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A SCHWARZ LEMMA FOR HARMONIC FUNCTIONS IN THE REAL UNIT BALL*

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Abstract We establish a precise Schwarz lemma for real-valued and bounded harmonic functions in the real unit ball of dimension n. This extends Chen's Schwarz-Pick lemma for real-valued and bounded planar harmonic mapping.

Key words harmonic functions; Schwarz-Pick lemma; unit ball

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1 Introduction

Let $n > 1$ be a positive integer, $\mathbb{R}^n$ be the real space of dimension n. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $|x| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$. The unit ball in $\mathbb{R}^n$ and its boundary are denoted by

$$\mathbb{B}_n = \{x \in \mathbb{R}^n : |x| < 1\},$$
$$S_n = \{x \in \mathbb{R}^n : |x| = 1\}.$$

respectively. A twice continuously differentiable real-valued function F defined on $\mathbb{B}_n$ is called a harmonic function if $\Delta F \equiv 0$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplacian.

There have been a lot of researches on harmonic mappings on the unit disk $\mathbb{D}$ in the complex plane [1–4]. The classical Schwarz-Pick lemma of holomorphic mappings on the unit disk has been generalized to bounded planar harmonic mappings [5–15]. Recently, H. Chen obtained a precise version of the Schwarz-Pick lemma for real-valued and bounded planar harmonic mappings [7]:

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If F be a real-valued harmonic mapping of $\mathbb{D}$ into the open interval $I=(-1,1)$, then

$$\frac{|\nabla F(z)|}{\cos\frac{F(z)\pi}{2}}\leq\frac{4}{\pi}\frac{1}{1-|z|^2}\tag{1.1}$$

holds for $z\in\mathbb{D}$, where ∇F is the gradient of F. (1.1) is precise for any $z\in\mathbb{D}$ and any value of $F(z)$, and the equality occurs for some point and some value if and only if $F(z)=\frac{4}{\pi}\mathrm{Re}\{\arctan\varphi(z)\}$ for $z\in\mathbb{D}$ with a Mobiüs transformation φ of $\mathbb{D}$ onto itself.

In this short article, we try to extend Chen's result to bounded harmonic functions on the real unit ball $\mathbb{B}_n$ and obtain the following Schwarz lemma. To formulate our result, we introduce some notations. The Euclidian volume measure on $\mathbb{R}^n$ is denoted by V_n. By σ_n, we denote the area measure on S_n such that $\sigma_n(S_n)=1$. For $t\in(-1,1)$, let

$$S_{n,t}=\{x\in S_n: x_n=t\},\qquad S^+_{n,t}=\{x\in S_n: x_n>t\}$$

and

$$S^-_{n,t}=\{x\in S_n: x_n<t\}.$$

For $a\in(-1,1)$, let $t_a\in(-1,1)$ denote the unique real number such that $\sigma_n(S^+_{n,t_a})=\frac{1+a}{2}$ and $\sigma_n(S^-_{n,t_a})=\frac{1-a}{2}$, and let

$$u_a(\omega)=\begin{cases}1, & \omega\in S^+_{n,t_a},\\ 0, & \omega\in S_{n,t_a},\\ -1, & \omega\in S^-_{n,t_a},\end{cases}$$

and

$$U_a(x)=\int_{S_n}\frac{1-|x|^2}{|x-\omega|^n}u_a(\omega)\mathrm{d}\sigma_n(\omega)\quad\text{for}\quad x\in\mathbb{B}_n.$$

Then, $U_a(0)=a$ and $|U_a(x)|<1$ for $x\in\mathbb{B}_n$. In the next section, we will prove

$$\frac{\partial U_a(0)}{\partial x_n}=\frac{2(1-t_a^2)^{\frac{n-1}{2}}V_{n-1}(\mathbb{B}_{n-1})}{V_n(\mathbb{B}_n)},\quad a\geq 0.\tag{1.2}$$

Theorem 1.1 Let F be a real-valued harmonic function on $\mathbb{B}_n$, $F(0)=a\in(-1,1)$ and $|F(x)|<1$ for $x\in\mathbb{B}_n$. Then,

$$|\nabla F(0)|\leq\frac{2(1-t_a^2)^{\frac{n-1}{2}}V_{n-1}(\mathbb{B}_{n-1})}{V_n(\mathbb{B}_n)}.\tag{1.3}$$

The equality holds if and only if $F=U_a\circ T$ with a orthogonal transformation T.

