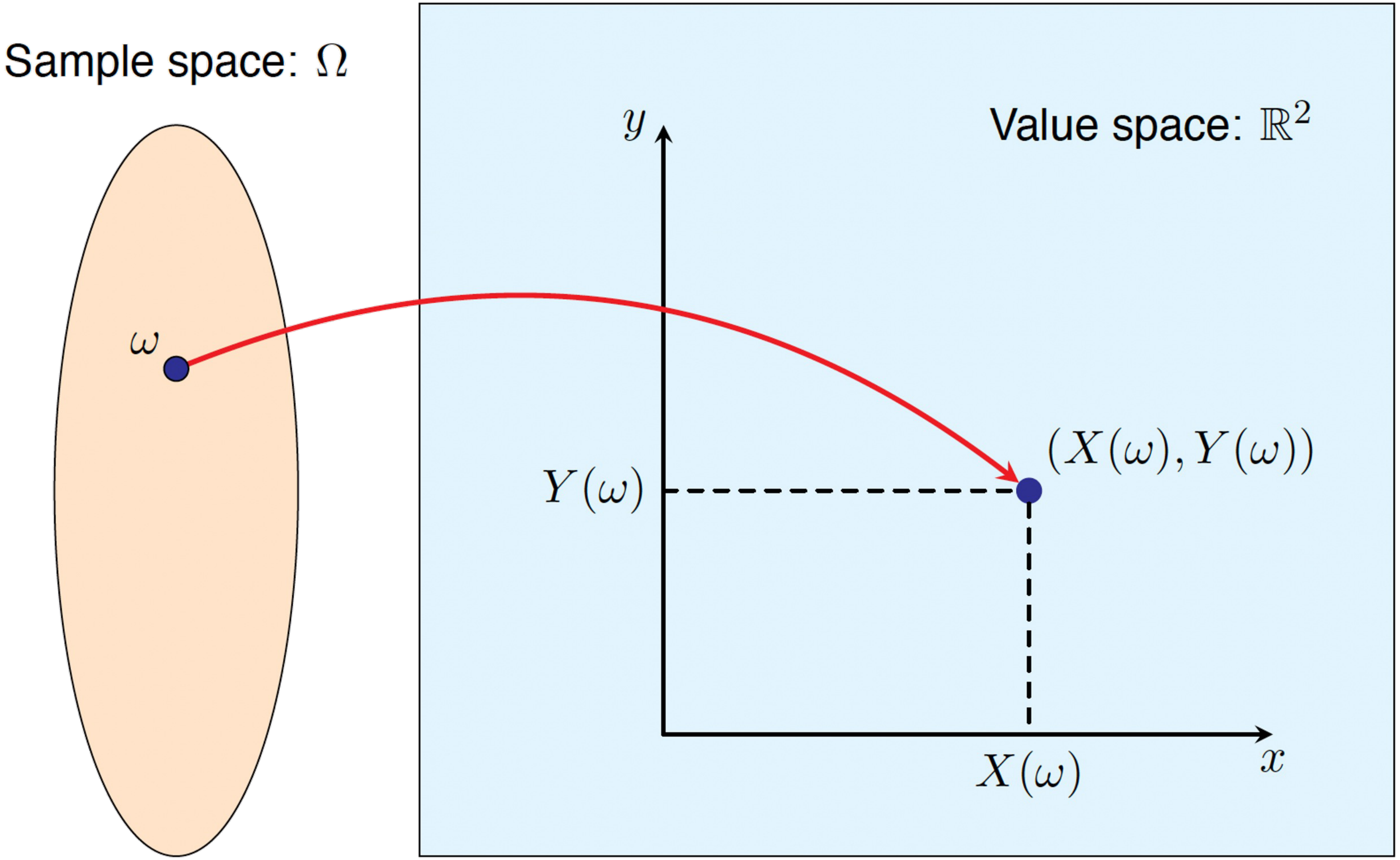
3.1 Random Vector and Joint Distribution

- We have been considering a single random variable each time, however, often we need to deal with several random variables. E.g.,
  - The height and weight of a randomly selected person.
  - The temperature, humidity, wind, precipitation of Singapore on a randomly selected day.
- Since different metrics of the same object are typically associated with each other, they should be considered simultaneously.
- Here we will talk about how to study two random variables X and Y simultaneously, all the concepts can be extended to n random variables $X_1, X_2, \dots, X_n$.

