

CONCAVITY PROPERTY OF MINIMAL L^2 INTEGRALS WITH LEBESGUE MEASURABLE GAIN VI — FIBRATIONS OVER PRODUCTS OF OPEN RIEMANN SURFACES

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ABSTRACT. In this article, we present characterizations of the concavity property of minimal L^2 integrals degenerating to linearity in the case of fibrations over products of open Riemann surfaces. As applications, we obtain characterizations of the holding of equality in optimal jets L^2 extension problem from fibers over products of analytic subsets to fibrations over products of open Riemann surfaces, which implies characterizations of the equality parts of Saitoh conjecture and extended Saitoh conjecture for fibrations over products of open Riemann surfaces.

1. INTRODUCTION

The strong openness property of multiplier ideal sheaves [36] (2-dim [41]) i.e. $\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \bigcup_{\epsilon > 0} \mathcal{I}((1 + \epsilon)\varphi)$ (conjectured by Demailly [11]) has opened the door to new types of approximation techniques, which was used in the study of several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [36, 42, 5, 6, 17, 7, 55, 39, 3, 56, 57, 18, 43, 8]), where φ is a plurisubharmonic function of a complex manifold M (see [9]), and the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is defined as the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is locally integrable (see e.g. [52, 45, 48, 12, 13, 11, 14, 44, 49, 50, 10, 40]).

When $\mathcal{I}(\varphi) = \mathcal{O}$, the strong openness property degenerates to the openness property conjectured by Demailly-Kollar [13]. Berndtsson [2] (2-dim by Favre-Jonsson [15]) proved the openness property by establishing an effectiveness result of the openness property. Stimulated by Berndtsson's effectiveness result, and continuing the proof of the strong openness property [36], Guan-Zhou [38] established an effectiveness result of the strong openness property by considering the minimal L^2 integral on the pseudoconvex domain D .

Considering the minimal L^2 integrals on the sublevel sets of the weight φ , Guan [22] obtained a sharp version of Guan-Zhou's effectiveness result, and established a concavity property of the minimal L^2 integrals on the sublevel sets of the weight φ (with constant gain). The concavity property was applied to study the upper bound of the Bergman kernel i.e. a proof of Saitoh's conjecture for conjugate Hardy H^2 kernels [23], and equisingular approximations for the multiplier ideal sheaves

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i.e. the sufficient and necessary condition of the existence of decreasing equisingular approximations with analytic singularities for the multiplier ideal sheaves with weights $\log(|z_1|^{a_1} + \cdots + |z_n|^{a_n})$ [24].

For smooth gain, Guan [21] (see also [25]) presented the concavity property on Stein manifolds (the weakly pseudoconvex Kähler case was obtained by Guan-Mi[26]). The concavity property [21] (see also [25]) was applied by Guan-Yuan to deduce an optimal support function related to the strong openness property [29] and an effectiveness result of the strong openness property in L^p [30].

For Lebesgue measurable gain, Guan-Yuan [28] obtained the concavity property on Stein manifolds (the weakly pseudoconvex Kähler case was obtained by Guan-Mi-Yuan [27]). The concavity property [28] was applied by Guan-Yuan to deduce a twisted L^p version of the strong openness property [31].

As the linearity is a degenerate case of concavity, a natural problem was posed in [32]:

Problem 1.1 ([32]). *How to characterize the concavity property degenerating to linearity?*

For 1-dim case, Guan-Yuan [28] gave an answer to Problem 1.1 for single point, i.e. for weights may not be subharmonic (the case of subharmonic weights was answered by Guan-Mi [25]), and Guan-Yuan [32] gave an answer to Problem 1.1 for finite points. For products of open Riemann surfaces, Guan-Yuan [33] gave answers to Problem 1.1 for products of analytic subsets. Recently, Bao-Guan-Yuan [1] gave an answer to Problem 1.1 for fibrations over open Riemann surfaces.

In the present article, we give answers to Problem 1.1 for fibrations over products of open Riemann surfaces.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leq j \leq n_1$. Let Y be an n_2 -dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y . Let $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$ be an n -dimensional complex manifold, where $n = n_1 + n_2$. Let $\pi_1, \pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1 \leq j \leq n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M .

Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \dots, n_1\}$, and denote that $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y \subset M$. For any $j \in \{1, \dots, n_1\}$, let φ_j be a subharmonic function on Ω_j such that $\varphi_j(z) > -\infty$ for any $z \in Z_j$. Let φ_Y be a plurisubharmonic function on Y , and denote that $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$. Let ψ be a plurisubharmonic function on M such that $\{\psi < -t\} \setminus Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$ and $\psi(z) = -\infty$ for any $z \in Z_0$. Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$, $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $c(-\psi)$ has a positive lower bound on any compact subset of $M \setminus Z_0$. Let f be a holomorphic $(n, 0)$ form on a neighborhood of Z_0 . Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\}$$

by $G(t; c)$ (without misunderstanding, we denote $G(t; c)$ by $G(t)$), where $t \in [0, +\infty)$ and $|f|^2 := \sqrt{-1}^{n^2} f \wedge \bar{f}$ for any $(n, 0)$ form f .

Recall that $G(h^{-1}(r))$ is concave with respect to r [27], where $h(t) = \int_t^{+\infty} c(s)e^{-s}ds$ for any $t \geq 0$.

In the following section, we present the characterizations of the concavity of $G(h^{-1}(r))$ degenerating to linearity.

1.1. Main results.

We recall some notations (see [19], see also [37, 28, 27]). Let $P_j : \Delta \rightarrow \Omega_j$ be the universal covering from unit disc Δ to Ω_j . we call the holomorphic function f (resp. holomorphic $(1, 0)$ form F) on Δ a multiplicative function (resp. multiplicative differential (Prym differential)), if there is a character χ , which is the representation of the fundamental group of Ω_j , such that $g^*(f) = \chi(g)f$ (resp. $g^*(F) = \chi(g)F$), where $|\chi| = 1$ and g is an element of the fundamental group of Ω . Denote the set of such kinds of f (resp. F) by $\mathcal{O}^\chi(\Omega_j)$ (resp. $\Gamma^\chi(\Omega_j)$).

It is known that for any harmonic function u on Ω_j , there exists a $\chi_{j,u}$ (called character associate to u) and a multiplicative function $f_u \in \mathcal{O}^{\chi_{j,u}}(\Omega_j)$, such that $|f_u| = P_j^*(e^u)$. If $u_1 - u_2 = \log |f|$, then $\chi_{j,u_1} = \chi_{j,u_2}$, where u_1 and u_2 are harmonic functions on Ω_j and f is a holomorphic function on Ω_j . Let $z_j \in \Omega_j$. Recall that for the Green function $G_{\Omega_j}(z, z_j)$, there exist a χ_{j,z_j} and a multiplicative function $f_{z_j} \in \mathcal{O}^{\chi_{j,z_j}}(\Omega_j)$, such that $|f_{z_j}(z)| = P_j^*\left(e^{G_{\Omega_j}(z, z_j)}\right)$ (see [51]).

Let $Z_0 = \{z_0\} \times Y = \{(z_1, \dots, z_{n_1})\} \times Y \subset M$. Let

$$\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\},$$

where p_j is positive real number for $1 \leq j \leq n_1$. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1 \leq j \leq n_1} V_{z_j}$, and $w := (w_1, \dots, w_{n_1})$ is a local coordinate on V_0 of $z_0 \in \prod_{1 \leq j \leq n_1} \Omega_j$. Denote that $E := \left\{(\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0}\right\}$. Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y \subset M$.

We present a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the case $Z_0 = \{z_0\} \times Y$.

Theorem 1.2. *Assume that $G(0) \in (0, +\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

- (1) $f = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + g_0$ on $V_0 \times Y$, where g_0 is a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying $(g_0, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and f_α is a holomorphic $(n_2, 0)$ form on Y such that $\sum_{\alpha \in E} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \in (0, +\infty)$;
- (2) $\varphi_j = 2 \log |g_j| + 2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j) \neq 0$ and u_j is a harmonic function on Ω_j for any $1 \leq j \leq n_1$;
- (3) $\chi_{j,z_j}^{\alpha_j + 1} = \chi_{j,-u_j}$ for any $j \in \{1, 2, \dots, n\}$ and $\alpha \in E$ satisfying $f_\alpha \not\equiv 0$.

Let $c_j(z)$ be the logarithmic capacity (see [47]) on Ω_j , which is locally defined by

$$c_j(z_j) := \exp \lim_{z \rightarrow z_j} (G_{\Omega_j}(z, z_j) - \log |w_j(z)|).$$

Remark 1.3. *Lemma 2.26 shows that the above result also holds when we replace that sheaf $\mathcal{I}(\varphi + \psi)$ (in the definition of $G(t)$ and statement (1) in Theorem 1.2) by $\mathcal{I}(\psi)$.*

Remark 1.4. When the three statements in Theorem 1.2 hold,

$$\sum_{\alpha \in E} c_\alpha \left(\wedge_{1 \leq j \leq n_1} \pi_{1,j}^* \left(g_j(P_j)_* \left(f_{u_j} f_{z_j}^{\alpha_j} df_{z_j} \right) \right) \right) \wedge \pi_2^*(f_\alpha)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, z) \in (\mathcal{O}(K_M))_z \otimes \mathcal{I}(\varphi + \psi)_z$ for any $z \in Z_0$ and

$$\begin{aligned} G(t) &= \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \\ &= \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \end{aligned}$$

for any $t \geq 0$, where f_{u_j} is a holomorphic function on Δ such that $|f_{u_j}| = P_j^*(e^{u_j})$ for any $j \in \{1, \dots, n_1\}$, f_{z_j} is a holomorphic function on Δ such that $|f_{z_j}| = P_j^*(e^{G_{\Omega_j}(\cdot, z_j)})$ for any $j \in \{1, \dots, n_1\}$ and c_α is a constant such that $c_\alpha = \prod_{1 \leq j \leq n_1} \left(\lim_{z \rightarrow z_j} \frac{w_j^{\alpha_j} dw_j}{g_j(P_j)_* (f_{u_j} f_{z_j}^{\alpha_j} df_{z_j})} \right)$ for any $\alpha \in E$. We prove the remark in Section 3.1.

Let $Z_j = \{z_{j,1}, \dots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \dots, n_1\}$, where m_j is a positive integer. Let

$$\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

where $p_{j,k}$ is a positive real number. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $I_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j \leq m_j \text{ for any } j \in \{1, \dots, n_1\}\}$, $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \dots, \beta_{n_1}) \in I_1$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$ satisfying $w_\beta(z_\beta) = 0$.

Let $\beta^* = (1, \dots, 1) \in I_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geq 0}^{n_1}$. Denote that $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} > \sum_{1 \leq j \leq n_1} \frac{\alpha_{\beta^*,j} + 1}{p_{j,1}} \right\}$. Let f be a holomorphic $(n, 0)$ form on $\cup_{\beta \in I_1} V_\beta \times Y$ satisfying $f = \pi_1^* \left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2, 0)$ forms on Y .

We present a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the case Z_j is a set of finite points.

Theorem 1.5. Assume that $G(0) \in (0, +\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s) e^{-s} ds]$ if and only if the following statements hold:

(1) $\varphi_j = 2 \log |g_j| + 2u_j$ for any $j \in \{1, \dots, n_1\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k}) \neq 0$ for any $k \in \{1, \dots, m_j\}$;

(2) There exists a nonnegative integer $\gamma_{j,k}$ for any $j \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, m_j\}$, which satisfies that $\prod_{1 \leq k \leq m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ and $\sum_{1 \leq j \leq n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ for any $\beta \in I_1$;

(3) $f = \pi_1^* \left(c_\beta \left(\prod_{1 \leq j \leq n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}} \right) dw_{1,\beta_1} \wedge \dots \wedge dw_{n,\beta_n} \right) \wedge \pi_2^*(f_0) + g_\beta$ on $V_\beta \times Y$ for any $\beta \in I_1$, where c_β is a constant, $f_0 \not\equiv 0$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$, and g_β is a holomorphic $(n, 0)$ form on $V_\beta \times Y$ such that $(g_\beta, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$;

(4) $c_\beta \prod_{1 \leq j \leq n_1} \left(\lim_{z \rightarrow z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_* \left(f_{u_j} \left(\prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left(\sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right)} \right) = c_0$ for any $\beta \in I_1$, where $c_0 \in \mathbb{C} \setminus \{0\}$ is a constant independent of β , f_{u_j} is a holomorphic function Δ such that $|f_{u_j}| = P_j^*(e^{u_j})$ and $f_{z_{j,k}}$ is a holomorphic function on Δ such that $|f_{z_{j,k}}| = P_j^* \left(e^{G_{\Omega_j}(\cdot, z_{j,k})} \right)$ for any $j \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, m_j\}$.

Denote that

$$c_{j,k} := \exp \lim_{z \rightarrow z_{j,k}} \left(\frac{\sum_{1 \leq k_1 \leq m_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, m_j\}$.

Remark 1.6. When the four statements in Theorem 1.5 hold,

$$c_0 \left(\wedge_{1 \leq j \leq n_1} \pi_{1,j}^* \left(g_j(P_j)_* \left(f_{u_j} \left(\prod_{k=1}^{m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left(\sum_{k=1}^{m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right) \right) \right) \wedge \pi_2^*(f_0)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, z) \in (\mathcal{O}(K_M))_z \otimes \mathcal{I}(\varphi + \psi)_z$ for any $z \in Z_0$ and

$$\begin{aligned} G(t) &= \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \\ &= \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in I_1} \frac{|c_\beta|^2 (2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\gamma_{j,\beta_j} + 1) c_{j,\beta_j}^{2\gamma_{j,\beta_j} + 2}} \int_Y |f_0|^2 e^{-\varphi_Y} \end{aligned}$$

for any $t \geq 0$. We prove the remark in Section 4.

Let $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \dots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. Let $p_{j,k}$ be a positive number for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$ such that $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \neq -\infty$ for any j . Let

$$\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}.$$

Assume that $\limsup_{t \rightarrow +\infty} c(t) < +\infty$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \dots, n_1\}$ and $1 \leq k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$, $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$.

Let $\beta^* = (1, \dots, 1) \in I_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geq 0}^{n_1}$. Denote that $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} > \sum_{1 \leq j \leq n_1} \frac{\alpha_{\beta^*,j} + 1}{p_{j,1}} \right\}$. Let f be a holomorphic $(n, 0)$ form on $\cup_{\beta \in I_1} V_\beta \times Y$ satisfying $f = \pi_1^* \left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge$

$\pi_2^*(f_{\alpha_{\beta^*}}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2, 0)$ forms on Y .

We present that $G(h^{-1}(r))$ is not linear when there exists $j_0 \in \{1, \dots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$ as follows.

Theorem 1.7. *If $G(0) \in (0, +\infty)$ and there exists $j_0 \in \{1, \dots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$, then $G(h^{-1}(r))$ is not linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$.*

Let $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \dots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. Let $p_{j,k}$ be a positive number for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$ such that $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j . Let

$$\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}.$$

Let $M_1 \subset M$ be an n -dimensional weakly pseudoconvex Kähler manifold satisfying that $Z_0 \subset M_1$. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset M_1$ of Z_0 . Replace M in the definition of $G(t)$ by M_1 .

Proposition 1.8. *If $G(0) \in (0, +\infty)$ and $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$, we have $M_1 = M$.*

1.2. Applications.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leq j \leq n_1$. Let Y be an n_2 -dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y . Let $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$ be an n -dimensional complex manifold, where $n = n_1 + n_2$. Let $\pi_1, \pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1 \leq j \leq n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M . Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \dots, n_1\}$, and denote that $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y$. Let $M_1 \subset M$ be an n -dimensional complex manifold satisfying that $Z_0 \subset M_1$, and let K_{M_1} be the canonical (holomorphic) line bundle on M_1 .

In this section, we present the characterizations of the holding of equality in optimal jets L^2 extension problem from Z_0 to M_1 .

Let $Z_0 = \{z_0\} \times Y \subset M_1$, where $z_0 = (z_1, \dots, z_{n_1}) \in \prod_{1 \leq j \leq n_1} \Omega_j$. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1 \leq j \leq n_1} V_{z_j}$, and $w := (w_1, \dots, w_{n_1})$ is a local coordinate on V_0 of $z_0 \in \prod_{1 \leq j \leq n_1} \Omega_j$. Let $\Psi \leq 0$ be a plurisubharmonic function on $\prod_{1 \leq j \leq n_1} \Omega_j$, and let φ_j be a Lebesgue measurable function on Ω_j such that $\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$ is plurisubharmonic on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j . Let φ_Y be a plurisubharmonic function on Y . Denote that

$$\psi := \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\} + \pi_1^*(\Psi)$$

and $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on M , where p_j is a positive real number for $1 \leq j \leq n_1$. Denote that $E := \{(\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0}\}$

and $\tilde{E} := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} \geq 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$. Let

$$f = \sum_{\alpha \in \tilde{E}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha)$$

be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset (V_0 \times Y) \cap M_1$ of Z_0 , where f_α is a holomorphic $(n_2, 0)$ form on Y . Let $c_j(z)$ be the logarithmic capacity (see [47]) on Ω_j , which is locally defined by

$$c_j(z_j) := \exp \lim_{z \rightarrow z_j} (G_{\Omega_j}(z, z_j) - \log |w_j(z)|).$$

We obtain a characterization of the holding of equality in optimal jets L^2 extension problem for the case $Z_0 = \{z_0\} \times Y$.

Theorem 1.9. *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that*

$$\sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \in (0, +\infty).$$

Then there exists a holomorphic $(n, 0)$ form F on M_1 satisfying that $(F - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^(G_{\Omega_j}(\cdot, z_j))\}))_z$ for any $z \in Z_0$ and*

$$\begin{aligned} & \int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}. \end{aligned}$$

Moreover, equality $\inf \left\{ \int_{M_1} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(M_1, \mathcal{O}(K_{M_1})) \& (\tilde{F} - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^(G_{\Omega_j}(\cdot, z_j))\}))_z \right\} = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \times \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}$ holds if and only if the following statements hold:*

- (1) $M_1 = \left(\prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y$ and $\Psi \equiv 0$;
- (2) $\varphi_j = 2 \log |g_j| + 2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j) \neq 0$ and u_j is a harmonic function on Ω_j for any $1 \leq j \leq n_1$;
- (3) $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$ for any $j \in \{1, 2, \dots, n\}$ and $\alpha \in E$ satisfying $f_\alpha \not\equiv 0$.

Remark 1.10. *If $(f_\alpha, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ for any $y \in Y$ and $\alpha \in \tilde{E} \setminus E$, the above result also holds when we replace the ideal sheaf $\mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\})$ by $\mathcal{I}(\varphi + \psi)$. We prove the remark in Section 6.2.*

Remark 1.11. *Let f be a holomorphic $(n, 0)$ form on a neighborhood of Z_0 . It follows from Lemma 2.23 that there exists a sequence of holomorphic $(n_2, 0)$ form $\{f_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha)$ on a neighborhood of Z_0 . In the setting of Theorem 1.9, we assume that $f_\alpha \equiv 0$ for $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$ satisfying $\sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} < 1$.*

Remark 1.12. *Let $\tilde{\psi} = \max_{1 \leq j \leq n_1} \{2n_1 \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$. It follows from Lemma 2.18 that $(H_1 - H_2, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\tilde{\psi}))_z$ for any $z \in Z_0$ if and only if $(H_1 -$*

$H_2)|_{Z_0} = 0$, where H_1 and H_2 are holomorphic $(n, 0)$ forms on a neighborhood of Z_0 . Thus, Theorem 1.9 gives a characterization of the holding of equality in optimal L^2 extension theorem when $p_j = n_1$ for any $1 \leq j \leq n_1$.

Let $Z_j = \{z_{j,1}, \dots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \dots, n_1\}$, where m_j is a positive integer. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $I_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j \leq m_j \text{ for any } j \in \{1, \dots, n_1\}\}$, $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$ for any $\beta = (\beta_1, \dots, \beta_{n_1}) \in I_1$. Then $Z_0 = \{(z_\beta, y) : \beta \in I_1 \& y \in Y\} \subset M_1$.

Let $\Psi \leq 0$ be a plurisubharmonic function on $\prod_{1 \leq j \leq n_1} \Omega_j$, and let φ_j be a Lebesgue measurable function on Ω_j such that $\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$ is plurisubharmonic on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j . Let φ_Y be a plurisubharmonic function on Y . Denote that

$$\psi := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} + \pi_1^*(\Psi)$$

and $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on M , where $p_{j,k}$ is a positive real number for $1 \leq j \leq n_1$ and $1 \leq k \leq m_j$.

Denote that $E_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$ and $\tilde{E}_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \geq 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$ for any $\beta \in I_1$. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset M_1$ of Z_0 such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in I_1$. Let $\beta^* = (1, \dots, 1) \in I_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in E_{\beta^*}$. Denote that $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,1}} > 1 \right\}$. Assume that $f = \pi_1^* \left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^* (f_{\alpha_{\beta^*}, \beta^*}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha,\beta})$ on $U_0 \cap (V_{\beta^*} \times Y)$. Denote that

$$c_{j,k} := \exp \lim_{z \rightarrow z_{j,k}} \left(\frac{\sum_{1 \leq k_1 \leq m_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m_j\}$.

We obtain a characterization of the holding of equality in optimal jets L^2 extension problem for the case that Z_j is finite.

Theorem 1.13. *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that*

$$\sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \in (0, +\infty).$$

Then there exists a holomorphic $(n, 0)$ form F on M_1 satisfying that $(F - f, z) \in \left(\mathcal{O}(K_{M_1}) \otimes \mathcal{I} \left(\max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^(G_{\Omega_j}(\cdot, z_{j,k})) \right\} \right) \right)_z$ for any $z \in Z_0$*

and

$$\int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}.$$

Moreover, equality $\inf \left\{ \int_{M_1} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(M_1, \mathcal{O}(K_{M_1})) \& (\tilde{F} - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I} \left(\max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} \right))_z \text{ for any } z \in Z_0 \right\} = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}$ holds if

and only if the following statements hold:

$$(1) M_1 = \left(\prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y \text{ and } \Psi \equiv 0;$$

(2) $\varphi_j = 2 \log |g_j| + 2u_j$ for any $j \in \{1, \dots, n_1\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k}) \neq 0$ for any $k \in \{1, \dots, m_j\}$;

(3) There exists a nonnegative integer $\gamma_{j,k}$ for any $j \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, m_j\}$, which satisfies that $\prod_{1 \leq k \leq m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ and $\sum_{1 \leq j \leq n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ for any $\beta \in I_1$;

(4) $f_{\alpha,\beta} = c_\beta f_0$ holds for $\alpha = (\gamma_{1,\beta_1}, \dots, \gamma_{n_1,\beta_{n_1}})$ and $f_{\alpha,\beta} \equiv 0$ holds for any $\alpha \in E_\beta \setminus \{(\gamma_{1,\beta_1}, \dots, \gamma_{n_1,\beta_{n_1}})\}$, where $\beta \in I_1$, c_β is a constant and $f_0 \not\equiv 0$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$;

$$(5) c_\beta \prod_{1 \leq j \leq n_1} \left(\lim_{z \rightarrow z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_* \left(f_{u_j} \left(\prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left(\sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right)} \right) = c_0 \text{ for any } \beta \in I_1, \text{ where } c_0 \in \mathbb{C} \setminus \{0\} \text{ is a constant independent of } \beta, f_{u_j} \text{ is a holomorphic function } \Delta \text{ such that } |f_{u_j}| = P_j^*(e^{u_j}) \text{ and } f_{z_{j,k}} \text{ is a holomorphic function on } \Delta \text{ such that } |f_{z_{j,k}}| = P_j^* \left(e^{G_{\Omega_j}(\cdot, z_{j,k})} \right) \text{ for any } j \in \{1, \dots, n_1\} \text{ and } k \in \{1, \dots, m_j\}.$$

Remark 1.14. If $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ holds for any $y \in Y$, $\alpha \in \tilde{E}_\beta \setminus E_\beta$ and $\beta \in I_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I} \left(\max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} \right)$ by $\mathcal{I}(\varphi + \psi)$. We prove the remark in Section 7.2.

Let $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \dots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$, $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$ for any $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$. Then $Z_0 = \{(z_\beta, y) : \beta \in \tilde{I}_1 \& y \in Y\} \subset M_1$.

Let $\Psi \leq 0$ be a plurisubharmonic function on $\prod_{1 \leq j \leq n_1} \Omega_j$, and let φ_j be a Lebesgue measurable function on Ω_j such that $\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$ is plurisubharmonic on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j .

Let φ_Y be a plurisubharmonic function on Y . Let $p_{j,k}$ be a positive number for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$, which satisfies that $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \leq j \leq n_1$. Denote that

$$\psi := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} + \pi_1^*(\Psi)$$

and $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on M .

Denote that $E_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$ and $\tilde{E}_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} \geq 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$ for any $\beta \in \tilde{I}_1$. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset M_1$ of Z_0 such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$. Let $\beta^* = (1, \dots, 1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in E_{\beta^*}$. Denote that $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} > 1 \right\}$. Assume that $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1}\right) \wedge \pi_2^*(f_{\alpha_{\beta^*},\beta^*}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha,\beta})$ on $U_0 \cap (V_{\beta^*} \times Y)$. Denote that

$$c_{j,k} := \exp \lim_{z \rightarrow z_{j,k}} \left(\frac{\sum_{1 \leq k_1 < \tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \dots, n\}$ and $1 \leq k < \tilde{m}_j$ (following from Lemma 2.12 and Lemma 2.13, we get that the above limit exists).

We obtain that the equality in optimal jets L^2 extension problem could not hold when there exists $j_0 \in \{1, \dots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$.

Theorem 1.15. *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that*

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \in (0, +\infty)$$

and there exists $j_0 \in \{1, \dots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$.

