Assignment 4

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T1. $f(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$

$$= \begin{cases} 15x^2y & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$P(X>0.5) = 1 - P(X \leq 0.5)$

$$= 1 - \int_0^{0.5}\int_x^1 f(x,y)\,dy\,dx$$

$$= 1 - \int_0^{0.5} \left(\frac{15}{2}x^2y^2\right)\Big]_x^1 dx$$

$$= 1 - \int_0^{0.5} \left(\frac{15}{2}x^2 - \frac{15}{2}x^4\right) dx$$

$$= 1 - \left(\frac{5}{2}x^3 - \frac{3}{2}x^5\right)\Big]_0^{0.5}$$

$$= 1 - \left(\frac{5}{16} - \frac{3}{64}\right) = \frac{47}{64}$$

$P(X>0.5) = \frac{47}{64}$

T2. $\sum_i \sum_j P_{ij} = 1$

$$\Rightarrow a + \frac{1}{9} + c + \frac{1}{9} + b + \frac{1}{3} = 1$$

$$a+b+c = \frac{4}{9}$$

for X and Y are independent

$f(x,y) = f_X(x) \cdot f_Y(y)$

$$\frac{1}{9} = \left(\frac{1}{9}+b+\frac{1}{3}\right)\left(a+\frac{1}{9}\right)$$

$$\frac{1}{9} = \left(a+\frac{1}{9}+c\right)\left(\frac{1}{9}+b\right) \quad \Rightarrow \left(a+\frac{1}{9}\right)3 = c+\frac{1}{3}$$

$$3a = c$$

$$\frac{1}{3} = \left(\frac{1}{9}+\frac{1}{3}+b\right)\left(c+\frac{1}{3}\right)$$

$$\Rightarrow \left(a+\frac{1}{9}\right)\left(\frac{4}{9}+b\right) = \frac{1}{9}$$

$$\left(4a+\frac{1}{9}\right)\left(\frac{1}{9}+b\right) = \frac{1}{9}$$

$$\Rightarrow ab = \frac{1}{81}$$

$$\begin{cases} a = \frac{1}{18} \\ b = \frac{2}{9} \\ c = \frac{1}{6} \end{cases}$$

T3.

$$\mathrm{Cov}(U,V) = \mathrm{Cov}(2X+Y, 2X-Y)$$

$$= \mathrm{Cov}(2X, 2X-Y) + \mathrm{Cov}(Y, 2X-Y)$$

$$= \mathrm{Cov}(2X,2X) + \mathrm{Cov}(2X,-Y) + \mathrm{Cov}(Y,2X) + \mathrm{Cov}(Y,-Y)$$

$$= 4(E(X^2)-E(X)E(X)) - (E(Y^2)-E(Y)E(Y)) + \cancel{\mathrm{Cov}(2X,Y) - \mathrm{Cov}(2X,Y)}$$

$$= 4\mathrm{Var}(X) - \mathrm{Var}(Y) = 4\lambda - \lambda = 3\lambda$$

$$\mathrm{Var}(U) = \mathrm{Var}(2X) + \mathrm{Var}(Y) = 4\lambda + \lambda = 5\lambda$$

$$\mathrm{Var}(V) = \mathrm{Var}(2X) + \mathrm{Var}(Y) = 5\lambda$$

$$\rho_{UY} = \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \frac{3\lambda}{5\lambda} = \frac{3}{5}$$

T4. $Y = n - X$

$$Cov(X,Y) = Cov(X, n-X) = Cov(X,n) - Cov(X,X)$$
$$= -Var(X)$$
$$= -np(1-p) = -\frac{1}{4}n$$

$$Var(X) = Var(Y) = -\frac{1}{4}n$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-\frac{1}{4}n}{|-\frac{1}{4}n|} = -1$$

