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Kengo Hirachi
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The Bergman Kernel and Related Topics

Hayama Symposium on SCV XXIII, Kanagawa, Japan, July 2022

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Editors

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Preface

The present volume is the proceedings of the 23rd Hayama symposium on complex analysis of several variables which was held 2022 July 23–26 at RecTore Hayama and July 27–28 at The University of Tokyo, in Japan. Among various topics of several complex variables, which range from partial differential equations and complex differential geometry to algebraic geometry, the talks given in this workshop are mostly on the questions related to the Bergman kernel function. The reason for such a choice is to recognize that the Bergman kernel became 100 years old in 2022 and still a subject of active research. The activity has been even accelerated in this decade by the solution of Suita's conjecture, which was asked in 1972 as an extension of Riemann's mapping theorem in the context of the theory of conformal mappings. N. Suita asked it in the presence of S. Bergman and his good friend M. Schiffer, but passed away in 2004. In the workshop, major contributors of recent development gave talks and many of them wrote up papers. This volume may be regarded as a supporting evidence of an assertion that geometric potential theory remains strongly as the main character of several complex variables.

For the success of the conference, the organizers would like to thank all the participants and the supporters who attended the meeting in the particularly difficult period of COVID-19.

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October 2023	

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In Memory of Jean-Pierre Demailly (1957–2022)

Shijie Bao, Qi'an Guan, Zhitong Mi, and Zheng Yuan

Abstract In this article, we present characterizations of the concavity property of minimal L^2 integrals with negligible weights degenerating to linearity on the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces. As applications, we obtain characterizations of the holding of equality in optimal jets L^2 extension problem with negligible weights on the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces.

Keywords Plurisubharmonic function · Minimal L^2 integral · Optimal L^2 extension theorem

1 Introduction

Recall that the strong openness property of multiplier ideal sheaves [35], i.e. $\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \bigcup_{\epsilon>0} \mathcal{I}((1+\epsilon)\varphi)$ (conjectured by Demailly [11]) has been widely used and discussed in several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [4, 6–9, 17, 18, 35, 38, 40, 41, 51–53]), where multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2e^{-\varphi}$ is locally integrable (see e.g. [11–15, 39, 42, 43, 45–47, 49]), and φ is a plurisubharmonic function on a complex manifold M (see [10]).

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When $\mathcal{I}(\varphi) = \mathcal{O}$, the strong openness property is the openness property (conjectured by Demailly-Kollár [14]). Berndtsson [3] established an effectiveness result of the openness property, and obtained the openness property. Stimulated by Berndtsson's effectiveness result, and continuing the solution of the strong openness property [35], Guan-Zhou [37] established an effectiveness result of the strong openness property by considering the minimal L^2 integral on the pseudoconvex domain D.

Considering the minimal L^2 integrals on the sub-level sets of the weight φ, Guan [21] obtained a sharp version of Guan-Zhou's effectiveness result, and established a concavity property of the minimal L^2 integrals on the sublevel sets of the weight φ (with constant gain), which was applied to give a proof of Saitoh's conjecture for conjugate Hardy H^2 kernels [22], and the sufficient and necessary condition of the existence of decreasing equisingular approximations with analytic singularities for the multiplier ideal sheaves with weights $\log(|z_1|^{a_1} + \cdots + |z_n|^{a_n})$ [23].

For smooth gain, Guan [20] (see also [24]) obtained the concavity property on Stein manifolds (weakly pseudoconvex Kähler case was obtained by Guan-Mi [25]), which was applied by Guan-Yuan to give an optimal support function related to the strong openness property [33] and an effectiveness result of the strong openness property in L^p [29]. For Lebesgue measurable gain, Guan-Yuan [28] obtained the concavity property on Stein manifolds (weakly pseudoconvex Kähler case was obtained by Guan-Mi-Yuan [26]), which deduced a twisted L^p strong openness property [30].

Note that the linearity is a degenerate concavity. A natural problem was posed in [31]:

? Questions

How can one characterize the concavity property degenerating to linearity?

For open Riemann surfaces, Guan-Yuan gave an answer to Problem 1 for single points [28] (for the case of subharmonic weights, see Guan-Mi [24]), and gave an answer to Problem 1 for finite points [31]. For products of open Riemann surfaces, Guan-Yuan [32] gave an answer to Problem 1 for products of finite points.

For fibrations over open Riemann surfaces, Bao-Guan-Yuan [1] gave an answer to Problem 1 with negligible weights pulled back from the open Riemann surfaces. For fibrations over products of open Riemann surfaces, Bao-Guan-Yuan [2] gave an answer to Problem 1 with negligible weights vanishing identically.

In this article, for the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces, we give answers to Problem 1 with negligible weights on fibrations.

We would like to recall the definition of minimal L^2 integral as follows.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \le j \le n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = \left(\prod_{1\le j\le n_1} \Omega_j\right) \times Y$ be an $n-$dimensional complex manifold, where $n = n_1 + n_2$.

Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\le j\le n_1}\Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M.

Let Z_j be a (closed) analytic subset of Ω_j for any $j\in\{1,\ldots,n_1\}$, and denote that $Z_0 := \left(\prod_{1\le j\le n_1} Z_j\right)\times Y\subset M$.

Let $\psi<0$ be a plurisubharmonic function on M such that $\{\psi<-t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$ and $Z_0\subset\{\psi=-\infty\}$. Let φ_1 be a Lebesgue measurable function on $\left(\prod_{1\le j\le n_1}\Omega_j\right)$ such that $\pi_1^*(\varphi_1)+\psi$ is a plurisubharmonic function on M. Let φ_2 be a plurisubharmonic function on Y. Denote $\varphi:=\pi_1^*(\varphi_1)+\pi_1^*(\varphi_2)$. Let $\mathcal{F}_{(z,y)}\supset\mathcal{I}(\varphi+\psi)_{(z,y)}$ be an ideal of $\mathcal{O}_{(z,y)}$ for any $(z,y)\in Z_0$. Let f be a holomorphic $(n,0)$ form on a neighborhood U of Z_0. Let $c(t)$ be a positive Lebesgue measurable function on $(0,+\infty)$. Denote

$$
\begin{aligned}
\inf\Bigg\{\int_{\{\psi<-t\}}|\tilde{f}|^2e^{-\varphi}c(-\psi) :&\big(\tilde{f}-f,(z,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{F}\big)_{(z,y)}\\
&\text{for any }(z,y)\in Z_0,\\
&\&\ \tilde{f}\in H^0\big(\{\psi<-t\},\mathcal{O}(K_M)\big).\Bigg\}
\end{aligned}
$$

by $G(t;c,f,\varphi,\psi,\mathcal{F})$ for any $t\in[0,+\infty)$. Here $|f|^2:=(\sqrt{-1})^{n^2}f\wedge\bar{f}$ for any $(n,0)$ form f. We simply denote $G(t;c,f,\varphi,\psi,\mathcal{F})$ by $G(t)$ when there is no misunderstands and denote $G(t;c,\varphi,\psi,\mathcal{F})$ by $G(t;c)$, $G(t;\varphi)$, $G(t;\psi)$ and $G(t;\mathcal{F})$ when we focus on various choices of $c(t)$, φ, ψ and $\mathcal{F}$ respectively.

We generally assume that $c(t)$ is a positive function on $(0,+\infty)$ such that $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$, $c(t)e^{-t}$ is decreasing with respect to t on $(0,+\infty)$ and $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $M\backslash Z_0$ in this paper (when other assumption for $c(t)$ is used, we introduce it explicitly). Then $G\big(h^{-1}(r)\big)$ is concave with respect to $r\in[0,\int_0^{+\infty}c(t_1)e^{-t_1}dt_1]$ (see [26], see also Theorem 10), where $h(t)=\int_t^{+\infty}c(s)e^{-s}ds$ for any $t\in[0,+\infty)$.

1.1 Main Results

In this section, we present characterizations of the concavity property of minimal L^2 integrals with negligible weights degenerating to linearity on the fibrations over open Riemann surfaces and products of open Riemann surfaces.

1.1.1 Linearity of the Minimal L^2 Integrals on Fibrations over Open Riemann Surfaces

In this section, we present characterizations of the concavity property of minimal L^2 integrals degenerating to linearity on the fibrations over open Riemann surface.

To state our results, we firstly recall the following notations (see [16], see also [26, 28, 36]).

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω. A character χ on $\pi_1(\Omega)$ is a homomorphism from $\pi_1(\Omega)$ to $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ which takes values in the unit circle $\{z \in \mathbb{C}; |z| = 1\}$.

Let $P : \Delta \to \Omega$ be the universal covering from unit disc $\Delta \subset \mathbb{C}$ to Ω. We call the holomorphic function f (resp. holomorphic $(1, 0)$ form F) on Δ is a multiplicative function $\big($resp. multiplicative differential (Prym differential)$\big)$ if there is a character χ on $\pi_1(\Omega)$, such that $g^* f = \chi(g) f$ (resp. $g^* F = \chi(g) F$) for every $g \in \pi_1(\Omega)$ which naturally acts on the universal covering of Ω. Denote the set of such kinds of f (resp. F) by $\mathcal{O}^\chi(\Omega)$ $\big($resp. $\Gamma^\chi(\Omega)\big)$.

As P is a universal covering, then for any harmonic function h on Ω, there exists a character χ_h associated to h and a multiplicative function $f_h \in \mathcal{O}^{\chi_h}(\Omega)$, such that $|f_h| = P^* e^h$. And if $g \in \mathcal{O}(\Omega)$ and g has no zero points on Ω, then we have $\chi_h = \chi_{h+\log|g|}$.

For Green function $G_\Omega(\cdot, z_0)$, one can find a χ_{z_0} and a multiplicative function $f_{z_0} \in \mathcal{O}^{\chi_{z_0}}(\Omega)$, such that $|f_{z_0}| = P^* e^{G_\Omega(\cdot, z_0)}$ (see [50]).

Now we assume that $n_1 = 1$ and then $M = \left(\prod_{1 \le j \le n_1} \Omega_j\right) \times Y = \Omega \times Y$, where Y is an $(n-1)-$dimensional weakly pseudoconvex Kähler manifold. Denote $Z_\Omega = \{z_j : 1 \le j < \gamma\}$ be a subset of Ω of discrete points, where $\gamma > 1$ is a positive integer or $\gamma = +\infty$. Denote $Z_0 := Z_\Omega \times Y$. Denote $Z_j := \{z_j\} \times Y$ for any j.

Let ψ be a plurisubharmonic function on M such that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$ and $Z_0 \subset \{\psi = -\infty\}$. It follows from Siu's decomposition theorem that

$$dd^c\psi = \sum_{j\ge1} 2p_j[Z_j] + \sum_{i\ge1} \lambda_i[A_i] + R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $(n-1)-$dimensional analytic set, and where R is a closed positive current with the property that $dim E_c(R) < n-1$ for every $c > 0$, where $E_c(R) = \{x \in M : v(R, x) \ge c\}$ is the upperlevel sets of Lelong number. We assume that $p_j > 0$ for any $1 \le j < \gamma$.

Then $N := \psi - \pi_1^*\big(\sum\limits_{j\ge1} 2p_j G_\Omega(z, z_j)\big)$ is a plurisubharmonic function on M, where $\pi_1 : M \to \Omega$ be the natural projection. We assume that $N \le 0$ and $N|_{Z_j}$ is not identically $-\infty$ for any j.

Let φ_1 be a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on M. By Siu's decomposition theorem, we have

$$dd^c(\psi+\pi_1^*(\varphi))=\sum_{j\geq 1}2\tilde{q}_j[Z_j]+\sum_{i\geq 1}\tilde{\lambda}_i[\tilde{A}_i]+\tilde{R},$$

where $\tilde{q}_j\geq 0$ for any $1\leq j<\gamma$.

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function g on Ω such that $ord_{z_j}(g)=q_j:=[\tilde{q}_j]$ for any $z_j\in Z_\Omega$ and $g(z)\neq 0$ for any $z\notin Z_\Omega$, where $[q]$ equals to the integral part of the nonnegative real number q. Then we know that there exists a plurisubharmonic function $\tilde{\psi}_2\in Psh(M)$ such that

$$\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\log|g|)+\tilde{\psi}_2.$$

Let $\varphi_2\in Psh(Y)$. Denote $\pi_2:M\to Y$ be the natural projection and $\varphi:=\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)$.

For any $1\leq j<\gamma$, let $\tilde{z}_j$ be a local coordinate on a neighborhood V_j of z_j satisfying $\tilde{z}_j(z_0)=0$ and $V_j\cap V_k=\emptyset$, for any $j\neq k$. Denote $V_0:=\cup_{1\leq j<\gamma}V_j$. Let f be a holomorphic $(n,0)$ form on $V_0\times Y$ which is a neighborhood of Z_0. Denote $\mathcal{F}_{(z_j,y)}=\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Let $G(t)$ be the minimal L^2 integral on $\{\psi<-t\}$ with respect to φ, f, $\mathcal{F}$ and c for any $t\geq 0$.

Let $\gamma=m+1$ is an positive integer, i.e., $Z_\Omega=\{z_j:1\leq j<\gamma\}$ contains m points. We obtain the following characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity on the fibrations over open Riemann surfaces.

Theorem 1 *Assume that $G(0)\in(0,+\infty)$. Then $G\big(h^{-1}(r)\big)$ is linear with respect to $r\in(0,\int_0^{+\infty}c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1) $N\equiv 0$ and $\psi=\pi_1^*\left(2\sum\limits_{j=1}^m p_jG_\Omega(\cdot,z_j)\right)$;

(2) for any $j\in\{1,2,\ldots,m\}$, $f=\pi_1^*(a_j\tilde{z}_j^{k_j}d\tilde{z}_j)\wedge\pi_2^*(f_Y)+f_j$ on $V_j\times Y$, where $a_j\in\mathbb{C}\setminus\{0\}$ is a constant, k_j is a nonnegative integer, f_Y is a holomorphic $(n-1,0)$ form on Y such that $\int_Y|f_Y|^2e^{-\varphi_2}\in(0,+\infty)$, and $\big(f_j,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\big)_{(z_j,y)}\otimes\mathcal{I}(\varphi+\psi)_{(z_j,y)}$ for any $j\in\{1,2,\ldots,m\}$ and $y\in Y$;

(3) $\varphi_1+2\sum\limits_{j=1}^m p_jG_\Omega(\cdot,z_j)=2\log|g|+2\sum\limits_{j=1}^m G_\Omega(\cdot,z_j)+2u$, where g is a holomorphic function on Ω such that $ord_{z_j}(g)=k_j$ and u is a harmonic function on Ω;

(4) $\prod\limits_{j=1}^m\chi_{z_j}=\chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot,z_j)$ respectively;

(5) for any $j\in\{1,2,\ldots,m\}$,

$$\lim_{z\to z_j}\frac{a_j\tilde{z}_j^{k_j}d\tilde{z}_j}{gP_*\left(f_u\left(\prod\limits_{l=1}^m f_{z_l}\right)\left(\sum\limits_{l=1}^m p_l\frac{df_{z_l}}{f_{z_l}}\right)\right)}=c_0,\tag{1}$$

where $c_0 \in \mathbb{C} \setminus \{0\}$ *is a constant independent of* j, f_u *is a holomorphic function* Δ *such that* $|f_u| = P^*(e^u)$ *and* f_{z_l} *is a holomorphic function on* Δ *such that* $|f_{z_l}| = P^*\left(e^{G_\Omega(\cdot,z_l)}\right)$ *for any* $l \in \{1, \dots, m\}$.

Remark 1 When $N \equiv \pi_1^*(N_1)$, it follows from $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ (see Lemma 23) that Theorem 1 reduces to Theorem 1.4 in [1] (see also Theorem 11), where $N_1 \le 0$ is a subharmonic function on Ω and $N_1(z_j) > -\infty$ for any j.

Let $\gamma = +\infty$, i.e., $Z_\Omega = \{z_j : 1 \le j < \gamma\}$ is an infinite subset of Ω of discrete points. Assume that $2\sum\limits_{j\ge 1} p_j G_\Omega(\cdot, z_j) \not\equiv -\infty$. We present a necessary condition such that $G(h^{-1}(r))$ is linear as follows.

Proposition 1 *Assume that* $G(0) \in (0, +\infty)$. *If* $G(h^{-1}(r))$ *is linear with respect to* $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$, *then the following statements hold:*

(1) $N \equiv 0$ *and* $\psi = \pi_1^*\left(2\sum\limits_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j)\right)$;

(2) for any $j \in \mathbb{N}_+$, $f = \pi_1^*(a_j \tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(f_Y) + f_j$ *on* $V_j \times Y$, *where* $a_j \in \mathbb{C} \setminus \{0\}$ *is a constant,* k_j *is a nonnegative integer,* f_Y *is a holomorphic* $(n-1, 0)$ *form on* Y *such that* $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$, *and* $\big(f_j, (z_j, y)\big) \in \big(\mathcal{O}(K_M)\big)_{(z_j,y)} \otimes \mathcal{I}(\varphi+\psi)_{(z_j,y)}$ *for any* $j \in \mathbb{N}_+$ *and* $y \in Y$;

(3) $\varphi_1 + 2\sum\limits_{j=1}^{m} p_j G_\Omega(\cdot, z_j) = 2\log|g|$, *where* g *is a holomorphic function on* Ω *such that* $ord_{z_j}(g) = k_j + 1$ *for any* $j \in \mathbb{N}_+$;

(4) for any $j \in \mathbb{N}_+$,

$$\frac{p_j}{ord_{z_j} g} \lim_{z \to z_j} \frac{dg}{a_j \tilde{z}_j^{k_j} d\tilde{z}_j} = c_0, \tag{2}$$

where $c_0 \in \mathbb{C} \setminus \{0\}$ *is a constant independent of* j;

(5) $\sum\limits_{j \in \mathbb{N}_+} p_j < +\infty$.

Remark 2 When $N \equiv \pi_1^*(N_1)$, Proposition 1 is Proposition 1.6 in [1] (see also Theorem 4), where $N_1 \le 0$ is a subharmonic function on Ω and $N_1(z_j) > -\infty$ for any j.

Let $\tilde{M} \subset M$ be an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0 \subset \tilde{M}$. Let f be a holomorphic $(n, 0)$ form on a neighborhood $\tilde{U}_0$ of Z_0 in $\tilde{M}$.

Let ψ be a plurisubharmonic function on $\tilde{M}$ such that $\{\psi < -t\} \setminus Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$. It follows from Siu's decomposition theorem that

$$dd^c\psi = \sum_{j\ge 1} 2p_j[Z_j] + \sum_{i\ge 1} \lambda_i[A_i] + R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $(n-1)-$dimensional analytic set, and where R is a closed positive current with the property that $dimE_c(R) < n-1$ for every $c > 0$, where $E_c(R) = \{x \in \tilde{M} : v(R, x) \geq c\}$ is the upperlevel sets of Lelong number. We assume that $p_j > 0$ for any $1 \leq j < \gamma$.

Then $N := \psi - \pi_1^*(\sum_{j\geq 1} 2p_j G_\Omega(z, z_j))$ is a plurisubharmonic function on $\tilde{M}$. We assume that $N \leq 0$ and $N|_{Z_j}$ is not identically $-\infty$ for any j.

Let φ_1 be a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on $\tilde{M}$. By Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j\geq 1} 2\tilde{q}_j[Z_j] + \sum_{i\geq 1} \tilde{\lambda}_i[\tilde{A}_i] + \tilde{R},$$

where $\tilde{q}_j \geq 0$ for any $1 \leq j < \gamma$.

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function g on Ω such that $ord_{z_j}(g) = q_j := [\tilde{q}_j]$ for any $z_j \in Z_\Omega$ and $g(z) \neq 0$ for any $z \notin Z_\Omega$, where $[q]$ equals to the integral part of the nonnegative real number q. Then we know that there exists a plurisubharmonic function $\tilde{\psi}_2 \in Psh(\tilde{M})$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2.$$

Let $\varphi_2 \in Psh(Y)$. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Let $c(t)$ be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$, $c(t)e^{-t}$ is decreasing with respect to t on $(0, +\infty)$ and $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $\tilde{M}\backslash Z_0$.

Denote

$$\inf\Bigg\{\int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi}c(-\psi) : (\tilde{f} - f, (z, y)) \in \Big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)\Big)_{(z,y)}$$
$$\text{for any } (z, y) \in Z_0,$$
$$\& \ \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_{\tilde{M}})).\Bigg\}$$

by $\tilde{G}(t)$ for any $t \in [0, +\infty)$. Note that $\tilde{G}(t)$ is a minimal L^2 integrals on $\tilde{M}$, where $\tilde{M}$ is a submanifold of M.

We present a necessary condition such that $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ as follows.

Proposition 2 *Assume that $\tilde{G}(0) \in (0, +\infty)$. If $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$, then $\tilde{M} = M$.*

Remark 3 When $N = \pi_1^*(N_1)|_{\tilde{M}}$, Proposition 2 is Proposition 1.7 in [1], where $N_1 \leq 0$ is a subharmonic function on Ω and $N_1(z_j) > -\infty$ for any j.

1.1.2 Linearity of the Minimal L^2 Integrals on Fibrations over Products of Open Riemann Surfaces

In this section, we present characterizations of the concavity property of minimal L^2 integrals degenerating to linearity on the fibrations over products of open Riemann surfaces.

When $n_1 \geq 1$, $M = \left(\prod_{1\leq j\leq n_1}\Omega_j\right) \times Y$ is an $n-$dimensional complex manifold, where $n = n_1 + n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\leq j\leq n_1}\Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M. Denote $P_j : \Delta \to \Omega_j$ be the universal covering from unit disc Δ to Ω_j for $1 \leq j \leq n_1$.

Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, and denote that $Z_0 := \left(\prod_{1\leq j\leq n_1} Z_j\right) \times Y \subset M$. Let $N \leq 0$ be a plurisubharmonic function on M satisfying $N|_{Z_0} \not\equiv -\infty$. For any $j \in \{1, \ldots, n_1\}$, let φ_j be an upper semi-continuous function on Ω_j such that $\varphi_j(z) > -\infty$ for any $z \in Z_j$. Assume that $\sum_{1\leq j\leq n_1}\pi_{1,j}^*(\varphi_j) + N$ is a plurisubharmonic function on M. Let φ_Y be a plurisubharmonic function on Y, and denote that $\varphi := \sum_{1\leq j\leq n_1}\pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$.

Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$, $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $c(-\psi)$ has a positive lower bound on any compact subset of $M\backslash Z_0$. Let f be a holomorphic $(n, 0)$ form on a neighborhood of Z_0.

Assume that $Z_0 = \{z_0\} \times Y = \{(z_1, \ldots, z_{n_1})\} \times Y \subset M$. Denote

$$\hat{G} := \max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*\big(G_{\Omega_j}(\cdot, z_j)\big)\right\},$$

where p_j is positive real number for $1 \leq j \leq n_1$. Denote

$$\psi := \max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*\big(G_{\Omega_j}(\cdot, z_j)\big)\right\} + N.$$

We assume that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$. Denote $\mathcal{F}_{(z_0,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_0,y)}$ for any $(z_0, y) \in Z_0$. Let $G(t)$ be the minimal L^2 integral on $\{\psi < -t\}$ with respect to φ, f, $\mathcal{F}$ and c for any $t \geq 0$.

Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1\leq j\leq n_1} V_{z_j}$, and $w := (w_1, \ldots, w_{n_1})$ is a local coordinate on V_0 of $z_0 \in \prod_{1\leq j\leq n_1}\Omega_j$. Denote that $E := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_j} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geq 0}\right\}$. Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y \subset M$.

We present a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the case $Z_0 = \{z_0\} \times Y$.

Theorem 2 *Assume that $G(0) \in (0, +\infty)$. $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1) $N \equiv 0$ *and* $\psi = \max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}$;

(2) $f = \sum_{\alpha\in E} \pi_1^*\left(w^\alpha dw_1 \wedge \ldots \wedge dw_{n_1}\right) \wedge \pi_2^*(f_\alpha) + g_0$ *on* $V_0 \times Y$*, where* g_0 *is a holomorphic* $(n, 0)$ *form on* $V_0 \times Y$ *satisfying* $(g_0, z) \in \left(\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)\right)_z$ *for any point* $z \in Z_0$ *and* f_α *is a holomorphic* $(n_2, 0)$ *form on* Y *such that* $\sum_{\alpha\in E} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \in (0, +\infty)$*;*

(3) $\varphi_j = 2\log|g_j| + 2u_j$*, where* g_j *is a holomorphic function on* Ω_j *such that* $g_j(z_j) \neq 0$ *and* u_j *is a harmonic function on* Ω_j *for any* $1 \leq j \leq n_1$*;*

(4) $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$ *for any* $j \in \{1, 2, ..., n\}$ *and* $\alpha \in E$ *satisfying* $f_\alpha \not\equiv 0$*, where* $\chi_{j,-u_j}$ *be the character associated to* $-u_j$ *on* Ω_j *and* χ_{j,z_j} *be the character associated to* $G_{\Omega_j}(\cdot, z_j)$ *on* Ω_j*.*

Remark 4 When $N \equiv 0$ (φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \leq j \leq n_1$), it follows from $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ (see Lemma 24) that Theorem 2 can be referred to Theorem 1.2 in [2] (see also Theorem 14).

Let $Z_j = \{z_{j,1}, \ldots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \ldots, n_1\}$, where m_j is a positive integer. Denote $Z_0 := (\prod_{1\leq j\leq n_1} Z_j) \times Y$. Let $N \leq 0$ be a plurisubharmonic function on M satisfying $N|_{Z_0} \not\equiv -\infty$. Denote

$$\hat{G} := \max_{1\leq j\leq n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\leq k\leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

where $p_{j,k}$ is a positive real number. Denote

$$\psi := \max_{1\leq j\leq n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\leq k\leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\} + N.$$

We assume that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \left\{(\beta_1, \ldots, \beta_{n_1}) : 1 \leq \beta_j \leq m_j \text{ for any } j \in \{1, \ldots, n_1\}\right\}$, $V_\beta := \prod_{1\leq j\leq n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1\leq j\leq n_1} \Omega_j$ satisfying $w_\beta(z_\beta) = 0$.

Let $\beta^* = (1, \ldots, 1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \ldots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geq 0}^{n_1}$. Denote that $E' := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{1\leq j\leq n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. Let f be a holomorphic $(n, 0)$ form on $\cup_{\beta\in\tilde{I}_1} V_\beta \times Y$ satisfying $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*}}\right) + \sum_{\alpha\in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2, 0)$ forms on Y. Denote $\mathcal{F}_{(z,y)} = \mathcal{I}\left(\hat{G} + \pi_2^*(\varphi_Y)\right)_{(z,y)}$ for any point

$(z, y) \in Z_0$. Let $G(t; c)$ be the minimal L^2 integral on $\{\psi < -t\}$ with respect to φ, f and $\mathcal{F}$ for any $t \geq 0$.

We present a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the case that Z_j is a set of finite points.

Theorem 3 *Assume that $G(0) \in (0, +\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ if and only if the following statements hold:*

(1) *$N \equiv 0$ and $\psi = \max_{1\leq j\leq n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\leq k\leq m_j} p_{j,k}G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$;*

(2) *$\varphi_j = 2\log|g_j| + 2u_j$ for any $j \in \{1, \ldots, n_1\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k}) \neq 0$ for any $k \in \{1, \ldots, m_j\}$;*

(3) *There exists a nonnegative integer $\gamma_{j,k}$ for any $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, m_j\}$, which satisfies that $\prod_{1\leq k\leq m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ and $\sum_{1\leq j\leq n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ for any $\beta \in \tilde{I}_1$;*

(4) *$f = \pi_1^*\left(c_\beta\left(\prod_{1\leq j\leq n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}}\right)dw_{1,\beta_1}\wedge\ldots\wedge dw_{n,\beta_n}\right)\wedge\pi_2^*(f_0) + g_\beta$ on $V_\beta \times Y$ for any $\beta \in \tilde{I}_1$, where c_β is a constant, $f_0 \not\equiv 0$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$, and g_β is a holomorphic $(n, 0)$ form on $V_\beta \times Y$ such that $(g_\beta, z) \in \big(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi)\big)_z$ for any $z \in \{z_\beta\}\times Y$;*

(4) *$c_\beta\prod_{1\leq j\leq n_1}\left(\lim_{z\to z_{j,\beta_j}}\frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}}dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\leq k\leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\leq k\leq m_j} p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)}\right) = c_0$ for any $\beta \in \tilde{I}_1$, where $c_0 \in \mathbb{C}\backslash\{0\}$ is a constant independent of β, f_{u_j} is a holomorphic function Δ such that $|f_{u_j}| = P_j^*(e^{u_j})$ and $f_{z_{j,k}}$ is a holomorphic function on Δ such that $|f_{z_{j,k}}| = P_j^*\left(e^{G_{\Omega_j}(\cdot,z_{j,k})}\right)$ for any $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, m_j\}$.*

Remark 5 When $N \equiv 0$ (φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_{j,\beta_j}) > -\infty$ for any $1 \leq j \leq n_1$ and $1 \leq \beta_j \leq m_j$), it follows from $\mathcal{I}(\varphi+\psi)_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$ for any $(z, y) \in Z_0$ (see Lemma 24) that Theorem 3 can be referred to Theorem 1.5 in [2] (see also Theorem 15).

Let $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2}\cup\{+\infty\}$. Denote $Z_0 := (\prod_{1\leq j\leq n_1} Z_j)\times Y$. Let $p_{j,k}$ be a positive number for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$ such that $\sum_{1\leq k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j. Let

$$\hat{G} = \max_{1\leq j\leq n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\leq k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot, z_{j,k})\right)\right\},$$

and

$$\psi = \max_{1\leq j\leq n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\leq k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot, z_{j,k})\right)\right\} + N.$$

We assume that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$ and $\limsup_{t\to+\infty} c(t) < +\infty$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1,\ldots,n_1\}$ and $1 \le k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \ne k'$. Denote that $\tilde{I}_1 := \{(\beta_1,\ldots,\beta_{n_1}) : 1 \le \beta_j < \tilde{m}_j$ for any $j \in \{1,\ldots,n_1\}\}$, $V_\beta := \prod_{1\le j\le n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1,\ldots,\beta_{n_1}) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}}) \in \prod_{1\le j\le n_1}\Omega_j$.

Let $\beta^* = (1,\ldots,1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1}) \in \mathbb{Z}^{n_1}_{\ge 0}$. Denote that $E' := \left\{\alpha \in \mathbb{Z}^{n_1}_{\ge 0} : \sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,1}} > \sum_{1\le j\le n_1}\frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. Let f be a holomorphic $(n,0)$ form on $\cup_{\beta\in\tilde{I}_1} V_\beta \times Y$ satisfying $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1}\wedge\ldots\wedge dw_{n_1,1}\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*}}\right) + \sum_{\alpha\in E'}\pi_1^*(w^\alpha dw_{1,1}\wedge\ldots\wedge dw_{n_1,1})\wedge\pi_2^*(f_\alpha)$ on $V_{\beta^*}\times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2,0)$ forms on Y. Denote $\mathcal{F}_{(z,y)} = \mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_Y)\big)_{(z,y)}$ for any $(z,y) \in Z_0$. Let $G(t)$ be the minimal L^2 integral on $\{\psi < -t\}$ with respect to φ, f, $\mathcal{F}$ and c for any $t \ge 0$.

When there exists $j_0 \in \{1,\ldots,n_1\}$ such that $\tilde{m}_{j_0} = +\infty$, we present that $G\big(h^{-1}(r)\big)$ is not linear.

Theorem 4 *If $G(0) \in (0,+\infty)$ and there exists $j_0 \in \{1,\ldots,n_1\}$ such that $\tilde{m}_{j_0} = +\infty$, then $G\big(h^{-1}(r)\big)$ is not linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$.*

Remark 6 When $N \equiv 0$ (φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_{j,\beta_j}) > -\infty$ for any $1 \le j \le n_1$ and $1 \le \beta_j \le m_j$), it follows from $\mathcal{I}(\varphi+\psi)_{(z,y)} = \mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))_{(z,y)}$ for any $(z,y) \in Z_0$ (see Lemma 24) that Theorem 4 can be referred to Theorem 1.7 in [2] (see also Theorem 16).

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1,\ldots,n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2}\cup\{+\infty\}$. Denote $Z_0 := (\prod_{1\le j\le n_1} Z_j)\times Y$.

Let $\tilde{M} \subset M$ be an $n-$dimensional weakly pseudoconvex Kähler manifold satisfying that $Z_0 \subset \tilde{M}$. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0.

Let $N \le 0$ be a plurisubharmonic function on $\tilde{M}$ satisfying $N|_{Z_0} \not\equiv -\infty$. For any $j \in \{1,\ldots,n_1\}$, let φ_j be an upper semi-continuous function on Ω_j such that $\varphi_j(z) > -\infty$ for any $z \in Z_j$. Assume that $\sum_{1\le j\le n_1}\pi_{1,j}^*(\varphi_j) + N$ is a plurisubharmonic function on $\tilde{M}$. Let φ_Y be a plurisubharmonic function on Y, and denote that $\varphi := \sum_{1\le j\le n_1}\pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$.

Let $p_{j,k}$ be a positive number for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$ such that $\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j. Denote

$$\hat{G} = \max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\},$$

and

$$\psi = \max_{1 \le j \le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1 \le k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\} + N.$$

We assume that $\{\psi < -t\} \backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$.

Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$, $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $c(-\psi)$ has a positive lower bound on any compact subset of $\tilde{M} \backslash Z_0$. Let f be a holomorphic $(n, 0)$ form on a neighborhood of Z_0.

Denote

$$\begin{aligned} \inf \Bigg\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : &\big(\tilde{f} - f, (z, y)\big) \in \Big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big) \Big)_{(z,y)} \\ &\text{for any } (z, y) \in Z_0, \\ &\& \ \tilde{f} \in H^0\big(\{\psi < -t\}, \mathcal{O}(K_{\tilde{M}})\big). \Bigg\} \end{aligned}$$

by $\tilde{G}(t)$ for any $t \in [0, +\infty)$. Note that $\tilde{G}(t)$ is a minimal L^2 integrals on $\tilde{M}$, where $\tilde{M}$ is a submanifold of M.

We present a necessary condition such that $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ as follows.

Proposition 3 *If $\tilde{G}(0) \in (0, +\infty)$ and $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$, we have $\tilde{M} = M$.*

Remark 7 When $N \equiv 0$ (φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_{j,\beta_j}) > -\infty$ for any $1 \le j \le n_1$ and $1 \le \beta_j \le m_j$), it follows from $\mathcal{I}(\varphi + \psi)_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$ for any $(z, y) \in Z_0$ (see Lemma 24) that Proposition 3can be referred to Proposition 1.8 in [2].

1.2 Applications

In this section, we present characterizations of the holding of equality in optimal jets L^2 extension problem with the negligible weights on fibrations.

1.2.1 Background: Equality in Optimal Jets L^2 Extension Problem

Let Ω be an open Riemann surface with a nontrivial Green function G_Ω. Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let $c_\beta(z)$ be the logarithmic capacity (see [?]) on Ω, i.e.

$$c_\beta(z_0) := \exp \lim_{\xi \to z_0} G_\Omega(z, z_0) - \log|w(z)|.$$

Let $B_\Omega(z_0)$ be the Bergman kernel function on Ω. An open question was posed by Sario-Oikawa [?]: find a relation between the magnitudes of the quantities $\sqrt{\pi B_\Omega(z)}$, $c_\beta(z)$.

In [48], Suita conjectured: $\pi B_\Omega(z_0) \geq \left(c_\beta(z_0)\right)^2$ holds, and the equality holds if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

The inequality part of Suita conjecture for bounded planar domains was proved by Błocki [5], and original form of the inequality part was proved by Guan-Zhou [34]. The equality part of Suita conjecture was proved by Guan-Zhou [36], then Suita conjecture was completed proved.

It follows from the extremal property of the Bergman kernel function that the holding of the following two equalities are equivalent

(1) $\pi B_\Omega(z_0) = (c_\beta(z_0))^2$;

(2) $\inf\{\int_\Omega |F|^2 : F$ is a holomorphic $(1, 0)$ form on Ω such that $F(z_0) = dw\} = \frac{2\pi}{(c_\beta(z_0))^2}$.

Note that (2) is equivalent to the holding of the equality in optimal 0-jet L^2 extension problem for open Riemann surface Ω and single point z_0 with trivial weights $\varphi \equiv 0$ and trivial gain $c \equiv 1$. Then it is natural to ask

? Questions

How can one characterize the holding of the equality in optimal k-jets L^2 extension problem, where k is a nonnegative integer?

For open Riemann surfaces and single points, when the weights are harmonic and gain is constant, Guan-Zhou [36] gave an answer to 0-jet version of Problem 1.2.1, i.e. a proof of the extended Suita conjecture posed by Yamada [50].

For open Riemann surfaces and single points, Guan-Yuan [28] gave an answer to 0-jet version of Problem 1.2.1 (when the weights are subharmonic and gain is smooth, Guan-Mi [24] gave an answer to 0-jet version of Problem 1.2.1), and Guan-Mi-Yuan [26] gave an answer to Problem 1.2.1.

For open Riemann surfaces and analytic sets (with finite or infinite points), Guan-Yuan [31] gave an answer to Problem 1.2.1. For products of open Riemann surfaces and products of analytic sets, Guan-Yuan [32] gave an answer to Problem 1.2.1.

For fibrations over open Riemann surfaces, Bao-Guan-Yuan [1] gave an answer to Problem 1.2.1 with negligible weights pulled back from the open Riemann surfaces. For fibrations over products of open Riemann surfaces, Bao-Guan-Yuan [2] gave an answer to Problem 1.2.1 with negligible weights pulled back from the products of open Riemann surfaces.

In the following sections, for fibrations over open Riemann surfaces and fibrations over products of open Riemann surfaces, we give answers to Problem 1.2.1 with negligible weights on fibrations.

1.2.2 Fibrations over Open Riemann Surfaces

In this section, we give characterizations of the holding of equality in optimal jets L^2 extension problem with negligible weights from fibers over analytic subsets to fibrations over open Riemann surfaces.

Let Ω be an open Riemann surface with nontrivial Green functions. Let Y be an $(n-1)-$dimensional weakly pseudoconvex Kähler manifold. Denote $M = \Omega \times Y$. Let K_M be the canonical line bundle on M. Let π_1 and π_2 be the natural projections from M to Ω and Y respectively.

Let $Z_\Omega = \{z_j : j \in \mathbb{N}_+ \& 1 \leq j < \gamma\}$ be a subset of Ω of discrete points. Denote $Z_0 := Z_\Omega \times Y$. Denote $Z_j := \{z_j\} \times Y$ for any $j \geq 1$.

Assume that $\tilde{M} \subset M$ is an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0 \subset \tilde{M}$.

Let ψ be a plurisubharmonic function on $\tilde{M}$ such that $\{\psi < -t\} \backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$ and $Z_0 \subset \{\psi = -\infty\}$. It follows from Siu's decomposition theorem that

$$dd^c\psi = \sum_{j\geq 1} 2p_j[Z_j] + \sum_{i\geq 1} \lambda_i[A_i] + R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $n-$dimensional analytic set, and where R is a closed positive current with the property that $dim E_c(R) < n$ for every $c > 0$, where $E_c(R) = \{x \in M : v(R, x) \geq c\}$ is the upper-level sets of Lelong number. We assume that $p_j \geq 0$ is a positive number for any $1 \leq j < \gamma$ and $2\sum_{j\geq 1} p_j G_\Omega(\cdot, z_j) \not\equiv -\infty$.

Then $N := \psi - \pi_1^*(\sum_{j\geq 1} 2p_j G_\Omega(z, z_j))$ is a plurisubharmonic function on $\tilde{M}$. We assume that $N \leq 0$.

Let k_j be a nonnegative integer for any $1 \leq j < \gamma$. Let φ_1 be a Lebesgue measurable function on Ω such that $\pi_1^*(\varphi_1) + \psi$ is a plurisubharmonic function on $\tilde{M}$. We also assume that there exists a holomorphic function $g \in O(\Omega)$ and a plurisubharmonic function $\tilde{\psi}_2 \in Psh(\tilde{M})$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2,$$

where $ord_{z_j}(g) = k_j + 1$, for any $1 \leq j < \gamma$.

Let φ_2 be a plurisubharmonic funciton on Y. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_1^*(\varphi_2)$.

For any $1 \le j < \gamma$, let $\tilde{z}_j$ be a local coordinate on a neighborhood V_j of z_j satisfying $\tilde{z}_j(z_j) = 0$ and $V_j \cap V_k = \emptyset$, for any $j \ne k$. We assume that $g = d_j \tilde{z}_j^{k_j+1} h_j(\tilde{z}_j)$ on each V_j, where $h_j(z_j) = 1$. Denote $V_0 := \cup_{1 \le j < \gamma} V_j$.

Let $\gamma = m + 1$ be a positive integer. We give an application of Theorem 1 as below.

Theorem 5 *Let ψ, φ_1 and φ_2 be as above. Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n, 0)$ form on Y. Assume that*

$$\sum_{j=1}^{m} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} \in (0, +\infty).$$

Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying that $f = \pi_1^*(a_j \tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(F_j)$ on $V_j \times Y$. Then there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^{m} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (3)$$

Moreover, equality $\left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^{m} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} = \inf\Big\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F}$ is a holomorphic $(n, 0)$ form on $\tilde{M}$ such that $(\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))_{(z_j,y)}$ for any $(z_j, y) \in Z_0\Big\}$ holds if and only if the following statements hold:

(1) $N \equiv 0$ and $\psi = \pi_1^*(2 \sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j))$;

(2) $\varphi_1 + 2 \sum_{j=1}^{m} k_j G_\Omega(\cdot, z_j) = 2\log|g| + 2 \sum_{j=1}^{m} (k_j + 1) G_\Omega(\cdot, z_j) + 2u$, where g is a holomorphic function on Ω such that $g(z_j) \ne 0$ for any $j \in \{1, 2, \ldots, m\}$ and u is a harmonic function on Ω;

(3) $\prod_{j=1}^{m} \chi_{z_j}^{k_j+1} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_j)$ respectively;

(4) for any $j \in \{1, 2, \ldots, m\}$,

$$\lim_{z \to z_j} \frac{a_j \tilde{z}_j^{k_j} d\tilde{z}_j}{g P_* \left(f_u \left(\prod_{l=1}^{m} f_{z_l}^{k_l+1} \right) \left(\sum_{l=1}^{m} p_l \frac{d f_{z_l}}{f_{z_l}} \right) \right)} = c_j \in \mathbb{C} \setminus \{0\}, \quad (4)$$

and there exist $c_0 \in \mathbb{C} \setminus \{0\}$ and a holomorphic $(n-1,0)$ form F_Y on Y which are independent of j such that $c_0 F_Y = c_j F_j$ for any $j \in \{1, 2, \ldots, m\}$;

(5) $\tilde{M} = M$.

Remark 8 When $N \equiv \pi_1^*(N_1)$, it follows from $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))_{(z_j,y)}$ (see Lemma 23) that Theorem 5 can be referred to Theorem 1.8 in [1] (see also Theorem 12), where $N_1 \le 0$ is a subharmonic function on Ω.

When $\gamma = +\infty$, i.e., $Z_\Omega = \{z_j : 1 \le j < \gamma\}$ is an infinite subset of Ω of discrete points. Let k_j be a nonnegative integer for any $j \in \mathbb{N}^+$. Assume that $p_j = k_j + 1$, i.e., we have $\psi = \pi_1^*\big(\sum_{j=1}^{+\infty} 2(k_j+1)G_\Omega(z, z_j)\big) + N$, where $N \le 0$ is a plurisubharmonic function on $\tilde{M}$.

We give an L^2 extension result from fibers over analytic subsets to fibrations over open Riemann surfaces, where the analytic subsets are infinite subsets of discrete points on open Riemann surfaces.

Theorem 6 *Let ψ, φ_1 and φ_2 be as above. Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n, 0)$ form on Y. Assume that*

$$\sum_{j=1}^{+\infty} \frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} \in (0, +\infty).$$

Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying that $f = \pi_1^*(a_j \tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(F_j)$ on $V_{z_j} \times Y$. Then there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ such that $\big(F - f, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) < \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{+\infty} \frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \tag{5}$$

Remark 9 When $N \equiv \pi_1^*(N_1)$, it follows from $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))_{(z_j,y)}$ (see Lemma 23) that Theorem 6 can be referred to Theorem 1.10 in [1] (see also Theorem 13), where $N_1 \le 0$ is a subharmonic function on Ω.

1.2.3 Fibrations over Products of Open Riemann Surfaces

In this section, we present characterizations of the holding of equality in optimal jets L^2 extension problem with negligible weights from fibers over analytic subsets to fibrations over products of open Riemann surfaces.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \le j \le n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = \left(\prod_{1\le j\le n_1} \Omega_j\right) \times Y$ be an $n-$dimensional complex manifold, where $n = n_1 + n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\le j\le n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M. Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \dots, n_1\}$, and denote that $Z_0 := \left(\prod_{1\le j\le n_1} Z_j\right) \times Y$. Let $\tilde{M} \subset M$ be an $n-$dimensional complex manifold satisfying that $Z_0 \subset \tilde{M}$, and let $K_{\tilde{M}}$ be the canonical (holomorphic) line bundle on $\tilde{M}$.

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \dots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \ne k'$. Denote that $\tilde{I}_1 := \big\{(\beta_1, \dots, \beta_{n_1}) : 1 \le \beta_j < \tilde{m}_j$ for any $j \in \{1, \dots, n_1\}\big\}$, $V_\beta := \prod_{1\le j\le n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1\le j\le n_1} \Omega_j$ for any $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$. Then $Z_0 = \{(z_\beta, y) : \beta \in \tilde{I}_1 \,\&\, y \in Y\} \subset \tilde{M}$.

Let $N \le 0$ be a plrusubharmonic function on $\tilde{M}$ and let φ_j be a Lebesgue measurable function on Ω_j such that $N + \sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j)$ is a plurisubharmonic function on $\tilde{M}$ satisfying $\left(N + \sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j)\right)|_{Z_0} \not\equiv -\infty$. Let φ_Y be a plurisubharmonic function on Y.

Let $p_{j,k}$ be a positive number for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, which satisfies that $\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \le j \le n_1$. Denote

$$\hat{G} := \max_{1\le j\le n_1} \left\{ 2 \sum_{1\le k<\tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\},$$

and denote that

$$\psi := \max_{1\le j\le n_1} \left\{ 2 \sum_{1\le k<\tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} + N.$$

We assume that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$. Let $\varphi := \sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on $\tilde{M}$.

Denote that $E_\beta := \left\{(\alpha_1, \dots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ and $\tilde{E}_\beta := \left\{(\alpha_1, \dots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \ge 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ for any $\beta \in \tilde{I}_1$.

Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta}) \tag{6}$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$.

Denote that

$$c_{j,k} := \exp \lim_{z \to z_{j,k}} \left(\frac{\sum_{1 \le k_1 < \tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \ldots, n\}$ and $1 \le k < \tilde{m}_j$ (following from Lemma 3 and Lemma 4, we get that the above limit exists).

Let $c_j(z)$ be the logarithmic capacity (see [?]) on Ω_j, which is locally defined by

$$c_j(z_j) := \exp \lim_{z \to z_j} (G_{\Omega_j}(z, z_j) - \log |w_j(z)|).$$

For the case $Z_0 = \{z_0\} \times Y \subset \tilde{M}$, where $z_0 = (z_1, \ldots, z_{n_1}) \in \prod_{1 \le j \le n_1} \Omega_j$, we denote that $E := \left\{ (\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1 \le j \le n_1} \frac{\alpha_j + 1}{p_j} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0} \right\}$. Let f_α be the holomorphic $(n_2, 0)$ form on Y which comes from formula (6). We obtain a characterization of the holding of equality in optimal jets L^2 extension problem.

Theorem 7 *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that*

$$\sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - \left(N + \pi_{1,j}^*(\sum_{1 \le j \le n_1} \varphi_j)\right)(z_0, w)}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \in (0, +\infty).$$

Then there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $(F - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \le j \le n_1} \left\{ 2 p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j)) \right\} \right) \right)_z$ for any $z \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - \left(N + \pi_{1,j}^*(\sum_{1 \le j \le n_1} \varphi_j)\right)(z_o, w)}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}.$$

Moreover, equality $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \,\&\, (\tilde{F} - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \le j \le n_1} \left\{ 2 p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j)) \right\} \right) \right)_z \right.$ *for any* $\left. z \in Z_0 \right\} = \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \times \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - \left(N + \pi_{1,j}^*(\sum_{1 \le j \le n_1} \varphi_j)\right)(z_0, w)}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}$ *holds if and only if the following statements hold:*

(1) $\tilde{M} = \left(\prod_{1 \le j \le n_1} \Omega_j \right) \times Y$ *and* $N \equiv 0$*;*

(2) $\varphi_j = 2\log|g_j| + 2u_j$, *where* g_j *is a holomorphic function on* Ω_j *such that* $g_j(z_j) \neq 0$ *and* u_j *is a harmonic function on* Ω_j *for any* $1 \le j \le n_1$*;*

(3) $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$ *for any* $j \in \{1, 2, ..., n\}$ *and* $\alpha \in E$ *satisfying* $f_\alpha \not\equiv 0$.

Remark 10 If $(f_\alpha, y) \in \big(\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y)\big)_y$ for any $y \in Y$ and $\alpha \in \tilde{E}\backslash E$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*\big(G_{\Omega_j}(\cdot, z_j)\big)\right\}\right)$ by $\mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_2)\big)$. We prove the remark in Sect. 7.

Let $Z_j = \{z_{j,1}, \ldots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \ldots, n_1\}$, where m_j is a positive integer. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$. Let $\beta^* = (1, \ldots, 1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \ldots, \alpha_{\beta^*,n_1}) \in E_{\beta^*}$. Denote that $E' := \left\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,1}} > 1\right\}$. Assume that $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right) + \sum_{\alpha\in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha,\beta})$ on $U_0 \cap (V_{\beta^*} \times Y)$.

We obtain a characterization of the holding of equality in optimal jets L^2 extension problem for the case that Z_j is finite.

