Phy 500 Final Project

The Simple Pendulum with Air Drag

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Abstract

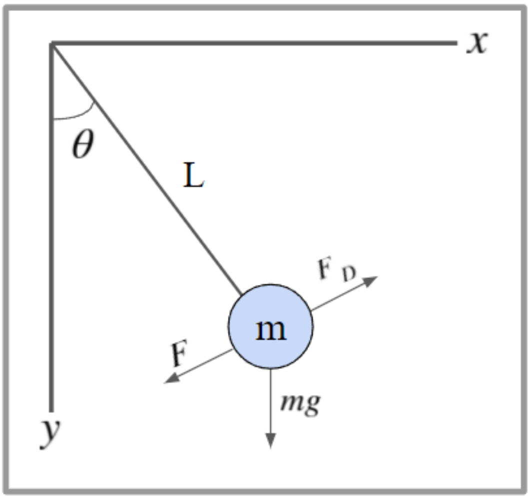
The simple pendulum is one of the common topics in physics. In this paper, I introduce the simple pendulum with air drag. For simulation, the equations of motion of a simple pendulum that doesn't consider the air drag is calculated by Lagrangian and derive the equations of motion of the pendulum with air drag by adding drag force, $-kv$, to the calculated equation. The Euler method and fourth-order Runge-Kutta method is used for solving the differential equation of the pendulum motion. As a result, I describe the comparison between Euler method and Runge-Kutta method, pendulum simulation according to various k values and pendulum simulation according to various initial angles.

1 Introduction

A simple pendulum is a simple example but one of the most common topics in physics. It is a system that moves in an oscillatory motion and it is made to oscillate by attaching a mass to the end of the string, and the oscillatory motion is repeated by gravity force. In a vacuum, the simple pendulum will continue to move without stopping. However, the simple pendulum with air resistance stops at some point since it will lose energy due to air resistance. In this paper, I will deal with the simple pendulum motion with air drag. After deriving the equations of motions, the numerical solutions will be obtained from the differential equation using the Euler method and fourth-order Runge-Kutta method.

2 Theoretical Background

2.1 Simple Pendulum with Air Drag



[Figure 1: Pendulum with Air Drag]

The simple pendulum consists of a mass **m** hanging from length **L** string and the angle between the string and the y-axis is defined as θ. The tension of the string will be ignored since the magnitude of tension of the string is the same as the force by gravity along the string direction, and the directions of the two forces are completely opposite. Thus, the angle θ will be decreased and increased by only the gravitational force along the arc direction **F**. F is given by below, where g is the gravity:

$$F = -mg\sin\theta$$

The simple pendulum with air drag is performed by adding drag force F_D in opposite direction of F in the same setting above. F_D is given by:

$$F_D = -kv$$

,where k is a positive constant; the damping parameter and v is the velocity of θ. In this simulation, the pendulum motion will stop at some point because the magnitude of velocity is decreased by the air resistance. Since mass **m** is initially assumed to be placed at some angle, the initial value is as follows:

$$\theta(0) = \theta_0$$

$$\dot{\theta}(0) = 0$$

2.2 Euler Method

Euler method is a first order numerical procedure for solving ordinary differential equations. Euler method can be written by following equation:

$$y_{n+1} = y_n + \Delta t g_n + \mathcal{O}\left((\Delta t)^2\right)$$

where $\Delta t = t_{n+1} - t_n$. For coding, each step of it can be performed by following:
For the number of steps N

$$k1 = f(y, t)$$
$$y = y + h \cdot k1$$
$$t = t + h$$

where $f(y, t)$ is y', a differential equation and h is the step size, Δt.

2.3 Fourth-Order Runge-Kutta Method

Runge-Kutta method is also used to approximate solutions of ordinary differential equations. Fourth-order Runge-Kutta method is performed by:

$$k_{n1} = f\left(y_n, t_n\right)$$
$$k_{n2} = f\left(y_n + \frac{1}{2} h k_{n1}, t_n + \frac{1}{2} h\right)$$
$$k_{n3} = f\left(y_n + \frac{1}{2} h k_{n2}, t_n + \frac{1}{2} h\right)$$
$$k_{n4} = f\left(y_n + h k_{n3}, t_n + h\right)$$
$$y_{n+1} = y_n + \frac{h}{6}\left(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}\right)$$

where $f(y, t)$ is y', a differential equation and h is the step size, Δt. Since it's fourth-order Runge-Kutta, this method is more accurate than the Euler method.

