Notes on Mean Testing

By Paulo F. Naibert

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1 Mean Testing

Based on Ledoit & Wolf (2008, 2011)
T pairs: $(r_{1i}, r_{1n})', \ldots, (r_{Ti}, r_{Tn})'$.
Bivariate return distribution:

$$\mu = \begin{bmatrix} \mu_i \\ \mu_n \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_i^2 & \sigma_{ni} \\ \sigma_{in} & \sigma_n^2 \end{bmatrix}.$$

Any quantity x when denoted as $\hat{x}$ simply means its sample counterpart, e.g. $\hat{\mu}$ denotes the sample counterpart of μ.

Define the difference of returns as $r_{td} = r_{ti} - r_{tn}$ with moments:

$$\begin{aligned}
E(r_{td}) &= \mu_d = \mu_i - \mu_n \\
V(r_{dt}) &= E[(r_{ti} - r_{tn})^2] - [E(r_{ti} - r_{tn})]^2 \\
V(r_{dt}) &= E(r_{ti}^2) + E(r_{tn}^2) - 2E(r_{ti} r_{tn}) - \mu_i^2 + \mu_n^2 - 2\mu_i \mu_n \\
V(r_{dt}) &= V(r_{ti}) + V(r_{tn}) - 2Cov(r_{ti}, r_{tn}) \\
\sigma_d^2 &= \sigma_i^2 + \sigma_n^2 - 2\sigma_{in}
\end{aligned}$$

What we want to test is if the difference in means is different from zero statistically significant.

$$H_0 : \hat{\mu}_d = 0 \text{ vs. } H_1 : \hat{\mu}_d \neq 0.$$

The classical T-test:
Define the test statistic as:

$$\Theta = T^{1/2} \frac{\hat{\mu}_d}{\hat{\sigma}_d} = T^{1/2} \frac{\hat{\mu}_i - \hat{\mu}_n}{\hat{\sigma}_i^2 + \hat{\sigma}_n^2 - 2\hat{\sigma}_{in}}. \tag{1}$$

Further, let's define $t_\lambda(k)$ as the λ-quantile of $t(k)$, or the t distribution with k degrees of freedom. The test rejects H_0 at a significance level α iff:

$$|\Theta| > t_{1-\alpha/2}(T-1).$$

2 Reformulation

What we are going to test is whether the difference in mean, Δ, is zero or not. In other terms:

$$H_0 : \Delta = 0 \text{ vs. } H_1 : \Delta \neq 0.$$

Define Δ as a function of $v = (\mu_i, \mu_n)'$:

$$\Delta = f(v) = \mu_i - \mu_n.$$

The gradient of the function $f(v)$ is:

$$\nabla' f(v) = \left(\frac{\partial f(v)}{\partial \mu_i}, \frac{\partial f(v)}{\partial \mu_n} \right) = (1, -1)\,. \tag{2}$$

Now, assume that:

$$T^{1/2}(v - \hat{v}) \rightarrow^d N(0, \Psi), \tag{3}$$

where Ψ is an unknown symmetric PSD matrix.

If we apply a function on the vector v of parameters, the Taylor expansion (Delta method) implies:

$$T^{1/2}[f(v) - f(\hat{v})] \rightarrow^d N\left(0; \nabla' f(v) \Psi \nabla f(v)\right).$$

Well, we will use $f(\cdot)$ as defined earlier, and we denote $f(v) = \Delta$, so we have:

$$T^{1/2}(\Delta - \hat{\Delta}) \rightarrow^d N\left(0; \nabla' f(v) \Psi \nabla f(v)\right).$$

Now, if a consistent estimator $\hat{\Psi}$ of Ψ is available, then $se(\hat{\Delta})$ is given by:

$$se(\hat{\Delta}) = \sqrt{T^{-1} \nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v})}. \tag{4}$$

$$\begin{aligned} \nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v}) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{\psi}_{11} & \hat{\psi}_{12} \\ \hat{\psi}_{21} & \hat{\psi}_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \hat{\psi}_{11} + \hat{\psi}_{22} - 2\hat{\psi}_{12} \end{aligned}$$

If we use $\hat{\Psi}$ as the sample covariance matrix:

$$\nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v}) = \hat{\sigma}_1^2 + \hat{\sigma}_{22}^2 - 2\hat{\sigma}_{12}.$$

And we have the same case as the classical t-test. So we use HAC consistend methods to estimate $\hat{\Psi}$.

3 Log Reformulation

Reformulating the problem with a log transformation:

$$\Delta = \log(\Theta) = \log(1 + \mu_i) - \log(1 + \mu_n) = \log[(1 + \mu_i)/(1 + \mu_n)]. \tag{5}$$

OBS: Log transformation, see Efron and Tibshirani 1993, sec 12.6.

What we are going to test is whether the difference in log variances, Δ, is zero or not. In other terms:

$$H_0 : \Delta = 0 \text{ vs. } H_1 : \Delta \neq 0.$$

Define Δ as a function of $v = (\mu_i, \mu_n)'$:

$$\Delta = f(v) = \log(1 + \mu_i) - \log(1 + \mu_n) = \log\left(\frac{1 + \mu_i}{1 + \mu_n}\right).$$

The gradient of the function $f(v)$ is:

$$\nabla' f(v) = \left(\frac{\partial f(v)}{\partial \mu_i}, \frac{\partial f(v)}{\partial \mu_n}\right) = \left(\frac{1}{1 + \mu_i}, \frac{-1}{1 + \mu_n}\right). \tag{6}$$

Now, assume that:

$$T^{1/2}(v - \hat{v}) \to^d N(0, \Psi), \tag{7}$$

where Ψ is an unknown symmetric PSD matrix.

If we apply a function on the vector v of parameters, the Taylor expansion (Delta method) implies:

$$T^{1/2}[f(v) - f(\hat{v})] \to^d N\left(0; \nabla' f(v)\Psi\nabla f(v)\right).$$

Well, we will use $f(\cdot)$ as defined earlier, and we denote $f(v) = \Delta$, so we have:

$$T^{1/2}(\Delta - \hat{\Delta}) \to^d N\left(0; \nabla' f(v)\Psi\nabla f(v)\right).$$

Now, if a consistent estimator $\hat{\Psi}$ of Ψ is available, then $se(\hat{\Delta})$ is given by:

$$se(\hat{\Delta}) = \sqrt{T^{-1}\nabla' f(\hat{v})\hat{\Psi}\nabla f(\hat{v})}. \tag{8}$$

$$\nabla' f(v)\Psi\nabla f(v) = \begin{bmatrix} \frac{1}{1+\mu_i} & \frac{-1}{1+\mu_n} \end{bmatrix} \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{1+\mu_i} \\ \frac{-1}{1+\mu_n} \end{bmatrix}$$

$$\nabla' f(v)\Psi\nabla f(v) = \frac{\psi_{11}}{(1 + \mu_i)^2} + \frac{\psi_{22}}{(1 + \mu_n)^2} - \frac{2\psi_{21}}{(1 + \mu_i)(1 + \mu_n)}$$

References

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