Theorem 8 *Let c be a positive function on* $(0, +\infty)$ *such that* $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ *and* $c(t)e^{-t}$ *is decreasing on* $(0, +\infty)$. *Assume that*

$$\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \left(N + \pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta, w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \in (0, +\infty).$$

Then there exists a holomorphic $(n, 0)$ *form* F *on* $\tilde{M}$ *satisfying that* $(F - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\big(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\big)\right)_z$ *for any* $z \in Z_0$ *and*

$$\begin{aligned}
&\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\
&\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \left(N + \pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta, w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}$$

Moreover, equality $\inf\Big\{\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}}))\ \&\ (\tilde{F} - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\big(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\big)\right)_z$ *for any*

$z \in Z_0 \Big\} = \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$

holds if and only if the following statements hold:

(1) $\tilde{M} = \left(\prod_{1\le j\le n_1}\Omega_j\right) \times Y$ *and* $N \equiv 0$*;*

(2) $\varphi_j = 2\log|g_j| + 2u_j$ *for any* $j \in \{1,\ldots,n_1\}$*, where* u_j *is a harmonic function on* Ω_j *and* g_j *is a holomorphic function on* Ω_j *satisfying* $g_j(z_{j,k}) \ne 0$ *for any* $k \in \{1,\ldots,m_j\}$*;*

(3) *There exists a nonnegative integer* $\gamma_{j,k}$ *for any* $j \in \{1,\ldots,n_1\}$ *and* $k \in \{1,\ldots,m_j\}$*, which satisfies that* $\prod_{1\le k\le m_j}\chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ *and* $\sum_{1\le j\le n_1}\frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ *for any* $\beta \in \tilde{I}_1$*;*

(4) $f_{\alpha,\beta} = c_\beta f_0$ *holds for* $\alpha = (\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})$ *and* $f_{\alpha,\beta} \equiv 0$ *holds for any* $\alpha \in E_\beta\backslash\{(\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})\}$*, where* $\beta \in \tilde{I}_1$*,* c_β *is a constant and* $f_0 \not\equiv 0$ *is a holomorphic* $(n_2, 0)$ *form on* Y *satisfying* $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$*;*

(5) $c_\beta \prod_{1\le j\le n_1}\left(\lim_{z\to z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\le k\le m_j} f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\le k\le m_j} p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)}\right) = c_0$ *for any* $\beta \in \tilde{I}_1$*, where* $c_0 \in \mathbb{C}\backslash\{0\}$ *is a constant independent of* β*,* f_{u_j} *is a holomorphic function* Δ *such that* $|f_{u_j}| = P_j^*(e^{u_j})$ *and* $f_{z_{j,k}}$ *is a holomorphic function on* Δ *such that* $|f_{z_{j,k}}| = P_j^*\left(e^{G_{\Omega_j}(\cdot,z_{j,k})}\right)$ *for any* $j \in \{1,\ldots,n_1\}$ *and* $k \in \{1,\ldots,m_j\}$*.*

Remark 11 If $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ holds for any $y \in Y, \alpha \in \tilde{E}_\beta\backslash E_\beta$ and $\beta \in \tilde{I}_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)$ by $\mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_2)\big)$. We prove the remark in Section 7.

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1,\ldots,n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$. Let $\beta^* = (1,\ldots,1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1}) \in E_{\beta^*}$. Denote that $E' := \left\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,1}} > 1\right\}$. Assume that $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right) + \sum_{\alpha\in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha,\beta})$ on $U_0 \cap (V_{\beta^*} \times Y)$.

When there exists $j_0 \in \{1,\ldots,n_1\}$ such that $\tilde{m}_{j_0} = +\infty$, we obtain that the equality in optimal jets L^2 extension problem *could not hold.*

Theorem 9 *Let* c *be a positive function on* $(0, +\infty)$ *such that* $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ *and* $c(t)e^{-t}$ *is decreasing on* $(0, +\infty)$*. Assume that*

$$\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\in(0,+\infty)$$

and there exists $j_0\in\{1,\dots,n_1\}$ *such that* $\tilde{m}_{j_0}=+\infty$.

Then there exists a holomorphic $(n,0)$ *form* F *on* $\tilde{M}$ *satisfying that* $(F-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ *for any* $z\in Z_0$ *and*

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$<\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Remark 12 If $(f_{\alpha,\beta},y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ holds for any $y\in Y$, $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)$ by $\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_2)\big)$. We prove the remark in Section 7.

Remark 13 We note that, by using our recent progress (see [27]) on minimal L^2 integrals related to boundary points, the main results in the present article hold without assuming the condition "$\{\psi<-t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$".

2 Preparations I: Minimal L^2 Integrals

In this section, we recall and present some lemmas related to minimal L^2 integrals.

2.1 *Minimal L^2 Integrals on Weakly Pseudoconvex Kähler Manifolds*

In this section, we recall some results about the concavity property of minimal L^2 integrals on weakly pseudoconvex Kähler manifolds in [26].

Let M be an $n-$dimensional complex manifold. Let X and Z be closed subsets of M. A triple (M,X,Z) satisfies condition (A), if the following statements hold:

(1) X is a closed subset of M and X is locally negligible with respect to L^2 holomorphic functions, i.e., for any coordinated neighborhood $U\subset M$ and for any L^2 holomorphic function f on $U\setminus X$, there exists an L^2 holomorphic function $\tilde{f}$ on U such that $\tilde{f}|_{U\setminus X}=f$ with the same L^2 norm;

(2) Z is an analytic subset of M and $M \setminus (X \cup Z)$ is a weakly pseudoconvex Kähler manifold.

Let (M, X, Z) be a triple satisfying condition (A). Let K_M be the canonical line bundle on M. Let ψ be a plurisubharmonic function on M such that $\{\psi < -t\}\setminus(X \cup Z)$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$. Let φ be a Lebesgue measurable function on M such that $\varphi + \psi$ is a plurisubharmonic function on M. Denote $T = -\sup_M \psi$.

We recall the concept of "gain" in [26]. A positive measurable function c on $(T, +\infty)$ is in the class $\mathcal{P}_{T,M}$ if the following two statements hold:

(1) $c(t)e^{-t}$ is decreasing with respect to t;

(2) there is a closed subset E of M such that $E \subset Z \cap \{\psi = -\infty\}$ and for any compact subset $K \subset M \setminus E$, $e^{-\varphi}c(-\psi)$ has a positive lower bound on K.

Let Z_0 be a subset of M such that $Z_0 \cap \mathrm{Supp}(\mathcal{O}/\mathcal{I}(\varphi + \psi)) \neq \emptyset$. Let $U \supset Z_0$ be an open subset of M, and let f be a holomorphic $(n, 0)$ form on U. Let $\mathcal{F}_{z_0} \supset \mathcal{I}(\varphi + \psi)_{z_0}$ be an ideal of $\mathcal{O}_{z_0}$ for any $z_0 \in Z_0$.

Denote that

$$G(t; c) := \inf \left\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0}) \right\}, \tag{7}$$

and

$$\mathcal{H}^2(c, t) := \left\{ \tilde{f} : \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) < +\infty, \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right. \\ \left. \& (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0}) \right\}, \tag{8}$$

where $t \in [T, +\infty)$, and c is a nonnegative measurable function on $(T, +\infty)$. Here $|\tilde{f}|^2 := \sqrt{-1}^{n^2} \tilde{f} \wedge \bar{\tilde{f}}$ for any $(n, 0)$ form $\tilde{f}$, and $(\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ means that $(\tilde{f} - f, z_0) \in (\mathcal{O}(K_M) \otimes \mathcal{F})_{z_0}$ for any $z_0 \in Z_0$. If there is no holomorphic $(n, 0)$ form $\tilde{f}$ on $\{\psi < -t\}$ satisfying $(\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$, we set $G(t; c) = +\infty$.

In [26], Guan-Mi-Yuan obtained the following concavity of $G(t; c)$.

Theorem 10 ([26]) *Let $c \in \mathcal{P}_{T,M}$ such that $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. If there exists $t \in [T, +\infty)$ satisfying that $G(t) < +\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt)$, $\lim_{t\to T+0} G(t) = G(T)$ and $\lim_{t\to+\infty} G(t) = 0$, where $h(t) = \int_t^{+\infty} c(t_1)e^{-t_1}dt_1$.*

Lemma 1 ([26]) *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. Assume that $G(t) < +\infty$ for some $t \in [T, +\infty)$. Then there exists a unique holomorphic $(n, 0)$*

form F_t *on* $\{\psi < -t\}$ *satisfying* $(F_t - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ *and* $\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} c(-\psi) = G(t)$. *Furthermore, for any holomorphic* $(n, 0)$ *form* $\hat{F}$ *on* $\{\psi < -t\}$ *satisfying* $(\hat{F} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ *and* $\int_{\{\psi<-t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi) < +\infty$, *we have the following equality*

$$\begin{aligned} &\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} c(-\psi) + \int_{\{\psi<-t\}} |\hat{F} - F_t|^2 e^{-\varphi} c(-\psi) \\ =&\int_{\{\psi<-t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi). \end{aligned} \tag{9}$$

Guan-Mi-Yuan also obtained the following corollary of Theorem 10, which is a necessary condition for the concavity degenerating to linearity.

Lemma 2 ([26]) *Let* $c(t) \in \mathcal{P}_{T,M}$ *such that* $\int_T^{+\infty} c(s) e^{-s} ds < +\infty$. *If* $G(t) \in (0, +\infty)$ *for some* $t \geq T$ *and* $G(h^{-1}(r))$ *is linear with respect to* $r \in (0, \int_T^{+\infty} c(s)e^{-s} ds)$, *where* $h(t) = \int_t^{+\infty} c(l)e^{-l} dl$, *then there exists a unique holomorphic* $(n, 0)$ *form* F *on* M *satisfying* $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$, *and* $G(t; c) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)$ *for any* $t \geq T$.

Furthermore, we have

$$\int_{\{-t_2 \leq \psi < -t_1\}} |F|^2 e^{-\varphi} a(-\psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(t)e^{-t} dt} \int_{t_1}^{t_2} a(t)e^{-t} dt, \tag{10}$$

for any nonnegative measurable function a *on* $(T, +\infty)$, *where* $T \leq t_1 < t_2 \leq +\infty$.

Especially, if $\mathcal{H}^2(\tilde{c}, t_0) \subset \mathcal{H}^2(c, t_0)$ *for some* $t_0 \geq T$, *where* $\tilde{c}$ *is a nonnegative measurable function on* $(T, +\infty)$, *we have*

$$G(t_0; \tilde{c}) = \int_{\{\psi<-t_0\}} |F|^2 e^{-\varphi} \tilde{c}(-\psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(s)e^{-s} ds} \int_{t_0}^{+\infty} \tilde{c}(s)e^{-s} ds. \tag{11}$$

Remark 14 ([26]) Let $c(t) \in \mathcal{P}_{T,M}$. If $\mathcal{H}^2(\tilde{c}, t_1) \subset \mathcal{H}^2(c, t_1)$, then $\mathcal{H}^2(\tilde{c}, t_2) \subset \mathcal{H}^2(c, t_2)$, where $t_1 > t_2 > T$. In the following, we give some sufficient conditions of $\mathcal{H}^2(\tilde{c}, t_0) \subset \mathcal{H}^2(c, t_0)$ for $t_0 > T$:

(1) $\tilde{c} \in \mathcal{P}_{T,M}$ and $\lim\limits_{t\to+\infty} \frac{\tilde{c}(t)}{c(t)} > 0$. Especially, $\tilde{c} \in \mathcal{P}_{T,M}$, c and $\tilde{c}$ are smooth on $(T, +\infty)$ and $\frac{d}{dt}(\log \tilde{c}(t)) \geq \frac{d}{dt}(\log c(t))$;

(2) $\tilde{c} \in \mathcal{P}_{T,M}$, $\mathcal{H}^2(c, t_0) \neq \emptyset$ and there exists $t > t_0$ such that $\{\psi < -t\} \Subset \{\psi < -t_0\}$, $\{z \in \overline{\{\psi < -t\}} : \mathcal{I}(\varphi + \psi)_z \neq \mathcal{O}_z\} \subset Z_0$ and $\mathcal{F}|_{\overline{\{\psi<-t\}}} = \mathcal{I}(\varphi + \psi)|_{\overline{\{\psi<-t\}}}$.

2.2 The Sufficient and Necessary Conditions of the Concavity of $G\big(h^{-1}(r)\big)$ Degenerating to Linearity

In this section, we recall some result on the characterizations of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity on the fibrations over open Riemann surfaces and products of open Riemann surfaces.

The following result can be referred to [1].

Let $Z_0^1 := \{z_j : j \in \mathbb{N}\&1 \le j \le m\}$ be a finite subset of the open Riemann surface Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y and $Z_0 := \pi_1^{-1}(Z_0^1)$. Let ψ_1 be a subharmonic function on Ω such that $p_j = \frac{1}{2}v(dd^c\psi_1, z_j) > 0$, and let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω. Let φ_2 be a plurisubharmonic function on Y. Denote that $\psi := \pi_1^*(\psi_1), \varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \Subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0^1$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any $j, k,\ j \neq k$. Denote that $V_0 := \bigcup_{1\le j\le m} V_{z_j}$. Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y$. Denote $\mathcal{F}_{(z_j,y)} = \mathcal{I}(\psi + \varphi)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Let $G(t; c)$ be the minimal L^2 integral on $\{\psi < -t\}$ with respect to φ, f and $\mathcal{F}$ for any $t \ge 0$.

We recall a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the fibers over sets of finite points as follows.

Theorem 11 ([1]) *Assume that $G(0) \in (0, +\infty)$ and $\big(\psi_1 - 2p_jG_\Omega(\cdot, z_j)\big)(z_j) > -\infty$, where $p_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$ for any $j \in \{1, 2, \ldots, m\}$. Then $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1) $\psi_1 = 2\sum_{j=1}^{m} p_jG_\Omega(\cdot, z_j)$;

(2) for any $j \in \{1, 2, \ldots, m\}$, $f = \pi_1^*(a_jw_j^{k_j}dw_j) \wedge \pi_2^*(f_Y) + f_j$ on $V_{z_j} \times Y$, where $a_j \in \mathbb{C} \setminus \{0\}$ is a constant, k_j is a nonnegative integer, f_Y is a holomorphic $(n-1, 0)$ form on Y such that $\int_Y |f_Y|^2e^{-\varphi_2} \in (0, +\infty)$, and $\big(f_j, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)\big)_{(z_j,y)}$ for any $j \in \{1, 2, \ldots, m\}$ and $y \in Y$;

(3) $\varphi_1 + \psi_1 = 2\log|g| + 2\sum_{j=1}^{m} G_\Omega(\cdot, z_j) + 2u$, where g is a holomorphic function on Ω such that $ord_{z_j}(g) = k_j$ and u is a harmonic function on Ω;

(4) $\prod_{j=1}^{m} \chi_{z_j} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_j)$ respectively;

(5) for any $j \in \{1, 2, \ldots, m\}$,

$$\lim_{z\to z_j} \frac{a_jw_j^{k_j}dw_j}{gP_*\left(f_u\left(\prod_{l=1}^{m} f_{z_l}\right)\left(\sum_{l=1}^{m} p_l\frac{df_{z_l}}{f_{z_l}}\right)\right)} = c_0, \tag{12}$$

where $c_0 \in \mathbb{C} \setminus \{0\}$ *is a constant independent of* j, f_u *is a holomorphic function* Δ *such that* $|f_u| = P^*(e^u)$ *and* $f_{z_{j,k}}$ *is a holomorphic function on* Δ *such that* $|f_{z_l}| = P^*\left(e^{G_\Omega(\cdot, z_l)}\right)$ *for any* $l \in \{1, \ldots, m\}$.

When $Z_0^1 := \{z_j : j \in \mathbb{N} \& 1 \le j < +\infty\}$ is an infinite subset of the open Riemann surface Ω of discrete set. Assume that $2\sum_{j=1}^{+\infty} p_j G_\Omega(\cdot, z_j) \not\equiv -\infty$. We recall the following necessary condition such that $G\big(h^{-1}(r)\big)$ is linear.

Proposition 4 ([1]) *Assume that* $G(0) \in (0, +\infty)$ *and* $\big(\psi_1 - 2p_j G_\Omega(\cdot, z_j)\big)(z_j) > -\infty$, *where* $p_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$ *for any* $j \in \mathbb{N}_+$. *Assume that* $G\big(h^{-1}(r)\big)$ *is linear with respect to* $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$, *then the following statements hold:*

(1) $\psi_1 = 2\sum_{j=1}^{\infty} p_j G_\Omega(\cdot, z_j)$;

(2) for any $j \in \mathbb{N}_+$, $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$ *on* $V_{z_j} \times Y$, *where* $a_j \in \mathbb{C} \setminus \{0\}$ *is a constant,* k_j *is a nonnegative integer,* f_Y *is a holomorphic* $(n-1, 0)$ *form on* Y *such that* $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$, *and* $\big(f_j, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)\big)_{(z_j, y)}$ *for any* $j \in \mathbb{N}_+$ *and* $y \in Y$;

(3) $\varphi_1 + \psi_1 = 2\log|g|$, *where* g *is a holomorphic function on* Ω *such that* $ord_{z_j}(g) = k_j + 1$ *for any* $j \in \mathbb{N}_+$;

(4). for any $j \in \mathbb{N}_+$,

$$\frac{p_j}{ord_{z_j} g} \lim_{z \to z_j} \frac{dg}{a_j w_j^{k_j} dw_j} = c_0, \tag{13}$$

where $c_0 \in \mathbb{C} \setminus \{0\}$ *is a constant independent of* j;

(5) $\sum_{j \in \mathbb{N}_+} p_j < +\infty$.

Let $Z_0^1 := \{z_j : j \in \mathbb{N} \& 1 \le j \le m\}$ be a finite subset of the open Riemann surface Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y, and $Z_0 := \pi_1^{-1}(Z_0^1)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \Subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0^1$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any j, k, $j \ne k$. Let $c_\beta(z)$ be the logarithmic capacity (see [?]) on Ω which is defined by

$$c_\beta(z_j) := \exp \lim_{z \to z_0} (G_\Omega(z, z_j) - \log|w_j(z)|).$$

Denote that $V_0 := \bigcup_{1 \le j \le m} V_{z_j}$. Assume that $\tilde{M} \subset M$ is an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0 \subset \tilde{M}$.

We recall the following characterization of the holding of the equality in optimal L^2 extension from fibers over analytic subsets to fibrations over open Riemann surfaces (see [1]).

Theorem 12 ([1]) *Let k_j be a nonnegative integer for any $j \in \{1, 2, ..., m\}$. Let ψ_1 be a negative subharmonic function on Ω satisfying that $\frac{1}{2}v(dd^c\psi_1, z_j) = p_j > 0$ for any $j \in \{1, 2, ..., m\}$. Denote $\psi := \pi_1^*(\psi_1)$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω, $\frac{1}{2}v(dd^c(\varphi_1 + \psi_1), z_j) = k_j + 1$ and $\alpha_j := \big(\varphi_1 + \psi_1 - 2(k_j + 1)G_\Omega(\cdot, z_j)\big)(z_j) > -\infty$ for any j. Let φ_2 be a plurisubharmonic function on Y. Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n - 1, 0)$ form on Y such that $\int_Y |F_j|^2e^{-\varphi_2} < +\infty$ for any j.*

Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying that $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_j)$ on $V_{z_j} \times Y$. Then there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2e^{-\varphi}c(-\psi) \le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{m} \frac{2\pi|a_j|^2e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2e^{-\varphi_2}. \quad (14)$$

Moreover, equality $\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{m} \frac{2\pi|a_j|^2e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2e^{-\varphi_2} = \inf\Big\{ \int_M |\tilde{F}|^2e^{-\varphi}c(-\psi) : \tilde{F}$ is a holomorphic $(n, 0)$ form on $\tilde{M}$ such that $\big(\tilde{F} - f, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0\Big\}$ holds if and only if the following statements hold:

(1) $\psi_1 = 2\sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j)$;

(2). $\varphi_1 + \psi_1 = 2\log|g| + 2\sum_{j=1}^{m}(k_j + 1)G_\Omega(\cdot, z_j) + 2u$, where g is a holomorphic function on Ω such that $g(z_j) \ne 0$ for any $j \in \{1, 2, \dots, m\}$ and u is a harmonic function on Ω;

(3) $\prod_{j=1}^{m} \chi_{z_j}^{k_j+1} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_j)$ respectively;

(4) for any $j \in \{1, 2, \dots, m\}$,

$$\lim_{z\to z_j} \frac{a_j w_j^{k_j} dw_j}{g P_*\left(f_u\left(\prod_{l=1}^{m} f_{z_l}^{k_l+1}\right)\left(\sum_{l=1}^{m} p_l \frac{df_{z_l}}{f_{z_l}}\right)\right)} = c_j \in \mathbb{C} \setminus \{0\}, \quad (15)$$

and there exist $c_0 \in \mathbb{C} \setminus \{0\}$ and a holomorphic $(n - 1, 0)$ form F_Y on Y which are independent of j such that $c_0 F_Y = c_j F_j$ for any $j \in \{1, 2, \dots, m\}$;

(5) $\tilde{M} = M$.

Let $Z_0^1 := \{z_j : 1 \le j < +\infty\}$ be an infinite discrete subset of the open Riemann surface Ω. Let Y be an $n - 1$ dimensional weakly pseudoconvex Kähler manifold.

Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y, and $Z_0 := \pi_1^{-1}(Z_0^1)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \Subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0^1$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any j, k, $j \neq k$. Denote that $V_0 := \bigcup_{j=1}^{\infty} V_{z_j}$.

We recall the following L^2 extension result from fibers over analytic subsets to fibrations over open Riemann surfaces, where the analytic subsets are infinite points on open Riemann surfaces (see [1]).

Theorem 13 ([1]) *Let k_j be a nonnegative integer for any $j \in \mathbb{N}_+$. Let ψ_1 be a negative subharmonic function on Ω satisfying that $\frac{1}{2}v(dd^c\psi_1, z_j) = k_j + 1 > 0$ for any $j \in \mathbb{N}_+$. Denote $\psi := \pi_1^*(\psi_1)$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω, $\frac{1}{2}v(dd^c(\varphi_1 + \psi_1), z_j) = k_j + 1$ and $\alpha_j := \big(\varphi_1 + \psi_1 - 2(k_j + 1)G_\Omega(\cdot, z_j)\big)(z_j) > -\infty$ for any j. Let φ_2 be a plurisubharmonic function on Y. Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n-1, 0)$ form on Y such that $\int_Y |F_j|^2 e^{-\varphi_2} < +\infty$ for any j.*

Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying that $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_j)$ on $V_{z_j} \times Y$. If

$$\sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j + 1)c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} < +\infty,$$

then there exists a holomorphic $(n, 0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) < \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j + 1)c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \tag{16}$$

The following results can be referred to [2].

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \le j \le n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = \left(\prod_{1 \le j \le n_1} \Omega_j\right) \times Y$ be an $n-$dimensional complex manifold, where $n = n_1 + n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1 \le j \le n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M. Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, and denote that $Z_0 := \left(\prod_{1 \le j \le n_1} Z_j\right) \times Y$.

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \big\{(\beta_1, \ldots, \beta_{n_1}) : 1 \le \beta_j <$

$\tilde{m}_j$ for any $j \in \{1, \ldots, n_1\}\big\}$, $V_\beta := \prod_{1\le j\le n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1\le j\le n_1} \Omega_j$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$. Then $Z_0 = \{(z_\beta, y) : \beta \in \tilde{I}_1 \,\&\, y \in Y\} \subset \tilde{M}$.

Let φ_j be a subharmonic function on Ω_j such that $\varphi_j(z_{j,k}) > -\infty$ for any $1 \le k \le \tilde{m}_j$. Let φ_Y be a plurisubharmonic function on Y.

Let $p_{j,k}$ be a positive number for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, which satisfies that $\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \le j \le n_1$. Denote that

$$\psi := \max_{1\le j\le n_1} \left\{ 2 \sum_{1\le k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$$

and $\varphi := \sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on M.

Denote that $E_\beta := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ and $\tilde{E}_\beta := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \ge 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ for any $\beta \in \tilde{I}_1$.

Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$.

Let $Z_0 = \{z_0\} \times Y = \{(z_1, \ldots, z_{n_1})\} \times Y \subset M$. Let

$$\psi = \max_{1\le j\le n_1} \left\{2p_j\pi_{1,j}^*\big(G_{\Omega_j}(\cdot, z_j)\big)\right\},$$

where p_j is positive real number for $1 \le j \le n_1$. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1\le j\le n_1} V_{z_j}$, and $w := (w_1, \ldots, w_{n_1})$ is a local coordinate on V_0 of $z_0 \in \prod_{1\le j\le n_1} \Omega_j$. Denote that $E := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_j} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$. Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y \subset M$.

We recall a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the case $Z_0 = \{z_0\} \times Y$.

Theorem 14 ([2]) *Assume that $G(0) \in (0, +\infty)$. $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1) $f = \sum_{\alpha\in E} \pi_1^*\left(w^\alpha dw_1 \wedge \ldots \wedge dw_{n_1}\right) \wedge \pi_2^*(f_\alpha) + g_0$ on $V_0 \times Y$, where g_0 is a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying $(g_0, z) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)\big)_z$ for any $z \in Z_0$ and f_α is a holomorphic $(n_2, 0)$ form on Y such that $\sum_{\alpha\in E} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \in (0, +\infty)$;

(2) $\varphi_j = 2\log|g_j| + 2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j) \ne 0$ and u_j is a harmonic function on Ω_j for any $1 \le j \le n_1$;

(3) $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$ *for any* $j \in \{1, 2, ..., n\}$ *and* $\alpha \in E$ *satisfying* $f_\alpha \not\equiv 0$.

Let $Z_j = \{z_{j,1}, \ldots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \ldots, n_1\}$, where m_j is a positive integer. Let

$$\psi = \max_{1 \le j \le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1 \le k \le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

where $p_{j,k}$ is a positive real number. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \big\{(\beta_1, \ldots, \beta_{n_1}) : 1 \le \beta_j \le m_j \text{ for any } j \in \{1, \ldots, n_1\}\big\}$, $V_\beta := \prod_{1 \le j \le n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \le j \le n_1} \Omega_j$ satisfying $w_\beta(z_\beta) = 0$.

Let $\beta^* = (1, \ldots, 1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \ldots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\ge 0}^{n_1}$. Denote that $E' := \left\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1 \le j \le n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{1 \le j \le n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. Let f be a holomorphic $(n, 0)$ form on $\cup_{\beta \in \tilde{I}_1} V_\beta \times Y$ satisfying $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*}}\right) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2, 0)$ forms on Y.

We recall a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the case Z_j is a set of finite points.

Theorem 15 ([2]) *Assume that* $G(0) \in (0, +\infty)$. $G\big(h^{-1}(r)\big)$ *is linear with respect to* $r \in (0, \int_0^{+\infty} c(s)e^{-s} ds]$ *if and only if the following statements hold:*

(1) $\varphi_j = 2\log|g_j| + 2u_j$ *for any* $j \in \{1, \ldots, n_1\}$, *where* u_j *is a harmonic function on* Ω_j *and* g_j *is a holomorphic function on* Ω_j *satisfying* $g_j(z_{j,k}) \neq 0$ *for any* $k \in \{1, \ldots, m_j\}$;

(2) *There exists a nonnegative integer* $\gamma_{j,k}$ *for any* $j \in \{1, \ldots, n_1\}$ *and* $k \in \{1, \ldots, m_j\}$, *which satisfies that* $\prod_{1 \le k \le m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ *and* $\sum_{1 \le j \le n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ *for any* $\beta \in \tilde{I}_1$;

(3) $f = \pi_1^*\left(c_\beta \left(\prod_{1 \le j \le n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}}\right) dw_{1,\beta_1} \wedge \ldots \wedge dw_{n,\beta_n}\right) \wedge \pi_2^*(f_0) + g_\beta$ *on* $V_\beta \times Y$ *for any* $\beta \in \tilde{I}_1$, *where* c_β *is a constant,* $f_0 \not\equiv 0$ *is a holomorphic* $(n_2, 0)$ *form on* Y *satisfying* $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$, *and* g_β *is a holomorphic* $(n, 0)$ *form on* $V_\beta \times Y$ *such that* $(g_\beta, z) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)\big)_z$ *for any* $z \in \{z_\beta\} \times Y$;

(4) $c_\beta \prod_{1 \le j \le n_1} \left(\lim_{z \to z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1 \le k \le m_j} f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1 \le k \le m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)} \right) = c_0$ *for any* $\beta \in \tilde{I}_1$, *where* $c_0 \in \mathbb{C}\backslash\{0\}$ *is a constant independent of* β, f_{u_j} *is a holomorphic function* Δ *such that* $|f_{u_j}| = P_j^*(e^{u_j})$ *and* $f_{z_{j,k}}$ *is a holomorphic function on* Δ *such that* $|f_{z_{j,k}}| = P_j^*\left(e^{G_{\Omega_j}(\cdot, z_{j,k})}\right)$ *for any* $j \in \{1, \ldots, n_1\}$ *and* $k \in \{1, \ldots, m_j\}$.

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \dots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Let $p_{j,k}$ be a positive number for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$ such that $\sum_{1 \le k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j. Let

$$\psi = \max_{1 \le j \le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1 \le k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}.$$

Assume that $\limsup_{t \to +\infty} c(t) < +\infty$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \dots, n_1\}$ and $1 \le k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \ne k'$. Denote that $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \le \beta_j < \tilde{m}_j$ for any $j \in \{1, \dots, n_1\}\}$, $V_\beta := \prod_{1 \le j \le n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \le j \le n_1} \Omega_j$.

Let $\beta^* = (1, \dots, 1) \in I_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\ge 0}^{n_1}$. Denote that $E' := \left\{ \alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1 \le j \le n_1} \frac{\alpha_j + 1}{p_{j,1}} > \sum_{1 \le j \le n_1} \frac{\alpha_{\beta^*,j} + 1}{p_{j,1}} \right\}$. Let f be a holomorphic $(n, 0)$ form on $\cup_{\beta \in I_1} V_\beta \times Y$ satisfying $f = \pi_1^* \left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^* \left(f_{\alpha_{\beta^*}} \right) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2, 0)$ forms on Y.

We recall that $G\big(h^{-1}(r)\big)$ is not linear when there exists $j_0 \in \{1, \dots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$ as follows.

Theorem 16 ([2]) *If $G(0) \in (0, +\infty)$ and there exists $j_0 \in \{1, \dots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$, then $G\big(h^{-1}(r)\big)$ is not linear with respect to $r \in (0, \int_0^{+\infty} c(s) e^{-s} ds]$.*

Let $\tilde{M} \subset M$ be an $n-$dimensional complex manifold satisfying that $Z_0 \subset \tilde{M}$, and let $K_{\tilde{M}}$ be the canonical (holomorphic) line bundle on $\tilde{M}$.

Let $\Psi \le 0$ be a plurisubharmonic function on $\prod_{1 \le j \le n_1} \Omega_j$, and let φ_j be a Lebesgue measurable function on Ω_j such that $\Psi + \sum_{1 \le j \le n_1} \tilde{\pi}_j^*(\varphi_j)$ is plurisubharmonic on $\prod_{1 \le j \le n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \le j \le n_1} \Omega_j$ to Ω_j. Let φ_Y be a plurisubharmonic function on Y. Let $p_{j,k}$ be a positive number for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, which satisfies that $\sum_{1 \le k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \le j \le n_1$. Denote that

$$\psi := \max_{1 \le j \le n_1} \left\{ 2 \sum_{1 \le k < \tilde{m}_j} p_{j,k} \pi_{1,j}^* \big(G_{\Omega_j}(\cdot, z_{j,k}) \big) \right\} + \pi_1^*(\Psi)$$

and $\varphi := \sum_{1 \le j \le n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on M.

Denote that $E_\beta := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ and $\tilde{E}_\beta := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \ge 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ for any $\beta \in \tilde{I}_1$. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$. Denote that

$$c_{j,k} := \exp \lim_{z\to z_{j,k}} \left(\frac{\sum_{1\le k_1<\tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \ldots, n\}$ and $1 \le k < \tilde{m}_j$ (following from Lemmas 3 and 4, we get that the above limit exists).

When $Z_0 = \{z_0\} \times Y \subset \tilde{M}$, where $z_0 = (z_1, \ldots, z_{n_1}) \in \prod_{1\le j\le n_1} \Omega_j$.

We recall a characterization of the holding of equality in optimal jets L^2 extension problem for the case $Z_0 = \{z_0\} \times Y$.

Theorem 17 ([2]) *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that*

$$\sum_{\alpha\in E} \frac{(2\pi)^{n_1} e^{-\left(\Psi+\sum_{1\le j\le n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1\le j\le n_1} (\alpha_j + 1)c_j(z_j)^{2\alpha_j+2}} \in (0, +\infty).$$

Then there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $(F - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}\right)\right)_z$ for any $z \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)$$
$$\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\alpha\in E} \frac{(2\pi)^{n_1} e^{-\left(\Psi+\sum_{1\le j\le n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1\le j\le n_1} (\alpha_j + 1)c_j(z_j)^{2\alpha_j+2}}.$$

Moreover, equality $\inf\left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \,\&\, (\tilde{F} - f, z) \in (\mathcal{O}\left(K_{\tilde{M}}\right) \otimes \mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}\right)\right)_z \text{ for any } z \in Z_0\right\} = \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \times \sum_{\alpha\in E} \frac{(2\pi)^{n_1} e^{-\left(\Psi+\sum_{1\le j\le n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1\le j\le n_1} (\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}$ *holds if and only if the following statements hold:*

(1) $\tilde{M} = \left(\prod_{1\le j\le n_1} \Omega_j\right) \times Y$ *and* $\Psi \equiv 0$;

(2) $\varphi_j = 2\log|g_j| + 2u_j$, *where* g_j *is a holomorphic function on* Ω_j *such that* $g_j(z_j) \neq 0$ *and* u_j *is a harmonic function on* Ω_j *for any* $1 \leq j \leq n_1$*;*

(3) $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$ *for any* $j \in \{1, 2, ..., n\}$ *and* $\alpha \in E$ *satisfying* $f_\alpha \not\equiv 0$.

Remark 15 ([2]) If $(f_\alpha, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ for any $y \in Y$ and $\alpha \in \tilde{E}\backslash E$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}\right)$ by $\mathcal{I}(\varphi+\psi)$.

Let $Z_j = \{z_{j,1}, \ldots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \ldots, n_1\}$, where m_j is a positive integer. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$. Let $\beta^* = (1, \ldots, 1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \ldots, \alpha_{\beta^*,n_1}) \in E_{\beta^*}$. Assume that $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right) + \sum_{\alpha\in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha,\beta})$ on $U_0 \cap (V_{\beta^*} \times Y)$.

We recall a characterization of the holding of equality in optimal jets L^2 extension problem for the case that Z_j is finite.

Theorem 18 ([2]) *Let* c *be a positive function on* $(0, +\infty)$ *such that* $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ *and* $c(t)e^{-t}$ *is decreasing on* $(0, +\infty)$*. Assume that*

$$\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi+\sum_{1\leq j\leq n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \in (0, +\infty).$$

Then there exists a holomorphic $(n, 0)$ *form* F *on* $\tilde{M}$ *satisfying that* $(F - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k\leq m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\right)_z$ *for any* $z \in Z_0$ *and*

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi+\sum_{1\leq j\leq n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Moreover, equality $\inf\Bigg\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}}))\ \&\ (\tilde{F} - f, z)$ $\in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k\leq m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\right)_z$ *for any* $z \in Z_0\Bigg\}$

$= \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi+\sum_{1\le j\le n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ *holds if and only if the following statements hold:*

(1) $\tilde{M} = \left(\prod_{1\le j\le n_1} \Omega_j\right) \times Y$ *and* $\Psi \equiv 0$*;*

(2) $\varphi_j = 2\log|g_j| + 2u_j$ *for any* $j \in \{1,\ldots,n_1\}$*, where* u_j *is a harmonic function on* Ω_j *and* g_j *is a holomorphic function on* Ω_j *satisfying* $g_j(z_{j,k}) \neq 0$ *for any* $k \in \{1,\ldots,m_j\}$*;*

(3) *There exists a nonnegative integer* $\gamma_{j,k}$ *for any* $j \in \{1,\ldots,n_1\}$ *and* $k \in \{1,\ldots,m_j\}$*, which satisfies that* $\prod_{1\le k\le m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ *and* $\sum_{1\le j\le n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ *for any* $\beta \in \tilde{I}_1$*;*

(4) $f_{\alpha,\beta} = c_\beta f_0$ *holds for* $\alpha = (\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})$ *and* $f_{\alpha,\beta} \equiv 0$ *holds for any* $\alpha \in E_\beta\backslash\{(\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})\}$*, where* $\beta \in \tilde{I}_1$*,* c_β *is a constant and* $f_0 \not\equiv 0$ *is a holomorphic* $(n_2,0)$ *form on* Y *satisfying* $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$*;*

(5) $c_\beta \prod_{1\le j\le n_1} \left(\lim_{z\to z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\le k\le m_j} f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\le k\le m_j} p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)}\right) = c_0$ *for any* $\beta \in \tilde{I}_1$*, where* $c_0 \in \mathbb{C}\backslash\{0\}$ *is a constant independent of* β*,* f_{u_j} *is a holomorphic function* Δ *such that* $|f_{u_j}| = P_j^*(e^{u_j})$ *and* $f_{z_{j,k}}$ *is a holomorphic function on* Δ *such that* $|f_{z_{j,k}}| = P_j^*\left(e^{G_{\Omega_j}(\cdot,z_{j,k})}\right)$ *for any* $j \in \{1,\ldots,n_1\}$ *and* $k \in \{1,\ldots,m_j\}$*.*

Remark 16 ([2]) If $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ holds for any $y \in Y, \alpha \in \tilde{E}_\beta\backslash E_\beta$ and $\beta \in \tilde{I}_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)$ by $\mathcal{I}(\varphi+\psi)$.

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1,\ldots,n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$. Let $\beta^* = (1,\ldots,1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1}) \in E_{\beta^*}$. Assume that $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right) + \sum_{\alpha\in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha,\beta})$ on $U_0 \cap (V_{\beta^*} \times Y)$.

We recall that the equality in optimal jets L^2 extension problem could not hold when there exists $j_0 \in \{1,\ldots,n_1\}$ such that $\tilde{m}_{j_0} = +\infty$.

Theorem 19 ([2]) *Let* c *be a positive function on* $(0,+\infty)$ *such that* $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ *and* $c(t)e^{-t}$ *is decreasing on* $(0,+\infty)$*. Assume that*

$$\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\le j\le n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\in(0,+\infty)$$

and there exists $j_0\in\{1,\ldots,n_1\}$ such that $\tilde{m}_{j_0}=+\infty$.

Then there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ for any $z\in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$<\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\le j\le n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Remark 17 ([2]) If $(f_{\alpha,\beta},y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ holds for any $y\in Y,\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)$ by $\mathcal{I}(\varphi+\psi)$.

2.3 Basic Properties of the Green Functions

In this section, we recall some basic properties of the Green functions. Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω, and let $z_0\in\Omega$.

Lemma 3 (see [44]) *Let w be a local coordinate on a neighborhood of z_0 satisfying $w(z_0)=0$. $G_\Omega(z,z_0)=\sup_{v\in\Delta_\Omega^*(z_0)}v(z)$, where $\Delta_\Omega^*(z_0)$ is the set of negative subharmonic function on Ω such that $v-\log|w|$ has a locally finite upper bound near z_0. Moreover, $G_\Omega(\cdot,z_0)$ is harmonic on $\Omega\backslash\{z_0\}$ and $G_\Omega(\cdot,z_0)-\log|w|$ is harmonic near z_0.*

Lemma 4 (see [31]) *Let $K=\{z_j:j\in\mathbb{Z}_{\ge1}\,\&\,j<\gamma\}$ be a discrete subset of Ω, where $\gamma\in\mathbb{Z}_{>1}\cup\{+\infty\}$. Let ψ be a negative subharmonic function on Ω such that $\frac{1}{2}v(dd^c\psi,z_j)\ge p_j$ for any j, where $p_j>0$ is a constant. Then $2\sum_{1\le j<\gamma}p_jG_\Omega(\cdot,z_j)$ is a subharmonic function on Ω satisfying that $2\sum_{1\le j<\gamma}p_jG_\Omega(\cdot,z_j)\ge\psi$ and $2\sum_{1\le j<\gamma}p_jG_\Omega(\cdot,z_j)$ is harmonic on $\Omega\backslash K$.*

Lemma 5 (see [28]) *For any open neighborhood U of z_0, there exists $t>0$ such that $\{G_\Omega(z,z_0)<-t\}$ is a relatively compact subset of U.*

Lemma 6 see [31]) *There exists a sequence of open Riemann surfaces $\{\Omega_l\}_{l\in\mathbb{Z}^+}$ such that $z_0\in\Omega_l\Subset\Omega_{l+1}\Subset\Omega$, $\cup_{l\in\mathbb{Z}^+}\Omega_l=\Omega$, Ω_l has a smooth boundary $\partial\Omega_l$ in Ω and $e^{G_{\Omega_l}(\cdot,z_0)}$ can be smoothly extended to a neighborhood of $\overline{\Omega_l}$ for any $l\in\mathbb{Z}^+$, where G_{Ω_l} is the Green function of Ω_l. Moreover, $\{G_{\Omega_l}(\cdot,z_0)-G_\Omega(\cdot,z_0)\}$ is decreasingly convergent to 0 on Ω with respect to l.*

Let $M = (\prod_{1\le j\le n_1} \Omega_j) \times Y$, where Ω_j is an open Riemann surface and Y is an n_2-dimensional complex manifold and $n_1 + n_2 = n$. Let $\pi_1, \pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\le j\le n_1} \Omega_j$, Ω_j and Y respectively. Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $1 \le j \le n_1$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Denote that $Z_0 := \left(\prod_{1\le j\le n_1} Z_j\right) \times Y$.

Let $G = \max_{1\le j\le n_1} \left\{\tilde{\pi}_j^* \left(2 \sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$ be a plurisubharmonic function on $\prod_{1\le j\le n_1} \Omega_j$, where $\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $j \in \{1, .., n\}$ and $\tilde{\pi}_j$ is the natural projection from $\prod_{1\le j\le n_1} \Omega_j$ to Ω_j. Let $\tilde{M} \subset M$ be an
$n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0 \subset \tilde{M}$.

Let $N \le 0$ be a plurisubharmonic function on $\tilde{M}$. Denote $\psi = \pi_1^*(G) + N$.

Lemma 7 *Let $l(t)$ be a positive Lebesgue measurable function on $(0, +\infty)$ satisfying that l is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} l(t)dt < +\infty$.*

If $N \not\equiv 0$, then there exists a Lebesgue measurable subset $\tilde{V}$ of $\tilde{M}$ such that $l(-\psi) < l\big(-\pi_1^*(G)\big)$ on $\tilde{V}$ and $\mu(\tilde{V}) > 0$, where μ is the Lebesgue measure on $\tilde{M}$.

Proof Let $V_1 \Subset \prod_{1\le j\le n_1} \Omega_j \backslash \{z_\beta : \exists j \in \{1, 2, \ldots, n_1\}\, s.t. 2 \le \beta_j \le \tilde{m}_j\}$ be an open neighborhood of z_{β^*}, where $\beta^* = (1, 1, \ldots, 1)$. It follows from Lemma 5 that there exists $t_0 > 0$ such that $V_1 \cap \{G < -t_0\} \Subset V_1$.

As $l(t)$ is decreasing and $\int_0^{+\infty} l(t)dt < +\infty$, there exists $t_1 > t_0$ such that $l(t) < l(t_1)$ holds for any $t > t_1$.

For any $w \in Y$, let $U \Subset Y$ be an open neighborhood of w in Y such that $V_1 \times U \subset \tilde{M}$.

As ψ is upper semi-continuous function and $N \not\equiv 0$ is a plurisubharmonic function on $\tilde{M}$, we have

$$\sup_{(z,w)\in(V_1\cap\{G\le -t_1\})\times U} \psi(z) < -t_1,$$

which implies that there exists $t_2 \in (t_0, t_1)$ such that

$$\sup_{(z,w)\in(V_1\cap\{G\le -t_2\})\times \tilde{U}} \psi(z) < -t_1,$$

where $\tilde{U} \Subset U$ is open.

Denote $t_3 = -\sup_{(z,w)\in(V_1\cap\{G\le -t_2\})\times \tilde{U}} \psi(z)$. Then we know $-t_3 < -t_1$.

Let $V = \{-t_1 < G < -t_2\} \cap V_1$, then $\mu(V \times \tilde{U}) > 0$. As $l(t)$ is decreasing with respect to t, for any $(z, w) \in V \times \tilde{U}$, we have

$$l(-\psi) \le l(t_3) < l(t_1) \le l\big(\pi_1^*(-G)\big).$$

Lemma 7 is proved.

2.4 Other Lemmas

We call a positive measurable function c on $(S, +\infty)$ in class $\tilde{\mathcal{P}}_S$ if $\int_S^s c(l)e^{-l}dl < +\infty$ for some $s > S$ and $c(t)e^{-t}$ is decreasing with respect to t.

Lemma 8 (see [26]) *Let $B \in (0, +\infty)$ and $t_0 \geq S$ be arbitrarily given. Let (M, ω) be an $n-$dimensional weakly pseudoconvex Kähler manifold. Let $\psi < -S$ be a plurisubharmonic function on M. Let φ be a plurisubharmonic function on M. Let F be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\}$ such that*

$$\int_{K\cap\{\psi<-t_0\}} |F|^2 < +\infty,$$

for any compact subset K of M, and

$$\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2e^{-\varphi} \leq C < +\infty.$$

Then there exists a holomorphic $(n, 0)$ form $\widetilde{F}$ on X, such that

$$\int_M |\widetilde{F} - (1 - b_{t_0,B}(\psi))F|^2 e^{-\varphi+v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi)) \leq C\int_S^{t_0+B} c(t)e^{-t}dt. \quad (17)$$

where $b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B}\mathbb{I}_{\{-t_0-B<s<-t_0\}}ds$, $v_{t_0,B}(t) = \int_{-t_0}^t b_{t_0,B}(s)ds - t_0$ and $c(t) \in \tilde{\mathcal{P}}_S$.

Lemma 9 (see [26]) *Let M be a complex manifold. Let S be an analytic subset of M. Let $\{g_j\}_{j=1,2,...}$ be a sequence of nonnegative Lebesgue measurable functions on M, which satisfies that g_j are almost everywhere convergent to g on M when $j \to +\infty$, where g is a nonnegative Lebesgue measurable function on M. Assume that for any compact subset K of $M\backslash S$, there exist $s_K \in (0, +\infty)$ and $C_K \in (0, +\infty)$ such that*

$$\int_K {g_j}^{-s_K}dV_M \leq C_K$$

for any j, where dV_M is a continuous volume form on M.

Let $\{F_j\}_{j=1,2,...}$ be a sequence of holomorphic $(n, 0)$ form on M. Assume that $\liminf_{j\to+\infty} \int_M |F_j|^2 g_j \leq C$, where C is a positive constant. Then there exists a subsequence $\{F_{j_l}\}_{l=1,2,...}$, which satisfies that $\{F_{j_l}\}$ is uniformly convergent to a holomorphic $(n, 0)$ form F on M on any compact subset of M when $l \to +\infty$, such that

$$\int_M |F|^2 g \leq C.$$

Lemma 10 (see [19]) *Let N be a submodule of $\mathcal{O}^q_{\mathbb{C}^n,o}$, $1 \leq q < \infty$, let $f_j \in \mathcal{O}_{\mathbb{C}^n}(U)^q$ be a sequence of q-tuples holomorphic function in an open neighborhood U of the* *origin o. Assume that the f_j converges uniformly in U towards a $q-$tuples $f \in \mathcal{O}_{\mathbb{C}^n}(U)^q$, assume furthermore that all germs (f_j, o) belong to N. Then $(f, o) \in N$.*

Lemma 11 (see [32]) *Let $\psi = \max_{1\le j\le n}\{2p_j \log|w_j|\}$ be a plurisubharmonic function on $\mathbb{C}^n$, where $p_j > 0$. Let $f = \sum_{\alpha\in\mathbb{Z}^n_{\ge 0}} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $\{\psi < -t_0\}$, where $t_0 > 0$. Denote that $q_\alpha := \sum_{1\le j\le n} \frac{\alpha_j+1}{p_j} - 1$ for any $\alpha \in \mathbb{Z}^n_{\ge 0}$ and $E_1 := \{\alpha \in \mathbb{Z}^n_{\ge 0} : q_\alpha = 0\}$. Let k_1 be a constant satisfying $k_1 + 1 > 0$.Then*

$$\int_{\{-t-1-k_1<\psi<-t\}} |f|^2 e^{-\psi} d\lambda_n = \sum_{\alpha\in E_1} \frac{|b_\alpha|^2\pi^n(1+k_1)}{\prod_{1\le j\le n}(\alpha_j+1)} + \sum_{\alpha\notin E_1} \frac{|b_\alpha|^2\pi^n(q_\alpha+1)(e^{-q_\alpha t} - e^{-q_\alpha(t+1+k_1)})}{q_\alpha \prod_{1\le j\le n}(\alpha_j+1)}$$

for any $t > t_0$.

Remark 18 Lemma 11 is stated in [32] in the case $k_1 = 0$, the same proof as in [32] shows that Lemma 11 holds for k_1 which satisfying $k_1 + 1 > 0$.

Lemma 12 *Let $\psi = \max_{1\le j\le n}\{2p_j \log|w_j|\}$ be a plurisubharmonic function on $\mathbb{C}^n$, where $p_j > 0$. Let $f = \sum_{\alpha\in\mathbb{Z}^n_{\ge 0}} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $\{\psi < -t_0\}$, where $t_0 > 0$. Let $c(t)$ be a nonnegative measurable function on $(t_0, +\infty)$. Denote that $q_\alpha := \sum_{1\le j\le n} \frac{\alpha_j+1}{p_j} - 1$ for any $\alpha \in \mathbb{Z}^n_{\ge 0}$. Let A be a real constant such that $t_0 + A > 0$. Then*

$$\int_{\{\psi+A<-t\}} |f|^2 c(-\psi - A) d\lambda_n = \sum_{\alpha\in\mathbb{Z}^n_{\ge 0}} \left(\int_t^{+\infty} c(s)e^{-(q_\alpha+1)s} ds\right) e^{-(q_\alpha+1)A} \frac{(q_\alpha+1)|b_\alpha|^2\pi^n}{\prod_{1\le j\le n}(\alpha_j+1)}$$

holds for any $t \ge t_0$.