Random Vector

We say that (X, Y) is a **random vector (随机向量)** if $\omega \in \Omega \longmapsto (X(\omega), Y(\omega))$ is a function valued on $\mathbb{R}^2$. (X, Y) can also be called a two-dimensional random variable (二维随机变量).

3.1 Random Vector and Joint Distribution



3.1 Random Vector and Joint Distribution

- The major concepts that will be introduced for random vectors are:
  - (Joint) CDF/PMF/PDF;
  - Marginal CDF/PMF/PDF;
  - Conditional PMF/PDF.
- First, for a discrete/continuous random vector, we can describe its distribution using the (joint) CDF (cumulative distribution function).

Joint Cumulative Distribution Function

For a random vector (X, Y), either discrete or continuous, its cumulative distribution function (CDF, 累积分布函数) is defined as

$$F(x, y) = P(X \leq x, Y \leq y), \forall x, y \in \mathbb{R}.$$

$F(x, y)$ is also called the joint CDF (联合累积分布函数) of X and Y.

- You can treat $\{X \leq x\}$ as an event A, and $\{Y \leq y\}$ as an event B, then

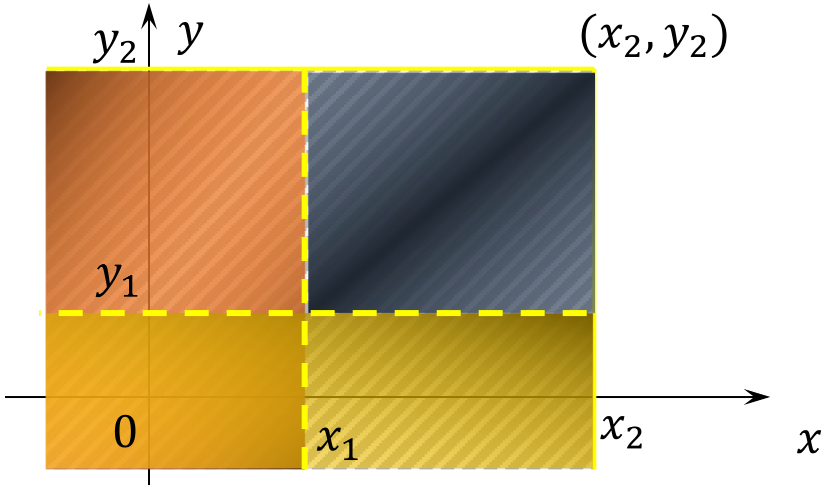
$$F(x, y) = P(A \cap B).$$

- It follows that for $\forall x, y \in \mathbb{R}$, we have $0 \leq F(x, y) \leq 1$ and

$$F(+\infty, +\infty) = 1, F(-\infty, -\infty) = 0,$$

$$F(-\infty, y) = 0, F(x, -\infty) = 0.$$

3.1 Random Vector and Joint Distribution



- The joint CDF can be used to compute $P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$ for any $-\infty < x_1 < x_2 < \infty, -\infty < y_1 < y_2 < \infty$.

$$P\{x_1 < X \leq x_2,\ \ y_1 < Y \leq y_2\}$$
$$= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$
$$\geq 0.$$

- With the joint CDF, it is straight forward to define the marginal CDF.

Marginal CDF

Let $F(x, y)$ be the joint CDF of (X, Y), $F_X(x)$ be the CDF of X without considering Y, and $F_Y(y)$ be the CDF of Y without considering X, then for $\forall x, y \in \mathbb{R}$:

$$F_X(x) = F(x, \infty), F_Y(y) = F(\infty, y).$$

F_X (F_Y) is also called the marginal CDF (边缘累积分布函数) of X (Y).

Note: While the joint CDF uniquely determines the marginal CDFs, the reverse is not true.

