Inventiones math. 39, 129–141 (1977)

Inventiones mathematicae

Pseudoconvex Domains: Bounded Strictly Plurisubharmonic Exhaustion Functions

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Introduction

The complex analysis of strictly pseudoconvex domains in $\mathbb{C}^n$ is rather well known, especially since the boundary regularity properties of solutions of the inhomogeneous Cauchy-Riemann equations $\bar{\partial}\alpha=\beta$ on such domains have been described more and more precisely and important consequences from this have been derived (for a survey of the results in this direction known until 1972 (see [8]), more recent results are for instance contained in [15, 23]). In contrast to this situation, many fundamental questions with respect to the most "natural" domains of complex analysis, namely arbitrary pseudoconvex domains in $\mathbb{C}^n$, are still open. What is the reason for this discrepancy?

One possible answer to this question can be seen in the following observation. In almost all important proofs concerning the analysis of strictly pseudoconvex domains, at least one out of three elementary properties of these domains play a fundamental role. They are:

1) Strict pseudoconvexity is stable with respect to small C^2-perturbations.

2) A strictly pseudoconvex domain is locally biholomorphically equivalent to strictly convex domains.

3) If $\Omega\subset\subset\mathbb{C}^n$ is strictly pseudoconvex with a smooth C^2-boundary, then there is a neighborhood U of $\bar{\Omega}$ and a strictly plurisubharmonic C^2 function ρ on U such that $d\rho(p)\neq 0$ for $p\in b\Omega$ and

$$\Omega=\{p/\rho(p)<0\}.$$

In particular, the domains

$$\Omega_\varepsilon=\{p/\rho(p)<\varepsilon\}, \quad |\varepsilon| \text{ small enough},$$

are strictly pseudoconvex and approximate Ω in a very nice way from the inside ($\varepsilon<0$) and the outside ($\varepsilon>0$).

* During the preparation of this paper the second author has been partially supported by an NSF grant and by an NAVF grant from Norway

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What can, however, be said about analogues of these properties for arbitrary pseudoconvex domains with smooth boundary in $\mathbb{C}^n$? Obviously, arbitrarily small bumps and dents may destroy pseudoconvexity, such that there is no analogue to 1). The question whether 2) can be replaced by local biholomorphic equivalence to weakly convex domains has been open until J.J. Kohn and L. Nirenberg constructed in 1973 (see [13]) an example of a pseudoconvex boundary $M \subset \mathbb{C}^2$, $O \in M$, which is even strictly pseudoconvex on $M \setminus \{0\}$, but, nevertheless, is not locally equivalent to something convex.

This and some subsequent papers of the authors ([3, 4]) deal with the analogy to 3), i.e. with approximation properties of arbitrary bounded pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$ from the inside and the outside. The most direct way to do this would of course consist in proving the following statement, which has been conjectured for quite a long time and is f.i. mentioned as a problem by R.O. Wells in [25] (Problem 3.5, p. 415):

For any pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$ with smooth C^2 boundary there is a neighborhood U of $b\Omega$ and a plurisubharmonic C^2 function ρ on U, such that $d\rho(p) \neq 0$ for $p \in b\Omega$ and $\Omega \cap U = \{p \in U / \rho(p) < 0\}$.

Unfortunately, such a function does not exist in general, as is shown in [3]. Instead, we will construct in this paper a bounded strictly plurisubharmonic exhaustion function $\hat{\rho}$ on any relatively compact pseudoconvex domain Ω with smooth boundary on an arbitrary Stein manifold X (§1, Theorem I). In §2 we show that in case of $X = \mathbb{C}^n$ one cannot take as $\hat{\rho}$ the negative of any l-th root of the euclidean distance function $\operatorname{dist}(z, b\Omega)$ on Ω. As a consequence of Theorem I and by using a theorem of J.L. Stehlé ([22]), the Serre conjecture on locally trivial holomorphic fiber spaces over Stein spaces is verified for the case of the typical fiber being a domain for which Theorem I holds (or an intersection and/or cartesian product of such domains) (§3).

The results of this paper were announced in [1].

We want to thank the referee of a first version of this paper very much for communicating to us the elegant and short proof of Theorem I given in §1. It replaces our much longer proof of the first version. On the other hand, the original proof contains some more geometrical insight and gives additional information, which might be useful for applications. Therefore we will publish it in a subsequent paper under a somewhat different point of view.

This joint research was done, while the first author was a visitor in the Department of Mathematics of Princeton University. He expresses his sincere thanks for the hospitality and generosity of the mathematicians in this department with its stimulating mathematical atmosphere.