Then there exists a holomorphic $(n, 0)$ form F on M_1 satisfying that $(F - f, z) \in \left(\mathcal{O}(K_{M_1}) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} \right)\right)_z$ for any $z \in Z_0$ and

$$\begin{aligned} & \int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \\ & < \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned}$$

Remark 1.16. *If $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ holds for any $y \in Y$, $\alpha \in \tilde{E}_\beta \setminus E_\beta$ and $\beta \in \tilde{I}_1$, the above result also holds when we replace the ideal sheaf*

$\mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)$ by $\mathcal{I}(\varphi + \psi)$. We prove the remark in Section 8.2.

1.2.1. *Suiza conjecture and extended Suiza conjecture.*

In this section, we present characterizations of the equality parts of Suiza conjecture and extended Suiza conjecture for fibrations over products of open Riemann surfaces.

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω . Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let κ_Ω be the Bergman kernel for holomorphic $(1,0)$ form on Ω . We define that

$$B_\Omega(z)dw \otimes \overline{dw} := \kappa_\Omega|_{V_{z_0}}.$$

Let $c_\beta(z)$ be the logarithmic capacity (see [47]) which is locally defined by

$$c_\beta(z_0) := \exp \lim_{z \rightarrow z_0} (G_\Omega(z, z_0) - \log |w(z)|)$$

on Ω . In [51], Suiza stated a conjecture as below.

Conjecture 1.17. $c_\beta(z_0)^2 \leq \pi B_\Omega(z_0)$ holds for any $z_0 \in \Omega$, and equality holds if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

The inequality part of Suiza conjecture for bounded planar domain was proved by Błocki [4], and original form of the inequality was proved by Guan-Zhou [34]. The equality part of Suiza conjecture was proved by Guan-Zhou [37], which completed the proof of Suiza conjecture.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leq j \leq n_1$. Let Y be an n_2 -dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y . Let $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$ be an n -dimensional complex manifold, where $n = n_1 + n_2$. Let $\pi_1, \pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1 \leq j \leq n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M .

Denote the space of L^2 integrable holomorphic section of K_M (resp. K_Y) by $A^2(M, K_M, dV_M^{-1}, dV_M)$ (resp. $A^2(Y, K_Y, dV_Y^{-1}, dV_Y)$). Let $\{\sigma_l\}_{l=1}^{+\infty}$ (resp. $\{\pi_l\}_{l=1}^{+\infty}$) be a complete orthogonal system of $A^2(M, K_M, dV_M^{-1}, dV_M)$ (resp. $A^2(Y, K_Y, dV_Y^{-1}, dV_Y)$) satisfying $(\sqrt{-1})^{n^2} \int_M \frac{\sigma_i}{\sqrt{2^n}} \wedge \frac{\bar{\sigma}_j}{\sqrt{2^n}} = \delta_i^j$. Put $\kappa_M = \sum_{l=1}^{+\infty} \sigma_l \otimes \bar{\sigma}_l \in C^\omega(M, K_M \otimes \overline{K_M})$ and $\kappa_Y = \sum_{l=1}^{+\infty} \pi_l \otimes \bar{\pi}_l \in C^\omega(Y, K_Y \otimes \overline{K_Y})$.

Let $z_0 = (z_1, \dots, z_{n_1}) \in \prod_{1 \leq j \leq n_1} \Omega_j$, and let $y_0 \in Y$. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1 \leq j \leq n_1} V_{z_j}$, and $w := (w_1, \dots, w_{n_1})$ is a local coordinate on V_0 of z_0 . Let $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_{n_2})$ be a local coordinate on a neighborhood U_0 of y_0 satisfying $\tilde{w}(y_0) = 0$. We define

$B_M((z, y))dw_1 \wedge \dots \wedge dw_{n_1} \wedge d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2} \otimes \overline{dw_1 \wedge \dots \wedge dw_{n_1} \wedge d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}} := \kappa_M$
on $V_0 \times U_0$ and

$$B_Y(y)d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2} \otimes \overline{d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}} := \kappa_Y$$

on U_0 . Let $c_j(z_j)$ be the logarithmic capacity which is locally defined by

$$c_j(z_j) := \exp \lim_{z \rightarrow z_j} (G_{\Omega_j}(z, z_j) - \log |w_j(z)|).$$

Assume that $B_Y(y_0) > 0$. Theorem 1.9 gives a characterization of the holding of equality in Saito conjecture for fibrations over products of open Riemann surfaces.

Theorem 1.18. $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) \leq \pi^{n_1} B_M((z_0, y_0))$ holds, and equality holds if and only if Ω_j is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero for any $j \in \{1, \dots, n\}$.

Let $M_1 \subset M$ be an n -dimensional complex manifold satisfying that $\{z_0\} \times Y \subset M_1$. Similar to M , we can define the Bergman kernel B_{M_1} . Theorem 1.18 implies the following result.

Remark 1.19. $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) \leq \pi^{n_1} B_{M_1}((z_0, y_0))$ holds, and equality holds if and only if $M_1 = M$ and Ω_j is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero for any $j \in \{1, \dots, n\}$.

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω , and let K_Ω be the canonical (holomorphic) line bundle on Ω . Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let $\rho = e^{-2u}$ on Ω , where u is a harmonic function on Ω . We define that

$$B_{\Omega, \rho} dw \otimes \overline{dw} := \sum_{l=1}^{+\infty} (\sigma_l \otimes \overline{\sigma}_l)|_{V_{z_0}} \in C^\omega(V_{z_0}, K_\Omega \otimes \overline{K_\Omega}),$$

where $\{\sigma_l\}_{l=1}^{+\infty}$ are holomorphic $(1, 0)$ forms on Ω satisfying $\sqrt{-1} \int_\Omega \rho \frac{\sigma_i}{\sqrt{2}} \wedge \frac{\overline{\sigma}_j}{\sqrt{2}} = \delta_i^j$ and $\{F \in H^0(\Omega, K_\Omega) : \int_\Omega \rho |F|^2 < +\infty \text{ & } \int_\Omega \rho \sigma_l \wedge \overline{F} = 0 \text{ for any } l \in \mathbb{Z}_{>0}\} = \{0\}$.

In [54], Yamada stated a conjecture as below (so-called extended Saito conjecture).

Conjecture 1.20. $c_\beta(z_0)^2 \leq \pi \rho(z_0) B_{\Omega, \rho}(z_0)$ holds for any $z_0 \in \Omega$, and equality holds if and only if $\chi_{-u} = \chi_{z_0}$, where χ_{-u} and χ_{z_0} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_0)$ respectively.

The inequality part of extended Saito conjecture was proved by Guan-Zhou [35]. The equality part of extended Saito conjecture was proved by Guan-Zhou [37].

Let $\rho = e^{-2 \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(u_j)}$ on M , where u_j is a harmonic function on Ω_j for any $j \in \{1, \dots, n\}$. We define that

$$B_{M, \rho} dw_1 \wedge \dots \wedge dw_{n_1} \wedge d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2} \otimes \overline{dw_1 \wedge \dots \wedge dw_{n_1} \wedge d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}} := \sum_{l=1}^{+\infty} e_l \otimes \overline{e}_l$$

on $V_0 \times Y$, where $\{e_l\}_{l=1}^{+\infty}$ are holomorphic $(n, 0)$ forms on M satisfying $(\sqrt{-1})^{n^2} \int_M \rho \frac{e_i}{\sqrt{2^n}} \wedge \frac{\overline{e}_j}{\sqrt{2^n}} = \delta_i^j$ and $\{F \in H^0(M, K_M) : \int_M \rho |F|^2 < +\infty \text{ & } \int_M \rho e_l \wedge \overline{F} = 0 \text{ for any } l \in \mathbb{Z}_{>0}\} = \{0\}$.

Assume that $B_Y(y_0) > 0$. Theorem 1.9 gives a characterization of the holding of equality in the extended Saito conjecture for fibrations over products of open Riemann surfaces.

Theorem 1.21. $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) \leq \pi^{n_1} \rho(z_0) B_{M, \rho}(z_0)$ holds, and equality holds if and only if $\chi_{j, -u_j} = \chi_{j, z_j}$ for any $j \in \{1, \dots, n\}$, where $\chi_{j, -u_j}$ and χ_{j, z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_0)$ respectively.

Let $M_1 \subset M$ be an n -dimensional complex manifold satisfying that $\{z_0\} \times Y \subset M_1$. Similar to M , we can define the Bergman kernel $B_{M_1, \rho}$. Theorem 1.21 implies the following result.

Remark 1.22. $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) \leq \pi^{n_1} B_{M_1, \rho}((z_0, y_0))$ holds, and equality holds if and only if $M_1 = M$ and $\chi_{j, -u_j} = \chi_{j, z_j}$ for any $j \in \{1, \dots, n\}$.

2. PREPARATION

2.1. Concavity property of minimal L^2 integrals.

In this section, we recall some results about the concavity property of minimal L^2 integrals (see [27, 33]).

Let M be a complex manifold. Let X and Z be closed subsets of M . We say that a triple (M, X, Z) satisfies condition (A), if the following statements hold:

I. X is a closed subset of M and X is locally negligible with respect to L^2 holomorphic functions; i.e., for any local coordinate neighborhood $U \subset M$ and for any L^2 holomorphic function f on $U \setminus X$, there exists an L^2 holomorphic function \tilde{f} on U such that $\tilde{f}|_{U \setminus X} = f$ with the same L^2 norm;

II. Z is an analytic subset of M and $M \setminus (X \cup Z)$ is a weakly pseudoconvex Kähler manifold.

Let M be an n -dimensional complex manifold, and let (M, X, Z) satisfy condition (A). Let K_M be the canonical line bundle on M . Let ψ be a plurisubharmonic function on M such that $\{\psi < -t\} \setminus (X \cup Z)$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$, and let φ be a Lebesgue measurable function on M such that $\psi + \varphi$ is a plurisubharmonic function on M . Denote $T = -\sup_M \psi$.

Definition 2.1. We call a positive measurable function c on $(T, +\infty)$ in class $P_{T,M}$ if the following two statements hold:

- (1) $c(t)e^{-t}$ is decreasing with respect to t ;
- (2) there is a closed subset E of M such that $E \subset Z \cap \{\psi(z) = -\infty\}$ and for any compact subset $K \subset M \setminus E$, $e^{-\varphi}c(-\psi)$ has a positive lower bound on K .

Let Z_0 be a subset of $\{\psi = -\infty\}$ such that $Z_0 \cap \text{Supp}(\mathcal{O}/\mathcal{I}(\varphi + \psi)) \neq \emptyset$. Let $U \supset Z_0$ be an open subset of M , and let f be a holomorphic $(n, 0)$ form on U . Let $\mathcal{F}_{z_0} \supset \mathcal{I}(\varphi + \psi)_{z_0}$ be an ideal of \mathcal{O}_{z_0} for any $z_0 \in Z_0$.

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0}) \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\}$$

by $G(t; c)$ ($G(t)$ for short), where $t \in [T, +\infty)$, c is a nonnegative function on $(T, +\infty)$, $|f|^2 := \sqrt{-1}^{n^2} f \wedge \bar{f}$ for any $(n, 0)$ form f and $(\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ means $(\tilde{f} - f, z) \in \mathcal{O}(K_M)_z \otimes \mathcal{F}_z$ for all $z \in Z_0$.

The following Theorem shows the concavity for $G(t)$.

Theorem 2.2 ([27]). Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)e^{-s} ds < +\infty$. If there exists $t \in [T, +\infty)$ satisfying that $G(t) < +\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(s)e^{-s} ds)$, $\lim_{t \rightarrow T+0} G(t) = G(T)$ and $\lim_{t \rightarrow +\infty} G(t) = 0$, where $h(t) = \int_t^{+\infty} c(s)e^{-s} ds$.

Denote that

$$\begin{aligned} \mathcal{H}^2(c, t) := \left\{ \tilde{f} : \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) < +\infty, (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0}) \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\}, \end{aligned}$$

where $t \in [T, +\infty)$ and c is a nonnegative measurable function on $(T, +\infty)$.

Corollary 2.3 ([27]). *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. If $G(t) \in (0, +\infty)$ for some $t \geq T$ and $G(h^{-1}(r))$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)e^{-s}ds]$, then there is a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ and $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq T$. Furthermore,*

$$\int_{\{-t_1 \leq \psi < -t_2\}} |F|^2 e^{-\varphi} a(-\psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(t)e^{-t}dt} \int_{t_2}^{t_1} a(t)e^{-t}dt \quad (2.1)$$

for any nonnegative measurable function a on $(T, +\infty)$, where $+\infty \geq t_1 > t_2 \geq T$ and $T_1 > T$.

Especially, if $\mathcal{H}^2(\tilde{c}, t_0) \subset \mathcal{H}^2(c, t_0)$ for some $t_0 \geq T$, where \tilde{c} is a nonnegative measurable function on $(T, +\infty)$, we have

$$G(t_0; \tilde{c}) = \int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} \tilde{c}(-\psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{t_0}^{+\infty} \tilde{c}(s)e^{-s}ds. \quad (2.2)$$

The following lemma is a characterization of $G(t) = 0$, where $t \geq T$.

Lemma 2.4 ([27]). *The following two statements are equivalent:*

- (1) $(f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$.
- (2) $G(t) = 0$.

Lemma 2.5 ([27]). *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. Assume that $G(t) < +\infty$ for some $t \in [T, +\infty)$. Then there exists a unique holomorphic $(n, 0)$ form F_t on $\{\psi < -t\}$ satisfying $(F_t - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ and $\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) = G(t)$. Furthermore, for any holomorphic $(n, 0)$ form \hat{F} on $\{\psi < -t\}$ satisfying $(\hat{F} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ and $\int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi) < +\infty$, we have the following equality*

$$\begin{aligned} & \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) + \int_{\{\psi < -t\}} |\hat{F} - F_t|^2 e^{-\varphi} c(-\psi) \\ &= \int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi). \end{aligned} \quad (2.3)$$

The following result will be used in the proof of Theorem 1.9.

Lemma 2.6. *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. Assume $G(t) \in (0, +\infty)$ for some $t \geq T$ and $G(h^{-1}(r))$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)e^{-s}ds]$. Let \tilde{c} be a nonnegative function on $(T, +\infty)$, and let $t_0 \geq T$. If there is a holomorphic $(n, 0)$ form $\tilde{F} \in \mathcal{H}^2(\tilde{c}, t_0)$ such that*

$$G(t_0; \tilde{c}) = \int_{\{\psi < -t_0\}} |\tilde{F}|^2 e^{-\varphi} \tilde{c}(-\psi)$$

and $\tilde{F} \in \mathcal{H}^2(c, t_0)$, then we have

$$G(t_0; \tilde{c}) = \int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} \tilde{c}(-\psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(s) e^{-s} ds} \int_{t_0}^{+\infty} \tilde{c}(s) e^{-s} ds,$$

where $T_1 > T$.

Proof. Using Corollary 2.3, we know there is a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ and $G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) = \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s) e^{-s} ds} \int_t^{+\infty} c(s) e^{-s} ds$ for any $t \geq T$. It follows from the dominated convergence theorem that

$$\int_{\{z \in M : -\psi(z) \in N\}} |F|^2 e^{-\varphi} = 0 \quad (2.4)$$

holds for any $N \subset \subset (T, +\infty)$ satisfying $\mu(N) = 0$, where μ is the Lebesgue measure on \mathbb{R} . As $\tilde{F} \in \mathcal{H}^2(c, t_0)$, It follows from Lemma 2.5 that

$$\begin{aligned} \int_{\{\psi < -t\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) &= \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \\ &\quad + \int_{\{\psi < -t\}} |\tilde{F} - F|^2 e^{-\varphi} c(-\psi) \end{aligned}$$

for any $t \geq t_0$, then

$$\begin{aligned} \int_{\{-t_3 \leq \psi < -t_4\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) &= \int_{\{-t_3 \leq \psi < -t_4\}} |F|^2 e^{-\varphi} c(-\psi) \\ &\quad + \int_{\{-t_3 \leq \psi < -t_4\}} |\tilde{F} - F|^2 e^{-\varphi} c(-\psi) \end{aligned} \quad (2.5)$$

holds for any $t_3 > t_4 \geq t_0$. It follows from the dominated convergence theorem, equality (2.4), equality (2.5) and $c(t) > 0$ for any $t > T$, that

$$\int_{\{z \in M : -\psi(z) = t\}} |\tilde{F}|^2 e^{-\varphi} = \int_{\{z \in M : -\psi(z) = t\}} |\tilde{F} - F|^2 e^{-\varphi} \quad (2.6)$$

holds for any $t > t_0$.

Choosing any closed interval $[t'_4, t'_3] \subset (t_0, +\infty) \subset (T, +\infty)$. Note that $c(t)$ is uniformly continuous and have positive lower bound and upper bound on $[t'_4, t'_3] \setminus U_k$, where $\{U_k\}_{k \in \mathbb{Z}_{\geq 1}}$ is a decreasing sequence of open subsets of $(T, +\infty)$, such that c is continuous on $(T, +\infty) \setminus U_k$ and $\lim_{k \rightarrow +\infty} \mu(U_k) = 0$ (As $c(t)e^{-t}$ is decreasing, $\{U_k\}_{k \in \mathbb{Z}_{\geq 1}}$ exists). Take $N = \cap_{k=1}^{+\infty} U_k$. Note that

$$\begin{aligned} &\int_{\{-t'_3 \leq \psi < -t'_4\}} |\tilde{F}|^2 e^{-\varphi} \\ &= \lim_{n \rightarrow +\infty} \sum_{i=0}^{n-1} \int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |\tilde{F}|^2 e^{-\varphi} + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \cap U_k\}} |\tilde{F}|^2 e^{-\varphi} \\ &\leq \limsup_{n \rightarrow +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{I_{n,i} \setminus U_k} c(t)} \int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\ &\quad + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \cap U_k\}} |\tilde{F}|^2 e^{-\varphi}, \end{aligned} \quad (2.7)$$

where $I_{n,i} = (t'_4 - (i+1)\alpha_n, t'_3 - i\alpha_n]$ and $\alpha_n = \frac{t'_3 - t'_4}{n}$. It follows from equality (2.4), equality (2.5), equality (2.6) and the dominated convergence theorem that

$$\begin{aligned} & \int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\ &= \int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |F|^2 e^{-\varphi} c(-\psi) + \int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |\tilde{F} - F|^2 e^{-\varphi} c(-\psi). \end{aligned} \quad (2.8)$$

As $c(t)$ is uniformly continuous and have positive lower bound and upper bound on $[t'_4, t'_3] \setminus U_k$, following from equality (2.8), we have

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{I_{n,i} \setminus U_k} c(t)} \int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\ &= \limsup_{n \rightarrow +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{I_{n,i} \setminus U_k} c(t)} \left(\int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |F|^2 e^{-\varphi} c(-\psi) \right. \\ & \quad \left. + \int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |\tilde{F} - F|^2 e^{-\varphi} c(-\psi) \right) \\ &\leq \limsup_{n \rightarrow +\infty} \sum_{i=0}^{n-1} \frac{\sup_{I_{n,i} \setminus U_k} c(t)}{\inf_{I_{n,i} \setminus U_k} c(t)} \left(\int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |F|^2 e^{-\varphi} \right. \\ & \quad \left. + \int_{\{z \in M : -\psi(z) \in I_{n,i} \setminus U_k\}} |\tilde{F} - F|^2 e^{-\varphi} \right) \\ &= \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \setminus U_k\}} |F|^2 e^{-\varphi} + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \setminus U_k\}} |\tilde{F} - F|^2 e^{-\varphi}. \end{aligned} \quad (2.9)$$

It follows from inequality (2.7) and (2.9) that

$$\begin{aligned} & \int_{\{-t'_3 \leq \psi < -t'_4\}} |\tilde{F}|^2 e^{-\varphi} \\ &\leq \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \setminus U_k\}} |F|^2 e^{-\varphi} + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \setminus U_k\}} |\tilde{F} - F|^2 e^{-\varphi} \\ & \quad + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \cap U_k\}} |\tilde{F}|^2 e^{-\varphi}. \end{aligned} \quad (2.10)$$

It follows from $\tilde{F} \in \mathcal{H}^2(c, t_0)$ that $\int_{\{-t'_3 \leq \psi < -t'_4\}} |\tilde{F}|^2 e^{-\varphi} < +\infty$. Letting $k \rightarrow +\infty$, it follows from equality (2.4), inequality (2.10) and the dominated convergence theorem that

$$\begin{aligned} \int_{\{-t'_3 \leq \psi < -t'_4\}} |\tilde{F}|^2 e^{-\varphi} &\leq \int_{\{-t'_3 \leq \psi < -t'_4\}} |F|^2 e^{-\varphi} \\ & \quad + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \setminus N\}} |\tilde{F} - F|^2 e^{-\varphi} \\ & \quad + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \cap N\}} |\tilde{F}|^2 e^{-\varphi}. \end{aligned} \quad (2.11)$$

Following from a similar discussion, we can obtain that

$$\begin{aligned} \int_{\{-t'_3 \leq \psi < -t'_4\}} |\tilde{F}|^2 e^{-\varphi} &\geq \int_{\{-t'_3 \leq \psi < -t'_4\}} |F|^2 e^{-\varphi} \\ &\quad + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \setminus N\}} |\tilde{F} - F|^2 e^{-\varphi} \\ &\quad + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \cap N\}} |\tilde{F}|^2 e^{-\varphi}, \end{aligned}$$

then combining inequality (2.11) we have

$$\begin{aligned} \int_{\{-t'_3 \leq \psi < -t'_4\}} |\tilde{F}|^2 e^{-\varphi} &= \int_{\{-t'_3 \leq \psi < -t'_4\}} |F|^2 e^{-\varphi} \\ &\quad + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \setminus N\}} |\tilde{F} - F|^2 e^{-\varphi} \\ &\quad + \int_{\{z \in M : -\psi(z) \in (t'_4, t'_3] \cap N\}} |\tilde{F}|^2 e^{-\varphi}. \end{aligned} \tag{2.12}$$

Using equality (2.4), equality (2.6), equality (2.12) and the monotone convergence theorem, we have

$$\begin{aligned} \int_{\{z \in M : -\psi(z) \in U\}} |\tilde{F}|^2 e^{-\varphi} &= \int_{\{z \in M : -\psi(z) \in U\}} |F|^2 e^{-\varphi} \\ &\quad + \int_{\{z \in M : -\psi(z) \in U \setminus N\}} |\tilde{F} - F|^2 e^{-\varphi} \\ &\quad + \int_{\{z \in M : -\psi(z) \in U \cap N\}} |\tilde{F}|^2 e^{-\varphi} \end{aligned}$$

holds for any open set $U \subset \subset (t_0, +\infty)$, and

$$\begin{aligned} \int_{\{z \in M : -\psi(z) \in V\}} |\tilde{F}|^2 e^{-\varphi} &= \int_{\{z \in M : -\psi(z) \in V\}} |F|^2 e^{-\varphi} \\ &\quad + \int_{\{z \in M : -\psi(z) \in V \setminus N\}} |\tilde{F} - F|^2 e^{-\varphi} \\ &\quad + \int_{\{z \in M : -\psi(z) \in V \cap N\}} |\tilde{F}|^2 e^{-\varphi} \end{aligned}$$

holds for any compact set $V \subset (t_0, +\infty)$. For any measurable set $E \subset \subset (t_0, +\infty)$, there exists a sequence of compact sets $\{V_l\}$, such that $V_l \subset V_{l+1} \subset E$ for any l and $\lim_{l \rightarrow +\infty} \mu(V_l) = \mu(E)$, hence

$$\begin{aligned} \int_{\{\psi < -t_0\}} |\tilde{F}|^2 e^{-\varphi} \mathbb{I}_E(-\psi) &\geq \lim_{l \rightarrow +\infty} \int_{\{\psi < -t_0\}} |\tilde{F}|^2 e^{-\varphi} \mathbb{I}_{V_l}(-\psi) \\ &\geq \lim_{l \rightarrow +\infty} \int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} \mathbb{I}_{V_l}(-\psi) \\ &= \int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} \mathbb{I}_E(-\psi). \end{aligned} \tag{2.13}$$

It is clear that for any $t > t_0$, there exists a sequence of functions $\left\{ \sum_{j=1}^{n_i} \mathbb{I}_{E_{ij}} \right\}_{i=1}^{+\infty}$ defined on $(t, +\infty)$, satisfying $E_{ij} \subset \subset (t, +\infty)$, $\sum_{j=1}^{n_{i+1}} \mathbb{I}_{E_{i+1,j}}(s) \geq \sum_{j=1}^{n_i} \mathbb{I}_{E_{ij}}(s)$,

and $\lim_{i \rightarrow +\infty} \sum_{j=1}^{n_i} \mathbb{E}_{E_{ij}}(s) = \tilde{c}(s)$ for any $s > t$. Combining the monotone convergence theorem and inequality (2.13), we have

$$\int_{\{\psi < -t_0\}} |\tilde{F}|^2 e^{-\varphi} \tilde{c}(-\psi) \geq \int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} \tilde{c}(-\psi).$$

By the definition of $G(t_0, \tilde{c})$, we have $G(t_0, \tilde{c}) = \int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} \tilde{c}(-\psi)$. Thus, Lemma 2.6 holds. \square

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leq j \leq n$. Let $M = \prod_{1 \leq j \leq n} \Omega_j$ be an n -dimensional complex manifold, and let π_j be the natural projection from M to Ω_j . Let K_M be the canonical (holomorphic) line bundle on M . Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \dots, n\}$, and let $Z_0 = \prod_{1 \leq j \leq n} Z_j$. For any $j \in \{1, \dots, n\}$, let φ_j be a subharmonic function on Ω_j such that $\varphi_j(z) > -\infty$ for any $z \in Z_j$, and let $\varphi = \sum_{1 \leq j \leq n} \pi_j^*(\varphi_j)$. Let ψ be a plurisubharmonic function on M such that $\psi(z) = -\infty$ for any $z \in Z_0$ and ψ is continuous on $M \setminus Z_0$. Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t) e^{-t} dt < +\infty$ and $c(t) e^{-t}$ is decreasing on $(0, +\infty)$. Let $\mathcal{F}_z = \mathcal{I}(\psi)_z$ for any $z \in Z_0$.