3.1 Random Vector and Joint Distribution

Example 3.1

- Suppose that the joint CDF of random vector (X,Y) is (a,b,c are constants)

$$F(x,y)=a\left(b+\arctan\frac{x}{2}\right)\left(c+\arctan\frac{y}{2}\right),-\infty<x,y<\infty.$$

- 1. Determine the value of a,b,c. 2. Calculate $P(-2<X\leq 2,-2<Y\leq 2)$.
- 3. Obtain the marginal CDFs of X and Y.

Solution

- 1. By the basic properties of the joint CDF, we have:

$$F(+\infty,+\infty)=a\left(b+\frac{\pi}{2}\right)\left(c+\frac{\pi}{2}\right)=1,\ F(-\infty,+\infty)=a\left(b-\frac{\pi}{2}\right)\left(c+\frac{\pi}{2}\right)=0,$$

$$F(+\infty,-\infty)=a\left(b+\frac{\pi}{2}\right)\left(c-\frac{\pi}{2}\right)=0.\quad\Rightarrow b=\frac{\pi}{2},c=\frac{\pi}{2},a=\frac{1}{\pi^2}.$$

- Therefore, the joint CDF is

$$F(x,y)=\frac{1}{\pi^2}\left(\frac{\pi}{2}+\arctan\frac{x}{2}\right)\left(\frac{\pi}{2}+\arctan\frac{y}{2}\right).$$

3.1 Random Vector and Joint Distribution

Solution

- 2. By the definition and property of the joint CDF, we have:

$$P(-2<X\le 2,-2<Y\le 2)=F(2,2)-F(2,-2)-F(-2,2)+F(-2,-2)$$

$$=\frac{1}{\pi^2}\left[\left(\frac{\pi}{2}+\arctan(1)\right)^2-2\left(\frac{\pi}{2}+\arctan(1)\right)\left(\frac{\pi}{2}+\arctan(-1)\right)+\left(\frac{\pi}{2}+\arctan(-1)\right)^2\right]$$

$$=\frac{1}{\pi^2}\left[\left(\frac{\pi}{2}+\arctan(1)\right)-\left(\frac{\pi}{2}+\arctan(-1)\right)\right]^2=\frac{1}{\pi^2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]^2=\frac{1}{4}.$$

- 3. By the definition of the marginal distribution, we have:

$$F_X(x)=F(x,+\infty)=\frac{1}{\pi^2}\left(\frac{\pi}{2}+\arctan\frac{x}{2}\right)\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x}{2},$$

$$F_Y(y)=F(+\infty,y)=\frac{1}{2}+\frac{1}{\pi}\arctan\frac{y}{2}.$$

3.1 Random Vector and Joint Distribution

- Then for discrete random vectors, we introduce the joint and marginal PMF.

Joint and Marginal PMF for Discrete Random Vector

- For a random vector (X, Y), let $S_X = \{x_1, x_2, \dots\}$ and $S_Y = \{y_1, y_2, \dots\}$ be the support of X and Y, respectively. Then the joint PMF (联合概率质量函数) of (X, Y) is defined as

$$p(x_i, y_j) = P(X = x_i, Y = y_j) \triangleq p_{ij}, i, j = 1, 2, \dots.$$

- The marginal PMF (边缘概率质量函数) of X is the PMF of X without considering Y:

$$p_X(x_i) = P(X = x_i) = \sum_{j=1}^{\infty} p_{ij} \triangleq p_{i\cdot}, i = 1, 2, \cdots.$$

- Similarly, the marginal PMF of Y is

$$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{\infty} p_{ij} \triangleq p_{\cdot j}, j = 1, 2, \dots.$$

- The joint PMF satisfies:
- Non-negativity: $p_{ij} \geq 0, i, j = 1, 2, \dots;$
- Normalization: $\sum_i \sum_j p_{ij} = 1.$

- The joint PMF is typically displayed in a tabular format:

$X \backslash Y$	y_1	y_2	$\cdots$	y_j	$\cdots$
x_1	p_{11}	p_{12}	$\cdots$	p_{1j}	$\cdots$
x_2	p_{21}	p_{22}	$\cdots$	p_{2j}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
x_i	p_{i1}	p_{i2}	$\cdots$	p_{ij}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$

3.1 Random Vector and Joint Distribution

Example 3.2

- Two dice are tossed independently. Let X be the smaller number of points and Y be the larger number of points. If both dice show the same number, say, z points, then $X = Y = z$.
- 1. Find the joint PMF of (X, Y); 2. Find the marginal PMF of X.

