

# Notes on Ledoit and Wolf (2008, 2011)

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## 1 Sharpe Testing

[Ledoit & Wolf \(2008\)](#)

$T$  pairs:  $(r_{1i}, r_{1n})', \dots, (r_{Ti}, r_{Tn})'$ .

Bivariate return distribution:

$$\mu = \begin{bmatrix} \mu_i \\ \mu_n \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_i^2 & \sigma_{ni} \\ \sigma_{in} & \sigma_n^2 \end{bmatrix}.$$

Any quantity  $x$  when denoted as  $\hat{x}$  simply means its sample counterpart, e.g.  $\hat{\mu}$  denotes the sample counterpart of  $\mu$ .

Define the difference in Sharpe Ratios as:

$$\Delta = SR_i - SR_n = \frac{\mu_i}{\sigma_i} - \frac{\mu_n}{\sigma_n}, \quad (1)$$

further, define  $\Delta$  as a function of  $u = (\mu_i, \mu_n, \sigma_i^2, \sigma_n^2)'$ . So,

$$\Delta = f(u) = SR_i - SR_n = \frac{\mu_i}{\sqrt{\sigma_i^2}} - \frac{\mu_n}{\sqrt{\sigma_n^2}}. \quad (2)$$

The gradient of  $f(u)$  is:

$$\nabla' f(u) = \left( \frac{\partial f(u)}{\partial \mu_i}, \frac{\partial f(u)}{\partial \mu_n}, \frac{\partial f(u)}{\partial \sigma_i^2}, \frac{\partial f(u)}{\partial \sigma_n^2} \right)$$

Assume that:

$$T^{1/2}(u - \hat{u}) \rightarrow^d N(0, \Omega),$$

where  $\Omega$  is an unknown symmetric PSD matrix.

If we apply a function on the vector  $u$  of parameters, the Taylor expansion (Delta method) implies:

$$T^{1/2}[f(u) - f(\hat{u})] \rightarrow^d N(0; \nabla' f(u) \Omega \nabla f(u)).$$

Well, we will use  $f(\cdot)$  as defined earlier, and we denote  $f(u) = \Delta$ , so we have:

$$T^{1/2}(\Delta - \hat{\Delta}) \rightarrow^d N(0; \nabla' f(u) \Omega \nabla f(u)).$$

Now, if a consistent estimator  $\hat{\Omega}$  of  $\Omega$  is available, then  $se(\hat{\Delta})$  is given by:

$$se(\hat{\Delta}) = \sqrt{T^{-1} \nabla' f(\hat{u}) \hat{\Omega} \nabla f(\hat{u})}. \quad (3)$$

The problem is that [Jobson & Korkie \(1981\)](#) uses a formula for  $\Omega$  that crucially relies on *iid* return data for a bivariate normal distribution:

$$\Omega = \begin{bmatrix} \sigma_i & \sigma_n & 0 & 0 \\ \sigma_i & \sigma_n & 0 & 0 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_n^4 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_n^4 \end{bmatrix}$$

Further information can be found in: [Lo \(2002\)](#); [Jobson & Korkie \(1981\)](#).

## 1.1 Ledoit and Wolf (2008)

Define the uncentered second moments as:

$$\gamma_i = E(r_{it}^2), \quad \gamma_n = E(r_{nt}^2),$$

and define the vector

$$v = (\mu_i, \mu_n, \gamma_i, \gamma_n)'$$

Then, we define  $\Delta$  as a function of  $v$ :

$$\Delta = f(v) = SR_i - SR_n = \frac{\mu_i}{\sqrt{\sigma_i^2}} - \frac{\mu_n}{\sqrt{\sigma_n^2}}. \quad (4)$$

The gradient of  $f(v)$  is:

$$\begin{aligned} \nabla' f(v) &= \left( \frac{\partial f(v)}{\partial \mu_i}, \frac{\partial f(v)}{\partial \mu_n}, \frac{\partial f(v)}{\partial \gamma_i}, \frac{\partial f(v)}{\partial \gamma_n} \right) \\ &= \left( \frac{\gamma_i}{(\gamma_i - \mu_i^2)^{3/2}}, \frac{-\gamma_n}{(\gamma_n - \mu_n^2)^{3/2}}, \frac{-\mu_i}{2(\gamma_i - \mu_i^2)^{3/2}}, \frac{\mu_n}{2(\gamma_n - \mu_n^2)^{3/2}} \right). \end{aligned}$$

Assume that:

$$T^{1/2}(v - \hat{v}) \rightarrow^d N(0, \Psi),$$

where  $\Psi$  is an unknown symmetric PSD matrix.