§ 0. Definitions and Notations

Let Ω be a domain on a complex manifold X and $q \in b\Omega$ fixed. We call $b\Omega$ a C^r-boundary near q, $r \geqq 1$ integer, if there is a neighborhood $U \subset X$ of q and a C^r function $\rho: U \to \mathbb{R}$, such that $d\rho(p) \neq 0$ for $p \in b\Omega \cap U$ and $\Omega \cap U = \{p \in U / \rho(p) < 0\}$. Any such function ρ is called a *local defining function* of Ω at q. If U is a neighborhood of $b\Omega$, then ρ is called a *defining function* of Ω. By $T_p^{10} b\Omega$

the complex tangent space to $b\Omega$ at p is denoted and $T_p^{10}X$ (resp. T_pX) is the complex tangent space (resp. real tangent space) to X at p.

If φ is a C^2 function on a neighborhood of $p\in X$, the Leviform of φ at p applied to t, $\hat{t}\in T_p^{10}X$ is written as $\mathcal{L}_\varphi(p;t,\hat{t})=\mathcal{L}_\varphi(t,\hat{t})$. If $t=\hat{t}$, we also write $\mathcal{L}(t)$ instead of $\mathcal{L}_\varphi(p;t,t)$.

If $\Omega\subset X$ is a domain, $q\in b\Omega$, U a neighborhood of q and $-\infty\leqq a<b\leqq\infty$ are given, a continuous function $\varphi\colon\Omega\cap U\to(a,b)$ is called a *local exhaustion function* of Ω at q, if for all c, d with $a<c<d<b$

$$\overline{\varphi^{-1}([c,d])}\cap b\Omega=\emptyset.$$

An *exhaustion function* φ on Ω with values (a,b) is a continuous proper map $\varphi\colon\Omega\to(a,b)$.

Finally, we define for a domain $\Omega\subset\subset\mathbb{C}^n$

$$\delta_\Omega(p)=\begin{cases}-\operatorname{dist}(p,b\,\Omega) & \text{for } p\in\Omega\\ \operatorname{dist}(p,b\,\Omega) & \text{for } p\notin\Omega,\end{cases}$$

where dist means the euclidean distance.

§ 1. Bounded Strictly Plurisubharmonic Exhaustion Functions

In this section we will show that the following theorem holds:

Theorem I. *Let X be a Stein manifold and $\Omega\subset X$ a relatively compact pseudoconvex domain with C^r-boundary, $2\leqq r\leqq\infty$. Then there is a C^r defining function ρ on a neighborhood U of $\bar{\Omega}$, such that for any number η with $0<\eta<1$ and η small enough, the function $\hat{\rho}=-(-\rho)^\eta$ is a strictly plurisubharmonic bounded exhaustion function on Ω.*

In this section Ω will always denote a domain as in Theorem I. Furthermore, we fix once and for all a C^∞ hermitian metric $d\mu^2$ on X. All tangent vectors will be measured in this metric. For the proof of Theorem I we will need the following consequence of Okas lemma:

Lemma 1. *There is a C^r defining function σ of Ω on a neighborhood U of $b\Omega$ in X, such that the function $-\log(-\sigma)$ is a plurisubharmonic exhaustion function of Ω on $U\cap\Omega$. For any such function there is, eventually after shrinking U, a constant $C>0$ with*

$$\mathcal{L}_\sigma(p;t)\geqq -C\,|t|_p\,|\langle\partial\sigma_p,t\rangle| \tag{1}$$

for all $t\in T_p^{10}X$ and all $p\in U\cap\Omega$.

Proof. 1) We embed X as a closed complex submanifold into some $\mathbb{C}^N$ and choose a Stein tubular neighborhood V of X in $\mathbb{C}^N$ with its holomorphic retraction $\pi\colon V\to X$. The domain $\pi^{-1}(\Omega)=\hat{\Omega}$ is Stein, $b\Omega\subset b\hat{\Omega}$ and $b\hat{\Omega}$ is a smooth C^r-boundary near $b\Omega$. Therefore, there is a neighborhood $\hat{U}$ of $b\Omega$ in X, such that the function $\delta_{\hat{\Omega}}$ is a C^r defining function of $\hat{\Omega}$ on U and, according to Oka's lemma, $-\log(-\delta_{\hat{\Omega}})$ is plurisubharmonic on $\hat{\Omega}$. Since, finally, X and $b\hat{\Omega}$ intersect each other transversally, the function $\sigma=\delta_{\hat\Omega}|U$ with $U=\hat U\cap X$ has the desired properties.