3 Numerical Details

3.1 Equations of Motion

For a **simple pendulum**, we can derive the E.O.M from the Lagrangian. The Lagrangian is defined as $\mathscr{L} = T - U$ and kinetic energy T and potential energy is given by:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$
$$U = mgy$$

Thus, if we calculate T and U in simple pendulum,

$$x = L\sin\theta,\ y = -L\cos\theta$$
$$\dot{x} = L\dot{\theta}\cos\theta,\ \dot{y} = L\dot{\theta}\sin\theta$$
$$\dot{x}^2 + \dot{y}^2 = L^2\dot{\theta}^2$$
$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}mL^2\dot{\theta}^2$$
$$U = mgy = -mgL\cos\theta$$
$$\mathscr{L} = \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

Then, we can find equations of motion using :

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\theta}} - \frac{\partial \mathscr{L}}{\partial \theta} = 0$$

Calculation:

$$\frac{\partial \mathscr{L}}{\partial \dot{\theta}} = mL^2\dot{\theta}$$
$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta}$$
$$\frac{\partial \mathscr{L}}{\partial \theta} = -mgL\sin\theta$$

$$mL^2\ddot{\theta} + mgL\sin\theta = 0$$

Then the E.O.M of the simple pendulum is given by:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

For **simple pendulum with air drag**, we simply add the drag force F_D to the right hand side of E.O.M of the simple pendulum. Thus, E.O.M of the simple pendulum with air drag is given by:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = -k\dot{\theta}$$

$$\ddot{\theta} + k\dot{\theta} + \frac{g}{L}\sin\theta = 0$$

For numerical methods, set the output vector y as:

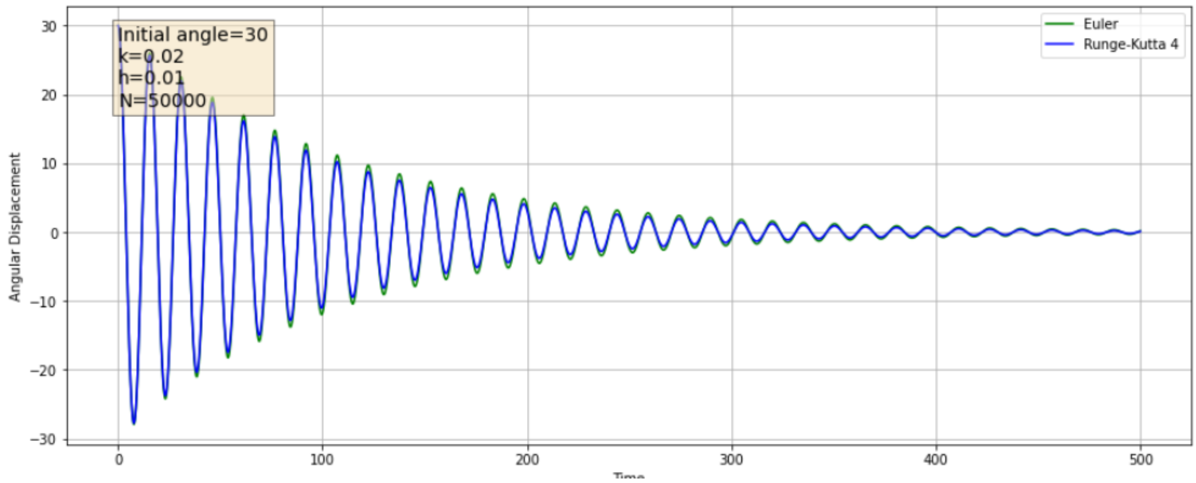
$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{y} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -k\dot{\theta} - \frac{g}{L}\theta \end{bmatrix} = \begin{bmatrix} y_2 \\ -k \cdot y_2 - \frac{g}{L}y_1 \end{bmatrix}$$