Solution

- 1. Since the two dice are tossed independently, it is not difficult to obtain the joint PMF of (X, Y):

$X \backslash Y$	1	2	3	4	5	6
1	1/36	1/18	1/18	1/18	1/18	1/18
2	0	1/36	1/18	1/18	1/18	1/18
3	0	0	1/36	1/18	1/18	1/18
4	0	0	0	1/36	1/18	1/18
5	0	0	0	0	1/36	1/18
6	0	0	0	0	0	1/36

- 2. With the joint PMF of (X, Y), the marginal PMF of X can be directly obtained as:

Value	1	2	3	4	5	6
Prob.	$\frac{11}{36}$	$\frac{1}{4}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{1}{12}$	$\frac{1}{36}$

3.1 Random Vector and Joint Distribution

- Then for continuous random vectors, we introduce the joint PDF.

Joint PDF for Continuous Random Vector

- (X, Y) is said to be a continuous random vector if there exists a non-negative function $f(x, y)$, defined for all $(x, y) \in \mathbb{R}^2$, satisfies that for any $D \subset \mathbb{R}^2$,

$$P\big((X, Y) \in D\big) = \iint_{(x,y)\in D} f(x, y)dxdy.$$

  $f(x, y)$ is called the joint PDF (联合概率密度函数) of (X, Y)

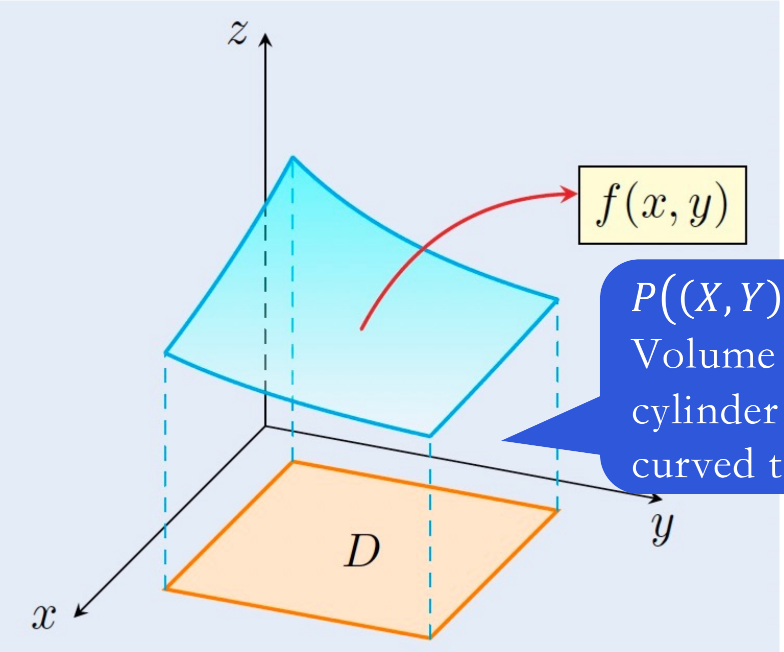
- Particularly, we have the joint CDF of (X, Y) to be

