Notes on Ledoit and Wolf (2008, 2011)

By Paulo F. Naibert

October 16, 2019

1 Sharpe Testing

Ledoit & Wolf (2008)
T pairs: $(r_{1i}, r_{1n})', \dots, (r_{Ti}, r_{Tn})'$.
Bivariate return distribution:

$$\mu = \begin{bmatrix} \mu_i \\ \mu_n \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_i^2 & \sigma_{ni} \\ \sigma_{in} & \sigma_n^2 \end{bmatrix}.$$

Any quantity x when denoted as $\hat{x}$ simply means its sample counterpart, e.g. $\hat{\mu}$ denotes the sample counterpart of μ.

Define the difference in Sharpe Ratios as:

$$\Delta = SR_i - SR_n = \frac{\mu_i}{\sigma_i} - \frac{\mu_n}{\sigma_n}, \tag{1}$$

further, define Δ as a function of $u = (\mu_i, \mu_n, \sigma_i^2, \sigma_n^2)'$. So,

$$\Delta = f(u) = SR_i - SR_n = \frac{\mu_i}{\sqrt{\sigma_i^2}} - \frac{\mu_n}{\sqrt{\sigma_n^2}}. \tag{2}$$

The gradient of $f(u)$ is:

$$\nabla' f(u) = \left(\frac{\partial f(u)}{\partial \mu_i}, \frac{\partial f(u)}{\partial \mu_n}, \frac{\partial f(u)}{\partial \sigma_i}, \frac{\partial f(u)}{\partial \sigma_n} \right)$$

Assume that:

$$T^{1/2}(u - \hat{u}) \to^d N(0, \Omega),$$

where Ω is an unknown symmetric PSD matrix.

If we apply a function on the vector u of parameters, the Taylor expansion (Delta method) implies:

$$T^{1/2}[f(u) - f(\hat{u})] \to^d N\left(0; \nabla' f(u) \Omega \nabla f(u)\right).$$

Well, we will use $f(\cdot)$ as defined earlier, and we denote $f(u) = \Delta$, so we have:

$$T^{1/2}(\Delta - \hat{\Delta}) \to^d N\left(0; \nabla' f(u) \Omega \nabla f(u)\right).$$

Now, if a consistent estimator $\hat{\Omega}$ of Ω is available, then $se(\hat{\Delta})$ is given by:

$$se(\hat{\Delta}) = \sqrt{T^{-1} \nabla' f(\hat{u}) \hat{\Omega} \nabla f(\hat{u})}. \tag{3}$$

The problem is that Jobson & Korkie (1981) uses a formula for $\hat{\Omega}$ that crucially relies on *iid* return data for a bivariate normal distribution:

$$\Omega = \begin{bmatrix} \sigma_i & \sigma_n & 0 & 0 \\ \sigma_i & \sigma_n & 0 & 0 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_n^4 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_n^4 \end{bmatrix}$$

Further information can be found in: Lo (2002); Jobson & Korkie (1981).

1.1 Ledoit and Wolf (2008)

Define the uncentered second moments as:

$$\gamma_i = E(r_{it}^2),\ \gamma_n = E(r_{nt}^2),$$

and define the vector

$$v = (\mu_i, \mu_n, \gamma_i, \gamma_n)'.$$

Then, we define Δ as a function of v:

$$\Delta = f(v) = SR_i - SR_n = \frac{\mu_i}{\sqrt{\sigma_i^2}} - \frac{\mu_n}{\sqrt{\sigma_n^2}}. \tag{4}$$

The gradient of $f(v)$ is:

$$\begin{aligned}
\nabla' f(v) &= \left(\frac{\partial f(v)}{\partial \mu_i}, \frac{\partial f(v)}{\partial \mu_n}, \frac{\partial f(v)}{\partial \gamma_i}, \frac{\partial f(v)}{\partial \gamma_n} \right) \\
&= \left(\frac{\gamma_i}{(\gamma_i - \mu_i^2)^{3/2}}, \frac{-\gamma_n}{(\gamma_n - \mu_n^2)^{3/2}}, \frac{-\mu_i}{2(\gamma_i - \mu_i^2)^{3/2}}, \frac{\mu_n}{2(\gamma_n - \mu_n^2)^{3/2}} \right).
\end{aligned}$$

Assume that:

$$T^{1/2}(v - \hat{v}) \to^d N(0, \Psi),$$

where Ψ is an unknown symmetric PSD matrix.