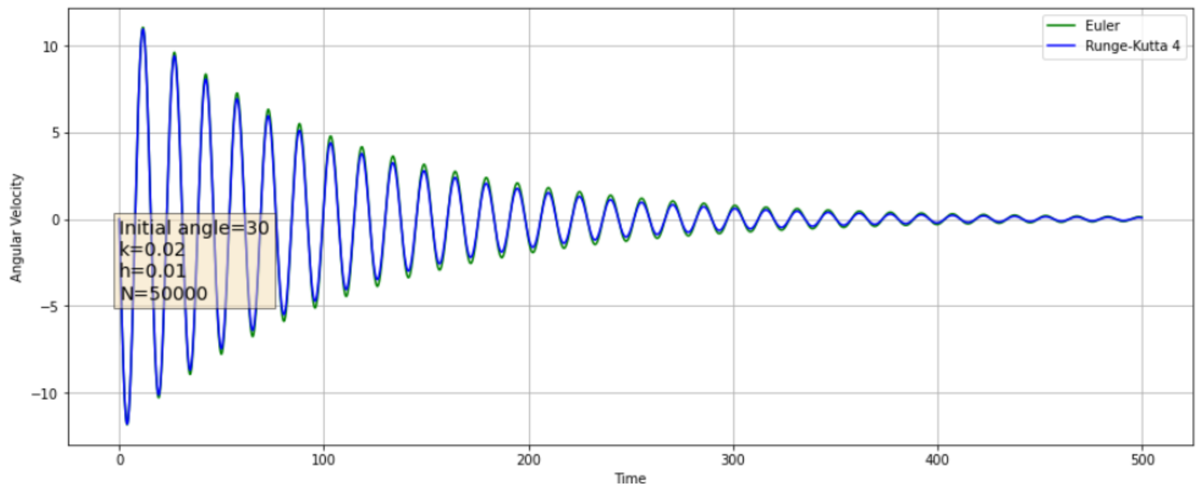
So $f(y,t)$ is defined as:

$$f(y,t) = \left(y_2,\ -k \cdot y_2 - \frac{g}{L}y_1\right)$$

4 Results and Discussion



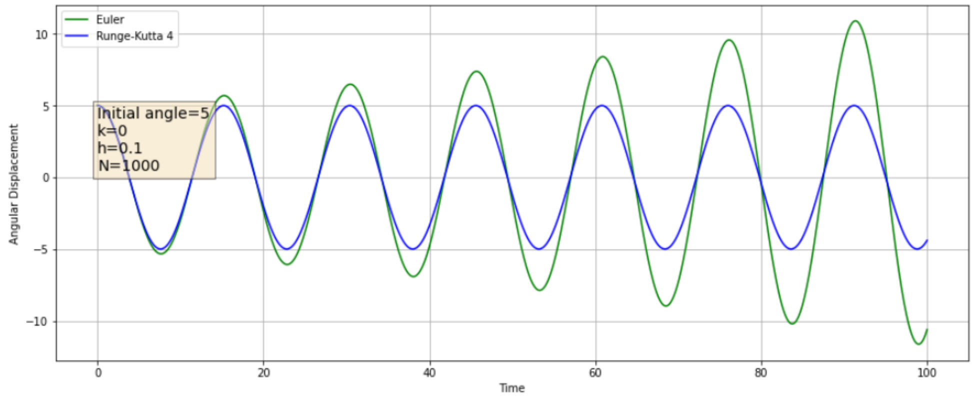
[Figure 2: Angular displacement of simple pendulum with air drag]



[Figure 3: Angular velocity of simple pendulum with air drag]

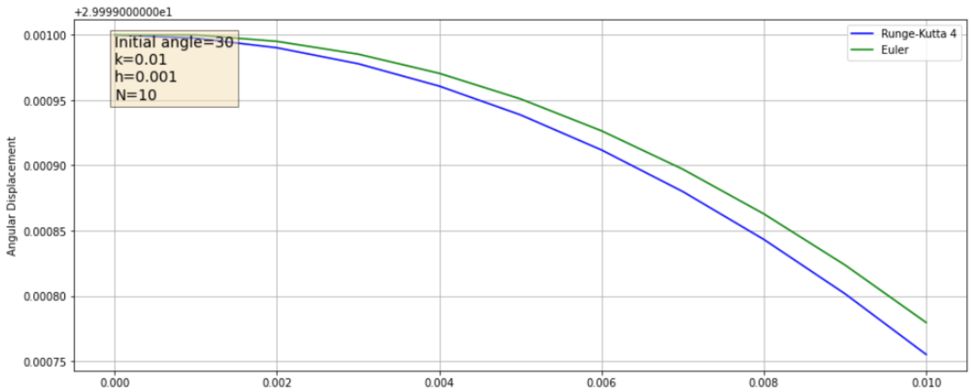
Figure 2 and Figure 3 show that angular movement and angular velocity in a simple pendulum are reduced by air resistance. The angle starts to move at the initial angle and the velocity starts at zero, we can know that both values are converging to zero.