In the following, we recall some results about the concavity of $G(h^{-1}(r))$ degenerating to linearity.

Let $Z_0 = \{z_0\} = \{(z_1, \dots, z_n)\} \subset M$. Let $\psi = \max_{1 \leq j \leq n} \{2p_j \pi_j^*(G_{\Omega_j}(\cdot, z_j))\}$, where p_j is positive real number. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1 \leq j \leq n} V_{z_j}$, and $w := (w_1, \dots, w_n)$ is a local coordinate on V_0 of $z_0 \in M$. Let f be a holomorphic $(n, 0)$ form on V_0 . Denote that $E := \left\{ (\alpha_1, \dots, \alpha_n) : \sum_{1 \leq j \leq n} \frac{\alpha_j + 1}{p_j} = 1 \text{ \& } \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$.

We recall a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the case Z_0 is a single point set as follows.

Theorem 2.7 ([33]). *Assume that $G(0) \in (0, +\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t) e^{-t} dt]$ if and only if the following statements hold:*

- (1) $f = (\sum_{\alpha \in E} d_\alpha w^\alpha + g_0) dw_1 \wedge \dots \wedge dw_n$ on V_0 , where $d_\alpha \in \mathbb{C}$ such that $\sum_{\alpha \in E} |d_\alpha| \neq 0$ and g_0 is a holomorphic function on V_0 such that $(g_0, z_0) \in \mathcal{I}(\psi)_{z_0}$;
- (2) $\varphi_j = 2 \log |g_j| + 2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j) \neq 0$ and u_j is a harmonic function on Ω_j for any $1 \leq j \leq n$;
- (3) $\chi_{j, z_j}^{\alpha_j + 1} = \chi_{j, -u_j}$ for any $j \in \{1, 2, \dots, n\}$ and $\alpha \in E$ satisfying $d_\alpha \neq 0$, χ_{j, z_j} and $\chi_{j, -u_j}$ are the characters associated to functions $G_{\Omega_j}(\cdot, z_j)$ and $-u_j$ respectively.

Let $c_j(z)$ be the logarithmic capacity (see [47]) on Ω_j , which is locally defined by

$$c_j(z_j) := \exp \lim_{z \rightarrow z_j} (G_{\Omega_j}(z, z_j) - \log |w_j(z)|).$$

Remark 2.8 ([33]). *When the three statements in Theorem 2.7 hold,*

$$\sum_{\alpha \in E} \tilde{d}_\alpha \wedge_{1 \leq j \leq n} \pi_j^* \left(g_j(P_j)_* \left(f_{u_j} f_{z_j}^{\alpha_j} df_{z_j} \right) \right)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, z_0) \in (\mathcal{O}(K_M))_{z_0} \otimes \mathcal{I}(\psi)_{z_0}$ and

$$G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{|d_\alpha|^2 (2\pi)^n e^{-\varphi(z_0)}}{\prod_{1 \leq j \leq n} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}$$

for any $t \geq 0$, where $P_j : \Delta \rightarrow \Omega_j$ is the universal covering, f_{u_j} is a holomorphic function on Δ such that $|f_{u_j}| = P_j^*(e^{u_j})$ for any $j \in \{1, \dots, n\}$, f_{z_j} is a holomorphic function on Δ such that $|f_{z_j}| = P_j^* \left(e^{G_{\Omega_j}(\cdot, z_j)} \right)$ for any $j \in \{1, \dots, n\}$ and \tilde{d}_α is a constant such that $\tilde{d}_\alpha = \lim_{z \rightarrow z_0} \frac{d_\alpha w^\alpha dw_1 \wedge \dots \wedge dw_n}{\wedge_{1 \leq j \leq n} \pi_j^*(g_j(P_j)_*(f_{u_j} f_{z_j}^{\alpha_j} df_{z_j}))}$ for any $\alpha \in E$.

Let $Z_j = \{z_{j,1}, \dots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \dots, n\}$, where m_j is a positive integer. Let $\psi = \max_{1 \leq j \leq n} \left\{ \pi_j^* \left(2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $I_1 := \{(\beta_1, \dots, \beta_n) : 1 \leq \beta_j \leq m_j \text{ for any } j \in \{1, \dots, n\}\}$, $V_\beta := \prod_{1 \leq j \leq n} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \dots, \beta_n) \in I_1$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n,\beta_n})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n,\beta_n}) \in M$. Let f be a holomorphic $(n, 0)$ form on $\cup_{\beta \in I_1} V_\beta$ such that $f = w_\beta^{\alpha_\beta} dw_{1,1} \wedge \dots \wedge dw_{n,1}$ on V_{β^*} , where $\beta^* = (1, \dots, 1) \in I_1$.

We recall a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the case Z_j is a set of finite points as follows.

Theorem 2.9 ([33]). *Assume that $G(0) \in (0, +\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s) e^{-s} ds]$ if and only if the following statements hold:*

(1) $\varphi_j = 2 \log |g_j| + 2u_j$ for any $j \in \{1, \dots, n\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k}) \neq 0$ for any $k \in \{1, \dots, m_j\}$;

(2) There exists a nonnegative integer $\gamma_{j,k}$ for any $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m_j\}$, which satisfies that $\prod_{1 \leq k \leq m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ and $\sum_{1 \leq j \leq n} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ for any $\beta \in I_1$, where $\chi_{j,z_{j,k}}$ and $\chi_{j,-u_j}$ are the characters associated to $G_{\Omega_j}(\cdot, z_{j,k})$ and $-u_j$ respectively;

(3) $f = \left(c_\beta \prod_{1 \leq j \leq n} w_{j,\beta_j}^{\gamma_{j,\beta_j}} + g_\beta \right) dw_{1,\beta_1} \wedge \dots \wedge dw_{n,\beta_n}$ on V_β for any $\beta \in I_1$, where c_β is a constant and g_β is a holomorphic function on V_β such that $(g_\beta, z_\beta) \in \mathcal{I}(\psi)_{z_\beta}$;

(4) $\lim_{z \rightarrow z_\beta} \frac{c_\beta \prod_{1 \leq j \leq n} w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{1,\beta_1} \wedge \dots \wedge dw_{n,\beta_n}}{\wedge_{1 \leq j \leq n} \pi_j^*(g_j(P_j)_*(f_{u_j} (\prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1}) (\sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}}))))} = c_0$ for any $\beta \in I_1$, where $P_j : \Delta \rightarrow \Omega_j$ is the universal covering, $c_0 \in \mathbb{C} \setminus \{0\}$ is a constant independent of β , f_{u_j} is a holomorphic function on Δ such that $|f_{u_j}| = P_j^*(e^{u_j})$ and $f_{z_{j,k}}$ is a holomorphic function on Δ such that $|f_{z_{j,k}}| = P_j^*(e^{G_{\Omega_j}(\cdot, z_{j,k})})$ for any $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m_j\}$.

Denote that

$$c_{j,k} := \exp \lim_{z \rightarrow z_{j,k}} \left(\frac{\sum_{1 \leq k_1 \leq m_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m_j\}$.

Remark 2.10 ([33]). *When the four statements in Theorem 2.9 hold,*

$$c_0 \wedge_{1 \leq j \leq n} \pi_j^* \left(g_j(P_j)_* \left(f_{u_j} \left(\prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left(\sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right) \right)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, z_\beta) \in (\mathcal{O}(K_M))_{z_\beta} \otimes \mathcal{I}(\psi)_{z_\beta}$ for any $\beta \in I_1$ and

$$G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in I_1} \frac{|c_\beta|^2 (2\pi)^n e^{-\varphi(z_\beta)}}{\prod_{1 \leq j \leq n} (\gamma_{j,\beta_j} + 1) c_{j,\beta_j}^{2\gamma_{j,\beta_j} + 2}}$$

for any $t \geq 0$.

Let $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \dots, n\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. Let $p_{j,k}$ be a positive number for any $1 \leq j \leq n$ and $1 \leq k < \tilde{m}_j$ such that $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j . Let $\psi = \max_{1 \leq j \leq n} \left\{ \pi_j^* \left(2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$. Assume that $\limsup_{t \rightarrow +\infty} c(t) < +\infty$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \dots, n\}$ and $1 \leq k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \{(\beta_1, \dots, \beta_n) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n\}\}$, $V_\beta := \prod_{1 \leq j \leq n} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \dots, \beta_n) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n,\beta_n})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n,\beta_n}) \in M$. Let f be a holomorphic $(n, 0)$ form on $\cup_{\beta \in \tilde{I}_1} V_\beta$ such that $f = w_\beta^{\alpha\beta*} dw_{1,1} \wedge \dots \wedge dw_{n,1}$ on V_{β^*} , where $\beta^* = (1, \dots, 1) \in \tilde{I}_1$.

We recall that $G(h^{-1}(r))$ is not linear when there exists $j_0 \in \{1, \dots, n\}$ such that $\tilde{m}_{j_0} = +\infty$ as follows.

Theorem 2.11 ([33]). *If $G(0) \in (0, +\infty)$ and there exists $j_0 \in \{1, \dots, n\}$ such that $\tilde{m}_{j_0} = +\infty$, then $G(h^{-1}(r))$ is not linear with respect to $r \in (0, \int_0^{+\infty} c(s) e^{-s} ds]$.*

2.2. Some basic properties of the Green functions.

In this Section, we recall some basic properties of the Green functions. Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω , and let $z_0 \in \Omega$.

Lemma 2.12 (see [47], see also [53]). *Let w be a local coordinate on a neighborhood of z_0 satisfying $w(z_0) = 0$. $G_\Omega(z, z_0) = \sup_{v \in \Delta_\Omega^*(z_0)} v(z)$, where $\Delta_\Omega^*(z_0)$ is the set of negative subharmonic function on Ω such that $v - \log|w|$ has a locally finite upper bound near z_0 . Moreover, $G_\Omega(\cdot, z_0)$ is harmonic on $\Omega \setminus \{z_0\}$ and $G_\Omega(\cdot, z_0) - \log|w|$ is harmonic near z_0 .*

Lemma 2.13 (see [32]). *Let $K = \{z_j : j \in \mathbb{Z}_{\geq 1} \& j < \gamma\}$ be a discrete subset of Ω , where $\gamma \in \mathbb{Z}_{>1} \cup \{+\infty\}$. Let ψ be a negative subharmonic function on Ω such that $\frac{1}{2}v(dd^c\psi, z_j) \geq p_j$ for any j , where $p_j > 0$ is a constant. Then $2 \sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j)$ is a subharmonic function on Ω satisfying that $2 \sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j) \geq \psi$ and $2 \sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j)$ is harmonic on $\Omega \setminus K$.*

Lemma 2.14 (see [28]). *For any open neighborhood U of z_0 , there exists $t > 0$ such that $\{G_\Omega(z, z_0) < -t\}$ is a relatively compact subset of U .*

Lemma 2.15 (see [32]). *There exists a sequence of open Riemann surfaces $\{\Omega_l\}_{l \in \mathbb{Z}^+}$ such that $z_0 \in \Omega_l \Subset \Omega_{l+1} \Subset \Omega$, $\cup_{l \in \mathbb{Z}^+} \Omega_l = \Omega$, Ω_l has a smooth boundary $\partial\Omega_l$ in Ω and $e^{G_{\Omega_l}(\cdot, z_0)}$ can be smoothly extended to a neighborhood of $\overline{\Omega_l}$ for any $l \in \mathbb{Z}^+$, where G_{Ω_l} is the Green function of Ω_l . Moreover, $\{G_{\Omega_l}(\cdot, z_0) - G_\Omega(\cdot, z_0)\}$ is decreasingly convergent to 0 on Ω with respect to l .*

Let Ω_j be an open Riemann surface for any $1 \leq j \leq n$, which admits a nontrivial Green function G_Ω . Let $\{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $1 \leq j \leq n$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. The following lemma will be used in the proof of the applications.

Lemma 2.16 (see [33]). *Let $\psi = \max_{1 \leq j \leq n} \left\{ \pi_j^* \left(2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$ be a plurisubharmonic function on $\prod_{1 \leq j \leq n} \Omega_j$, where $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $j \in \{1, \dots, n\}$. Let $\Psi \leq 0$ be a plurisubharmonic function on $\prod_{1 \leq j \leq n} \Omega_j$, and denote that $\tilde{\psi} := \psi + \Psi$. Let $l(t)$ be a positive Lebesgue measurable function on $(0, +\infty)$ satisfying that $l(t)$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} l(t) dt < +\infty$. If $\Psi \not\equiv 0$ on M , there exists a Lebesgue measurable subset V of $\prod_{1 \leq j \leq n} \Omega_j$ such that $l(-\tilde{\psi}(z)) < l(-\psi(z))$ for any $z \in V$ and $\mu(V) > 0$, where μ is the Lebesgue measure on $\prod_{1 \leq j \leq n} \Omega_j$.*

2.3. Some results related to $\max_{1 \leq j \leq n} \{2p_j \log |w_j|\}$.

In this section, we recall some basic property related to $\max_{1 \leq j \leq n} \{2p_j \log |w_j|\}$. In the following lemma, we recall a closedness of the submodules of $\mathcal{O}_{\mathbb{C}^n, o}^q$.

Lemma 2.17 (see [20]). *Let N be a submodule of $\mathcal{O}_{\mathbb{C}^n, o}^q$, $1 \leq q < +\infty$, let $f_j \in \mathcal{O}_{\mathbb{C}^n}(U)^q$ be a sequence of q -tuples holomorphic in an open neighborhood U of the origin o . Assume that the f_j converge uniformly in U towards a q -tuples $f \in \mathcal{O}_{\mathbb{C}^n}(U)^q$, assume furthermore that all germs (f_j, o) belong to N . Then $(f, o) \in N$.*

Let $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $D = \{w \in \mathbb{C}^n : |w_j| < r_0 \text{ for any } j \in \{1, \dots, n\}\}$, where $r_0 > 0$. Let

$$\psi = \max_{1 \leq j \leq n} \{2p_j \log |w_j|\}$$

be a plurisubharmonic function on \mathbb{C}^n , where $n_1 \leq n$ and $p_j > 0$ is a constant for any $j \in \{1, \dots, n_1\}$. We recall a characterization of $\mathcal{I}(\psi)_o$, where o is the origin in \mathbb{C}^n .

Lemma 2.18 (see [24]). *$(f, o) \in \mathcal{I}(\psi)_o$ if and only if $\sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} > 1$ for any $\alpha \in \mathbb{Z}_{\geq 0}^n$ satisfying $b_\alpha \neq 0$.*

Proof. For the convenience of the reader, we recall the proof.

Let $V = \{w \in \mathbb{C}^n : \max_{n_1+1 \leq j \leq n} \{|w_j|\} < s\}$, where $s \in (0, r_0)$. There exists $r_1 > 0$ such that $\{\psi < \log r_1\} \cap V \Subset D$. If $(f, o) \in \mathcal{I}(\psi)_o$, we have

$$\int_{\{\psi < \log r_1\} \cap V} |f|^2 e^{-\psi} d\lambda_n < +\infty, \quad (2.14)$$

where $d\lambda_n$ is the Lebesgue measure on \mathbb{C}^n . Note that

$$\begin{aligned} & \int_{\{\psi < \log r_1\} \cap V} |f|^2 e^{-\psi} d\lambda_n \\ &= \lim_{\epsilon \rightarrow 0+0} \int_{\left\{ \epsilon < |w_1| < r_1^{\frac{1}{2p_1}} \right\} \cap \dots \cap \left\{ \epsilon < |w_{n_1}| < r_1^{\frac{1}{2p_{n_1}}} \right\} \cap V} |f|^2 e^{-\psi} d\lambda_n \\ &= \lim_{\epsilon \rightarrow 0+0} \left(\sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} \int_{\left\{ \epsilon < |w_1| < r_1^{\frac{1}{2p_1}} \right\} \cap \dots \cap \left\{ \epsilon < |w_{n_1}| < r_1^{\frac{1}{2p_{n_1}}} \right\} \cap V} |b_\alpha w^\alpha|^2 e^{-\psi} d\lambda_n \right) \\ &= \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} |b_\alpha|^2 \int_{\{\psi < \log r_1\} \cap V} |w^\alpha|^2 e^{-\psi} d\lambda_n. \end{aligned}$$

Inequality (2.14) implies that

$$\int_{\{\psi < \log r_1\} \cap V} |w^\alpha|^2 e^{-\psi} d\lambda_n < +\infty \quad (2.15)$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^n$ satisfying $b_\alpha \neq 0$. Note that

$$\begin{aligned} \int_{\{\psi < \log r_1\} \cap V} |w^\alpha|^2 e^{-\psi} d\lambda_n &= \int_{\{\psi < \log r_1\} \cap V} |w^\alpha|^2 \left(\int_0^{+\infty} \mathbb{I}_{\{l < e^{-\psi}\}} dl \right) d\lambda_n \\ &= \int_0^{r_1} \left(\int_{\{\psi < \log r\} \cap V} |w^\alpha|^2 d\lambda_n \right) r^{-2} dr \\ &\quad + \frac{1}{r_1} \int_{\{\psi < \log r_1\} \cap V} |w^\alpha|^2 d\lambda_n \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \int_{\{\psi < \log r\} \cap V} |w^\alpha|^2 d\lambda_n &= \int_{\left\{ |w_1| < r^{\frac{1}{2p_1}} \right\} \cap \dots \cap \left\{ |w_{n_1}| < r^{\frac{1}{2p_{n_1}}} \right\} \cap V} \left| \prod_{1 \leq j \leq n} w_j^{\alpha_j} \right|^2 d\lambda_n \\ &= \pi^{n_1} \frac{r^{\sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_j}}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_V |w_{n_1+1}^{\alpha_{n_1+1}} \dots w_n^{\alpha_n}|^2 d\lambda_{n-n_1}. \end{aligned} \quad (2.17)$$

It follows from inequality (2.15), equality (2.16) and equality (2.17) that

$$\sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} > 1$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^n$ satisfying $b_\alpha \neq 0$.

If $\sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_j} > 1$ for any $\alpha \in \mathbb{Z}_{\geq 0}^n$ satisfying $b_\alpha \neq 0$, it follows from equality (2.16) and equality (2.17) that

$$\int_{\{\psi < \log r_1\}} |w^\alpha|^2 e^{-\psi} d\lambda_n < +\infty,$$

i.e. $(w^\alpha, o) \in \mathcal{I}(\psi)_o$ for any $\alpha \in \mathbb{Z}_{\geq 0}^n$ satisfying $b_\alpha \neq 0$. Using Lemma 2.17, we have $(f, o) \in \mathcal{I}(\psi)_o$. \square

For any $y \in D' = \{y \in \mathbb{C}^{n-n_1} : |y_k| < r_0 \text{ for } 1 \leq k \leq n - n_1\}$, denote that $f_y = f(\cdot, y)$ is a holomorphic function on $D'' = \{x \in \mathbb{C}^{n_1} : |x_j| < r_0 \text{ for any } j \in \{1, \dots, n_1\}\}$. It follows from Lemma 2.18 that the following lemma holds.

Lemma 2.19. $(f, (o_1, y)) \in \mathcal{I}(\psi)_{(o_1, y)}$ for any $y \in D'$ if and only if $(f_y, o_1) \in \mathcal{I}(\psi)_{o_1}$ for any $y \in D'$, where o_1 is the origin in \mathbb{C}^{n_1} .

The following lemma will be used in the proof of Lemma 2.29.

Lemma 2.20. Let $\psi = \max_{1 \leq j \leq n} \{2p_j \log |w_j|\}$ be a plurisubharmonic function on \mathbb{C}^n , where $p_j > 0$. Let $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $\{\psi < -t_0\}$, where $t_0 > 0$. Let $c(t)$ be a nonnegative measurable function on $(t_0, +\infty)$. Denote that $q_\alpha := \sum_{1 \leq j \leq n} \frac{\alpha_j + 1}{p_j} - 1$ for any $\alpha \in \mathbb{Z}_{\geq 0}^n$. Then

$$\int_{\{\psi < -t\}} |f|^2 c(-\psi) d\lambda_n = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} \left(\int_t^{+\infty} c(s) e^{-(q_\alpha + 1)s} ds \right) \frac{(q_\alpha + 1) |b_\alpha|^2 \pi^n}{\prod_{1 \leq j \leq n} (\alpha_j + 1)}$$

holds for any $t \geq t_0$.

Proof. By direct calculations, we obtain that

$$\begin{aligned} & \int_{\{\psi < -t\}} |w^\alpha|^2 c(-\psi) d\lambda_n \\ &= (2\pi)^n \int_{\left\{ \max_{1 \leq j \leq n} \left\{ s_j^{p_j} \right\} < e^{-\frac{t}{2}} \& s_j > 0 \right\}} \prod_{1 \leq j \leq n} s_j^{2\alpha_j + 1} \cdot c \left(-\log \max_{1 \leq j \leq n} \left\{ s_j^{2p_j} \right\} \right) ds_1 ds_2 \dots ds_n \\ &= (2\pi)^n \frac{1}{\prod_{1 \leq j \leq n} p_j} \\ & \quad \times \int_{\left\{ \max_{1 \leq j \leq n} \{r_j\} < e^{-\frac{t}{2}} \& r_j > 0 \right\}} \prod_{1 \leq j \leq n} r_j^{\frac{2\alpha_j + 2}{p_j} - 1} \cdot c \left(-\log \max_{1 \leq j \leq n} \{r_j^2\} \right) dr_1 dr_2 \dots dr_n. \end{aligned} \tag{2.18}$$

By the Fubini's theorem, we have

$$\begin{aligned} & \int_{\left\{ \max_{1 \leq j \leq n} \{r_j\} < e^{-\frac{t}{2}} \& r_j > 0 \right\}} \prod_{1 \leq j \leq n} r_j^{\frac{2\alpha_j + 2}{p_j} - 1} \cdot c \left(-\log \max_{1 \leq j \leq n} \{r_j^2\} \right) dr_1 dr_2 \dots dr_n \\ &= \sum_{j'=1}^n \int_0^{e^{-\frac{t}{2}}} \left(\int_{\{0 \leq r_j < r_{j'}, j \neq j'\}} \prod_{j \neq j'} r_j^{\frac{2\alpha_j + 2}{p_j} - 1} \cdot \wedge_{j \neq j'} dr_j \right) r_{j'}^{\frac{2\alpha_{j'} + 2}{p_{j'}} - 1} c(-2 \log r_{j'}) dr_{j'} \\ &= \sum_{j'=1}^n \left(\prod_{j \neq j'} \frac{p_j}{2\alpha_j + 2} \right) \int_0^{e^{-\frac{t}{2}}} r_{j'}^{\sum_{1 \leq k \leq n} \frac{2\alpha_k + 2}{p_k} - 1} c(-2 \log r_{j'}) dr_{j'} \\ &= (q_\alpha + 1) \left(\int_t^{+\infty} c(s) e^{-(q_\alpha + 1)s} ds \right) \prod_{1 \leq j \leq n} \frac{p_j}{2\alpha_j + 2}. \end{aligned} \tag{2.19}$$

Following from $\int_{\{\psi < -t\}} |f|^2 c(-\psi) d\lambda_n = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} |b_\alpha|^2 \int_{\{\psi < -t\}} |w^\alpha|^2 c(-\psi) d\lambda_n$, equality (2.18) and equality (2.19), we obtain that

$$\int_{\{\psi < -t\}} |f|^2 d\lambda_n = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} \left(\int_t^{+\infty} c(s) e^{-(q_\alpha+1)s} ds \right) \frac{(q_\alpha + 1) |b_\alpha|^2 \pi^n}{\prod_{1 \leq j \leq n} (\alpha_j + 1)}.$$

□

The following lemma will be used in the proof of Proposition 2.37.

Lemma 2.21 (see [33]). *Let $\psi = \max_{1 \leq j \leq n} \{2p_j \log |w_j|\}$ be a plurisubharmonic function on \mathbb{C}^n , where $p_j > 0$. Let $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $\{\psi < -t_0\}$, where $t_0 > 0$. Denote that $q_\alpha := \sum_{1 \leq j \leq n} \frac{\alpha_j + 1}{p_j} - 1$ for any $\alpha \in \mathbb{Z}_{\geq 0}^n$ and $E_1 := \{\alpha \in \mathbb{Z}_{\geq 0}^n : q_\alpha = 0\}$. Then*

$$\begin{aligned} \int_{\{-t-1 < \psi < -t\}} |f|^2 e^{-\psi} d\lambda_n &= \sum_{\alpha \in E_1} \frac{|b_\alpha|^2 \pi^n}{\prod_{1 \leq j \leq n} (\alpha_j + 1)} \\ &\quad + \sum_{\alpha \notin E_1} \frac{|b_\alpha|^2 \pi^n (q_\alpha + 1) (e^{-q_\alpha t} - e^{-q_\alpha(t+1)})}{q_\alpha \prod_{1 \leq j \leq n} (\alpha_j + 1)} \end{aligned}$$

for any $t > t_0$.

2.4. Some results about fibrations.

In this section, we discuss the fibrations.