T5. $f_X(x) = \int_{-x}^{x} 1 \, dy = 2x$

$$E(X) = \int_0^1 x f_X(x) dx = \frac{2}{3}x^3 \Big]_0^1 = \frac{2}{3}$$

$$f_Y(y) = \int_{|y|}^{1} 1 \, dx = 1 - |y|$$

$$E(Y) = \int_{-1}^{1} y f_Y(y) dy = \int_0^1 (y - y^2) dy + \int_{-1}^{0} (y + y^2) dy$$
$$= (\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3)\Big]_0^1 + (\tfrac{1}{2}y^2 + \tfrac{1}{3}y^3)\Big]_{-1}^{0}$$
$$= \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{3} = 0$$

$E(XY) = \iint xy f(x,y) dxdy$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = E(XY)$$
$$= \int_{-1}^{1}\int_{|y|}^{1} xy f(x,y) \, dxdy$$
$$= \int_{-1}^{1} y \left(\tfrac{1}{2}x^2\right)\Big]_{|y|}^{1} dy = \int_{-1}^{1} y\left(\tfrac{1}{2} - \tfrac{1}{2}y^2\right) dy$$
$$= \left(\tfrac{1}{4}y^2 - \tfrac{1}{8}y^4\right)\Big]_{-1}^{1}$$
$$= 0$$

(2) No,

$$f_X(x) = \int_{-x}^{x} f(x,y) dy = y\Big]_{-x}^{x} = 2x \quad 0<x<1$$

$$f_Y(y) = \int_{|y|}^{1} f(x,y) dx = x\Big]_{|y|}^{1} = 1 - |y| \quad |y| < x$$

$f_X(x) \cdot f_Y(y) = 2x - 2x|y|$

$f(x,y) = 1$ when $|y| < x, 0 < x < 1$

$f_X(x) \cdot f_Y(y) = f(x,y)$ not always holds

T6、 $f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ $\quad f_Y(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$f_T(t) = \int_{-\infty}^{\infty} f(t-y, y)\,dy$

X and Y is independent, then

$f_T(t) = \int_{-\infty}^{\infty} f_X(t-y) f_Y(y)\,dy$

when $0 \le t < 1$ $\quad f_T(t) = \int_0^t 1 \cdot 1\,dy = t$

when $1 \le t \le 2$ $\quad f_T(t) = \int_{t-1}^1 1 \cdot 1\,dy = 2 - t$

$\Rightarrow f_T(t) = \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$

T7、 $X_1 \sim N(45, 10)$ $\quad X_2 \sim N(50, 12)$ $\quad X_3 \sim N(75, 14)$

X_1、X_2、X_3 are independent

$\Rightarrow$ let $T = X_1 + X_2 + X_3$

$T \sim N(45+50+75,\ 10+12+14)$

$T \sim (170, 36)$

$P(T \le 180) = P\left(\frac{T-170}{\sqrt{36}} \le \frac{180-170}{\sqrt{36}}\right)$

$= \Phi(1.67) = 0.9525$

T8、 let X_i be the ith buld's lifespan

$X_i \sim Exp\left(\frac{1}{25}\right)$

let $T = X_1+X_2+\cdots+X_{40}$

$E(T)=25\times40=1000$ $\quad Var(T)=25^2\times40=10^3\times25$

for Central Limit Theorm

$T \overset{approx}{\sim} N(1000, 25000)$

$P(T\geqslant 900)=P\left(\frac{T-1000}{25\sqrt{40}}\geqslant\frac{900-1000}{25\sqrt{40}}\right)$

$=1-\Phi(-0.63)=0.7357$

T9. let X be the number of occupied room

$X\sim B(0.8, 500)$

$np=0.8\times500=400\geqslant5$ $\quad n(1-p)=0.2\times500=100\geqslant5$

use normal distribution to approximate

$E(X)=np=400$ $\quad Var(X)=np(1-p)=80$

$X\overset{approx}{\sim}N(400, 80)$

$P(X\leq x)=0.99$

$P\left(\frac{X-400}{\sqrt{80}}\leq\frac{x+0.5-400}{\sqrt{80}}\right)=0.99$

$\frac{x+0.5-400}{\sqrt{80}}=2.33$ $\quad x=421.3\approx421$

$\Rightarrow 421\times2=842\ kW$