4.1 The Euler method and Runge-Kutta 4

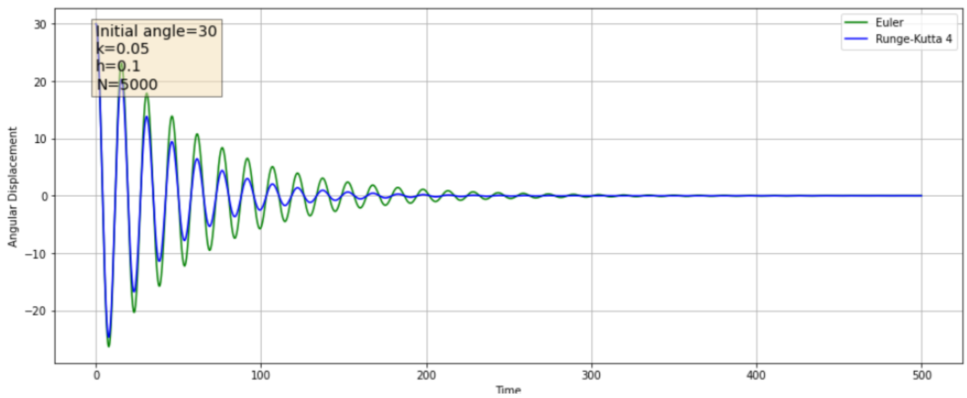


[Figure 4: Comparison Euler method and RK4 : no air drag]

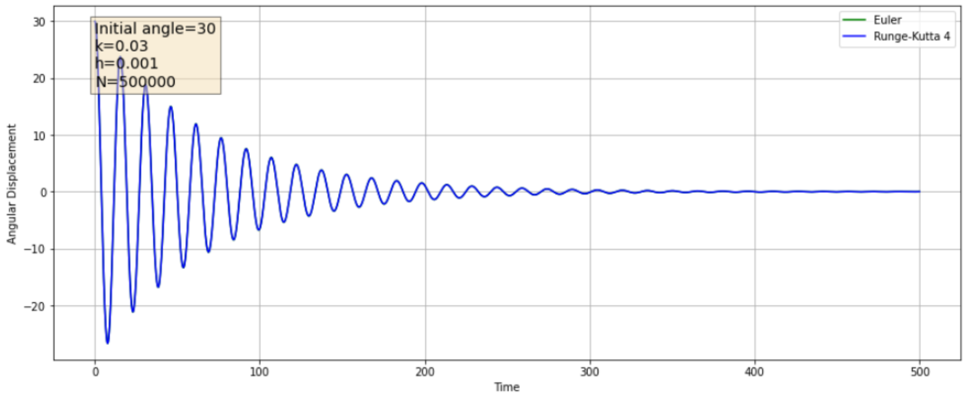
In Figure 4, the amplitude of the Runge-Kutta method is constant and it's expected behavior for the pendulum with no air drag. However, the Euler method draws an increasing graph. It tells us that the Runge-Kutta method is more accurate than the Euler method.



[Figure 5: Comparison Euler method and RK4 : small number of steps]



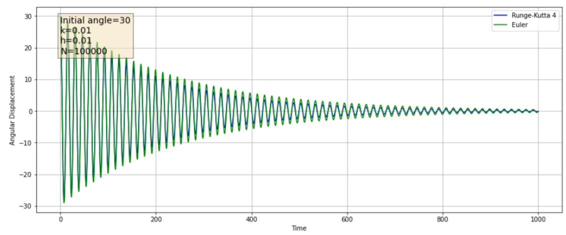
[Figure 6: Comparison Euler method and RK4 : step size = 0.1, step number = 5 x 1.0e3]



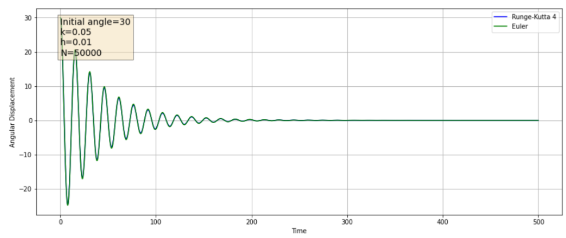
[Figure 7: Comparison Euler method and RK4 : step size=0.03, step number = 5 x 1.0e5]

Figure 5 shows the difference in calculated angle between Euler method and Runge-Kutta method. As we know, the error rate of the fourth-order Runge-Kutta method is less than the Euler method and we can see that the difference between the calculated values of the Euler method and Runge-Kutta method is large in Figure 6 and is small in Figure 7.