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v)dudv.$$

- It follows that

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y).$$

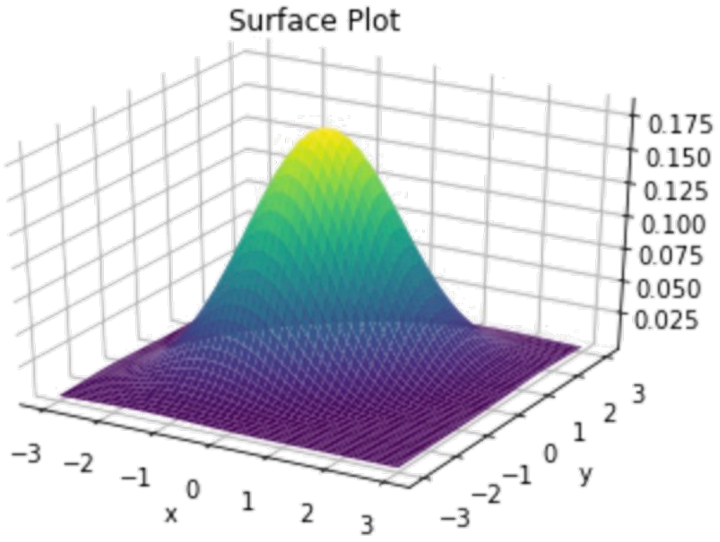
- The joint PDF satisfies:
- Non-negativity: $f(x, y) \geq 0, \forall (x, y) \in \mathbb{R}$;
- Normalization: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$.

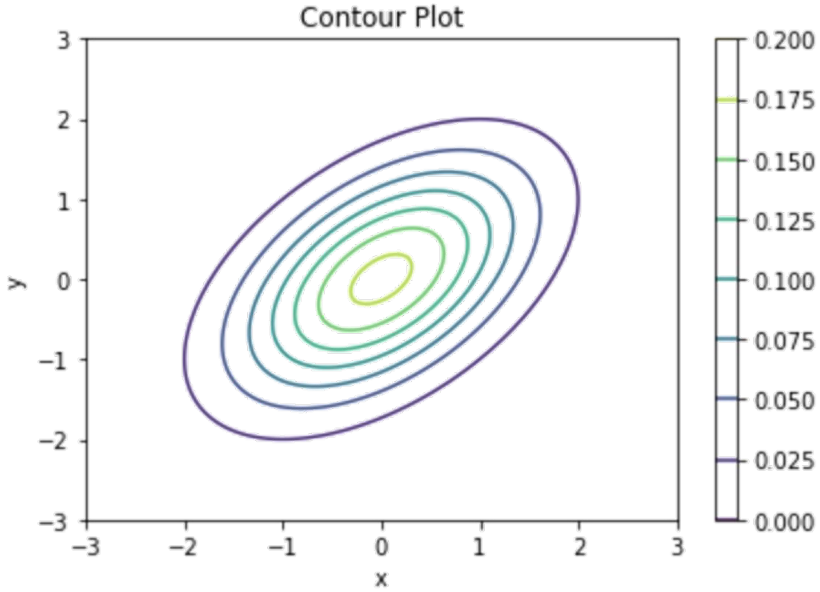


$P\big((X, Y) \in D\big) =$ Volume of the cylinder with a curved top

3.1 Random Vector and Joint Distribution

- Similar to the PDF of a single r.v., the joint PDF $f(x, y) \neq P(X = x, Y = y)$. Instead, it reflects the degree to which the probability is concentrated around (x, y).
- The joint PDF are typically visualized with the **surface plot (曲面图)** or the **contour plot (等高线图)** which help in intuitively understanding the distribution and relationship between X and Y.
  - **Surface plot**: a 3D plot where the height represents the value of the joint PDF at each point (x, y).
  - **Contour plot**: a 2D plot showing level curves where the joint PDF has constant values.