2) To prove inequality (1) we calculate at first

$$\mathcal{L}_{-\log(-\sigma)}(p;t)=\sigma^{-2}(p)\,[(-\sigma(p))\,\mathcal{L}_\sigma(p;t)+|\langle\partial\sigma_p,t\rangle|^2].$$

Since this is nonnegative for all $p\in U\cap\Omega$ and $t\in T_p^{10}$, we get in particular

$$\mathcal{L}_\sigma(p;t)\geqq 0 \tag{2}$$

for all $t\in T_p^1=\{t\in T_p^{10}\,|\,\langle\partial\sigma_p,t\rangle=0\}$, $p\in U\cap\Omega$. Let T_p^2 be the orthogonal complement of T_p^1 in T_p^{10} with respect to $d\mu_p^2$ and write $t=t'+t''\in T_p^1\oplus T_p^2$. Then (2) gives

$$\mathcal{L}_\sigma(p;t,t)\geqq 2\,\mathrm{Re}\,\mathcal{L}_\sigma(p;t',t'')+\mathcal{L}_\sigma(p;t'',t''). \tag{3}$$

Since σ is C^r on U there is obviously a constant $C_1>0$ such that, possibly after shrinking U,

$$|\mathcal{L}_\sigma(p;\hat t,t'')|\leqq C_1\,|\hat t|_p\,|t''|_p$$

for all $\hat t\in T_p^{10}$, $t''\in T_p^2$, $p\in U$. Furthermore, there is a constant $C_2>0$ such that

$$|\langle\partial\sigma_p,t\rangle|=|\langle\partial\sigma_p,t''\rangle|\geqq C_2\,|t''|_p$$

for $t=t'+t''\in T_p^1\oplus T_p^2$, $p\in U$. Together with (3) we therefore get (1). □

Now, we can give the

Proof of Theorem 1. 1) Let σ and U be chosen as in Lemma 1. Furthermore, fix a strictly plurisubharmonic C^∞ function ψ on X and define on U

$$\rho=\sigma\,e^{-L\psi}$$

and $\hat\rho=-(-\rho)^\eta$ with constants η, $0<\eta<1$, and $L>0$, which will be specified later (L will have to be large and η small). In any case, ρ is a C^r defining function of Ω on $U\cap\Omega$. A simple straightforward calculation leads to the following formula on $U\cap\Omega$:

$$\begin{aligned}\mathcal{L}_{\hat\rho}(t)=\eta\,(-\sigma)^{\eta-2}\,e^{-\eta L\psi}\,&[L\,\sigma^2(\mathcal{L}_\psi(t)-\eta\,L\,|\langle\partial\psi,t\rangle|^2)\\&+(-\sigma)\,(\mathcal{L}_\sigma(t)-2\eta\,L\,\mathrm{Re}\,\langle\partial\sigma,t\rangle\,\overline{\langle\partial\psi,t\rangle})+(1-\eta)\,|\langle\partial\sigma,t\rangle|^2]\end{aligned} \tag{4}$$

for $t\in T^{10}(U\cap\Omega)$. Therefore, $\mathcal{L}_{\hat\rho}$ will be positive definite on $T^{10}(U\cap\Omega)$ if the expression in [ ], for which we write $D(t)$, is strictly positive there. By Lemma 1 and by taking $\eta<\eta_0(L)$, $D(t)$ can be estimated from below by

$$D(t)\geqq LC_1\,\sigma^2\,|t|^2-C_2(-\sigma)\,|t|\,|\langle\partial\sigma,t\rangle|+\tfrac{1}{2}\,|\langle\partial\sigma,t\rangle|^2$$

for $t\in T^{10}(U\cap\Omega)$ with positive constants C_1, C_2 independent of L, η and t. From this, we finally get on $T^{10}(U\cap\Omega)$

$$D(t)\geqq\tfrac{3}{4}\,LC_1\,\sigma^2\,|t|^2+\left(\frac{1}{2}-\frac{C_2^2}{LC_1}\right)|\langle\partial\sigma,t\rangle|^2.$$

If we therefore take $L > 2C_2^2 C_1^{-1}$ and then $\eta < \eta_0(L)$, $\hat{\rho}$ becomes strictly plurisubharmonic on $U \cap \Omega$.

2) We still have to fill in the possible hole $\Omega \setminus U$ in the definition of ρ. This is easily done by replacing the function σ in the above arguments by $-\exp(-\lambda(\log(-\sigma)^{-1}))$, where λ is a convex increasing function on the real axis with $\lambda(s) = s$ for large s and λ constant before a suitable value of s. □

Remarks. a) The function $\hat{\rho}$ is Hölder continuous with exponent η on $\bar{\Omega}$. It is therefore of considerable interest to know how close to 1 the number η can possibly be taken. In [3], the authors, however, show, that in general η must be taken arbitrarily close to 0. On the other hand, under certain additional hypothesis on $b\Omega$, which guarantee, that the set of degeneracy of the Leviform of σ on $b\Omega$ is rather tame, it can be shown, that η can be taken arbitrarily close to 1 (see [4]).

b) For a fixed point $q \in b\Omega$ and a fixed η, $0 < \eta < 1$, there is always a neighborhood U and a C^r defining function ρ of Ω on U, such that $-(-\rho)^\eta$ is strictly plurisubharmonic on $\Omega \cap U$. This can be derived directly from (4) by chosing the strictly plurisubharmonic function ψ such that $d\psi(q) = 0$. Furthermore, the original proof of Theorem 1 shows, that such a defining function ρ can always be obtained by lifting the δ-function from a suitable biholomorphic image of Ω (or rather $\hat{\Omega}$ as in the proof of Lemma 1).