Proof For any $t \ge t_0$, by direct calculations, we obtain that (note that $t + A > 0$)

$$\begin{aligned}
&\int_{\{\psi+A<-t\}} |w^\alpha|^2 c(-\psi - A) d\lambda_n \\
=&(2\pi)^n \int_{\left\{\max_{1\le j\le n}\left\{s_j^{p_j}\right\}<e^{-\frac{t+A}{2}} \,\&\, s_j>0\right\}} \prod_{1\le j\le n} s_j^{2\alpha_j+1} \cdot c\left(-\log \max_{1\le j\le n}\left\{s_j^{2p_j}\right\} - A\right) ds_1 ds_2 \dots ds_n \\
=&(2\pi)^n \frac{1}{\prod_{1\le j\le n} p_j} \\
&\times \int_{\left\{\max_{1\le j\le n}\{r_j\}<e^{-\frac{t+A}{2}} \,\&\, r_j>0\right\}} \prod_{1\le j\le n} r_j^{\frac{2\alpha_j+2}{p_j}-1} \cdot c\left(-\log \max_{1\le j\le n}\left\{r_j^2\right\} - A\right) dr_1 dr_2 \dots dr_n.
\end{aligned} \tag{18}$$

By the Fubini's theorem, we have

$$\begin{aligned}
&\int_{\left\{\max_{1\le j\le n}\{r_j\}<e^{-\frac{t+A}{2}}\ \&\, r_j>0\right\}} \prod_{1\le j\le n} r_j^{\frac{2\alpha_j+2}{p_j}-1}\cdot c\left(-\log\max_{1\le j\le n}\left\{r_j^2\right\}-A\right)dr_1dr_2\ldots dr_n\\
=&\sum_{j'=1}^{n}\int_0^{e^{-\frac{t+A}{2}}}\left(\int_{\{0\le r_j<r_{j'},j\ne j'\}}\prod_{j\ne j'} r_j^{\frac{2\alpha_j+2}{p_j}-1}\cdot\wedge_{j\ne j'}dr_j\right) r_{j'}^{\frac{2\alpha_{j'}+2}{p_{j'}}-1}c(-2\log r_{j'}-A)dr_{j'}\\
=&\sum_{j'=1}^{n}\left(\prod_{j\ne j'}\frac{p_j}{2\alpha_j+2}\right)\int_0^{e^{-\frac{t+A}{2}}} r_{j'}^{\sum_{1\le k\le n}\frac{2\alpha_k+2}{p_k}-1}c(-2\log r_{j'}-A)dr_{j'}\\
=&(q_\alpha+1)e^{-(q_\alpha+1)A}\left(\int_t^{+\infty}c(s)e^{-(q_\alpha+1)s}ds\right)\prod_{1\le j\le n}\frac{p_j}{2\alpha_j+2}.
\end{aligned}\tag{19}$$

Following from $\int_{\{\psi<-t\}}|f|^2c(-\psi)d\lambda_n=\sum_{\alpha\in\mathbb{Z}^n_{\ge 0}}|b_\alpha|^2\int_{\{\psi<-t\}}|w^\alpha|^2c(-\psi)d\lambda_n$, equality (18) and equality (19), we obtain that

$$\int_{\{\psi<-t\}}|f|^2d\lambda_n=\sum_{\alpha\in\mathbb{Z}^n_{\ge 0}}\left(\int_t^{+\infty}c(s)e^{-(q_\alpha+1)s}ds\right)e^{-(q_\alpha+1)A}\frac{(q_\alpha+1)|b_\alpha|^2\pi^n}{\prod_{1\le j\le n}(\alpha_j+1)}.$$

Lemma 13 *Let $\Delta^n\subset\mathbb{C}^n$ be a polydisc. Let φ be a bounded subharmonic function on Δ^n. Assume that v is a nonnegative continuous real function on Δ^n. Denote*

$$I_t:=\int_{\{z\in\Delta^n:-t-1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t+k_1\}}v(z)\frac{\prod_{j=1}^n|z_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z_1,\ldots,z_n)}dz\wedge d\bar z,$$

$$J_t:=\int_{\{z\in\Delta^n:-t-1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t+k_1\}}v(z)\frac{\prod_{j=1}^n|z_j|^{2\beta_j}}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z_1,\ldots,z_n)}dz\wedge d\bar z,$$

where $p_j,\alpha_j,\beta_j>0$ are constants for any j satisfying $\sum_{1\le j\le n}\frac{\alpha_j+1}{p_j}=1$ and $\sum_{1\le j\le n}\frac{\beta_j+1}{p_j}>1$, k_1,k_2 are constants satisfying $k_1+k_2+1>0$.
Then

$$\limsup_{t\to+\infty}I_t\le\frac{(2\pi)^n(1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)}v(0)e^{-\varphi(0)}.\tag{20}$$

and

$$\lim_{t\to+\infty}J_t=0.$$

Proof The idea of the proof can be referred to Lemma 3.3 in [52]. For the convenience of the readers, we give a proof below.

Denote

$$I_{t+k_1} := \int_{\{z\in\Delta^n: -t-1-k_1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t\}} v(z)\frac{\prod_{j=1}^n |z_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z_1,\ldots,z_n)}dz\wedge d\bar{z},$$

Note that

$$\limsup_{t\to+\infty} I_t = \limsup_{t\to+\infty} I_{t+k_1},$$

then we only need to prove (20) for I_{t+k_1}.

Denote

$$B_{\delta,t} := \{z\in\Delta^n : \varphi(e^{\frac{-t}{2p_1}}z_1, e^{\frac{-t}{2p_2}}z_2,\ldots,e^{\frac{-t}{2p_n}}z_n) < (1+\delta)\varphi(0)\},$$

where $\delta\in(0,+\infty)$ and $t\in(0,+\infty)$.

Let $\lambda(B_{\delta,t})$ be the $2n-$dimensional Lebesgue measure of $B_{\delta,t}$.

Since the computation is local, we may assume that φ is a negative upper semi-continuous function on Δ^n. Note that $\varphi(0)>-\infty$. For any $\epsilon\in(0,1)$, there exists $t_\epsilon>0$ such that

$$\varphi(e^{\frac{-t}{2p_1}}z_1, e^{\frac{-t}{2p_2}}z_2,\ldots,e^{\frac{-t}{2p_n}}z_n) \le (1-\epsilon)\varphi(0)$$

for any $z\in\Delta^n$, when $t\ge t_\epsilon$. We denote $\varphi(e^{\frac{-t}{2p_1}}z_1, e^{\frac{-t}{2p_2}}z_2,\ldots,e^{\frac{-t}{2p_n}}z_n)$ by $\varphi(e^{\frac{-t}{2p}}z)$ when there is no misunderstanding.

Note that for fixed t, $\varphi(e^{\frac{-t}{2p}}z)$ is subharmonic on Δ^n with respect to z. It follows from mean value inequality that, for all $t\ge t_\epsilon$, we have

$$\begin{aligned}
\varphi(0) &\le \frac{1}{\pi}\int_{z_1\in\Delta}\varphi(e^{\frac{-t}{2p_1}}z_1,0\ldots,0)d\lambda_{z_1}\\
&\le \frac{1}{\pi^n}\int_{z\in\Delta^n}\varphi(e^{\frac{-t}{2p_1}}z_1, e^{\frac{-t}{2p_2}}z_2,\ldots,e^{\frac{-t}{2p_n}}z_n)d\lambda_z\\
&= \frac{1}{\pi^n}\int_{\Delta^n\setminus B_{\delta,t}}\varphi(e^{\frac{-t}{2p}}z)d\lambda_z + \frac{1}{\pi}\int_{B_{\delta,t}}\varphi(e^{\frac{-t}{2p}}z)d\lambda_z\\
&\le \frac{(1-\epsilon)\varphi(0)}{\pi^n}(\pi^n-\lambda(B_{\delta,t})) + \frac{(1+\delta)\varphi(0)}{\pi^n}\lambda(B_{\delta,t})\\
&= \varphi(0)(1-\epsilon+\frac{\delta+\epsilon}{\pi^n}\lambda(B_{\delta,t})).
\end{aligned}$$

As $\varphi(0)<0$, we have

$$\lambda(B_{\delta,t}) \le \frac{\pi^n\epsilon}{\delta+\epsilon} \le \frac{\pi^n\epsilon}{\delta}$$

for any $t\ge t_\epsilon$. Hence

$$\lim_{t\to+\infty}\lambda(B_{\delta,t}) = 0.$$

Since φ is bounded, we have $e^{-\varphi} \le C_1$ for some $C_1 > 0$. As $v(t)$ is continuous, when t is large enough, we have

$$\sup_{\{z\in\Delta^n: -t-1-k_1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t\}} v(z) \le C_2,$$

where $C_2 > 0$ is a constant independent of t. We denote $v(e^{\frac{-t}{2p_1}}z_1, e^{\frac{-t}{2p_2}}z_2, \ldots, e^{\frac{-t}{2p_n}}z_n)$ by $v(e^{\frac{-t}{2p}}z)$ when there is no misunderstandings.

Let $z_i = e^{\frac{-t}{2p_i}}w_i$. We denote $v(e^{\frac{-t}{2p_1}}w_1, e^{\frac{-t}{2p_2}}w_2, \ldots, e^{\frac{-t}{2p_n}}w_n)$ by $v(e^{\frac{-t}{2p}}w)$ and denote $\varphi(e^{\frac{-t}{2p_1}}w_1, e^{\frac{-t}{2p_2}}w_2, \ldots, e^{\frac{-t}{2p_n}}w_n)$ by $\varphi(e^{\frac{-t}{2p}}w)$ for simplicity. By Lemma 11, we have

$$\begin{aligned}
I_{t+k_1} &= \int_{\{z\in\Delta: -t-1-k_1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t\}} v(z)\frac{\prod_{j=1}^n|z_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z)}dz\wedge d\bar z\\
&= \int_{\{w\in\Delta^n: e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} v(e^{\frac{-t}{2p}}w)\frac{\prod_{j=1}^n|w_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}}e^{-\varphi(e^{\frac{-t}{2p}}w)}dw\wedge d\bar w\\
&= \int_{\{w\in\Delta^n: e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}\cap B_{\delta,t}} v(e^{\frac{-t}{2p}}w)\frac{\prod_{j=1}^n|w_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}}e^{-\varphi(e^{\frac{-t}{2p}}w)}dw\wedge d\bar w+\\
&\int_{\{w\in\Delta^n: e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}\setminus B_{\delta,t}} v(e^{\frac{-t}{2p}}w)\frac{\prod_{j=1}^n|w_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}}e^{-\varphi(e^{\frac{-t}{2p}}w)}dw\wedge d\bar w\\
&\le CC_1C_2\lambda(B_{\delta,t}) + \left(\sup_{\{e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} v(e^{\frac{-t}{2p}}w)\right)e^{-(1+\delta)\varphi(0)}\frac{(2\pi)^n(1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)}.
\end{aligned}\tag{21}$$

Hence we know

$$\limsup_{t\to+\infty} I_{t+k_1} = \frac{(2\pi)^n(1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)}e^{-(1+\delta)\varphi(0)}v(0).$$

By the arbitrariness of δ, we know (20) holds for I_{t+k_1}.

For J_t, we know when t is large enough, the function $v(z)$ and $e^{-\varphi(z)}$ are uniformly bounded by some constant $M > 0$ with respect to t. Then it follows from Lemma 11 that

$$\begin{aligned}
\limsup_{t\to+\infty} J_{t+k_1} \le& M\limsup_{t\to+\infty}\int_{\{z\in\Delta^n: -t-1-k_1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t\}}\frac{\prod_{j=1}^n|z_j|^{2\beta_j}}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}dz\wedge d\bar z\\
=& M\lim_{t\to+\infty}\frac{\pi^n(q_\alpha+1)(e^{-q_\alpha t}-e^{-q_\alpha(t+1+k_1+k_2)})}{q_\alpha\prod_{1\le j\le n}(\alpha_j+1)},
\end{aligned}$$

where $q_\alpha = \frac{\beta_j+1}{p_j} - 1$. As $k_1+k_2+1 > 0$, we have $\lim\limits_{t\to+\infty} J_t = 0$.

Lemma 13 is proved.

Lemma 14 *Let $\Delta^n \subset \mathbb{C}^n$ be a polydisc. Let $f = \sum_{\alpha\in E} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on Δ^n, where $E := \left\{\alpha = (\alpha_1, \ldots, \alpha_n) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_j} = 1\right\}$. Assume that v is a nonnegative continuous real function on Δ^n. Denote*

$$S_t := \int_{\{z\in\Delta^n: -t-1-k_2 < \max\limits_{1\le j\le n}\{2p_j \log|z_j|\} < -t+k_1\}} v(z)\frac{|f|^2}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}} e^{-\varphi(z_1,\ldots,z_n)} dz\wedge d\bar{z}.$$

Then we have

$$\limsup_{t\to+\infty} S_t \le \sum_{\alpha\in E} \frac{|b_\alpha|^2 (2\pi)^n (1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)} v(0)e^{-\varphi(0)}.$$

Proof In the proof of Lemma 14, we follow the notations we used in the proof of Lemma 13. Let $z_i = e^{\frac{-t}{2p_i}} w_i$, by Lemma 11, we have

$$\begin{aligned}
S_t &= \int_{\{z\in\Delta^n: -t-1-k_2 < \max\limits_{1\le j\le n}\{2p_j \log|z_j|\} < -t+k_1\}} v(z)\frac{|f|^2}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}} e^{-\varphi(z_1,\ldots,z_n)} dz\wedge d\bar{z} \\
&= \int_{\{w\in\Delta^n: e^{-1-k_1-k_2} < \max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}\cap B_{\delta,t}} v(e^{\frac{-t}{2p}}w)\frac{|f(w)|^2}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}} e^{-\varphi(e^{\frac{-t}{2p}}w)} dw\wedge d\bar{w} + \\
&\int_{\{w\in\Delta^n: e^{-1-k_1-k_2} < \max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}\setminus B_{\delta,t}} v(e^{\frac{-t}{2p}}w)\frac{|f(w)|^2}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}} e^{-\varphi(e^{\frac{-t}{2p}}w)} dw\wedge d\bar{w} \\
&\le CC_1C_2\lambda(B_{\delta,t}) + \left(\sup_{\{e^{-1-k_1-k_2} < \max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} v(e^{\frac{-t}{2p}}w)\right) e^{-(1+\delta)\varphi(0)} \times \\
&\int_{\{e^{-1-k_1-k_2} < \max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} \frac{|f(w)|^2}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}} dw\wedge d\bar{w} \\
&= CC_1C_2\lambda(B_{\delta,t}) + \left(\sup_{\{e^{-1-k_1-k_2} < \max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} v(e^{\frac{-t}{2p}}w)\right) e^{-(1+\delta)\varphi(0)} \times \\
&\sum_{\alpha\in E}\int_{\{e^{-1-k_1-k_2} < \max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} \frac{|b_\alpha|^2\prod_{j=1}^{n_1}|w_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}} dw\wedge d\bar{w}.
\end{aligned}$$

Hence we know

$$\limsup_{t\to+\infty} S_t \le \sum_{\alpha\in E} \frac{|b_\alpha|^2 (2\pi)^n (1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)} v(0)e^{-\varphi(0)}.$$

Lemma 15 (see [32]) *Let $c(t)$ be a positive measurable function on $(0, +\infty)$, and let $a \in \mathbb{R}$. Assume that $\int_t^{+\infty} c(s)e^{-s}ds \in (0, +\infty)$ when t near $+\infty$. Then we have*

(1) $\lim\limits_{t\to+\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = 1$ *if and only if $a = 1$,*

(2) $\lim_{t\to+\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = 0$ *if and only if* $a > 1$,

(3) $\lim_{t\to+\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = +\infty$ *if and only if* $a < 1$.

Lemma 16 (see [31]) *If* $c(t)$ *is a positive measurable function on* $(T, +\infty)$ *such that* $c(t)e^{-t}$ *is decreasing on* $(T, +\infty)$ *and* $\int_{T_1}^{+\infty} c(s)e^{-s}ds < +\infty$ *for some* $T_1 > T$*, then there exists a positive measurable function* $\tilde{c}$ *on* $(T, +\infty)$ *satisfying the following statements:*

(1) $\tilde{c} \geq c$ *on* $(T, +\infty)$*;*

(2) $\tilde{c}(t)e^{-t}$ *is strictly decreasing on* $(T, +\infty)$ *and* $\tilde{c}$ *is increasing on* $(a, +\infty)$*, where* $a > T$ *is a real number;*

(3) $\int_{T_1}^{+\infty} \tilde{c}(s)e^{-s}ds < +\infty$.

Moreover, if $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$ *and* $c \in \mathcal{P}_T$*, we can choose* $\tilde{c}$ *satisfying the above conditions,* $\int_T^{+\infty} \tilde{c}(s)e^{-s}ds < +\infty$ *and* $\tilde{c} \in \mathcal{P}_T$.

3 Preparations II: Multiplier Ideal Sheaves and Optimal L^2 Extensions

In this section, we recall and present some lemmas related to multiplier ideal sheaves and optimal L^2 extensions.

3.1 Multiplier Ideal Sheaves

We need the following lemmas in the local cases.

Let $\Delta \subset \mathbb{C}$ be the unit disc. Let $X = \Delta^m$ and let $Y = \Delta^{n-m}$. Denote $M = X \times Y$. Let π_1 and π_2 be the natural projections from M to X and Y respectively.

Let ψ_1 be a plurisubharmonic function on X. Let $\psi = \pi_1^*(\psi_1)$. Let φ_1 be a Lebesgue measurable function on X and $\varphi_2 \in Psh(Y)$. Denote $\varphi = \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Lemma 17 *Assume that*

$$\int_M |f|^2 e^{-\varphi} c(-\psi) < +\infty.$$

Then for any $w \in \Delta^{n-m}$,

$$\int_{z\in\Delta^m} |f(z, w)|^2 e^{-\varphi_1} c(-\psi_1) < +\infty.$$

Proof According to the Fubini's Theorem, we have

$$\begin{aligned}
&\int_{z\in\Delta^m}|f(z,w)|^2e^{-\varphi_1}c(-\psi_1)\\
\le&\frac{1}{(\pi r^2)^{n-m}}\int_{w'\in\Delta^{n-m}(w,r)}\left(\int_{z\in\Delta^m}|f(z,w')|^2e^{-\varphi_1}c(-\psi_1)\right)\\
\le&\frac{e^T}{(\pi r^2)^{n-m}}\int_{w'\in\Delta^{n-m}(w,r)}\left(\int_{z\in\Delta^m}|f(z,w')|^2e^{-\varphi_1}c(-\psi_1)\right)e^{-\varphi_2}\\
\le&\frac{e^T}{(\pi r^2)^{n-m}}\int_{\Delta^n}|f|^2e^{-\varphi}c(-\psi)<+\infty,
\end{aligned}$$

where $r>0$ such that $\Delta^{n-m}(w,r)\Subset\Delta^{n-m}$, and $T:=-\sup_{w'\in\Delta^{n-m}(w,r)}\varphi_2(w')$.

Lemma 18 *Let $f_1(z)$ be a holomorphic function on X such that $(f_1,o)\in\mathcal{I}(\varphi_1+\psi_1)_o$, and $f_2(w)$ be a holomorphic function on Y such that $(f_2,w)\in\mathcal{I}(\varphi_2)_w$ for any $w\in Y$. Let $\tilde{f}(z,w)=f_1(z)f_2(w)$ on M, then $\left(\tilde{f},(o,w)\right)\in\mathcal{I}(\varphi+\psi)_{(o,w)}$ for any $(o,w)\in Y$.*

Proof According to $(f_1,o)\in\mathcal{I}(\varphi_1+\psi_1)_o$ and $(f_2,w)\in\mathcal{I}(\varphi_2)_w$, we can find some $r>0$ such that

$$\int_{\Delta^m(o,r)}|f_1|^2e^{-\varphi_1-\psi_1}<+\infty,$$

and

$$\int_{\Delta^{n-m}(w,r)}|f_2|^2e^{-\varphi_2}<+\infty.$$

Then using Fubini's Theorem, we get

$$\int_{\Delta^n((o,w),r)}|\tilde{f}|^2e^{-\varphi-\psi}=\int_{\Delta^m(o,r)}|f_1|^2e^{-\varphi_1-\psi_1}\int_{\Delta^{n-m}(w,r)}|f_2|^2e^{-\varphi_2}<+\infty,$$

which means that $\left(\tilde{f},(o,w)\right)\in\mathcal{I}(\varphi+\psi)_{(o,w)}$.

Let $\Delta^{n_1}=\left\{w\in\mathbb{C}^{n_1}:|w_j|<1 \text{ for any } j\in\{1,\dots,n_1\}\right\}$ be product of the unit disks. Let Y be an n_2-dimensional complex manifold, and let $M=\Delta^{n_1}\times Y$. Denote $n=n_1+n_2$. Let π_1 and π_2 be the natural projections from M to Δ^{n_1} and Y respectively. Let ρ_1 be a nonnegative Lebesgue measurable function on Δ^{n_1} satisfying that $\rho_1(w)=\rho_1(|w_1|,\dots,|w_{n_1}|)$ for any $w\in\Delta^{n_1}$ and the Lebesgue measure of $\{w\in\Delta^{n_1}:\rho_1(w)>0\}$ is positive. Let ρ_2 be a nonnegative Lebesgue measurable function on Y, and denote that $\rho=\pi_1^*(\rho_1)\times\pi_2^*(\rho_2)$ on M.

Lemma 19 (see [2]) *For any holomorphic* $(n,0)$ *form* F *on* M*, there exists a unique sequence of holomorphic* $(n_2,0)$ *forms* $\{F_\alpha\}_{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}}$ *on* Y *such that*

$$F=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}}\pi_1^*(w^\alpha dw_1\wedge\ldots\wedge dw_{n_1})\wedge\pi_2^*(F_\alpha), \tag{22}$$

where the right term of the above equality is uniformly convergent on any compact subset of M*. Moreover, if* $\int_M |F|^2\rho<+\infty$*, we have*

$$\int_Y |F_\alpha|^2\rho_2<+\infty \tag{23}$$

for any $\alpha\in\mathbb{Z}^{n_1}_{\geq 0}$*.*

Let $\tilde{M}\subset M$ be an $n-$dimensional complex manifold satisfying that $\{o\}\times Y\subset\tilde{M}$, where o is the origin in Δ^{n_1}.

Lemma 20 (see [2]) *For any holomorphic* $(n,0)$ *form* F *on* $\tilde{M}$*, there exist a unique sequence of holomorphic* $(n_2,0)$ *forms* $\{F_\alpha\}_{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}}$ *on* Y *and a neighborhood* $M_2\subset\tilde{M}$ *of* $\{o\}\times Y$*, such that*

$$F=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}}\pi_1^*(w^\alpha dw_1\wedge\ldots\wedge dw_{n_1})\wedge\pi_2^*(F_\alpha)$$

on M_2*, where the right term of the above equality is uniformly convergent on any compact subset of* M_2*. Moreover, if* $\int_{\tilde{M}}|F|^2\rho<+\infty$*, we have*

$$\int_K |F_\alpha|^2\rho_2<+\infty$$

for any compact subset K *of* Y *and* $\alpha\in\mathbb{Z}^{n_1}_{\geq 0}$*.*

Let $f=\sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $D=\{w\in\mathbb{C}^n : |w_j|<r_0$ for any $j\in\{1,\ldots,n\}\}$, where $r_0>0$. Let

$$\psi=\max_{1\leq j\leq n_1}\left\{2p_j\log|w_j|\right\}$$

be a plurisubharmonic function on $\mathbb{C}^n$, where $n_1\leq n$ and $p_j>0$ is a constant for any $j\in\{1,\ldots,n_1\}$. We recall a characterization of $\mathcal{I}(\psi)_o$, where o is the origin in $\mathbb{C}^n$.

Lemma 21 (see [2]) $(f,o)\in\mathcal{I}(\psi)_o$ *if and only if* $\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_j}>1$ *for any* $\alpha\in\mathbb{Z}^n_{\geq 0}$ *satisfying* $b_\alpha\neq 0$*.*

Let $\Omega=\Delta\subset\mathbb{C}$ be an unit disk. Let $Y=\Delta^n$. Denote $M=\Omega\times Y$. Let π_1 and π_2 be the natural projections from M to Ω and Y respectively.

Let $\psi = \pi_1^*(2p \log |z|) + N$ be a plurisubharmonic function on M, where $N \le 0$ is a plurisubharmonic function on M and $N|_{\{0\}\times\Delta^n} \not\equiv -\infty$. Assume that there exist a holomorphic function g on Δ and a function $\tilde{\psi}_2 \in Psh(M)$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log |g|) + \tilde{\psi}_2,$$

where $ord_0(g) = q$. We assume that $g = dz^q h(z)$ on Δ, where d is a constant, $h(z)$ is a holomorphic function on Δ and $h(0) = 1$.

Let $\varphi_2 \in Psh(Y)$. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Let F be a holomorphic $(n, 0)$ form on M, where

$$F = \sum_{j=k}^{\infty} \pi_1^*(z^j dz) \wedge \pi_2^*(F_j)$$

according to Lemma 19. Here $k \in \mathbb{N}$ and F_j is a holomorphic $(n, 0)$ form on Y for any $j \ge k$.

Assume that $\int_M |F|^2 e^{-\varphi} c(-\psi) < +\infty$ and $c(t)$ is increasing near $+\infty$. As $\psi = \pi_1^*(2p \log |z|) + N \le \pi_1^*(2p \log |z|)$, when t is large enough, we have

$$S := \int_{\{\pi_1^*(2p\log|z|)<-t\}} |F|^2 e^{-\varphi} c(-\pi_1^*(2p \log |z|)) \le \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty.$$

Lemma 22 *Let c be a positive measurable function on $(0, +\infty)$ such that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, c is increasing near $+\infty$, and $\int_0^{+\infty} c(s)e^{-s} ds < +\infty$. Assume that $k \ge q$, and*

$$S = \int_{\{\pi_1^*(2p\log|z|)<-t\}} |F|^2 e^{-\varphi} c\big(-\pi_1^*(2p \log |z|)\big) < +\infty.$$

Then

$$\big(F, (0, y)\big) \in \big(\mathcal{O}(K_M)\big) \otimes \mathcal{I}(\pi_1^*(2\log |g|) + \pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $y \in Y$.

Proof It follows from M is a Stein manifold that there exist smooth plurisubharmonic functions $\tilde{\psi}_{2,l}$ on M such that $\tilde{\psi}_{2,l}$ are decreasingly convergent to $\tilde{\psi}_2$. Since the computation is local, we assume that $|\tilde{\psi}_{2,l}(z, w) - \tilde{\psi}_{2,l}(0, w)| \le \epsilon$ for any $(z, w) \in M = \Delta \times \Delta^n$.

We also assume that $F = z^k \tilde{h}(z, w) dz \wedge dw$ on M and $|h(z) - h(0)| < \epsilon$ for any $z \in \Delta$.

$$
\begin{aligned}
S &= \int_{\{\pi_1^*(2p\log|z|)<-t\}} |F|^2 e^{-\pi_1^*(2\log|g|)-\tilde{\psi}_2+\pi_1^*(2p\log|z|)+N-\pi_2^*(\varphi_2)} c(-\pi_1^*(2p\log|z|)) \\
&\geq \int_{\{\pi_1^*(2p\log|z|)<-t\}} \frac{|z|^{2k+2p}|\tilde{h}(z,w)|^2}{d|z|^{2q}|h(z)|^2} e^{-\tilde{\psi}_{2,l}+N-\pi_2^*(\varphi_2)} c(-\pi_1^*(2p\log|z|)) \\
&\geq \int_{w\in\Delta^n}\int_{\{2p\log|z|<-t\}} |z|^{2k+2p-2q} e^N \frac{|\tilde{h}(z,w)|^2}{d|1+\epsilon|^2} e^{-\tilde{\psi}_{2,l}(0,w)-\epsilon-\pi_2^*(\varphi_2)} c(-\pi_1^*(2p\log|z|)) \\
&= \int_{w\in\Delta^n}\left(2\int_{\{2p\log|r|<-t\}}\int_0^{2\pi} |r|^{2k+2p-2q+1} e^{N(re^{i\theta},w)}|\tilde{h}(re^{i\theta},w)|^2 c(-2p\log r)d\theta dr\right)\times \\
&\quad \frac{e^{-\tilde{\psi}_{2,l}(0,w)-\epsilon-\varphi_2(w)}}{d|1+\epsilon|^2} \\
&\geq \frac{4\pi}{d|1+\epsilon|^2}e^{-\epsilon}\int_{\{2p\log|r|<-t\}} r^{2k+2p-2q+1}c(-2p\log r)dr\int_{w\in\Delta^n}|\tilde{h}(0,w)|^2e^{N(0,w)-\tilde{\psi}_{2,l}(0,w)}e^{-\varphi_2(w)} \\
&= \frac{2\pi}{pd|1+\epsilon|^2}e^{-\epsilon}\int_t^{+\infty} c(s)e^{-(\frac{k-q+1}{p}+1)s}ds\int_{w\in\Delta^n}|\tilde{h}(0,w)|^2e^{N(0,w)-\tilde{\psi}_{2,l}(0,w)}e^{-\varphi_2(w)}.
\end{aligned} \tag{24}
$$

Since $k \geq q$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$, we have

$$
\int_t^{+\infty} c(s)e^{-\left(\frac{k-q+1}{p}+1\right)s}ds < +\infty.
$$

Letting $l \to +\infty$ in (24), it follows from $S < +\infty$ and Levi's Theorem that we have

$$
\frac{2\pi}{pd|1+\epsilon|^2}e^{-\epsilon}\int_t^{+\infty} c(s)e^{-(\frac{k-q+1}{p}+1)s}ds\int_{w\in\Delta^n}|\tilde{h}(0,w)|^2e^{N(0,w)-\tilde{\psi}_2(0,w)}e^{-\varphi_2(w)} < +\infty. \tag{25}
$$

It follows from inequality (25) that we have

$$
\int_{w\in\Delta^n}|\tilde{h}(0,w)|^2e^{N(0,w)-\tilde{\psi}_2(0,w)}e^{-\varphi_2(w)} < +\infty. \tag{26}
$$

Note that $N|_{\{0\}\times\Delta^n} \not\equiv -\infty$. It follows from (26) that there must exist $w_1 \in Y$ such that $\tilde{\psi}_2(0,w_1) > -\infty$.

Since $k \geq q$, we know $z^k \in \mathcal{I}(2\log|g|)_0$. It follows from Lemma 18 that

$$
\left(\pi_1^*(z^k dz)\wedge\pi_2^*(F_k),(0,y)\right) \in \left(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\right)_{(0,y)}, \tag{27}
$$

for any $y \in Y$.

Let $0 < \delta < 1$ be a constant such that $w_1 \in \Delta_\delta^n$. It follows from Fubini's Theorem and $c(t)e^{-t}$ is decreasing with respect to t that

$$\begin{aligned}&\int_{\Delta_\delta\times\Delta_\delta^n}|\pi_1^*(z^kdz)\wedge\pi_2^*(F_k)|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))\\=&\int_{\Delta_\delta}|z^k|^2e^{-\varphi_1}c(-2p\log|z|)|dz|^2\cdot\int_{\Delta_\delta^n}|F_k|^2e^{-\varphi_2}\\\le& C\int_{\Delta_\delta}|z^k|^2e^{-\varphi_1-2p\log|z|}|dz|^2\cdot\int_{\Delta_\delta^n}|F_k|^2e^{-\varphi_2},\end{aligned}\tag{28}$$

where C is a positive constant independent of F.

Consider

$$\begin{aligned}I:=&\int_{\Delta_\delta}|z|^{2k}e^{-\varphi_1-2p\log|z|}|dz|^2\\=&\int_{\Delta_\delta}|z|^{2k}e^{-2\log|g|-u(z)}|dz|^2\\=&\int_{\Delta_\delta}|z|^{2k}e^{-2\log|g|+N(z,w)-\tilde{\psi}_2(z,w)}|dz|^2.\end{aligned}\tag{29}$$

As N is a plurisubharmonic function on M, e^N has a upper bound C_1 on $\Delta_\delta\times\Delta_\delta^n$ (especially, C_1 is independent of w). Hence

$$I\le C_1\int_{\Delta_\delta}|z|^{2k}e^{-2\log|g|-\tilde{\psi}_2(z,w)}|dz|^2=C_1\int_{\Delta_\delta}|z|^{2k-2q}|h(z)|^2e^{-\tilde{\psi}_2(z,w)}|dz|^2.$$

Denote $M(w)=\int_{\Delta_\delta}|z|^{2k-2q}|h(z)|^2e^{-\tilde{\psi}_2(z,w)}|dz|^2$. We have $I\le M(w)$ for any $w\in\Delta_\delta^n$, especially $I\le M(w_1)$.

Next we prove $M(w_1)<+\infty$. Note that $M(w_1)=\int_{\Delta_\delta}|z|^{2k-2q}|h(z)|^2e^{-\tilde{\psi}_2(z,w_1)}|dz|^2$. As $e^{-\tilde{\psi}_2(0,w_1)}>-\infty$, $k\ge q$ and $h(z)$ is a holomorphic function on Δ_δ, by Hölder inequality, we have

$$\begin{aligned}M(w_1)=&\int_{\Delta_\delta}|z|^{2k-2q}|h(z)|^2e^{-\tilde{\psi}_2(z,w_1)}|dz|^2\\\le&\left(\int_{\Delta_\delta}|z|^{s(2k-2q)}|h(z)|^{2s}|dz|^2\right)^{\frac{1}{s}}\left(\int_{\Delta_\delta}e^{-\frac{s}{s-1}\tilde{\psi}_2(z,w_1)}|dz|^2\right)^{\frac{s-1}{s}}\\<&+\infty,\end{aligned}$$

where $s>1$ is a real number. Hence we know $I<+\infty$. Then

$$\int_{\Delta_\delta\times\Delta_\delta^n}|\pi_1^*(z^kdz)\wedge\pi_2^*(F_k)|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))<+\infty.$$

As $S=\int_{\Delta_\delta\times\Delta_\delta^n}|F|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))<+\infty$, we have

$$\int_{\Delta_\delta\times\Delta_\delta^n}|F-\pi_1^*(z^k dz)\wedge\pi_2^*(F_k)|^2 e^{-\varphi}c(-\pi_1^*(2p\log|z|))<+\infty.$$

Note that

$$F-\pi_1^*(z^k dz)\wedge\pi_2^*(F_k)=\sum_{j=k+1}^{\infty}\pi_1^*(z^j dz)\wedge\pi_2^*(F_j)$$

and $k+1>q$. Using the same method as above, we can get that

$$\big(\pi_1^*(z^{k+1}dz)\wedge\pi_2^*(F_{k+1}),(0,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $y\in Y$ and

$$\int_{\Delta_\delta\times\Delta_\delta^n}\Big|F-\pi_1^*(z^k dz)\wedge\pi_2^*(F_k)-\pi_1^*(z^{k+1}dz)\wedge\pi_2^*(F_{k+1})\Big|^2 e^{-\varphi}c(-\pi_1^*(2p\log|z|))<+\infty.$$

By induction, we know that

$$\big(\pi_1^*(z^j dz)\wedge\pi_2^*(F_j),(0,y)\big)\in\big(\mathcal{O}(K_M))\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $j\geq k$, $y\in Y$. Then it follows from Lemma 10 that

$$\big(F,(0,y)\big)=\big(\sum_{j=k}^{\infty}\pi_1^*(z^j dz)\wedge\pi_2^*(F_j),(0,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $y\in Y$.

Let $\Omega=\Delta$ be the unit disk in $\mathbb{C}$, where the coordinate is z. Let $Y=\Delta^n$ be the unit polydisc in $\mathbb{C}^n$, where the coordinate is $w=(w_1,\ldots,w_n)$. Let $M=\Omega\times Y$. Let π_1,π_2 be the natural projections from M to Ω and Y.

Let $\psi_1=2p\log|z|+\psi_0$ on Ω, where $p>0$ and ψ_0 is a negative subharmonic function on Ω with $\psi_0(0)>-\infty$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1+\psi$ is a subharmonic function on Ω. It follows from the Weierstrass Theorem on open Riemann surfaces (see [16]) and the Siu's Decomposition Theorem, that $\varphi_1+\psi_1=2\log|g|+2u$, where g is a holomorphic function on Ω with $ord(g)_0=q\in\mathbb{N}$, u is a subharmonic function on Ω such that $v(dd^c u,z)\in[0,1)$ for any $z\in\Omega$. Let φ_2 be a plurisubharmonic function on Y. Denote $\varphi:=\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)$ on M.

Lemma 23 *$\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}=\mathcal{I}\big(\pi_1^*(\psi_1)+\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$ for any $y\in Y$.*

Proof It is easy to see that $\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}\supset\mathcal{I}\big(\pi_1^*(\psi_1+\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$ for any $y\in Y$.

Now we prove $\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}\subset\mathcal{I}\big(\pi_1^*(\psi_1+\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$ for any $y\in Y$.

Let $F \in \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(0,y)}$. Then by Lemma 19 (although Lemma 19 is stated for the holomorphic $(n, 0)$ forms, since our case is local, the decomposition still holds for holomorphic functions), we know $F = \sum_{j=k}^{+\infty} z^j F_j$ on M, where the right hand side is uniformly convergent to F on M, $F_j(w)$ is a holomorphic function on Y for any $j \geq k$ and $F_k \not\equiv 0$.

Since the case is local, we also assume that $F = z^k h_1(z, w)$ on M, where $h_1(z, w)$ is a holomorphic function on M satisfying $h_1(0, w) \neq 0$. $F \in \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(0,y)}$ implies that $k \geq q$ and

$$\int_{\Delta\times\Delta^n} |z|^{2k-2q}|h_1(z, w)|e^{-\pi_2^*(\varphi(w))} < +\infty.$$

By Fubini's theorem and sub-mean value inequality of subharmonic functions, we have

$$\begin{aligned}\int_{\Delta\times\Delta^n} |z|^{2k-2q}|h_1(z, w)|e^{-\pi_2^*(\varphi_2(w))} &= \int_{w\in\Delta^n}\int_{z\in\Delta} |z|^{2k-2q}|h_1(z, w)|e^{-\pi_2^*(\varphi_2(w))} \\ &\geq C\int_{w\in\Delta^n} |h_1(0, w)|e^{-\varphi_2(w)} \\ &= C\int_{w\in\Delta^n} |F_k|^2 e^{-\varphi_2(w)},\end{aligned}$$

where $C > 0$ is a constant. Hence we have $(F_k, y) \in \mathcal{I}(\varphi_2)_y$ for any $y \in Y$.

As $v(dd^c u, 0) \in [0, 1)$, we have $(z^k, 0) \in \mathcal{I}(2\log|g|)_0 = \mathcal{I}(2\log|g| + 2u)_0 = \mathcal{I}(\psi_1 + \varphi_1)_0$. It follows from Lemma 18 that we know $\big(z^k F_k, (0, y)\big) \in \mathcal{I}\big(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2)\big)_{(0,y)}$. Then we know

$$\int_{\Delta\times\Delta^n} |z^k F_k|^2 e^{\pi_1^*(-2\log|g|+2u)-\pi_2^*(\varphi_2)} < +\infty,$$

which implies

$$\int_{\Delta\times\Delta^n} |z^k F_k|^2 e^{\pi_1^*(-2\log|g|)-\pi_2^*(\varphi_2)} < +\infty.$$

Hence we have $\big(F - z^k F_k, (0, y)\big) \in \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(0,y)}$.

Denote $\tilde{F}_{k+1} = F - z^k F_k$ on $\Delta \times \Delta^n$. Note that $\tilde{F}_{k+1} = \sum_{j=k+1}^{+\infty} z^j F_j$ on M and $\big(\tilde{F}_{k+1}, (0, y)\big) \in \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(0,y)}$.

By using similar discussion as above, we know

$$\big(z^{k+1} F_{k+1}, (0, y)\big) \in \mathcal{I}\big(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2)\big)_{(0,y)},$$

and

$$\int_{\Delta\times\Delta^n} |z^{k+1} F_{k+1}|^2 e^{\pi_1^*(-2\log|g|)-\pi_2^*(\varphi_2)} < +\infty.$$

Hence $\big(\tilde{F}_{k+1} - z^{k+1}F_{k+1}, (0, y)\big) \in \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(0,y)}$. Denote $\tilde{F}_{k+2} = \tilde{F}_{k+1} - z^{k+1}F_{k+1}$ on $\Delta \times \Delta^n$. Note that $\tilde{F}_{k+2} = \sum_{j=k+2}^{+\infty} z^j F_j$ on M and $\big(\tilde{F}_{k+2}, (0, y)\big) \in \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(0,y)}$.

By induction, we know that for any $j \geq k$,

$$\big(z^j F_j, (0, y)\big) \in \mathcal{I}\big(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2)\big)_{(0,y)}$$

holds. Then it follows from Lemma 10 that we know

$$\big(F, (0, y)\big) \in \mathcal{I}\big(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $y \in Y = \Delta^n$.

We recall a well known result about multiplier ideal sheaves.

Lemma 24 (see [2]) *Let Φ_1 and Φ_2 be plurisubharmonic functions on Δ^n satisfying $\Phi_2(o) > -\infty$, where $n \in \mathbb{Z}_{>0}$ and o is the origin in Δ^n. Then $\mathcal{I}(\Phi_1)_o = \mathcal{I}(\Phi_1 + \Phi_2)_o$.*

3.2 Optimal Jet L^2 Extensions

Let Ω be an open Riemann surface with nontrivial Green function. Let $Z_\Omega = \{z_j : j \in \mathbb{N}_+ \& j < \gamma\}$ be a subset of Ω of discrete points, where $\gamma \in \mathbb{Z}_{\geq 2}^+$ or $\gamma = +\infty$. Let Y be an $n-$dimensional weakly pseudoconvex Kähler manifold. Denote $M = \Omega \times Y$. Let π_1 and π_2 be the natural projections from M to Ω and Y respectively. Denote $Z_0 := Z_\Omega \times Y$. Denote $Z_j := \{z_j\} \times Y$.

Let $\tilde{M} \subset M$ be an $n-$dimensional weakly pseudoconvex Kähler manifold satisfying that $Z_0 \subset \tilde{M}$. Let F be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0.

Let ψ be a plurisubharmonic function on $\tilde{M}$. It follows from Siu's decomposition theorem that

$$dd^c\psi = \sum_{j\geq 1} 2p_j[Z_j] + \sum_{i\geq 1} \lambda_i[A_i] + R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $(n-1)-$dimensional analytic set, and where R is a closed positive current with the property that $dim E_c(R) < n - 1$ for every $c > 0$, where $E_c(R) = \{x \in \tilde{M} : v(R, x) \geq c\}$ is the upperlevel sets of Lelong number. We assume that $p_j > 0$ for any $1 \leq j < \gamma$.

Then $N := \psi - \pi_1^*\big(\sum_{j\geq 1} 2p_j G_\Omega(z, z_j)\big)$ is a plurisubharmonic function on $\tilde{M}$. We assume that $N \leq 0$.

Let φ_1 be a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on $\tilde{M}$. With similar discussion as above, by Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j\geq 1} 2\tilde{q}_j[Z_j] + \sum_{i\geq 1} \tilde{\lambda}_i[\tilde{A}_i] + \tilde{R},$$

where $\tilde{q}_j \geq 0$ for any $1 \leq j < \gamma$.

By Weierstrass theorem on open Riemann surface, there exists a holomorphic function g on Ω such that $ord_{z_j}(g) = q_j := [\tilde{q}_j]$ for any $z_j \in Z_\Omega$ and $g(z) \neq 0$ for any $z \notin Z_\Omega$, where $[q]$ equals to the integer part of the nonnegative real number q. Then we know that there exists a plurisubharmonic function $\tilde{\psi}_2 \in Psh(\tilde{M})$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2.$$

Let $\varphi_2 \in Psh(Y)$. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

For $1 \leq j < \gamma$, let $(V_j, \tilde{z}_j)$ be a local coordinated open neighborhood of z_j in Ω satisfying $V_j \Subset \Omega$, $\tilde{z}_j(z_j) = 0$ under the local coordinate and $V_j \cap V_k = \emptyset$ for any $j \neq k$. Denote $V_0 := \cup_{1\leq j<\gamma} V_j$. We assume that $g = d_j\tilde{z}_j^{q_j}h_j(z)$ on V_j, where d_j is a constant, $h_j(z)$ is a holomorphic function on V_j and $h(z_j) = 1$.

Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$.

We have the following lemma.

Lemma 25 *Let F be a holomorphic $(n, 0)$ form on U_0 such that for $1 \leq j < \gamma$, $F = \pi_1^*(\tilde{z}_j^{k_j} f_j dz_j) \wedge \pi_2^*(F_j)$ on $U_j \Subset U_0 \cap (V_j \times Y)$, where U_j is an open neighborhood of Z_j in $\tilde{M}$, k_j is a nonnegative integer, f_j is a holomorphic function on V_j satisfying $f_j(z_j) = a_j \neq 0$ and F_j is a holomorphic $(n-1, 0)$ form on Y.*

Denote $I_F := \{j : 1 \leq j < \gamma \,\&\, k_j + 1 - q_j \leq 0\}$. Assume that $k_j + 1 = q_j$ for any $j \in I_F$ and $\tilde{\psi}_2|_{Z_j}$ is not identically $-\infty$ on Z_j. If

$$\sum_{j\in I_F} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty \tag{30}$$

and

$$\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty, \tag{31}$$

for any $j \notin I_F$. Then there exists a holomorphic $(n, 0)$ form $\tilde{F}$ on $\tilde{M}$ such that $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)))_{(z_j,y)}$ for any $1 \leq j < \gamma$ and $y \in Y$ and

$$\int_{\tilde{M}} |\tilde{F}|^2 c(-\psi) e^{-\varphi} \leq (\int_0^{+\infty} c(s)e^{-s}ds) \sum_{j \in I_F} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty. \tag{32}$$

Proof Denote $G = \sum_{j \geq 1} 2p_j G_\Omega(z, z_j)$. Note that $\psi \leq \pi_1^*(G)$ and $c(t)e^{-t}$ is decreasing. We have

$$c(-\psi)e^{-\varphi} \leq c(-\pi_1^*(G))e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)}.$$

To prove Lemma 25, it suffice to prove

$$\begin{aligned} &\int_{\tilde{M}} |\tilde{F}|^2 c\big(-\pi_1^*(G)\big)e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)} \\ \leq &(\int_0^{+\infty} c(s)e^{-s}ds) \sum_{j \in I_F} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty. \end{aligned} \tag{33}$$

The following remark shows that we only need to prove formula (33) when Z_Ω is a finite set.

Remark 19 It follows from Lemma 6 that there exists a sequence of open Riemann surfaces $\{\Omega_l\}_{l \in \mathbb{Z}^+}$ such that $\Omega_l \Subset \Omega_{l+1} \Subset \Omega$, $\cup_{l \in \mathbb{Z}^+} \Omega_l = \Omega$ and $\{G_{\Omega_l}(\cdot, z_0) - G_\Omega(\cdot, z_0)\}$ is decreasingly convergent to 0 on Ω with respect to l for any $z_0 \in \Omega$.

Denote $Z_l := \Omega_l \cap Z_0$. As Z_Ω is a subset of Ω of discrete points, Z_l is a set of finite points.

Denote

$$G_l := \sum_{z_j \in Z_l} 2p_j G_\Omega(z, z_j)$$

and

$$\varphi_{1,l} := \varphi_1 + G - G_l.$$

Then we have $N + \pi_1^*(G_l) + \pi_1^*(\varphi_{1,l}) = N + \pi_1^*(G) + \pi_1^*(\varphi_1) = \psi + \pi_1^*(\varphi_1)$.

Let $I_l := I_F \cap \{j : z_j \in Z_l\}$. Denote $\tilde{M}_l := (\Omega_l \times Y) \cap \tilde{M}$. We note that $\tilde{M}_l$ is weakly pseudoconvex Kähler. Now we assume that the formula (33) holds on $\tilde{M}_l$, i.e. we have

$$\begin{aligned} &\int_{\tilde{M}_l} |\tilde{F}_l|^2 c\big(-\pi_1^*(G_l)\big)e^{-N-\pi_1^*(\varphi_{1,l})-\pi_2^*(\varphi_2)} \\ \leq &(\int_0^{+\infty} c(s)e^{-s}ds) \sum_{j \in I_l} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty, \end{aligned}$$

where F_l is a holomorphic $(n, 0)$ form on M_l satisfying $(F_l - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g_0|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$ for any l, $z_j \in Z_l$, and $y \in Y$.