Let $\Delta^{n_1} = \{w \in \mathbb{C}^{n_1} : |w_j| < 1 \text{ for any } j \in \{1, \dots, n_1\}\}$ be product of the unit disks. Let Y be an n_2 -dimensional complex manifold, and let $M = \Delta^{n_1} \times Y$. Denote $n = n_1 + n_2$. Let π_1 and π_2 be the natural projections from M to Δ^{n_1} and Y respectively. Let ρ_1 be a nonnegative Lebesgue measurable function on Δ^{n_1} satisfying that $\rho_1(w) = \rho_1(|w_1|, \dots, |w_{n_1}|)$ for any $w \in \Delta^{n_1}$ and the Lebesgue measure of $\{w \in \Delta^{n_1} : \rho_1(w) > 0\}$ is positive. Let ρ_2 be a nonnegative Lebesgue measurable function on Y , and denote that $\rho = \pi_1^*(\rho_1) \times \pi_2^*(\rho_2)$ on M .

Lemma 2.22. *For any holomorphic $(n, 0)$ form F on M , there exists a unique sequence of holomorphic $(n_2, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that*

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha), \quad (2.20)$$

where the right term of the above equality is uniformly convergent on any compact subset of M . Moreover, if $\int_M |F|^2 \rho < +\infty$, we have

$$\int_Y |F_\alpha|^2 \rho_2 < +\infty \quad (2.21)$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$.

Proof. Firstly, we consider the local case. Assume that $Y = \Delta^{n_2}$, and the coordinate is $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_{n_2})$. Then there exists a holomorphic function $\tilde{F}(w, \tilde{w})$ on Δ^n such that

$$F = \tilde{F}(w, \tilde{w}) dw_1 \wedge \dots \wedge dw_{n_1} \wedge d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}.$$

Let

$$F_\alpha = \frac{1}{\alpha!} \left(\left(\frac{\partial}{\partial w} \right)^\alpha \tilde{F} \right) \Big|_{w=0} d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}$$

be a holomorphic $(n_2, 0)$ form on Y . Considering the Taylor's expansion of \tilde{F} , we can assume that

$$\tilde{F}(w, \tilde{w}) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}, \tilde{\alpha} \in \mathbb{Z}_{\geq 0}^{n_2}} d_{\alpha, \tilde{\alpha}} w^\alpha \tilde{w}^{\tilde{\alpha}} = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \frac{1}{\alpha!} \left(\left(\frac{\partial}{\partial w} \right)^\alpha \tilde{F} \right) \Big|_{w=0} \cdot w^\alpha,$$

where the summations are uniformly convergent on any compact subset of M , then we have

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha).$$

Secondly, we need to prove that the gluing is independent of the choices of the local coordinates of Y . Assume that $y = (y_1, \dots, y_{n_2})$ is another coordinate on $Y = \Delta^{n_2}$, and $F = \tilde{F}_0(w, y) dw_1 \wedge \dots \wedge dw_{n_1} \wedge dy_1 \wedge \dots \wedge dy_{n_2}$, thus we have $\tilde{F}(w, \tilde{w}(y)) \frac{\partial(\tilde{w}_1, \dots, \tilde{w}_{n_2})}{\partial(y_1, \dots, y_{n_2})} = \tilde{F}_0(w, y)$. By direct calculations, we have

$$\begin{aligned} F_\alpha &= \frac{1}{\alpha!} \left(\left(\frac{\partial}{\partial w} \right)^\alpha \tilde{F} \right) \Big|_{w=0} dw_1 \wedge \dots \wedge dw_{n_2} \\ &= \frac{1}{\alpha!} \left(\left(\frac{\partial}{\partial w} \right)^\alpha \tilde{F} \right) \Big|_{w=0} \frac{\partial(\tilde{w}_1, \dots, \tilde{w}_{n_2})}{\partial(y_1, \dots, y_{n_2})} dy_1 \wedge \dots \wedge dy_{n_2} \\ &= \frac{1}{\alpha!} \left(\left(\frac{\partial}{\partial w} \right)^\alpha \tilde{F}_0 \right) \Big|_{w=0} dy_1 \wedge \dots \wedge dy_{n_2}, \end{aligned}$$

which means that F_α is independent of the choices of the coordinates for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$. For general Y , we can find holomorphic $(n_2, 0)$ forms F_α on Y such that $F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha)$.

Then, for the uniqueness, it suffices to prove the local case $Y = \Delta^{n_2}$. There exists a holomorphic function $\tilde{F}(w, \tilde{w})$ on Δ^n such that $F = \tilde{F}(w, \tilde{w}) dw_1 \wedge \dots \wedge dw_{n_1} \wedge d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}$. If

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha)$$

for a holomorphic $(n_2, 0)$ form F_α on Y , we have

$$F_\alpha = \frac{1}{\alpha!} \left(\left(\frac{\partial}{\partial w} \right)^\alpha \tilde{F} \right) \Big|_{w=0} d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}.$$

Thus, the uniqueness holds.

Finally, we prove inequality (2.21). Let $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} b_\alpha w^\alpha$ be a holomorphic function on Δ^{n_1} . As $\rho(w) = \rho(|w_1|, \dots, |w_{n_1}|)$ for any $w \in \Delta^{n_1}$, we have

$$\begin{aligned} &\int_{\Delta^{n_1}} |f|^2 \rho_1 d\lambda_{n_1} \\ &= \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} (2\pi)^{n_1} |b_\alpha|^2 \int_{\{0 \leq r_1 \leq 1\} \times \dots \times \{0 \leq r_{n_1} \leq 1\}} \left(\prod_{1 \leq j \leq n_1} r_j^{2\alpha_j} \right) \rho_1(r_1, \dots, r_{n_1}) dr_1 \dots dr_{n_1} \\ &= \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} |b_\alpha|^2 \int_{\Delta^{n_1}} |w^\alpha|^2 \rho_1 d\lambda_{n_1}. \end{aligned} \tag{2.22}$$

It follows from equality (2.20), equality (2.22) and the Fubini's theorem that

$$\begin{aligned} \int_M |F|^2 \rho &= \int_{\Delta^{n_1} \times Y} \left| \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha) \right|^2 \pi_1^*(\rho_1) \pi_2^*(\rho_2) \\ &= \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \left(\int_{\Delta^{n_1}} |w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}|^2 \rho_1 \right) \left(\int_Y |F_\alpha|^2 \rho_2 \right). \end{aligned} \quad (2.23)$$

As $\int_M |F|^2 \rho < +\infty$ and the Lebesgue measure of $\{w \in \Delta^{n_1} : \rho_1(w) > 0\}$ is a positive number, equality (2.23) implies that $\int_Y |F_\alpha|^2 \rho_2 < +\infty$ for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$. \square

Let $M_1 \subset M$ be an n -dimensional complex manifold satisfying that $\{o\} \times Y \subset M_1$, where o is the origin in Δ^{n_1} .

Lemma 2.23. *For any holomorphic $(n, 0)$ form F on M_1 , there exist a unique sequence of holomorphic $(n_2, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y and a neighborhood $M_2 \subset M_1$ of $\{o\} \times Y$, such that*

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha)$$

on M_2 , where the right term of the above equality is uniformly convergent on any compact subset of M_2 . Moreover, if $\int_{M_1} |F|^2 \rho < +\infty$, we have

$$\int_K |F_\alpha|^2 \rho_2 < +\infty$$

for any compact subset K of Y and $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$.

Proof. For any open subset V of Y satisfying $V \Subset Y$, there exists $s_V \in (0, 1)$ such that $\Delta_{s_V}^{n_1} \times V \subset M_1$, where $\Delta_{s_V} = \{w \in \mathbb{C} : |w| < s_V\}$. It follows from Lemma 2.22 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{V,\alpha}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on V such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_{V,\alpha})$$

on $\Delta_{s_V}^{n_1} \times V$, where the right term of the above equality is uniformly convergent on any compact subset of $\Delta_{s_V}^{n_1} \times V$. If $\int_{M_1} |F|^2 \rho < +\infty$, Lemma 2.22 shows that

$$\int_V |F_{V,\alpha}|^2 \rho_2 < +\infty.$$

Following from the uniqueness of decomposition in Lemma 2.22, we get that there exists a unique sequence of holomorphic $(n_2, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y and a neighborhood $M_2 \subset M_1$ of $\{o\} \times Y$, such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha) \quad (2.24)$$

on M_2 , where the right term of the above equality is uniformly convergent on any compact subset of M_2 . Moreover, if $\int_{M_1} |F|^2 \rho < +\infty$, we have

$$\int_K |F_\alpha|^2 \rho_2 < +\infty$$

for any compact subset K of Y and $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$. \square

Let $M = X \times Y$ be n -dimensional complex manifold, and let K_M be the canonical (holomorphic) line bundle on M , where X is an n_1 -dimensional weakly pseudoconvex Kähler manifold, Y is an n_2 -dimensional weakly pseudoconvex Kähler manifold, and $n = n_1 + n_2$. Let K_X and K_Y be the canonical (holomorphic) line bundles on X and Y respectively. Let π_X and π_Y be the natural projections from M to X and Y respectively. It is clear that $(M, \emptyset, \emptyset)$ satisfies condition (A).

Let ψ_1 be a plurisubharmonic function on X , and let φ_1 be a Lebesgue measurable function on X such that $\varphi_1 + \psi_1$ is plurisubharmonic. Let φ_2 be a plurisubharmonic function on Y . Denote that $\varphi := \pi_X^*(\varphi_1) + \pi_Y^*(\varphi_2)$ and $\psi := \pi_X^*(\psi_1)$ on M . Let $T = -\sup_M \psi$, and let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$.

Let $Z_0 \subset X$ be a subset of $\{\psi_1 = -\infty\}$ such that $Z_0 \cap \text{Supp}(\mathcal{O}_X/\mathcal{I}(\varphi_1 + \psi_1)) \neq \emptyset$, and let $\tilde{Z}_0 = Z_0 \times Y \subset M$. Let $U \supset Z_0$ be an open subset of X , and let f_1 be a holomorphic $(n_1, 0)$ form on U . Let f_2 be a holomorphic $(n_2, 0)$ form on Y , and let $f = \pi_X^*(f_1) \wedge \pi_Y^*(f_2)$ on $U \times Y$. Let $\mathcal{F}_x \supset \mathcal{I}(\varphi_1 + \psi_1)_x$ be an ideal of $\mathcal{O}_{X,x}$ for any $x \in Z_0$. Let $\tilde{\mathcal{F}}_z \supset \mathcal{I}(\varphi + \psi)_z$ be an ideal of $\mathcal{O}_{M,z}$ for any $z \in \tilde{Z}_0$. For any $x \in Z_0$ and any holomorphic function g , assume that $(g, (x, y)) \in \tilde{\mathcal{F}}_{(x,y)}$ for any $y \in Y$ if and only if $(g(\cdot, y), x) \in \mathcal{F}_x$ for any $y \in Y$.

Denote

$$\inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\varphi_1} c(-\psi_1) : (\tilde{f} - f_1) \in H^0(Z_0, (\mathcal{O}(K_X) \otimes \mathcal{F})|_{Z_0}) \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_X)) \right\}$$

by $G_X(t)$, where $t \in [T, +\infty)$, $|f|^2 := \sqrt{-1}^{n_1^2} f \wedge \bar{f}$ for any $(n_1, 0)$ form f and $(\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ means $(\tilde{f} - f, x) \in \mathcal{O}(K_X)_x \otimes \mathcal{F}_x$ for all $x \in Z_0$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f) \in H^0(\tilde{Z}_0, (\mathcal{O}(K_M) \otimes \tilde{\mathcal{F}})|_{\tilde{Z}_0}) \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\}$$

by $G_M(t)$, where $t \in [T, +\infty)$.

Theorem 2.2 shows that $G_X(h^{-1}(r))$ and $G_M(h^{-1}(r))$ are concave with respect to r , where $h(t) = \int_t^{+\infty} c(s)e^{-s}ds$. The following Proposition gives a property of the minimal L^2 integrals on fibration, which implies that $G_M(h^{-1}(r))$ is linear with respect to r if and only if $G_X(h^{-1}(r))$ is linear with respect to r .

Proposition 2.24. $G_M(t) = G_X(t) \int_Y |f_2|^2 e^{-\varphi_2}$ holds for any $t \geq T$. Moreover, if $G_X(t) < +\infty$, there exists a holomorphic $(n_1, 0)$ form F_1 on $\{\psi_1 < -t\}$ such that $(F_{1,t} - f_1) \in H^0(Z_0, (\mathcal{O}(K_X) \otimes \mathcal{F})|_{Z_0})$, $G_X(t) = \int_{\{\psi_1 < -t\}} |F_{1,t}|^2 e^{-\varphi_1} c(-\psi_1)$ and $G_M(t) = \int_{\{\psi < -t\}} |\pi_X^*(F_{1,t}) \wedge \pi_Y^*(f_2)|^2 e^{-\varphi} c(-\psi)$.

Proof. Let \tilde{f}_1 be a holomorphic $(n_1, 0)$ form on $\{\psi_1 < -t\}$ satisfying $(\tilde{f}_1 - f_1) \in H^0(Z_0, (\mathcal{O}(K_X) \otimes \mathcal{F})|_{Z_0})$, where $t \geq T$. As $f = \pi_X^*(f_1) \wedge \pi_Y^*(f_2)$ and $\tilde{Z}_0 = Z_0 \times Y$, it follows from the relationship between $\tilde{\mathcal{F}}$ and \mathcal{F} that $(\pi_X^*(\tilde{f}_1) \wedge \pi_Y^*(f_2) - f) \in$

$H^0(\tilde{Z}_0, (\mathcal{O}(K_M) \otimes \tilde{\mathcal{F}})|_{\tilde{Z}_0})$. By the definitions of $G_X(t)$ and $G_M(t)$, we obtain that

$$G_M(t) \leq G_X(t) \int_Y |f_2|^2 e^{-\varphi_2} \quad (2.25)$$

for any $t \geq T$.

Let $t \geq T$. If $G_M(t) = +\infty$, inequality (2.25) implies that $G_X(t) \int_Y |f_2|^2 e^{-\varphi_2} = G_M(t) = +\infty$. Thus, assume that $G_M(t) < +\infty$. Lemma 2.5 shows that there exists a holomorphic $(n, 0)$ form F_t on $\{\psi < -t\}$ such that $(F_t - f) \in H^0(\tilde{Z}_0, (\mathcal{O}(K_M) \otimes \tilde{\mathcal{F}})|_{\tilde{Z}_0})$ and $G_M(t) = \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi)$. For any $y_0 \in Y$, let $w = (w_1, \dots, w_{n_2})$ be a coordinate on a neighborhood U of y satisfying $w(y_0) = 0$ and $w(U) = \Delta^{n_2}$. Lemma 2.22 implies that $F_t|_{U \times Y} = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_2}} \pi_X^*(f_\alpha) \wedge \pi_Y^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_2})$, where f_α is a holomorphic $(n_1, 0)$ form on $\{\psi_1 < -t\}$ for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_2}$. There exists a holomorphic function $\tilde{f}_2(w)$ on U such that $f_2 = \tilde{f}_2(w) dw_1 \wedge \dots \wedge dw_{n_2}$ on U . As $(g, (x, y)) \in \tilde{\mathcal{F}}_{(x, y)}$ for any $y \in Y$ if and only if $(h(\cdot, y), x) \in \mathcal{F}_x$ for any $y \in Y$, where $x \in Z_0$ and g is a holomorphic function, it follows from $(F_t - f) \in H^0(\tilde{Z}_0, (\mathcal{O}(K_M) \otimes \tilde{\mathcal{F}})|_{\tilde{Z}_0})$ and $f = \pi_X^*(f_1) \wedge \pi_Y^*(f_2)$ that $(\sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_2}} w^\alpha f_\alpha - \tilde{f}_2(w) f_1) \in H^0(Z_0, (\mathcal{O}(K_X) \otimes \mathcal{F})|_{Z_0})$ for any $w \in \Delta^{n_2}$. Let U_1 be an open subset of U , and let $V = w(U_1) \subset \Delta^{n_2}$. Following the Fubini's theorem and the definition of $G_X(t)$, we have

$$\begin{aligned} & \int_{\{\psi_1 < -t\} \times U_1} |F_t|^2 e^{-\varphi} c(-\psi) \\ &= \int_V \left(\int_{\{\psi_1 < -t\}} \left| \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_2}} w^\alpha f_\alpha \right|^2 e^{-\varphi_1} c(-\psi_1) \right) e^{-\varphi_2} |dw_1 \wedge \dots \wedge dw_{n_2}|^2 \\ &\geq G_X(t) \int_V |\tilde{f}_2(w) dw_1 \wedge \dots \wedge dw_{n_2}|^2 e^{-\varphi_2} \\ &= G_X(t) \int_{U_1} |f_2|^2 e^{-\varphi_2}, \end{aligned}$$

which implies $G_M(t) = \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) \geq G_X(t) \int_Y |f_2|^2 e^{-\varphi_2}$. Thus, we have $G_M(t) = G_X(t) \int_Y |f_2|^2 e^{-\varphi_2}$ for any $t \geq T$. If $G_X(t) < +\infty$, it follows from Lemma 2.5 that there exists a holomorphic $(n_1, 0)$ form $F_{1,t}$ on $\{\psi_1 < -t\}$ satisfying that $(F_{1,t} - f_1) \in H^0(Z_0, (\mathcal{O}(K_X) \otimes \mathcal{F})|_{Z_0})$ and $G_X(t) = \int_{\{\psi < -t\}} |F_{1,t}|^2 e^{-\varphi_1} c(-\psi_1)$, hence $G_M(t) = G_X(t) \int_Y |f_2|^2 e^{-\varphi_2} = \int_{\{\psi < -t\}} |\pi_X^*(F_{1,t}) \wedge \pi_Y^*(f_2)|^2 e^{-\varphi} c(-\psi)$. \square

We recall a result about multiplier ideal sheaves.

Lemma 2.25. *Let Φ_1 and Φ_2 be plurisubharmonic functions on Δ^n satisfying $\Phi_2(o) > -\infty$, where $n \in \mathbb{Z}_{>0}$ and o is the origin in Δ^n . Then $\mathcal{I}(\Phi_1)_o = \mathcal{I}(\Phi_1 + \Phi_2)_o$.*

Proof. For convenience of the reader, we give the proof. It is clear that $\mathcal{I}(\Phi_1 + \Phi_2)_o \subset \mathcal{I}(\Phi_1)_o$. Let f be a holomorphic function on a neighborhood of o satisfying $(f, o) \in \mathcal{I}(\Phi_1)_o$. Following from the strong openness property of multiplier ideal sheaves ([36]) and $\Phi_2(o) > -\infty$, there exist $s > 1$ and $r > 0$ such that

$$\int_{|w| < r} |f|^{2s} e^{-s\Phi_1} d\lambda_n < +\infty \quad (2.26)$$

and

$$\int_{|w|< r} e^{-\frac{s}{s-1}\Phi_2} d\lambda_n < +\infty, \quad (2.27)$$

where $d\lambda_n$ is the Lebesgue measure on \mathbb{C}^n . Combining inequality (2.26), inequality (2.27) and the Hölder inequality, we have

$$\begin{aligned} & \int_{|w|< r} |f|^2 e^{-\Phi_1-\Phi_2} d\lambda_n \\ & \leq \left(\int_{|w|< r} |f|^{2s} e^{-s\Phi_1} d\lambda_n \right)^{\frac{1}{s}} \left(\int_{|w|< r} e^{-\frac{s}{s-1}\Phi_2} d\lambda_n \right)^{1-\frac{1}{s}} \\ & < +\infty, \end{aligned}$$

which implies that $(f, o) \in \mathcal{I}(\Phi_1 + \Phi_2)_o$. Thus, we have $\mathcal{I}(\Phi_1)_o = \mathcal{I}(\Phi_1 + \Phi_2)_o$. \square

In the following, we consider fibrations over products of open Riemann surfaces. Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leq j \leq n_1$. Let Y be an n_2 -dimensional weakly pseudoconvex Kähler manifold. Let $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$ be an n -dimensional complex manifold, where $n = n_1 + n_2$. Let $\pi_1, \pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1 \leq j \leq n_1} \Omega_j, \Omega_j$ and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M .

Let $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \dots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. Denote that $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y \subset M$. Let $p_{j,k}$ be a positive number such that $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j , and let

$$\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on M . For any $j \in \{1, \dots, n_1\}$, let φ_j be a subharmonic function on Ω_j such that $\varphi_j(z) > -\infty$ for any $z \in Z_j$. Let φ_Y be a plurisubharmonic function on Y , and denote that $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$. Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t) e^{-t} dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Let f be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\}$ satisfying $\int_{\{\psi < -t_0\}} |f|^2 e^{-\varphi} c(-\psi) < +\infty$, where $t_0 > 0$ is constant.

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\}$$

by $G(t)$, where $t \in [0, +\infty)$, and denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\}$$

by $\tilde{G}(t)$, where $t \in [0, +\infty)$.

Lemma 2.26. *Let $t \geq 0$. If $\tilde{G}(t) < +\infty$, there exists a unique holomorphic $(n, 0)$ form F_t on $\{\psi < -t\}$ satisfying that $(F_t - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and $G(t) = \tilde{G}(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$.*

Proof. As $\mathcal{I}(\varphi + \psi) \subset \mathcal{I}(\psi)$, we have $\tilde{G}(t) \leq G(t)$. It follows from Lemma 2.5 that there exists a unique holomorphic $(n, 0)$ form F_t on $\{\psi < -t\}$ satisfying that $\tilde{G}(t) = \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi)$ and $(F_t - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$.

Let $z_0 = (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$, where $1 \leq \beta_j < \tilde{m}_j$ for any $1 \leq j \leq n_1$. It follows from Lemma 2.12 and Lemma 2.13 that there exists a local coordinate w_j on a neighborhood $V_{z_{j,\beta_j}} \Subset \Omega_j$ of $z_{j,\beta_j} \in \Omega_j$ satisfying $w_j(z_{j,\beta_j}) = 0$ and

$$\log |w_j| = \frac{1}{p_{j,\beta_j}} \sum_{1 \leq k \leq \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})$$

on $V_{z_{j,\beta_j}}$ for any $j \in \{1, \dots, n_1\}$. Denote that $V_0 := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ and $w := (w_1, \dots, w_{n_1})$ on V_0 . Thus, there exists $t_1 > \max\{t, t_0\}$ such that

$$\left\{ z \in \Omega_j : 2 \sum_{1 \leq k \leq \tilde{m}_j} p_{j,k} G_{\Omega_j}(z, z_{j,k}) < -t_1 \right\} \cap V_{z_{j,\beta_j}} \Subset V_{z_{j,\beta_j}}$$

for any $1 \leq j \leq n_1$. Let $\psi_1 = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left(2 \sum_{1 \leq k \leq \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$ on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j . Note that

$$\{\psi < -t_1\} = \{\psi_1 < -t_1\} \times Y$$

and

$$\{\psi_1 < -t_1\} \cap V_0 = \prod_{1 \leq j \leq n_1} \left\{ |w_j| < e^{-\frac{t_1}{2p_{j,\beta_j}}} \right\}.$$

As φ_j is a subharmonic function on Ω_j , $\int_{\{\psi < -t_1\}} |f|^2 e^{-\varphi} c(-\psi) \leq \int_{\{\psi < -t_0\}} |f|^2 e^{-\varphi} c(-\psi) < +\infty$ implies that $\int_{\{\psi < -t_1\}} |f|^2 e^{-\pi_2^*(\varphi_Y)} c(-\psi) < +\infty$ and $\int_{\{\psi < -t_1\}} |F_t|^2 e^{-\varphi} c(-\psi) < +\infty$ implies that $\int_{\{\psi < -t_1\}} |F_t|^2 e^{-\pi_2^*(\varphi_Y)} c(-\psi) < +\infty$. It follows from Lemma 2.22 that there exist a sequence of holomorphic $(n_2, 0)$ forms $\{f_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y and a sequence of holomorphic $(n_2, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that

$$f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) \quad (2.28)$$

on $(\{\psi_1 < -t_1\} \cap V_0) \times Y$,

$$F_t = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha) \quad (2.29)$$

on $(\{\psi_1 < -t_1\} \cap V_0) \times Y$,

$$\int_Y |f_\alpha|^2 e^{-\varphi_Y} < +\infty \quad (2.30)$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$ and

$$\int_Y |F_\alpha|^2 e^{-\varphi_Y} < +\infty \quad (2.31)$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$, where the right terms of the equalities (2.28) and (2.29) are uniformly convergent on any compact subset of $(\{\psi_1 < -t_1\} \cap V_0) \times Y$. As $(F_t - f, (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_0, y)}$ for any $y \in Y$, it follows from Lemma 2.18 that

$$f_\alpha = F_\alpha$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$ satisfying $\sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j, \beta_j} \leq 1$. Denote that

$$R := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j, \beta_j} > 1 \right\}.$$

Lemma 2.18 shows that $(w^\alpha, z_0) \in \mathcal{I}(\psi_1)_{z_0}$ for any $\alpha \in R$. It follows from inequality (2.30) and inequality (2.31) that $(\pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha), (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi + \pi_2^*(\varphi_Y)))_{(z_0, y)}$ and $(\pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha), (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi + \pi_2^*(\varphi_Y)))_{(z_0, y)}$ for any $y \in Y$ and any $\alpha \in R$. As $\varphi_j(z_{j, \beta_j}) > -\infty$, using Lemma 2.25, we obtain that

$$(\pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha), (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_0, y)}$$

and

$$(\pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha), (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_0, y)}$$

for any $y \in Y$ and any $\alpha \in R$. It follows from equality (2.28), equality (2.29) and Lemma 2.17 that

$$\begin{aligned} (f - F_t, (z_0, y)) &= \left(\sum_{\alpha \in R} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha - F_\alpha), (z_0, y) \right) \\ &\in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_0, y)} \end{aligned}$$

holds for any $y \in Y$. Hence we have $(F_t - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$, which implies that $G(t) \leq \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) = \tilde{G}(t)$. Thus, we obtain that $G(t) = \tilde{G}(t) = \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi)$. \square

The following two lemmas will be used in the proof of Lemma 2.29.