§ 2. Two Counterexamples

1. It is easy to see that some smoothness conditions on $b\Omega$ are necessary if one wants to have a bounded plurisubharmonic exhaustion function on Ω. A counterexample in the nonsmooth case is given by the so-called Hartogs triangle

$$\Omega_2 = \{(z, w) \in \mathbb{C}^2 \,|\, |z| < |w| < 1\}.$$

We have the

Proposition 1. *The domain Ω_2 is a domain of holomorphy, but there is no bounded plurisubharmonic exhaustion function on Ω_2.*

Proof. Assume that φ is such a function and put

$$L = \sup_{p \in \Omega_2} \varphi(p)$$

and $\varphi_0(w) = \varphi(0, w)$. Then φ_0 is subharmonic on $B' = \{w \,|\, 0 < |w| < 1\}$, $\varphi_0 \leqq L$ and $\lim_{w \to 0} \varphi_0(w) = L$. Therefore, φ_0 can be extended by $\varphi_0(0) = L$ to a subharmonic function on the unit disc (see [9], Theorem 7.7). Because of the maximum principle it must be constant. This is a contradiction to the exhaustion property of φ. □

Remark. It is not known, whether one really needs a smooth C^2-boundary for Theorem 1 to hold. It might be possible to generalize the theorem to the case of domains with smooth C^1-boundaries and all the other properties of Theorem 1. This would be interesting to know because of some applications.

2. Oka's lemma says that in any pseudoconvex domain $\Omega \subset \mathbb{C}^n$ the function $-\log(-\delta)$ is plurisubharmonic. Because the function $\lambda(t) = -(-t)^{1/l}$, $l \in N$, has similar convexity properties to $\hat{\lambda}(t) = -\log(-t)$ for $t \in (-\varepsilon, 0)$, $\varepsilon > 0$, it has been asked, *whether the function* $-(-\delta)^{1/l}$ *is plurisubharmonic near* $b\Omega$ for any pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$ with smooth C^∞-boundary, if l is chosen large enough (depending on Ω). We now want to show, that this is in general not true, i.e., that in Theorem 1 one can in general not take $\rho = \delta$. We define

$$\tilde{\Omega} = \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w < -4(|z|^4 + |z|^2 |w|^2) + 2(\operatorname{Re} w)(\operatorname{Im} z)\}. \tag{5}$$

(The set $\tilde{\Omega}$ is unbounded, but we will be interested in it only near $0 \in b\tilde{\Omega}$.)

Lemma 2. *The boundary $b\tilde{\Omega}$ is strictly pseudoconvex (from the side of $\tilde{\Omega}$) at all points $p \neq 0$, which are close enough to 0, and (weakly) pseudoconvex at 0.*

Proof. A defining function of $\tilde{\Omega}$ is

$$\rho(z, w) = \frac{i}{2}(\bar{w} - w) + 4 z^2 \bar{z}^2 + 4 z \bar{z} w \bar{w} + \frac{i}{2}(w + \bar{w})(z - \bar{z}).$$

The vector $t(p) = -\rho_w(p) \dfrac{\partial}{\partial z} + \rho_z \dfrac{\partial}{\partial w}$ spans $T_p^{10} b\tilde{\Omega}$ at $p = (z, w) \in b\tilde{\Omega}$ and after a few calculations one obtains

$$\begin{aligned}\mathscr{L}_\rho(p; t) &= 4|z|^2 + |w|^2 + \frac{i}{4}(\bar{z} - z)(w + \bar{w}) + O(|(z, w)|^3) \\ &= y u + 4x^2 + 4y^2 + u^2 + v^2 + O(|(z, w)|^3)\end{aligned} \tag{6}$$

in real coordinates defined by $z = x + iy$, $w = u + iv$. Now (5) gives on $b\tilde{\Omega}$

$$v = 2yu - 4x^4 - 4y^4 - 8x^2y^2 - 4(x^2 + y^2)u^2 - 4(x^2 + y^2)v^2. \tag{17}$$