As $G \leq G_l$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, we have

$$
\begin{aligned}
&\int_{\tilde{M}_l} |\tilde{F}_l|^2 c\big(-\pi_1^*(G)\big)e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)} \\
\leq &\int_{\tilde{M}_l} |\tilde{F}_l|^2 c\big(-\pi_1^*(G_l)\big)e^{-N-\pi_1^*(\varphi_{1,l})-\pi_2^*(\varphi_2)} \\
\leq &\Big(\int_0^{+\infty} c(s)e^{-s}ds\Big)\sum_{j\in I_l}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty.
\end{aligned}
\tag{34}
$$

Note that $\pi_1^*(G)$ is continuous on $\tilde{M}\backslash Z_0$, where Z_0 is a closed analytic subset of $\tilde{M}$ and $N + \pi_1^*(G) + \pi_1^*(\varphi_1) = \psi + \pi_1^*(\varphi_1)$ is a plurisubharmonic function on M. For any compact subset K of $\tilde{M}\backslash Z_0$, there exist $\hat{l}$ (depending on the choice of K) such that $K \subset\subset \tilde{M}_{\hat{l}}$ and $C_K > 0$ such that $\frac{e^{N+\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)}}{c(-\pi_1^*(G))} = \frac{e^{N+\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)+\pi_1^*(G)}}{c(-\pi_1^*(G))e^{\pi_1^*(G)}} \leq C_K$ on K. It follows from Lemma 9 and diagonal method that there exists a subsequence of $\{F_l\}$ (also denoted by $\{F_l\}$), which is compactly convergent to a holomorphic $(n, 0)$ form $\tilde{F}$ on $\tilde{M}$. Combining formula (34) and Fatou's lemma, we have

$$
\begin{aligned}
&\int_{\tilde{M}_l} |\tilde{F}|^2 c(-\pi_1^*(G))e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)} \\
\leq &\liminf_{l\to+\infty}\int_{\tilde{M}_l} |\tilde{F}_l|^2 c(-\pi_1^*(G))e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)} \\
\leq &\liminf_{l\to+\infty}\int_{\tilde{M}_l} |\tilde{F}_l|^2 c(-\pi_1^*(G_l))e^{-N-\pi_1^*(\varphi_{1,l})-\pi_2^*(\varphi_2)} \\
\leq &\liminf_{l\to+\infty}\Big(\int_0^{+\infty} c(s)e^{-s}ds\Big)\sum_{j\in I_l}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} \\
\leq &\Big(\int_0^{+\infty} c(s)e^{-s}ds\Big)\sum_{j\in I_F}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty.
\end{aligned}
$$

As $\{F_l\}$ is compactly convergent to $\tilde{F}$ on $\tilde{M}$ and $(F_l - F, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g_0|) + \pi_2^*(\varphi_2)))_{(z_j,y)}$ for any l, $z_j \in Z_l$, and $y \in Y$. It follows from Lemma 10 that $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g_0|) + \pi_2^*(\varphi_2)))_{(z_j,y)}$ for any $z_j \in Z_\Omega$, and $y \in Y$.

We continue to prove Lemma 25. Now we assume that $\gamma = m + 1$ i.e. $Z_\Omega = \{z_j : 1 \leq j \leq m\}$ and $I_F = \{1, 2, \ldots, m_1\}$, where $m_1 \leq m$.

Denote $\tilde{\psi}_{2,l} = \max\{\tilde{\psi}_2, -l\}$, where l is a positive integer. As $\tilde{\psi}_2$ is plurisubharmonic, we know $\{\tilde{\psi}_{2,l}\}_{l=1}^{+\infty}$ is a sequence of plurisubharmonic functions on $\tilde{M}$ decreasingly convergent to $\tilde{\psi}_2$. We also note that every $\tilde{\psi}_{2,l}$ is lower bounded.

When t_0 is large enough, we know that $\{G < -t\} \Subset V_0$, for any $t > t_0$. As $\tilde{M}$ is a weakly pseudoconvex Kähler manifold, there exists a sequence of weakly pseudoconvex Kähler manifolds $\tilde{M}_s$ satisfying $\tilde{M}_1 \Subset \tilde{M}_2 \cdots \Subset \tilde{M}_s \Subset \cdots \tilde{M}$ and $\cup_{s\in\mathbb{N}^+}\tilde{M}_s = \tilde{M}$.

It is easy to verify that $\int_{\{\pi_1^*(G)<-t\}\cap\tilde{M}_s}|F|^2 < +\infty$, and it follows from formula (31) and Fubini's theorem that

$$\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2 e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)} < +\infty.$$

It follows from Lemma 8 that there exists a holomorphic $(n, 0)$ form $F_{t,l,s}$ on M such that

$$\begin{aligned}&\int_{\tilde{M}_s}|F_{t,l,s}-(1-b_{t,1}(\pi_1^*(G)))F|^2 e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+v_{t,1}(\pi_1^*(G))}c(-v_{t,1}(\pi_1^*(G)))\\ \le&(\int_0^{t+1}c(s)e^{-s}ds)\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2 e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)} < +\infty,\end{aligned} \tag{35}$$

for any $t \ge t_0$.

As $b_{t,1}(\hat{t}) = 0$ for $\hat{t}$ largely enough, then we know $(F_{t,l,s} - F, (z_0, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g_0|) + \pi_2^*(\varphi_2)))_{(z_j,y)}$ for any $(z_j, y) \in Z_0 \cap \tilde{M}_s$.

For any $\epsilon > 0$, there exists $t_1 > t_0$ such that

(1) for any $j \in \{1, 2, \ldots, m\}$
$$\sup_{\tilde{z}_j\in\{G<-t_1\}\cap V_j}|g_1(\tilde{z}) - g_1(z_j)| < \epsilon,$$
where $g_1(z)$ is a smooth function on V_0 satisfying $g_1(\tilde{z})|_{V_j} := G - 2p_j\log|\tilde{z}_j|$.

(2) for any $j \in \{1, 2, \ldots, m\}$
$$\sup_{\tilde{z}_j\in\{G<-t_1\}\cap V_j}|f_j(\tilde{z}_j) - a_j| < \epsilon.$$

(3) for any $j \in \{1, 2, \ldots, m\}$
$$\sup_{\tilde{z}_j\in\{G<-t_1\}\cap V_j}|h_j(\tilde{z}_j) - 1| < \epsilon.$$

For any $(z_j, y) \in Z_0 \cap \tilde{M}_s$, letting (U_y, w) be a small local coordinated open neighborhood of y and shrinking V_j if necessary, we have $V_j \times U_y \Subset U_j$ for any j. Recall that $V_0 := \cup_{j\ge1}V_j$. Assume that $F_j = \tilde{h}_j(w)dw$ on U_y, where $\tilde{h}_j$ is a holomorphic function on U_y and $dw = dw_1 \wedge dw_2 \wedge \ldots \wedge dw_n$. There exists $t_2 > t_1$ such that when $t > t_2$, $(\{G < -t\} \times U_y) \cap (V_j \times U_y) \Subset V_j \times U_y$, for any j.

When $t > t_2$, direct calculation shows

$$\begin{aligned}&\int_{V_0\times U_y}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2 e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}\\ =&\sum_{j=1}^m\int_{\{-t-1<G<-t\}\times U_y}\frac{|\tilde{z}_j|^{2k_j}|f_j|^2|F_j|^2}{|d_j|^2|\tilde{z}|^{2q_j}|h_j|^2}e^{-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}|d\tilde{z}_j|^2|dw|^2\\ \le&\sum_{j=1}^m\int_{y\in U_y}\left(\int_{\{-t-1-\epsilon-g_1(z_0)<2p_j\log|\tilde{z}|<-t+\epsilon-g_1(z_0)\}}|\tilde{z}_j|^{2k_j-2q_j}\frac{|f_j|^2}{|d_j|^2|h_j|^2}e^{-\tilde{\psi}_{2,l}(z,w)}|d\tilde{z}_j|^2\right)\times\\ &|\tilde{h}_j|^2 e^{-\varphi_2}|dw|^2.\end{aligned}$$

Note that $2k_j - 2q_j = -2$ for any $1 \le j \le m_1$ and $2k_j - 2q_j \ge 0$ for $m_1 < j \le m$. When t is laege enough, for any j, the integral

$$\int_{\{-t-1-\epsilon-g_1(z_0)<2p_j \log|\tilde{z}|<-t+\epsilon-g_1(z_0)\}} |\tilde{z}_j|^{2k_j-2q_j} \frac{|f_j|^2}{|d_j|^2|h_j|^2} e^{-\tilde{\psi}_{2,l}(z,w)}|d\tilde{z}_j|^2$$

is uniformly bounded with respect to t. It follows from (31) that $\int_{U_y} |\tilde{h}_j|^2 e^{-\varphi_2}|dw|^2 < +\infty$ for any j. Then, by Fatou's lemma and Lemma 13 (we use Lemma 13 for the case $n = 1$), we have

$$\begin{aligned}
&\limsup_{t\to+\infty} \int_{V_0\times U_y} \mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2 e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)} \\
\le &\limsup_{t\to+\infty} \sum_{j=1}^{m} \int_{y\in U_y} \Big(\int_{\{-t-1-\epsilon-g_1(z_0)<2p_j \log|\tilde{z}|<-t+\epsilon-g_1(z_0)\}} |\tilde{z}_j|^{2k_j-2q_j} \times \\
&\frac{|f_j|^2}{|d_j|^2|h_j|^2} e^{-\tilde{\psi}_{2,l}(z,w)}|d\tilde{z}_j|^2\Big)|\tilde{h}_j|^2 e^{-\varphi_2}|dw|^2 \\
\le &\sum_{j=1}^{m} \int_{y\in U_y} \limsup_{t\to+\infty}\Big(\int_{\{-t-1-\epsilon-g_1(z_0)<2p_j \log|\tilde{z}|<-t+\epsilon-g_1(z_0)\}} |\tilde{z}_j|^{2k_j-2q_j} \times \\
&\frac{|f_j|^2}{|d_j|^2|h_j|^2} e^{-\tilde{\psi}_{2,l}(z,w)}|d\tilde{z}_j|^2\Big)|\tilde{h}_j|^2 e^{-\varphi_2}|dw|^2 \\
\le &\sum_{j=1}^{m_1} \int_{y\in U_y} \frac{2\pi(1+2\epsilon)|a_j|^2}{p_j|d_j|^2} e^{-\tilde{\psi}_{2,l}(z_0,w)}|\tilde{h}_j|^2 e^{-\varphi_2}|dw|^2.
\end{aligned}$$

Let $\epsilon \to +\infty$, we have

$$\begin{aligned}
&\limsup_{t\to+\infty} \int_{V_0\times U_y} \mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2 e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)} \\
\le &\sum_{j=1}^{m_1} \int_{y\in U_y} \frac{2\pi|a_j|^2}{p_j|d_j|^2} e^{-\tilde{\psi}_{2,l}(z_0,w)}|\tilde{h}_j|^2 e^{-\varphi_2}|dw|^2.
\end{aligned}$$

As y and U_y are arbitrarily chosen, we have

$$\begin{aligned}
&\limsup_{t\to+\infty} \int_{\tilde{M}_s} \mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2 e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)} \\
&\quad \le \sum_{j=1}^{m_1} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_{(\{z_j\}\times Y)\cap\tilde{M}_s} |F_j|^2 e^{-\tilde{\psi}_{2,l}(z_0,w)-\varphi_2}.
\end{aligned} \tag{36}$$

Since $v_{t,1}(\pi_1^*(G)) \ge \pi_1^*(G)$ and $c(t)e^{-t}$ is decreasing with respect to t, it follows from (35) and (36) that

$$
\begin{aligned}
&\limsup_{t\to+\infty}\int_{\tilde{M}_s}|F_{t,l,s}-(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\
\le&\limsup_{t\to+\infty}\int_{\tilde{M}_s}|F_{t,l,s}-(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+v_{t,1}(\pi_1^*(G))}c(-v_{t,1}(\pi_1^*(G)))\\
\le&\limsup_{t\to+\infty}(\int_0^{t+1}c(t_1)e^{-t_1}dt_1)\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}\\
\le&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_{2,l}(z_0,w)-\varphi_2}.
\end{aligned}
\tag{37}
$$

Note that $k_j-q_j=-1$ for $1\le j\le m_1$, $k_j-q_j\ge 0$ for $m_1<j\le m$, when t is large enough, we have

$$
\begin{aligned}
&\int_{(\{\pi_1^*(G)<-t\}\times Y)\cap\tilde{M}_s}|F|^2e^{\pi_1^*(-2\log|g|+G)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}c(-\pi_1^*(G))\\
\le&\sum_{j=1}^{m}C_1\int_{\{G<-t\}}|z_j|^{2k_j}|f_j|^2e^{-2\log|g_0|}e^{-G}c(-G)\int_{\{z_j\}\times Y}|F_j|^2e^{-\varphi_2}\\
\le&C_2\sum_{j=1}^{m}\int_t^{+\infty}c(t_1)e^{-t_1}dt_1\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\varphi_2}<+\infty,
\end{aligned}
$$

where C_1 and C_2 are constants. Hence we have

$$
\limsup_{t\to+\infty}\int_{\tilde{M}_s}|(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))<+\infty.
$$

Combining with (37), we know

$$
\limsup_{t\to+\infty}\int_{\tilde{M}_s}|F_{t,l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))<+\infty.
$$

By Lemma 9, we know there exists a subsequence of $\{F_{t,l,s}\}_{t\to+\infty}$ (still denoted by $\{F_{t,l,s}\}_{t\to+\infty}$) compactly convergent to a holomorphic $(n,0)$ form $F_{l,s}$ on M_s. It follows from (37) and Fatou's lemma that

$$
\begin{aligned}
&\int_{\tilde{M}_s}|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\
\le&\liminf_{t\to+\infty}\int_{\tilde{M}_s}|F_{t,l,s}-(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\
\le&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_{2,l}(z_0,w)-\varphi_2}<+\infty.
\end{aligned}
\tag{38}
$$

As $\tilde{\psi}_{2,l}$ is decreasingly convergent to $\tilde{\psi}_2$, when $l\to+\infty$, and $\int_Y|F_j|^2 e^{-\tilde{\psi}_2(z_j,w)-\pi_2^*(\varphi_2)}<+\infty$ for any j,

$$\begin{aligned}&\int_{\tilde{M}_s}|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \le&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}<+\infty.\end{aligned}\tag{39}$$

As $\tilde{\psi}_{2,l}$ is decreasingly convergent to $\tilde{\psi}_2$, for any compact subset $K\subset\tilde{M}_s$, we have

$$\inf_l\inf_K e^{-\tilde{\psi}_{2,l}}\ge\inf_K e^{-\tilde{\psi}_{2,1}}>0,$$

then it follows from (39) that

$$\sup_l\int_K|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))<+\infty.$$

Hence it follows from Lemma 9 and diagonal method that there exists a subsequence of $\{F_{l,s}\}_{l\to+\infty}$ (still denoted by $\{F_{l,s}\}_{l\to+\infty}$) compactly convergent to a holomorphic $(n,0)$ form F_s on $\tilde{M}_s$. It follows from (31), (39) and Fatou's lemma that

$$\begin{aligned}&\int_{\tilde{M}_s}|F_s|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_2-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \le&\liminf_{l\to+\infty}\int_{\tilde{M}_s}|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \le&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}\\ \le&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}<+\infty.\end{aligned}\tag{40}$$

Again using Lemma 9 and diagonal method, we know that there exists a subsequence of $\{F_s\}_{s\to+\infty}$ (still denoted by $\{F_s\}_{s\to+\infty}$) compactly convergent to a holomorphic $(n,0)$ form $\tilde{F}$ on $\tilde{M}$. It follows from (40) and Fatou's lemma that

$$\begin{aligned}&\int_{\tilde{M}}|\tilde{F}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_2-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \le&\liminf_{s\to+\infty}\int_{M_s}|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \le&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}<+\infty.\end{aligned}\tag{41}$$

Note that $N+\pi_1^*(G)+\pi_1^*(\varphi_1)=2\log|g|+\tilde{\psi}_2$. We have

$$\begin{aligned}&\int_{\tilde{M}}|\tilde{F}|^2e^{-\pi_1^*(\varphi_1)-N-\pi_2^*(\varphi_2)}c(-\pi_1^*(G))\\ \le&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}<+\infty.\end{aligned} \tag{42}$$

It follows from $(F_{t,l,s}-F,(z_j,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$ for any $(z_j,y)\in Z_0\cap\tilde{M}_s$, Lemma 10 and the compactly convergence of all the sequences, we know that

$$(\tilde{F}-F,(z_j,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$$

for any $y\in Y$.

Lemma 25 is proved.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1\le j\le n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M=\left(\prod_{1\le j\le n_1}\Omega_j\right)\times Y$ be an $n-$dimensional complex manifold, where $n=n_1+n_2$, and let K_M be the canonical (holomorphic) line bundle on M. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\le j\le n_1}\Omega_j$, Ω_j and Y respectively.

Let $\tilde{M}\subset M$ be an $n-$dimensional weakly pseudoconvex Kähler manifold satisfying that $Z_0\subset\tilde{M}$. Let F be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset\tilde{M}$ of Z_0.

Let $Z_j=\{z_{j,k}:1\le k<\tilde{m}_j\}$ be a discrete subset of Ω_j for any $j\in\{1,\ldots,n_1\}$, where $\tilde{m}_j\in\mathbb{Z}_{\ge2}\cup\{+\infty\}$. Denote that $Z_0:=\left(\prod_{1\le j\le n_1}Z_j\right)\times Y$.

Let $p_{j,k}$ be a positive number for any $1\le j\le n_1$ and $1\le k<\tilde{m}_j$, which satisfies that $\sum_{1\le k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\not\equiv-\infty$ for any $1\le j\le n_1$. Denote that

$$G:=\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}.$$

Let $N\le0$ be a plurisubharmonic function on $\tilde{M}$. Denote $\psi:=G+N$.

Let φ_X be a Lebesgue measurable function on $\prod_{1\le j\le n_1}\Omega_j$. Assume that $\pi_1^*(\varphi_X)+N$ is a plurisubharmonic function on $\tilde{M}$ and $(\pi_1^*(\varphi_X)+N)|_{Z_0}\not\equiv-\infty$. Denote $\Phi=\pi_1^*(\varphi_X)+N$. Let φ_Y be a plurisubharmonic function on Y, and

$$\varphi:=\pi_1^*(\varphi_X)+\pi_2^*(\varphi_Y)$$

on M.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}}\Subset\Omega_j$ of $z_{j,k}\in\Omega_j$ satisfying $w_{j,k}(z_{j,k})=0$ for any $1\le j\le n_1$ and $1\le k<\tilde{m}_j$, where $V_{z_{j,k}}\cap V_{z_{j,k'}}=\emptyset$ for any j and $k\ne k'$.

Denote that $\tilde{I}_1 := \{\beta = (\beta_1, \ldots, \beta_{n_1}) : 1 \le \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \ldots, n_1\}\}$, $V_\beta := \prod_{1 \le j \le n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \le j \le n_1} \Omega_j$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$. Denote that $E_\beta := \left\{\alpha = (\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1 \le j \le n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ and $\tilde{E}_\beta := \left\{\alpha = (\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1 \le j \le n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} \ge 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ for any $\beta \in \tilde{I}_1$. Let U_0 be an open neighborhood of Z_0 in $\tilde{M}$. Let f be a holomorphic $(n, 0)$ form on U_0 such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in \tilde{E}_\beta$ and $\beta \in \tilde{I}_1$. Denote that

$$c_{j,k} := \exp \lim_{z \to z_{j,k}} \left(\frac{\sum_{1 \le k_1 < \tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \ldots, n\}$ and $1 \le k < \tilde{m}_j$ (following from Lemmas 3 and 4, we get that the above limit exists).

Lemma 26 *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that $f_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)_y$ for any $y \in Y$, where $\alpha \in \tilde{E}_\beta \backslash E_\beta$ and $\beta \in \tilde{I}_1$, and*

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \left(N + \pi_1^*(\varphi_X)\right)(z_\beta, w')}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} < +\infty. \tag{43}$$

Then there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $\left(F - f, (z, y)\right) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y))\right)_{(z,y)}$ for any $(z, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \left(N + \pi_1^*(\varphi_X)\right)(z_\beta, w')}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}.$$

Remark 20 If we don't assume that $f_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)_y$ for any $y \in Y$, where $\alpha \in \tilde{E}_\beta \backslash E_\beta$ and $\beta \in \tilde{I}_1$, we can still find a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $\left(F - f, (z, y)\right) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(G)\right)_{(z,y)}$ for any $(z, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \left(N + \pi_1^*(\varphi_X)\right)(z_\beta, w')}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}$$

as long as (43) holds.

Proof (*Proof of Lemma* 26) As $\psi = G + N \le G$, we know that $c(-\psi)e^{-\varphi} \le c(-G)e^{-\varphi-N}$. To prove Lemma 26, it suffice to prove that there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $(F - f, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_z$ for any $z \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi-N} c(-G) \le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

The following Remark shows that it suffices to prove Proposition 26 for the case $\tilde{m}_j < +\infty$ for any $j \in \{1, \ldots, n_1\}$.

Remark 21 Assume that Proposition 26 holds for the case $\tilde{m}_j < +\infty$ for any $j \in \{1, \ldots, n_1\}$. For any $j \in \{1, \ldots, n_1\}$, it follows from Lemma 6 that there exists a sequence of Riemann surfaces $\{\Omega_{j,l}\}_{l\in\mathbb{Z}_{\ge1}}$, which satisfies that $\Omega_{j,l} \Subset \Omega_{j,l+1} \Subset \Omega_j$ for any l, $\cup_{l\in\mathbb{Z}_{\ge1}}\Omega_{j,l} = \Omega_j$ and $\{G_{\Omega_{j,l}}(\cdot, z) - G_{\Omega_j}(\cdot, z)\}_{l\in\mathbb{Z}_{\ge1}}$ is decreasingly convergent to 0 with respect to l for any $z \in \Omega_j$. As Z_j is a discrete subset of Ω_j, $Z_{j,l} := \Omega_{j,l} \cap Z_j$ is a set of finite points. Denote that $\tilde{M}_l := \left((\prod_{1\le j\le n_1}\Omega_{j,l}) \times Y\right) \cap \tilde{M}$ and $G_l := \max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(\sum_{z_{j,k}\in Z_{j,l}} 2p_{j,k}G_{\Omega_{j,l}}(\cdot, z_{j,k})\right)\right\}$ on $\tilde{M}_l$. Note that $\tilde{M}_l$ is weakly pseudoconvex Kähler manifold. Denote that

$$c_{j,k,l} = \exp \lim_{z\to z_{j,k}}\left(\frac{\sum_{z_{j,k_1}\in Z_{j,l}} p_{j,k_1}G_{\Omega_{j,l}}(z, z_{j,k_1})}{p_{j,k}} - \log|w_{j,k}(z)|\right)$$

for any $1 \le j \le n_1$, $l \in \mathbb{Z}_{\ge1}$ and $1 \le k < \tilde{m}_j$ satisfying $z_{j,k} \in Z_{j,l}$. Hence $c_{j,k,l}$ is decreasingly convergent to $c_{j,k}$ with respect to l, G_l is decreasingly convergent to G with respect to l and $\cup_{l\in\mathbb{Z}_{\ge1}}\tilde{M}_l = \tilde{M}$.

Then there exists a holomorphic $(n, 0)$ form F_l on $\tilde{M}_l$ such that $(F_l - f, (z_\beta, y)) \in (\mathcal{O}(K_{\tilde{M}_l}) \otimes (\mathcal{I}(G_l) + \pi_2^*(\varphi_Y))_{(z_\beta,y)} = (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z_\beta,y)}$ for any $\beta \in \{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1\le j\le n_1}\Omega_{j,l}\}$ and $y \in Y$, and F_l satisfies

$$\begin{aligned}
&\int_{\tilde{M}_l} |F_l|^2 e^{-\varphi-N} c(-G_l) \\
\le &\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\{\tilde{\beta}\in\tilde{I}_1 : z_{\tilde{\beta}}\in\prod_{1\le j\le n_1}\Omega_{j,l}\}} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j,l}^{2\alpha_j+2}} \\
\le &\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \\
< &+\infty.
\end{aligned}$$

Since $\psi \le \psi_l$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, we have

$$
\begin{aligned}
&\int_{\tilde{M}_l}|F_l|^2e^{-\varphi-N-G_l+G}c(-G)\\
\le&\int_{\tilde{M}_l}|F_l|^2e^{-\varphi-N}c(-G_l)\\
\le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_1^*(\varphi_X)\right)(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}
\tag{44}
$$

Note that ψ is continuous on $M\backslash Z_0$, ψ_l is continuous on $M_l\backslash Z_0$ and Z_0 is a closed complex submanifold of M. For any compact subset K of $M\backslash Z_0$, there exist $l_K>0$ such that $K\Subset M_{l_K}$ and $C_K>0$ such that $\frac{e^{\varphi+N+G_l-G}}{c(-G)}\le C_K$ for any $l\ge l_K$. It follows from Lemma 9 and the diagonal method that there exists a subsequence of $\{F_l\}$, denoted still by $\{F_l\}$, which is uniformly convergent to a holomorphic $(n,0)$ form F on any compact subset of M. It follows from Fatou's Lemma and inequality (44) that

$$
\begin{aligned}
\int_{\tilde{M}}|F|^2e^{-\varphi-N}c(-G)=&\int_{\tilde{M}}\lim_{l\to+\infty}|F_l|^2e^{-\varphi-N-G_l+G}c(-G)\\
\le&\liminf_{l\to+\infty}\int_{\tilde{M}_l}|F_l|^2e^{-\varphi-N-G_l+G}c(-G)\\
\le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}
$$

Since $\{F_l\}$ is uniformly convergent to F on any compact subset of $\tilde{M}$ and $(F_l-f,(z_\beta,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y)))_{(z_\beta,y)}$ for any $\beta\in\left\{\tilde{\beta}\in\tilde{I}_1:z_{\tilde{\beta}}\in\prod_{1\le j\le n_1}\Omega_{j,l}\right\}$ and $y\in Y$, it follows from Lemma 10 that $(F-f,(z_\beta,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y)))_{(z_\beta,y)}$ for any $\beta\in\tilde{I}_1$ and $y\in Y$.

In the following, we assume that $\tilde{m}_j<+\infty$ for any $1\le j\le n_1$. Denote that $m_j=\tilde{m}_j-1$.

As $\tilde{M}$ is a weakly pseudoconvex Kähler manifold, there exists a sequence of weakly pseudoconvex Kähler manifolds $\tilde{M}_s$ satisfying $\tilde{M}_1\Subset\tilde{M}_2\cdots\Subset\tilde{M}_s\Subset\cdots\tilde{M}$ and $\cup_{s\in\mathbb{N}^+}\tilde{M}_s=\tilde{M}$.

Recall that $\Phi=\pi_1^*(\varphi_X)+N\in Psh(M)$. Denote $\Phi_l=\max\{\Phi,-l\}$, where l is a positive integer. We note that Φ_l is a bounded plurisubharmonic function on $\tilde{M}$.

It follows from Lemmas 3 and 4 that there exists a local coordinate $\tilde{w}_{j,k}$ on a neighborhood $\tilde{V}_{z_{j,k}}\Subset V_{z_{j,k}}$ of $z_{j,k}$ satisfying $\tilde{w}_{j,k}(z_{j,k})=0$ and

$$
|\tilde{w}_{j,k}|=\exp\left(\frac{\sum_{1\le k_1\le m_j}p_{j,k_1}G_{\Omega_j}(\cdot,z_{j,k_1})}{p_{j,k}}\right)
$$

on $\tilde{V}_{z_{j,k}}$.

Denote that $\tilde{V}_\beta := \prod_{1\le j\le n_1} \tilde{V}_{j,\beta_j}$ for any $\beta \in \tilde{I}_1$. Let $\tilde{f}$ be a holomorphic $(n, 0)$ form on $(\cup_{\beta\in\tilde{I}_1} \tilde{V}_\beta \times Y) \cap U_0$ satisfying

$$\tilde{f} = \sum_{\alpha\in E_\beta} c_{\alpha,\beta}\pi_1^*(\tilde{w}_\beta^\alpha d\tilde{w}_{1,\beta_1} \wedge d\tilde{w}_{2,\beta_2} \wedge \ldots \wedge d\tilde{w}_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $(\tilde{V}_\beta \times Y) \cap U_0$, where $c_{\alpha,\beta} = \prod_{1\le j\le n_1}\left(\lim_{z\to z_{j,\beta_j}} \frac{w_{j,\beta_j}(z)}{\tilde{w}_{j,\beta_j}(z)}\right)^{\alpha_j+1}$. It follows from $f_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)_y$ for any $y \in Y$, where $\alpha \in \tilde{E}_\beta$ and $\beta \in \tilde{I}_1$, and Lemma 21 that

$$(f - \tilde{f}, (z, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z,y)} \tag{45}$$

for any $(z, y) \in Z_0$.

Denote that $G_1 := \max_{1\le j\le n_1}\left\{\tilde{\pi}_j^*\left(2\sum_{1\le k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$ on $\prod_{1\le j\le n_1}\Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\le j\le n_1}\Omega_j$ to Ω_j and $G = \pi_1^*(G_1)$. It follows from Lemmas 4 and 5 that there exists $t_0 > 0$ such that $\{G_1 < -t_0\} \Subset \cup_{\beta\in\tilde{I}_1}\tilde{V}_\beta$, which implies that $\int_{\{G<-t\}\cap\tilde{M}_s} |\tilde{f}|^2 < +\infty$. It follows from (43), Φ_l is bounded and Fubini's theorem that we know

$$\int_{\tilde{M}_s} \mathbb{I}_{\{-t-1<G<-t\}}|\tilde{f}|^2 e^{-G-\pi_2^*(\varphi_Y)-\Phi_l} < +\infty.$$

Using Lemma 8, there exists a holomorphic $(n, 0)$ form $F_{l,s,t}$ on $\tilde{M}_s$ such that

$$\begin{aligned}&\int_{\tilde{M}_s} |F_{l,s,t} - (1 - b_{t,1}(\psi))\tilde{f}|^2 e^{-G-\pi_2^*(\varphi_Y)-\Phi_l+v_{t,1}(G)}c(-v_{t,1}(G))\\&\le\left(\int_0^{t+1} c(t)e^{-t}dt\right)\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<G<-t\}}|\tilde{f}|^2 e^{-G-\pi_2^*(\varphi_Y)-\Phi_l},\end{aligned} \tag{46}$$

where $t \ge t_0$. Note that $b_{t,1}(t_1) = 0$ for large enough t_1, then $(F_{l,s,t} - \tilde{f}, (z, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z,y)}$ for any $(z, y) \in Z_0 \cap \tilde{M}_s$.

For any $(z_\beta, y) \in Z_0 \cap \tilde{M}_s$, letting (U_y, w) be a small local coordinated open neighborhood of y and shrinking $\tilde{V}_\beta$ if necessary, we have $\tilde{V}_\beta \times U_y \Subset U_0 \cap (V_\beta \times Y)$ for any $\beta \in \tilde{I}_1$. Assume that $F_j = \tilde{h}_j(w)dw$ on U_y, where $\tilde{h}_j$ is a holomorphic function on U_y and $dw = dw_1 \wedge dw_2 \wedge \ldots \wedge dw_n$. There exists $t_2 > t_1$ such that when $t > t_2$, $(\{G_1 < -t\} \times U_y) \cap (\tilde{V}_\beta \times U_y) \Subset (V_\beta \times Y) \cap \tilde{M}$, for any $\beta \in \tilde{I}_1$.

Now we consider

$$\begin{aligned}&\int_{(\cup_{\beta\in\tilde{I}_1}\tilde{V}_\beta)\times U_y}\mathbb{I}_{\{-t-1<G<-t\}}|\tilde{f}|^2 e^{-G-\pi_2^*(\varphi_Y)-\Phi_l}\\&=\sum_{\beta\in\tilde{I}_1}\int_{w'\in U_y}(\int_{\{-t-1<G_1<-t\}}|f(\tilde{w}, w')|^2\frac{e^{-\Phi_l(\tilde{w},w')}}{\max_{1\le j\le n}\{|\tilde{w}_{j,\beta_j}|^{2p_{j,\beta_j}}\}})e^{-\varphi_Y}.\end{aligned} \tag{47}$$

Note that $|c_{\alpha,\beta}| = \frac{1}{\prod_{1\le j\le n_1} c_{j,\beta_j}^{\alpha_j+1}}$ and $\int_{w'\in Y_s} |f_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$. It follows from (47), Fatou's Lemma and Lemma 14 that we have

$$
\begin{aligned}
&\limsup_{t\to+\infty} \int_{\cup_{\beta\in\tilde{I}_1}\tilde{V}_\beta\times U_y} \mathbb{I}_{\{-t-1<G<-t\}}|\tilde{f}|^2 e^{-G-\pi_2^*(\varphi_Y)-\Phi_l} \\
=&\limsup_{t\to+\infty}\sum_{\beta\in\tilde{I}_1}\int_{w'\in U_y}\Big(\int_{\{-t-1<G_1<-t\}}|f(\tilde{w},w')|^2\frac{e^{-\Phi_l(\tilde{w},w')}}{\max_{1\le j\le n}\{|\tilde{w}_{j,\beta_j}|^{2p_{j,\beta_j}}\}}\Big)e^{-\varphi_Y} \\
\le&\sum_{\beta\in\tilde{I}_1}\int_{w'\in U_y}\limsup_{t\to+\infty}\Big(\int_{\{-t-1<G_1<-t\}}|f(\tilde{w},w')|^2\frac{e^{-\Phi_l(\tilde{w},w')}}{\max_{1\le j\le n}\{|\tilde{w}_{j,\beta_j}|^{2p_{j,\beta_j}}\}}\Big)e^{-\varphi_Y} \\
\le&\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{U_y}|f_{\alpha,\beta}|^2 e^{-\varphi_Y-\Phi_l(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}
\tag{48}
$$

Note that y and U_y are arbitrarily chosen, $v_{t,1}(\psi)\ge\psi$ and $c(t)e^{-t}$ is decreasing. Combining inequalities (46) and (48), then we have

$$
\begin{aligned}
&\int_{\tilde{M}_s}|F_{l,s,t}-(1-b_{t,1}(\psi))\tilde{f}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l}c(-G) \\
\le&\int_{\tilde{M}_s}|F_{l,s,t}-(1-b_{t,1}(\psi))\tilde{f}|^2 e^{-G-\pi_2^*(\varphi_Y)-\Phi_l+v_{t,1}(G)}c(-v_{t,1}(G)) \\
\le&\left(\int_0^{t+1}c(t_1)e^{-t_1}dt_1\right)\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<G<-t\}}|\tilde{f}|^2 e^{-G-\pi_2^*(\varphi_Y)-\Phi_l} \\
\le&\left(\int_0^{t+1}c(t_1)e^{-t_1}dt_1\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s}|f_{\alpha,\beta}|^2 e^{-\varphi_Y-\Phi_l(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}
\tag{49}
$$

Note that G is continuous on $\tilde{M}\backslash Z_0$. For any open set $K\Subset\tilde{M}_s\backslash Z_0$, as $b_{t,1}(t_1)=1$ for any t_1 large enough and $c(t_2)e^{-t_2}$ is decreasing with respect to t_2, we get that there exists a constant $C_K>0$ such that

$$\int_K|(1-b_{t,1}(\psi))\tilde{f}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l}c(-G)\le C_K\int_{\{G<-t_1\}\cap K}|\tilde{f}|^2<+\infty$$

for any $t>t_1$, which implies that

$$\limsup_{t\to+\infty}\int_K|F_{l,s,t}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l}c(-G)<+\infty.$$

Using Lemma 9 and the diagonal method, we obtain that there exists a subsequence of $\{F_{l,s,t}\}_{t\to+\infty}$ denoted by $\{F_{l,s,t_m}\}_{m\to+\infty}$ uniformly convergent on any compact subset of $\tilde{M}_s\backslash Z_0$. As Z_0 is a closed complex submanifold of $\tilde{M}$, we obtain that $\{F_{l,s,t_m}\}_{m\to+\infty}$ is uniformly convergent to a holomorphic $(n,0)$ form $F_{l,s}$ on $\tilde{M}_s$ on any compact subset of $\tilde{M}_s$. Then it follows from inequality (49) and Fatou's Lemma that

$$\begin{aligned}
&\int_{\tilde{M}_s} |F_{l,s}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l} c(-G)\\
=&\int_{\tilde{M}_s} \liminf_{m\to+\infty} |F_{l,s,t_m}-(1-b_{t_m,1}(\psi))\tilde{f}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l} c(-G)\\
\le&\liminf_{m\to+\infty}\int_{\tilde{M}_s} |F_{l,s,t_m}-(1-b_{t_m,1}(\psi))\tilde{f}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l} c(-G)\\
\le&\left(\int_0^{+\infty} c(t_1)e^{-t_1}dt_1\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s}|f_{\alpha,\beta}|^2 e^{-\varphi_Y-\Phi_l(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\
<&+\infty.
\end{aligned}$$

As $\Phi_l(z_\beta,w')$ is decreasingly convergent to $\Phi(z_\beta,w')$ for any $\beta\in\tilde{I}_1$, then we have

$$\begin{aligned}
&\limsup_{l\to+\infty}\int_{\tilde{M}_s} |F_{l,s}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l} c(-G)\\
\le&\left(\int_0^{+\infty} c(t_1)e^{-t_1}dt_1\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s}|f_{\alpha,\beta}|^2 e^{-\varphi_Y-\Phi(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\
=&\left(\int_0^{+\infty} c(t_1)e^{-t_1}dt_1\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s}|f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_1^*(\varphi_X)\right)(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\
<&+\infty.
\end{aligned}\tag{50}$$

Note that G is continuous on $\tilde{M}\backslash Z_0$ and Z_0 is a closed complex submanifold of $\tilde{M}$. Using Lemma 9, we obtain that there exists a subsequence of $\{F_{l,s}\}_{l\to+\infty}$ (also denoted by $\{F_{l,s}\}_{l\to+\infty}$) uniformly convergent to a holomorphic $(n,0)$ form F_s on $\tilde{M}_s$ on any compact subset of $\tilde{M}_s$, which satisfies that

$$\begin{aligned}
&\int_{\tilde{M}_s} |F_s|^2 e^{-\pi_2^*(\varphi_Y)-N-\varphi_X} c(-G)\\
\le&\left(\int_0^{+\infty} c(t_1)e^{-t_1}dt_1\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s}|f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_1^*(\varphi_X)\right)(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}$$

As $\cup_{s\in\mathbb{Z}_{\geq 1}}\tilde{M}_s=\tilde{M}$, we have

$$\begin{aligned}
&\limsup_{s\to+\infty}\int_{\tilde{M}_s}|F_s|^2e^{-\varphi-N}c(-\psi)\\
&\leq\lim_{s\to+\infty}\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s}|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\varphi_X)(z_\beta,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\
&=\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_1^*(\varphi_X)\right)(z_\beta,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\
&<+\infty.
\end{aligned}\tag{51}$$

Note that ψ is continuous on $\tilde{M}\backslash Z_0$, Z_0 is a closed complex submanifold of $\tilde{M}$ and $\cup_{s\in\mathbb{Z}_{\geq 1}}\tilde{M}_s=\tilde{M}$. Using Lemma 9 and the diagonal method, we get that there exists a subsequence of $\{F_s\}$ (also denoted by $\{F_s\}$) uniformly convergent to a holomorphic $(n,0)$ form F on $\tilde{M}$ on any compact subset of $\tilde{M}$. Then it follows from inequality (51) and Fatou's Lemma that

$$\begin{aligned}
\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)&=\int_{\tilde{M}}\liminf_{s\to+\infty}\mathbb{I}_{\tilde{M}_s}|F_{l'}|^2e^{-\varphi}c(-\psi)\\
&\leq\liminf_{s\to+\infty}\int_{\tilde{M}_s}|F_{l'}|^2e^{-\varphi}c(-\psi)\\
&\leq\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_1^*(\varphi_X)\right)(z_\beta,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}$$

Following from Lemma 10, we have $(F-f,(z,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y)))_{(z,y)}$ for any $(z,y)\in Z_0$.

Thus, Proposition 26 holds.

Proof (*Proof of Remark* 20) If we don't assume that $f_{\alpha,\beta}\in\mathcal{I}(\varphi_Y)_y$ for any $y\in Y$, where $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$, it follows from Lemma 21 that for any $(z,y)\in Z_0$, we will have

$$(f-\tilde{f},(z,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(G))_{(z,y)}.\tag{52}$$

Replace the formula (45) by $(f-\tilde{f},(z,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(G))_{(z,y)}$. Then if (43) holds, by the same proof as Proposition 26, we know Remark 20 holds.

3.3 Other Calculations

Let Ω be an open Riemann surface with nontrivial Green functions. Let $Z_\Omega = \{z_j : j \in \mathbb{N}_+ \& j < \gamma\}$ be a subset of Ω of discrete points. Let Y be an $(n-1)-$dimensional weakly pseudoconvex Kähler manifold. Denote $M = \Omega \times Y$. Let π_1 and π_2 be the natural projections from M to Ω and Y respectively. Denote $Z_0 := Z_\Omega \times Y$. Denote $Z_j := \{z_j\} \times Y$.

Let ψ be a plurisubharmonic function on M. It follows from Siu's decomposition theorem that

$$dd^c\psi = \sum_{j\geq 1} 2p_j[Z_j] + \sum_{i\geq 1} \lambda_i[A_i] + R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $(n-1)-$dimensional analytic set, and where R is a closed positive current with the property that $dim E_c(R) < n-1$ for every $c > 0$. We assume that $p_j > 0$ for any $1 \leq j < \gamma$.

Then $N := \psi - \pi_1^*\big(\sum_{j\geq 1} 2p_j G_\Omega(z, z_j)\big)$ is a plurisubharmonic function on M. We assume that $N \leq 0$ and $N|_{Z_j}$ is not identically $-\infty$ for any j.

Let φ_1 be a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on M. With Similar discussion as above, by Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j\geq 1} 2\tilde{q}_j[Z_j] + \sum_{i\geq 1} \tilde{\lambda}_i[\tilde{A}_i] + \tilde{R},$$

where $\tilde{q}_j \geq 0$ for any $1 \leq j < \gamma$.

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function g on Ω such that $ord_{z_j}(g) = q_j := [\tilde{q}_j]$ for any $z_j \in Z_\Omega$ and $g(z) \neq 0$ for any $z \notin Z_\Omega$, where $[q]$ equals to the integer part of the nonnegative real number q. Then we know that there exists a plurisubharmonic function $\tilde{\psi}_2 \in Psh(M)$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2.$$

Let $\varphi_2 \in Psh(Y)$. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

For $1 \leq j < \gamma$, let $(V_j, \tilde{z}_j)$ be a local coordinated open neighborhood of z_j in Ω satisfying $V_j \Subset \Omega$, $\tilde{z}_j(z_j) = 0$ under the local coordinate and $V_j \cap V_k = \emptyset$ for any $j \neq k$. Denote $V_0 := \cup_{1\leq j<\gamma} V_j$. We assume that $g = d_j \tilde{z}_j^{q_j} h_j(z)$ on V_j, where d_j is a constant, $h_j(z)$ is a holomorphic function on V_j and $h_j(z_j) = 1$.

Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$.

Lemma 27 *Assume that $c(t)$ is increasing near $+\infty$. Let F be a holomorphic $(n, 0)$ form on M such that $F = \sum_{l=k_j}^{+\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$ on $V_j \times Y$ $(1 \leq j < \gamma)$, where k_j is a nonnegative integer, $F_{j,l}$ is a holomorphic $(n-1, 0)$ form on Y, for any l, j, and $F_j := F_{j,k_j} \not\equiv 0$ on Y. Denote that*

$$I_F := \{j : k_j + 1 - q_j \le 0 \& 1 \le j < \gamma\}.$$

Assume that

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} < +\infty.$$

Then $k_j + 1 - q_j = 0$ *for any* $j \in I_F$, *and*

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \ge \sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}. \tag{53}$$

Proof As $\tilde{\psi}_2$ and N are upper semi-continuous functions on M, there exists continuous functions $\tilde{\psi}_{2,\alpha}$ and N_β on M decreasingly convergent to $\tilde{\psi}_2$ and N respectively.

There exists some $t_1 \ge 0$ such that $c(t)$ is increasing on $[t_1, +\infty)$. Recall that $\psi = \pi_1^*(\sum_{j\ge1} 2p_j\pi_1^*(G_\Omega(z, z_j))) + N$ and denote $G = \sum_{j\ge1} 2p_j\pi_1^*(G_\Omega(z, z_j))$. For any $t > t_1$,

$$\begin{aligned}
&\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) \\
=&\int_{\{G+N<-t\}} |F|^2 e^{-2\pi_1^*(\log|g|)-\tilde{\psi}_2+\pi_1^*(G)+N-\pi_2^*(\varphi_2)} c(-(G+N)) \\
\ge&\int_{\{G+N_\beta<-t\}} |F|^2 e^{-2\pi_1^*(\log|g|)-\tilde{\psi}_{2,\alpha}+\pi_1^*(G)+N-\pi_2^*(\varphi_2)} c(-(G+N_\beta)).
\end{aligned} \tag{54}$$

For any $y \in Y$, let (U_y, w) be a local coordinated open neighborhood of y in Y satisfying $U_y \Subset Y$. We assume that $F = \tilde{z}_j^{k_j}\tilde{h}_j(\tilde{z}_j, w)d\tilde{z}_j \wedge dw$ on $V_j \times U_y$.

Let $m \in \mathbb{N}$. For any $0 < \epsilon < \frac{1}{2}$, let s_0 be large enough and shrink U_y if necessary such that,

(1) for any $j \in \{1, 2, \ldots, m\}$,
$$\tilde{V}_j := \{|\tilde{z}_j| < s_0, z \in V_j\} \Subset V_j,$$
(2) for any $j \in \{1, 2, \ldots, m\}$ and any $w \in U_y$, we have
$$\sup_{\tilde{z}\in\tilde{V}_j} |\tilde{\psi}_{2,\alpha}(\tilde{z}, w) - \tilde{\psi}_{2,\alpha}(z_0, w)| < \epsilon.$$
(3) Denote $H_j(\tilde{z}_j, w) := G - 2p_j \log|\tilde{z}_j| + N_\beta(\tilde{z}_j, w) + \epsilon$ on $\tilde{V}_j \times U_y$. Then for any $w \in U_y$, we have
$$\sup_{\tilde{z}\in\tilde{V}_j} |H(\tilde{z}_j, w) - H(z_j, w)| < \epsilon.$$
(4) Recall that $g = d_j\tilde{z}^{q_j}h_j(\tilde{z}_j)$ on V_j, where d_j is a constant, $h_j(\tilde{z})$ is a holomorphic function on V_j and $h(z_j) = 1$. We assume that
$$\sup_{\tilde{z}\in\tilde{V}_j} |h_j(\tilde{z}_j) - 1| < \epsilon.$$
(5) Denote that $2g_j(\tilde{z}_j) = G - 2p_j \log|\tilde{z}_j|$ on $\tilde{V}_j$. Note that g_j is a harmonic function on $\tilde{V}_j$.

It follows from Lemma 5 that there exists $t' > t_1$ such that

$$\Big(\{G + N_\beta < -t\} \cap (\tilde{V}_j \times U_y)\Big) \Subset \tilde{V}_j \times U_y,$$

for any $t > t'$. We also have $G + N_\beta \le 2p_j \log|\tilde{z}_j| + H_j(z_j, w)$ on $V_j \times U_y$.

Following from (54), for any $t > t'$, direct calculation shows that

$$\begin{aligned}
&\int_{\{\psi<-t\}\cap(V_0\times U_y)} |F|^2 e^{-\varphi} c(-\psi) \\
\geq &\int_{\{G+N_\beta<-t\}\cap(V_0\times U_y)} |F|^2 e^{-2\pi_1^*(\log|g|)-\tilde{\psi}_{2,\alpha}+\pi_1^*(G)+N-\pi_2^*(\varphi_2)} c(-(G+N_\beta)) \\
\geq &\sum_{j=1}^{m} \int_{\{2p_j\log|\tilde{z}_j|+H_j(z_j,w)<-t\}\cap(\tilde{V}_j\times U_y)} |\tilde{z}_j|^{2k_j+2p_j-2q_j} \frac{|\tilde{h}_j(\tilde{z}_j,w)|^2}{|d_j|^2|h_j(\tilde{z}_j)|^2} \times \\
&e^{-\tilde{\psi}_{2,l,\alpha}(z_j,w)-\epsilon+2g_j(\tilde{z}_j)+N-\pi_2^*(\varphi_2)} c(-2p_j\log|\tilde{z}_j| - H_j(z_j,w)) \\
\geq &\sum_{j=1}^{m} \int_{w\in U_y} (\int_{\{2p_j\log|\tilde{z}_j|+H_j(z_j,w)<-t\}} |\tilde{h}_j(\tilde{z}_j,w)|^2 |\tilde{z}_j|^{2k_j+2p_j-2q_j} e^{2g_j(\tilde{z}_j)+N} \times \\
&c(-2p_j\log|\tilde{z}_j| - H_j(z_j,w))|d\tilde{z}_j|^2) \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)}}{|d|^2|1+\epsilon|^2} |dw|^2 \\
= &\sum_{j=1}^{m} \int_{w\in U_y} (2\int_0^{e^{\frac{-t-H_j(z_j,w)}{2p_j}}} \int_0^{2\pi} |\tilde{h}_j(re^{-\theta},w)|^2 r^{2k_j+2p_j-2q_j+1} e^{2g_j(re^{i\theta})+N(re^{i\theta},w)} \times \\
&c(-2p_j\log r - H_j(z_j,w))d\theta dr) \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)}}{|d|^2|1+\epsilon|^2} |dw|^2 \\
\geq &\sum_{j=1}^{m} \int_{w\in U_y} 4\pi (\int_0^{e^{\frac{-t-H_j(z_j,w)}{2p_j}}} r^{2k_j+2p_j-2q_j+1} c(-2p_j\log r - H_j(z_j,w))) \times \\
&|\tilde{h}_j(z_j,w)|^2 e^{2g_j(z_j)+N(z_j,w)} \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)}}{|d|^2|1+\epsilon|^2} |dw|^2 \\
= &\sum_{j=1}^{m} \frac{2\pi}{p_j|d_j|^2|1+\epsilon|^2} \left(\int_t^{+\infty} c(s) e^{-(\frac{k_j+1-q_j}{p_j}+1)s} ds\right) \times \\
&\int_{w\in U_y} |\tilde{h}_j(z_j,w)|^2 e^{-(\frac{k_j+1-q_j}{p_j}+1)H_j(z_j,w)} e^{2g_j(z_j)+N(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)} |dw|^2.
\end{aligned} \tag{55}$$

As $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$, hence we have

$$\begin{aligned}&\frac{\int_{\{\psi<-t\}\cap(V_0\times U_y)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\\ \geq&\sum_{j=1}^{m}\frac{2\pi}{p_j|d_j|^2|1+\epsilon|^2}\left(\frac{\int_t^{+\infty}c(s)e^{-(\frac{k+1-q_2}{p}+1)s}ds}{\int_t^{+\infty}c(s)e^{-s}ds}\right)\times\\ &\int_{w\in U_y}|\tilde{h}_j(z_j,w)|^2e^{-(\frac{k_j+1-q_j}{p_j}+1)H_j(z_j,w)}e^{2g_j(z_j)+N(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)}|dw|^2.\end{aligned}\tag{56}$$

Denote

$$I_m:=\{1\leq j\leq m:k_j+1-q_j\leq 0\}.$$

Note that

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}<+\infty,$$

$N(z_j,w)\not\equiv-\infty$ and $|\tilde{h}(z_j,w)|^2\not\equiv 0$. It follows from Lemma 15 and (56) that we know $k_j+1-q_j=0$ for any $j\in I_m$ and

$$\begin{aligned}&\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(V_0\times U_y)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\\ \geq&\sum_{j\in I_m}\frac{2\pi}{p_j|d_j|^2}\int_{w\in U_y}|\tilde{h}_j(z_j,w)|^2e^{-H(z_j,w)}e^{2g_j(z_j)+N(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\varphi_2(w)}|dw|^2\\ =&\sum_{j\in I_m}\frac{2\pi}{p_j|d_j|^2}\int_{w\in U_y}|\tilde{h}_j(z_j,w)|^2e^{-2g_j(z_j,0)+N_\beta(z_j,w)}e^{2g_j(z_j)+N(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\varphi_2(w)}|dw|^2\\ =&\sum_{j\in I_m}\frac{2\pi}{p_j|d_j|^2}\int_{w\in U_y}|\tilde{h}_j(z_j,w)|^2e^{N(z_j,w)-N_\beta(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\varphi_2(w)}|dw|^2.\end{aligned}$$

By Monotone convergence theorem, letting $\alpha\to+\infty$ and $\beta\to+\infty$, we have

$$\begin{aligned}&\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(V_0\times U_y)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\\ \geq&\sum_{j\in I_m}\frac{2\pi}{p_j|d_j|^2}\int_{w\in U_y}|\tilde{h}_j(z_j,w)|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2(w)}|dw|^2.\end{aligned}\tag{57}$$

As y and U_y are arbitrarily chosen, it follows from Y is a weakly pseudoconvex Kähler manifold that we have

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds} \geq \sum_{j\in I_m}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2(w)}. \tag{58}$$

Let $m\to+\infty$ and we have

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\geq\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2(w)}. \tag{59}$$

Especially, for any $j\in I_F$, we have

$$\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2(w)}<+\infty.$$

Denote $M=\prod_{1\leq j\leq n_1}\Omega_j\times Y$. Let $p_{j,k}$ be a positive number for any $1\leq j\leq n_1$ and $1\leq k<\tilde{m}_j$, which satisfies that $\sum_{1\leq k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\not\equiv-\infty$ for any $1\leq j\leq n_1$. Recall that

$$G:=\max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}.$$

Let $N\leq 0$ be a plurisubharmonic function on M satisfying that $N|_{Z_0}\not\equiv-\infty$. Denote $\psi:=G+N$.