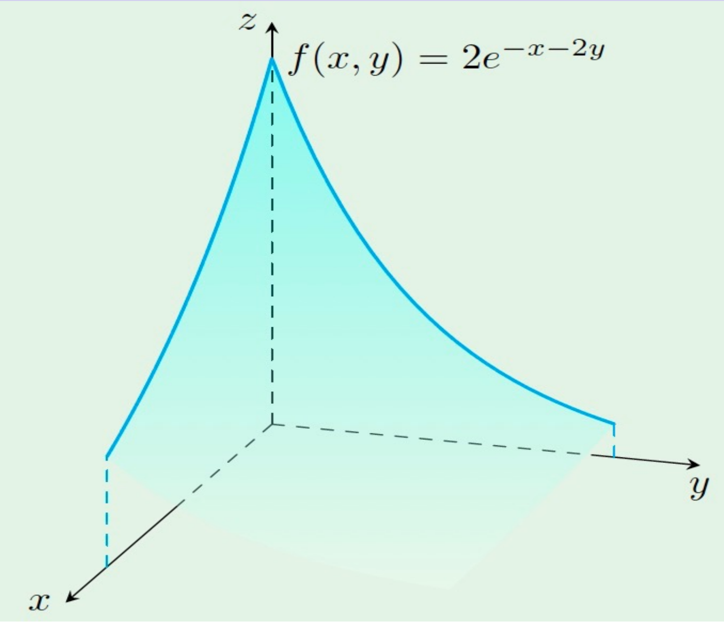




3.1 Random Vector and Joint Distribution

Example 3.3

- The lifetime (in years) of two electronic components of a randomly selected machine is denoted by r.v.s X and Y, which has a joint PDF
$$f(x,y)=\begin{cases}2e^{-x-2y}, & 0<x,y<\infty\\ 0, & \text{otherwise}\end{cases}.$$
- Compute: 1. $P(X<1, Y<1)$; 2. $P(X<Y)$.



3.1 Random Vector and Joint Distribution

Solution

- 1. By the definition of the joint PDF of (X, Y), we have:

$$P(X<1, Y<1)=\int_{-\infty}^{1}\left(\int_{-\infty}^{1} f(x, y)\, dx\right) dy=\int_{0}^{1}\left(\int_{0}^{1} e^{-x} dx\right) 2 e^{-2 y} dy$$

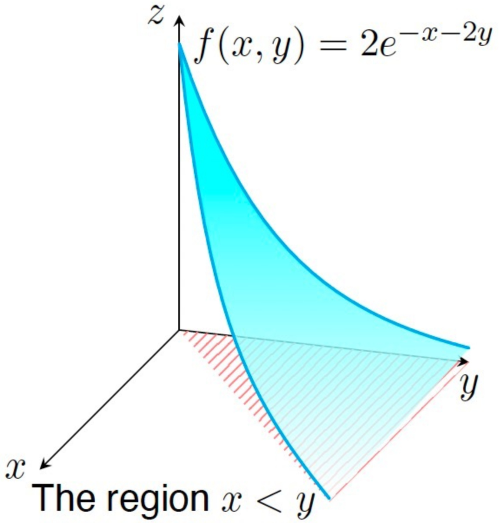
$$=\left(1-e^{-1}\right) \int_{0}^{1} 2 e^{-2 y} dy=\left(1-e^{-1}\right)\left(1-e^{-2}\right) \approx 0.5466.$$

- 2. Again, by the joint PMF of (X, Y):

$$P(X<Y)=\iint_{(x, y): x<y} f(x, y) dx dy=\int_{0}^{\infty}\left(\int_{0}^{y} 2 e^{-x-2 y} dx\right) dy$$

$$=\int_{0}^{\infty} 2 e^{-2 y}\left(1-e^{-y}\right) dy=\int_{0}^{\infty} 2 e^{-2 y} dy-\int_{0}^{\infty} 2 e^{-3 y} dy$$

$$=1-\frac{2}{3}=\frac{1}{3}.$$



3.1 Random Vector and Joint Distribution

- With the joint PDF, the marginal PDF can be determined.
- The marginal PDF mirrors the definition of the marginal PMF for the discrete case, except with sums replaced by integrals and the joint PMF replaced by the joint PDF.