Inserting this into (6) gives at once near 0

$$\begin{aligned}\mathscr{L}_\rho(p; t) &= y u + 4x^2 + 4y^2 + u^2 + O(|(x, u, y)|^3) \\ &\geqq 4x^2 + 3y^2 + \tfrac{3}{4}u^2 + O(|(x, u, y)|^3).\end{aligned}$$

Consequently, $\mathscr{L}_\rho(p; t) > 0$, if $p \in b\tilde{\Omega}$ is close to 0 and $p \neq 0$. □

Lemma 3. *Let δ denote the δ-function with respect to $\tilde{\Omega}$ Then there is an $\varepsilon > 0$, such that for all v with $-\varepsilon < v < 0$ and $p_0 = (0, 0, 0, v)$*

$$\mathscr{L}_\delta(p_0; t) = \frac{1}{1 - 4v^2}\left[(-v)|t^1|^2 + (-v)|t^2|^2 + \frac{i}{2} t^1 \bar{t}^2 - \frac{i}{2} \bar{t}^1 t^2\right]$$

with $t = t^1 \dfrac{\partial}{\partial z} + t^2 \dfrac{\partial}{\partial w}$.

Proof. 1) To simplify some notations we put $z = x^1 + i x^3$, $w = x^2 + i x^4$. We assume that we have represented $b\tilde{\Omega}$ near 0 in the form

$$b\tilde{\Omega} = \{x = (x^1, \dots, x^4) \mid x^4 = R(x') \text{ for } x' = (x^1, x^2, x^3) \in V'\} \tag{8}$$

with a C^∞ function R on a neighborhood V' of $0 \in \mathbb{R}^3$, such that $R(0)=0$, $R_i(0)=0$, $i=1,2,3$. We denote by π the projection on $b\tilde{\Omega}$ along the normals to $b\tilde{\Omega}$; it has the components $\pi=(\pi^1,\dots,\pi^4)$ and is well-defined on a neighborhood V of 0. Because of (8) one has $\pi^4(x)=R(\pi'(x))$ with $\pi'=(\pi^1,\pi^2,\pi^3)$. With these notations the δ-function can be written as

$$\delta(x)=-\left[\sum_{i=1}^{3}(x^i-\pi^i(x))^2+(x^4-R(\pi'(x)))^2\right]^{1/2}$$

for $x\in V\cap\tilde{\Omega}$. By differentiating this we get

$$\frac{\partial\delta(x)}{\partial x^j}=\frac{1}{\delta(x)}\left[\sum_{k=1}^{3}(x^k-\pi^k(x))\left(\delta_j^k-\frac{\partial\pi^k(x)}{\partial x^j}\right)\right.$$
$$\left.-(x^4-R(\pi'(x)))\sum_{k=1}^{2n-1}R_k(\pi'(x))\frac{\partial\pi^k(x)}{\partial x^j}\right]. \tag{9}$$

Another differentiation gives at $p_0=(0,0,0,x_0^4)$, $x_0^4<0$,

$$\frac{\partial^2\delta(p_0)}{\partial x^i\,\partial x^j}=\frac{1}{x_0^4}\left[\sum_{k=1}^{3}\left(\delta_i^k-\frac{\partial\pi^k(p_0)}{\partial x^i}\right)\left(\delta_j^k-\frac{\partial\pi^k(p_0)}{\partial x^j}\right)\right.$$
$$\left.-x_0^4\sum_{k,l=1}^{3}R_{kl}(0)\frac{\partial\pi^l(p_0)}{\partial x^i}\frac{\partial\pi^k(p_0)}{\partial x^j}\right] \tag{10}$$

for $1\leqq i,j\leqq 3$. Furthermore, it is clear that

$$\frac{\partial^2\delta(p_0)}{\partial x^4\,\partial x^j}=0 \qquad \text{for } 1\leqq j\leqq 4. \tag{11}$$

2) How can the values $\dfrac{\partial\pi^l(p_0)}{\partial x^i}$, $1\leqq i\leqq 3$, be computed? A basis of the real tangent space $T_q b\tilde{\Omega}$, $q\in b\tilde{\Omega}$, is given by the vectors

$$X_i(q)=\frac{\partial}{\partial x^i}+R_i(q)\frac{\partial}{\partial x^4}, \qquad 1\leqq i\leqq 3$$

and the vector $\pi(p)-p$ for $p\in V$ is orthogonal to $T_{\pi(p)}b\tilde{\Omega}_3$. Therefore, one has

$$(\pi(p)-p)\cdot X_i(\pi(p))=0 \qquad \text{for } 1\leqq i\leqq 3$$

where $\cdot$ denotes the euclidean inner product on $\mathbb{R}^4$. Differentiation of this equation yields at p_0

$$\frac{\partial\pi^i(p_0)}{\partial x^j}-x_0^4\sum_{k=1}^{3}R_{ik}(0)\frac{\partial\pi^k(p_0)}{\partial x^j}=\delta_j^i \qquad \text{for } 1\leqq i,j\leqq 3.$$