Let $\varphi_X:=\sum_{1\leq j\leq n_1}\varphi_j(z)$, where each φ_j is an upper semi-continuous function on Ω_j satisfying $\varphi_j(z_j)\neq-\infty$ for any $z_j\in\Omega_j$. We assume that $\pi_1^*(\varphi_X)+N$ is a plurisubharmonic function on M. Let φ_Y be a plurisubharmonic function on Y. Denote $\varphi:=\pi_1^*(\varphi_X)+\pi_2^*(\varphi_Y)$.

It follows from Lemmas 3 and 4 that there exists a local coordinate $w_{j,k}$ on a neighborhood $V_{z_{j,k}}\Subset\Omega_j$ of $z_{j,k}\in\Omega_j$ satisfying $w_{j,k}(z_{j,k})=0$ and

$$\log|w_{j,k}|=\frac{1}{p_{j,k}}\sum_{1\leq k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})$$

for any $j\in\{1,\ldots,n_1\}$ and $1\leq k<\tilde{m}_j$, where $V_{z_{j,k}}\cap V_{z_{j,k'}}=\emptyset$ for any j and $k\neq k'$.

Denote that $\tilde{I}_1:=\{(\beta_1,\ldots,\beta_{n_1}):1\leq\beta_j<\tilde{m}_j \text{ for any } j\in\{1,\ldots,n_1\}\}$, $V_\beta:=\prod_{1\leq j\leq n_1}V_{z_{j,\beta_j}}$ for any $\beta=(\beta_1,\ldots,\beta_n)\in\tilde{I}_1$ and $w_\beta:=(w_{1,\beta_1},\ldots,w_{n,\beta_n})$ is a local coordinate on V_β of $z_\beta:=(z_{1,\beta_1},\ldots,z_{n,\beta_n})\in M$.

Let

$$G_1 = \max_{1\leq j\leq n_1}\left\{\tilde{\pi}_j^*\left(2\sum_{1\leq k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$$

on $\prod_{1\leq j\leq n_1}\Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leq j\leq n_1}\Omega_j$ to Ω_j. Note that $G = \pi_1^*(G_1)$.

Let F be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\} \subset M$ for some $t_0 > 0$ satisfying $\int_{\{\psi<-t_0\}} |F|^2 e^{-\varphi}c(-\psi) < +\infty$. For any $\beta \in \tilde{I}_1$, it follows from Lemma 19 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on $V_\beta \times Y$.

Denote that $E_\beta := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}} = 1\right\}$, $E_{1,\beta} := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}} < 1\right\}$ and $E_{2,\beta} := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}} > 1\right\}$.

Assume that $c(t)$ is increasing near $+\infty$.

Lemma 28 *If* $\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds} < +\infty$, *we have* $F_{\alpha,\beta} \equiv 0$ *for any* $\alpha \in E_{1,\beta}$ *and* $\beta \in \tilde{I}_1$, *and*

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds} \geq \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}\int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}.$$

Proof As φ_X is an upper semi-continuous functions on $X := \prod_{1\leq j\leq n_1}\Omega_j$ and N is an upper semi-continuous function on M, there exist continuous functions $\varphi_{X,l}$ and N_γ on X and M decreasingly convergent to φ_X and N respectively.

When t is large enough, $c(t)$ is increasing, then we have

$$\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi) \geq \int_{\{G+N_\gamma<-t\}}|F|^2e^{-\pi_1^*(\varphi_{X,l})-\pi_2^*(\varphi_Y)}c(-G-N_\gamma).$$

As Y is a weakly pseudoconvex Kähler manifold, there exist open weakly pseudoconvex Kähler manifolds $Y_1 \Subset \ldots \Subset Y_s \Subset Y_{s+1} \Subset \ldots$ such that $\cup_{s\in\mathbb{Z}_{\geq 1}} Y_s = Y$.

Fix Y_s. For any $\beta \in \tilde{I}_1$, there exists $t_\beta > t_0$ such that $\{G + N_\gamma < -t_\beta\} \cap (V_\beta \times Y_s) \Subset (V_\beta \times Y_{s+1})$.

On $V_\beta \times Y_s$, for any $\epsilon > 0$, there exists t_ϵ large enough such that when $t > t_\epsilon$,

(1) for any $(w, w') \in \{G + N_\gamma < -t\}$,

$$|\varphi_{X,l}(w) - \varphi_{X,l}(z_\beta)| < \epsilon.$$

(2) Denote $H(w, w') := N_\gamma(w, w') + \epsilon$. For any $(w, w') \in \{G + N_\gamma < -t\}$,

$$|H(w, w') - H(z_\beta, w')| < \epsilon.$$

Then we have $G + H(z_\beta, w') \geq G + N_s(w, w')$, for any $(w, w') \in \{G + N_\gamma < -t_\beta\} \cap (V_\beta \times Y_s)$.

Let

$$G_1 = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left(2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j. Note that $G = \pi_1^*(G_1)$.

For any $t \geq \max\{t_\beta, t_\epsilon\}$, note that $\{G_1 < -t\} = \prod_{1 \leq j \leq n_1} \left\{ |w_{j,\beta_j}| < e^{-\frac{t}{2p_{j,\beta_j}}} \right\}$ and $F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$ on $\{G_1 < -t\} \times Y$, then we have

$$\begin{aligned}
&\int_{\{\psi < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi) \\
\geq &\int_{\{G + N_\gamma < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\pi_1^*(\varphi_{X,l}) - \pi_2^*(\varphi_Y)} c(-G - N_\gamma) \\
\geq &e^{-\epsilon} \int_{\{G + H(z_\beta, w') < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\pi_1^*(\varphi_{X,l}(z_\beta, w')) - \pi_2^*(\varphi_Y)} c(-G - H(z_\beta, w')) \\
= &e^{-\epsilon} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \int_{w' \in Y_s} \left(\int_{\{G_1 + H(z_\beta, w') < -t\}} |w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}|^2 c(-G - H(z_\beta, w')) \right) \times \\
&e^{-\varphi_{X,l}(z_\beta)} |F_{\alpha,\beta}|^2 e^{-\varphi_Y}.
\end{aligned} \tag{60}$$

Denote that $q_\alpha := \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} - 1$. It follows from Lemma 12 and inequality (60) that

$$\begin{aligned}
&\int_{\{\psi < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi) \\
\geq &e^{-\varphi_{X,l}(z_\beta) - \epsilon} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \left(\int_t^{+\infty} c(s) e^{-(q_\alpha + 1)s} ds \right) \frac{(q_\alpha + 1)(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_{Y_s} |F_{\alpha,\beta}|^2 e^{-\varphi_Y - (q_\alpha + 1) H(z_\beta, w')}.
\end{aligned}$$

It follows from $\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty$ and Lemma 15 that

$$F_{\alpha,\beta} \equiv 0$$

for any α satisfying $q_\alpha < 0$ and

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq e^{-\varphi_{X,l}(z_\beta) - \epsilon} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_{Y_s} |F_{\alpha,\beta}|^2 e^{-\varphi_Y - H(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)}.$$

Note that $H(w, w') := N_\gamma(w, w') + \epsilon$. Letting $\epsilon \to 0$, $\gamma \to +\infty$ and $s \to +\infty$, we have

$$\begin{aligned}&\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}\cap(V_\beta\times Y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds}\\ \geq &\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}.\end{aligned} \tag{61}$$

Note that $V_\beta \cap V_{\tilde\beta} = \emptyset$ for any $\beta \neq \tilde\beta$ and $\{\psi_1 < -t_\beta\} \cap V_\beta \Subset V_\beta$ for any $\beta \in \tilde I_1$. It follows from inequality (61) that

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}.$$

Thus, Lemma 28 holds.

Let $\tilde M$ be an open complex submanifold of M satisfying that $Z_0 = \{z_\beta : \beta \in \tilde I_1\} \times Y \subset \tilde M$, and let $K_{\tilde M}$ be the canonical (holomorphic) line bundle on M_1. Let F_1 be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\} \cap \tilde M$ for $t_0 > 0$ satisfying that $\int_{\{\psi<-t_0\}\cap\tilde M} |F_1|^2 e^{-\varphi} c(-\psi) < +\infty$. For any $\beta \in \tilde I_1$, it follows from Lemma 20 that there exist a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}}$ on Y and an open subset U_β of $\{\psi < -t_0\} \cap \tilde M \cap (V_\beta \times Y)$ such that

$$F_1 = \sum_{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on U_β and

$$\int_K |F_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$$

for any $\alpha \in \mathbb{Z}^{n_1}_{\geq 0}$ and compact subset K of Y. Using the similar method in Lemma 28, we have the following Remark.

Remark 22 If $\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}\cap\tilde M} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} < +\infty$, we have $F_{\alpha,\beta} \equiv 0$ for any $\alpha \in E_{1,\beta}$ and $\beta \in \tilde I_1$, and

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}\cap\tilde M} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N}.$$

Proof As φ_X is an upper semi-continuous functions on $X := \prod_{1\le j\le n_1}\Omega_j$ and N is an upper semi-continuous function on M, there exist continuous functions $\varphi_{X,l}$ and N_γ on X and M decreasingly convergent to φ_X and N respectively.

When t is large enough, $c(t)$ is increasing, then we have

$$\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\ge\int_{\{G+N_\gamma<-t\}\cap\tilde{M}}|F|^2e^{-\pi_1^*(\varphi_{X,l})-\pi_2^*(\varphi_Y)}c(-G-N_\gamma).$$

As Y is a weakly pseudoconvex Kähler manifold, there exist open weakly pseudoconvex Kähler manifolds $Y_1\Subset\ldots\Subset Y_s\Subset Y_{s+1}\Subset\ldots$ such that $\cup_{s\in\mathbb{Z}_{\ge1}}Y_s=Y$.

For any $\beta\in\tilde{I}_1$ and any Y_s , there exists open subset $\hat{V}_\beta\Subset V_\beta$ and $t_{\beta,s}>t_0$ such that $\{G+N_\gamma<-t_{\beta,s}\}\cap(\hat{V}_\beta\times Y_s)\Subset(\hat{V}_\beta\times Y_{s+1})\Subset U_\beta$.

On $\hat{V}_\beta\times Y_s$, for any $\epsilon>0$, there exists t_ϵ large enough such that when $t>t_\epsilon$,
(1) for any $(w,w')\in\{G+N_\gamma<-t\}$,

$$|\varphi_{X,l}(w)-\varphi_{X,l}(z_\beta)|<\epsilon.$$

(2) Denote $H(w,w'):=N_\gamma(w,w')+\epsilon$. For any $(w,w')\in\{G+N_\gamma<-t\}$,

$$|H(w,w')-H(z_\beta,w')|<\epsilon.$$

Then we have $G+H(z_\beta,w')\ge G+N_s(w,w')$, for any $(w,w')\in\{G+N_\gamma<-t_\beta\}\cap(\hat{V}_\beta\times Y_s)$.

Recall that $G_1=\max_{1\le j\le n_1}\left\{\tilde{\pi}_j^*\left(2\sum_{1\le k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$ on $\prod_{1\le j\le n_1}\Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\le j\le n_1}\Omega_j$ to Ω_j. Note that $G=\pi_1^*(G_1)$. For any $t\ge\max\{t_{\beta,s},t_\epsilon\}$, note that $\{G_1<-t\}=\prod_{1\le j\le n_1}\left\{|w_{j,\beta_j}|<e^{-\frac{t}{2p_{j,\beta_j}}}\right\}$ and $F=\sum_{\alpha\in\mathbb{Z}_{\ge0}^{n_1}}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(F_{\alpha,\beta})$ on U_β, then we have

$$\begin{aligned}&\int_{\{\psi<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)\\\ge&\int_{\{G+N_\gamma<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\pi_1^*(\varphi_{X,l})-\pi_2^*(\varphi_Y)}c(-G-N_\gamma)\\\ge&e^{-\epsilon}\int_{\{G+H(z_\beta,w')<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\pi_1^*(\varphi_{X,l}(z_\beta,w'))-\pi_2^*(\varphi_Y)}c(-G-H(z_\beta,w'))\\=&e^{-\epsilon}\sum_{\alpha\in\mathbb{Z}_{\ge0}^{n_1}}\int_{w'\in Y_s}\left(\int_{\{G_1+H(z_\beta,w')<-t\}}|w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}}|^2c(-G-H(z_\beta,w'))\right)\times\\&e^{-\varphi_{X,l}(z_\beta)}|F_{\alpha,\beta}|^2e^{-\varphi_Y}.\end{aligned}\tag{62}$$

Denote that $q_\alpha:=\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}-1$. It follows from Lemma 12 and inequality (62) that

$$\begin{aligned}&\int_{\{\psi<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)\\\ge&e^{-\varphi_{X,l}(z_\beta)-\epsilon}\sum_{\alpha\in\mathbb{Z}_{\ge0}^{n_1}}\left(\int_t^{+\infty}c(s)e^{-(q_\alpha+1)s}ds\right)\frac{(q_\alpha+1)(2\pi)^{n_1}}{\prod_{1\le j\le n_1}(\alpha_j+1)}\int_{Y_s}|F_{\alpha,\beta}|^2e^{-\varphi_Y-(q_\alpha+1)H(z_\beta,w')}.\end{aligned}$$

It follows from $\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}<+\infty$ and Lemma 15 that

$$F_{\alpha,\beta}\equiv 0$$

for any α satisfying $q_\alpha<0$ and

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\geq e^{-\varphi_{X,l}(z_\beta)-\epsilon}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{Y_s}|F_{\alpha,\beta}|^2e^{-\varphi_Y-H(z_\beta,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}.$$

Note that $H(w,w'):=N_\gamma(w,w')+\epsilon$. Letting $\epsilon\to 0$, $\gamma\to+\infty$ and $s\to+\infty$, we have

$$\begin{aligned}&\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\\ \geq&\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}.\end{aligned}\tag{63}$$

Note that $V_\beta\cap V_{\tilde{\beta}}=\emptyset$ implies that $\hat{V}_\beta\cap\hat{V}_{\tilde{\beta}}=\emptyset$ for any $\beta\neq\tilde{\beta}$. It follows from inequality (63) that

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\geq\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}.$$

Thus, Remark 22 holds.

4 Proofs of Theorem 1, Propositions 1 and 2

In this section, we prove Theorem 1, Propositions 1 and 2.

4.1 Proof of Theorem 1

Proof We firstly give the proof of the sufficiency in Theorem 1.

As $\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1$ is a subharmonic function on Ω. By the construction of $g\in\mathcal{O}_\Omega$, we have $2u(z):=2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1-2\log|g|$ is a subharmonic function on Ω which satisfies $v(dd^cu,z)\in[0,1)$ for any $z\in Z_\Omega$. Then it follows from Lemma 23 that we know $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=$

$\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y)\in Z_0$. It follows from Theorem 11 that we know the sufficiency part of Theorem 1 holds.

Now we prove the necessity part of Theorem 1.

Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$. Then according to Lemma 2, there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f)\in H^0(Z_0,\Big(\mathcal{O}(K_M)\otimes\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)\Big)|_{Z_0})$, and $G(t;c)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$ for any $t\geq 0$. Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that c is increasing near $+\infty$.

Following from Lemma 19, for any $j\in\{1,2,\ldots,m\}$, we assume that

$$F=\sum_{l=k_j}^{\infty}\pi_1^*(\tilde{z}_j^l d\tilde{z}_j)\wedge\pi_2^*(F_{j,l})$$

on $V_j\times Y$, where $k_j\in\mathbb{N}$, $F_{j,l}$ is a holomorphic $(n-1,0)$ form on Y for any $l\geq k_j$, and $\tilde{F}_j:=F_{j,k_j}\not\equiv 0$.

We firstly recall the construction of ψ and φ.

Recall that $\psi=\pi_1^*\big(\sum\limits_{j=1}^{m}2p_jG_\Omega(z,z_j)\big)+N$ is a plurisubharmonic function on M. We assume that $N\leq 0$ is a plurisubharmonic function on M and $N|_{Z_j}$ is not identically $-\infty$ for any j. φ_1 is a Lebesgue measurable function on Ω such that $\psi+\pi_1^*(\varphi)$ is a plurisubharmonic function on M. We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surfaces, we have

$$\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\log|g|)+\tilde{\psi}_2,$$

where g is a holomorphic function on Ω and $\tilde{\psi}_2\in Psh(M)$. Denote $ord_{z_j}(g)=p_j$ and $d_j:=\lim_{z\to z_j}(g/\tilde{z}_j^{q_j})(z)$.

Now we prove that $\psi=\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$. As $G(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 27, we can get that $k_j-q_j+1=0$ for any $j\in I_F$ and

$$\frac{\int_M|F|^2e^{-\varphi}c(-\psi)}{\int_0^{+\infty}c(s)e^{-s}ds}\geq\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)},\tag{64}$$

where $I_F:=\{j:k_j-q_j+1\leq 0\}$. Especially, $\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}<+\infty$ and $\tilde{\psi}_2$ is not identically $-\infty$ on Z_j for any $j\in I_F$.

Let $V_j\times U_y$ be an open neighborhood of (z_j,y). Assume that $F=\tilde{z}_j^{k_j}h(\tilde{z}_j,w)d\tilde{z}_j\wedge dw$ on $V_j\times U_y$, where $h(z_j,w)dw=\tilde{F}_j$. Note that $h(z_j,w)$ is not identically zero on $\{z_j\}\times U_y$, there must exist $\hat{w}\in U_y$ such that $h(z_j,\hat{w})\neq 0$. Then we know $|h(z_j,w)|^2$ has a positive lower bound on $\tilde{V}_j\times U_{\hat{w}}$, where $\tilde{V}_j\times U_{\hat{w}}$ is a small open neighborhood of $(z_j,\hat{w})$. Then, according to Lemma 17, $\psi\leq\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$ and $c(t)$ is increasing near $+\infty$, we know that for any

$w \in U_{\hat{w}}$, we have $\int_{\tilde{V}_j} |\tilde{z}_j^{k_j} d\tilde{z}_j|^2 e^{-\varphi_1} c(-\sum_{j=1}^m 2p_j G_\Omega(\cdot, z_j)) < +\infty$. It follows from Lemma 18 that we have

$$\int_{\tilde{V}_j \times U_{\hat{w}}} |\pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j)|^2 e^{-\varphi} c\big(-\pi_1^*(\sum_{j=1}^m 2p_j G_\Omega(\cdot, z_j))\big) < +\infty.$$

Then by Lemma 22, we have

$$\big(F - \pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j, y)}$$

for any $j \in I_F$, $y \in Y$. And according to Lemma 22 we also have

$$\big(F, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j, y)}$$

for any $j \in \{1, 2, \ldots, m\} \setminus I_F$, $y \in Y$.

Denote that $\tilde{\psi} := \pi_1^*\big(2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j)\big)$, $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$, and $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$. Then according to Lemma 25 $\big(F = \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j)$ on $V_j \times Y$ for $j \in I_F$ and $F \equiv 0$ on $V_j \times Y$ for $j \notin I_F$ in Lemma 25$\big)$, there exists a holomorphic $(n, 0)$ form $\tilde{F}$ on M such that $\big(\tilde{F} - \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j, y)}$ for any $j \in I_F$, $y \in Y$, $\big(\tilde{F}, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j, y)}$ for any $j \in \{1, 2, \ldots, m\} \setminus I_F$, $y \in Y$, and

$$\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \leq \left(\int_0^{+\infty} c(s) e^{-s} ds\right) \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \tag{65}$$

Then $\big(\tilde{F} - F, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$. Combining inequality (64) with inequality (65), we have that

$$\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \geq \int_M |F|^2 e^{-\varphi} c(-\psi) \geq \int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}). \tag{66}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi \leq \tilde{\psi}$, we have $\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \geq \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi)$. Then all "$\geq$" in (66) should be "$=$". It follows from Lemma 7 that $N = 0$ and $\psi = \pi_1^*\big(2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j)\big)$.

As $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1$ is a subharmonic function on Ω. By the construction of $g \in \mathcal{O}_\Omega$, we have $2u(z) := 2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1 - 2\log|g|$ is a subharmonic function on Ω which satisfies $v(dd^c u, z) \in [0, 1)$ for any $z \in Z_0$. Then it follows from Lemma 23 that we know $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$.

Then it follows from Theorem 11 that we know Theorem 1 holds.

4.2 Proof of Proposition 1

Proof (*Proof of Proposition* 1) Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2, there exists a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)|_{Z_0})$, and $G(t; c) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq 0$. Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that c is increasing near $+\infty$.

Following from Lemma 19, for any $j \geq 1$, we assume that

$$F = \sum_{l=k_j}^{\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$$

on $V_j \times Y$, where $k_j \in \mathbb{N}$, $F_{j,l}$ is a holomorphic $(n-1, 0)$ form on Y for any $l \geq k_j$, and $\tilde{F}_j := F_{j,k_j} \not\equiv 0$.

We firstly recall the construction of ψ and φ.

Recall that $\psi = \pi_1^*\big(\sum_{j\geq 1} 2p_j G_\Omega(z, z_j)\big) + N$ is a plurisubharmonic function on M. We assume that $N \leq 0$ is a plurisubharmonic function on M and $N|_{Z_j}$ is not identically $-\infty$ for any j. φ_1 is a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on M. We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surface, we have

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2,$$

where g is a holomorphic function on Ω and $\tilde{\psi}_2 \in Psh(M)$. Denote $ord_{z_j}(g) = p_j$ and $d_j := \lim_{z\to z_j}(g/\tilde{z}_j^{q_j})(z)$.

Now we prove that $\psi_1 = 2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j)$. As $G(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 27, we can get that $k_j - q_j + 1 = 0$ for any $j \in I_F$ and

$$\frac{\int_M |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}, \tag{67}$$

where $I_F := \{j : k_j - q_j + 1 \leq 0\}$. Especially, $\sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty$ and $\tilde{\psi}_2$ is not identically $-\infty$ on Z_j for any $j \in I_F$.

Let $V_j \times U_y$ be an open neighborhood of (z_j, y). Assume that $F = \tilde{z}_j^{k_j} h(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$ on $V_j \times U_y$, where $h(z_j, w)dw = \tilde{F}_j$. Note that $h(z_j, w)$ is not identically zero on $\{z_j\} \times U_y$, there must exist $\hat{w} \in U_y$ such that $h(z_j, \hat{w}) \neq 0$. Then we know $|h(z_j, w)|^2$ has a positive lower bound on $\tilde{V}_j \times U_{\hat{w}}$, where $\tilde{V}_j \times U_{\hat{w}}$ is a small open neighborhood of $(z_j, \hat{w})$. Then, according to Lemma 17, $\psi \leq \pi_1^*(\sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j))$ and $c(t)$ is increasing near $+\infty$, we know that for any $w \in U_{\hat{w}}$, we have

$\int_{\tilde{V}_j}|\tilde{z}_j^{k_j}d\tilde{z}_j|^2e^{-\varphi_1}c(-\sum_{j=1}^m 2p_jG_\Omega(\cdot,z_j))<+\infty$. It follows from Lemma 18 that we have

$$\int_{\tilde{V}_j\times U_{\hat{w}}}|\pi_1^*(\tilde{z}_j^{k_j}d\tilde{z}_j)\wedge\pi_2^*(\tilde{F}_j)|^2e^{-\varphi}c(-\pi_1^*(\sum_{j=1}^m 2p_jG_\Omega(\cdot,z_j)))<+\infty.$$

Then by Lemma 22, we have

$$\big(F-\pi_1^*(\tilde{z}_j^{k_j}d\tilde{z}_j)\wedge\pi_2^*(\tilde{F}_j),(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j\in I_F$, $y\in Y$. And according to Lemma 22 we also have

$$\big(F,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j\in\{1,2,\ldots,m\}\setminus I_F$, $y\in Y$.

Denote that $\tilde{\psi}:=\pi_1^*\big(2\sum_{j=1}^{\gamma}p_jG_\Omega(\cdot,z_j)\big)$, $\tilde{\varphi}_1:=\pi_1^*(\varphi_1)+\psi-\tilde{\psi}$, and $\tilde{\varphi}:=\tilde{\varphi}_1+\pi_2^*(\varphi_2)$. Then according to Lemma 25 $\big(F=\pi_1^*(\tilde{z}_j^{k_j}dw_j)\wedge\pi_2^*(\tilde{F}_j)$ on $V_j\times Y$ for $j\in I_F$ and $F\equiv 0$ on $V_j\times Y$ for $j\notin I_F$ in Lemma 25$\big)$, there exists a holomorphic $(n,0)$ form $\tilde{F}$ on M such that $\big(\tilde{F}-\pi_1^*(\tilde{z}_j^{k_j}dw_j)\wedge\pi_2^*(\tilde{F}_j),(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j\in I_F$, $y\in Y$, $\big(\tilde{F},(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j\in\{1,2,\ldots,m\}\setminus I_F$, $y\in Y$, and

$$\int_M|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\tag{68}$$

Then $\big(\tilde{F}-F,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Combining inequality (67) with inequality (68), we have that

$$\int_M|\tilde{F}|^2e^{-\varphi}c(-\psi)\ge\int_M|F|^2e^{-\varphi}c(-\psi)\ge\int_M|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi}).\tag{69}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi\le\tilde{\psi}$, we have $\int_M|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})\ge\int_M|\tilde{F}|^2e^{-\varphi}c(-\psi)$. Then all "$\ge$" in (69) should be "=". It follows from Lemma 7 that $N=0$ and $\psi=\pi_1^*\big(2\sum_{j=1}^{\gamma}p_jG_\Omega(\cdot,z_j)\big)$.

As $\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\sum_{j=1}^{\gamma}p_jG_\Omega(\cdot,z_j)+\varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^{\gamma}p_jG_\Omega(\cdot,z_j)+\varphi_1$ is a subharmonic function on Ω. By the construction of $g\in\mathcal{O}_\Omega$, we have $2u(z):=2\sum_{j=1}^{\gamma}p_jG_\Omega(\cdot,z_j)+\varphi_1-2\log|g|$ is a subharmonic function on Ω which satisfies $v(dd^cu,z)\in[0,1)$

for any $z \in Z_0$. Then it follows from Lemma 23 that we know $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$.

Then it follows from Proposition 4 that we know Proposition 1 holds.

4.3 *Proof of Proposition 2*

Proof Assume that $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2, there exists a unique holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)|_{Z_0})$, and $\tilde{G}(t; c) = \int_{\{\psi<-t\}} |F|^2e^{-\varphi}c(-\psi)$ for any $t \geq 0$. Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that c is increasing near $+\infty$.

We firstly recall the construction of ψ and φ.

Recall that $\psi = \pi_1^*\big(\sum_{j\geq 1} 2p_jG_{\Omega}(z, z_j)\big) + N$ is a plurisubharmonic function on $\tilde{M}$. We assume that $N \leq 0$ is a plurisubharmonic function on $\tilde{M}$ and $N|_{Z_j}$ is not identically $-\infty$ for any j. φ_1 is a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on $\tilde{M}$. We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surface, we have

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2,$$

where g is a holomorphic function on Ω and $\tilde{\psi}_2 \in Psh(\tilde{M})$. Denote $ord_{z_j}(g) = p_j$ and $d_j := \lim_{z\to z_j}(g/\tilde{z}_j^{q_j})(z)$.

Following from Lemma 20, for any $j \in \{1, 2, \ldots, m\}$, we assume that

$$F = \sum_{l=k_j}^{\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$$

on $U_j \Subset (V_j \times Y) \cap \tilde{M}$ is a neighborhood of $Z_j := \{z_j\} \times Y$ in $\tilde{M}$, and $k_j \in \mathbb{N}$, $F_{j,l}$ is a holomorphic $(n-1, 0)$ form on Y for any $l \geq k_j$, and $\tilde{F}_j := F_{j,k_j} \not\equiv 0$.

Let W be an open subset of Y such that $W \Subset Y$. Then for any j, $1 \leq j < \gamma$, there exists $r_{j,W} > 0$ such that $V_{j,W} \times W \subset \tilde{M}$, where $V_{j,W} := \{z \in \Omega : |\tilde{z}_j(z)| < r_{j,W}\}$.

As $\tilde{G}(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 27, we can get that $k_j - q_j + 1 = 0$ for any $j \in I_F$ and

$$\frac{\int_{\tilde{M}} |F|^2e^{-\varphi}c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_W |F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)},$$

where $I_F := \{j : 1 \leq j < \gamma\ \&\ \tilde{k}_j + 1 - k_j \leq 0\}$.

By the arbitrariness of W, we have $\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty$ for any $j \in I_F$, and

$$\frac{\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}. \tag{70}$$

Especially, we know $\sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty$ and $\tilde{\psi}_2$ is not identically $-\infty$ on Z_j for any $j \in I_F$.

Let $V_j \times U_y$ be an open neighborhood of (z_j, y) in $\tilde{M}$. Assume that $F = \tilde{z}_j^{k_j} h(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$ on $V_j \times U_y$, where $h(z_j, w)dw = \tilde{F}_j$. Note that $h(z_j, w)$ is not identically zero on $\{z_j\} \times U_y$, there must exist $\hat{w} \in U_y$ such that $h(z_j, \hat{w}) \neq 0$. Then we know $|h(z_j, w)|^2$ has a positive lower bound on $\tilde{V}_j \times U_{\hat{w}}$, where $\tilde{V}_j \times U_{\hat{w}} \Subset V_j \times U_y$ is a small open neighborhood of $(z_j, \hat{w})$. Then, according to Lemma 17, $\psi \leq \sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j)$ and $c(t)$ is increasing near $+\infty$, we know that for any $w \in U_{\hat{w}}$, we have $\int_{\tilde{V}_j} |\tilde{z}_j^{k_j} d\tilde{z}_j|^2 e^{-\varphi_1} c(-\sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j)) < +\infty$. It follows from Lemma 18 that we have

$$\int_{\tilde{V}_j\times U_{\hat{w}}} |\pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j)|^2 e^{-\varphi} c\big(-\pi_1^*(\sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j))\big) < +\infty.$$

Then by Lemma 22, we have

$$\big(F - \pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j \in I_F$, $y \in Y$. And according to Lemma 22 we also have

$$\big(F, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j \in \{1, 2, \ldots, m\} \setminus I_F$, $y \in Y$.

Now we prove that $N \equiv 0$. Denote that $\tilde{\psi} := \pi_1^*(2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j))$, $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$, and $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$. Then according to Lemma 25 $\big(F = \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j)$ on U_j for $j \in I_F$ and $F \equiv 0$ on U_j for $j \notin I_F$ in Lemma 25$\big)$, there exists a holomorphic $(n, 0)$ form $\tilde{F}$ on $\tilde{M}$ such that $\big(\tilde{F} - \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j \in I_F$, $y \in Y$, $\big(\tilde{F}, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j \geq 1 \& j \notin I_F$ and any $y \in Y$, and

$$\int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}. \tag{71}$$

Note that we have $\big(\tilde{F}-F,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

It follows from (70) and (73) that, we have

$$\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)\geq\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\geq\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi}). \tag{72}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi\leq\tilde{\psi}$, we have $\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})\geq\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)$. Then all "$\geq$" in (72) should be "=". It follows from Lemma 7 that $N=0$ and $\psi=\pi_1^*\big(2\sum_{j=1}^{\gamma}p_jG_\Omega(\cdot,z_j)\big)$.

Using Lemma 25 $\big(\tilde{M}\sim M$ and $F=\pi_1^*(\tilde{z}_j^{k_j}dw_j)\wedge\pi_2^*(\tilde{F}_j)$ for $j\in I_F$ and $F\equiv 0$ for $j\notin I_F$ in Lemma 25$\big)$, there exists a holomorphic $(n,0)$ form $\hat{F}$ on M such that $\big(\hat{F}-\pi_1^*(\tilde{z}_j^{k_j}dw_j)\wedge\pi_2^*(\tilde{F}_j),(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j\in I_F$, $y\in Y$, $\big(\hat{F},(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j\geq 1\&j\notin I_F$ and any $y\in Y$, and

$$\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}. \tag{73}$$

Then $\big(\hat{F}-F,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

Now According to the choice of F, we have that

$$\begin{aligned}\tilde{G}(0)&=\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\leq\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)\\&\leq\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)\\&\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned} \tag{74}$$

Combining inequality (70) with inequality (74), we get that

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)=\int_M|\hat{F}|^2e^{-\varphi}c(-\psi).$$

As $\tilde{F}\not\equiv 0$, the above equality implies that $\tilde{M}=M$.

5 Proofs of Theorems 2, 3, 4 and Proposition 3

In this section, we prove Theorems 2, 3, 4 and Proposition 3.

5.1 *Proof of Theorem 2*

Proof We firstly give the proof of the sufficiency of Theorem 2. It follows from Theorem 14 that $G\big(h^{-1}(r); \mathcal{I}(\varphi+\psi)\big)$ is linear with respect to r.

When $N \equiv 0$, then φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$, it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$.

Hence $G\big(h^{-1}(r); \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big)$ is linear with respect to r. The sufficiency of Theorem 2 is proved.

We prove the necessity part of Theorem 2.

Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2, there exists a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big)|_{Z_0})$, and $G(t; c) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \ge 0$. Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that c is increasing near $+\infty$.

It follows from Lemma 19 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\ge 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\ge 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \ldots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha)$$

on $V_0 \times Y$.

Denote that $E_0 := \Big\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1 \le j \le n_1} \frac{\alpha_j+1}{p_j} = 1\Big\}$, $E_1 := \Big\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1 \le j \le n_1} \frac{\alpha_j+1}{p_j} < 1\Big\}$ and $E_2 := \Big\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1 \le j \le n_1} \frac{\alpha_j+1}{p_j} > 1\Big\}$.

Now we prove $N \equiv 0$. It follows from Lemma 28 that we have $F_\alpha \equiv 0$ for any $\alpha \in E_1$, and

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \ge \sum_{\alpha \in E_0} \frac{(2\pi)^{n_1} e^{-\sum_{1 \le j \le n_1} \varphi_j(z_j)}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_j^{2\alpha_j+2}(z_j)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_0,w')}. \tag{75}$$

Note that $\psi = \hat{G} + N \le \hat{G}$, $c(t)$ is increasing near $+\infty$, φ_X is upper semi-continuous on $\prod_{j=1}^{n_1} \Omega_j$. When t is large enough, we have

$$\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)\geq\int_{\{\hat{G}<-t\}}|F|^2e^{-\pi_1^*\left(-\sum_{1\leq j\leq n_1}\varphi_j(z_j)\right)-\pi_2^*(\varphi_Y)}c(-\hat{G})$$
$$\geq C_0\int_{\{\hat{G}<-t\}}|F|^2e^{-\pi_2^*(\varphi_Y)}c(-\hat{G}),$$

where $C_0>0$ is a constant, then it follows from $\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)<+\infty$ and Lemma 19 that for any $\alpha\in E_2:=\left\{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_j}>1\right\}$, we know $F_\alpha\in\mathcal{I}(\varphi_Y)$ for any $y\in Y$.

Recall that

$$\hat{G}=\max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*\left(G_{\Omega_j}(\cdot,z_j)\right)\right\}.$$

Denote $\tilde{\varphi}:=\varphi+\psi-\hat{G}$. It follows from Lemma 26 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on M satisfying that $(\hat{F}-F,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\right)_z$ for any $z\in Z_0$ and

$$\int_M|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})$$
$$\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E_0}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_j)}\int_Y|f_\alpha|^2e^{-\varphi_Y-N(z_0,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}. \tag{76}$$

It follows from (75) and (76) that we have

$$\int_M|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\leq\int_M|F|^2e^{-\varphi}c(-\psi)\leq\int_M|\hat{F}|^2e^{-\varphi}c(-\psi) \tag{77}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi\leq\hat{G}$, we have $\int_M|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\geq\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)$. Then all "$\geq$" in (77) should be "$=$". It follows from Lemma 7 that we know $N\equiv 0$ and $\psi=\max_{1\leq j\leq n_1}\{2p_j\pi_{1,j}^*\left(G_{\Omega_j}(\cdot,z_j)\right)\}$.

As $N\equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\leq j\leq n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\left(\hat{G}+\pi_2^*(\varphi_Y)\right)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\leq j\leq n_1}\{2p_j\pi_{1,j}^*\left(G_{\Omega_j}(\cdot,z_j)\right)\}$. It follows from Theorem 14 that the necessity part of Theorem 2 holds.

5.2 *Proof of Theorem 3*

Proof We firstly give the proof of the sufficiency of Theorem 3. It follows from Theorem 15 that $G\left(h^{-1}(r);\mathcal{I}(\varphi+\psi)\right)$ is linear with respect to r.

When $N\equiv 0$, then φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\leq j\leq n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\left(\hat{G}+\pi_2^*(\varphi_Y)\right)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

Hence $G\big(h^{-1}(r); \mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)$ is linear with respect to r. The sufficiency of Theorem 3 is proved.

We prove the necessity part of Theorem 3.

Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2, there exists a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)|_{Z_0})$, and $G(t; c) = \int_{\{\psi<-t\}} |F|^2e^{-\varphi}c(-\psi)$ for any $t \geq 0$. Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that c is increasing near $+\infty$.

It follows from Lemma 19 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on $V_\beta \times Y$, for any $\beta \in \tilde{I}_1$. Denote that $E_\beta := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1\right\}$, $E_{\beta,1} := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} < 1\right\}$ and $E_{\beta,2} := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} > 1\right\}$.

Now we prove $N \equiv 0$. It follows from Lemma 28 that we have we have $F_\alpha \equiv 0$ for any $\alpha \in E_{\beta,1}$, and

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\leq j\leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1} (\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta,w')}. \tag{78}$$

Note that $\psi = \hat{G}+N \leq \hat{G}$, $c(t)$ is increasing near $+\infty$, φ_X is upper semi-continuous on $\prod_{j=1}^{n_1} \Omega_j$. When t is large enough, we have

$$\begin{aligned}\int_{\{\psi<-t\}} |F|^2e^{-\varphi}c(-\psi) \geq& \int_{\{\hat{G}<-t\}} |F|^2e^{-\pi_1^*\left(\sum_{1\leq j\leq n_1} \pi_{1,j}^*(\varphi_j)\right)-\pi_2^*(\varphi_Y)}c(-\hat{G}) \\ \geq& C_0 \int_{\{\hat{G}<-t\}} |F|^2e^{-\pi_2^*(\varphi_Y)}c(-\hat{G}),\end{aligned}$$

then it follows from $\int_{\{\psi<-t\}} |F|^2e^{-\varphi}c(-\psi) < +\infty$ and Lemma 19 that for any $\alpha \in E_{\beta,2} := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} > 1\right\}$ and any $\beta \in \mathcal{I}_1$, we know $F_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)$ for any $y \in Y$.

Recall that

$$\hat{G} := \max_{1\leq j\leq n_1} \left\{\pi_{1,j}^*\left(2\sum_{1\leq k\leq m_j} p_{j,k}G_{\Omega_j}(\cdot, z_{j,k})\right)\right\},$$

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Lemma 26 that there exists a holomorphic $(n, 0)$ form $\hat{F}$ on M satisfying that $(\hat{F} - F, z) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big)_z$ for any $z \in Z_0$ and

$$\begin{aligned}&\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G})\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\le j\le n_1} \varphi_j(z_{j,\beta_j})} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}}{\prod_{1\le j\le n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned} \tag{79}$$

It follows from (78) and (79) that we have

$$\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \le \int_M |F|^2 e^{-\varphi} c(-\psi) \le \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi). \tag{80}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi \le \hat{G}$, we have $\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \ge \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)$. Then all "$\ge$" in (80) should be "$=$". It follows from Lemma 7 that we know $N \equiv 0$ and $\psi = \max_{1\le j\le n_1} \left\{\pi_{1,j}^* \left(2 \sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1} \left\{\pi_{1,j}^* \left(2 \sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$. It follows from Theorem 15 that the necessity part of Theorem 3 holds.

5.3 *Proof of Theorem 4*

Proof We prove Theorem 4 by contradiction.

Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2, there exists a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big|_{Z_0})$, and $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \ge 0$. Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that c is increasing near $+\infty$.

It follows from Lemma 19 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\ge 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\ge 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on $V_\beta \times Y$ for any $\beta \in \tilde{I}_1$. Denote that $E_\beta := \left\{\alpha \in \mathbb{Z}^{n_1}_{\geq 0} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1\right\}$, $E_{\beta,1} := \left\{\alpha \in \mathbb{Z}^{n_1}_{\geq 0} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} < 1\right\}$ and $E_{\beta,2} := \left\{\alpha \in \mathbb{Z}^{n_1}_{\geq 0} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} > 1\right\}$.

Now we prove $N \equiv 0$. It follows from Lemma 28 that we have we have $F_\alpha \equiv 0$ for any $\alpha \in E_{\beta,1}$, and

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\leq j\leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1} (\alpha_j+1) c_{j,\beta_j}^{2\alpha_j+2}} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}. \tag{81}$$

Note that $\psi = \hat{G} + N \leq \hat{G}$, $c(t)$ is increasing near $+\infty$, $\sum_{1\leq j\leq n_1} \pi^*_{1,j}(\varphi_j)$ is upper semi-continuous on $\prod_{j=1}^{n_1} \Omega_j$. When t is large enough, we have

$$\begin{aligned}
&\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) \geq \int_{\{G<-t\}} |F|^2 e^{-\sum_{1\leq j\leq n_1} \pi^*_{1,j}(\varphi_j) - \pi_2^*(\varphi_Y)} c(-\hat{G}) \\
\geq &C_0 \int_{\{\hat{G}<-t\}} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\hat{G}),
\end{aligned}$$

where C_0 is a constant. Then it follows from $\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$ and Lemma 19 that for any $\alpha \in E_{\beta,2} := \left\{\alpha \in \mathbb{Z}^{n_1}_{\geq 0} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} > 1\right\}$ and any $\beta \in \mathcal{I}_1$, we know $F_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)$ for any $y \in Y$.

Recall that

$$\hat{G} := \max_{1\leq j\leq n_1} \left\{ \pi^*_{1,j} \left(2 \sum_{1\leq k\leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}.$$

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Lemma 26 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on M satisfying that $(\hat{F} - F, z) \in \left(\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\right)_z$ for any $z \in Z_0$ and

$$\begin{aligned}
&\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\
\leq &\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\leq j\leq n_1} \varphi_j(z_{j,\beta_j})} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}}{\prod_{1\leq j\leq n_1} (\alpha_j+1) c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned} \tag{82}$$

It follows from (81) and (82) that we have

$$\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \leq \int_M |F|^2 e^{-\varphi} c(-\psi) \leq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi). \tag{83}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi \le \hat{G}$, we have $\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \ge \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)$. Then all "$\ge$" in (83) should be "=". It follows from Lemma 7 that we know $N \equiv 0$ and $\psi = \max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$. It follows from Theorem 16 that we get a contradiction.

5.4 *Proof of Proposition 3*

Proof As $\tilde{G}(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$, then according to Lemma 2, there exists a unique holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big)|_{Z_0})$, and $G(t) = \int_{\{\psi<-t\}\cap\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \ge 0$. Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that c is increasing near $+\infty$.

It follows from Lemma 20 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha\in\mathbb{Z}_{\ge 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha\in\mathbb{Z}_{\ge 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on a open neighborhood $U_\beta \subset (V_\beta \times Y) \cap \tilde{M}$ of $\{z_\beta\} \times Y$ for any $\beta \in \tilde{I}_1$. Denote that $E_\beta := \left\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1\right\}$, $E_{\beta,1} := \left\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} < 1\right\}$ and $E_{\beta,2} := \left\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} > 1\right\}$.

It follows from Remark 22 that we have we have $F_{\alpha,\beta} \equiv 0$ for any $\alpha \in E_{\beta,1}$, and

$$\frac{G(0)}{\int_t^{+\infty} c(s)e^{-s}ds} \ge \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\le j\le n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}. \quad (84)$$

Note that $\psi = \hat{G} + N \le \hat{G}$, $c(t)$ is increasing near $+\infty$, $\sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j)$ is upper semi-continuous on $\prod_{j=1}^{n_1} \Omega_j$. When t is large enough, we have

$$\int_{\{\psi<-t\}\cap\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \ge \int_{\{\hat{G}<-t\}\cap\tilde{M}} |F|^2 e^{-\sum_{1\le j\le n_1}\pi_{1,j}^*(\varphi_j) - \pi_2^*(\varphi_Y)} c(-\hat{G})$$
$$\ge C_0 \int_{\{\hat{G}<-t\}\cap\tilde{M}} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\hat{G}),$$

then it follows from $\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)<+\infty$ and Lemma 20 that for any $\alpha\in E_{\beta,2}:=\left\{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}>1\right\}$ and any $\beta\in\mathcal{I}_1$, we know $F_{\alpha,\beta}\in\mathcal{I}(\varphi_Y)$ for any $y\in Y$.

Recall that $\hat{G}:=\max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}$ and $\tilde{\varphi}=\varphi+N$. It follows from Lemma 26 that there exists a holomorphic $(n,0)$ form $\tilde{F}$ on $\tilde{M}$ satisfying that $(\tilde{F}-F,z)\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y)))_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\\ \leq&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}\tag{85}$$

Combining with (84) and (85), we have

$$\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)\geq\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\geq\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\hat{G}).\tag{86}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi\leq\hat{G}$, we have $\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\geq\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)$. Then all "$\geq$" in (86) should be "$=$". It follows from Lemma 7 that $N=0$ and $\psi=\max_{1\leq j\leq n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\leq k\leq m_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$.

It follows from Lemma 26 ($\tilde{M}\sim M$) that there exists a holomorphic $(n,0)$ form $\hat{F}$ on M satisfying that $(\hat{F}-F,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\right)_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \leq&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}\tag{87}$$

We also have

$$\frac{\tilde{G}(0)}{\int_t^{+\infty}c(s)e^{-s}ds}=\frac{\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\leq\frac{\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}.\tag{88}$$

Combining (84), (87) and (88), we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)=\int_M|\hat{F}|^2e^{-\varphi}c(-\psi).$$

As $\hat{F}\neq 0$, the above equality shows that $M=\tilde{M}$.

6 Proofs of Theorems 5 and 6

In this section, we prove Theorems 5 and 6.

6.1 Proof of Theorem 5

Proof It follows from Lemma 25 that we know there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ such that $(F - f, (z_j, y)) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{m} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)}. \tag{89}$$

In the following, we prove the characterization of the holding of the equality $\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{m} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)} = \inf\big\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F}$ is a holomorphic $(n, 0)$ form on $\tilde{M}$ such that $\big(\tilde{F} - f, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0\big\}$.

We firstly give the proof of the sufficiency in Theorem 5.

As $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j) + \varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j) + \varphi_1$ is a subharmonic function on Ω. By the construction of $g \in \mathcal{O}_\Omega$, we have $2u(z) := 2\sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j) + \varphi_1 - 2\log|g|$ is a subharmonic function on Ω which satisfies $v(dd^c u, z) \in [0, 1)$ for any $z \in Z_\Omega$. Then it follows from Lemma 23 that we know $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. It follows from Theorem 12 that we know the sufficiency part of Theorem 5 holds.

Next we prove the necessity part of Theorem 5.

Denote that $\tilde{\psi} := \pi_1^*\big(2\sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j)\big)$, $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$, and $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$. It follows from Lemma 25 that there exists a holomorphic $(n, 0)$ form $\tilde{F}_1 \not\equiv 0$ on $\tilde{M}$ such that $\big(\tilde{F}_1 - f, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\begin{aligned}
&\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\tilde{\varphi}} c\left(-\tilde{\psi}\right) \\
&\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{m} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)}.
\end{aligned} \tag{90}$$

As $\left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y |F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} = \inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):$ $\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0\big\}$ holds, we know

$$\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\varphi}c\left(-\psi\right)\geq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi\leq\tilde{\psi}$. Hence we must have

$$\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\varphi}c\left(-\psi\right)=\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\tilde{\varphi}}c\left(-\tilde{\psi}\right).$$

It follows from Lemma 7 that we have $N\equiv 0$ and $\psi=\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$.