For $n=2$, $t_a=-\sin\frac{a\pi}{2}$, $V_1(\mathbb{B}_1)=2$, $V_2(\mathbb{B}_2)=\pi$, and (1.3) becomes (1.1). For $n=3$, $t_a=-a$, $V_3(\mathbb{B}_3)=\frac{4}{3}\pi$ and (1.3) can be written in

$$|\nabla F(0)|\leq\frac{3}{2}(1-a^2).$$

For general n, (1.3) improves a known result [1]:

$$|\nabla F(0)|\leq\frac{2V_{n-1}(\mathbb{B}_{n-1})}{V_n(\mathbb{B}_n)}.\tag{1.4}$$

Note that (1.3) is coincident with (1.4) if $a=0$, and better than (1.4) if $a\neq 0$.

For positive harmonic functions in $\mathbb{B}_n$, we have similar result. Let E_0 denote the set which consists one point $(0, \cdots, 0, 1) \in S_n$ only and μ_a the finite positive Borel measure such that $\mu_a(E_0) = a$ and $\mu_a(S_n \setminus E_0) = 0$. Such a measure is called a singleton. Define

$$\begin{aligned} U_a^+(x) &= \int_{S_n} \frac{1-|x|^2}{|x-\omega|^n} \mathrm{d}\mu_a(\omega) \\ &= a \cdot \frac{1-|x|^2}{\left(x_1^2 + \cdots, x_{n-1}^2 + (1-x_n)^2\right)^{n/2}} \quad \text{for } x \in \mathbb{B}_n. \end{aligned}$$

Theorem 1.2 Let F be a positive harmonic function on $\mathbb{B}_n$, $F(0) = a$. Then,

$$|\nabla F(0)| \leq na. \tag{1.5}$$

The equality holds if and only if $F = U_a^+ \circ T$ with a orthogonal transformation T.

2 The Extremal Function $U_a(x)$

Now, we proceed to prove (1.2). We will use the following well known formula for a spherical integral on S_n: if f be an integral Borel measurable function on S_n, then

$$\int_{S_n} f \mathrm{d}\sigma_n(\omega) = \frac{1}{nV_n(\mathbb{B}_n)} \int_{\mathbb{B}_{n-1}} \frac{f(x, x_n) + f(x, -x_n)}{x_n} \mathrm{d}V_{n-1}(x), \tag{2.1}$$

where $x_n = \sqrt{1-|x|^2}$ for $x = (x_1, \cdots, x_{n-1}) \in \mathbb{B}_{n-1}$.

Let $0 \leq a < 1$. We have

$$\frac{\partial U_a(0)}{\partial x_n} = n \int_{S_n} u_a(\omega)\omega_n \mathrm{d}\sigma_n(\omega),$$

because

$$\frac{\partial}{\partial x_n}\left(\frac{1-|x|^2}{|x-\omega|^n}\right)\bigg|_{x=0} = n\omega_n,$$

where $\omega = (\omega_1, \cdots, \omega_n) \in S_n$ and $x = (x_1, \cdots, x_n) \in \mathbb{B}_n$. Consequently, using (2.1) gives

$$\begin{aligned} \frac{\partial U_a(0)}{\partial x_n} &= \frac{1}{V_n(\mathbb{B}_n)} \int_{\mathbb{B}_{n-1}} \frac{u_a(x, x_n)x_n - u_a(x, -x_n)x_n}{x_n} \mathrm{d}V_{n-1}(x) \\ &= \frac{1}{V_n(\mathbb{B}_n)} \int_{\mathbb{B}_{n-1}} \left(u_a(x, x_n) - u_a(x, -x_n)\right) \mathrm{d}V_{n-1}(x). \end{aligned}$$

It is obvious that

$$u_a(x, x_n) = 1, \quad x \in \mathbb{B}_{n-1}$$

and

$$u_a(x, -x_n) = \begin{cases} -1, & |x| < \sqrt{1-t_a^2}, \\ 1, & |x| > \sqrt{1-t_a^2}. \end{cases}$$