If we apply a function on the vector  $v$  of parameters, the Taylor expansion (Delta method) implies:

$$T^{1/2}[f(v) - f(\hat{v})] \rightarrow^d N(0; \nabla' f(v) \Psi \nabla f(v)).$$

Well, we will use  $f(\cdot)$  as defined earlier, and we denote  $f(v) = \Delta$ , so we have:

$$T^{1/2}(\Delta - \hat{\Delta}) \rightarrow^d N(0; \nabla' f(v) \Psi \nabla f(v)).$$

Now, if a consistent estimator  $\hat{\Psi}$  of  $\Psi$  is available, then  $se(\hat{\Delta})$  is given by:

$$se(\hat{\Delta}) = \sqrt{T^{-1} \nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v})}.$$

## 2 Variance Testing

Ledoit & Wolf (2011)

$T$  pairs:  $(r_{1i}, r_{1n})', \dots, (r_{Ti}, r_{Tn})'$ .

Bivariate return distribution:

$$\mu = \begin{bmatrix} \mu_i \\ \mu_n \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_i^2 & \sigma_{ni} \\ \sigma_{in} & \sigma_n^2 \end{bmatrix}.$$

Any quantity  $x$  when denoted as  $\hat{x}$  simply means its sample counterpart, e.g.  $\hat{\mu}$  denotes the sample counterpart of  $\mu$ .

Define the reason of variances as:

$$\Theta = \sigma_i^2 / \sigma_n^2. \quad (5)$$

What we want to test is if the reason of variances is different from one statistically significant.

$$H_0 : \Theta = 1 \text{ vs. } H_1 : \Theta \neq 1.$$

**The classical  $F$ -test:**

Defining the test statistic as  $F = \hat{\sigma}_i^2 / \hat{\sigma}_n^2$ .  $F_{\lambda, k_1, k_2}$  is the  $\lambda$ -quantile of  $F_{k_1, k_2}$ , or the  $F$  distribution with  $k_1$  and  $k_2$  degrees of freedom. The  $F$ -test rejects  $H_0$  at a significance level  $\alpha$  iff:

$$F < F_{\alpha/2, T-1, T-1} \text{ or } F > F_{1-\alpha/2, T-1, T-1}.$$

### 2.1 Ledoit and Wolf (2011)

First, we reformulate the problem with a log transformation:

$$\Delta = \log(\Theta) = \log(\sigma_i^2) - \log(\sigma_n^2) = \log(\sigma_i^2 / \sigma_n^2). \quad (6)$$

**OBS:** Log transformation, see Efron and Tibshirani 1993, sec 12.6.

What we are going to test is whether the difference in log variances,  $\Delta$ , is zero or not. In other terms:

$$H_0 : \Delta = 0 \text{ vs. } H_1 : \Delta \neq 0.$$

Define the Uncentered Second Moments as:

$$\gamma_i = E(r_{it}^2), \quad \gamma_n = E(r_{nt}^2),$$

Then, we define  $\Delta$  as a function of  $v = (\mu_i, \mu_n, \gamma_i, \gamma_n)'$ :

$$\begin{aligned} \Delta &= f(v), \\ f(v) &= \log(\gamma_i - \mu_i^2) - \log(\gamma_n - \mu_n^2) \\ &= \log\left(\frac{\gamma_i - \mu_i^2}{\gamma_n - \mu_n^2}\right). \end{aligned}$$

The gradient of the function  $f(v)$  is:

$$\nabla' f(v) = \left( \frac{\partial f(v)}{\partial \mu_i}, \frac{\partial f(v)}{\partial \mu_n}, \frac{\partial f(v)}{\partial \gamma_i}, \frac{\partial f(v)}{\partial \gamma_n} \right) \quad (7)$$

$$= \left( \frac{-2\mu_i}{\gamma_i - \mu_i^2}, \frac{2\mu_n}{\gamma_n - \mu_n^2}, \frac{1}{\gamma_i - \mu_i^2}, \frac{1}{\gamma_n - \mu_n^2} \right). \quad (8)$$

Now, assume that:

$$T^{1/2}(v - \hat{v}) \rightarrow^d N(0, \Psi), \quad (9)$$

where  $\Psi$  is an unknown symmetric PSD matrix.

If we apply a function on the vector  $v$  of parameters, the Taylor expansion (Delta method) implies:

$$T^{1/2}[f(v) - f(\hat{v})] \rightarrow^d N(0; \nabla' f(v) \Psi \nabla f(v)).$$

Well, we will use  $f(\cdot)$  as defined earlier, and we denote  $f(v) = \Delta$ , so we have:

$$T^{1/2}(\Delta - \hat{\Delta}) \rightarrow^d N(0; \nabla' f(v) \Psi \nabla f(v)).$$

Now, if a consistent estimator  $\hat{\Psi}$  of  $\Psi$  is available, then  $se(\hat{\Delta})$  is given by:

$$se(\hat{\Delta}) = \sqrt{T^{-1} \nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v})}. \quad (10)$$

## References

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