It follows from Lemma 25 ($\tilde{M}\sim M$) that there exists a holomorphic $(n,0)$ form $\hat{F}\neq 0$ on M such that $\big(\hat{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\begin{aligned}&\int_M|\hat{F}|^2e^{-\varphi}c\left(-\psi\right)\\&\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}\tag{91}$$

As $\left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y |F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} = \inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):$ $\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0\big\}$ holds, we know

$$\begin{aligned}&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}\\&\leq\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c\left(-\psi\right)\\&\leq\int_M|\hat{F}|^2e^{-\varphi}c\left(-\psi\right)\\&\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}$$

As $\hat{F}\neq 0$, we have $\tilde{M}=M$.

Now $\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1)$ is a plurisubharmonic function on M. By definition, we know that $2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1$ is a subharmonic function on Ω. By the construction of $g \in \mathcal{O}_\Omega$, we have $2u(z) := 2\sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j) + \varphi_1 - 2\log|g|$ is a subharmonic function on Ω and satisfies $v(dd^c u, z) \in [0, 1)$ for any $z \in Z_\Omega$. Then it follows from Lemma 23 that we know $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$.

Then it follows from the necessity part of Theorem 12 that we know Theorem 5 holds.

6.2 *Proof of Theorem 6*

Proof It follows from Lemma 25 that we know there exists a holomorphic $(n, 0)$ form F on M such that $\big(F - f, (z_j, y)\big) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s} ds\right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}.$$

Now it suffices to show that the equality $\left(\int_0^{+\infty} c(s)e^{-s} ds\right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j+1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} = \inf\Big\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F}$ is a holomorphic $(n, 0)$ form on $\tilde{M}$ such that $\big(\tilde{F} - f, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j, y)}$ for any $(z_j, y) \in Z_0\Big\}$ can not hold. We assume that the equality holds to get a contradiction.

Denote that $\tilde{\psi} := \pi_1^*\big(2\sum_{j=1}^{m}(k_j + 1)G_\Omega(\cdot, z_j)\big)$, $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$, and $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$. It follows from Lemma 25 that there exists a holomorphic $(n, 0)$ form $\tilde{F}_1 \neq 0$ on $\tilde{M}$ such that $\big(\tilde{F}_1 - f, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\tilde{\varphi}} c\left(-\tilde{\psi}\right) \\ &\leq \left(\int_0^{+\infty} c(s)e^{-s} ds\right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}.\end{aligned} \tag{92}$$

As $\left(\int_0^{+\infty} c(s)e^{-s} ds\right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j+1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} = \inf\Big\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F}$ is a holomorphic $(n, 0)$ form on $\tilde{M}$ such that $\big(\tilde{F} - f, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j, y)}$ for any $(z_j, y) \in Z_0\Big\}$, we know

$$\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\varphi} c\left(-\psi\right) \geq \left(\int_0^{+\infty} c(s)e^{-s} ds\right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $\psi \le \tilde{\psi}$. Hence we must have

$$\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\tilde{\varphi}} c\left(-\tilde{\psi}\right).$$

Hence, by Lemma 7, we have $N \equiv 0$ and $\psi = \pi_1^*\big(2\sum_{j=1}^{+\infty}(k_j+1)G_\Omega(\cdot, z_j)\big)$.

It follows from Lemma 25 ($\tilde{M} \sim M$) that there exists a holomorphic $(n, 0)$ form $\hat{F} \not\equiv 0$ on M such that $\big(\hat{F} - f, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\begin{aligned}&\int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^m \frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned} \tag{93}$$

As $\left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} = \inf\big\{\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) :$ $\tilde{F}$ is a holomorphic $(n, 0)$ form on $\tilde{M}$ such that $\big(\tilde{F} - f, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0\big\}$ holds, we know

$$\begin{aligned}&\left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^m \frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}\\ &\le \int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi)\\ &\le \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^m \frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}$$

As $\hat{F} \not\equiv 0$, we have $\tilde{M} = M$.

Now $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\sum_{j=1}^{+\infty}(k_j+1)G_\Omega(\cdot, z_j) + \varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^m (k_j+1)G_\Omega(\cdot, z_j) + \varphi_1$ is a subharmonic function on Ω. By the construction of $g \in \mathcal{O}_\Omega$, we have $2u(z) := 2\sum_{j=1}^{+\infty}(k_j+1)G_\Omega(\cdot, z_j) + \varphi_1 - 2\log|g|$ is a subharmonic function on Ω and satisfies $v(dd^c u, z) \in [0, 1)$ for any $z \in Z_\Omega$. Then it follows from Lemma 23 that we know $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$.

Then it follows from Theorem 13 that we know Theorem 6 holds.

7 Proofs of Theorems 7 and 8 and 9

In this section, we prove Theorems 7 and 8 and 9.

7.1 Proofs of Theorem 7 and Remark 10

Proof (*Proof of Theorem* 7) It follows from Remark 20 that there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)\big)_z$ for any $z\in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}.$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 17 that we only need to show the necessity.

Denote $\tilde{\varphi}:=\varphi+\psi-\hat{G}$. It follows from Remark 20 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F}-F,z)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G})\big)_z$ for any $z\in Z_0$ and

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})$$
$$\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}. \tag{94}$$

When the equality $\inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,z)\in(\mathcal{O}\left(K_{\tilde{M}}\right)\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)\big)_z$ for any $z\in Z_0\Big\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\times\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}$ holds, we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)\ge\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi\le\hat{G}$. Hence we must have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c\left(-\psi\right)=\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c\left(-\hat{G}\right).$$

It follows from Lemma 7 that we have $N \equiv 0$ and $\psi = \max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}$. Then It follows from the necessity part of Theorem 17 that Theorem 7 holds.

Proof (*Proof of Remark* 10) As $(f_\alpha, y) \in \big(\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y)\big)_y$ for any $y \in Y$ and $\alpha \in \tilde{E}\backslash E$, it follows from Lemma 26 that there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $(F - f, (z, y)) \in \Big(\mathcal{O}(K_{\tilde{M}})\otimes \mathcal{I}\left(\hat{G} + \pi_2^*(\varphi_Y)\right)\Big)_{(z,y)}$ for any $(z, y) \in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y |f_\alpha|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}.\end{aligned}$$

In the following, we prove the characterization of the holding of the equality.

When $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}$. It follows from Remark 15 that we have the sufficiency of Remark 10.

Next we show the necessity of Remark 10.

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Remark 20 that there exists a holomorphic $(n, 0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F} - F, z) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big)_z$ for any $z \in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G})\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y |f_\alpha|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}.\end{aligned}\tag{95}$$

Now we know the equality $\inf\big\{\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}}))$ & $\big(\tilde{F} - f, (z, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big)_{(z,y)}$ for any $(z, y) \in Z_0\big\} = \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \times \sum_{\alpha\in E} \frac{(2\pi)^{n_1}\int_Y |f_\alpha|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}$ holds, then we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi)$$
$$\geq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - \left(N + \pi_{1,j}^*(\sum_{1\leq j\leq n_1} \varphi_j)\right)(z_0,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}. \tag{96}$$

It follows from (95) and (96) that we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c\,(-\psi) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c\left(-\hat{G}\right).$$

Note that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, $\psi \leq \hat{G}$. It follows from Lemma 7 that we have $N \equiv 0$ and $\psi = \max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \leq j \leq n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}$. Then It follows from Remark 15 that Remark 10 holds.

7.2 *Proofs of Theorem 8 and Remark 11*

Proof (*Proof of Theorem* 8) It follows from Remark 20 that there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $(F - f, z) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k\leq m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\big)_z$ for any $z \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)$$
$$\leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in \tilde{I}_1}\sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \left(N + \pi_{1,j}^*(\sum_{1\leq j\leq n_1} \varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 18 that we only need to show the necessity.

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Remark 20 that there exists a holomorphic $(n, 0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F} - F, z) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G})\big)_z$ for any $z \in Z_0$ and

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})$$
$$\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}. \tag{97}$$

When the equality $\inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,z)\in\Big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\Big(\max_{1\leq j\leq n_1}\Big\{2\sum_{1\leq k\leq m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\Big\}\Big)\Big)_z$ for any $z\in Z_0\Big\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ holds, we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)\geq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi\leq\hat{G}$. Hence we must have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c\left(-\psi\right)=\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c\left(-\hat{G}\right).$$

It follows from Lemma 7 that we have $N\equiv0$ and $\psi=\max_{1\leq j\leq n_1}\Big\{2\sum_{1\leq k\leq m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\Big\}$.

As $N\equiv0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\leq j\leq n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\leq j\leq n_1}\Big\{2\sum_{1\leq k\leq m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\Big\}$. Then It follows from the necessity part of Theorem 18 that Theorem 8 holds.

Proof (*Proof of Remark* 11) As $(f_{\alpha,\beta},y)\in\big(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y)\big)_y$ holds for any $y\in Y$, $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$, it follows from Lemma 26 that there exists a holomorphic $(n,0)$ form F on M satisfying that $\big(F-f,(z,y)\big)\in\Big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\hat{G}+\pi_2^*(\varphi_Y)\right)\Big)_{(z,y)}$ for any $(z,y)\in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

In the following, we prove the characterization of the holding of the equality.

When $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1}\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$. It follows from Remark 16 that we have the sufficiency of Remark 11.

Next we show the necessity of Remark 11.

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Remark 20 that there exists a holomorphic $(n, 0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F} - F, z) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big)_{(z,y)}$ for any $(z, y) \in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G})\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned} \tag{98}$$

Now we know the equality $\inf\big\{\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}}))$ & $(\tilde{F} - f, (z, y) \in (\mathcal{O}\left(K_{\tilde{M}}\right) \otimes \mathcal{I}\left(\hat{G} + \pi_2^*(\varphi_Y)\right)\Big)_{(z,y)}$ for any $(z, y) \in Z_0\big\} = \left(\int_0^{+\infty} c(s)e^{-s}ds\right)$ $\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ holds, then we have

$$\begin{aligned}&\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi)\\ &\ge \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned} \tag{99}$$

Note that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, $\psi \le \hat{G}$. It follows from (98) and (99) that we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi-N} c\left(-\hat{G}\right).$$

It follows from Lemma 7 that we have $N \equiv 0$ and $\psi = \max_{1\le j\le n_1}\Big\{2\sum_{1\le k\le m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\Big\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$, it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1}\Big\{2\sum_{1\le k\le m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\Big\}$. Then It follows from Remark 16 that Remark 11 holds.

7.3 Proofs of Theorem 9 and Remark 12

Proof (*Proof of Theorem* 9) It follows from Remark 20 that there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $(F - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\right)_z$ for any $z \in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 18 that we only need to show the necessity.

We show the equality $\inf\left\{\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}}))\ \&\ (\tilde{F} - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\right)_z \text{ for any } z \in Z_0\right\} = \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ can not hold. We assume that the equality holds to get a contradiction.

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Remark 20 that there exists a holomorphic $(n, 0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F} - F, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G})\right)_z$ for any $z \in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G})\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned} \tag{100}$$

As the equality holds, we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \ge \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, $\psi \le \hat{G}$. Hence we must have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c\left(-\psi\right) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c\left(-G\right).$$

It follows from Lemma 7 that we have $\psi = \max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(G + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}$. Then It follows from Theorem 19 that we get the contradiction and Theorem 9 is proved.

Proof (*Proof of Remark* 12) As $(f_{\alpha,\beta}, y) \in \big(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y)\big)_y$ holds for any $y \in Y$, $\alpha \in \tilde{E}_\beta\backslash E_\beta$ and $\beta \in \tilde{I}_1$, it follows Lemma 26 that there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $\big(F - f, (z, y)\big) \in \Big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\Big(\hat{G}+\pi_2^*(\varphi_Y)\Big)\Big)_{(z,y)}$ for any $(z, y) \in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

In the following, we assume that the equality $\inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi) : \tilde{F} \in H^0\big(\tilde{M}, \mathcal{O}(K_{\tilde{M}})\big)\,\&\,\big(\tilde{F} - f, (z, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y))\big)_{(z,y)}$ for any $(z, y) \in Z_0\Big\} = \left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ holds to get a contradiction.

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Remark 20 that there exists a holomorphic $(n, 0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $\big(\hat{F} - F, z\big) \in \big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)_{(z,y)}$ for any $(z, y) \in Z_0$ and

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})$$
$$\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}. \tag{101}$$

Since the equality holds, we have

$$
\begin{aligned}
&\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \\
&\geq \left(\int_0^{+\infty} c(s)e^{-s} ds\right) \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1} \varphi_j)\right)(z_\beta, w)}}{\prod_{1\le j\le n_1} (\alpha_j+1) c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}
\tag{102}
$$

Note that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, $\psi \le \hat{G}$. It follows from (101) and (102) that we have

$$
\int_{\tilde{M}} |F|^2 e^{-\varphi} c\left(-\psi\right) = \int_{\tilde{M}} |F|^2 e^{-\varphi - N} c\left(-\hat{G}\right).
$$

It follows from Lemma 7 that we have $N \equiv 0$ and $\psi = \max_{1\le j\le n_1} \left\{2\sum_{1\le k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 24 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1} \left\{2\sum_{1\le k<\tilde{m}_j} p_{j,k}\pi_{1,j}^* (G_{\Omega_j}(\cdot, z_{j,k}))\right\}$. Then It follows from Remark 17 that Remark 12 holds.

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M-harmonic Szegö Kernel on the Ball

Petr Blaschke and Miroslav Engliš

Abstract We give a description of the boundary singularity of the Szegö kernel of M-harmonic functions, i.e. functions annihilated by the invariant Laplacian, on the unit ball of the complex n-space, in terms of the Gauss hypergeometric functions.

Keywords M-harmonic function · Invariant Laplacian · Szegö kernel · Hypergeometric functions

1 Introduction

For a bounded domain $\Omega \subset \mathbf{C}^n$ with smooth boundary, the *Bergman space* $L^2_{\text{hol}}(\Omega) \equiv L^2_{\text{hol}}$ of Ω is the subspace in the standard Lebesgue space $L^2(\Omega)$ of all functions that are holomorphic on Ω. It follows from the mean value property of holomorphic functions that for each $z \in \Omega$, the evaluation functional $f \mapsto f(z)$ is bounded on L^2_{hol}, hence given by the inner product with some fixed element $K_z \in L^2_{\text{hol}}$:

$$f(z) = \langle f, K_z \rangle = \int_\Omega f(w)\overline{K_z(w)}\, dw, \qquad \forall f \in L^2_{\text{hol}}(\Omega).$$

The function of two variables, holomorphic in z and $\overline{w}$,

$$K(z, w) := \langle K_w, K_z \rangle = K_w(z) = \overline{K_z(w)}$$

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is known as the *Bergman kernel* of Ω. Following its first appearance in the paper of Bergman one hundred years ago [2] (after some precursory earlier observations of the reproducing property e.g. in the work of Zaremba [14]), and much more prominent treatment in the monograph [3] three decades later, the Bergman kernel has since played vital roles in complex analysis of several variables and in complex geometry. One reason for this is its transformation rule under holomorphic mappings: if $\phi : \Omega_1 \to \Omega_2$ is a biholomorphic map, then

$$K_{\Omega_1}(z, w) = J_\phi(z) K_{\Omega_2}(\phi(z), \phi(y)) \overline{J_\phi(w)},$$

where J_ϕ stands for the complex Jacobian of ϕ. It follows that the Hermitian metric

$$ds^2 = \sum_{j,k=1}^{n} g_{j\bar{k}}(z)\, dz_j\, d\overline{z}_k, \qquad g_{j\bar{k}}(z) := \frac{\partial^2 \log K(z, z)}{\partial z_j \partial \overline{z}_k}, \tag{1}$$

(called the *Bergman metric*) is invariant under biholomorphic maps, which makes it an extremely useful tool for studying the holomorphic equivalence problem in $\mathbf{C}^n$ (still unresolved to this day). Of particular importance in this connection is the boundary behavior of $K(z, w)$: one has

$$K(z, z) = \frac{a(z)}{\rho(z)^{n+1}} + b(z) \log \rho(z), \qquad \forall z \in \Omega,$$

with some functions a, b smooth on the closure $\overline{\Omega}$ of Ω and $\rho \in C^\infty(\overline{\Omega})$ a *defining function* for Ω, i.e. satisfying $\rho > 0$ on Ω and $\rho = 0 < \|\nabla \rho\|$ on $\partial\Omega$. There is also an off-diagonal version of this formula,

$$K(z, w) = \frac{a(z, w)}{\rho(z, w)^{n+1}} + b(z, w) \log \rho(z, w) \qquad \forall z, w \in \Omega, \tag{2}$$

where $a \in C^\infty(\overline{\Omega \times \Omega})$, $a(z, z) = a(z)$ and $\partial_w a(z, w)$, $\overline{\partial}_z a(z, w)$ vanish to infinite order on the diagonal $z = w$, and similarly for b and ρ. This celebrated result due to Fefferman [10], with later different proof by Boutet de Monvel and Sjöstrand [5], was an impetus for an overwhelming mass of later developments in complex analysis; see Hirachi and Komatsu [12] for a nice survey as of 1997.

Most of the above applies verbatim also to the Bergman space replaced by the *Hardy space* of holomorphic functions on Ω which are Poisson extensions of functions in L^2 on the boundary:

$$H^2(\Omega) \equiv H^2 := \{\mathbf{P}f : \ f \in L^2(\partial\Omega) \text{ and } \mathbf{P}f \text{ is holomorphic on } \Omega\},$$

and its reproducing kernel $S(z, w)$—the *Szegö* kernel of Ω, again holomorphic in z and $\overline{w}$—satisfying

$$\mathbf{P}f(z) = \int_{\partial\Omega} f(\zeta)S(z,\zeta)\,d\sigma(\zeta), \qquad \forall f \in H^2,\ \forall z \in \Omega;$$

here $L^2(\partial\Omega)$ is taken with respect to some surface measure $d\sigma$ on Ω with smooth density, and $\mathbf{P}$ stands for the Poisson extension operator. (Here we are abusing the notation slightly by denoting by the same letter also the radial boundary values $S(z,\zeta)$ of $S(z,w)$ on $\partial\mathbf{B}^n$.) The boundary behavior of $S(z,w)$ is again given by (2), only with the exponent $n+1$ replaced by n:

$$S(z,w) = \frac{a(z,w)}{\rho(z,w)^n} + b(z,w)\log\rho(z,w) \qquad \forall z,w \in \Omega, \tag{3}$$

(the functions a, b being different from the ones in (2), but again smooth on $\overline{\Omega}\times\overline{\Omega}$).

Despite the focus on the (difficult and beautiful) applications in complex analysis in several variables, the whole theory applies also in the context of elliptic boundary value problems on a domain $\Omega \subset \mathbf{R}^n$, and actually the excellent monograph by Bergman and Schiffer [6] on this topic appeared shortly after [3]. In particular, this applies to the setup where instead of holomorphic one considers *harmonic* functions, leading to the *harmonic Bergman space* $L^2_{\mathrm{harm}}(\Omega) \equiv L^2_{\mathrm{harm}}$ of all harmonic functions in $L^2(\Omega)$, and the associated *harmonic Bergman kernel* $K_{\mathrm{harm}}(z,w)$, which is harmonic in both variables, symmetric in z, w and satisfies

$$f(z) = \int_{\Omega} f(w)K_{\mathrm{harm}}(z,w)\,dw, \qquad \forall f \in L^2_{\mathrm{harm}},\ \forall z \in \Omega.$$

The analogue of the Hardy space in this setting is the space of Poisson extensions to Ω of all functions in $L^2(\partial\Omega)$:

$$H^2_{\mathrm{harm}}(\Omega) := \{\mathbf{P}f : f \in L^2(\partial\Omega)\}.$$

(The domain Ω is again assumed to be bounded and with C^∞ boundary, and $L^2(\partial\Omega)$ is taken with respect to some surface measure $d\sigma$ on $\partial\Omega$ with smooth density.) This space again has a reproducing kernel, the *harmonic Szegö kernel* $S_{\mathrm{harm}}(z,w)$, which is actually related in a rather simple way to the Poisson kernel of Ω: namely, if $P(z,\zeta)$ stands for the Poisson kernel, so that the Poisson operator $\mathbf{P}$ is given just by

$$\mathbf{P}f(z) = \int_{\partial\Omega} f(\zeta)P(z,\zeta)\,d\sigma(\zeta), \qquad z \in \Omega,$$

then

$$S_{\mathrm{harm}}(z,w) = \int_{\partial\Omega} P(z,\zeta)\overline{P(w,\zeta)}\,d\sigma(\zeta). \tag{4}$$

(As the Poisson kernel is real-valued, the complex conjugation is actually superfluous.)

Here the right tool for the study of the boundary behavior of $K_{\text{harm}}(z, w)$ and $S_{\text{harm}}(z, w)$ turns out to be the so-called calculus of boundary pseudodifferential operators, developed again by Boutet de Monvel [4]; see Grubb [11] for a recent exposition. The analogue of (2) is

$$K_{\text{harm}}(x, y) = |x - \tilde{y}|^{-n} a\Big(x, y, |x - \tilde{y}|, \frac{x - \tilde{y}}{|x - \tilde{y}|}\Big) + b(x, y) \log |x - \tilde{y}|$$

for x, y near the boundary, where $b \in C^\infty(\overline{\Omega} \times \overline{\Omega})$, $a \in C^\infty(\overline{\Omega} \times \overline{\Omega} \times \overline{\mathbf{R}_+} \times \mathbf{S}^{n-1})$, and $\tilde{y}$ denotes the "reflection" of y with respect to $\partial\Omega$. There is also an analogous formula for S_{harm} (i.e. the analogue of (3)), only the exponent $-n$ gets replaced by $1 - n$. We refer the reader to [8] for the details.

Associated to the Hermitian metric (1) on a domain $\Omega \subset \mathbf{C}^n$ is the Laplace-Beltrami operator (or *Bergman Laplacian*)

$$\widetilde{\Delta} := \sum_{j,k=1}^{n} g^{\overline{k}j}(z) \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \tag{5}$$

where $g^{\overline{k}j}$ denotes the inverse matrix to $g_{j\overline{k}}$. The functions annihilated by $\widetilde{\Delta}$ are called M-harmonic (or *invariantly harmonic*). This class of functions lies in a way on the crossroads between the holomorphic and the harmonic case: it resembles the latter in the sense that it is preserved by complex conjugation, while resembling the former by reflecting the complex structure inherent in the definition of the Hermitian metric (1) (we will see that in more detail below).

In this paper, we will be interested in the simplest situation when Ω is the unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$, so that the holomorphic Bergman kernel is just $K(z, w) = \frac{n!}{\pi^n}(1 - \langle z, w\rangle)^{-n-1}$, and $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta} = 4(1 - |z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \overline{z}_k) \frac{\partial^2}{\partial z_j \partial \overline{z}_k}.$$

One can again consider the corresponding M-harmonic *Bergman space* $L^2_{\text{Mh}}(\mathbf{B}^n) \equiv L^2_{\text{Mh}}$ of all M-harmonic functions in $L^2(\mathbf{B}^n)$, and the associated M-harmonic *Bergman kernel* $K_{\text{Mh}}(z, w)$; as well as the M-harmonic *Hardy space*

$$H^2_{\text{Mh}}(\mathbf{B}^n) := \{\mathbf{P}_{\text{Mh}} f : \ f \in L^2(\partial\mathbf{B}^n)\}$$

and its reproducing kernel, the M-harmonic *Szegö kernel* $S_{\text{Mh}}(z, w)$. Here $\mathbf{P}_{\text{Mh}}$ stands for the M-harmonic Poisson operator, i.e. the solution operator for the boundary value problem

$$\widetilde{\Delta}\mathbf{P}_{\text{Mh}} u = 0, \qquad \mathbf{P}_{\text{Mh}} u|_{\partial\mathbf{B}^n} = u.$$

Again, $\mathbf{P}_{\text{Mh}}$ is actually an integral operator

$$\mathbf{P}_{\mathrm{Mh}}u(z) = \int_{\partial\Omega} u(\zeta) P_{\mathrm{Mh}}(z,\zeta)\, d\sigma(\zeta),$$

with the M-harmonic *Poisson kernel* (often called *Poisson-Szegö kernel* in the literature) given explicitly by

$$P_{\mathrm{Mh}}(z,\zeta) = \frac{\Gamma(n)}{2\pi^n} \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}},$$

and the M-harmonic Szegö kernel is related to P_{Mh} as in (4):

$$S_{\mathrm{Mh}}(z,w) = \int_{\partial\Omega} P_{\mathrm{Mh}}(z,\zeta) P_{\mathrm{Mh}}(w,\zeta)\, d\sigma(\zeta).$$

Here and throughout the rest of this paper, we take for $d\sigma$ just the (unnormalized) surface measure on $\partial\mathbf{B}^n$.

The kernels $K_{\mathrm{Mh}}(z,w)$ and $S_{\mathrm{Mh}}(z,w)$ on $\mathbf{B}^n$ were recently studied in [9]. It was shown there, in particular, that most likely there is no "reasonable" formula for $K_{\mathrm{Mh}}(z,w)$ when $n > 1$ (even in the simplest case $n = 2$, its Taylor coefficients involve the value $\zeta(3)$ of the Riemann zeta function in a non-trivial way). For the M-harmonic Szegö kernel, the following formula was obtained,

$$S_{\mathrm{Mh}}(z,w) = \frac{\Gamma(n)}{2\pi^n}(1-|z|^2)^n(1-|w|^2)^n FD_1\left(\begin{matrix} n,n,n,n \\ n \end{matrix}\middle| |z|^2, \langle z,w\rangle, \langle w,z\rangle, |w|^2\right), \tag{6}$$

expressing it in terms of the hypergeometric function FD_1 of four variables

$$\begin{aligned} &FD_1\left(\begin{matrix} a,a',b_1,b_2 \\ c \end{matrix}\middle| x_1,x_2,y_1,y_2\right) = \\ &\qquad \sum_{i_1,i_2,j_1,j_2=0}^{\infty} \frac{(a)_{i_1+i_2}(a')_{j_1+j_2}(b_1)_{i_1+j_1}(b_2)_{i_2+j_2}}{(c)_{i_1+i_2+j_1+j_2}} \frac{x_1^{i_1}}{i_1!}\frac{x_2^{i_2}}{i_2!}\frac{y_1^{j_1}}{j_1!}\frac{y_2^{j_2}}{j_2!}, \end{aligned} \tag{7}$$

which generalizes the usual Gauss hypergeometric function

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\middle| z\right) = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{(c)_j} \frac{z^j}{j!}, \qquad |z| < 1.$$

Here $c \notin \{0,-1,-2,\dots\}$ while a, a', b, b_1, b_2 can be any complex numbers, and

$$(a)_j := a(a+1)\dots(a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}$$

stands for the Pochhammer symbol (rising factorial); the series (7) the converges for all x_1, x_2, y_1, y_2 in the unit disc. The right-hand side of (6) can be expressed in terms of ordinary ${}_2F_1$ functions:

$$S_{\mathrm{Mh}}(z,w)=\frac{\Gamma(n)}{2\pi^n}\frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}}\sum_{i_1=0}^{n}\sum_{i_2,j_1=0}^{n-i_1}\frac{(-n)_{i_1+i_2}(-n)_{i_1+j_1}(n)_{i_2}(n)_{j_1}}{i_1!i_2!j_1!(n)_{i_1+i_2+j_1}} \\ \times t_1^{i_1}t_2^{i_2}t_3^{j_1}\,{}_2F_1\left(\begin{matrix}i_2+n,\, j_1+n\\ i_1+i_2+j_1+n\end{matrix}\middle| t_4\right), \tag{8}$$

where

$$t_1=|z|^2,\quad t_2=\frac{|z|^2-\langle w,z\rangle}{1-\langle w,z\rangle},\quad t_3=\overline{t_2},\quad t_4=1-\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}.$$

See Corollary 4 and Theorem 6 in [9]. Note that using the formula [1, Sect. 2.10 (11)]

$${}_2F_1\left(\begin{matrix}k+1,k+m+1\\ k+m+l+2\end{matrix}\middle| z\right)=\frac{(k+m+l+1)!(-1)^{m+1}}{l!k!(m+k)!(m+l)!} \\ \times\frac{d^{k+m}}{dz^{k+m}}\left[(1-z)^{m+l}\frac{d^l}{dz^l}\frac{\log(1-z)}{z}\right],\qquad m,k,l=0,1,2,\dots,$$

and the elementary relation

$${}_2F_1\left(\begin{matrix}a,b\\ b\end{matrix}\middle| z\right)=(1-z)^{-a},$$

it is possible to express each ${}_2F_1$ in (8) in terms of $\log(1-t_4)$ and rational functions of t_4. The obvious drawback of (8), however, is that it completely obscures the symmetry $S_{\mathrm{Mh}}(z,w)=S_{\mathrm{Mh}}(w,z)$ of the M-harmonic Szegö kernel. This also makes it difficult to derive from it any reasonable description of the boundary singularity of $S_{\mathrm{Mh}}(z,w)$ like (2) and (3).

The aim of this paper is to obtain a simpler representation for $S_{\mathrm{Mh}}(z,w)$, and use it to get an analogue of (2) and (3) for the M-harmonic case.

In Sect. 2, we establish some useful facts about the FD_1 function which may be of interest in their own right, and use these to get a better formula for $S_{\mathrm{Mh}}(z,w)$ as the first main result (Theorem 1). This is then used to obtain a simple description of the singularity of $S_{\mathrm{Mh}}(z,w)$ at the boundary diagonal in Sect. 3 (Theorem 2). Section 4 contains some final remarks.

Throughout the rest of the paper, we abbreviate $\partial/\partial z$ etc. just to ∂_z etc., and similarly for $\overline{\partial}_z$. To make typesetting a little neater, the shorthand

$$\Gamma\left(\begin{matrix}a_1,a_2,\dots,a_k\\ b_1,b_2,\dots,b_m\end{matrix}\right):=\frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_k)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_m)}$$

is often employed throughout the paper. Finally, to make our notation the same as in [9], the M-harmonic Szegö kernel will be denoted K_{Sz} rather than S_{Mh} from now on.

2 Formula for the Szegö Kernel

Throughout this section, unless otherwise specified, the variables x_1, x_2, y_1, y_2 range in the unit disc and $t \in [0, 1]$. We will need the Appell hypergeometric function F_1 of two variables, defined by

$$F_1\left(\begin{matrix} a; b_1, b_2 \\ c \end{matrix}\middle| x, y\right) = \sum_{j,k=0}^{\infty} \frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c)_{j+k}} \frac{x^j}{j!} \frac{y^k}{k!}, \qquad |x| < 1, |y| < 1.$$

(Our notation and conventions for hypergeometric functions follow [1].)

Lemma 1 *For all $n \in \mathbf{N}$,*

$$FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ n \end{matrix}\middle| tx_1, tx_2, ty_1, ty_2\right) = \frac{(n + t\partial_t)_n}{(n)_n} FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ 2n \end{matrix}\middle| tx_1, tx_2, ty_1, ty_2\right).$$

Proof Note that $t\partial_t t^k = kt^k$, so

$$(n + t\partial_t)_n t^k = \frac{\Gamma(2n + k)}{\Gamma(n + k)} t^k. \tag{9}$$

The claim is thus immediate from the definition (7) of the function FD_1. □

Lemma 2 *For $c = b_1 + b_2$,*

$$FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = (1 - x_2)^{-a_1}(1 - y_2)^{-a_2} F_1\left(\begin{matrix} b_1; a_1, a_2 \\ c \end{matrix}\middle| \frac{x_1 - x_2}{1 - x_2}, \frac{y_1 - y_2}{1 - y_2}\right).$$

Proof Recall the integral representation [13, Formula 4.3.(8)]:

$$FD_1\left(\begin{matrix} a, a', b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = \Gamma\left(\begin{matrix} c \\ b_1, b_2, c - b_1 - b_2 \end{matrix}\right) \times \int_{\substack{u_1,u_2>0 \\ u_1+u_2<1}} \frac{u_1^{b_1-1} u_2^{b_2-1} (1 - u_1 - u_2)^{c-b_1-b_2-1}}{(1 - x_1u_1 - x_2u_2)^a (1 - y_1u_1 - y_2u_2)^{a'}} \, du_1 \, du_2,$$

valid for $c > b_1 + b_2$ and $b_1, b_2 > 0$. Making the change of variable $u_1 = u$, $u_2 = (1-u)v$, we obtain

$$FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = \Gamma\left(\begin{matrix} c \\ b_1, b_2, c-b_1-b_2 \end{matrix}\right)$$
$$\times \int_0^1 \int_0^1 \frac{u^{b_1-1} v^{b_2-1} (1-u)^{c-b_1-1} (1-v)^{c-b_1-b_2-1}}{(1-x_1 u - x_2(1-u)v)^{a_1} (1-y_1 u - y_2(1-u)v)^{a_2}} \, dv \, du.$$

Integrating with respect to v and using integral representation [1, Sect. 5.8 (5)]

$$F_1\left(\begin{matrix} a; b, b' \\ c \end{matrix}\middle| x, y\right) = \Gamma\left(\begin{matrix} c \\ a, c-a \end{matrix}\right) \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tx)^b (1-ty)^{b'}} \, dt, \qquad c > a > 0, \tag{10}$$

for the Appell F_1 function, we obtain

$$FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = \Gamma\left(\begin{matrix} c \\ b_1, c-b_1 \end{matrix}\right)$$
$$\times \int_0^1 \frac{u^{b_1-1}(1-u)^{c-b_1-1}}{(1-x_1 u)^{a_1} (1-y_1 u)^{a_2}} F_1\left(\begin{matrix} b_2; a_1, a_2 \\ c-b_1 \end{matrix}\middle| \frac{x_2(1-u)}{1-ux_1}, \frac{y_2(1-u)}{1-uy_1}\right) du.$$

This integral converges for $c > b_1 > 0$. We can therefore set $c := b_1 + b_2$ and using the obvious identity

$$F_1\left(\begin{matrix} a; b_1, b_2 \\ a \end{matrix}\middle| x, y\right) = (1-x)^{-b_1}(1-y)^{-b_2}, \tag{11}$$

we get (employing (10) one more time)

$$FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ b_1+b_2 \end{matrix}\middle| x_1, x_2, y_1, y_2\right) =$$
$$\Gamma\left(\begin{matrix} c \\ b_1, c-b_1 \end{matrix}\right) \int_0^1 \frac{u^{b_1-1}(1-u)^{c-b_1-1}}{(1-x_2-u(x_1-x_2))^{a_1} (1-y_2-u(y_1-y_2))^{a_2}} \, du$$
$$= (1-x_2)^{-a_1}(1-y_2)^{-a_2} F_1\left(\begin{matrix} b_1; a_1, a_2 \\ b_1+b_2 \end{matrix}\middle| \frac{x_1-x_2}{1-x_2}, \frac{y_1-y_2}{1-y_2}\right),$$

as claimed. $\square$

Corollary 1 *For $c = b_1 + b_2 = a_1 + a_2$ we have*

$$FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right)$$
$$= (1-x_2)^{b_1-a_1}(1-x_1)^{-b_1}(1-y_2)^{-a_2} {}_2F_1\left(\begin{matrix} a_2, b_1 \\ c \end{matrix}\middle| 1 - \frac{(1-x_2)(1-y_1)}{(1-x_1)(1-y_2)}\right).$$

Proof Straightforward from the well known identity

$$F_1\left(\begin{matrix}a; b_1, b_2\\ b_1+b_2\end{matrix}\middle| x, y\right) = (1-x)^{-a} {}_2F_1\left(\begin{matrix}a, b_2\\ b_1+b_2\end{matrix}\middle| \frac{x-y}{x-1}\right)$$

(see [1, Sect. 5.10 (1)]). □

Corollary 2

$$FD_1\left(\begin{matrix}\alpha,\alpha,\alpha,\alpha\\ 2\alpha\end{matrix}\middle| x_1, x_2, y_1, y_2\right) = (1-x_1)^{-\alpha}(1-y_2)^{-\alpha} {}_2F_1\left(\begin{matrix}\alpha,\alpha\\ 2\alpha\end{matrix}\middle| 1 - \frac{(1-x_2)(1-y_1)}{(1-x_1)(1-y_2)}\right).$$

Proof Take $a_1 = a_2 = b_1 = b_2 = \alpha$, $c = 2\alpha$ in Corollary 1. □

Here is our nicer formula for the M-harmonic Szegö kernel (upon evaluating at $t = 1$).

Theorem 1 *For $t \in [0, 1]$,*

$$\begin{aligned} K_{Sz}(z\sqrt{t}, w\sqrt{t}) = \frac{\Gamma(n)^2}{\Gamma(2n)2\pi^n}(1-t|z|^2)^n(1-t|w|^2)^n(n+t\partial_t)_n \\ (1-t|z|^2)^{-n}(1-t|w|^2)^{-n} {}_2F_1\left(\begin{matrix}n,n\\ 2n\end{matrix}\middle| 1 - \frac{|1-t\langle z,w\rangle|^2}{(1-t|z|^2)(1-t|w|^2)}\right). \end{aligned} \tag{12}$$

Proof By (6), the left-hand side equals

$$\frac{\Gamma(n)}{2\pi^n}(1-t|z|^2)^n(1-t|w|^2)^n FD_1\left(\begin{matrix}n,n,n,n\\ n\end{matrix}\middle| t|z|^2, t\langle z,w\rangle, t\langle w,z\rangle, t|w|^2\right).$$

The claim thus follows by combining Lemma 1 (with $a_1 = a_2 = b_1 = b_2 = n$) and Corollary 2 (with $\alpha = n$, $x_1 = |z|^2$, $x_2 = \langle z, w\rangle$, $y_1 = \langle w, z\rangle$, $y_2 = |w|^2$). □

Since for all integers $n \geq 1$,

$${}_2F_1\left(\begin{matrix}n,n\\ 2n\end{matrix}\middle| x\right) = -\frac{(2n-1)!}{(n-1)!^4}\partial_x^{n-1}(1-x)^{n-1}\partial_x^{n-1}\frac{\ln(1-x)}{x},$$

it is immediate, in particular, that $K_{Sz}(z, w)$ is an elementary function.

We also remark that, accidentally,

$$(n+t\partial_t)_n = t^{1-n}\partial_t^n t^{2n-1},$$

which however does not seem to lead to any simplifications in (12).

Likewise, one can rewrite (12) using the relation

$$\begin{aligned} &(1-t|z|^2)^n(1-t|w|^2)^n(n+t\partial_t)_n(1-t|z|^2)^{-n}(1-t|w|^2)^{-n} \\ &\qquad = \left(n\frac{1-t^2|z|^2|w|^2}{(1-t|z|^2)(1-t|w|^2)} + t\partial_t\right)_n, \end{aligned}$$

which again seems not to yield any significant advantage.

What is nice about (12) is, of course, that unlike (8) there is now manifest symmetry in the variables z, w.

3 Boundary Singularity

Let $\mathcal{F}_j$ denote the class of all functions $f(z)$ holomorphic in the cut complex plane $|\arg z| < \pi$ such that as $z \to 0$,

$$f(z) = a(z) + z^j b(z) \log z \qquad \text{with some } a, b \text{ holomorphic near } z = 0.$$

Note that by the formula for the analytic continuation of the ${}_2F_1$ function [1, Sect. 2.10 (12)],

$$ {}_2F_1\left(\begin{matrix} n, n \\ 2n \end{matrix} \middle| 1 - z\right) \in \mathcal{F}_0 \tag{13}$$

for any $n \in \mathbf{N}$, $n \geq 1$.

Theorem 2 *There exist $G_j \in \mathcal{F}_j$ and polynomials a_j on $\mathbf{C}^4$ of degree at most $4n$ such that*

$$FD_1\left(\begin{matrix} n, n, n, n \\ n \end{matrix} \middle| x_1, x_2, y_1, y_2\right) =$$
$$(1 - x_2)^{-2n}(1 - y_1)^{-2n} \sum_{j=0}^{n} a_j(x_1, x_2, y_1, y_2) G_j\left(\frac{(1 - x_1)(1 - y_2)}{(1 - x_2)(1 - y_1)}\right)$$

for all x_1, x_2, y_1, y_2 in the unit disc.

Feeding the last expression into (6), the theorem gives a fairly simple and explicit description of the boundary singularity of K_{Sz}:

$$K_{\mathrm{Sz}}(z, w) = \frac{\Gamma(n)}{2\pi^n} \frac{(1 - x_1)^n (1 - y_2)^n}{(1 - x_2)^{2n}(1 - y_1)^{2n}} \times$$
$$\sum_{j=0}^{n} a_j(x_1, x_2, y_1, y_2) G_j\left(\frac{(1 - x_1)(1 - y_2)}{(1 - x_2)(1 - y_1)}\right),$$

with $x_1 = |z|^2$, $x_2 = \langle z, w\rangle$, $y_1 = \langle w, z\rangle$, and $y_2 = |w|^2$, and a_j and G_j as in the theorem.

Proof Let again x_1, x_2, y_1, y_2 lie in the unit disc and t lie in the interval $[0, 1]$. As we saw in the last proof, by Lemma 1 and Corollary 2,

$$(n)_n FD_1\left(\begin{matrix} n, n, n, n \\ n \end{matrix} \middle| tx_1, tx_2, ty_1, ty_2\right) =$$

$$(n + t\partial_t)_n (1 - tx_1)^{-n} (1 - ty_2)^{-n} {}_2F_1\left(\begin{matrix} n, n \\ 2n \end{matrix} \middle| 1 - \frac{(1 - tx_2)(1 - ty_1)}{(1 - tx_1)(1 - ty_2)}\right).$$

Applying the Pfaff transform

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| y\right) = (1 - y)^{-a} {}_2F_1\left(\begin{matrix} a, c - b \\ c \end{matrix} \middle| \frac{y}{y - 1}\right),$$

the right-hand side becomes

$$(n + t\partial_t)_n (1 - tx_2)^{-n} (1 - ty_1)^{-n} {}_2F_1\left(\begin{matrix} n, n \\ 2n \end{matrix} \middle| 1 - \frac{(1 - tx_1)(1 - ty_2)}{(1 - tx_2)(1 - ty_1)}\right). \tag{14}$$

Let us temporarily denote

$$Z(t) := \frac{(1 - tx_1)(1 - ty_2)}{(1 - tx_2)(1 - ty_1)}, \qquad F(z) := {}_2F_1\left(\begin{matrix} n, n \\ 2n \end{matrix} \middle| 1 - z\right).$$

Let c denote a real constant, not necessarily the same one on each occurrence. Consider the expression

$$V_{pqrs,F} := (1 - tx_1)^{-p} (1 - tx_2)^{-q} (1 - ty_1)^{-r} (1 - ty_2)^{-s} F(Z(t)).$$

Note that

$$t\partial_t (1 - tz)^m = -\frac{mtz}{1 - tz} (1 - tz)^m = m\left(1 - \frac{1}{1 - tz}\right)(1 - tz)^m.$$

Similarly,

$$t\partial_t F(Z(t)) = t\frac{Z'(t)}{Z(t)} Z(t) F'(Z(t)) = V_{0100,TF} + V_{0010,TF} - V_{1000,TF} - V_{0001,TF},$$

where we have introduced the notation $TF(z) := zF'(z)$. Thus by the Leibniz rule

$$\begin{aligned}(c + t\partial_t) V_{pqrs,F} &= \\ cV_{pqrs,F} &+ cV_{p+1,q,r,s,F} + cV_{p,q+1,r,s,F} + cV_{p,q,r+1,s,F} + cV_{p,q,r,s+1,F} \\ &+ cV_{p+1,q,r,s,TF} + cV_{p,q+1,r,s,TF} + cV_{p,q,r+1,s,TF} + cV_{p,q,r,s+1,TF}.\end{aligned}$$

Iterating the last formula n times, we obtain

$$(n+t\partial_t)_n V_{0,n,n,0,F} = \sum_{\substack{m_1+m_2+m_3+m_4=m,\\ p+q+r+s+m\le n}} c\, V_{p+m_1,n+q+m_2,n+r+m_3,s+m_4,T^mF}.$$

Hence finally

$$\begin{aligned}(1-tx_1)^n(1-ty_2)^n FD_1\left(\begin{matrix}n,n,n,n\\ n\end{matrix}\middle| tx_1,tx_2,ty_1,ty_2\right)\\ = \sum_{\substack{m_1+m_2+m_3+m_4=m,\\ p+q+r+s+m\le n}} c\, V_{p+m_1-n,q+m_2+n,r+m_3+n,s+m_4-n,T^mF},\end{aligned}$$

i.e.

$$\begin{aligned}&(1-tx_1)^n(1-ty_2)^n FD_1\left(\begin{matrix}n,n,n,n\\ n\end{matrix}\middle| tx_1,tx_2,ty_1,ty_2\right)\\ &\quad = (1-tx_2)^{-2n}(1-ty_1)^{-2n}\sum_{m=0}^{n}\sum_{\substack{m_1+m_2+m_3+m_4=m,\\ p+q+r+s\le n-m}} c(1-tx_1)^{n-p-m_1}\\ &\qquad\times(1-tx_2)^{n-q-m_2}(1-ty_1)^{n-r-m_3}(1-ty_2)^{n-s-m_4}\,T^mF(Z(t))\\ &\qquad\qquad =: (1-tx_2)^{-2n}(1-ty_1)^{-2n}\sum_{m=0}^{n}\tilde a_m(tx_1,tx_2,ty_1,ty_2)\,T^mF(Z),\end{aligned} \tag{15}$$

for some polynomials $\tilde a_m$ of degree $\le 4n$ in the indicated variables. Observe now that $T^mF(z)$ comes as a sum of $cz^jF^{(j)}(z)$, $j\le m$, and a simple check also shows that $z^jF^{(j)}(z)\in\mathcal{F}_j$. Setting $t=1$ in (15), the theorem thus follows. □

Note that if z tends to a point on the boundary while w stays inside or tends to a different boundary point, then $Z(1)\equiv Z\to 0$ while $1/(1-x_2)$, $1/(1-y_1)$ stay bounded and smooth. Thus $K_{\mathrm{Sz}}=a+bZ^n\log Z$, with a,b smooth up to the boundary. Consequently, K_{Sz} is C^{n-1} up to the boundary away from the boundary diagonal, in full agreement with Proposition 7 in [9].

If, on the other hand, $z=w$, then $Z=1$ so K_{Sz} is a polynomial in $1/(1-|z|^2)$ of degree n. This is in full agreement with formula (99) in [9].

For z,w both approaching the same point on the boundary, Z can range over the entire interval $0<Z\le 1$, and it seems unclear whether (15) can be simplified or brought to a more tangible form.

4 Concluding Remarks

Using the standard formulas

$$\int_1^\infty e^{-tp} t^s \, dt = \begin{cases} \dfrac{\Gamma(s+1)}{p^{s+1}} + \mathcal{O}(p), & s \in \mathbf{C} \setminus \{-1, -2, \ldots\}, \\ \dfrac{(-1)^{k+1}}{k!} p^k (\log p + \mathcal{O}(p)), & s = -1-k, \ k \in \mathbf{N}, \end{cases} \tag{16}$$

valid for Re $p > 0$, where $\mathcal{O}(p)$ denotes a function of p which is smooth (in fact — holomorphic) in a neighborhood of the origin, the boundary singularity (3) of the holomorphic Szegö kernel S can also be rewritten as

$$S(x, y) \sim \int_0^\infty e^{-t\rho(x,y)} \, b(x, y, t) \, dt, \qquad x, y \in \Omega,$$

where b is a classical symbol in the Hörmander class $S^{n-1}(\overline{\Omega} \times \overline{\Omega} \times \mathbf{R}_+)$ with asymptotic expansion

$$b(x, y, t) \sim \sum_{j=0}^\infty t^{n-1-j} \, b_j(x, y) \qquad \text{for } t > 1,$$

with some functions $b_j \in C^\infty(\overline{\Omega} \times \overline{\Omega})$. In other words,

$$S(x, y) \approx \int_0^\infty \sum_{j=0}^\infty t^{n-1-j} \, e^{-t\rho(x,y)} \, b_j(x, y) \, dt \tag{17}$$

on $\overline{\Omega} \times \overline{\Omega}$. (Here the integrals need to be understood as "finite parts"; see [5] for the details.) For the harmonic Szegö kernel, the analogue of (3)—as mentioned in the Introduction—becomes

$$S_{\text{harm}}(x, y) \approx \int_0^\infty \sum_{j=0}^\infty t^{n-1-j} \, e^{-t|x-\tilde{y}|} \, b_j(x, |x-\tilde{y}|, \tfrac{x-\tilde{y}}{|x-\tilde{y}|}) \, dt \tag{18}$$

again on $\overline{\Omega} \times \overline{\Omega}$ (this time with Ω a bounded domain with smooth boundary in $\mathbf{R}^n$ rather than $\mathbf{C}^n$), now with $b_j \in C^\infty(\overline{\Omega} \times \overline{\mathbf{R}_+} \times \mathbf{S}^{n-1})$. In other words, for each fixed x, $\tilde{b}_j(x, x+w) := b_j(x, |w|, \frac{w}{|w|})$ comes as an asymptotic expansion of homogeneous distributions in w of higher and higher degree. The role of the "sesquiholomorphic extension" $\rho(x, y)$ of the defining function is thus played simply by the Euclidean distance function $|x - \tilde{y}|$.