Lemma 2.27 (see [33]). *Let $c(t)$ be a positive measurable function on $(0, +\infty)$, and let $a \in \mathbb{R}$. Assume that $\int_t^{+\infty} c(s)e^{-s} ds \in (0, +\infty)$ when t near $+\infty$. Then we have*

- (1) $\lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as} ds}{\int_t^{+\infty} c(s)e^{-s} ds} = 1$ if and only if $a = 1$;
- (2) $\lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as} ds}{\int_t^{+\infty} c(s)e^{-s} ds} = 0$ if and only if $a > 1$;
- (3) $\lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as} ds}{\int_t^{+\infty} c(s)e^{-s} ds} = +\infty$ if and only if $a < 1$.

Proof. For the convenience of the reader, we recall the proof.

If $a = 1$, it clear that $\lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as} ds}{\int_t^{+\infty} c(s)e^{-s} ds} = 1$.

If $a > 1$, then $c(s)e^{-as} \leq e^{(1-a)s_0} c(s)e^{-s}$ for $s \geq s_0 > 0$, which implies that $\limsup_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as} ds}{\int_t^{+\infty} c(s)e^{-s} ds} \leq e^{(1-a)s_0}$. Let $s_0 \rightarrow +\infty$, we have $\lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as} ds}{\int_t^{+\infty} c(s)e^{-s} ds} = 0$

If $a < 1$, then $c(s)e^{-as} \geq e^{(1-a)s_0}c(s)e^{-s}$ for $a > s_0 > 0$, which implies that $\liminf_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} \geq e^{(1-a)s_0}$. Let $s_0 \rightarrow +\infty$, we have $\lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = +\infty$. \square

The following Lemma belongs to Fornaess and Narasimhan on approximation property of plurisubharmonic functions of Stein manifolds.

Lemma 2.28 ([16]). *Let X be a Stein manifold and $\varphi \in PSH(X)$. Then there exists a sequence $\{\varphi_n\}_{n=1,\dots}$ of smooth strongly plurisubharmonic functions such that $\varphi_n \downarrow \varphi$.*

It follows from Lemma 2.12 and Lemma 2.13 that there exists a local coordinate $w_{j,k}$ on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ and

$$\log |w_{j,k}| = \frac{1}{p_{j,k}} \sum_{1 \leq k \leq \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})$$

for any $j \in \{1, \dots, n_1\}$ and $1 \leq k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$, $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \dots, \beta_n) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n,\beta_n})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n,\beta_n}) \in M$. Let

$$\psi_1 = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left(2 \sum_{1 \leq k \leq \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j . Note that $\psi = \pi_1^*(\psi_1)$.

Let F be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\} \subset M$ for some $t_0 > 0$ satisfying $\int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$. Without loss of generality, we can assume $\cup_{\beta \in \tilde{I}_1} V_\beta \times Y \subset \{\psi < -t_0\}$. For any $\beta \in \tilde{I}_1$, it follows from Lemma 2.22 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on $V_\beta \times Y$ and

$$\int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$. Denote that $E_\beta := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \right\}$, $E_{1,\beta} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} < 1 \right\}$ and $E_{2,\beta} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} > 1 \right\}$.

Lemma 2.29. *If $\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} < +\infty$, we have $F_{\alpha,\beta} \equiv 0$ for any $\alpha \in E_{1,\beta}$ and $\beta \in \tilde{I}_1$, and*

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Proof. As $\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$ is a plurisubharmonic function on $\prod_{1 \leq j \leq n_1} \Omega_j$, it follows from Lemma 2.28 that there exists a sequence of smooth plurisubharmonic functions $\{\Phi_l\}_{l \in \mathbb{Z}_{\geq 0}}$ on $\prod_{1 \leq j \leq n_1} \Omega_j$, which is decreasingly convergent to $\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$.

Let $\beta \in \tilde{I}_1$ and $l \in \mathbb{Z}_{\geq 0}$. For any $\epsilon > 0$, there exists $t_\beta > t_0$ such that $\{\psi_1 < -t_\beta\} \cap V_\beta \Subset V_\beta$ and

$$\sup_{z \in \{\psi_1 < -t_\beta\} \cap V_\beta} |\Phi_l(z) - \Phi(z_\beta)| < \epsilon.$$

For any $t \geq t_\beta$, note that $\{\psi_1 < -t\} = \prod_{1 \leq j \leq n_1} \left\{ |w_{j,\beta_j}| < e^{-\frac{t}{2p_{j,\beta_j}}} \right\}$ and $F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$ on $\{\psi_1 < -t\} \times Y$, then we have

$$\begin{aligned} & \int_{\{\psi_1 < -t\} \cap (V_\beta \times Y)} |F|^2 e^{-\varphi} c(-\psi) \\ & \geq \int_{\{\psi_1 < -t\} \cap (V_\beta \times Y)} |F|^2 e^{-\pi_1^*(\Phi_l) - \pi_2^*(\varphi_Y)} c(-\psi) \\ & \geq e^{-\Phi_l(z_\beta) - \epsilon} \int_{(\{\psi_1 < -t\} \cap V_\beta) \times Y} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\pi_1^*(\psi_1)) \\ & = e^{-\Phi_l(z_\beta) - \epsilon} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \int_{\{\psi_1 < -t\}} |w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}|^2 c(-\psi) \times \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned} \tag{2.32}$$

Denote that $q_\alpha := \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} - 1$. It follows from Lemma 2.20 and inequality (2.32) that

$$\begin{aligned} & \int_{\{\psi_1 < -t\} \cap (V_\beta \times Y)} |F|^2 e^{-\varphi} c(-\psi) \\ & \geq e^{-\Phi_l(z_\beta) - \epsilon} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \left(\int_t^{+\infty} c(s) e^{-(q_\alpha + 1)s} ds \right) \frac{(q_\alpha + 1)(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned}$$

It follows from $\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi_1 < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty$ and Lemma 2.27 that

$$F_{\alpha,\beta} \equiv 0$$

for any α satisfying $q_\alpha < 0$ and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi_1 < -t\} \cap (V_\beta \times Y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq e^{-\Phi_l(z_\beta) - \epsilon} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)}.$$

Letting $\epsilon \rightarrow 0$ and $l \rightarrow +\infty$, we have

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi_1 < -t\} \cap (V_\beta \times Y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned} \tag{2.33}$$

Note that $V_\beta \cap V_{\tilde{\beta}} = \emptyset$ for any $\beta \neq \tilde{\beta}$ and $\{\psi_1 < -t_\beta\} \cap V_\beta \Subset V_\beta$ for any $\beta \in \tilde{I}_1$. It follows from inequality (2.33) that

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Thus, Lemma 2.29 holds. \square

Let M_1 be an open complex submanifold of M satisfying that $Z_0 = \{z_\beta : \beta \in \tilde{I}_1\} \times Y \subset M_1$, and let K_{M_1} be the canonical (holomorphic) line bundle on M_1 . Let F_1 be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\} \cap M_1$ for $t_0 > 0$ satisfying that $\int_{\{\psi < -t_0\} \cap M_1} |F_1|^2 e^{-\varphi} c(-\psi) < +\infty$. For any $\beta \in \tilde{I}_1$, it follows from Lemma 2.23 that there exist a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y and an open subset U_β of $\{\psi < -t_0\} \cap M_1 \cap (V_\beta \times Y)$ such that

$$F_1 = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on U_β and

$$\int_K |F_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$ and compact subset K of Y .

Lemma 2.30. *If $\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap M_1} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty$, we have $F_{\alpha,\beta} \equiv 0$ for any $\alpha \in E_{1,\beta}$ and $\beta \in \tilde{I}_1$ and*

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap M_1} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned}$$

Proof. Note that $\psi_1 = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left(2 \sum_{1 \leq k < \tilde{m}_j} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$ on $\prod_{1 \leq j \leq n_1} \Omega_j$. For any $\beta \in \tilde{I}_1$ and any open subset V of Y satisfying $V \Subset Y$, it follows from Lemma 2.12 and Lemma 2.13 that there exists $t_{\beta,V} > t_0$ such that $\{\psi_1 < -t_{\beta,V}\} \times V \Subset U_\beta$. $\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap M_1} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty$ implies that

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi_1 < -t\} \times V} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty. \quad (2.34)$$

It follows from equality (2.34) and Lemma 2.29 that $F_{\alpha,\beta} \equiv 0$ on V for any $\alpha \in E_{1,\beta}$ and

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap U_\beta} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi_1 < -t\} \times V} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j, \beta_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_V |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned}$$

Following from the arbitrariness of V , we have

$$F_{\alpha,\beta} \equiv 0$$

on Y for any $\alpha \in E_{1,\beta}$ and

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap U_\beta} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j, \beta_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned} \tag{2.35}$$

$V_\beta \cap V_{\beta'} = \emptyset$ for any $\beta \neq \beta'$ implies that $U_\beta \cap U_{\beta'} = \emptyset$ for any $\beta \neq \beta'$. It follows from inequality (2.35) that

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap M_1} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j, \beta_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned}$$

Thus, Lemma 2.30 holds. \square

In the following, we consider the case that Z_j is a single point set. Let $M' = \prod_{1 \leq j \leq n_1} \Omega_j$ be an n_1 -dimensional complex manifold, and let $K_{M'}$ be the canonical (holomorphic) line bundle on M' . Let $z_j \in \Omega_j$ and $z_0 = (z_1, \dots, z_{n_1}) \in M'$. Let φ_j be subharmonic functions on Ω_j such that $\varphi_j(z_j) > -\infty$. Denote that

$$\psi_1 := \max_{1 \leq j \leq n_1} \{2p_j \tilde{\pi}_j^*(G_{\Omega_j}(\cdot, z_j))\}$$

and $\tilde{\varphi} := \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$ on M' , where p_j is a positive real number for $1 \leq j \leq n_1$ and $\tilde{\pi}_j$ is the natural projection from M' to Ω_j .

Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1 \leq j \leq n_1} V_{z_j}$, and $w := (w_1, \dots, w_{n_1})$ is a local coordinate on V_0 of $z_0 \in M'$. Take $E = \left\{(\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0}\right\}$.

Let $c_j(z)$ be the logarithmic capacity (see [47]) on Ω_j , which is locally defined by

$$c_j(z_j) := \exp \lim_{z \rightarrow z_j} (G_{\Omega_j}(z, z_j) - \log |w_j(z)|).$$

Lemma 2.31 (see [27]). *Let $c(t)$ be a positive function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. For any $\alpha \in E$, there exists a holomorphic $(n_1, 0)$ form F on M' , which satisfies that $(F - w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in (\mathcal{O}(K_{\Omega_j}) \otimes \mathcal{I}(\psi_1))_{z_0}$ and*

$$\int_{M'} |F|^2 e^{-\tilde{\varphi}} c(-\psi_1) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \frac{(2\pi)^{n_1} e^{-\tilde{\varphi}(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}.$$

As φ_j is subharmonic on Ω_j , it follows from Lemma 2.31 and Lemma 2.5 that there exists a holomorphic $(1, 0)$ form f_{j,α_j} on Ω_j such that $(f_{j,\alpha_j} - w_j^{\alpha_j} dw_j, z_j) \in (\mathcal{O}(K_{\Omega_j}) \otimes \mathcal{I}(2(\alpha_j + 1)G_{\Omega_j}(\cdot, z_j)))_{z_j}$ and $\int_{\Omega_j} |f_{j,\alpha_j}|^2 e^{-\varphi_j} = \inf \left\{ \int_{\Omega_j} |\tilde{f}|^2 e^{-\varphi_j} : \tilde{f} \in H^0(\Omega_j, \mathcal{O}(K_{\Omega_j})) \text{ & } (\tilde{f} - w_j^{\alpha_j} dw_j, z_j) \in (\mathcal{O}(K_{\Omega_j}) \otimes \mathcal{I}(2(\alpha_j + 1)G_{\Omega_j}(\cdot, z_j)))_{z_j} \right\} < +\infty$ for any $\alpha \in E$ and $j \in \{1, \dots, n_1\}$.

Lemma 2.32 (see [33]). *$F = \sum_{\alpha \in E} d_\alpha \prod_{1 \leq j \leq n_1} \tilde{\pi}_j^*(f_{j,\alpha_j})$ is a holomorphic $(n_1, 0)$ form on M' satisfying that $(F - \sum_{\alpha \in E} d_\alpha w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in \mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}$,*

$$\int_{M'} |F|^2 e^{-\tilde{\varphi}} = \sum_{\alpha \in E} |d_\alpha|^2 \int_{M'} \left| \prod_{1 \leq j \leq n_1} \pi_j^*(f_{j,\alpha_j}) \right|^2 e^{-\tilde{\varphi}}$$

and $\int_{M'} |F|^2 e^{-\tilde{\varphi}} = \inf \left\{ \int_{M'} |\tilde{F}|^2 e^{-\tilde{\varphi}} : \tilde{F} \text{ is a holomorphic } (n_1, 0) \text{ form on } M' \text{ such that } (F - \sum_{\alpha \in E} d_\alpha w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in \mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0} \right\}$, where d_α is a constant for any $\alpha \in E$.

Let φ_Y be a plurisubharmonic function on Y . Let f_α be a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |f_\alpha|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in E$. Let $f = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha)$ be a holomorphic $(n, 0)$ form on $V_0 \times Y \subset M = M' \times Y$. Denote that $\varphi := \pi_1^*(\tilde{\varphi}) + \pi_2^*(\varphi_Y)$ and $\psi := \pi_1^*(\psi_1)$ on M .

Lemma 2.33. *$F = \sum_{\alpha \in E} \pi_{1,1}^*(f_{1,\alpha_1}) \wedge \dots \wedge \pi_{1,n_1}^*(f_{n_1,\alpha_{n_1}}) \wedge \pi_2^*(f_\alpha)$ is a holomorphic $(n, 0)$ form on M , and satisfies that $(F - f, (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_0, y)}$ for any $y \in Y$,*

$$\int_M |F|^2 e^{-\varphi} = \sum_{\alpha \in E} \left(\int_Y |f_\alpha|^2 e^{-\varphi_Y} \right) \prod_{1 \leq j \leq n_1} \int_{\Omega_j} |f_{j,\alpha_j}|^2 e^{-\varphi_j}$$

and $\int_M |F|^2 e^{-\varphi} = \inf \left\{ \int_M |\tilde{F}|^2 e^{-\varphi} : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } M \text{ such that } (\tilde{F} - f, (z_0, y)) \in \mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_0, y)} \text{ for any } y \in Y \right\}$.

Proof. It follows from Lemma 2.19 that $(f, (z_0, y)) \in \mathcal{I}(\psi)_{(z_0, y)}$ for any $y \in Y$ if and only if $(f(\cdot, y), z_0) \in \mathcal{I}(\psi_1)_{z_0}$. For any $\alpha \in E$, using Proposition 2.24 and Lemma 2.32, we obtain that $F_\alpha = \pi_{1,1}^*(f_{1,\alpha_1}) \wedge \dots \wedge \pi_{1,n_1}^*(f_{n_1,\alpha_{n_1}}) \wedge \pi_2^*(f_\alpha)$ satisfies that $\int_M |F_\alpha|^2 e^{-\varphi} = (\int_Y |f_\alpha|^2 e^{-\varphi_Y}) \prod_{1 \leq j \leq n_1} \int_{\Omega_j} |f_{j,\alpha_j}|^2 e^{-\varphi_j} = \inf \left\{ \int_M |\tilde{F}|^2 e^{-\varphi} : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } M \text{ such that } (\tilde{F} - \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha), (z_0, y)) \in \mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_0, y)} \text{ for any } y \in Y \right\}$, i.e.

$$\int_M F_\alpha \wedge \overline{\tilde{F}} e^{-\varphi} = 0 \tag{2.36}$$

for any holomorphic $(n, 0)$ form \tilde{F} satisfying $\int_M |\tilde{F}|^2 e^{-\varphi} < +\infty$ and $(\tilde{F}, (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_0, y)}$ for any $y \in Y$. It follows from the Fubini's theorem and Lemma 2.32 that

$$\int_M F_\alpha \wedge \overline{F_{\bar{\alpha}}} e^{-\varphi} = 0 \tag{2.37}$$

for any $\alpha \neq \tilde{\alpha}$. Note that $F = \sum_{\alpha \in E} F_\alpha$ and $(F - f, (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_0, y)}$ for any $y \in Y$. It follows from equality (2.36) and equality (2.37) that

$$\int_M |F|^2 e^{-\varphi} = \sum_{\alpha \in E} \left(\int_Y |f_\alpha|^2 e^{-\varphi_Y} \right) \prod_{1 \leq j \leq n_1} \int_{\Omega_j} |f_{j, \alpha_j}|^2 e^{-\varphi_j}$$

and $\int_M |F|^2 e^{-\varphi} = \inf \left\{ \int_M |\tilde{F}|^2 e^{-\varphi} : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } M \text{ such that } (\tilde{F} - f, (z_0, y)) \in \mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_0, y)} \text{ for any } y \in Y \right\}$. \square

Let X be an n_1 -dimensional complex manifold, and let Y be an n_2 -dimensional complex manifold. Let $M = X \times Y$ be an n -dimensional complex manifold, where $n = n_1 + n_2$. Let π_1 and π_2 be the natural projections from M to X and Y respectively. We recall the following lemma.

Lemma 2.34 (see [1]). *Let $F \not\equiv 0$ be a holomorphic $(n, 0)$ form on M . Let f_1 be a holomorphic $(n_1, 0)$ form on an open subset U of X , and let f_2 be a holomorphic $(n_2, 0)$ form on an open subset V of Y . If*

$$F = \pi_1^*(f_1) \wedge \pi_2^*(f_2)$$

on $U \times V$, there exist a holomorphic $(n_1, 0)$ form F_1 on X and a holomorphic $(n_2, 0)$ form F_2 on Y such that $F_1 = f_1$ on U , $F_2 = f_2$ on V , and

$$F = \pi_1^*(F_1) \wedge \pi_2^*(F_2)$$

on M .

2.5. Optimal jets L^2 extension.

In this section, we give an optimal jets L^2 extension result, i.e. Proposition 2.37. We recall two lemmas, which will be used in the proof of Proposition 2.37.

Lemma 2.35 ([27]). *Let c be a positive function on $(0, +\infty)$, such that $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Let $B \in (0, +\infty)$ and $t_0 \geq 0$ be arbitrarily given. Let M be an n -dimensional weakly pseudoconvex Kähler manifold. Let $\psi < 0$ be a plurisubharmonic function on M . Let φ be a plurisubharmonic function on M . Let F be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\}$, such that*

$$\int_{K \cap \{\psi < -t_0\}} |F|^2 < +\infty \quad (2.38)$$

for any compact subset K of M , and

$$\int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B < \psi < -t_0\}} |F|^2 e^{-\varphi} \leq C < +\infty. \quad (2.39)$$

Then there exists a holomorphic $(n, 0)$ form \tilde{F} on M , such that

$$\int_M |\tilde{F} - (1 - b_{t_0, B}(\psi))F|^2 e^{-\varphi + v_{t_0, B}(\psi)} c(-v_{t_0, B}(\psi)) \leq C \int_0^{t_0+B} c(t)e^{-t} dt \quad (2.40)$$

where $b_{t_0, B}(t) = \int_{-\infty}^t \frac{1}{B} \mathbb{I}_{\{-t_0-B < s < -t_0\}} ds$ and $v_{t_0, B}(t) = \int_{-t_0}^t b_{t_0, B}(s) ds - t_0$.

It is clear that $\mathbb{I}_{(-t_0, +\infty)} \leq b_{t_0, B}(t) \leq \mathbb{I}_{(-t_0-B, +\infty)}$ and $\max\{t, -t_0 - B\} \leq v_{t_0, B}(t) \leq \max\{t, -t_0\}$.

Lemma 2.36 (see [28]). *Let M be a complex manifold. Let S be an analytic subset of M . Let $\{g_j\}_{j \in \mathbb{Z}_{\geq 1}}$ be a sequence of nonnegative Lebesgue measurable functions on M , which satisfies that g_j are almost everywhere convergent to g on M when $j \rightarrow +\infty$, where g is a nonnegative Lebesgue measurable function on M . Assume that for any compact subset K of $M \setminus S$, there exist $s_K \in (0, +\infty)$ and $C_K \in (0, +\infty)$ such that*

$$\int_K g_j^{-s_K} dV_M \leq C_K$$

for any j , where dV_M is a continuous volume form on M .

Let $\{F_j\}_{j \in \mathbb{Z}_{\geq 1}}$ be a sequence of holomorphic $(n, 0)$ form on M . Assume that $\liminf_{j \rightarrow +\infty} \int_M |F_j|^2 g_j \leq C$, where C is a positive constant. Then there exists a subsequence $\{F_{j_l}\}_{l \in \mathbb{Z}_{\geq 1}}$, which satisfies that $\{F_{j_l}\}$ is uniformly convergent to a holomorphic $(n, 0)$ form F on M on any compact subset of M when $l \rightarrow +\infty$, such that

$$\int_M |F|^2 g \leq C.$$

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leq j \leq n_1$. Let Y be an n_2 -dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y . Let $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$ be an n -dimensional complex manifold, where $n = n_1 + n_2$, and let K_M be the canonical (holomorphic) line bundle on M . Let $\pi_1, \pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1 \leq j \leq n_1} \Omega_j$, Ω_j and Y respectively. Let $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \dots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. Denote that $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y$.

Let φ_X be a plurisubharmonic function on $\prod_{1 \leq j \leq n_1} \Omega_j$ satisfying that $\varphi_X(z) > -\infty$ for any $z \in \prod_{1 \leq j \leq n_1} Z_j$, and let φ_Y be a plurisubharmonic function on Y . Let $p_{j,k}$ be a positive number for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$, which satisfies that $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \neq -\infty$ for any $1 \leq j \leq n_1$. Denote that

$$\psi := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$$

and

$$\varphi := \pi_1^*(\varphi_X) + \pi_2^*(\varphi_Y)$$

on M .

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$, $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$ for any $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$. Denote that $E_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$ and $\tilde{E}_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \geq 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$ for any $\beta \in \tilde{I}_1$. Let f be a holomorphic $(n, 0)$ form on a neighborhood U_0 of Z_0 such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in \tilde{E}_\beta$ and $\beta \in \tilde{I}_1$. Denote that

$$c_{j,k} := \exp \lim_{z \rightarrow z_{j,k}} \left(\frac{\sum_{1 \leq k_1 < \tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \dots, n\}$ and $1 \leq k < \tilde{m}_j$ (following from Lemma 2.12 and Lemma 2.13, we get that the above limit exists).

Proposition 2.37. *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that*

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} < +\infty.$$

Then there exists a holomorphic $(n, 0)$ form F on M satisfying that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}}.$$

Proof. The following Remark shows that it suffices to prove Proposition 2.37 for the case $\tilde{m}_j < +\infty$ for any $j \in \{1, \dots, n_1\}$.

Remark 2.38. *Assume that Proposition 2.37 holds for the case $\tilde{m}_j < +\infty$ for any $j \in \{1, \dots, n_1\}$. For any $j \in \{1, \dots, n_1\}$, it follows from Lemma 2.15 that there exists a sequence of Riemann surfaces $\{\Omega_{j,l}\}_{l \in \mathbb{Z}_{\geq 1}}$, which satisfies that $\Omega_{j,l} \Subset \Omega_{j,l+1} \Subset \Omega_j$ for any l , $\cup_{l \in \mathbb{Z}_{\geq 1}} \Omega_{j,l} = \Omega_j$ and $\{G_{\Omega_{j,l}}(\cdot, z) - G_{\Omega_j}(\cdot, z)\}_{l \in \mathbb{Z}_{\geq 1}}$ is decreasingly convergent to 0 with respect to l for any $z \in \Omega_j$. As Z_j is a discrete subset of Ω_j , $Z_{j,l} := \Omega_{j,l} \cap Z_j$ is a set of finite points. Denote that $M_l := \left(\prod_{1 \leq j \leq n_1} \Omega_{j,l} \right) \times Y$ and $\psi_l := \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left(\sum_{z_{j,k} \in Z_{j,l}} 2p_{j,k} G_{\Omega_{j,l}}(\cdot, z_{j,k}) \right) \right\}$ on M_l . Denote that*

$$c_{j,k,l} = \exp \lim_{z \rightarrow z_{j,k}} \left(\frac{\sum_{z_{j,k_1} \in Z_{j,l}} p_{j,k_1} G_{\Omega_{j,l}}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $1 \leq j \leq n_1$, $l \in \mathbb{Z}_{\geq 1}$ and $1 \leq k < \tilde{m}_j$ satisfying $z_{j,k} \in Z_{j,l}$. Hence $c_{j,k,l}$ is decreasingly convergent to $c_{j,k}$ with respect to l , ψ_l is decreasingly convergent to ψ with respect to l and $\cup_{l \in \mathbb{Z}_{\geq 1}} M_l = M$.