Marginal PDF for Continuous Random Vector

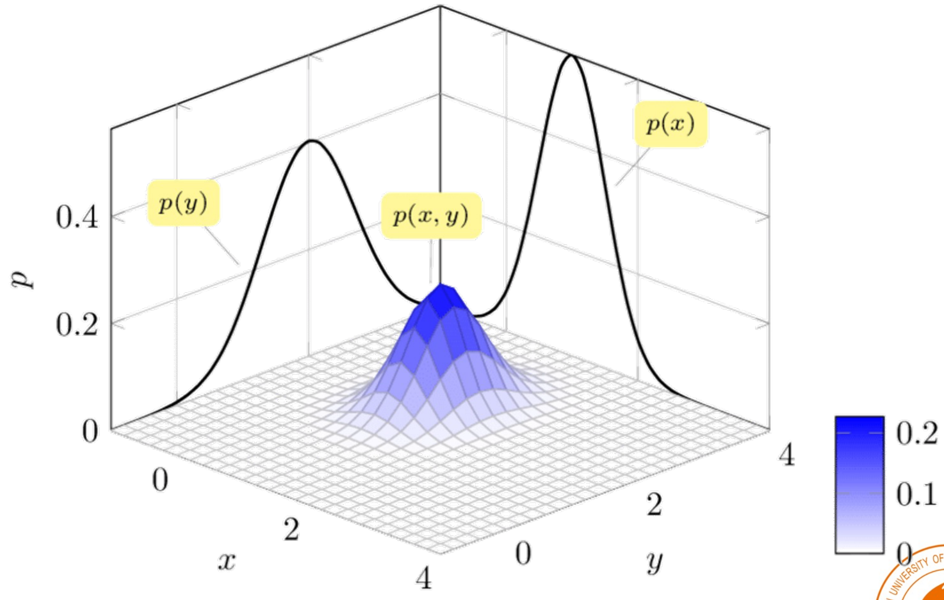
- (X, Y) a continuous random vector with joint PDF $f(x, y)$, then the **marginal PDF (边缘概率密度函数)** of X, i.e., the PDF of X without considering Y is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy .$$

- Likewise, the marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

- Note that a joint PDF uniquely defines the marginal PDFs, however, the reverse is not true.



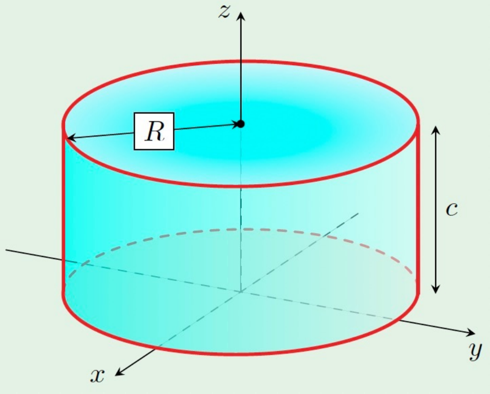
3.1 Random Vector and Joint Distribution

Example 3.4

- Suppose that the joint PDF of a random vector (X, Y) is given by

$$f(x,y) = \begin{cases} c, \text{if } x^2+y^2 \le R^2 \\ 0, \qquad \text{otherwise} \end{cases}$$

- 1. Determine the constant c;
- 2. Find the marginal PDF of X.



Solution

- 1. By the normalization property of the joint PDF:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1,$$

- it follows that

$$1 = \iint_{(x,y):x^2+y^2\le R^2} c\,dxdy \Longrightarrow c = \frac{1}{\pi R^2}.$$

- 2. With the joint PDF of (X, Y), the marginal PDF of X can be directly obtained as:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} c dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2},$$

- if $|x| \le R$; and $f_X(x) = 0$, otherwise.

3.1 Random Vector and Joint Distribution

- Finally, we talk about the conditional PMF/PDF.

Conditional PMF/PDF

- For a **discrete** random vector (X, Y) with joint PMF $p(x, y)$, the conditional PMF (条件概率质量函数) of X given $Y = y$ is defined as
$$p_{X|Y}(x|y) \triangleq P(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)},$$
for all values of y such that $p_Y(y) > 0$.

- For a **continuous** random vector (X, Y) with joint PDF $f(x, y)$, the conditional PDF (条件概率密度函数) of X given $Y = y$ is defined as
$$f_{X|Y}(x|y) \triangleq \frac{f(x, y)}{f_Y(y)},$$
for all values of y such that $f_Y(y) > 0$.

- The conditional PDF mirrors the definition of the conditional PMF for the discrete case, except with the joint/marginal PMF replaced by the joint/marginal PDF.

- The conditional PMF/PDFs also satisfy the non-negativity and normalization properties.

- Question: for the continuous case, $P(Y = y) = 0$, so that $P(X = x|Y = y)$ or $P(X \leq x|Y = y)$ is not defined. Then how to understand the conditional PDF of X given $Y = y$?