If we apply a function on the vector v of parameters, the Taylor expansion (Delta method) implies:

$$T^{1/2}[f(v) - f(\hat{v})] \to^d N\left(0; \nabla' f(v) \Psi \nabla f(v)\right).$$

Well, we will use $f(\cdot)$ as defined earlier, and we denote $f(v) = \Delta$, so we have:

$$T^{1/2}(\Delta - \hat{\Delta}) \to^d N\left(0; \nabla' f(v) \Psi \nabla f(v)\right).$$

Now, if a consistent estimator $\hat{\Psi}$ of Ψ is available, then $se(\hat{\Delta})$ is given by:

$$se(\hat{\Delta}) = \sqrt{T^{-1} \nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v})}.$$

2 Variance Testing

Ledoit & Wolf (2011)
T pairs: $(r_{1i}, r_{1n})', \ldots, (r_{Ti}, r_{Tn})'$.
Bivariate return distribution:

$$\mu = \begin{bmatrix} \mu_i \\ \mu_n \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_i^2 & \sigma_{ni} \\ \sigma_{in} & \sigma_n^2 \end{bmatrix}.$$

Any quantity x when denoted as $\hat{x}$ simply means its sample counterpart, e.g. $\hat{\mu}$ denotes the sample counterpart of μ.

Define the reason of variances as:

$$\Theta = \sigma_i^2/\sigma_n^2. \tag{5}$$

What we want to test is if the reason of variances is different from one statistically significant.

$$H_0 : \Theta = 1 \text{ vs. } H_1 : \Theta \neq 1.$$

The classical F-test:

Defining the test statistic as $F = \hat{\sigma}_i^2/\hat{\sigma}_n^2$. F_{λ,k_1,k_2} is the λ-quantile of $F_{k1,k2}$, or the F distribution with k_1 and k_2 degrees of freedom. The F-test rejects H_0 at a significance level α iff:

$$F < F_{\alpha/2,T-1,T-1} \text{ or, } F > F_{1-\alpha/2,T-1,T-1}.$$

2.1 Ledoit and Wolf (2011)

First, we reformulate the problem with a log transformation:

$$\Delta = \log(\Theta) = \log(\sigma_i^2) - \log(\sigma_n^2) = \log(\sigma_i^2/\sigma_n^2). \tag{6}$$

OBS: Log transformation, see Efron and Tibshirani 1993, sec 12.6.

What we are going to test is whether the difference in log variances, Δ, is zero or not. In other terms:

$$H_0 : \Delta = 0 \text{ vs. } H_1 : \Delta \neq 0.$$

Define the Uncentered Second Moments as:

$$\gamma_i = E(r_{it}^2), \; \gamma_n = E(r_{nt}^2),$$

Then, we define Δ as a function of $v = (\mu_i, \mu_n, \gamma_i, \gamma_n)'$:

$$\begin{aligned} \Delta &= f(v), \\ f(v) &= \log\left(\gamma_i - \mu_i^2\right) - \log\left(\gamma_n - \mu_n^2\right) \\ &= \log\left(\frac{\gamma_i - \mu_i^2}{\gamma_n - \mu_n^2}\right). \end{aligned}$$

The gradient of the function $f(v)$ is:

$$\nabla' f(v) = \left(\frac{\partial f(v)}{\partial \mu_i}, \frac{\partial f(v)}{\partial \mu_n}, \frac{\partial f(v)}{\partial \gamma_i}, \frac{\partial f(v)}{\partial \gamma_n} \right) \tag{7}$$

$$= \left(\frac{-2\mu_i}{\gamma_i - \mu_i^2}, \frac{2\mu_n}{\gamma_n - \mu_n^2}, \frac{1}{\gamma_i - \mu_i^2}, \frac{1}{\gamma_n - \mu_n^2} \right). \tag{8}$$

Now, assume that:

$$T^{1/2}(v - \hat{v}) \to^d N(0, \Psi), \tag{9}$$

where Ψ is an unknown symmetric PSD matrix.

If we apply a function on the vector v of parameters, the Taylor expansion (Delta method) implies:

$$T^{1/2}[f(v) - f(\hat{v})] \to^d N\left(0; \nabla' f(v) \Psi \nabla f(v)\right).$$

Well, we will use $f(\cdot)$ as defined earlier, and we denote $f(v) = \Delta$, so we have:

$$T^{1/2}(\Delta - \hat{\Delta}) \to^d N\left(0; \nabla' f(v) \Psi \nabla f(v)\right).$$

Now, if a consistent estimator $\hat{\Psi}$ of Ψ is available, then $se(\hat{\Delta})$ is given by:

$$se(\hat{\Delta}) = \sqrt{T^{-1} \nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v})}. \tag{10}$$

References

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