Then there exists a holomorphic $(n, 0)$ form F_l on M_l such that $(F_l - f, (z_\beta, y)) \in (\mathcal{O}(K_{M_l}) \otimes \mathcal{I}(\psi_l))_{(z_\beta, y)} = (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_\beta, y)}$ for any $\beta \in \{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1 \leq j \leq n_1} \Omega_{j,l}\}$ and $y \in Y$, and F_l satisfies

$$\begin{aligned} & \int_{M_l} |F_l|^2 e^{-\varphi} c(-\psi_l) \\ & \leq \left(\int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{\beta \in \{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1 \leq j \leq n_1} \Omega_{j,l}\}} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j,l}^{2\alpha_j+2}} \\ & \leq \left(\int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \\ & < +\infty. \end{aligned}$$

Since $\psi \leq \psi_l$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, we have

$$\begin{aligned} & \int_{M_l} |F_l|^2 e^{-\varphi - \psi_l + \psi} c(-\psi) \\ & \leq \int_{M_l} |F_l|^2 e^{-\varphi} c(-\psi_l) \\ & \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned} \quad (2.41)$$

Note that ψ is continuous on $M \setminus Z_0$, ψ_l is continuous on $M_l \setminus Z_0$ and Z_0 is a closed complex submanifold of M . For any compact subset K of $M \setminus Z_0$, there exist $l_K > 0$ such that $K \Subset M_{l_K}$ and $C_K > 0$ such that $\frac{e^{\varphi + \psi_l - \psi}}{c(-\psi)} \leq C_K$ for any $l \geq l_K$. It follows from Lemma 2.36 and the diagonal method that there exists a subsequence of $\{F_l\}$, denoted still by $\{F_l\}$, which is uniformly convergent to a holomorphic $(n, 0)$ form F on M on any compact subset of M . It follows from the Fatou's Lemma and inequality (2.41) that

$$\begin{aligned} \int_M |F|^2 e^{-\varphi} c(-\psi) &= \int_M \lim_{l \rightarrow +\infty} |F_l|^2 e^{-\varphi - \psi_l + \psi} c(-\psi) \\ &\leq \liminf_{l \rightarrow +\infty} \int_{M_l} |F_l|^2 e^{-\varphi - \psi_l + \psi} c(-\psi) \\ &\leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned}$$

Since $\{F_l\}$ is uniformly convergent to F on any compact subset of M and $(F_l - f, (z_\beta, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_\beta, y)}$ for any $\beta \in \{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1 \leq j \leq n_1} \Omega_{j,l}\}$ and $y \in Y$, it follows from Lemma 2.17 that $(F - f, (z_\beta, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_\beta, y)}$ for any $\beta \in \tilde{I}_1$ and $y \in Y$.

In the following, we assume that $\tilde{m}_j < +\infty$ for any $1 \leq j \leq n_1$. Denote that $m_j = \tilde{m}_j - 1$. As $\prod_{1 \leq j \leq n_1} \Omega_j$ is a Stein manifold, it follows from Lemma 2.28 that there exist smooth plurisubharmonic functions Φ_l on $\prod_{1 \leq j \leq n_1} \Omega_j$, which are decreasingly convergent to φ_X with respect to l . Denote that

$$\varphi_l := \pi_1^*(\Phi_l) + \pi_2^*(\varphi_Y).$$

As Y is a weakly pseudoconvex Kähler manifold, it is well-known that there exist open weakly pseudoconvex Kähler manifolds $D_1 \Subset \dots \Subset D_{l'} \Subset D_{l'+1} \Subset \dots$ such that $\cup_{l' \in \mathbb{Z}_{\geq 1}} D_{l'} = Y$. Denote that $M_{l'} := (\prod_{1 \leq j \leq n_1} \Omega_j) \times D_{l'}$.

It follows from Lemma 2.12 and Lemma 2.13 that there exists a local coordinate $\tilde{w}_{j,k}$ on a neighborhood $\tilde{V}_{z_{j,k}} \Subset V_{z_{j,k}}$ of $z_{j,k}$ satisfying $\tilde{w}_{j,k}(z_{j,k}) = 0$ and

$$|\tilde{w}_{j,k}| = \exp \left(\frac{\sum_{1 \leq k_1 \leq m_j} p_{j,k_1} G_{\Omega_j}(\cdot, z_{j,k_1})}{p_{j,k}} \right)$$

on $\tilde{V}_{z_{j,k}}$. Denote that $\tilde{V}_\beta := \prod_{1 \leq j \leq n_1} \tilde{V}_{j,\beta_j}$ for any $\beta \in \tilde{I}_1$. Let \tilde{f} be a holomorphic $(n, 0)$ form on $\cup_{\beta \in \tilde{I}_1} \tilde{V}_\beta \times Y$ satisfying

$$\tilde{f} = \sum_{\alpha \in E_\beta} c_{\alpha,\beta} \pi_1^*(\tilde{w}_\beta^\alpha d\tilde{w}_{1,\beta_1} \wedge d\tilde{w}_{2,\beta_2} \wedge \dots \wedge d\tilde{w}_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $\tilde{V}_\beta \times Y$, where $c_{\alpha,\beta} = \prod_{1 \leq j \leq n_1} \left(\lim_{z \rightarrow z_{j,\beta_j}} \frac{w_{j,\beta_j}(z)}{\tilde{w}_{j,\beta_j}(z)} \right)^{\alpha_j+1}$. It follows from Lemma 2.18 that

$$(f - \tilde{f}, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$$

for any $z \in Z_0$. Denote that

$$\psi_1 := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \tilde{\pi}_j^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$$

on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j . Note that $\psi = \pi_1^*(\psi_1)$. It follows from Lemma 2.14 and Lemma 2.13 that there exists $t_0 > 0$ such that $\{\psi_1 < -t_0\} \Subset \cup_{\beta \in \tilde{I}_1} \tilde{V}_\beta$, which implies that $\int_{\{\psi_1 < -t\} \times D_{l'}} |\tilde{f}|^2 < +\infty$.

Using Lemma 2.35, there exists a holomorphic $(n, 0)$ form $F_{l,l',t}$ on $M_{l'}$ such that

$$\begin{aligned} & \int_{M_{l'}} |F_{l,l',t} - (1 - b_{t,1}(\psi)) \tilde{f}|^2 e^{-\varphi_l - \psi + v_{t,1}(\psi)} c(-v_{t,1}(\psi)) \\ & \leq \left(\int_0^{t+1} c(s) e^{-s} ds \right) \int_{M_{l'}} \mathbb{I}_{\{-t-1 < \psi < -t\}} |\tilde{f}|^2 e^{-\varphi_l - \psi}, \end{aligned} \quad (2.42)$$

where $t \geq t_0$. Note that $b_{t,1}(s) = 0$ for large enough s , then $(F_{l,l',t} - \tilde{f}, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0 \cap M_{l'}$.

For any $\epsilon > 0$, there exists $t_1 > t_0$, such that $\sup_{z \in \{\psi_1 < -t_1\} \cap \tilde{V}_\beta} |\Phi_l(z) - \Phi_l(z_\beta)| < \epsilon$ for any $\beta \in \tilde{I}_1$. Note that $\varphi_l = \pi_1^*(\Phi_l) + \pi_2^*(\varphi_Y)$ and $|c_{\alpha,\beta}| = \frac{1}{\prod_{1 \leq j \leq n_1} c_{j,\beta_j}^{\alpha_j+1}}$ for any $\beta \in \tilde{I}_1$ and $\alpha \in E_\beta$. As $\{\psi_1 < -t_1\} \Subset \cup_{\beta \in \tilde{I}_1} \tilde{V}_\beta$, it follows from Lemma 2.21, the Fubini's theorem and

$$\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$$

that

$$\begin{aligned} \int_{M_{l'}} \mathbb{I}_{\{-t-1 < \psi < -t\}} |\tilde{f}|^2 e^{-\varphi_l - \psi} &= \int_{\{-t-1 < \psi_1 < -t\} \times D_{l'}} |\tilde{f}|^2 e^{-\pi_1^*(\Phi_l + \psi) - \pi_2^*(\varphi_Y)} \\ &\leq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\Phi_l(z_\beta) + \epsilon}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \int_{D_{l'}} |f_{\alpha,\beta}|^2 e^{-\varphi_Y} \end{aligned} \quad (2.43)$$

for $t > t_1$. Letting $t \rightarrow +\infty$ and $\epsilon \rightarrow 0$, inequality (2.43) implies that

$$\limsup_{t \rightarrow +\infty} \int_{M_{l'}} \mathbb{I}_{\{-t-1 < \psi < -t\}} |\tilde{f}|^2 e^{-\varphi_l - \psi} \leq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\Phi_l(z_\beta)} \int_{D_{l'}} |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}}. \quad (2.44)$$

As $v_{t,1}(\psi) \geq \psi$ and $c(t)e^{-t}$ is decreasing, Combining inequality (2.42) and (2.44), then we have

$$\begin{aligned}
& \limsup_{t \rightarrow +\infty} \int_{M_{l'}} |F_{l,l',t} - (1 - b_{t,1}(\psi))\tilde{f}|^2 e^{-\varphi_l} c(-\psi) \\
& \leq \limsup_{t \rightarrow +\infty} \int_{M_{l'}} |F_{l,l',t} - (1 - b_{t,1}(\psi))\tilde{f}|^2 e^{-\varphi_l - \psi + v_{t,1}(\psi)} c(-v_{t,1}(\psi)) \\
& \leq \limsup_{t \rightarrow +\infty} \left(\int_0^{t+1} c(s) e^{-s} ds \right) \int_{M_{l'}} \mathbb{I}_{\{-t-1 < \psi < -t\}} |\tilde{f}|^2 e^{-\varphi_l - \psi} \\
& \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\Phi_l(z_\beta)} \int_{D_{l'}} |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \\
& < +\infty.
\end{aligned} \tag{2.45}$$

Note that ψ is continuous on $M \setminus Z_0$. For any open set $K \Subset M_{l'} \setminus Z_0$, as $b_{t,1}(s) = 1$ for any $s \geq -t$ and $c(s)e^{-s}$ is decreasing with respect to s , we get that there exists a constant $C_K > 0$ such that

$$\int_K |(1 - b_{t,1}(\psi))\tilde{f}|^2 e^{-\varphi_l} c(-\psi) \leq C_K \int_{\{\psi < -t_1\} \cap K} |\tilde{f}|^2 < +\infty$$

for any $t > t_1$, which implies that

$$\limsup_{t \rightarrow +\infty} \int_K |F_{l,l',t}|^2 e^{-\varphi_l} c(-\psi) < +\infty.$$

Using Lemma 2.36 and the diagonal method, we obtain that there exists a subsequence of $\{F_{l,l',t}\}_{t \rightarrow +\infty}$ denoted by $\{F_{l,l',t_m}\}_{m \rightarrow +\infty}$ uniformly convergent on any compact subset of $M_{l'} \setminus Z_0$. As Z_0 is a closed complex submanifold of M , we obtain that $\{F_{l,l',t_m}\}_{m \rightarrow +\infty}$ is uniformly convergent to a holomorphic $(n, 0)$ form $F_{l,l'}$ on $M_{l'}$ on any compact subset of $M_{l'}$. Then it follows from inequality (2.45) and the Fatou's Lemma that

$$\begin{aligned}
& \int_{M_{l'}} |F_{l,l'}|^2 e^{-\varphi_l} c(-\psi) \\
& = \int_{M_{l'}} \liminf_{m \rightarrow +\infty} |F_{l,l',t_m} - (1 - b_{t_m,1}(\psi))\tilde{f}|^2 e^{-\varphi_l} c(-\psi) \\
& \leq \liminf_{m \rightarrow +\infty} \int_{M_{l'}} |F_{l,l',t_m} - (1 - b_{t_m,1}(\psi))\tilde{f}|^2 e^{-\varphi_l} c(-\psi) \\
& \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\Phi_l(z_\beta)} \int_{D_{l'}} |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \\
& < +\infty.
\end{aligned}$$

Note that $\lim_{l \rightarrow +\infty} \Phi_l(z_\beta) = \varphi_X(z_\beta) > -\infty$ for any $\beta \in I_1$, then we have

$$\begin{aligned}
& \limsup_{l \rightarrow +\infty} \int_{M_{l'}} |F_{l,l'}|^2 e^{-\varphi_l} c(-\psi) \\
& \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_{D_{l'}} |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \\
& < +\infty.
\end{aligned} \tag{2.46}$$

Note that ψ is continuous on $M \setminus Z_0$ and Z_0 is a closed complex submanifold of M . Using Lemma 2.36, we obtain that there exists a subsequence of $\{F_{l,l'}\}_{l \rightarrow +\infty}$ (also denoted by $\{F_{l,l'}\}_{l \rightarrow +\infty}$) uniformly convergent to a holomorphic $(n, 0)$ form $F_{l'}$ on $M_{l'}$ on any compact subset of $M_{l'}$, which satisfies that

$$\int_{M_{l'}} |F_{l'}|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_{D_{l'}} |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}}.$$

As $\cup_{l' \in \mathbb{Z}_{\geq 1}} D_{l'} = Y$, we have

$$\begin{aligned} & \limsup_{l' \rightarrow +\infty} \int_{M_{l'}} |F_{l'}|^2 e^{-\varphi} c(-\psi) \\ & \leq \lim_{l' \rightarrow +\infty} \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_{D_{l'}} |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \\ & = \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \\ & < +\infty. \end{aligned} \tag{2.47}$$

Note that ψ is continuous on $M \setminus Z_0$, Z_0 is a closed complex submanifold of M and $\cup_{l' \in \mathbb{Z}_{\geq 1}} M_{l'} = M$. Using Lemma 2.36 and the diagonal method, we get that there exists a subsequence of $\{F_{l'}\}$ (also denoted by $\{F_{l'}\}$) uniformly convergent to a holomorphic $(n, 0)$ form F on M on any compact subset of M . Then it follows from inequality (2.47) and the Fatou's Lemma that

$$\begin{aligned} \int_M |F|^2 e^{-\varphi} c(-\psi) &= \int_M \liminf_{l' \rightarrow +\infty} \mathbb{1}_{M_{l'}} |F_{l'}|^2 e^{-\varphi} c(-\psi) \\ &\leq \liminf_{l' \rightarrow +\infty} \int_{M_{l'}} |F_{l'}|^2 e^{-\varphi} c(-\psi) \\ &\leq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\varphi_X(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}}. \end{aligned}$$

Following from Lemma 2.17, we have $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$. Thus, Proposition 2.37 holds. \square

3. PROOFS OF THEOREM 1.2 AND REMARK 1.4

In this section, we prove Theorem 1.2 and Remark 1.4.

3.1. Proofs of the sufficiency part of Theorem 1.2 and Remark 1.4.

In this section, we prove the sufficiency part of Theorem 1.2 and Remark 1.4.

Denote that $M' := \prod_{1 \leq j \leq n_1} \Omega_j$, and let $\tilde{\pi}_j$ be the natural projection from M' to Ω_j . Denote that $\psi_1 := \max_{1 \leq j \leq n_1} \{\tilde{\pi}_j^*(2p_j G_{\Omega_j}(\cdot, z_j))\}$ and $\tilde{\varphi} := \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$ on M' . It follows from statements (2) and (3) in Theorem 1.2 that

$$\tilde{f}_\alpha = \wedge_{1 \leq j \leq n_1} \tilde{\pi}_j^* \left(g_j(P_j)_* \left(f_{u_j} f_{z_j}^{\alpha_j} df_{z_j} \right) \right)$$

is a (single-value) holomorphic $(n_1, 0)$ form on M' for any $\alpha \in E$ satisfying $f_\alpha \not\equiv 0$, where $P_j : \Delta \rightarrow \Omega_j$ is the universal covering, f_{u_j} is a holomorphic $(1, 0)$ form on Δ

satisfying $|f_{u_j}| = (P_j)^*(e^{u_j})$ and f_{z_j} is a holomorphic $(1,0)$ form on Δ satisfying $|f_{z_j}| = (P_j)^*\left(e^{G_{\Omega_j}(\cdot, z_j)}\right)$. Denote that $\tilde{E} := \{\alpha \in E : f_\alpha \not\equiv 0\}$. Let

$$F = \sum_{\alpha \in \tilde{E}} c_\alpha \pi_1^*(\tilde{f}_\alpha) \wedge \pi_2^*(f_\alpha)$$

be a holomorphic $(n,0)$ form on M , where $c_\alpha = \lim_{z \rightarrow z_0} \frac{w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}}{f_\alpha}$. As $\int_Y |f_\alpha|^2 e^{-\varphi_Y} < +\infty$ and $\tilde{\varphi}(z_0) > -\infty$, it follows Lemma 2.18 and Lemma 2.25 that

$$(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$$

for any $z \in Z_0$.

It follows from Remark 2.8 that $\sum_{\alpha \in \tilde{E}} c_\alpha d_\alpha \tilde{f}_\alpha$ is the unique holomorphic $(n_1, 0)$ form on M' such that $\left(\sum_{\alpha \in \tilde{E}} c_\alpha d_\alpha \tilde{f}_\alpha - \sum_{\alpha \in \tilde{E}} d_\alpha w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0\right) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}$, $\int_{\{\psi_1 < -t\}} |\sum_{\alpha \in \tilde{E}} c_\alpha d_\alpha \tilde{f}_\alpha|^2 e^{-\tilde{\varphi}} c(-\psi_1) = \inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\psi_1) : \tilde{F} \text{ is a holomorphic } (n_1, 0) \text{ form on } \{\psi_1 < -t\} \text{ satisfying that } (\tilde{F} - \sum_{\alpha \in \tilde{E}} d_\alpha w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in (\mathcal{O}(K_{M'}))_{z_0} \otimes \mathcal{I}(\psi_1)_{z_0} \right\}$ and

$$\begin{aligned} & \int_{\{\psi_1 < -t\}} \left| \sum_{\alpha \in \tilde{E}} c_\alpha d_\alpha \tilde{f}_\alpha \right|^2 e^{-\tilde{\varphi}} c(-\psi_1) \\ &= \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in \tilde{E}} \frac{|d_\alpha|^2 (2\pi)^{n_1} e^{-\tilde{\varphi}(z_0)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \end{aligned} \tag{3.1}$$

for any $t \geq 0$, where $c_j(z_j) = \exp \lim_{z \rightarrow z_j} (G_{\Omega_j}(z, z_j) - \log |w_j(z)|)$. Following from equality (3.1) and the Fubini's theorem, we obtain that

$$\begin{aligned} & \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \\ &= \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \\ &< +\infty \end{aligned} \tag{3.2}$$

for any $t \geq 0$. Thus, $G(t) \leq \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$ for any $t \geq 0$.

It follows from Lemma 2.5 that there exists a holomorphic $(n,0)$ form F_t on $\{\psi < -t\}$ satisfying that $(F_t - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and $G(t) = \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi)$. For any $y_0 \in Y$, let $u = (u_1, \dots, u_{n_2})$ be a coordinate on a neighborhood U of y satisfying $u(y_0) = 0$ and $u(U) = \Delta^{n_2}$. Lemma 2.22 implies that $F_t|_U = \sum_{\gamma \in \mathbb{Z}_{\geq 0}^{n_2}} \pi_1^*(f_{t,\gamma}) \wedge \pi_2^*(u^\gamma du_1 \wedge \dots \wedge du_{n_2})$, where $f_{t,\gamma}$ is a holomorphic $(n_1, 0)$ form on $\{\psi_1 < -t\}$ for any $\gamma \in \mathbb{Z}_{\geq 0}^{n_2}$. There exists a holomorphic function $f_{u,\alpha}$ on U such that $f_\alpha = f_{u,\alpha} du_1 \wedge \dots \wedge du_{n_2}$ on U for any $\alpha \in \tilde{E}$. Note that $f = \sum_{\alpha \in \tilde{E}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + g_0$ on $V_0 \times Y$, where g_0 is a holomorphic $(n,0)$ form on $V_0 \times Y$ satisfying $(g_0, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$. It follows from Lemma 2.19 and $(F_t - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ that $\left(\sum_{\gamma \in \mathbb{Z}_{\geq 0}^{n_2}} u^\gamma f_{t,\gamma} - \sum_{\alpha \in \tilde{E}} f_{u,\alpha}(u) w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}\right) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}$ for any $u \in \Delta^{n_2}$. Let U_1 be an open subset of U , and let $V = u(U_1) \subset \Delta^{n_2}$. Note that $\left(\sum_{\alpha \in \tilde{E}} c_\alpha f_{u,\alpha}(u) \tilde{f}_\alpha - \sum_{\alpha \in \tilde{E}} f_{u,\alpha}(u) w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}\right) \in (\mathcal{O}(K_{M'})) \otimes$

$\mathcal{I}(\psi_1))_{z_0}$ for any $u \in \Delta^{n_2}$. Following the Fubini's theorem and the minimal property of $\int_{\{\psi_1 < -t\}} |\sum_{\alpha \in \tilde{E}} c_\alpha f_{u,\alpha} \tilde{f}_\alpha|^2 e^{-\varphi} c(-\psi_1)$, we have

$$\begin{aligned} & \int_{\{\psi_1 < -t\} \times U_1} |F_t|^2 e^{-\varphi} c(-\psi) \\ &= \int_V \left(\int_{\{\psi_1 < -t\}} \left| \sum_{\gamma \in \mathbb{Z}_{\geq 0}^{n_2}} u^\gamma f_{t,\gamma} \right|^2 e^{-\tilde{\varphi}} c(-\psi_1) \right) e^{-\varphi_Y} |du_1 \wedge \dots \wedge du_{n_2}|^2 \\ &\geq \int_V \left(\int_{\{\psi_1 < -t\}} \left| \sum_{\alpha \in \tilde{E}} c_\alpha f_{u,\alpha}(u) \tilde{f}_\alpha \right|^2 e^{-\tilde{\varphi}} c(-\psi_1) \right) e^{-\varphi_Y} |du_1 \wedge \dots \wedge du_{n_2}|^2 \\ &= \int_{\{\psi_1 < -t\} \times U_1} \left| \sum_{\alpha \in \tilde{E}} c_\alpha \pi_1^*(\tilde{f}_\alpha) \wedge \pi_2^*(f_\alpha) \right|^2 e^{-\varphi} c(-\psi), \end{aligned}$$

which implies $G(t) = \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) \geq \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$. It follows from $G(t) \leq \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ and inequality (3.2) that

$$\begin{aligned} G(t) &= \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \\ &= \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}, \end{aligned}$$

hence $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s) e^{-s} ds]$. The uniqueness of F follows from Corollary 2.3.

Thus, the sufficiency part of Theorem 1.2 and Remark 1.4 hold.

3.2. Proof of the necessity part of Theorem 1.2.

In this section, we prove the necessity part of Theorem 1.2 in three steps.

Step 1. $f = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + g_0$.

Corollary 2.3 show that there is a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and $G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq 0$. It follows from Lemma 2.12 that there exists a local coordinate \tilde{w}_j on a neighborhood $\tilde{V}_{z_j} \Subset V_{z_j}$ of $z_j \in \Omega_j$ satisfying $\tilde{w}_j(z_j) = 0$ and

$$\log |\tilde{w}_j| = G_{\Omega_j}(\cdot, z_j)$$

on \tilde{V}_{z_j} for any $j \in \{1, \dots, n_1\}$. Denote that $\tilde{V}_0 := \prod_{1 \leq j \leq n_1} \tilde{V}_{z_j}$ and $\tilde{w} := (\tilde{w}_1, \dots, \tilde{w}_{n_1})$ on \tilde{V}_0 . Using Lemma 2.14, we get that there exists $t_0 > 0$ such that

$$\{2p_j G_{\Omega_j}(\cdot, z_j) < -t_0\} \Subset \tilde{V}_{z_j}$$

for any $1 \leq j \leq n_1$. As φ_j is a subharmonic function on Ω_j , $\int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$ implies that

$$\int_{\{\psi < -t_0\}} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\psi) < +\infty.$$

Note that

$$\{\psi < -t_0\} = \left(\prod_{1 \leq j \leq n_1} \left\{ |\tilde{w}_j| < e^{-\frac{t_0}{2p_j}} \right\} \right) \times Y.$$

It follows from Lemma 2.22 that there exists a unique sequence of holomorphic $(n_2, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(\tilde{w}^\alpha d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_1}) \wedge \pi_2^*(F_\alpha) \quad (3.3)$$

on $\{\psi < -t_0\}$ and

$$\int_Y |F_\alpha|^2 e^{-\varphi_Y} < +\infty, \quad (3.4)$$

where the right term of the above equality is uniformly convergent on any compact subset of M . As $\frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds}$ is a positive number independent of t , Lemma 2.29 implies that $F_\alpha \equiv 0$ for any $\alpha \in \mathbb{Z}_{\geq 0}$ satisfying $\sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} < 1$. Denote that $E_2 := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} > 1 \right\}$. Note that $\varphi(z_j) > -\infty$ for any $1 \leq j \leq n_1$. It follows from Lemma 2.18 and Lemma 2.25 that $(\pi_1^*(\tilde{w}^\alpha d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_1}) \wedge \pi_2^*(F_\alpha), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and $\alpha \in E_2$, thus

$$\left(\sum_{\alpha \in E_2} \pi_1^*(\tilde{w}^\alpha d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_1}) \wedge \pi_2^*(F_\alpha), z \right) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$$

for any $z \in Z_0$ (by using Lemma 2.17). As $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$, we have

$$\left(f - \sum_{\alpha \in E} \pi_1^*(\tilde{w}^\alpha d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_1}) \wedge \pi_2^*(F_\alpha), z \right) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$$

for any $z \in Z_0$. Denote that

$$\psi_1 := \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^*(2p_j G_{\Omega_j}(\cdot, z_j)) \right\}$$

on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j . Taking $c_\alpha = \prod_{1 \leq j \leq n_1} \left(\lim_{z \rightarrow z_j} \frac{\tilde{w}_j}{w_j} \right)^{\alpha_j + 1}$, it follows from Lemma 2.18 and Lemma 2.25 that $(\tilde{w}^\alpha d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_1} - c_\alpha w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in \mathcal{O}(K_{\prod_{1 \leq j \leq n_1} \Omega_j})_{z_0} \otimes \mathcal{I} \left(\sum_{1 \leq j \leq n_1} \tilde{\pi}_j(\varphi_j) + \psi_1 \right)_{z_0}$ for any $\alpha \in E$, which implies that $(\sum_{\alpha \in E} \pi_1^*(\tilde{w}^\alpha d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_1}) \wedge \pi_2^*(F_\alpha) - \sum_{\alpha \in E} \pi_1^*(c_\alpha w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$. Taking $f_\alpha = c_\alpha F_\alpha$, there exists a holomorphic $(n, 0)$ form g_0 on $V_0 \times Y$ such that

$$f = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + g_0$$

and $(g_0, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$. As $G(0) > 0$, we know that there exists $\alpha \in E$ such that $f_\alpha \not\equiv 0$.