From Theorem 2, using again (16), one can get an expansion akin to (17) and (18) also for our M-harmonic Szegö kernel K_{Sz} on $\mathbf{B}^n$:

$$K_{\mathrm{Sz}}(z,w) \approx |1-\langle z,w\rangle|^{-2n} \int_0^\infty \sum_{j=0}^\infty t^{-1-j}\, e^{-tZ(z,w)}\, a_j(z,w)\, dt, \tag{19}$$

with

$$Z(z,w) := \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \tag{20}$$

and $a_j(z,w)$ a polynomial in $z, \overline{z}, w, \overline{w}$ on $\mathbf{C}^n$ of degree at most $4n$. The marked difference is that now the coefficients of the asymptotic expansion are not smooth up to the closure $\overline{\mathbf{B}^n \times \mathbf{B}^n}$, due to the extra factor $|1-\langle z,w\rangle|^{-2n}$. Another difference is that the highest power of t in the integrand is independent of n. A drawback of (19), however, is that while the expansions (17) and (18) are "sharp" in the sense that the top order term b_0 is positive everywhere on $\partial\Omega \times \partial\Omega$ (in the holomorphic case, it is given by the Monge-Ampére determinant of $\rho(x)$, and a similar formula is available also in the harmonic case, see [8]), this is no longer clear for (19): the top order term there corresponds to $m=0$ and $Z=0$ in (15), hence is given by (after setting $t=1$)

$$(1-x_2)^{-2n}(1-y_1)^{-2n} \sum_{p+q+r+s\le n} c\,(1-x_1)^{n-p}(1-x_2)^{n-q}(1-y_1)^{n-r}(1-y_2)^{n-s}.$$

Now at least one of the exponents $n-q$ or $n-r$ is always positive, hence lowering the degree of singularity at $(x_1,x_2,y_1,y_2)=(1,1,1,1)$ by partially canceling the term in front of the sum. Or, put differently, the last sum is a polynomial of degree at least $3n$ in the variables $1-x_1, 1-x_2, 1-y_1, 1-y_2$, hence always vanishes at $(x_1,x_2,y_1,y_2)=(1,1,1,1)$. It is as yet unclear to the current authors how to remove this deficiency.

Here is an explicit formula for the "leading order" coefficient $a_0(z,w)$ in (19).

Proposition 1 *We have*

$$a_0(z,w) = \frac{1}{\Gamma(2n)2\pi^n}(1-|z|^2)^n(1-|w|^2)^n\, Q(\langle z,w\rangle, \langle w,z\rangle),$$

where Q is the polynomial

$$Q(x_2,y_1) := (1-x_2)^n(1-y_1)^n F_1\left(\begin{matrix}-n;\, n,\, n\\ n\end{matrix}\,\middle|\, \frac{x_2}{x_2-1}, \frac{y_1}{y_1-1}\right).$$

Proof By (16) for $s=-1$ and (12), a_0 equals $-\frac{\Gamma(n)^2}{\Gamma(2n)2\pi^n}$ times the coefficient at $Z^0 \log Z$ in the sum in (15). Since $T^m(z^k) = kz^k$ and $T^m(z^k\log z) = kz^k\log z + z^k$, this coefficient is nonzero only in the term $m=0$; hence, looking back at (14), it equals

$$(1-tx_1)^n(1-ty_2)^n(1-tx_2)^{2n}(1-ty_1)^{2n}(n+t\partial_t)_n(1-tx_2)^{-n}(1-ty_1)^{-n}\Big|_{t=1} \tag{21}$$

times the coefficient at $Z^0 \log Z$ in $F(Z)$, which according to [1, Sect. 2.10 (12)] is equal to $-1/\Gamma(n)^2$. To compute (21), revoke again (11) (taking $a = n$ there) and (9) to conclude that

$$\begin{aligned}(n+t\partial_t)_n(1-tx_2)^{-n}(1-ty_1)^{-n} &= (n+t\partial_t)_n F_1\binom{n;\, n, n}{n}\Big| tx_2, ty_1\Big) \\ &= F_1\binom{2n;\, n, n}{n}\Big| tx_2, ty_1\Big) \\ &= (1-tx_2)^{-n}(1-ty_1)^{-n} F_1\binom{-n;\, n, n}{n}\Big| \frac{tx_2}{tx_2-1}, \frac{ty_1}{ty_1-1}\Big),\end{aligned}$$

where on the last line we have used [1, Sect. 5.11 (1)]. Putting everything together and setting $t = 1$, we thus get

$$a_0(z, w) = \frac{1}{\Gamma(2n)2\pi^n}(1-x_1)^n(1-y_2)^n(1-x_2)^n(1-y_1)^n F_1\binom{-n;\, n, n}{n}\Big| \frac{x_2}{x_2-1}, \frac{y_1}{y_1-1}\Big)$$

with $x_1 = |z|^2$, $x_2 = \langle z, w\rangle$, $y_1 = \langle w, z\rangle$ and $y_2 = |w|^2$, and the assertion follows. □

Example For $n = 2$, the polynomial Q in the last proposition is given by

$$\begin{aligned}Q(x_2, y_1) = \frac{1}{3}\Big[&3(1-x_2)^2 + 4(1-x_2)(1-y_1) - 4(1-x_2)^2(1-y_1) \\ &+ 3(1-y_1)^2 - 4(1-x_2)(1-y_1)^2 + (1-x_2)^2(1-y_1)^2\Big].\end{aligned}$$

It is not clear how to express this in a simpler manner as $(x_2, y_1) \to (1, 1)$. □

Probably all one can say in general is that $\frac{a_j(z,w)}{(1-|z|^2)^n|1-\langle z,w\rangle|^{2n}(1-|w|^2)^n}$ is a polynomial of total degree $\le n$ in the variables $\frac{1}{1-|z|^2}$, $\frac{1}{1-\langle z,w\rangle}$, $\frac{1}{1-\langle w,z\rangle}$ and $\frac{1}{1-|w|^2}$.

We remark that, returning from the ball to a general domain $\Omega \subset \mathbf{C}^n$, one is tempted to speculate that the general analogue of (19) may be

$$K_{\mathrm{Sz}}(z, w) \approx |\rho(z, w)|^{-2n} \int_0^\infty \sum_{j=0}^\infty t^{-1-j}\, e^{-tZ(z,w)}\, a_j(z, w)\, dt,$$

with

$$Z(z, w) := \frac{\rho(z, z)\rho(w, w)}{|\rho(z, w)|^2}$$

and $a_j \in C^\infty(\overline{\Omega} \times \overline{\Omega})$. Note that $Z(z, w) = e^{D(z,w)}$, where

$$D(z, w) := \log \rho(z, z) + \log \rho(w, w) - \log \rho(z, w) - \log \rho(w, z)$$

resembles the famous *Calabi diastasis function*, which was introduced in [7] in connection with isometric imbeddings of complex manifolds and which also plays a prominent role e.g. in some quantization procedures on Kähler manifolds; however, these applications do not involve its behavior near the boundary diagonal.

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Some Aspects of the $p-$Bergman Theory

Bo-Yong Chen, Yuanpu Xiong, and Liyou Zhang

Abstract This survey presents various results on the $p-$Bergman theory: basic properties, regularity of the $p-$Bergman kernel, geometric properties of the $p-$Bergman metric, analysis of the $p-$Bergman space.

Keywords $p-$Bergman kernel · $p-$Bergman metric · $p-$Bergman space

1 Introduction and Preliminaries

In contrast with the classical Bergman theory, the L^p-analogue is relatively new. Historically, the (on-diagonal) $p-$Bergman kernel (corresponding to pluricanonical sections over compact complex manifolds) first appeared in the work of Narasimhan-Simha [14] on Hausdorff property of the muduli space of compact complex manifolds with ample canonical bundle. A few years later, the $p-$Bergman kernel was used by Sakai [16] to study Kodaira dimensions of complements of divisors in a compact complex manifold. Yet it is through the work of Siu [18, 19] and Tsuji [22, 22, 23] on the geometry of pluricanonical line bundles that the $p-$Bergman kernel became widely known, and it was studied by many authors (cf. [2, 8, 15, 20, 24]).

Recently, there arose some interests in developing a general $p-$Bergman theory (cf. [4, 6]). The goal of this survey is to present in a concise manner various recent results and open problems concerning the foundation of the $p-$Bergman theory. Most of the materials are borrowed from [4, 6] and we refer the reader to the original papers for detailed proofs. The lack of complete orthonormal bases causes real difficulties in generalizing the classical Bergman theory to the L^p case. In order to make the ideas

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as transparent as possible, we shall stick to the simplest case of bounded domains with trivial line bundle.

Given a bounded domain $\Omega \subset \mathbb{C}^n$, define the $p-$Bergman space to be

$$A^p(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \|f\|_p^p := \int_\Omega |f|^p < \infty \right\}.$$

The mean-value inequality implies

$$|f(z)|^p \leq \frac{1}{\pi^n \delta(z)^{2n}/n!} \|f\|_p^p, \quad f \in A^p(\Omega), \tag{1}$$

where δ denotes the Euclidean boundary distance. From (1) one concludes that $A^p(\Omega)$ is a Banach space for $p \geq 1$, while $A^p(\Omega)$ is only a complete metric space with respect to the metric $d(f_1, f_2) := \|f_1 - f_2\|_p^p$ for $0 < p < 1$. Moreover, $A^p(\Omega)$ is uniformly convex for $1 < p < \infty$ in view of the celebrated Clarkson inequality. The space $A^1(\Omega)$ is not uniformly convex; nevertheless, it is strictly convex. To see this, consider $f, g \in A^1(\Omega)$ with $\|f\|_1 = \|g\|_1 = 1$ and $\|(f+g)/2\|_1 = 1$. Since

$$\|f + g\|_1 = \|f\|_1 + \|g\|_1,$$

it follows that $f = \lambda g$ a.e. for suitable function $\lambda \geq 0$. This means that the meromorphic function f/g only takes real values a.e., which has to be a constant. Since $\|f\|_1 = \|g\|_1 = 1$, we conclude that $\lambda = 1$, a.e. on Ω.

The starting point of the $p-$Bergman theory is the following minimizing problem:

$$m_p(z) = m_{\Omega,p}(z) = \inf \left\{ \|f\|_p : f \in A^p(\Omega), f(z) = 1 \right\}. \tag{2}$$

A normal family argument yields that there exists at least one minimizer for $p > 0$. Moreover, as $A^p(\Omega)$ is strictly convex for $1 \leq p < \infty$, we see that the minimizer is unique in this case. The $p-$Bergman kernel is then given by

$$K_p(z) := m_p(z)^{-p} = \sup \left\{ |f(z)|^p : f \in A^p(\Omega), \|f\|_p = 1 \right\}, \quad 0 < p < \infty,$$

and the off-diagonal $p-$Bergman kernel is defined as

$$K_p(z, w) := m_p(z, w) K_p(w), \quad 1 \leq p < \infty.$$

Note that $m_p(z, z) = 1$ and $K_p(z, z) = K_p(z)$.

The following two basic properties follow easily from the extremal property.

Proposition 1 (Transformation rule) *Let $F : \Omega_1 \to \Omega_2$ be a biholomorphic mapping between bounded simply-connected domains. Let J_F denote the complex Jacobian of F. Then*

$$K_{\Omega_1,p}(z) = K_{\Omega_2,p}(F(z))|J_F(z)|^2, \quad 0 < p < \infty,$$
$$K_{\Omega_1,p}(z,w) = K_{\Omega_2,p}(F(z),F(w))J_F(z)^{\frac{2}{p}} J_F(w)^{1-\frac{2}{p}}\, \overline{J_F(w)}, \quad 1 \le p < \infty.$$

Moreover, equalities hold for arbitrary bounded domains when $2/p \in \mathbb{Z}^+$.

Remark 1 Note that the simply-connected hypothesis can not be removed (see [15, Remark 2.3]).

Proposition 2 (Product rule) *Let Ω' and Ω'' be bounded domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Set $\Omega = \Omega' \times \Omega''$ and $z = (z', z'')$. Then*

$$K_{\Omega,p}(z) = K_{\Omega',p}(z') \cdot K_{\Omega'',p}(z''), \quad 0 < p < \infty,$$
$$K_{\Omega,p}(z,w) = K_{\Omega',\,p}(z',w') \cdot K_{\Omega'',\,p}(z'',w''), \quad 1 \le p < \infty.$$

By Propositions 1 and 2, one may compute the $p-$Bergman kernels of the unit ball $\mathbb{B}^n$ and the unit polydisc Δ^n explicitly:

$$K_{\mathbb{B}^n,\,p}(z,w) = \frac{n!}{\pi^n}\,\frac{(1-|w|^2)^{(n+1)(\frac{2}{p}-1)}}{(1-\langle z,w\rangle)^{\frac{2(n+1)}{p}}},$$

$$K_{\Delta^n,\,p}(z,w) = \frac{1}{\pi^n}\prod_{j=1}^{n}\frac{(1-|w_j|^2)^{\frac{4}{p}-2}}{(1-\overline{w}_j z_j)^{\frac{4}{p}}}.$$

Here $\langle z,w\rangle := \sum_{j=1}^n z_j \bar{w}_j$.

However, the most fundamental property is the following reproducing formula:

Proposition 3 (cf. [6]) *For any $f \in A^p(\Omega)$ with $1 \le p < \infty$, one has*

$$f(z) = m_p(z)^{-p}\int_\Omega |m_p(\cdot,z)|^{p-2}\,\overline{m_p(\cdot,z)}\,f = \int_\Omega |m_p(\cdot,z)|^{p-2}\,\overline{K_p(\cdot,z)}\,f. \quad (3)$$

A proof is sketched as follows. Given $f \in A^p(\Omega)$ with $f(z) = 0$, consider the following holomorphic family

$$f_t = m_p(\cdot,z) + tf \in A^p(\Omega), \quad t \in \mathbb{C}.$$

Since $f_t(z) = 1$, so the function $J(t) := \|f_t\|_p^p$ attains the minimum at $t = 0$. Since $\frac{\partial|f_t|^p}{\partial t} = \frac{p}{2}|f_t|^{p-2}\bar{f}_t f$ holds outside the proper analytic set $f_t^{-1}(0) \subset \Omega$ (whose Lebesgue measure is zero), one has

$$\left|\frac{\partial|f_t|^p}{\partial t}\right| = \frac{p}{2}|f_t|^{p-1}|f| \le \frac{p}{2}|f|(|m_p(\cdot,z)| + |f|)^{p-1} =: \phi$$

whenever $|t| \leq 1$. Analogously, one may verify that $\left|\frac{\partial |f_t|^p}{\partial t}\right| \leq \phi$. Hölder's inequality implies $\phi \in L^1(\Omega)$, so that the dominated convergence theorem gives

$$0 = \frac{\partial J}{\partial t}(0) = \int_\Omega \frac{\partial |f_t|^p}{\partial t}\bigg|_{t=0} = \frac{p}{2}\int_\Omega |m_p(\cdot, z)|^{p-2}\,\overline{m_p(\cdot, z)}\, f. \qquad (4)$$

To get (3), it suffices to apply (4) with f replaced by $f - f(z)$.

The reproducing formula (**RF**) shows that the evaluation functional $T_{p,z} : f \mapsto f(z)$ is represented by

$$g_{p,z} = |m_p(\cdot, z)|^{p-2} K_p(\cdot, z) \in L^q(\Omega)$$

with $\|g_{p,z}\|_q = \|T_{p,z}\|$, where $\frac{1}{p} + \frac{1}{q} = 1$. **RF** also implies the following geometric interpretations: $m_p(z)$ is the distance between hyperplanes $H^c_{p,z}$ and $H^{c+1}_{p,z}$ in $A^p(\Omega)$, where

$$H^c_{p,z} = \{f \in A^p(\Omega) : f(z) = c\}, \quad c \in \mathbb{C};$$

$m_p(\cdot, z)$ is orthogonal in the sense of Birkhoff-James to $H^0_{p,z}$, i.e.,

$$\|m_p(\cdot, z)\|_p \leq \|m_p(\cdot, z) - f\|_p, \quad \forall f \in H^0_{p,z}.$$

Moreover, one may deduce from **RF** the following inequalities:

$$|K_p(\zeta, z)| \leq K_p(\zeta)^{\frac{1}{p}} K_p(z)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

$$\mathrm{Re}(K_p(\zeta, z) + K_p(z, \zeta)) \leq K_p(\zeta) + K_p(z).$$

In both inequalities, equality holds if and only if $\zeta = z$.

It is a little bit surprising that the $p-$Bergman kernel is related to certain weighted L^2 Bergman kernel when $1 \leq p < 2$. Given a weight φ, define the L^2 Bergman space with weight φ by

$$A^2(\Omega, \varphi) := \left\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 e^{-\varphi} < \infty \right\}.$$

Let $K_{\Omega,\varphi}$ denote the corresponding weighted L^2 Bergman kernel.

Proposition 4 (cf. [4]) *Let* $1 \leq p \leq 2$. *Then one has*

$$K_p(\cdot, z) = K_{2,p,z}(\cdot, z) := K_{\Omega,(2-p)\log|m_p(\cdot,z)|}(\cdot, z), \quad \forall z \in \Omega.$$

The idea of the proof is as follows. First of all, Hölder's inequality implies

$$A^2_{p,z}(\Omega) := A^2(\Omega, (2-p)\log|m_p(\cdot, z)|) \subset A^p(\Omega).$$

Apply **RF** for $A^p(\Omega)$ and $A^2_{p,z}(\Omega)$ respectively, one gets

$$f(z) = \int_\Omega |m_p(\cdot, z)|^{p-2}\overline{K_p(\cdot, z)}f$$
$$f(z) = \int_\Omega |m_p(\cdot, z)|^{p-2}\overline{K_{2,p,z}(\cdot, z)}f$$

for all $f \in A^2_{p,z}(\Omega)$. Thus

$$\int_\Omega |m_p(\cdot, z)|^{p-2}\overline{(K_p(\cdot, z) - K_{2,p,z}(\cdot, z))}f = 0, \quad \forall f \in A^2_{p,z}(\Omega). \tag{5}$$

Since

$$\int_\Omega |m_p(\cdot, z)|^{p-2}|K_p(\cdot, z)|^2 = K_p(z)^2 \int_\Omega |m_p(\cdot, z)|^p = K_p(z),$$

so $K_p(\cdot, z) \in A^2_{p,z}(\Omega)$. Substitute $f := K_p(\cdot, z) - K_{2,p,z}(\cdot, z)$ into (5), one immediately gets $f = 0$.

Proposition 4 shows that the powerful L^2 method of the $\bar{\partial}-$equation is still applicable in the $p-$Bergman theory. For instance, one may verify that $K_p(\cdot, z) \in L^q(\Omega)$ for any $z \in \Omega$ and $q < \frac{2pn}{2n-\alpha(\Omega)}$ provided $1 \le p \le 2$ and $\alpha(\Omega) > 0$, where $\alpha(\Omega)$ is the hyperconvexity index defined to be the supremum of those $\alpha \ge 0$ such that there exists a negative continuous psh function ρ on Ω with $-\rho \lesssim \delta^\alpha$. On the other hand, for each $2 < p < \infty$, there exists a bounded domain $\Omega \subset \mathbb{C}$ and a point $z \in \Omega$ such that $K_p(\cdot, z) \neq K_{2,p,z}(\cdot, z)$; there also exists a bounded domain $\Omega \subset \mathbb{C}$ with $\alpha(\Omega) > 0$ such that for each $p' > p$ there is a point $z \in \Omega$ with $K_p(\cdot, z) \notin L^{p'}(\Omega)$. We refer to [4] for more details.

Given a vector $X = \sum_j X_j \partial/\partial z_j$, define the p-Bergman metric to be

$$B_p(z; X) = B_{\Omega,p}(z; X) := K_p(z)^{-\frac{1}{p}} \cdot \sup_f |Xf(z)|, \quad 0 < p < \infty,$$

where the supremum is taken over all $f \in A^p(\Omega)$ with $f(z) = 0$ and $\|f\|_p = 1$. In a fashion analogous to Proposition 1, one can show that $B_p(z; X)$ is a biholomorphically invariant metric for bounded simply-connected domains, and the simply-connected hypothesis may be removed when $2/p \in \mathbb{Z}^+$. From this, one may compute the $p-$Bergman metric for the unit ball explicitly:

$$B_{\mathbb{B}^n,p}(z; X) = c_{n,p}\left(\frac{|X|^2}{1-|z|^2} + \frac{|\sum_{j=1}^n z_j X_j|^2}{(1-|z|^2)^2}\right)^{\frac{1}{2}}$$

where

$$c_{n,p} = (\pi^n/n!)^{\frac{1}{p}} \cdot \sup\left\{ \frac{|f(0)|}{\|z_1 f\|_p} : f \in A^p(\mathbb{B}^n) \right\}.$$

Recall that the Carathéodory metric is given by

$$C(z; X) = C_\Omega(z; X) := \sup_f |Xf(z)|$$

where the supremum is taken over all $f \in A^\infty(\Omega)$ with $f(z) = 0$ and $\|f\|_\infty = 1$.

Proposition 5 (cf. [6])

$$C(z; X) \le B_p(z; X) \to C(z; X) \quad (p \to \infty).$$

The proof of $C(z; X) \le B_p(z; X)$ is similar to that in the standard case $p = 2$, which goes back to Qi-Keng Lu [13]. The proof of $B_p(z; X) \to C(z; X)$ runs as follows. It suffices to verify

$$C(z; X) \ge \limsup_{p\to\infty} B_p(z; X). \tag{6}$$

To see this, first take a sequence $p_j \to \infty$ such that

$$\lim_{j\to\infty} B_{p_j}(z; X) = \limsup_{p\to\infty} B_p(z; X).$$

Choose $f_j \in A^{p_j}(\Omega)$ with $\|f_j\|_{p_j} = 1$, $f_j(z) = 0$ and $B_{p_j}(z; X) = |Xf_j(z)|/K_{p_j}(z)^{\frac{1}{p_j}}$ for every j. It is clear that $\{f_j\}$ forms a normal family, so that there is a subsequence $\{f_{j_k}\}$ converging locally uniformly to some $f_\infty \in \mathcal{O}(\Omega)$ which satisfies $f_\infty(z) = 0$ and $|f_\infty(\zeta)| = \lim_{k\to\infty} |f_{j_k}(\zeta)| \le 1$ for any $\zeta \in \Omega$ in view of (1). Since $K_p(z) \ge 1/|\Omega|$, it follows from (1) that $\lim_{p\to\infty} K_p(z)^{\frac{1}{p}} = 1$. Thus

$$\begin{aligned} C(z; X) \ge |Xf_\infty(z)| &= \lim_{k\to\infty} |Xf_{j_k}(z)| \cdot \lim_{k\to\infty} K_{p_{j_k}}(z)^{-\frac{1}{p_{j_k}}} \\ &= \lim_{k\to\infty} B_{p_{j_k}}(z; X) = \limsup_{p\to\infty} B_p(z; X). \end{aligned}$$

Question 1 Is it possible to give quantitative estimates of $|B_p(z; X) - C(z; X)|$?

The $p-$Bergman projection will be defined via the metric projection. For a metric space (X, d) and $Y \subset X$, a point $y_0 \in Y$ is said to be the best approximation of $x \in X$ if $d(x, y_0) = \inf_{y\in Y} d(x, y)$. Let $\mathcal{P}_Y(x)$ denote the set of all best approximations of x. If $\mathcal{P}_Y(x) \ne \emptyset$ (respectively $\mathcal{P}_Y(x)$ is a singleton) for all $x \in X$, then we say that Y is proximal (respectively Chebyshev). Note that $A^p(\Omega)$ is always proximal in $L^p(\Omega)$ for $0 < p \le \infty$. Since $L^p(\Omega)$ is uniformly convex for $1 < p < \infty$, it follows that $A^p(\Omega)$ is Chebyshev. One may define the $p-$Bergman projection $P_p(f)$ of $f \in L^p(\Omega)$ to be the (unique) best approximation of f in $A^p(\Omega)$ for $1 < p < \infty$.

Proposition 6 (cf. [6]) *The $p-$Bergman projection enjoys the following properties:*

1. *If $f \in L^p(\Omega)$ is a function such that $\bar\partial f$ is well-defined, then $Q_p(f) := f - P_p(f)$ is the L^p-minimal solution of $\bar\partial u = \bar\partial f$.*
2. *$L^p(\Omega)$ admits a decomposition $L^p(\Omega) = A^p(\Omega) \oplus P_p^{-1}(\Omega)$ in the sense that $f = P_p(f) + Q_p(f)$.*
3. *$P_p^{-1}(0)$ is $\mathbb{C}-$starlike in the following sense:*

$$f \in P_p^{-1}(0) \Rightarrow cf \in P_p^{-1}(0), \quad \forall c \in \mathbb{C}.$$

4. *P_p is a linear operator if and only if $P_p^{-1}(0)$ is a linear subspace of $L^p(\Omega)$.*
5. *There exists a constant $C_p > 0$ such that for all $f, g \in L^p(\Omega)$,*

$$\|P_p(f) - P_p(g)\|_p \le \begin{cases} C_p(\|f\|_p + \|f-g\|_p)^{1-1/p}\|f-g\|_p^{1/p}, & 2 < p < \infty; \\ C_p(\|f\|_p + \|f-g\|_p)^{1/2}\|f-g\|_p^{1/2}, & 1 < p \le 2. \end{cases}$$

6. *For the unit disc, P_p is nonlinear if $p \neq 2$.*

Question 2 What is the relationship between the $p-$Bergman kernel and the $p-$Bergman projection?

2 Regularity of the $p-$Bergman Kernel

Regularity in a minimizing problem is classical and goes back to the famous problem-list of Hilbert. Since the minimizer $m_p(\zeta, z)$ in (2) is already holomorphic in ζ, so we are interested in investigating the following properties:

1. Regularity of $m_p(z)$ and $K_p(z)$ in z for $0 < p < \infty$.
2. Regularity of $m_p(\zeta, z)$ and $K_p(\zeta, z)$ in z for $1 \le p < \infty$.
3. Regularity of m_p and K_p in p.

An elementary normal family argument immediately yields

(i) $m_p(z)$ and $K_p(z)$ are locally Lipschitz continuous.
(ii) $m_p(\zeta, z)$ and $K_p(\zeta, z)$ are continuous on $\Omega \times \Omega$.

A higher-order regularity result for off-diagonal cases is given as follows.

Theorem 1 (cf. [6]) *Let $U \subset\subset \Omega$ be any given open set. The the following properties hold:*

1. *For each $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$|m_p(z, w) - m_p(z, w')| \le C|w - w'|^{\frac{1}{2}}, \quad \forall z, w, w' \in U.$$

2. *There exists a constant* $C > 0$ *such that*

$$|m_1(z,w) - m_1(z,w')| \le C|w - w'|^{\frac{1}{2(n+2)}}, \quad \forall\, z, w, w' \in U.$$

Moreover, for every open set U *with* $w \in U \subset\subset \Omega\backslash S_w$*, where* $S_w := \{m_1(\cdot, w) = 0\}$*, there exists a constant* $C > 0$ *such that*

$$|m_1(z,w) - m_1(z,w')| \le C|w - w'|^{\frac{1}{2}}, \quad \forall\, z, w' \in U.$$

The same conclusions also hold for K_p.[1]

Why does the exponent $\frac{1}{2}$ appear? Even for $p = 2$, it seems not easy to verify that $K_2(\zeta, \cdot)$ is Lipschitz continuous without using Hermitian symmetry. A natural argument runs as follows

$$\begin{aligned}
&\int_\Omega |K_2(\cdot, z) - K_2(\cdot, z')|^2 = \int_\Omega |K_2(\cdot, z)|^2 + \int_\Omega |K_2(\cdot, z')|^2 \\
&- \int_\Omega K_2(\cdot, z)\overline{K_2(\cdot, z')} - \int_\Omega K_2(\cdot, z')\overline{K_2(\cdot, z)} \\
&= [K_2(z) - K_2(z', z)] + [K_2(z') - K_2(z, z')] \ \ (\text{by } \mathbf{RF}) \\
&\lesssim |z - z'|.
\end{aligned}$$

By mean value inequality, one concludes that $K_2(\zeta, \cdot)$ is $\frac{1}{2}-$Hölder continuous.

The proof of Theorem 1 for the case $2 < p < \infty$ is essentially similar, but more technical: Due to the nonlinear factor $|m_p(\cdot, z)|^{p-2}$ in **RF**, one needs to borrow some nonlinear techniques from the theory of the $p-$Laplacian (cf. [12]). Even so, one only gets a weaker conclusion that $m_p(\zeta, \cdot)$ is locally $\frac{1}{p}-$Hölder continuous. In order to improve the order from $\frac{1}{p}$ into $\frac{1}{2}$, holomorphicity of $m_p(\cdot, z)$ is used with the aid of the semicontinuity theorem of Demailly-Kollár [7] for log canonical thresholds.

For on-diagonal cases one has the following

Theorem 2 (cf. [4]) *For each* $1 < p < \infty$, $K_p(z) \in C^{1,\frac{1}{2}}(\Omega)$ [2]*; moreover,*

$$\frac{\partial K_p}{\partial x_j}(z) = p\mathrm{Re}\left.\frac{\partial K_p(\cdot, z)}{\partial x_j}\right|_z, \quad 1 \le j \le 2n,$$

where $z_j = x_j + i x_{n+j}$ *for* $1 \le j \le n$.

The proof is based on the duality method from functional analysis. Consider bounded linear functionals:

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[1] Actually, the order $1/2$ of Hölder continuity can be improved substantially to 1 for $p > 1$ (cf. [5]).

[2] The regularity can be improved to $K_p(z) \in C^{1,1}_{\mathrm{loc}}(\Omega)$ for $p \ge 1$ (cf. [5]).

$$T_{p,z} : f \in A^p(\Omega) \mapsto f(z), \quad T^j_{p,z} : f \in A^p(\Omega) \mapsto \frac{\partial f}{\partial x_j}(z).$$

The Banach-Steinhaus theorem gives

$$\|T_{p,z+te_j} - T_{p,z} - tT^j_{p,z}\| = O(|t|^2) \ \ (t \to 0). \tag{7}$$

Recall that $g_{p,z} := |m_p(\cdot, z)|^{p-2}K_p(\cdot, z)$ represents $T_{p,z}$ with $\|g_{p,z}\|_q = \|T_{p,z}\|$. It follows from the Hahn-Banach theorem and the Riesz representation theorem that there exists $g^j_{p,z} \in L^q(\Omega)$ represents $T^j_{p,z}$ with $\|g^j_{p,z}\|_q = \|T^j_{p,z}\|$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, the following inequalities hold:

$$\|T_{p,z} + tT^j_{p,z}\| \le \|g_{p,z} + tg^j_{p,z}\|_q \tag{8}$$

$$\|T_{p,z+te_j} - tT^j_{p,z}\| \le \|g_{p,z+te_j} - tg^j_{p,z}\|_q. \tag{9}$$

The crucial point is $K_p(z) = \|T_{p,z}\|^p$, so that (7) $\sim$ (9) give

$$K_p(z+te_j)^{\frac{q}{p}} - K_p(z)^{\frac{q}{p}} \le qt\mathrm{Re}\int_\Omega |g_{p,z}|^{q-2}\overline{g_{p,z}}g^j_{p,z} + qt\omega_1(t) + O(|t|^2)$$
$$K_p(z+te_j)^{\frac{q}{p}} - K_p(z)^{\frac{q}{p}} \ge qt\mathrm{Re}\int_\Omega |g_{p,z}|^{q-2}\overline{g_{p,z}}g^j_{p,z} + qt\omega_2(t) + O(|t|^2)$$

for suitable $\omega_i(t) \to 0$, $i = 1, 2$. Thus $K_p(z)^{q/p} \in C^1(\Omega)$ and

$$\frac{\partial K_p^{q/p}}{\partial x_j}(z) = q\mathrm{Re}\int_\Omega |g_{p,z}|^{q-2}\overline{g_{p,z}}g^j_{p,z}$$

$$\Rightarrow \quad \frac{\partial K_p}{\partial x_j}(z) = p\mathrm{Re}\left.\frac{\partial K_p(\cdot, z)}{\partial x_j}\right|_z \quad (\text{by } \mathbf{RF} \text{ of } g^j_{p,z}).$$

It remains to be verified that

$$z \mapsto \partial K_p(\cdot, z)/\partial x_j|_z \in C^{1/2}(\Omega). \tag{10}$$

Given two points $z, z' \in \Omega' \subset\subset \Omega$, the function

$$\widehat{h} := \frac{K_p(\cdot, z) - K_p(\cdot, z')}{|z - z'|^{1/2}}$$

is holomorphic on Ω and satisfies $\sup_{\Omega'} |\widehat{h}| \le C$ in view of Theorem 1. It follows from Cauchy's estimates that for each $z' \in \Omega'' \subset\subset \Omega'$,

$$\left|\frac{\partial \widehat{h}}{\partial x_j}(z')\right| \le MC,$$

where the constant M depends only on Ω', Ω'' and Ω. On the other hand, since $\partial K_p(\cdot, z)/\partial x_j$ is a holomorphic function on Ω, so

$$\left|\frac{\partial K_p(\cdot, z)}{\partial x_j}\bigg|_z - \frac{\partial K_p(\cdot, z)}{\partial x_j}\bigg|_{z'}\right| \le C'|z - z'|,$$

where $C' = C'(\Omega'', \Omega) > 0$. Hence

$$\begin{aligned}\left|\frac{\partial K_p(\cdot, z)}{\partial x_j}\bigg|_z - \frac{\partial K_p(\cdot, z')}{\partial x_j}\bigg|_{z'}\right| &\le \left|\frac{\partial K_p(\cdot, z)}{\partial x_j}\bigg|_z - \frac{\partial K_p(\cdot, z)}{\partial x_j}\bigg|_{z'}\right| + \left|\frac{\partial \widehat{h}}{\partial x_j}(z')\right| |z - z'|^{1/2} \\ &\le C'|z - z'| + MC|z - z'|^{1/2} \lesssim |z - z'|^{1/2},\end{aligned}$$

i.e., (10) holds.

In sharp contrast with the standard case $p = 2$, there are bounded domains on which $K_p(z)$ is not a **real-analytic** function for some $p > 2$. To be precise, one sets

$$\begin{aligned}&\mathcal{F}(\Omega) := \{z \in \Omega : K_2(\cdot, z) \text{ is zero-free}\}, \quad \mathcal{N}(\Omega) := \Omega \backslash \mathcal{F}(\Omega), \\ &\mathcal{A}_p(\Omega) := \left\{z \in \Omega : K_p(\cdot) \text{ is not real-analytic at } z\right\}.\end{aligned}$$

The Hurwitz theorem implies that $\mathcal{F}(\Omega)$ is closed in Ω.

Theorem 3 (cf. [6]) *If $\Omega \subset \mathbb{C}^n$ is a bounded simply-connected domain such that both $\mathcal{N}(\Omega)$ and $\mathcal{F}(\Omega)^\circ$ are non-empty, then*

$$\partial \mathcal{F}(\Omega)^\circ \cap \Omega \subset \mathcal{A}_{2k}(\Omega)$$

holds for any integer $k > 2$.

It is a famous problem due to Lu Qi-Keng [13] on whether $K_2(\cdot, \cdot)$ is zero-free. Many domains on which $K_2(\cdot, \cdot)$ is not zero-free have been found (cf. [3]). Most of these domains also satisfy the hypothesis of Theorem 3, e.g., the convex domain define by

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| + |z_2| + |z_3| < 1\}.$$

The idea of the proof of Theorem 3 is as follows. The first step is to verify the following inequalities for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$:

$$\begin{aligned}m_r(z) &\le m_p(z) \cdot m_q(z), \\ K_r(z)^{\frac{1}{r}} &\ge K_p(z)^{\frac{1}{p}} \cdot K_q(z)^{\frac{1}{q}}.\end{aligned}$$

To see this, take holomorphic functions f_p and f_q on Ω with $f_p(z) = f_q(z) = 1$, $\|f_p\|_p = m_p(z)$ and $\|f_q\|_q = m_q(z)$. Clearly, $f_r := f_p f_q$ is a holomorphic function on Ω satisfying $f_r(z) = 1$ and Hölder's inequality gives

$$\|f\|_r \leq \|f_p\|_p \cdot \|f_q\|_q = m_p(z) \cdot m_q(z).$$

By definition of $m_r(z)$ one immediately gets the first inequality. The second inequality follows directly from the first.

The second step is to verify that for $p \geq 1$ and $k \in \mathbb{Z}^+$, $K_p(z) = K_{pk}(z)$ if and only if $m_p(\cdot, z) = m_{pk}(\cdot, z)^k$. For the only if part, it suffices to note that $f_k := m_{pk}(\cdot, z)^k$ is a holomorphic function satisfying $f_k(z) = 1$ and

$$\int_\Omega |f_k|^p = \int_\Omega |m_{pk}(\cdot, z)|^{pk} = m_{pk}(z)^{pk} = m_p(z)^p,$$

so that $f_k = m_p(\cdot, z)$ in view of uniqueness of the minimizer. The if part follows from

$$m_p(z)^p = \int_\Omega |m_p(\cdot, z)|^p = \int_\Omega |m_{pk}(\cdot, z)|^{pk} = m_{pk}(z)^{pk}.$$

The third step is to show that if Ω is a bounded simply-connected domain in $\mathbb{C}^n$ such that $m_p(\cdot, z)$ is zero-free for some $p \geq 1$ and $z \in \Omega$, then $K_s(z) = K_p(z)$ for any $s \geq p$. To see this, first note that $f_{p,s} := m_p(\cdot, z)^{\frac{p}{s}}$ defines a holomorphic function on Ω satisfying $f_{p,s}(z) = 1$. Since

$$\int_\Omega |f_{p,s}|^s = \int_\Omega |m_p(\cdot, z)|^p = m_p(z)^p,$$

it follows that

$$K_s(z) \geq \frac{|f_{p,s}(z)|^s}{\|f_{p,s}\|_s^s} = \frac{1}{m_p(z)^p} = K_p(z).$$

This combined with the first step yields that if $\frac{1}{s} + \frac{1}{t} = \frac{1}{p}$ then

$$K_p(z)^{\frac{1}{p}} \geq K_s(z)^{\frac{1}{s}} \cdot K_t(z)^{\frac{1}{t}} \geq K_s(z)^{\frac{1}{s}} \cdot K_p(z)^{\frac{1}{t}}$$

since $t \geq p$. Thus $K_p(z) \geq K_s(z)$.

The fourth step concludes the proof. Suppose on the contrary that there exists $w_0 \in (\partial\mathcal{F}(\Omega)^\circ \cap \Omega) \backslash \mathcal{A}_{2k}(\Omega)$. Then for some $\varepsilon > 0$,

$$B(w_0, \varepsilon) \cap \mathcal{F}(\Omega)^\circ \neq \emptyset, \quad B(w_0, \varepsilon) \cap \mathcal{N}(\Omega) \neq \emptyset,$$

and $K_{2k}(\cdot) \in C^\omega(B(w_0, \varepsilon))$. The third step combined with **unique continuation** of C^ω functions yields $K_{2k}(z) = K_2(z)$ for all $z \in B(w_0, \varepsilon)$. By the second step, we obtain $m_2(\cdot, z) = m_{2k}(\cdot, z)^k$, $\forall z \in B(w_0, \varepsilon)$. Take $z_0 \in B(w_0, \varepsilon)$ and $\zeta_0 \in \Omega$ such that $\mathrm{ord}_{z_0} K_2(\zeta_0, \cdot) = 1$. If $k > 2$, then

$$\mathrm{ord}_{z_0} m_2(\zeta_0, \cdot) = 1 < \mathrm{ord}_{z_0} m_{2k}(\zeta_0, \cdot)^k,$$

since $m_{2k}(\zeta_0, \cdot)$ is $\frac{1}{2}-$Hölder continuous in view of Theorem 1, a contradiction!

It is natural to propose the following

Question 3 Is it possible to conclude that $K_p(z) \in C^\infty(\Omega)$? If not, what is the optimal regularity of $K_p(z)$?

For parameter dependence of m_p and K_p, one has the following

Theorem 4 (cf. [4, 6])

1. $m_p(\zeta, z) = \lim_{s\to p-} m_s(\zeta, z)$ *for each* $1 < p < \infty$.
2. $\lim_{s\to p+} m_s(\zeta, z)$ *exists for each* $1 \le p < \infty$.
3. *If* $\alpha(\Omega) > 0$, *then* $|m_s(z) - m_p(z)| \lesssim |s-p||\log|s-p||$ *for each* $1 \le p \le 2$.
4. *For each* $2 < p < \infty$, *there exists a bounded domain* $\Omega \subset \mathbb{C}$ *with* $\alpha(\Omega) > 0$, *such that* $m_s(z)$ *is not continuous at* $s = p$ *for some* $z \in \Omega$.

The same conclusions also hold for K_p.

Parameter dependence of $K_p(z)$ for pluricanonical sections over compact complex manifolds was also suggested by Yau [24].

3 Geometric Properties of the $p-$Bergman Metric

Recall that the generalized complex Laplacian of an upper semicontinuous function u on a domain $\Omega \subset \mathbb{C}$ is given by

$$\Box u(z) := \liminf_{r\to 0+} \frac{1}{r^2}\left\{\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta - u(z)\right\}.$$

The generalized Levi form of an upper semicontinuous function u on a domain $\Omega \subset \mathbb{C}^n$ is defined to be

$$i\partial\bar{\partial}u(z; X) := \Box u(z+tX)|_{t=0}.$$

It is well-known that $i\partial\bar{\partial}\log K_2(z; X) = B_2^2(z; X)$. Thus a basic question is

Question 4 What is the relationship between $i\partial\bar{\partial}\log K_p(z; X)$ and $B_p^2(z; X)$?

A partial answer is given as follows.

Theorem 5 (cf. [6])

$$i\partial\bar{\partial}\log K_p(z;X) \ge B_p(z;X)^2 \quad \textit{for}\ \ 2<p<\infty, \tag{11}$$

$$i\partial\bar{\partial}\log K_p(z;X) \ge C(z;X)^2 \quad \textit{for}\ \ 0<p<2. \tag{12}$$

Actually, the next result shows that (11) can be improved. Define

$$\widetilde{B}_p(z;X) = \widetilde{B}_{\Omega,p}(z;X) := K_p(z)^{-\frac{1}{2}} \cdot \sup_f |Xf(z)|,$$

where the supremum is taken over all $f \in A^p(\Omega)$ with $f(z)=0$ and

$$\int_\Omega |m_p(\cdot,z)|^{p-2}|f|^2 = 1.$$

We claim that $\widetilde{B}_p(z;X) > B_p(z;X)$ for $2<p<\infty$ and $X \neq 0$. To see this, first take $f \in A^p(\Omega)$ with $f(z)=0$, $\|f\|_p = 1$ and $|Xf(z)| = K_p(z)^{\frac{1}{p}} B_p(z;X)$. Hölder's inequality gives

$$\int_\Omega |m_p(\cdot,z)|^{p-2}|f|^2 \le \left(\int_\Omega |m_p(\cdot,z)|^p\right)^{1-\frac{2}{p}} \left(\int_\Omega |f|^p\right)^{\frac{2}{p}} = m_p(z)^{p-2} = K_p(z)^{\frac{2}{p}-1},$$

and equality holds if and only if $|f| = a|m_p(\cdot,z)|$ for some positive constant a. This can not happen since $f(z)=0$, $m_p(z,z)=1$ and $f \not\equiv 0$. Thus

$$\widetilde{B}_p(z;X)^2 \ge \frac{1}{K_p(z)} \cdot \frac{|Xf(z)|^2}{\int_\Omega |m_p(\cdot,z)|^{p-2}|f|^2} > B_p(z;X)^2.$$

Theorem 6 $i\partial\bar{\partial}\log K_p(z;X) \ge \widetilde{B}_p(z;X)^2$ *for all* $2<p<\infty$.

The proof of Theorem 6, which is essentially implicit in [6], runs as follows. Fix r,θ for a moment. Take $f \in A^p(\Omega)$ with $f(z)=0$ and $\int_\Omega |m_p(\cdot,z)|^{p-2}|f|^2 = 1$. Consider the holomorphic family

$$f_t := m_p(\cdot,z) + tf, \quad t \in \mathbb{C},\ |t| = O(r).$$

One has

$$|f_t(z+re^{i\theta}X)|^p = |m_p(z+re^{i\theta}X,z)|^p \left(1 + p\mathrm{Re}\left\{tre^{i\theta}Xf(z)\right\} + O(r^3)\right)$$

since $m_p(z+re^{i\theta}X,z) = 1+O(r)$. Set $J(t) := \|f_t\|_p^p$. By the calculus of variations, one can show

$$\frac{\partial J}{\partial t}(0) = \frac{\partial J}{\partial \bar{t}}(0) = 0.$$

$$\frac{\partial^2 J}{\partial t^2}(0) = \int_\Omega \frac{\partial^2 |f_t|^p}{\partial t^2}(0) = \frac{p(p-2)}{4} \int_\Omega |m_p(\cdot, z)|^{p-4} \left(\overline{m_p(\cdot, z)}\, f\right)^2$$

$$\frac{\partial^2 J}{\partial t \partial \bar{t}}(0) = \int_\Omega \frac{\partial^2 |f_t|^p}{\partial t \partial \bar{t}}(0) = \frac{p^2}{4} \int_\Omega |m_p(\cdot, z)|^{p-2} |f|^2 = \frac{p^2}{4}.$$

Thus one has

$$\begin{aligned} J(t) &= J(0) + \mathrm{Re}\left\{ \frac{\partial^2 J}{\partial t^2}(0) t^2 \right\} + \frac{\partial^2 J}{\partial t \partial \bar{t}}(0)|t|^2 + o(|t|^2) \\ &= m_p(z)^p + \mathrm{Re}\left\{ \frac{\partial^2 J}{\partial t^2}(0) t^2 \right\} + \frac{p^2}{4}|t|^2 + o(|t|^2). \end{aligned}$$

Take $t = \varepsilon r e^{-i\theta} \overline{Xf(z)}$. Then one has

$$\begin{aligned} & K_p(z + re^{i\theta} X) \\ \geq & \frac{|m_p(z + re^{i\theta} X, z)|^p}{m_p(z)^p} \\ & \cdot \frac{1 + p\varepsilon r^2 |Xf(z)|^2 + O(r^3)}{1 + \mathrm{Re}\left\{ \frac{\partial^2 J}{\partial t^2}(0)(\varepsilon r e^{-i\theta}\overline{Xf(z)})^2 / m_p(z)^p \right\} + \frac{p^2}{4}\varepsilon^2 r^2 |Xf(z)|^2 / m_p(z)^p + o(r^2)}. \end{aligned}$$

Since

$$\int_0^{2\pi} \mathrm{Re}\left\{ \frac{\partial^2 J}{\partial t^2}(0)(\varepsilon r e^{-i\theta}\overline{Xf(z)})^2 / m_p(z)^p \right\} d\theta = 0,$$

it follows that

$$i\partial\bar{\partial} \log K_p(z; X) \geq p\varepsilon |Xf(z)|^2 - \frac{p^2}{4}\varepsilon^2 K_p(z) |Xf(z)|^2 = |Xf(z)|^2 / K_p(z)$$

provided $\varepsilon = \frac{2}{pK_p(z)}$. Take supremum over f, we conclude the proof.

Question 5 Is it possible to conclude that $i\partial\bar{\partial} \log K_p(z; X) = \widetilde{B}_p(z; X)^2$ for all $2 < p < \infty$?

By computation, one can show that the answer is affirmative for the unit disc. An immediate consequence of Theorem 5 is

Corollary 1 $\log K_p(z)$ *is a continuous strictly plurisubharmonic function on* Ω.

In particular, one sees that the minimal set of $K_p(z)$ defined by

$$\mathrm{Min}_p(\Omega) := \{z \in \Omega : K_p(z) = \inf_{\zeta \in \Omega} K_p(\zeta)\}$$

is either empty or a totally real subset of Ω. Moreover, $\mathrm{Min}_p(\Omega)$ is a nonempty compact subset of Ω if $K_p(z)$ is an exhaustion function. It seems interesting to investigate the metric/analytic structure of $\mathrm{Min}_p(\Omega)$.

Question 6 Is it possible to characterize those bounded domains Ω such that $\mathrm{Min}_p(\Omega)$ is a singleton?

Under the condition that $\min_{z\in\Omega} K_p(z) = 1/|\Omega|$ for some p, the following rigidity results hold:

1. Ω is the complement of a closed polar subset in a disc if $p = 2$ (cf. [9]).
2. Ω is a disc if $0 < p < 2$ (cf. [4]).
3. $\overline{\Omega}$ is a closed disc if $2 < p < \infty$ and $|\overline{\Omega}^{\circ} \setminus \Omega| = 0$ (cf. [4]).

It is expected that analogous results hold for high dimensional cases where discs are replaced by translations of complete circular domains. However, this is wrong. A counterexample is given in [4].

The holomorphic sectional curvature of $B_p(z; X)$ is given by

$$\mathrm{HSC}_p(z; X) := \sup_{\sigma} \left\{ \frac{\Box \log B_p^2(\sigma; \sigma')(0)}{-B_p^2(z; X)} \right\}$$

where $\sigma \in \mathcal{O}(\Delta_r, \Omega)$ with $\sigma(0) = z$ and $\sigma'(0) = X$.

Theorem 7 (cf. [4]) *For each* $2 \le p < \infty$*, one has*

$$\mathrm{HSC}_p(z; X) \le \frac{2}{p} \cdot \frac{i\partial\bar{\partial} \log K_p(z; X)}{B_p^2(z; X)} + \frac{p}{2}.$$

Remark 2 The case $p = 2$ reduces to the well-known fact that $\mathrm{HSC}_2(z; X) \le 2$.

4 Analysis of the $p-$Bergman Space

Recall that the $p-$Bergman projection can be defined through the metric projection for $1 < p < \infty$. The same idea does not work for $0 < p \le 1$.

Theorem 8 (cf. [4]) *For each* $0 < p \le 1$ *and bounded domain* $\Omega \subset \mathbb{C}^n$*,* $A^p(\Omega)$ *is always non-Chebyshev in* $L^p(\Omega)$*.*

For the proof of Theorem 8, the following concept is needed.