Thus,

$$\int_{\mathbb{B}_{n-1}} u_a(x, x_n) \mathrm{d}V_{n-1}(x) = V_{n-1}(\mathbb{B}_{n-1})$$

$$\int_{\mathbb{B}_{n-1}} u_a(x, -x_n) \mathrm{d}V_{n-1}(x)$$

$$= \int_{|x|<\sqrt{1-t_a^2}} u_a(x,-x_n)\mathrm{d}V_{n-1}(x) + \int_{|x|>\sqrt{1-t_a^2}} u_a(x,-x_n)\mathrm{d}V_{n-1}(x)$$
$$= -2V_{n-1}(\tilde{\mathbb{B}}_{n-1}) + V_{n-1}(\mathbb{B}_{n-1}),$$

where

$$\tilde{\mathbb{B}}_{n-1} = \{x \in \mathbb{R}^{n-1} : |x| < \sqrt{1-t_a^2}\}.$$

This shows that

$$\frac{\partial U_a(0)}{\partial x_n} = \frac{2V_{n-1}(\tilde{\mathbb{B}}_{n-1})}{V_n(\mathbb{B}_n)}.$$

(1.2) is proved because

$$V_{n-1}(\tilde{\mathbb{B}}_{n-1}) = (1-t_a^2)^{\frac{n-1}{2}} V_{n-1}(\mathbb{B}_{n-1}).$$

3 Proof of Theorem 1.1

Without loss of generality, we may assume that

$$F(0) = a \geq 0, \quad |\nabla F(0)| = \partial F(0)/\partial x_n \tag{3.1}$$

by passing through an orthogonal transformation of variable x and consider $-F$ if necessary. There is a real-valued measurable function $u \in L^\infty(S_n)$ such that $||u||_\infty \leq 1$ and F is the Poisson integral of the function u:

$$F(x) = \int_{S_n} \frac{1-|x|^2}{|x-\omega|^n} u(\omega)\mathrm{d}\sigma_n(\omega), \quad x \in \mathbb{B}_n.$$

Note that

$$\int_{S_n} u(\omega)\mathrm{d}\sigma_n(\omega) = \int_{S_n} u_a(\omega)\mathrm{d}\sigma_n(\omega) = a,$$

where $u_a(\omega)$ is defined in the introduction.

It is indicated above that

$$\frac{\partial U_a(0)}{\partial x_n} = n\int_{S_n} u_a(\omega)\omega_n \mathrm{d}\sigma_n(\omega).$$

By the same reason,

$$\frac{\partial F(0)}{\partial x_n} = n\int_{S_n} u(\omega)\omega_n \mathrm{d}\sigma_n(\omega).$$

Thus,

$$\frac{1}{n}\left(\frac{\partial F(0)}{\partial x_n} - \frac{\partial U_a(0)}{\partial x_n}\right) = \int_{S_n} (u(\omega)-u_a(\omega))\omega_n \mathrm{d}\sigma_n(\omega)$$
$$= \int_{S_n} (u(\omega)-u_a(\omega))(\omega_n - t_a)\mathrm{d}\sigma_n(\omega).$$

Because

$$u(\omega) - u_a(\omega) = \begin{cases} u(\omega) - 1 \leq 0, & \omega_n > t_a, \\ u(\omega) + 1 \geq 0, & \omega_n < t_a, \end{cases}$$

we have $(u(\omega)-u_a(\omega))(\omega_n - t_a) \leq 0$ for $\omega \in S_n$. Thus,

$$\frac{\partial F(0)}{\partial x_n} \leq \frac{\partial U_a(0)}{\partial x_n}.$$

This shows (1.3).

If the equality in (1.3) holds, then $u(\omega) = u_a(w)$ a.e., and $F = U_a$ under the assumption (3.1). Generally, we have $F = U_a \circ T$ with a orthogonal transformation T. Conversely, If $F = U_a \circ T$ with a orthogonal transformation T, then

$$|\nabla F(0)| = |\nabla U_{|a|}(0)| \geq \frac{\partial U_{|a|}(0)}{\partial x_n} = \frac{2(1-t_a^2)^{\frac{n-1}{2}} V_{n-1}(\mathbb{B}_{n-1})}{V_n(\mathbb{B}_n)},$$

because of $U_a(x) = -U_{-a}(-x)$, $t_a = -t_{-a}$ and (1.2). On the other hand, a converse inequality can be obtained by applying the conclusion (1.3) which we have proved, because $U_{|a|}(0) = |a|$ and $|U_{|a|}(x)| < 1$ for $x \in \mathbb{B}_n$. Thus, the equality in (1.3) holds for $F = U_a \circ T$. The theorem is proved.