By solving this system one obtains

$$\frac{\partial\pi^i(p_0)}{\partial x^j}=\frac{\det\Delta_j^i}{\det\Delta}, \qquad 1\leqq i,j\leqq 3. \tag{12}$$

Here Δ is the 3×3-matrix $\Delta = I - x_0^4 \cdot (R_{ij}(0))_{1\leqq i,j\leqq 3}$ and Δ_j^i is the matrix obtained from Δ by replacing the i-th column by $(\delta_j^l)_{l=1}^3$.

3) It is clear from (7) that one has $R(x') = 2x^3 x^2 + O(|x'|^3)$. Therefore, $R_{23}(0) = 2$ and all the other $R_{ij}(0)$ vanish. This means that $\det \Delta = 1 - 4(x_0^4)^2$, $\det \Delta_1^1 = 1 - 4(x_0^4)^2$, $\det \Delta_2^2 = \det \Delta_3^3 = 1$, $\det \Delta_3^2 = \det \Delta_2^3 = 2x_0^4$ and all the other $\det \Delta_j^i = 0$. By putting all this into (12) and then into (10), and rewriting everything a little bit, one finally gets

$$\frac{\partial^2 \delta(p_0)}{\partial z\, \partial \bar{z}} = \frac{x_0^4}{1-4(x_0^4)^2} = \frac{\partial^2 \delta(p_0)}{\partial w\, \partial \bar{w}},$$

$$\frac{\partial^2 \delta(p_0)}{\partial z\, \partial \bar{w}} = \frac{i}{2}\, \frac{1}{1-4(x_0^4)^2}$$

for $p_0 = (0,0,0,x_0^4)$ with $x_0^4 < 0$ and $|x_0^4|$ small enough. This gives the wanted formula for $\mathscr{L}_\delta$. $\square$

Now, we can show, that there are smooth bounded pseudoconvex domains, for which no function

$$-(-\delta)^{1/l}, \qquad l \in \mathbb{N}$$

is plurisubharmonic everywhere near the boundary. More precisely, we show:

Theorem 2. *Let $\Omega \subset \mathbb{C}^2$ be any bounded smooth pseudoconvex domain with the following property:*

$0 \in b\Omega$ and there is a neighborhood V of 0 such that

$$\Omega \cap V = \{(z,w) \in V \mid \operatorname{Im} w < -4(|z|^4 + |z|^2 |w|^2) + 2(\operatorname{Re} w)(\operatorname{Im} z)\}.$$

Assume that for a certain $\varepsilon > 0$ a C^2-function

$$\lambda\colon (-\varepsilon, 0) \to \mathbb{R}$$

with $\lambda'(\tau_0) \neq 0$ for a $\tau_0 \in (-\varepsilon, 0)$ close enough to 0 is given such that the function

$$\rho = \lambda \circ \delta$$

is plurisubharmonic on $U_\varepsilon = \{p \in \Omega \mid -\delta(p) < \varepsilon\}$. Then there exists an ε_1 with $0 < \varepsilon_1 \leqq \varepsilon$ and a positive constant K, such that

$$\lambda(\tau) \geqq -K \log(-\tau)$$

for all $\tau \in (-\varepsilon_1, 0)$. In particular

$$\lim_{p \to b\Omega} \rho(p) = \infty.$$

Remark. Because of Lemma 2, Ω can even be chosen such that $b\Omega$ is strictly pseudoconvex at all points $\neq 0$.

Proof. The Leviform of ρ for $p \in U_\varepsilon$ is given by

$$\mathscr{L}_\rho(p; (t^1, t^2)) = \lambda' \circ \delta(p)\, \mathscr{L}_\delta(p; (t^1, t^2)) + \lambda'' \circ \delta(p) \left| \frac{\partial \delta(p)}{\partial z} t^1 + \frac{\partial \delta(p)}{\partial w} t^2 \right|^2. \tag{13}$$

Furthermore, there is a neighborhood $\hat{V} \subset V$ of 0, such that on $\hat{V} \cap \Omega$ the δ-function of Ω agrees with the δ-function of $\tilde{\Omega}$. We may assume that ε is $<\frac{1}{2}$ and so small that

$$S_\varepsilon = \{p = (0, i\,v) | -\varepsilon < v < 0\} \subset \Omega_3 \cap \hat{V}.$$