Step 2. $G(-\log r; \tilde{c} \equiv 1)$ is linear with respect to r .

It follows from Corollary 2.3 that $G(t; \tilde{c} \equiv 1) \leq \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} = \frac{G(0; c)}{\int_0^{+\infty} c(s) e^{-s} ds} e^{-t} < +\infty$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} : (\tilde{f} - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\}$$

by $\tilde{G}(t)$, where $t \geq 0$. It follows from Lemma 2.26 that $G(t; \tilde{c} \equiv 1) = \tilde{G}(t)$ for any $t \geq 0$. Denote that $M' := \prod_{1 \leq j \leq n_1} \Omega_j$, and let $K_{M'}$ be the canonical (holomorphic) line bundle on M' . Using Lemma 2.26, Lemma 2.32 and Lemma 2.33, we obtain that there exists a unique holomorphic $(n, 0)$ form $F_t = \sum_{\alpha \in E} \pi_1^*(h_{t,\alpha}) \wedge \pi_2^*(f_\alpha)$ on $\{\psi < -t\}$ satisfying

$$G(t; \tilde{c} \equiv 1) = \tilde{G}(t) = \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} = \sum_{\alpha \in E} \int_{\{\psi < -t\}} |\pi_1^*(h_{t,\alpha}) \wedge \pi_2^*(f_\alpha)|^2 e^{-\varphi}, \quad (3.5)$$

where $h_{t,\alpha}$ is a holomorphic $(n_1, 0)$ form on $\{\psi_1 < -t\}$ satisfying

$$(h_{t,\alpha} - w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}$$

and $\int_{\{\psi_1 < -t\}} |h_{t,\alpha}|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)} = \inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{F}|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)} : \tilde{F} \text{ is a holomorphic } (n_1, 0) \text{ form on } \{\psi_1 < -t\} \text{ satisfying } (\tilde{F} - w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0} \right\} < +\infty$. It follows from Lemma 2.31 that there exists a holomorphic $(n_1, 0)$ form \tilde{h}_α on M' such that $\int_{M'} |\tilde{h}_\alpha|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)} c(-\psi_1) < +\infty$ and $(\tilde{h}_\alpha - w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}$. As $\varphi_j(z_j) > -\infty$ for any $1 \leq j \leq n_1$, it follows from Lemma 2.25 that there exists $t_1 > t$ such that

$$\int_{\{\psi_1 < -t_1\}} |h_{t,\alpha} - \tilde{h}_\alpha|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j) - \psi_1} < +\infty$$

for any $\alpha \in E$. As $c(s)e^{-s}$ is a positive decreasing function on $(0, +\infty)$, for any $t > 0$, we obtain that

$$\begin{aligned} & \int_{\{\psi_1 < -t\}} |h_{t,\alpha}|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)} c(-\psi_1) \\ & \leq C \int_{\{\psi_1 < -t_1\}} |h_{t,\alpha} - \tilde{h}_\alpha|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j) - \psi_1} \\ & \quad + \int_{\{\psi_1 < -t_1\}} |\tilde{h}_\alpha|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)} c(-\psi_1) \\ & \quad + \sup_{s \in (t, t_1]} c(s) \times \int_{\{-t_1 \leq \psi_1 < -t\}} |h_{t,\alpha}|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)} \\ & < +\infty \end{aligned}$$

for any $\alpha \in E$, which implies that

$$\begin{aligned} & \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) \\ & \leq C \sum_{\alpha \in E} \int_{\{\psi_1 < -t\}} |h_{t,\alpha}|^2 e^{-\sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)} c(-\psi_1) \times \int_Y |f_\alpha|^2 e^{-\varphi_Y} \\ & < +\infty. \end{aligned} \quad (3.6)$$

It follows from Lemma 2.6 and inequality (3.6) that

$$G(t; \tilde{c} \equiv 1) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} = \frac{G(0; c)}{\int_0^{+\infty} c(s) e^{-s} ds} e^{-t}$$

for any $t > 0$. Theorem 2.2 shows that $\lim_{t \rightarrow 0^+} G(t; \tilde{c} \equiv 1) = G(0; \tilde{c} \equiv 1)$, hence we get $G(-\log r; \tilde{c} \equiv 1)$ is linear with respect to $r \in (0, 1]$.

Step 3. proofs of statements (2) and (3) in Theorem 1.2.

Denote

$$\inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\varphi} : (\tilde{f} - w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}, z_0) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0} \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_{M'})) \right\}$$

by $G_\alpha(t)$, where $t \geq 0$. Lemma 2.4 and Lemma 2.18 show that $G_\alpha(t) \neq 0$ for any $\alpha \in E$. It follows from equality (3.5) that

$$G(t; \tilde{c} \equiv 1) = \sum_{\alpha \in E} G_\alpha(t) \int_Y |f_\alpha|^2 e^{-\varphi}. \quad (3.7)$$

Theorem 2.2 tells us that $G_\alpha(-\log r)$ is concave with respect to r . It follows from the linearity of $G(-\log r; \tilde{c})$ and equality (3.7) that $G_\alpha(-\log r)$ is linear with respect to r for any $\alpha \in E$ satisfying $f_\alpha \not\equiv 0$. It follows from Theorem 2.7 and the linearity of $G_\alpha(-\log r)$ that statements (2) and (3) in Theorem 1.2 hold.

Thus, the necessity part of Theorem 1.2 holds.

4. PROOFS OF THEOREM 1.5 AND REAMRK 1.6

In this section, we prove Theorem 1.5 and Remark 1.6.

Denote that $M' := \prod_{1 \leq j \leq n_1} \Omega_j$, and let $K_{M'}$ be the conanical (holomorphic) line bundle on M' . Denote that

$$\psi_1 := \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left(2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on M' , where $\tilde{\pi}_j$ is the natural projection from M' to Ω_j . For any $\beta \in I_1$ and any holomorphic function h , it follows from Lemma 2.19 that $(h, (z_\beta, y)) \in \mathcal{I}(\psi)(z_\beta, y)$ for any $y \in Y$ if and only if $(h(\cdot, y), z_\beta) \in \mathcal{I}(\psi_1)_{z_\beta}$ for any $y \in Y$. The sufficiency part of Theorem 1.5 follows from Proposition 2.24, Theorem 2.9 and Lemma 2.26. In the following, we prove the necessity part of Theorem 1.5 and Remark 1.6.

Following from the linearity of $G(h^{-1}(r))$ and Corollary 2.3, there exists a holomorphic $(n, 0)$ form F on M , such that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and

$$G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi). \quad (4.1)$$

It follows from Lemma 2.13 and Lemma 2.14 that there exists $t_0 > 0$ such that $\{\psi_1 < -t_0\} \Subset \cup_{\beta \in I_1} V_\beta$ and $\left\{ z \in \Omega_j : 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(z, z_{j,k}) < -t_0 \right\} \cap V_{z_{j,k}}$ is

simply connected for any $j \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, m_j\}$. For any $\beta \in I_1$, denote

$$\inf \left\{ \int_{\{\psi < -t\} \cap (V_\beta \times Y)} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\} \cap (V_\beta \times Y), \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - f, (z_\beta, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_\beta, y)}, \forall y \in Y \right\}$$

by $G_\beta(t)$, where $t \in [t_0, +\infty)$. Note that $\{\psi < -t\} = \cup_{\beta \in I_1} (\{\psi < -t\} \cap (V_\beta \times Y))$ for any $t \geq t_0$. Following from the definition of $G(t)$ and $G_\beta(t)$, we have $G(t) = \sum_{\beta \in I_1} G_\beta(t)$ for $t \geq t_0$. Thus, we have

$$G_\beta(t) = \int_{\{\psi < -t\} \cap (V_\beta \times Y)} |F|^2 e^{-\varphi} c(-\psi)$$

for any $t \geq t_0$. Theorem 2.2 tells us that $G_\beta(h^{-1}(r))$ is concave with respect to $r \in (0, \int_{t_0}^{+\infty} c(s) e^{-s} ds]$. As $G(h^{-1}(r))$ is linear with respect to r , we have $G_\beta(h^{-1}(r))$ is linear with respect to $r \in (0, \int_{t_0}^{+\infty} c(s) e^{-s} ds]$.

Note that $f = \pi_1^*(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $E' = \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{j=1}^{n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{j=1}^{n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}} \right\}$. As $\frac{1}{2p_{j,1}} \left(2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) + t_0 \right)$ is the Green function on $\{z \in \Omega_j : 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(z, z_{j,k}) < -t_0\} \cap V_{z_{j,1}}$, it follows from Theorem 1.2 that $(f - \sum_{\alpha \in E_{\beta^*}} \pi_1^*(w_{\beta^*}^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(\tilde{f}_\alpha), (z_{\beta^*}, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_{\beta^*}, y)}$ for any $y \in Y$, where $E_{\beta^*} = \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta^*}} = 1 \right\}$ and \tilde{f}_α is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_\alpha|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in E_{\beta^*}$. Following from Lemma 2.18 and Lemma 2.19, we have $\alpha_{\beta^*} \in E_{\beta^*}$, $f_{\alpha_{\beta^*}} = \tilde{f}_{\alpha_{\beta^*}}$ and $\tilde{f}_\alpha \equiv 0$ for any $\alpha \neq \alpha_{\beta^*}$. Using Theorem 1.2 and Remark 1.4, we obtain that there exists a holomorphic $(n_1, 0)$ form h_0 on $\{\psi_1 < -t_0\} \cap V_{\beta^*}$ such that

$$F = \pi_1^*(h_0) \wedge \pi_2^*(f_{\alpha_{\beta^*}})$$

on $(\{\psi_1 < -t_0\} \cap V_{\beta^*}) \times Y$. It follows from Lemma 2.34 that there exists a holomorphic $(n_1, 0)$ form h_1 on M' such that

$$F = \pi_1^*(h_1) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) \tag{4.2}$$

on M and $h_0 = h_1$ on $\{\psi_1 < -t_0\} \cap V_{\beta^*}$.

Denote that $\tilde{\varphi} = \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$ on M' . Denote

$$\inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\tilde{\varphi}} c(-\psi_1) : (\tilde{f} - h_1, z_\beta) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_\beta}, \forall \beta \in I_1 \right. \\ \left. \& \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_{M'})) \right\}$$

by $G'(t)$, where $t \in [0, +\infty)$. Note that $f_{\alpha_{\beta^*}} = \tilde{f}_{\alpha_{\beta^*}}$ satisfies $\int_Y |f_{\alpha_{\beta^*}}|^2 e^{-\varphi_Y} < +\infty$. For any $\beta \in I_1$ and any holomorphic function h , note that $(h, (z_\beta, y)) \in \mathcal{I}(\psi)_{(z_\beta, y)}$ for any $y \in Y$ if and only if $(h(\cdot, y), z_\beta) \in \mathcal{I}(\psi_1)_{z_\beta}$ for any $y \in Y$. Following

from Lemma 2.26, equality (4.2) and Proposition 2.24, we get that $G'(0) < +\infty$, $G'(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ and

$$G'(t) = \int_{\{\psi_1 < -t\}} |h_1|^2 e^{-\tilde{\varphi}} c(-\psi_1) \quad (4.3)$$

for any $t \geq 0$. Theorem 2.9 tells us that the following statements hold:

- (1) $\varphi_j = 2 \log |g_j| + 2u_j$ for any $j \in \{1, \dots, n\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k}) \neq 0$ for any $k \in \{1, \dots, m_j\}$;
- (2) There exists a nonnegative integer $\gamma_{j,k}$ for any $j \in \{1, \dots, n_1\}$ and $k \in \{1, \dots, m_j\}$, which satisfies that $\prod_{1 \leq k \leq m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ and $\sum_{1 \leq j \leq n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ for any $\beta \in I_1$;
- (3) $h_1 = (c_\beta \prod_{1 \leq j \leq n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}} + \tilde{g}_\beta) dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}$ on V_β for any $\beta \in I_1$, where c_β is a constant and g_β is a holomorphic function on V_β such that $(g_\beta, z_\beta) \in \mathcal{I}(\psi_1)_{z_\beta}$;

$$(4) \lim_{z \rightarrow z_\beta} \frac{c_\beta \prod_{1 \leq j \leq n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}}{\wedge_{1 \leq j \leq n_1} \tilde{\pi}_j^* \left(g_j(P_j)_* \left(f_{u_j} \left(\prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left(\sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right) \right)} = c_0$$

for any $\beta \in I_1$, where $c_0 \in \mathbb{C} \setminus \{0\}$ is a constant independent of β , f_{u_j} is a holomorphic function Δ such that $|f_{u_j}| = P_j^*(e^{u_j})$ and $f_{z_{j,k}}$ is a holomorphic function on Δ such that $|f_{z_{j,k}}| = P_j^* \left(e^{G_{\Omega_j}(\cdot, z_{j,k})} \right)$ for any $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m_j\}$.

As $\int_Y |f_{\alpha_{\beta^*}}|^2 e^{-\varphi_Y} < +\infty$ and $\tilde{\varphi}(z_\beta) > -\infty$ for any $\beta \in I_1$, it follows from Lemma 2.25 that $\pi_1^*(\tilde{g}_\beta dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha_{\beta^*}}), z \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$. As $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and $F = \pi_1^*(h_1) \wedge \pi_2^*(f_{\alpha_{\beta^*}})$, we have

$$f = \pi_1^* \left(c_\beta \left(\prod_{1 \leq j \leq n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}} \right) dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}} \right) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) + g_\beta$$

on $V_\beta \times Y$ for any $\beta \in I_1$, where g_β is a holomorphic $(n, 0)$ form on $V_\beta \times Y$ such that $(g_\beta, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$. Take $f_0 = f_{\alpha_{\beta^*}}$. Thus, Theorem 1.5 holds.

Note that $G'(h^{-1}(r))$ is linear with respect to r . Following from Theorem 2.9, Remark 2.10 and equality (4.3), we have

$$h_1 = c_0 \wedge_{1 \leq j \leq n_1} \tilde{\pi}_j^* \left(g_j(P_j)_* \left(f_{u_j} \left(\prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left(\sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right) \right)$$

and

$$\begin{aligned} G'(t) &= \int_{\{\psi_1 < -t\}} |h_1|^2 e^{-\tilde{\varphi}} c(-\psi_1) \\ &= \left(\int_t^{+\infty} c(s)e^{-s} ds \right) \sum_{\beta \in I_1} \frac{|c_\beta|^2 (2\pi)^{n_1} e^{-\tilde{\varphi}(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\gamma_{j,\beta_j} + 1) c_{j,\beta_j}^{2\gamma_{j,\beta_j} + 2}}. \end{aligned}$$

Thus, we have

$$F = c_0 \left(\wedge_{1 \leq j \leq n_1} \tilde{\pi}_j^* \left(g_j(P_j)_* \left(f_{u_j} \left(\prod_{k=1}^{m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left(\sum_{k=1}^{m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right) \right) \right) \wedge \pi_2^*(f_0)$$

and

$$\begin{aligned} G(t) &= \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \\ &= \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in I_1} \frac{|c_\beta|^2 (2\pi)^{n_1} e^{-\tilde{\varphi}(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\gamma_{j,\beta_j} + 1) c_{j,\beta_j}^{2\gamma_{j,\beta_j}+2}} \int_Y |f_0|^2 e^{-\varphi_Y}. \end{aligned}$$

The uniqueness of F follows from Corollary 2.3. Thus, Remark 1.6 holds.

5. PROOFS OF THEOREM 1.7 AND PROPOSITION 1.8

In this section, we prove Theorem 1.7 and Proposition 1.8.

5.1. Proof of Theorem 1.7.

In this section, we prove Theorem 1.7 by contradiction. Assume that $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s) e^{-s} ds]$.

Denote that $M' := \prod_{1 \leq j \leq n_1} \Omega_j$, and let $K_{M'}$ be the canonical (holomorphic) line bundle on M' . Denote that

$$\psi_1 := \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left(2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on M' , where $\tilde{\pi}_j$ is the natural projection from M' to Ω_j . Following from the linearity of $G(h^{-1}(r))$ and Corollary 2.3, there exists a holomorphic $(n, 0)$ form F on M , such that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and

$$G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi). \quad (5.1)$$

For any $\beta \in \tilde{I}_1$, it follows from Lemma 2.13 and Lemma 2.14 that there exists $t_\beta > 0$ such that $\{\psi_1 < -t_\beta\} \cap V_\beta \Subset V_\beta$ and $\{z \in \Omega_j : 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(z, z_{j,k}) < -t_0\} \cap V_{z_{j,k}}$ is simply connected for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$. For any $\beta \in \tilde{I}_1$, denote

$$\begin{aligned} \inf \left\{ \int_{\{\psi < -t\} \cap (V_\beta \times Y)} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\} \cap (V_\beta \times Y), \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - f, (z_\beta, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_\beta, y)}, \forall y \in Y \right\} \end{aligned}$$

by $G_\beta(t)$, where $t \in [t_\beta, +\infty)$, and denote

$$\begin{aligned} \inf \left\{ \int_{\{\psi < -t\} \setminus (V_\beta \times Y)} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\} \setminus (V_\beta \times Y), \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z, \forall z \in (\tilde{I}_1 \setminus \{\beta\}) \times Y \right\} \end{aligned}$$

by $\tilde{G}_\beta(t)$, where $t \in [t_\beta, +\infty)$. By the definition of $G(t)$, $G_\beta(t)$ and $\tilde{G}_\beta(t)$, we have $G(t) = G_\beta(t) + \tilde{G}_\beta(t)$ for $t \geq t_\beta$. Thus, we have

$$G_\beta(t) = \int_{\{\psi < -t\} \cap (V_\beta \times Y)} |F|^2 e^{-\varphi} c(-\psi)$$

for any $t \geq t_\beta$. Theorem 2.2 tells us that $G_\beta(h^{-1}(r))$ and $\tilde{G}_\beta(h^{-1}(r))$ are concave with respect to $r \in (0, \int_{t_\beta}^{+\infty} c(s)e^{-s}ds]$. As $G(h^{-1}(r))$ is linear with respect to r , we have $G_\beta(h^{-1}(r))$ is linear with respect to $r \in (0, \int_{t_\beta}^{+\infty} c(s)e^{-s}ds]$.

Following from Lemma 2.12 and 2.13, we know $\frac{1}{2p_{j,1}} \left(2 \sum_{1 \leq k < \bar{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) + t_{\beta^*} \right)$ is the Green function on $\{z \in \Omega_j : 2 \sum_{1 \leq k < \bar{m}_j} p_{j,k} G_{\Omega_j}(z, z_{j,k}) < -t_{\beta^*}\} \cap V_{z_{j,1}}$. Note that $f = \pi_1^* \left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $E' = \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{j=1}^{n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{j=1}^{n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}} \right\}$. It follows from Theorem 1.2 that $(f - \sum_{\alpha \in E_{\beta^*}} \pi_1^*(w_{\beta^*}^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(\tilde{f}_\alpha), (z_{\beta^*}, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_{\beta^*}, y)}$ for any $y \in Y$, where $E_{\beta^*} = \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta^*}} = 1 \right\}$ and \tilde{f}_α is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_\alpha|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in E_{\beta^*}$. Following from Lemma 2.18 and Lemma 2.19, we have $\alpha_{\beta^*} \in E_{\beta^*}$, $f_{\alpha_{\beta^*}} = \tilde{f}_{\alpha_{\beta^*}}$ and $\tilde{f}_\alpha \equiv 0$ for any $\alpha \neq \alpha_{\beta^*}$. Using Theorem 1.2 and Remark 1.4, we obtain that there exists a holomorphic $(n_1, 0)$ form h_0 on $\{\psi_1 < -t_{\beta^*}\} \cap V_{\beta^*}$ such that

$$F = \pi_1^*(h_0) \wedge \pi_2^*(f_{\alpha_{\beta^*}})$$

on $(\{\psi_1 < -t_{\beta^*}\} \cap V_{\beta^*}) \times Y$. It follows from Lemma 2.34 that there exists a holomorphic $(n_1, 0)$ form h_1 on M' such that

$$F = \pi_1^*(h_1) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) \tag{5.2}$$

on M and $h_0 = h_1$ on $\{\psi_1 < -t_{\beta^*}\} \cap V_{\beta^*}$.

Denote that $\tilde{\varphi} = \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$ on M' . Denote

$$\begin{aligned} \inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\tilde{\varphi}} c(-\psi_1) : (\tilde{f} - h_1, z_\beta) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_\beta}, \forall \beta \in \tilde{I}_1 \right. \\ \left. \quad \& \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_{M'})) \right\} \end{aligned}$$

by $G'(t)$, where $t \in [0, +\infty)$. Note that $f_{\alpha_{\beta^*}} = \tilde{f}_{\alpha_{\beta^*}}$ satisfies $\int_Y |f_{\alpha_{\beta^*}}|^2 e^{-\varphi_Y} < +\infty$. For any $\beta \in \tilde{I}_1$ and any holomorphic function h , it follows from Lemma 2.19 that $(h, (z_\beta, y)) \in \mathcal{I}(\psi)_{(z_\beta, y)}$ for any $y \in Y$ if and only if $(h(\cdot, y), z_\beta) \in \mathcal{I}(\psi_1)_{z_\beta}$ for any $y \in Y$. Following from Lemma 2.26, equality (5.2) and Proposition 2.24, we get that $G'(0) < +\infty$ and $G'(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$, which contradicts to Theorem 2.11.

Thus, we obtain that $G(h^{-1}(r))$ is not linear.

5.2. Proof of Proposition 1.8.

It follows from Corollary 2.3 that there exists a holomorphic $(n, 0)$ form F on M_1 , which satisfies that $(F - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and

$$G(t) = \int_{\{\psi < -t\} \cap M_1} |F|^2 e^{-\varphi} c(-\psi) \quad (5.3)$$

for any $t \geq 0$.

It follows from Lemma 2.12 and Lemma 2.13 that there exists a local coordinate $w_{j,k}$ on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ and

$$\log |w_{j,k}| = \frac{1}{p_{j,k}} \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})$$

for any $j \in \{1, \dots, n_1\}$ and $1 \leq k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$, $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in M$. It follows from Lemma 2.23 that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on a neighborhood $U_\beta \subset (V_\beta \times Y) \cap M_1$ of $\{z_\beta\} \times Y$ for any $\beta \in \tilde{I}_1$, where $F_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y . Following from Lemma 2.30 and equality (5.3), we obtain that

$$F_{\alpha,\beta} \equiv 0$$

for any $\alpha \in \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} < 1 \right\}$ and $\beta \in \tilde{I}_1$, and we have

$$\frac{G(0)}{\int_0^{+\infty} c(s) e^{-s} ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}, \quad (5.4)$$

where $E_\beta = \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \right\}$ for any $\beta \in \tilde{I}_1$. Proposition 2.37 shows that there exists a holomorphic $(n, 0)$ form F_1 on M such that $(F_1 - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$ and

$$\begin{aligned} & \int_M |F_1|^2 e^{-\varphi} c(-\psi) \\ & \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned} \quad (5.5)$$

Denote that $\tilde{E}_\beta := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} \geq 1 \right\}$ for any $\beta \in \tilde{I}_1$. As $(F_1 - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$. It follows from Lemma 2.22 and Lemma 2.18 that

$$\begin{aligned} F_1 &= \sum_{\alpha \in E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta}) \\ &\quad + \sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(\tilde{F}_{\alpha,\beta}) \end{aligned}$$

on a neighborhood of $\{z_\beta\} \times Y$ for any $\beta \in \tilde{I}_1$, where $\tilde{F}_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y . It follows from Lemma 2.23 that $(F_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ and $(\tilde{F}_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ for any $y \in Y$. Using Lemma 2.18 and Lemma 2.25,

we obtain that $(F_1 - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$. Combining inequality (5.4) and (5.5), we have

$$\frac{G(0)}{\int_0^{+\infty} c(s)e^{-s}ds} = \int_M |F_1|^2 e^{-\varphi} c(-\psi) = \int_{M_1} |F_1|^2 e^{-\varphi} c(-\psi),$$

which implies that $M_1 = M$.

6. PROOFS OF THEOREM 1.9 AND REMARK 1.10

In this section, we prove Theorem 1.9 and Remark 1.10.

6.1. Proof of Theorem 1.9.

As $c(t)e^{-t}$ is decreasing and $\Psi \leq 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M , which satisfies that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}))_z$ for any $z \in Z_0$ and

$$\begin{aligned} & \int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)) \\ & \leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_0)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}. \end{aligned} \quad (6.1)$$

If $\Psi \equiv 0$, as $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}))_z$ for any $z \in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + \sum_{\alpha \in \tilde{E} \setminus E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(\tilde{f}_\alpha)$ on $V_0 \times Y$, where \tilde{f}_α is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_\alpha|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in \tilde{E} \setminus E$. Note that $(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_0) > -\infty$. It follows from Lemma 2.18, Lemma 2.25 and Lemma 2.17 that $(\sum_{\alpha \in \tilde{E} \setminus E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(\tilde{f}_\alpha), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$.

In the following, we prove the characterization of the holding of the equality in Theorem 1.9.