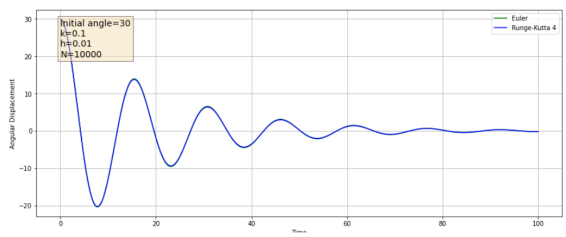
4.2 Various K



[Figure 8: Numerical method : k = 0.01]



[Figure 9: Numerical method : k = 0.05]



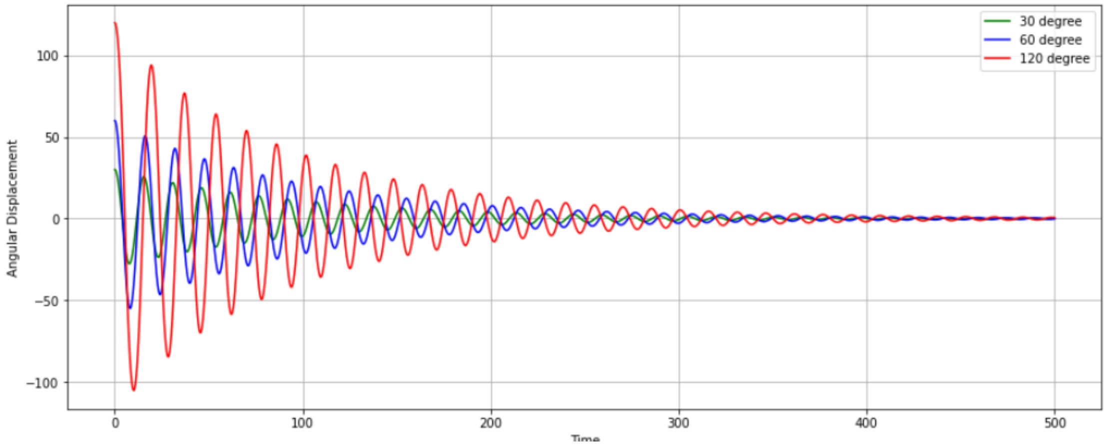
[Figure 10: Numerical method : k = 0.1]

	k = 0.01	k = 0.05	k = 0.1
t by Euler	820.72	500.00	100.00
t by RK4	805.03	500.00	100.00
Euler Time	29.75	49.24	53.07
RK4 Time	1.638523e+09	1.638524e+09	1.638524e+09

[Figure 11: Table for the time at angular velocity is almost zero. The threshold(1.03-5) is applied.]

By experiments according to various k values, the coefficient of the drag force, pendulum motion is more disturbed as the magnitude of air drag force increases. Therefore, the larger the k value, the faster the velocity decreases and the pendulum stops early. Figure 11 is about the time for the pendulum to stop. When we compare the time for those k values, the pendulum that has low air drag force, k value, stops earlier. Also, by Figure 11, we can know that The Runge-Kutta method took much longer time than the Euler method in every k.

4.3 Various Initial Angle



[Figure 12: Pendulum simulation for various initial angles : k = 0.02]

	Initial Angle 30 degree	Initial Angle 60 degree	Initial Angle 120 degree
Euler	605.01	744.46	647.31
RK4	650.58	653.06	669.32

[Figure 13: Time for pendulum to stop for various initial angles : k = 0.02]

Figure 12 shows the motion of the pendulum in 30, 60 and 120 degrees and Figure 13 shows the time that the pendulum stops for each degree. The values from the Euler method and from the Runge-Kutta method are different but we already know that the Runge-Kutta method is more accurate than the Euler method. Thus, According to results from the Runge-Kutta method, the time that the pendulum stops increases, not linearly, by increasing the initial angle.

5 Conclusions

I verified that air resistance on a simple pendulum decreases the angle and angular velocity. By several experiments, the magnitude of drag force, k value, affected the decreasing angular velocity and the time that pendulum motion stops. The initial angle also had an effect on the converging time. Thus, the pendulum that has a small magnitude of air drag and small initial angle will stop earlier. For the numerical methods, the fourth-order Runge-Kutta took longer time than the Euler method but it gives us more accurate value for the differential equation.

References

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