Therefore, for $p \in S_\varepsilon$, $\mathscr{L}_\delta$ is given by Lemma 3 and

$$\frac{\partial \delta(p)}{\partial z} = 0, \qquad \frac{\partial \delta(p)}{\partial w} = -\frac{i}{2}.$$

This gives with (13)

$$\mathscr{L}_\rho(p; (t^1, t^2)) = \frac{1}{1-4v^2} [\lambda' |v| (|t^1|^2 + |t^2|^2) + \lambda' \operatorname{Im}(\bar{t}^1 t^2) + \tfrac{1}{4} \lambda''(1-4v^2) |t^2|^2]$$

where the arguments in λ' and λ'' are $\delta(p) = v$ for $p \in S_\varepsilon$. The matrix of the hermitian form in [...] being

$$\begin{pmatrix} |v|\,\lambda' & \frac{i}{2}\lambda' \\ -\frac{i}{2}\lambda' & |v|\,\lambda' + \frac{1}{4}(1-4v^2)\,\lambda'' \end{pmatrix}$$

ρ can be plurisubharmonic on S_ε only if the following determinants are non-negative:

$$\lambda'(\tau) \geqq 0, \tag{14}$$

$$(1-4\tau^2)\,\lambda'(\tau)\,(|\tau|\,\lambda''(\tau) - \lambda'(\tau)) \geqq 0 \tag{15}$$

for all $\tau \in (-\varepsilon, 0)$.

The assumptions on λ and (14) enforce $\lambda'(\tau_0) > 0$ for a certain $\tau_0 \in (-\varepsilon, 0)$ and because of (15) $\lambda'(\tau) > 0$ implies $\lambda''(\tau) > 0$, thus showing that $\lambda'(\tau) > 0$ for $\tau_0 \leqq \tau < 0$. Therefore (15) finally gives

$$\frac{\lambda''(\tau)}{\lambda'(\tau)} \geqq \frac{1}{|\tau|} \qquad \text{for all } \tau \in (\tau_0, 0).$$

By integrating this differential inequality twice, one immediately obtains the claim. □

§ 3. Applications

1. A Special Case of the Serre Conjecture

In 1953 J-P. Serre asked the following question in [19]:

Let T, B, Y be complex spaces and $T \xrightarrow{\pi} B$ be a locally trivial holomorphic fiber space with typical fiber Y. Is T Stein, if B and Y are Stein?

The conjecture is, that the answer is always in the affirmative, but it is proved only in the special cases [5–7, 10, 11, 14, 16, 17, 19, 20–22, 24].

In particular, the somewhat most natural case of Y being just a domain of holomorphy in some $\mathbb{C}^n$ is in full generality not known until now. But a part of this question can be settled quite easily by using Theorem 1.

In his note [22] J.L. Stehlé defines:

Definition. A Stein space Y is called hyperconvex, if there exists a plurisubharmonic exhaustion function $\rho: Y \to \mathbb{R}^-$.

The first part of Theorem 1 can therefore be restated by using this terminology as saying:

Any relatively compact pseudoconvex domain Ω with C^2-boundary on an arbitrary Stein manifold X is hyperconvex.

On the other hand, the example Ω_2 in § 2 shows, that not every bounded Stein space is hyperconvex. But we can easily generalize the result on the hyperconvexity of bounded pseudoconvex domains with C^2-boundary, because we have

Proposition 2. a) *Let Y_1, Y_2 be hyperconvex spaces. Then $Y = Y_1 \times Y_2$ is also hyperconvex.*

b) *Let Y_1, Y_2 be open hyperconvex subspaces of a complex space X. Then $Y_1 \cap Y_2$ is also hyperconvex.*

Proof. Let $\varphi_k: Y_k \to \mathbb{R}^-$ be plurisubharmonic exhaustion functions, $k = 1, 2$. In Case a) we put

$$\varphi(p, q) = \max\{\varphi_1(p), \varphi_2(q)\}$$

for $(p, q) \in Y_1 \times Y_2 = Y$, and in Case b) we put

$$\varphi(p) = \max\{\varphi_1(p), \varphi_2(p)\}.$$

for $p \in Y = Y_1 \cap Y_2$. Then, in both cases, φ becomes a bounded plurisubharmonic exhaustion function on the respective space Y. □

Definition. With $\mathfrak{G}$ we denote the family of all open subsets of Stein manifolds, which can be obtained by starting with relatively compact pseudoconvex domains with C^2-boundary on Stein manifolds and taking successively finite intersections and finite cartesian products (in arbitrary order).

Because of Proposition 2 and Theorem 1 we have

Corollary. *The open subsets in $\mathfrak{G}$ are hyperconvex.*

All this is of interest, because J.L. Stehlé [22] gave a beautiful and very simple proof for the following fact:

Proposition. Let $T \xrightarrow{\pi} B$ be a locally trivial holomorphic fiber space with typical fiber Y. If B is Stein and Y is hyperconvex, then T is also Stein.