Firstly, we prove the necessity. Using inequality (6.1), we have

$$\int_{M_1} |F|^2 e^{-\varphi} c(-\psi) = \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)).$$

Note that $c(t)e^{-t}$ is decreasing. As $F \not\equiv 0$, we get that

$$M_1 = M = \left(\prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y.$$

As $\Psi \leq 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1 \leq j \leq n_1} \{ \pi_{1,j}^*(2p_j G_{\Omega_j}(\cdot, z_j)) \}.$$

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right\}$$

by $G(t)$, where $t \geq 0$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right\}$$

by $\tilde{G}(t)$, where $t \geq 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geq 0$. Let $t \geq 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here \sim means the former replaced by the latter) that there exists a holomorphic $(n, 0)$ form F_t on $\{\psi < -t\}$ satisfying that $(F_t - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$ and

$$\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) \\ \leq \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}. \quad (6.2)$$

Following from inequality (6.2), we have

$$\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s) e^{-s} ds} \leq \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}$$

holds for any $t \geq 0$. Note that

$$\tilde{G}(0) = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}.$$

Combining Theorem 2.2, we obtain that $\tilde{G}(h^{-1}(r))$ is linear with respect to r , which implies that $G(h^{-1}(r))$ is linear with respect to r , where $h(t) = \int_t^{+\infty} c(s) e^{-s} ds$. It follows from Theorem 1.2 that statements (2) and (3) in Theorem 1.9 hold.

Now, we prove the sufficiency. Following from Remark 1.4 and $G(0) = \tilde{G}(0)$, we obtain that

$$\tilde{G}(0) = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}.$$

Thus, Theorem 1.9 holds.

6.2. Proof of Remark 1.10.

Note that $\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j) \right)(z_0) > -\infty$. As $(f_\alpha, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ for any $y \in Y$ and $\alpha \in \tilde{E} \setminus E$, following from Lemma 2.25, Lemma 2.18 and Lemma 2.17, we get that $\left(\sum_{\alpha \in \tilde{E} \setminus E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha), z \right) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$.

As $c(t)e^{-t}$ is decreasing and $\Psi \leq 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M , which satisfies that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}))_z$ for any $z \in Z_0$ and

$$\begin{aligned} & \int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)) \\ & \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_0)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}. \end{aligned} \quad (6.3)$$

If $\Psi \equiv 0$, as $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}))_z$ for any $z \in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that $F = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + \sum_{\alpha \in \tilde{E} \setminus E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(\tilde{f}_\alpha)$ on $V_0 \times Y$, where \tilde{f}_α is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_\alpha|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in \tilde{E} \setminus E$. Note that $(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_0) > -\infty$. It follows from Lemma 2.18, Lemma 2.25 and Lemma 2.17 that $(\sum_{\alpha \in \tilde{E} \setminus E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(\tilde{f}_\alpha), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$. Thus, we have $(F - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$.

In the following, we prove the characterization of the holding of the equality (replacing the ideal sheaf $\mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\})$ by $\mathcal{I}(\varphi + \psi)$) in Theorem 1.9.

Firstly, we prove the necessity. Using inequality (6.3), we have

$$\int_{M_1} |F|^2 e^{-\varphi} c(-\psi) = \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)).$$

Note that $c(t)e^{-t}$ is decreasing. As $F \not\equiv 0$, we get that

$$M_1 = M = \left(\prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y.$$

As $\Psi \leq 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1 \leq j \leq n_1} \{\pi_{1,j}^*(2p_j G_{\Omega_j}(\cdot, z_j))\}.$$

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right\}$$

by $G(t)$, where $t \geq 0$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right\}$$

by $\tilde{G}(t)$, where $t \geq 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geq 0$. Let $t \geq 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here \sim means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s)e^{-s}ds} \leq \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}.$$

Note that

$$G(0) = \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}.$$

Combining Theorem 2.2, we obtain that $G(h^{-1}(r))$ is linear with respect to r , where $h(t) = \int_t^{+\infty} c(s)e^{-s}ds$. It follows from Theorem 1.2 that statements (2) and (3) in Theorem 1.9 hold.

Now, we prove the sufficiency. Following from Remark 1.4, we obtain that

$$G(0) = \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 e^{-\varphi_Y}.$$

Thus, Remark 1.10 holds.

7. PROOFS OF THEOREM 1.13 AND REMARK 1.14

In this section, we prove Theorem 1.13 and Remark 1.14.

7.1. Proof of Theorem 1.13.

As $c(t)e^{-t}$ is decreasing and $\Psi \leq 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M , which satisfies that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right))_z$ for any $z \in Z_0$ and

$$\begin{aligned} & \int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)) \\ & \leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned} \tag{7.1}$$

If $\Psi \equiv 0$, as $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right))_z$ for any $z \in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F = \sum_{\alpha \in E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta}) + \sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(\tilde{f}_{\alpha,\beta})$ on $V_\beta \times Y$, where $\tilde{f}_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in \tilde{E}_\beta \setminus E_\beta$ and $\beta \in I_1$. Note that $\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta) > -\infty$. It follows from Lemma 2.18, Lemma 2.25 and Lemma 2.17 that $\left(\sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(\tilde{f}_{\alpha,\beta}), z\right) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$, where $\beta \in I_1$.

In the following, we prove the characterization of the holding of the equality in Theorem 1.13.

Firstly, we prove the necessity. Using inequality (7.1), we have

$$\int_{M_1} |F|^2 e^{-\varphi} c(-\psi) = \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)).$$

Note that $c(t)e^{-t}$ is decreasing. As $F \not\equiv 0$, we get that

$$M_1 = M = \left(\prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y.$$

As $\Psi \leq 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}.$$

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right\}$$

by $G(t)$, where $t \geq 0$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right\}$$

by $\tilde{G}(t)$, where $t \geq 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geq 0$. Let $t \geq 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here \sim means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s) e^{-s} ds} \leq \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Note that

$$\tilde{G}(0) = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Combining Theorem 2.2, we obtain that $\tilde{G}(h^{-1}(r))$ is linear with respect to r , which implies that $G(h^{-1}(r))$ is linear with respect to r , where $h(t) = \int_t^{+\infty} c(s) e^{-s} ds$. As $f_{\alpha,\beta^*} \equiv 0$ for any $\alpha \neq \alpha_{\beta^*}$ satisfying $\sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} = 1$, where $\beta^* = (1, \dots, 1) \in I_1$, it follows from Theorem 1.5 that statements (2), (3), (4) and (5) in Theorem 1.13 hold.

Now, we prove the sufficiency. Following from Remark 1.6 and $G(0) = \tilde{G}(0)$, we obtain that

$$\tilde{G}(0) = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Thus, Theorem 1.13 holds.

7.2. Proof of Remark 1.14.

Note that $\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta) > -\infty$ for any $\beta \in I_1$. As $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ for any $y \in Y$, $\alpha \in \tilde{E}_\beta \setminus E_\beta$ and $\beta \in I_1$, following from Lemma 2.25, Lemma 2.18 and Lemma 2.17, we get that $(\sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta}), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$, where $\beta \in I_1$.

As $c(t)e^{-t}$ is decreasing and $\Psi \leq 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M , which satisfies that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right))_z$ for any $z \in Z_0$ and

$$\begin{aligned} & \int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)) \\ & \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned} \quad (7.2)$$

If $\Psi \equiv 0$, as $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F = \sum_{\alpha \in E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta}) + \sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(\tilde{f}_{\alpha,\beta})$ on $V_\beta \times Y$, where $\beta \in I_1$ and $\tilde{f}_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in \tilde{E}_\beta \setminus E_\beta$. Note that $(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta) > -\infty$ for any $\beta \in I_1$. Following from Lemma 2.18, Lemma 2.25 and Lemma 2.17, we obtain that $(\sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(\tilde{f}_{\alpha,\beta}), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$. Thus, we have $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$.

In the following, we prove the characterization of the holding of the equality (replacing the ideal sheaf $\mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)$ by $\mathcal{I}(\varphi + \psi)$) in Theorem 1.13.

Firstly, we prove the necessity. Using inequality (7.2), we have

$$\int_{M_1} |F|^2 e^{-\varphi} c(-\psi) = \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)).$$

Note that $c(t)e^{-t}$ is decreasing. As $F \not\equiv 0$, we get that

$$M_1 = M = \left(\prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y.$$

As $\Psi \leq 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}.$$

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right\}$$

by $G(t)$, where $t \geq 0$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right\}$$

by $\tilde{G}(t)$, where $t \geq 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geq 0$. Let $t \geq 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here \sim means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s)e^{-s} ds} \leq \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j, \beta_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y}.$$

Note that

$$G(0) = \left(\int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j, \beta_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y}.$$

Combining Theorem 2.2, we obtain that $G(h^{-1}(r))$ is linear with respect to r , where $h(t) = \int_t^{+\infty} c(s)e^{-s} ds$. It follows from Theorem 1.2 that statements (2), (3), (4) and (5) in Theorem 1.13 hold.

Now, we prove the sufficiency. Following from Remark 1.6, we obtain that

$$G(0) = \left(\int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{\beta \in I_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j, \beta_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y}.$$

Thus, Remark 1.14 holds.

8. PROOFS OF THEOREM 1.15 AND REMARK 1.16

In this section, we prove Theorem 1.15 and Remark 1.16.

8.1. Proof of Theorem 1.15.

As $c(t)e^{-t}$ is decreasing and $\Psi \leq 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M , which satisfies that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}))_z$ for any $z \in Z_0$

and

$$\begin{aligned}
& \int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \\
& \leq \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)) \\
& \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y}.
\end{aligned} \tag{8.1}$$

If $\Psi \equiv 0$, as $(F - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F = \sum_{\alpha \in E_\beta} \pi_1^*(w_\beta^\alpha dw_{1, \beta_1} \wedge \dots \wedge dw_{n_1, \beta_{n_1}}) \wedge \pi_2^*(f_{\alpha, \beta}) + \sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1, \beta_1} \wedge \dots \wedge dw_{n_1, \beta_{n_1}}) \wedge \pi_2^*(\tilde{f}_{\alpha, \beta})$ on $V_\beta \times Y$, where $\tilde{f}_{\alpha, \beta}$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_{\alpha, \beta}|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in \tilde{E}_\beta \setminus E_\beta$ and $\beta \in \tilde{I}_1$. Note that $(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta) > -\infty$. For any $\beta \in \tilde{I}_1$, it follows from Lemma 2.18, Lemma 2.25 and Lemma 2.17 that $(\sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1, \beta_1} \wedge \dots \wedge dw_{n_1, \beta_{n_1}}) \wedge \pi_2^*(\tilde{f}_{\alpha, \beta}), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$.

Denote that $\tilde{\psi} := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$. Now, we assume $\left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y} = \inf \left\{ \int_{M_1} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } M_1 \text{ such that } (\tilde{F} - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\tilde{\psi}))_z \text{ for any } z \in Z_0 \right\}$ to get a contradiction.

Using inequality (8.1), we have

$$\int_{M_1} |F|^2 e^{-\varphi} c(-\psi) = \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)).$$

Note that $c(t)e^{-t}$ is decreasing. As $F \not\equiv 0$, we get that

$$M_1 = M = \left(\prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y.$$

As $\Psi \leq 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}.$$

Denote

$$\begin{aligned}
& \inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\
& \quad \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right\}
\end{aligned}$$

by $G(t)$, where $t \geq 0$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right.$$

$$\left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right\}$$

by $\tilde{G}(t)$, where $t \geq 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geq 0$. Let $t \geq 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here \sim means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s)e^{-s}ds} \leq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j+2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Note that

$$\tilde{G}(0) = \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j+2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Combining Theorem 2.2, we obtain that $\tilde{G}(h^{-1}(r))$ is linear with respect to r , which implies that $G(h^{-1}(r))$ is linear with respect to r , where $h(t) = \int_t^{+\infty} c(s)e^{-s}ds$. As $f_{\alpha,\beta^*} \equiv 0$ for any $\alpha \neq \alpha_{\beta^*}$ satisfying $\sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,1}} = 1$, where $\beta^* = (1, \dots, 1) \in \tilde{I}_1$, the linearity of $G(h^{-1}(r))$ contradicts to Theorem 1.7. Thus, we obtain that there exists a holomorphic $(n, 0)$ form \tilde{F} on M_1 , which satisfies that $(\tilde{F} - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}))_z$ for any $z \in Z_0$ and

$$\begin{aligned} & \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\ & < \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j+2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned}$$

8.2. Proof of Remark 1.16.

Note that $(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta) > -\infty$ for any $\beta \in \tilde{I}_1$. As $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ for any $y \in Y$, $\alpha \in \tilde{E}_\beta \setminus E_\beta$ and $\beta \in \tilde{I}_1$, following from Lemma 2.25, Lemma 2.18 and Lemma 2.17, we get that $(\sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta}), z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$, where $\beta \in \tilde{I}_1$.

As $c(t)e^{-t}$ is decreasing and $\Psi \leq 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M , which satisfies that $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}))_z$ for any $z \in Z_0$

and

$$\begin{aligned}
& \int_{M_1} |F|^2 e^{-\varphi} c(-\psi) \\
& \leq \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)) \\
& \leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y}.
\end{aligned} \tag{8.2}$$

If $\Psi \equiv 0$, as $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $\tilde{F} = \sum_{\alpha \in E_\beta} \pi_1^*(w_\beta^\alpha dw_{1, \beta_1} \wedge \dots \wedge dw_{n_1, \beta_{n_1}}) \wedge \pi_2^*(f_{\alpha, \beta}) + \sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1, \beta_1} \wedge \dots \wedge dw_{n_1, \beta_{n_1}}) \wedge \pi_2^*(\tilde{f}_{\alpha, \beta})$ on $V_\beta \times Y$, where $\tilde{f}_{\alpha, \beta}$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_{\alpha, \beta}|^2 e^{-\varphi_Y} < +\infty$ for any $\alpha \in \tilde{E}_\beta \setminus E_\beta$ and $\beta \in \tilde{I}_1$. Note that $\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta) > -\infty$. Following from Lemma 2.18, Lemma 2.25 and Lemma 2.17, we obtain that $\left(\sum_{\alpha \in \tilde{E}_\beta \setminus E_\beta} \pi_1^*(w_\beta^\alpha dw_{1, \beta_1} \wedge \dots \wedge dw_{n_1, \beta_{n_1}}) \wedge \pi_2^*(\tilde{f}_{\alpha, \beta}), z\right) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$, where $\beta \in \tilde{I}_1$. Hence, we have $(F - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$.

In the following, we assume that $\inf \left\{ \int_{M_1} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } M_1 \text{ such that } (\tilde{F} - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right\} = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y}$ to get a contradiction.

Using inequality (8.2), we have

$$\int_{M_1} |F|^2 e^{-\varphi} c(-\psi) = \int_M |F|^2 e^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)).$$

Note that $c(t)e^{-t}$ is decreasing. As $F \not\equiv 0$, we get that

$$M_1 = M = \left(\prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y.$$

As $\Psi \leq 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j, k} \pi_{1, j}^*(G_{\Omega_j}(\cdot, z_{j, k})) \right\}.$$

Denote

$$\begin{aligned}
& \inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\
& \quad \left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right\}
\end{aligned}$$

by $G(t)$, where $t \geq 0$. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right.$$

$$\left. \& (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right\}$$

by $\tilde{G}(t)$, where $t \geq 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geq 0$. Let $t \geq 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here \sim means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s) e^{-s} ds} \leq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Note that

$$G(0) = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}.$$

Combining Theorem 2.2, we obtain that $G(h^{-1}(r))$ is linear with respect to r , which implies that $G(h^{-1}(r))$ is linear with respect to r , where $h(t) = \int_t^{+\infty} c(s) e^{-s} ds$. As $f_{\alpha,\beta^*} \equiv 0$ for any $\alpha \neq \alpha_{\beta^*}$ satisfying $\sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} = 1$, where $\beta^* = (1, \dots, 1) \in \tilde{I}_1$, the linearity of $G(h^{-1}(r))$ contradicts to Theorem 1.7. Thus, we obtain that there exists a holomorphic $(n, 0)$ form \tilde{F} on Ω such that $(\tilde{F} - f, z) \in (\varphi + \psi)_z$ for any $z \in Z_0$ and

$$\begin{aligned} & \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\ & < \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned}$$

9. PROOFS OF THEOREM 1.18, REMARK 1.19, THEOREM 1.21 AND REMARK 1.22

In this section, we prove Theorem 1.18, Remark 1.19, Theorem 1.21 and Remark 1.22.

9.1. Proofs of Theorem 1.18 and Remark 1.19.

Let $f_1 = dw_1 \wedge \dots \wedge dw_{n_1} \wedge d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}$ on $V_0 \times U_0$, and let $f_2 = d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}$ on U_0 . Let $\psi = \max_{1 \leq j \leq n_1} \{\pi_{1,j}^*(2n_1 G_{\Omega_j}(\cdot, z_j))\}$. Following from Lemma 2.18, we get that $(H_1 - H_2, (z_0, y)) \in \mathcal{I}(\psi)_{(z_0, y)}$ for any $y \in Y$ if and only if $(H_1 - H_2)|_{\{z_0\} \times Y} = 0$, where H_1 and H_2 are holomorphic $(n, 0)$ form on a neighborhood of $\{z_0\} \times Y$. Let f be a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |f|^2 < +\infty$. It follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M such that $F|_{\{z_0\} \times Y} = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f)$ and

$$\int_M |F|^2 \leq \frac{(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |f|^2.$$

Note that

$$B_Y(y_0) = \frac{2^{n_2}}{\inf \left\{ \int_Y |f|^2 : f \in H^0(Y, \mathcal{O}(K_Y)) \& f(y_0) = f_2(y_0) \right\}}$$

and

$$B_M((z_0, y_0)) = \frac{2^n}{\inf \left\{ \int_M |F|^2 : F \in H^0(M, \mathcal{O}(K_M)) \text{ & } F((z_0, y_0)) = f_1((z_0, y_0)) \right\}}.$$

Thus, we have $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) \leq \pi^{n_1} B_M((z_0, y_0))$.

In the following, we prove the characterization of the holding of the equality $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} B_M((z_0, y_0))$.

There exists a holomorphic $(n_2, 0)$ form f_0 on Y such that $f_0(y_0) = f_2(y_0)$ and

$$B_Y(y_0) = \frac{2^{n_2}}{\int_Y |f_0|^2} > 0.$$

It follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F_0 on M such that $F_0 = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_0)$ and

$$\int_M |F_0|^2 \leq \frac{(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |f_0|^2. \quad (9.1)$$

Firstly, we prove the necessity. Note that $B_M((z_0, y_0)) \geq \frac{2^n}{\int_M |\tilde{F}|^2}$ for any holomorphic $(n, 0)$ form \tilde{F} on M satisfying that $\tilde{F} = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_0)$ on $\{z_0\} \times Y$. Combining $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} B_M((z_0, y_0))$, $B_Y(y_0) = \frac{2^{n_2}}{\int_Y |f_0|^2}$ and inequality (9.1), we obtain that $\frac{(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |f_0|^2 = \inf \left\{ \int_M |\tilde{F}|^2 : \tilde{F} \in H^0(M, \mathcal{O}(K_M)) \text{ & } \tilde{F}|_{\{z_0\} \times Y} = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_0) \right\}$. It follows from Theorem 1.9 that $\chi_{j, z_j} = 1$ for any $1 \leq j \leq n_1$. $\chi_{j, z_j} = 1$ implies that there exists a holomorphic function f_j on Ω_j such that $|f_j| = e^{G_{\Omega_j}(\cdot, z_j)}$, thus Ω_j is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero (see [51], see also [54] and [35]).

Now, we prove the sufficiency. As Ω_j is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero, we have $\chi_{j, z_j} = 1$. We prove $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} B_M((z_0, y_0))$ by contradiction: if not, there exists a holomorphic $(n, 0)$ form \tilde{F}_0 on M such that $\tilde{F}_0((z_0, y_0)) = f_1((z_0, y_0))$ and

$$\int_M |\tilde{F}_0|^2 < \frac{(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |f_0|^2. \quad (9.2)$$

There exists a holomorphic $(n_2, 0)$ form \tilde{f}_0 on Y such that $\tilde{F}_0 = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(\tilde{f}_0)$ on $\{z_0\} \times Y$. Hence $\tilde{f}_0(y_0) = f_2(y_0) = f_0(y_0)$, which implies that $\int_Y |\tilde{f}_0|^2 \geq \int_Y |f_0|^2$. Combining inequality (9.2), we have $\inf \left\{ \int_M |\tilde{F}|^2 : \tilde{F} \in H^0(M, \mathcal{O}(K_M)) \text{ & } \tilde{F}|_{\{z_0\} \times Y} = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(\tilde{f}_0) \right\} < \frac{(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |\tilde{f}_0|^2$, which contradicts to Theorem 1.9, hence $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} B_M((z_0, y_0))$.

Thus, Theorem 1.18 holds.

Note that $B_{M_1}((z_0, y_0)) \geq B_M((z_0, y_0)) > 0$ and $B_{M_1}((z_0, y_0)) = B_M((z_0, y_0))$ if and only if $M = M_1$, thus Theorem 1.18 shows Remark 1.19 holds.

9.2. Proofs of Theorem 1.21 and Remark 1.22.

Let $f_1 = dw_1 \wedge \dots \wedge dw_{n_1} \wedge d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}$ on $V_0 \times U_0$, and let $f_2 = d\tilde{w}_1 \wedge \dots \wedge d\tilde{w}_{n_2}$ on U_0 . Let $\psi = \max_{1 \leq j \leq n_1} \{\pi_{1,j}^*(2n_1 G_{\Omega_j}(\cdot, z_j))\}$. Following from Lemma 2.18, we get that $(H_1 - H_2, (z_0, y)) \in \mathcal{I}(\psi)_{(z_0, y)}$ for any $y \in Y$ if and only if $(H_1 - H_2)|_{\{z_0\} \times Y} = 0$, where H_1 and H_2 are holomorphic $(n, 0)$ form on a neighborhood of $\{z_0\} \times Y$. Let f be a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |f|^2 < +\infty$. It

follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M such that $F|_{\{z_0\} \times Y} = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f)$ and

$$\int_M |F|^2 \rho \leq \frac{(2\pi)^{n_1} \rho(z_0)}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |f|^2.$$

Note that

$$B_Y(y_0) = \frac{2^{n_2}}{\inf \left\{ \int_Y |f|^2 : f \in H^0(Y, \mathcal{O}(K_Y)) \& f(y_0) = f_2(y_0) \right\}}$$

and

$$B_{M, \rho}((z_0, y_0)) = \frac{2^n}{\inf \left\{ \int_M |F|^2 \rho : F \in H^0(M, \mathcal{O}(K_M)) \& F((z_0, y_0)) = f_1((z_0, y_0)) \right\}}.$$

Thus, we have $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) \leq \pi^{n_1} \rho(z_0) B_{M, \rho}((z_0, y_0))$.

In the following, we prove the characterization of the holding of the equality $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} \rho(z_0) B_{M, \rho}((z_0, y_0))$.

There exists a holomorphic $(n_2, 0)$ form f_0 on Y such that $f_0(y_0) = f_2(y_0)$ and

$$B_Y(y_0) = \frac{2^{n_2}}{\int_Y |f_0|^2} > 0.$$

It follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F_0 on M such that $F_0 = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_0)$ and

$$\int_M |F_0|^2 \rho \leq \frac{(2\pi)^{n_1} \rho(z_0)}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |f_0|^2. \quad (9.3)$$

Firstly, we prove the necessity. Note that $B_{M, \rho}((z_0, y_0)) \geq \frac{2^n}{\int_M |\tilde{F}|^2 \rho}$ for any holomorphic $(n, 0)$ form \tilde{F} on M satisfying that $\tilde{F} = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_0)$ on $\{z_0\} \times Y$. Combining $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} \rho(z_0) B_{M, \rho}((z_0, y_0))$, $B_Y(y_0) = \frac{2^{n_2}}{\int_Y |f_0|^2}$ and inequality (9.3), we obtain that $\frac{(2\pi)^{n_1} \rho(z_0)}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |f_0|^2 = \inf \left\{ \int_M |\tilde{F}|^2 \rho : \tilde{F} \in H^0(M, \mathcal{O}(K_M)) \& \tilde{F}|_{\{z_0\} \times Y} = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_0) \right\}$. It follows from Theorem 1.9 that $\chi_{j, z_j} = \chi_{j, -u_j}$ for any $1 \leq j \leq n_1$.

Now, we prove $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} \rho(z_0) B_{M, \rho}((z_0, y_0))$ by contradiction: if not, there exists a holomorphic $(n, 0)$ form \tilde{F}_0 on M such that $\tilde{F}_0((z_0, y_0)) = f_1((z_0, y_0))$ and

$$\int_M |\tilde{F}_0|^2 \rho < \frac{(2\pi)^{n_1} \rho(z_0)}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |f_0|^2. \quad (9.4)$$

There exists a holomorphic $(n_2, 0)$ form \tilde{f}_0 on Y such that $\tilde{F}_0 = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(\tilde{f}_0)$ on $\{z_0\} \times Y$. Hence $\tilde{f}_0(y_0) = f_2(y_0) = f_0(y_0)$, which implies that $\int_Y |\tilde{f}_0|^2 \geq \int_Y |f_0|^2$. Combining inequality (9.4), we have $\inf \left\{ \int_M |\tilde{F}|^2 \rho : F \in H^0(M, \mathcal{O}(K_M)) \& \tilde{F}|_{\{z_0\} \times Y} = \pi_1^*(dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(\tilde{f}_0) \right\} < \frac{(2\pi)^{n_1} \rho(z_0)}{\prod_{1 \leq j \leq n_1} c_j(z_j)^2} \int_Y |\tilde{f}_0|^2$, which contradicts to Theorem 1.9, hence $\prod_{1 \leq j \leq n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} \rho(z_0) B_{M, \rho}((z_0, y_0))$.

Thus, Theorem 1.21 holds.

Note that $B_{M_1, \rho}((z_0, y_0)) \geq B_{M, \rho}((z_0, y_0)) > 0$ and $B_{M_1, \rho}((z_0, y_0)) = B_{M, \rho}((z_0, y_0))$ if and only if $M = M_1$, thus Theorem 1.21 shows Remark 1.22 holds.

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