4 The Proof of Theorem 1.2

Without loss of generality, assume that

$$|\nabla F(0)| = \frac{\partial F(0)}{\partial x_n}. \tag{4.1}$$

According the theory of H_p spaces, there exists a finite positive Borel measure μ on S_n such that $\mu(S_n) = a$ and

$$F(x) = \int_{S_n} \frac{1-|x|^2}{|x-\omega|^n} \mathrm{d}\mu(\omega) \quad \text{for} \quad x \in \mathbb{B}_n.$$

As in the proof of Theorem 1.1, we have

$$\frac{\partial U_a^+(0)}{\partial x_n} = n\int_{S_n} \omega_n \mathrm{d}\mu_a(\omega) = na,$$

and

$$\frac{\partial F(0)}{\partial x_n} = n\int_{S_n} \omega_n \mathrm{d}\mu(\omega).$$

Thus,

$$\begin{aligned}\frac{\partial F(0)}{\partial x_n} &= n\int_{E_0} \omega_n \mathrm{d}\mu(\omega) + n\int_{S_n\setminus E_0} \omega_n \mathrm{d}\mu(\omega)\\ &\leq n\mu(E_0) + n\mu(S_n \setminus E_0)\\ &= n\mu(S_n) = na.\end{aligned}$$

This shows (1.5).

If the equality in (1.5) holds, then $\mu(S_n \setminus E_0) = 0$ and, consequently, $\mu = \mu_a$ and $F = U_a^+$ under the assumption (4.1). Generally, we have $F = U_a^+ \circ T$ with a orthogonal transformation T. Conversely, the argument at the end of the proof of Theorem 1.1 shows that the equality (1.5) holds for $F = U_a^+ \circ T$ with a orthogonal transformation T. The theorem is proved.

References

[1] Axler S, Bourdon P, Ramey W. Harmonic Function Theory. Graduate Texts in Mathematics. 2nd ed. New York: Springer-Verlag, 2001

[2] Duren P. Harmonic Mappings in the Plane. Cambridge Tracts in Mathematics, Vol 137. New York: Springer-Verlag, 1992

[3] Bshouty D, Hengartner W. Univalent harmonic mappings in the plane. Ann Univ Mariae Curie-Sklodowska Sect A, 1994, **48**: 12–42

[4] Chen H, Gauthier P M, Hengartner W. Bloch constants for planar harmonic mappings. Proc Amer Math Soc, 2000, **128**: 3231–3240

[5] Chen H. The Schwarz-Pick lemma for planar harmonic mappings. Sci China Math, 2011, **54**: 1101–1118

[6] Heinz E. On one-to-one harmonic mappings. Pacific J Math, 1959, **9**: 101–105

[7] Chen H. The Schwarz-Pick lemma and Julia lemma for real planar harmonic mappings. Sci China Math, 2013, **56**: 2327–2334

[8] Kalaj D. Harmonic Functions and Harmonic Quasiconformal Mappings Between Convex Domains[D]. Beograd, 2002

[9] Knežević M, Mateljević M. On the quasi-isometries of harmonic quasiconformal mappings. J Math Anal Appl, 2007, **334**: 404–413

[10] Pavlović M. A Schwarz lemma for the modulus of a vector-valued analiytic function. Proc Amer Math Soc, 2011, **139**: 969–973

[11] Kalaj D, Vuorinen M. On harmonic functions and the Schwarz lemma. Proc Amer Math Soc, 2012, **140**: 161–165

[12] Liu Y, Chen Z, Pan Y. A boundary Schwarz lemma for holomorphic mappings on the polydisc. Chin Ann Math, Series B, 2018, **39**(1): 9–16

[13] Liu Y, Chen Z, Pan Y. Boundary Schwarz lemma for nonequidimensional holomorphic mappings and its application. Pacific J Math, 2018, **295**(2): 463–476

[14] Liu Y, Dai S, Pan Y. Boundary Schwarz lemma for pluriharmonic mappings between unit balls. J Math Anal Appl, 2016, **433**(1): 487–495

[15] Chen S, Ponnusamy S, Rasila A, Wang X. Linear connectivity, Schwarz-Pick lemma and univalency criteria for planar harmonic mapping. Acta Math Sin (English Series), 2017, **32**(3): 297–308