Because of the above Corollary we get the following immediate consequence:

Theorem 2. *Let $T \xrightarrow{\pi} B$ be a locally trivial holomorphic fiber space with typical fiber belonging to $\mathfrak{G}$. Suppose that B is a Stein space. Then T is also a Stein space.*

Remarks. 1) Theorem 2 contains the case of the Serre conjecture in which the fiber is a bounded domain of holomorphy with C^2-boundary in $\mathbb{C}^n$. For the proof given here, no information on the boundary behavior of the biholomorphic automorphisms of the typical fiber is needed.

2) Y.-T. Siu recently has verified the Serre conjecture in [21] for the case of the typical fiber being a relatively compact domain Ω with zero first Betti number on a Stein manifold X with trivial canonical line bundle.

The paper of Y.-T. Siu also contains a survey of what is known until now with respect to the Serre conjecture.

2. An Embedding Lemma

As a second application of Theorem 1 we prove a statement, which allows sometimes to reduce an investigation on pseudoconvex domains on Stein manifolds to questions on pseudoconvex domains in $\mathbb{C}^n$.

Theorem 3. *Let $\Omega \subset\subset X$ be a pseudoconvex domain with C^2-boundary on the Stein manifold X. Let X be embedded as a closed submanifold into some $\mathbb{C}^N$. Let $\pi\colon U \to X$ be a holomorphic retraction from a Stein neighborhood U of X onto X. Then there is a bounded pseudoconvex domain $\hat{\Omega} \subset\subset U$ with C^2-boundary such that $\hat{\Omega} \cap X = \Omega$ and $\pi(b\hat{\Omega}) = \bar{\Omega}$. The domain $\hat{\Omega}$ can be chosen to be strictly pseudoconvex outside X.*

Proof. We follow the argument of H. Rossi in [18]. Let $f_1, \ldots, f_s$ be holomorphic functions on $\mathbb{C}^N$, which generate the ideal sheaf of X on $\mathbb{C}^N$. Then we have

$$\sum_{i=1}^{s} \left| \sum_{k=1}^{N} \frac{\partial f_i(q)}{\partial z^k} t^k \right|^2 > 0 \tag{16}$$

for all $q \in X$ and all vectors $t = \sum_{k=1}^{N} t^k \frac{\partial}{\partial z^k}$ in q not tangential to X. By shrinking U, we therefore can assume that (16) holds at all $q \in U$ and for all t tangential to $\pi^{-1} \circ \pi(q)$ at q.

Let ρ be a C^2 defining function of Ω on a neighborhood U of $\bar{\Omega}$ together with an exponent $\eta = 1/l$, $l \in \mathbb{N}$, as given by Theorem 1. We put $\hat{\rho} = -(-\rho)^{1/l}$ on Ω and

$$\varphi = \hat{\rho} \circ \pi + L \sum_{i=1}^{s} |f_i|^2$$

on $\pi^{-1}(\Omega)$ with a constant $L > 0$ which will be specified later. The Leviform of φ is at $q \in \pi^{-1}(\Omega)$

$$\mathcal{L}_\varphi(q; t) = \mathcal{L}_{\hat{\rho}}(\pi(q); \pi_* t) + L \sum_{i=1}^{s} \left| \sum_{k=1}^{N} \frac{\partial f_i(q)}{\partial z^k} t^k \right|^2.$$

This shows because of the strict plurisubharmonicity of $\hat{\rho}$ and (16) that φ is strictly plurisubharmonic on $\pi^{-1}(\Omega)$. For L chosen large enough we get

$$\hat{\Omega}=\{p\in\pi^{-1}(\Omega)|\varphi(p)<0\}\subset\subset U$$

and $d\varphi=\pi^*(d\hat{\rho})+L\sum_{i=1}^{s}(\bar{f}_i\,\partial f_i+f_i\,\partial\bar{f}_i)\neq 0$ on $b\hat{\Omega}\setminus X$. To check that $b\hat{\Omega}$ is also smooth at $b\,\Omega$, we observe that $b\hat{\Omega}$ is described by the C^2-function

$$\psi=\rho\circ\pi+L^l\left(\sum_{i=1}^{s}|f_i|^2\right)^l,$$

for which $d\psi=\pi^*(d\rho)\neq 0$ on $b\,\Omega$. This completes the proof. □

3. *Remark.* Using Theorem 1, N. Kerzman has given a simple proof showing that the domains satisfying the suppositions of Theorem 1 are taut in the sense of H. Wu [26].

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Received March 11, 1976; Revised Version December 6, 1976