3.1 Random Vector and Joint Distribution

- Here we talk about how to understand the conditional PDF of X given $Y = y$.
- Conditioning on $Y = y$ can be understood as conditioning on $\{y \le Y \le y + \varepsilon\}$ where $\varepsilon \to 0$.
- Consider the conditional CDF:

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$$P\{X \le x | y < Y \le y + \varepsilon\} = \frac{P\{X \le x, y < Y \le y + \varepsilon\}}{P\{y < Y \le y + \varepsilon\}} = \frac{\int_{-\infty}^{x} \int_{y}^{y+\varepsilon} f(u, v) dv du}{\int_{y}^{y+\varepsilon} f_Y(y) dy}$$

By the mean value theorem of integrals (积分中值定理)

$$= \frac{\varepsilon \int_{-\infty}^{x} f(u, y_\varepsilon) du}{\varepsilon f_Y(\tilde{y}_\varepsilon)} \to \int_{-\infty}^{x} \frac{f(u, y)}{f_Y(y)} du \quad (\varepsilon \to 0).$$

The conditional PDF

- By the definition of the conditional PDF, we have

$$f(x, y) = f_{X|Y}(x|y) f_Y(y).$$

- Take the integration w.r.t. y on both sides, the marginal distribution of X can be expressed as

连续情况 $$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y) dy$$

The law of total probability under the continuous case

离散情况 $P_X(x) = \sum_{i=1}^{n} P_{X|Y}(x|y_i) P_Y(y_i)$

3.1 Random Vector and Joint Distribution

Example 3.5

- For a randomly selected person in an automobile accident, let X be his/her extent of injury and Y be the type of safety restraint he/she was wearing at the time of the accident. The joint PMF of X and Y is

$X \backslash Y$	1 (None)	2 (Belt Only)	3 (Belt and Harness)	$p_X(x)$
1 (None)	0.065	0.075	0.060	0.20
2 (Minor)	0.165	0.160	0.125	0.45
3 (Major)	0.145	0.10	0.055	0.30
4 (Death)	0.025	0.015	0.010	0.05
$p_Y(y)$	0.40	0.35	0.25	1.00

- 1. What is the PMF of extent of injury for a randomly selected person with no restraint?
- 2. What is the PMF of extent of injury for a randomly selected person with belt and harness?

3.1 Random Vector and Joint Distribution

Solution

- 1. By the definition of conditional PMF, we have the conditional distribution of X given $Y = 1$ is

Value	1 (None)	2 (Minor)	3 (Major)	4 (Death)
Prob.	$\frac{0.065}{0.4} = 0.1625$	$\frac{0.165}{0.4} = 0.4125$	$\frac{0.145}{0.4} = 0.3625$	$\frac{0.025}{0.4} = 0.0625$

- 2. Similarly, the conditional distribution of X given $Y = 3$ is

Value	1 (None)	2 (Minor)	3 (Major)	4 (Death)
Prob.	$\frac{0.06}{0.25} = 0.24$	$\frac{0.125}{0.25} = 0.50$	$\frac{0.055}{0.25} = 0.22$	$\frac{0.01}{0.25} = 0.04$

- With belt and harness, the probability of a major injury or death is 16.5% lower than the case without any safety restraint.

Example 3.4 (Continued)

- Determine the conditional PDF of X given $Y = y$ (where $|y| \le R$).

Solution

- Similar with 2, we obtain the marginal PDF of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \frac{2\sqrt{R^2 - y^2}}{\pi R^2}, \text{if } |y| \le R \\ 0, \quad \text{otherwise} \end{cases}.$$

- Then, by the definition, the conditional PDF of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{2\sqrt{R^2 - y^2}}, \text{if } |x| \le \sqrt{R^2 - y^2} \\ 0, \quad \text{otherwise} \end{cases}.$$

- This suggest that given $Y = y$, X follows $\text{Uniform}[-\sqrt{R^2 - y^2}, \sqrt{R^2 - y^2}]$.
- Since $f_{X|Y}(x|y) \ne f_X(x)$, we say that X is not independent of Y.