Definition 1 Let Ω be a domain in $\mathbb{C}^n$ and E a measurable set in Ω. For $0 < p < \infty$, the $p-$Schwarz content of E relative to Ω is defined by

$$s_p(E, \Omega) := \sup \left\{ \frac{\int_E |f|^p}{\int_\Omega |f|^p} : f \in A^p(\Omega) \setminus \{0\} \right\}.$$

Here are a few basic properties of the $p-$Schwarz content:

1. $s_p(E,\Omega)\le 1$ and $s_p(\Omega,\Omega)=1$;
2. $E_1\subset E_2$ implies $s_p(E_1,\Omega)\le s_p(E_2,\Omega)$, and $\Omega_1\subset\Omega_2$ implies $s_p(E,\Omega_1)\ge s_p(E,\Omega_2)$;
3. Subadditivity: $s_p(\bigcup_{j=1}^{\infty} E_j,\Omega)\le\sum_{j=1}^{\infty} s_p(E_j,\Omega)$;
4. $s_p(E,\Omega)\ge |E|/|\Omega|$;
5. $s_p(E,\Omega)\le C_n|E|/d^{2n}$ where $d=d(E,\partial\Omega)$;
6. For each compact set $E\subset\Omega$ with $|E|>0$, there exists $f_E\in A^p(\Omega)\setminus\{0\}$ such that
$$s_p(E,\Omega)=\frac{\int_E |f_E|^p}{\int_\Omega |f_E|^p}.$$
7. If $F:\Omega\to\mathbb{C}^n$ is a holomorphic injective mapping, then
$$s_p(E,\Omega)=s_p(F(E),F(\Omega))$$
when Ω is simply-connected or $2/p\in\mathbb{Z}^+$.
8. If $\Omega'\subset\Omega$ are two bounded domains in $\mathbb{C}^n$, then
$$\frac{K_{\Omega,p}(z)}{K_{\Omega',p}(z)}\le s_p(\Omega',\Omega),\quad \forall z\in\Omega'.$$

The idea of the proof of Theorem 8 is as follows. By standard normal family arguments, it is not difficult to verify that if $\{\Omega_t\}_{0<t\le\delta_0}$ is a continuous family of open subsets in Ω so that $|\partial\Omega_t|=0$, then $s_p(\Omega_t,\Omega)$ is continuous in t. One starts from some compact set $K\subset\Omega$ with $s_p(K,\Omega)\ge|K|/|\Omega|>1/2$, then take balls $B(z_k,\delta_0)$ in Ω, $1\le k\le N$, which cover K. Consider the continuous family $\Omega_t:=\bigcup_{k=1}^{N} B(z_k,t)$, $0<t\le\delta_0$. Since $s_p(\Omega_{\delta_0},\Omega)>1/2$ and $\lim_{t\to 0+}s_p(\Omega_t,\Omega)=0$, it follows from the intermediate value theorem that there exists $0<t_0<\delta_0$ such that $s_p(\Omega_{t_0},\Omega)=1/2$. Take $f_0\in A^p(\Omega)$ such that $s_p(\Omega_{t_0},\Omega)=\|f_0\|^p_{L^p(\Omega_{t_0})}/\|f_0\|^p_{L^p(\Omega)}$, i.e.,
$$\int_{\Omega_{t_0}}|f_0|^p=\int_{\Omega\setminus\Omega_{t_0}}|f_0|^p=:I.$$

Since
$$\frac{1}{2}=s_p(\Omega_{t_0},\Omega)\ge\frac{\int_{\Omega_{t_0}}|f|^p}{\int_\Omega|f|^p}=\frac{\int_{\Omega_{t_0}}|f|^p}{\int_{\Omega_{t_0}}|f|^p+\int_{\Omega\setminus\Omega_{t_0}}|f|^p},\quad\forall f\in A^p(\Omega),$$

so $\int_{\Omega_{t_0}}|f|^p\le\int_{\Omega\setminus\Omega_{t_0}}|f|^p$. Given $h_1\in A^p(\Omega)$, define $h_2:=h_1+f_0\in A^p(\Omega)$ and

$$g(z) := \begin{cases} h_1(z), & z \in \Omega_{t_0}, \\ h_2(z), & z \in \Omega \setminus \Omega_{t_0}. \end{cases}$$

Clearly, $g \in L^p(\Omega)$. Moreover, for any $h \in A^p(\Omega)$,

$$\begin{aligned} \int_\Omega |g-h|^p &= \int_{\Omega_{t_0}} |h_1 - h|^p + \int_{\Omega\setminus\Omega_{t_0}} |h_2 - h|^p \\ &\geq \int_{\Omega_{t_0}} |h_1 - h|^p + \int_{\Omega_{t_0}} |h_2 - h|^p \\ &\geq \int_{\Omega_{t_0}} |h_1 - h_2|^p \ = \ \int_{\Omega_{t_0}} |f_0|^p \ = \ I, \end{aligned}$$

$$\int_\Omega |g-h_1|^p = \int_{\Omega\setminus\Omega_{t_0}} |h_2 - h_1|^p = \int_{\Omega\setminus\Omega_{t_0}} |f_0|^p = I,$$

$$\int_\Omega |g-h_2|^p = \int_{\Omega_{t_0}} |h_1 - h_2|^p = \int_{\Omega_{t_0}} |f_0|^p = I.$$

Thus h_1, h_2 are both best approximations of g, while $h_1 \neq h_2$.

The $p-$Schwarz content has another application on quantitative estimates of the Banach-Mazur distance between $p-$Bergman spaces of different domains. Recall that the Banach-Mazur distance between two Banach spaces X, Y is defined to be

$$d_{\text{BM}}(X, Y) := \inf \left\{ \|T\| \cdot \|T^{-1}\| : T \in \mathcal{L}(X, Y) \right\}$$

where $\mathcal{L}(X, Y)$ denotes the set of continuous isomorphisms from X to Y. Set $d_{\text{BM}}(X, Y) = \infty$ if X is not isomorphic to Y.

Suppose $\Omega' \subset \Omega$ are bounded domains in $\mathbb{C}^n$ such that every $f \in A^p(\Omega')$ extends to an element in $A^p(\Omega)$. This is satisfied for instance, when $n \geq 2$ and $\Omega\setminus\Omega'$ is compact in Ω, in view of the Hartogs extension theorem. Banach's open mapping theorem implies that the identity mapping $I : A^p(\Omega) \to A^p(\Omega')$ is an isomorphism, so that $d_{\text{B}M}(A^p(\Omega'), A^p(\Omega)) < \infty$. Furthermore, one has

Proposition 7 (cf. [4]) *For* $1 \leq p < \infty$, *one has*

$$d_{\text{BM}}(A^p(\Omega'), A^p(\Omega)) \leq \left(1 - s_p(\Omega\setminus\Omega', \Omega)\right)^{-1/p}. \tag{13}$$

The identity mapping $I : A^p(\Omega) \to A^p(\Omega'),\ f \mapsto f|_{\Omega'}$, provides a continuous isomorphism, which satisfies $\|I\| \leq 1$ and

$$\|I^{-1}\| = \sup \left\{ \frac{\|f\|_{L^p(\Omega)}}{\|f\|_{L^p(\Omega')}} : f \in A^p(\Omega)\setminus\{0\} \right\}.$$

It follows from the definition of the Banach-Mazur distance that

$$d_{\mathrm{BM}}(A^p(\Omega'), A^p(\Omega)) \le \sup\left\{\frac{\|f\|_{L^p(\Omega)}}{\|f\|_{L^p(\Omega')}} : f \in A^p(\Omega)\backslash\{0\}\right\}.$$

Since

$$\int_\Omega |f|^p = \int_{\Omega'} |f|^p + \int_{\Omega\backslash\Omega'} |f|^p \le \int_{\Omega'} |f|^p + s_p(\Omega\backslash\Omega', \Omega)\int_\Omega |f|^p,$$

one immediately gets (13).

We are left to find upper bounds of the $p-$Schwarz content, which is of independent interest.

Proposition 8 (cf. [4]) *Let Ω be a bounded domain in $\mathbb{C}^n$ and E a measurable relatively compact subset in Ω. Set $d := d(E, \partial\Omega)$. Then the following properties hold:*

1. *For $0 < p < \infty$, one has $s_p(E, \Omega) \le \frac{128/\lambda_1(\Omega)}{128/\lambda_1(\Omega)+d^2}$, where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian on Ω.*
2. *For $1 < p < \infty$ and $n \ge 2$, one has $s_p(E, \Omega) \le \frac{C_{n,p}(\Omega)}{C_{n,p}(\Omega)+d^{p/\tilde{p}}}$, where $\tilde{p}$ stands for the largest integer smaller than p.*

The proof makes use of some classical integral inequalities, such as the Caccioppoli inequality, the Sobolev inequality, and L^p-boundedness of the Beurling-Ahlfors operator (cf. [1]).

It may happen that for a pair $\Omega' \subset \Omega$ of bounded domains, $A^p(\Omega)$ is isomorphic to $A^p(\Omega')$ but functions in $A^p(\Omega')$ do not necessary extend to elements in $A^p(\Omega)$. A simple example is the pair $(A^1(\Delta^*), A^1(\Delta))$ where Δ^* denotes the punctured unit disc, since

$$T : A^1(\Delta^*) \to A^1(\Delta), \quad f \mapsto zf$$

provides an isomorphism. It seems interesting to compute $d_{\mathrm{B}M}(A^1(\Delta^*), A^1(\Delta))$ explicitly.

From the definition of the Banach-Mazur distance, one knows that

$$d_{\mathrm{B}M}(A^p(\Omega'), A^p(\Omega)) = 1$$

whenever there exists a linear isometry between $A^p(\Omega')$ and $A^p(\Omega)$. This is satisfied, for instance, when Ω' is simply-connected and biholomorphic to Ω. On the other hand, one has

Theorem 9 (cf. [8]) *Let $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ be hyperconvex domains. Suppose that there exists $p > 0$, $p \ne 2, 4, 6, \cdots$, such that*

(1) *there exists a linear isometry $T : A^p(\Omega_1) \to A^p(\Omega_2)$,*
(2) *both $K_{\Omega_1,p}(z)$ and $K_{\Omega_2,p}(z)$ are exhaustion functions.*

Then $m = n$ and Ω' is biholomorphic to Ω. Moreover, the assumption of hyperconvexity can be removed when $n = 1$.

We refer the reader to the original paper for the proof. It is also known that $K_{\Omega,p}(z)$ is an exhaustion function provided one of the following condtions holds:

1. $\Omega \subset \mathbb{C}^n$ is a bounded pseudoconvex domain and $0 < p < 2$ (cf. [15]).
2. $\Omega \subset \mathbb{C}^n$ is a bounded hyperconvex domain and $p = 2$ (cf. [17]).
3. $\Omega \subset \mathbb{C}^n$ is a bounded pseudoconvex domain with C^2-boundary and $2 < p < 2 + \frac{2}{n}$ (cf. [6]).
4. $\Omega \subset \mathbb{C}^n$ is a simply-connected uniformly squeenzing domain or smooth strictly pseudoconvex domain and $2 < p < \infty$ (cf. [8]).

Question 7 Is it possible to conclude that $K_{\Omega,p}(z)$ is an exhaustion function if Ω is a bounded pseudoconvex domain with C^2-boundary and $2 < p < \infty$?

Note that complete orthonormal bases of $A^2(\Omega)$ play a fundamental role in the classical Bergman theory. Therefore, the following question is basic.

Question 8 Does $A^p(\Omega)$ admit a Schauder basis for each $1 \le p < \infty$?

Here a Schauder basis $\{x_j\}$ for a normed linear space X means that each $x \in X$ can written uniquely as $x = \sum_{j=1}^{\infty} c_j x_j$, where $c_j \in \mathbb{C}$ and the series converges in norm to x. It is not difficult to show that $\{z^j\}_{j=0}^{\infty}$ is a Schauder basis of $A^p(\Delta)$ for each $1 < p < \infty$ by using the argument in [11].

A possible way of constructing Schauder bases in $A^p(\Omega)$ for general bounded domains is sketched as follows. Set

$$\mathcal{H}_{p,z}^{\alpha} := \left\{ f \in A^p(\Omega) : \ \partial^{(\alpha)} f(z) = 1 \text{ and } \partial^{(\beta)} f(z) = 0, \ \forall \beta \prec \alpha \right\}.$$

Here, $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ are multi-indices, and by $\beta \prec \alpha$ we mean that β precedes α under the lexicographical order. For $1 \le p < \infty$, the following minimizing problem

$$m_p^{(\alpha)}(z) := \inf\{\|f\|_p : \ f \in \mathcal{H}_{p,z}^{\alpha}\}$$

also admits a unique minimizer $m_p^{(\alpha)}(\cdot, z)$. It is a beautiful observation of Stefan Bergman that $\{m_2^{(\alpha)}(\cdot, z)/\|m_2^{(\alpha)}(\cdot, z)\|_2\}_\alpha$ forms a complete orthonormal basis of $A^2(\Omega)$. It is natural to ask the following

Question 9 Is $\{m_p^{(\alpha)}(\cdot, z)/\|m_p^{(\alpha)}(\cdot, z)\|_p\}_\alpha$ a Schauder basis of $A^p(\Omega)$ for $1 \le p < \infty$?

A sequence $\{x_j\}$ is called basic if it is a basis for its closed linear span $\overline{\text{span}(\{x_j\})}$. A mimic of Mazur's construction of basic sequences in an infinite-dimensional Banach space gives the following

Proposition 9 (cf. [6]) *There exists a subsequence $\alpha^{j_1} \prec \alpha^{j_2} \prec \cdots$ and functions $f_{j_k} \in \mathcal{H}_{p,z}^{\alpha^{j_k}}$, $k = 1, 2, \cdots$, such that $\{f_{j_k}\}$ is a Schauder basic sequence.*

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On Semiclassical Ohsawa-Takegoshi Extension Theorem

Siarhei Finski

Abstract For a fixed complex submanifold in a complex manifold, we consider the operator which associates to a given holomorphic section of a positive line bundle over the submanifold the holomorphic extension of it to the ambient manifold with the minimal L^2-norm. When the tensor power of the line bundle tends to infinity, we obtain an explicit asymptotic formula for this extension operator and derive several consequences of this study.

Keywords Complex manifolds · Continuation of analytic objects in several complex variables · Integral representations; canonical kernels · Toeplitz operators · Section rings

1 Introduction

In the seminal paper [33], Ohsawa and Takegoshi gave a sufficient condition under which a holomorphic section of a vector bundle on a submanifold extends to a holomorphic section over an ambient manifold with a reasonable bound on the L^2-norm of the extension in terms of the L^2-norm of the section itself. The main goal of this survey paper, where we deliberately decided to omit the technical aspects of the theory, is to review a refinement of this result in the semiclassical setting and to explain various consequences of it. Semiclassical setting means that the vector bundle we consider is a tensor power of a fixed ample line bundle, and we are mostly interested in the asymptotic properties when the tensor power tends to infinity.

The semiclassical extension theorem has several applications. Among those are the proof of the asymptotically optimal L^∞-bound on the extension of holomorphic sections [18, Theorem 1.10], refining previous results of Zhang [43, Theorem 2.2] and Bost [9, Theorem A.1], the proof of higher derivative bounds on the optimal extension operator, see [18, Corollary 1.3], lacking in Demailly's approach to the invariance of plurigenera for Kähler families, see [17, (4.19)], and the proof of asymptotic isometry

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between two natural norms on the space of holomorphic jets: one induced from the ambient manifold and another from the submanifold, [22, Theorem 1.3].

In its simplest form, semiclassical extensions have already been studied in the paper of Tian [40], where he introduced *peak sections*. In fact, peak sections can be seen as images of the *optimal extension operator*, see (2.5), applied for a submanifold given by a point. In this perspective, our work [18] can be seen as a generalization of the result of Tian from points to general submanifolds of higher dimension, and our subsequent work [22] is a further generalization where holomorphic sections are replaced by holomorphic jets. We survey the results of these papers in Sect. 2.

In [19], we study further the geometry of optimal extensions and show that for a tower of submanifolds, in the semi-classical setting, the extension operators associated to the tower satisfy transitivity property up to some small defect. We survey the results of this paper in Sect. 3.

Then in [20], we use our methods from [18, 19, 22], to study for a polarized projective manifold the relation between metric and algebraic structures on the associated section ring. More precisely, we prove that on the section ring, the L^2-norms behave additively under a multiplication, i.e. once the kernel is factored out, the multiplication operator becomes an approximate isometry (up to a normalization) with respect to the L^2-norms. We, moreover, show that L^2-norms associated with continuous plurisubharmonic metrics are actually characterized by the multiplicativity properties of this type. As an application, we refine a theorem of Phong-Sturm [34] and Berndtsson [6] about quantization of Mabuchi geodesics from weaker level of Fubini-Study convergence to stronger level of norm equivalences. We survey these results in Sect. 4.

Another application of the later techniques is given in [21], where we study the set of submultiplicative norms on section rings. As an application, we study the convex hulls of Narasimhan-Simha pseudonorms on pluricanonical sections, refining previous result of Berman-Demailly [5]. We survey these developments in Sect. 5.

2 Holomorphic Jet Extensions in the Semiclassical Setting

In this section, we review the results from [18, 22] on asymptotics of the L^2-optimal holomorphic extensions of holomorphic jets along submanifolds associated with high tensor powers of a positive line bundle. We will then briefly survey the proofs from [18, 22] of these results, underlying the differences between the two approaches from these papers.

Let us fix two complex manifolds X, Y of dimensions n and m respectively. For the sake of simplicity, we assume that X and Y are compact, although our results work in a more general setting of manifolds and embeddings of bounded geometry. We fix also a complex embedding $\iota : Y \to X$, a positive line bundle (L, h^L) over X and an arbitrary Hermitian vector bundle (F, h^F) over X. In particular, we assume that for the curvature R^L of the Chern connection on (L, h^L), the closed real $(1, 1)$-

differential form $\omega := \frac{\sqrt{-1}}{2\pi} R^L$ is positive. We denote by g^{TX} the Riemannian metric on X so that its Kähler form coincides with ω. We denote by g^{TY} the induced metric on Y and by dv_X, dv_Y the associated Riemannian volume forms on X and Y.

Let TX, TY be the holomorphic tangent bundles of X and Y. We identify the (holomorphic) normal bundle $N = TX/TY$ of Y in X as orthogonal complement of TY in TX, so that we have the (smooth) orthogonal decomposition $TX|_Y \to TY \oplus N$. We denote by g^N the metric on N induced by g^{TX}. The metric g^N induces the metric on $\mathrm{Sym}^k N^*$ as usual.

For any smooth sections f, f' of $L^p \otimes F$, $p \in \mathbb{N}$, over X, we define the L^2-scalar product using the pointwise scalar product $\langle \cdot, \cdot \rangle_h$, induced by h^L and h^F as

$$\langle f, f' \rangle_{L^2(X)} := \int_X \langle f(x), f'(x) \rangle_h dv_X(x). \tag{2.1}$$

Similarly, using the volume form dv_Y and the metric on $\mathrm{Sym}^k N^*$, we define the L^2-product for smooth sections of $\mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F)$ over Y.

We denote by $\mathcal{J}_Y$ the ideal sheaf of holomorphic germs on X, which vanish along Y. For $k \in \mathbb{N}$, we endow the space $H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$ with the L^2-norm induced by the natural inclusion $H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k) \hookrightarrow H^0(X, L^p \otimes F)$.

From Weierstrass division theorem, for any $k \in \mathbb{N}$, a k-jet associated with a section from $H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$ is holomorphic. Hence, we have a well-defined operator

$$\begin{aligned} &\mathrm{Res}_{k,p} : H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k) \to H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F)), \\ &\mathrm{Res}_{k,p}(f) = (\nabla^k f)|_Y, \end{aligned} \tag{2.2}$$

where ∇ is any fixed connection on $L^p \otimes F$. Note that the above definition doesn't depend on the choice of the connection due to the vanishing of f over Y.

A standard argument based on short exact sequences and Serre vanishing theorem implies that for any $k \in \mathbb{N}$, there is $p_0 \in \mathbb{N}$, such that for any $p \geq p_0$, the operator $\mathrm{Res}_{k,p}$ is surjective. Indeed, consider the short exact sequence of sheaves

$$0 \to \mathscr{O}_X(L^p \otimes F) \otimes \mathcal{J}_Y^{k+1} \to \mathscr{O}_X(L^p \otimes F) \otimes \mathcal{J}_Y^k \to \mathscr{O}_X(L^p \otimes F) \otimes \mathcal{J}_Y^k / \mathcal{J}_Y^{k+1} \to 0. \tag{2.3}$$

The associated long exact sequence in cohomology gives us

$$\begin{aligned} \cdots \to H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k) \to H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k / \mathcal{J}_Y^{k+1}) \\ \to H^1(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1}) \to \cdots . \end{aligned} \tag{2.4}$$

By Serre vanishing theorem, for p big enough, the cohomology $H^1(X, L^p \otimes F \otimes \mathcal{J}_Y^{k+1})$ vanishes by the ampleness of L. This finishes the proof of the surjectivity of the restriction morphism, $\mathrm{Res}_{k,p}$, as the first map of the above long exact sequence corresponds to $\mathrm{Res}_{k,p}$ under the natural isomorphism $H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k / \mathcal{J}_Y^{k+1}) \simeq H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$, and by (2.4) the vanishing of the first cohomology group means exactly that the map is surjective.

By the surjectivity of $\mathrm{Res}_{k,p}$, we can define the *optimal extension operator*

$$\mathrm{E}_{k,p} : H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F)) \to H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k), \tag{2.5}$$

by putting $\mathrm{E}_{k,p} g = f$, where $\mathrm{Res}_{k,p}(f) = g$, and f has the minimal L^2-norm among those $f' \in H^0(X, L^p \otimes F \otimes \mathcal{J}_Y^k)$ satisfying $\mathrm{Res}_{k,p}(f') = g$. Clearly, the minimizing f exists and it is unique. Moreover, the operator $\mathrm{E}_{k,p}$ is linear since the minimality of the L^2-norm among different extensions is characterized by a linear condition, requiring the image to be orthogonal to the space of holomorphic sections vanishing along Y up to order at least $k+1$. The main goal of [18, 22] is to find an explicit asymptotic expansion of the operator $\mathrm{E}_{k,p}$, as $p \to \infty$.

To describe the first term of the asymptotic expansion, we introduce some trivializations. For $y \in Y$, $Z_N \in N_y$, let $\mathbb{R} \ni t \mapsto \exp_y^X(tZ_N) \in X$ be the geodesic in X in direction Z_N. For $r_\perp > 0$ small enough, this map induces a diffeomorphism of $r_\perp$-neighborhood of the zero section in N with a tubular neighborhood U of Y in X. We use this identification, called *geodesic normal coordinates*, implicitly. We denote by $\pi : U \to Y$ the natural projection $(y, Z_N) \mapsto y$. Over U, we identify L, F to $\pi^*(L|_Y)$, $\pi^*(F|_Y)$ by the parallel transport with respect to the Chern connections along the geodesic $[0,1] \ni t \mapsto (y, tZ_N) \in X$, $|Z_N| < r_\perp$.

We fix a smooth function $\rho : \mathbb{R}_+ \to [0,1]$, satisfying

$$\rho(x) = \begin{cases} 1, & \text{for } x < \frac{1}{4}, \\ 0, & \text{for } x > \frac{1}{2}. \end{cases} \tag{2.6}$$

For $g \in \mathscr{C}^\infty(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$, $k \in \mathbb{N}$, we define using the above isomorphisms over U the section $\{g\} \in \mathscr{C}^\infty(X, L^p \otimes F)$ as follows

$$\{g\}(y, Z_N) := \rho\Big(\frac{|Z_N|}{r_\perp}\Big) \cdot g(y) \cdot Z_N^{\otimes k}, \tag{2.7}$$

where the norm $|Z_N|$, is taken with respect to g^N. Away from U, we extend $\{g\}$ by zero.

We define the *trivial extension operator* $\mathrm{E}_{k,p}^0 : H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F)) \to L^2(X, L^p \otimes F)$ as follows. For $g \in H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$, we let $(\mathrm{E}_p^0 g)(x) = 0$ for $x \notin U$, and in U, we define $\mathrm{E}_{k,p}^0 g$ using the geodesic normal coordinates as follows

$$(\mathrm{E}_{k,p}^0 g)(y, Z_N) = \{g\}(y, Z_N) \cdot \exp\Big(-p\frac{\pi}{2}|Z_N|^2\Big). \tag{2.8}$$

The Gaussian integral calculation gives us for any $f \in H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$, as $p \to \infty$, the following asymptotics

$$\big\|\mathrm{E}_{k,p}^0 f\big\|_{L^2(X)} \sim \frac{1}{p^{\frac{n-m+k}{2}}} \cdot \frac{1}{\sqrt{k! \cdot (2\pi)^k}} \cdot \big\|f\big\|_{L^2(Y)}. \tag{2.9}$$

In particular, as $p \to \infty$, we obtain that

$$\left\| \mathrm{E}^0_{k,p} \right\| \sim \frac{1}{p^{\frac{n-m+k}{2}}} \cdot \frac{1}{\sqrt{k! \cdot (2\pi)^k}}, \tag{2.10}$$

where $\| \cdot \|$ is the operator norm, calculated with respect to the L^2-norms.

Now, the section $\mathrm{E}^0_{k,p} g$ satisfies $(\mathrm{E}^0_{k,p} g)|_Y = g$, but it is not holomorphic over X (unless g is null). Nevertheless, as our main result of [18, 22] says, $\mathrm{E}^0_{k,p} g$ approximates very well the holomorphic section $\mathrm{E}_{k,p} g$. More precisely, we have the following result.

Theorem 1 *For any $k \in \mathbb{N}$, there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have*

$$\left\| \mathrm{E}_{k,p} - \mathrm{E}^0_{k,p} \right\| \leq \frac{C}{p^{\frac{n-m+k+1}{2}}}. \tag{2.11}$$

Remark 1 (a) By (2.10), Theorem 1 tells us that the principal asymptotic term of the optimal extension operator is given by the trivial extension operator.

(b) Theorem 1 refines Randriambololona [36, Théorème 3.1.10], stating that for any $\epsilon > 0$, there is $p_1 \in \mathbb{N}^*$, such that $\left\| \mathrm{E}_{k,p} \right\| \leq \exp(\epsilon p)$ for $p \geq p_1$.

(c) Clearly, Theorem 1 and (2.10) imply that for any $\epsilon > 0, k \in \mathbb{N}$, there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, we have

$$\| \mathrm{E}_{k,p} \| \leq \frac{1+\epsilon}{p^{\frac{n-m+k}{2}}} \cdot \frac{1}{\sqrt{k! \cdot (2\pi)^k}}. \tag{2.12}$$

This statement can be rephrased in the usual language of extension theorem as follows. For any $\epsilon > 0$, $k \in \mathbb{N}$, there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, $f \in H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$, there is $\tilde{f} \in H^0(X, L^p \otimes F \otimes \mathcal{J}^k_Y)$, verifying $\mathrm{Res}_{k,p}(\tilde{f}) = f$, and such that

$$\| \tilde{f} \|_{L^2(X)} \leq \frac{1+\epsilon}{p^{\frac{n-m+k}{2}}} \cdot \frac{1}{\sqrt{k! \cdot (2\pi)^k}} \cdot \| f \|_{L^2(Y)}. \tag{2.13}$$

Remark, however, that Theorem 1 and (2.9) say even more: the optimal extension operator is an asymptotic isometry modulo multiplication by an explicit constant, see [22, Theorem 1.3] for a precise statement. This observation lies in the foundation of [20], which we survey in Sect. 4.

Theorem above appears in [18, Theorem 1.1] for $k = 0$ and in [22, Theorem 1.1] for $k \geq 0$ as almost direct consequence of more precise results about the asymptotics of the *Schwartz kernel* $\mathrm{E}_{k,p}(x, y) \in (\mathrm{Sym}^k N^* \otimes L^p \otimes F)_x \otimes (\mathrm{Sym}^k N^* \otimes L^p \otimes F)^*_y$, $x \in X$, $y \in Y$, of $\mathrm{E}_{k,p}$ with respect to dv_Y. To describe these results, recall first that the *Schwartz kernel* $\mathrm{E}_{k,p}(x, y)$ is defined so that for any $g \in L^2(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$, $x \in X$, we have

$$(\mathrm{E}_{k,p}g)(x) = \int_Y \mathrm{E}_{k,p}(x,y)\cdot g(y)dv_Y(y), \tag{2.14}$$

where we extended the domain of $\mathrm{E}_{k,p}$ from $H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$ to $L^2(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$ by precomposing it with the orthogonal projection onto $H^0(Y, \mathrm{Sym}^k N^* \otimes \iota^*(L^p \otimes F))$.

Then the first result needed for the proof of Theorem 1 shows that $\mathrm{E}_{k,p}(x,y)$ has exponential decay with respect to the distance between the parameters, see [18, Theorem 1.5] and [22, Theorem 5.1]. This means that there are $c > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, $x \in X$, $y \in Y$, the following estimate holds

$$\left|\mathrm{E}_{k,p}(x,y)\right| \leq Cp^{m-\frac{k}{2}} \exp\left(-c\sqrt{p}\cdot \mathrm{dist}(x,y)\right). \tag{2.15}$$

From (2.15), we see that to understand fully the asymptotics of $\mathrm{E}_{k,p}(x,y)$, it suffices to do so for x, y in a neighborhood of a fixed point $(y_0, y_0) \in Y \times Y$ in $X \times Y$. In [18, Theorem 1.6] and [22, Theorem 5.2], we show that after a reparametrization, given by a homothety with factor $\sqrt{p}$ in the so-called Fermi coordinates around (y_0, y_0), the Schwartz kernel $\mathrm{E}_{k,p}(x,y)$ admits a complete asymptotic expansion in integer powers of $\sqrt{p}$, as $p \to \infty$. The first two terms of this expansion can be easily calculated explicitly, and the first term corresponds to the Schwartz kernel of the optimal extension operator of the Bargmann space.

More precisely, we fix a point $y_0 \in Y$ and an orthonormal frame $(e_1, \ldots, e_{2m})$ (resp. $(e_{2m+1}, \ldots, e_{2n})$) in $(T_{y_0}Y, g^{TY}_{y_0})$ (resp. in $(N_{y_0}, g^N_{y_0})$), such that for $i = 1, \ldots, n$, $Je_{2i-1} = e_{2i}$, where J is the complex structure of X. For $Z \in \mathbb{R}^{2n}$, we denote by z_i, $i = 1, \ldots, n$, the induced complex coordinates $z_i = Z_{2i-1} + \sqrt{-1}Z_{2i}$. We frequently use the decomposition $Z = (Z_Y, Z_N)$, where $Z_Y = (Z_1, \ldots, Z_{2m})$ and $Z_N = (Z_{2m+1}, \ldots, Z_{2n})$ and implicitly identify Z (resp. Z_Y, Z_N) to an element in T_yX (resp. T_yY, N_y) by

$$Z = \sum_{i=1}^{2n} Z_i e_i, \quad Z_Y = \sum_{i=1}^{2m} Z_i e_i, \quad Z_N = \sum_{i=2m+1}^{2n} Z_i e_i. \tag{2.16}$$

We fix $r_Y > 0$ small enough and $r_\perp$ as before (2.7). We define the coordinate system $\psi_{y_0} : B_0^{\mathbb{R}^{2m}}(r_Y) \times B_0^{\mathbb{R}^{2(n-m)}}(r_\perp) \to X$, for $Z = (Z_Y, Z_N)$, $Z_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $Z_Y = (Z_1, \ldots, Z_{2m})$, $Z_N = (Z_{2m+1}, \ldots, Z_{2n})$, $|Z_Y| < r_Y$, $|Z_N| < r_\perp$, by

$$\psi_{y_0}(Z_Y, Z_N) := \exp^X_{\exp^Y_{y_0}(Z_Y)}(Z_N(Z_Y)), \tag{2.17}$$

where $Z_N(Z_Y) \in N_{\exp^Y_{y_0}(Z_Y)}$ is the parallel transport of Z_N along the geodesic $\exp^Y_{y_0}(tZ_Y)$, $t = [0,1]$, with respect to the connection ∇^N on N given by the projection of the Levi-Civita connection on N, and $B_0^{\mathbb{R}^k}(\epsilon)$, $\epsilon > 0$ means the euclidean ball of radius ϵ around $0 \in \mathbb{R}^k$. The coordinates ψ_{y_0} are called the *Fermi coordinates* at y_0. Fermi coordinates, in particular, provide a trivialization of the normal bundle, N, in a neighborhood of y_0.

We will now introduce a trivialization of the vector bundles F, L using Fermi coordinates. We fix an orthonormal frame $f_1, \dots, f_r \in F_{y_0}$, and define the orthonormal frame $\tilde{f}_1, \dots, \tilde{f}_r$ by taking the parallel transport of $f_1, \dots, f_r$ with respect to the Chern connection ∇^F of (F, h^F), done first along the path $\psi_{y_0}(tZ_Y, 0), t \in [0, 1]$, and then along the path $\psi_{y_0}(Z_Y, tZ_N)$, $t \in [0, 1]$, $Z_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $|Z_Y| < r_Y$, $|Z_N| < r_\perp$. Similarly, we trivialize L in the neighborhood of y_0. These frames, the induced frames of the dual vector bundles as well as the trivialization of the normal bundle allow us to interpret $\mathrm{E}_{k,p}(x, y)$ as an element of $\mathrm{End}(F_{y_0}) \otimes (\mathrm{Sym}^k N^*)^*_{y_0}$ for $x \in X$, $y \in Y$ in a $\min(r_\perp, r_Y)$-neighborhood of y_0.

Using the identifications similar to the ones before (2.16), we view the space $\mathbb{C}^{n-m}$ as a holomorphic normal bundle of $\mathbb{C}^m$ in $\mathbb{C}^n$ with basis $\frac{\partial}{\partial z_i}$, $i = m+1, \dots, n$. We define the function $\mathscr{E}^k_{n,m} : \mathbb{R}^{2n} \times \mathbb{R}^{2m} \to \mathrm{Sym}^k \mathbb{C}^{n-m}$ for $Z \in \mathbb{R}^{2n}$, $Z'_Y \in \mathbb{R}^{2m}$ as follows

$$\begin{aligned} \mathscr{E}^k_{n,m}(Z, Z'_Y) = \exp\Big(- \frac{\pi}{2} \sum_{i=1}^{m} \big(|z_i|^2 + |z'_i|^2 - 2 z_i \overline{z}'_i \big) - \frac{\pi}{2} \sum_{i=m+1}^{n} |z_i|^2 \Big) \cdot \\ \cdot \sum_{\substack{\beta \in \mathbb{N}^{n-m} \\ |\beta| = k}} \frac{1}{\beta!} \cdot z_N^\beta \cdot \big(\frac{\partial}{\partial z_N} \big)^{\odot \beta}. \end{aligned} \tag{2.18}$$

The rationale behind the function (2.18) is that it corresponds precisely to the Schwartz kernel, written in Fermi coordinates, of the optimal extension operator from an m-dimensional linear subspace of the n-dimensional Bargmann space to the entire Bargmann space, see [18, Sect. 3.2] and [22, Sect. 3.1] for a justification. The results [18, Theorem 1.6] and [22, Theorem 5.2] say that the general embedding of a submanifold in a manifold is not too far from this model one.

More precisely, these results say that for any $k \in \mathbb{N}$, there are $\epsilon, c, C, Q > 0$, $p_1 \in \mathbb{N}^*$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z = (Z_Y, Z_N)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, $Z_N \in \mathbb{R}^{2(n-m)}$, $|Z|, |Z'_Y| \leq \epsilon$, we have

$$\begin{aligned} \Big| \frac{1}{p^{m - \frac{k}{2}}} \mathrm{E}_{k,p}\big(\psi_{y_0}(Z), \psi_{y_0}(Z'_Y)\big) - \mathscr{E}^0_{n,m}(\sqrt{p} Z, \sqrt{p} Z'_Y) \Big| \\ \leq \frac{C}{p^{\frac{1}{2}}} \cdot \Big(1 + \sqrt{p}|Z| + \sqrt{p}|Z'_Y| \Big)^Q \exp\Big(- c\sqrt{p}\big(|Z_Y - Z'_Y| + |Z_N| \big) \Big). \end{aligned} \tag{2.19}$$

Let us now explain how (2.15) and (2.19) imply Theorem 1. As we explain in [18, (5.124) and (5.125)], directly from the off-diagonal asymptotic expansion of the Bergman kernel due to Dai-Liu-Ma [12], we deduce that the Schwartz kernel of the trivial extension operator, $\mathrm{E}^0_{k,p}(x, y)$, $x \in X$, $y \in Y$ has an asymptotic expansion as in (2.19) (in particular, with the same first term). From this, (2.15), (2.19) and the exponential decay of the Bergman kernel proved by Ma-Marinescu [29], we deduce that the Schwartz kernel $K^0_{k,p}(x, y)$, $x \in X$, $y \in Y$, of $K_{k,p} := \mathrm{E}_{k,p} - \mathrm{E}^0_{k,p}$ satisfies the following bound, see [18, (5.127)] and [22, (5.15)]. For any $k \in \mathbb{N}$, there are $c, C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, $x \in X$, $y \in Y$, the following estimate holds

$$\Big|K_{k,p}(x, y)\Big| \leq Cp^{m-\frac{k+1}{2}} \exp\big(-c\sqrt{p}\mathrm{dist}(x, y)\big). \tag{2.20}$$

From (2.20), we conclude that for any $k \in \mathbb{N}$, there are $c, C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, $y_0, y_1 \in Y$, the Schwartz kernel $G_{k,p}(y_0, y_1)$ of the operator $G_{k,p} := K^*_{k,p} \circ K_{k,p}$ satisfies the following estimate

$$\int_Y \big|G_{k,p}(y_0, y)\big| dv_Y(y) \leq \frac{C}{p^{n-m+k+1}}, \qquad \int_Y \big|G_{k,p}(y, y_0)\big| dv_Y(y) \leq \frac{C}{p^{n-m+k+1}}. \tag{2.21}$$

Directly from (2.21) and Young's inequality for integral operators, cf. [39, Theorem 0.3.1] applied for $p, q = 2$, $r = 1$ in the notations of [39], we obtain that $\|G_{k,p}\| \leq \frac{C}{p^{n-m+k+1}}$, which implies that

$$\|K_{k,p}\| \leq \Big(\frac{C}{p^{n-m+k+1}}\Big)^{\frac{1}{2}}. \tag{2.22}$$

But the latter statement is a restatement of Theorem 1 by the very definition of $K_{k,p}$.

As explained above, Theorem 1 follows from (2.15) and (2.19). On their turn, the proofs of (2.15) and (2.19) for $k = 0$ in [18] and for $k \geq 0$ in [22] both rely on the asymptotic refinement of the Ohsawa-Takegoshi extension theorem, [18, Theorem 4.4] and [22, Theorem 4.4], and on the use of the so-called *orthogonal Bergman projector*, defined below. Otherwise the approaches from the papers [18, 22] are quite different.

The approach from [18] for the proofs of (2.15) and (2.19) for $k = 0$ is based on some techniques from spectral analysis, inspired by [7], on the asymptotic expansion of the Bergman kernel due to Dai-Liu-Ma [12], and on some technical results, inspired by the work of Ma-Marinescu [27], about algebras of operators with Taylor-type expansion of the Schwartz kernel, cf. [25, Sect. 7]. The approach from [22] for the proofs of (2.15) and (2.19) for $k \geq 0$ is based on the refinement of the theory of Toeplitz operators and on the asymptotic criteria for those Toeplitz-like operators along the lines of Ma-Marinescu [26, Theorem 4.9].

To explain the main features of each of those approaches more precisely, we define the *orthogonal Bergman projector of order* k, $B^{\perp,k}_p$, as the orthogonal projector from the L^2-space $L^2(X, L^p \otimes F)$ onto the subspace of elements from $H^0(X, L^p \otimes F \otimes \mathcal{J}^k_Y)$, which are orthogonal to $H^0(X, L^p \otimes F \otimes \mathcal{J}^{k+1}_Y)$.

For simplicity of the notation, we denote $B^{\perp}_p := B^{\perp,0}_p$ and similarly $\mathrm{E}_p := \mathrm{E}_{0,p}$, $\mathrm{Res}_p := \mathrm{Res}_{0,p}$. We also denote by B^X_p (resp. B^Y_p) the Bergman projector from

$L^2(X, L^p \otimes F)$ (resp. $L^2(Y, \iota^*(L^p \otimes F))$) to $H^0(X, L^p \otimes F)$ (resp. $H^0(Y, \iota^*(L^p \otimes F))$).

The core idea from [18] for the proofs of (2.15) and (2.19) for $k = 0$ is to find an algebraic expression for E_p in terms of $B_p^{\perp}$, B_p^Y and Res_p. For this, we define the operators $G_p : L^2(Y, \iota^*(L^p \otimes F)) \to L^2(Y, \iota^*(L^p \otimes F))$ and $I_p : L^2(Y, \iota^*(L^p \otimes F)) \to L^2(X, L^p \otimes F)$ as follows

$$G_p := \mathrm{Res}_Y \circ I_p - B_p^Y, \qquad I_p := B_p^{\perp} \circ \mathrm{E}_p^0 \circ B_p^Y. \tag{2.23}$$

From the basic properties of the asymptotics of the Bergman kernel, in [18, Lemma 5.13], we establish that there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have

$$\left\| G_p \right\| \leq \frac{C}{p}. \tag{2.24}$$

The bound (2.24) implies that there is $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, the infinite sum

$$T_p := \sum_{i=1}^{\infty} (-1)^i G_p^i. \tag{2.25}$$

is well-defined as an operator on $L^2(Y, \iota^*(L^p \otimes F))$. The crux of the matter is to realize, cf. [18, Lemma 5.14], that the following identity holds

$$\mathrm{E}_p = I_p + I_p \circ T_p. \tag{2.26}$$

To establish (2.26), it suffices to verify that the restriction of an operator from the right-hand side of (2.26) to Y is an identity operator and that its image lies in the space of holomorphic sections orthogonal to the sections vanishing at Y. The second property is trivial, and the first one follows from a calculation of a telescopic sum.

Then due to (2.24), we see that in (2.26) only the first summand matters for the first asymptotic term in the asymptotic expansion of E_p. From (2.23), we see that the study of the first summand, on its turn, reduces to the study of the operator $B_p^{\perp}$. It turns out, see [18, Theorems 1.7, 1.10], that it is possible to prove that for the operator $B_p^{\perp}$, the analogues of (2.15) and (2.19) hold. For completeness, we state those results more generally in the jet setting (i.e. for any $k \in \mathbb{N}$) as they appear in [22, Theorems 5.1 and 5.5], although for the needs of [18] we only need them for sections (i.e. for $k = 0$). More precisely, [18, Theorem 1.7] and [22, Theorem 5.1b)] say that for any $k \in \mathbb{N}$, there are $c, C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, $x_1, x_2 \in X$, the following estimate holds

$$\left| B_p^{\perp,k}(x_1, x_2) \right| \leq C p^n \exp\Big(- c\sqrt{p}\big(\mathrm{dist}(x_1, x_2) + \mathrm{dist}(x_1, Y) + \mathrm{dist}(x_2, Y)\big)\Big). \tag{2.27}$$

Hence, as in (2.19), for the asymptotics of the Schwartz kernel of $B_p^{\perp,k}$, it is only left to study it in the neighborhood of a fixed point $(y_0, y_0) \in Y \times Y$ in $X \times X$.

For this, we define the function $\mathscr{P}^{\perp,k}_{n,m} : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ for $Z, Z' \in \mathbb{R}^{2n}$, as follows

$$\mathscr{P}^{\perp,k}_{n,m}(Z,Z') = \exp\Big(-\frac{\pi}{2}\sum_{i=1}^{m}\big(|z_i|^2 + |z'_i|^2 - 2z_i\overline{z}'_i\big) - \frac{\pi}{2}\sum_{i=m+1}^{n}\big(|z_i|^2 + |z'_i|^2\big)\Big)\cdot \\ \cdot \pi^k \cdot \sum_{\substack{\beta\in\mathbb{N}^{n-m} \\ |\beta|=k}} \frac{z_N^\beta \cdot (\overline{z}'_N)^\beta}{\beta!}. \quad (2.28)$$

Similarly to (2.18), the rationale behind the function (2.28) is that it corresponds precisely to the Schwartz kernel, written in Fermi coordinates, of the orthogonal Bergman kernel of order $k \in \mathbb{N}$ of a linear subspace of dimension m of the n-dimensional Bargmann space inside the entire Bargmann space, see [18, Sect. 3.2] and [22, Sect. 3.1] for a justification. The results [18, Theorem 1.8] and [22, Theorem 5.5] say that the orthogonal Bergman kernel of order $k \in \mathbb{N}$ of a general embedding is not too far from this model one. More precisely, the following result holds. For any $k \in \mathbb{N}$, there are $\epsilon, c, C, Q > 0$, $p_1 \in \mathbb{N}^*$, such that for any $y_0 \in Y$, $p \geq p_1$, $Z = (Z_Y, Z_N)$, $Z' = (Z'_Y, Z'_N)$, $Z_Y, Z'_Y \in \mathbb{R}^{2m}$, $Z_N, Z'_N \in \mathbb{R}^{2(n-m)}$, $|Z|, |Z'| \leq \epsilon$, we have

$$\Big|\frac{1}{p^n} B^{\perp,k}_p\big(\psi_{y_0}(Z), \psi_{y_0}(Z')\big) - \mathscr{P}^{\perp,k}_{n,m}(\sqrt{p}Z, \sqrt{p}Z')\Big| \\ \leq \frac{C}{p^{\frac{1}{2}}}\cdot\Big(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|\Big)^Q \exp\Big(-c\sqrt{p}\big(|Z_Y - Z'_Y| + |Z_N| + |Z'_N|\big)\Big). \quad (2.29)$$

Once (2.27) and (2.29) are established, the verification of (2.15) and (2.19) for $k = 0$ becomes a routine exercise due to the relation (2.26), see [18, proofs of Theorems 1.5, 1.6 in Sect. 5.4]. We will now describe the main ideas behind the proofs of (2.27) and (2.29).

The main idea of the proof is to reduce the study of the orthogonal Bergman projector $B^{\perp}_p$ to the study of the "usual" Bergman projector B^X_p. For this, we define the "approximate projection" $B^{\perp,a}_p : L^2(X, L^p \otimes F) \to L^2(X, L^p \otimes F)$ as follows. We let $C_p := \mathrm{E}^0_p \circ \mathrm{Res}_Y \circ B^X_p$, and then define $B^{\perp,a}_p$ through the following formula

$$B^{\perp,a}_p := C^*_p \circ C_p. \quad (2.30)$$

The operator $B^{\perp,a}_p$ has an advantage over the operator $B^{\perp}_p$ since it has an explicit formula in terms of the Bergman kernel and some simple operators (as restriction or trivial extension operator). From this, we easily see that the analogues of (2.27) and (2.29) for $B^{\perp,a}_p$ follow from the analogous results for B^X_p due to Dai-Liu-Ma [12] and Ma-Marinescu [29], see [18, (5.5)] and [18, Lemma 5.10 and (5.61)] for details.

Now, it is possible to deduce (2.27) and (2.29) from their analogues for $B_p^{\perp,a}$ because, as we explain below, those operators are "spectrally related".

More precisely, from (2.30), we have

$$\ker B_p^{\perp,a} = \ker B_p^{\perp}. \tag{2.31}$$

A much deeper analysis, see [18, (5.4)], shows that the operators $B_p^{\perp,a}$, $p \in \mathbb{N}$, posses a *uniform spectral gap*, i.e. there are $a, b > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have

$$\mathrm{Spec}(B_p^{\perp,a}) \subset \{0\} \cup [a, b]. \tag{2.32}$$

Remark also that since $B_p^{\perp}$ is an orthogonal projection, it satisfies

$$\mathrm{Spec}(B_p^{\perp}) \subset \{0, 1\}. \tag{2.33}$$

A combination of (2.31), (2.32) and (2.33) is the spectral relation between $B_p^{\perp,a}$ and $B_p^{\perp}$.

Now, as we explain in [18, Sects. 5.1, 5.3 proof of Theorems 1.7, 1.8], the spectral relations imply that we can use the techniques developed by Bismut-Lebeau [7] to transfer the asymptotic results from $B_p^{\perp,a}$ to the corresponding statements on $B_p^{\perp}$.

The only statement which is, thus, left unproved is the uniform spectral gap, (2.32). Using (2.9), it can be easily seen that (2.32) can be restated as follows, see [18, (5.4)] for details. There are $c, C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have

$$cp^{\frac{n-m}{2}} \leq \left\| \mathrm{Res}_p \right\| \leq Cp^{\frac{n-m}{2}}. \tag{2.34}$$

The statement (2.34) is established in [18, Theorem 4.1]. Its proof is based on the asymptotic refinement of the Ohsawa-Takegoshi extension theorem, [18, Theorem 4.4]. The later statement is inspired by the works of Ohsawa-Takegoshi [33] and Demailly [13] and relies on Hörmander's L^2-estimates and the precise calculation of the first two terms of the Taylor expansion of the $\overline{\partial}$-operator in a shrinking tubular neighborhood of Y, [18, Theorem 4.3], following the previous works of Bismut-Lebeau [7, Theorem 8.18] and Dai-Liu-Ma [12, Theorem 4.6].

Now, let us explain the main idea of [22] for the proof of (2.15) and (2.19) for $k \geq 0$. The central point here is to study a sequence of operators, relating the extension operator with the adjoint of the restriction operator. This sequence of operators, which we call the *multiplicative defect*, was introduced in [19] to study asymptotic transitivity properties for optimal holomorphic extensions, which we survey in Sect. 3.

More precisely, in [22, Theorem 4.1], we prove that there is $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, there is a unique operator $A_p \in \mathrm{End}(H^0(Y, \iota^*(L^p \otimes F)))$, verifying

$$(\mathrm{Res}_p \circ B_p^X)^* = \mathrm{E}_p \circ A_p. \tag{2.35}$$

In fact, to establish this statement, it suffices to verify that the kernels and the images of the operators $(\mathrm{Res}_p \circ B_p^X)^*$ and E_p coincide for any $p \geq p_1$. This fact is an easy consequence of the surjectivity of Res_p, which was established in (2.4).

From (2.35), we see that the following explicit formula for the operators A_p holds

$$A_p = \mathrm{Res}_p \circ (\mathrm{Res}_p \circ B_p^X)^*. \tag{2.36}$$

From the formula (2.36), the off-diagonal expansion of the Bergman kernel and the asymptotic criteria for Toeplitz operators of Ma-Marinescu [26, Theorem 4.9], we can then see that the sequence of operators A_p is a Toeplitz operator in the sense of [25, Sect. 8], see [22, Theorem 5.7] for details. From this observation and a small calculation, we obtain in [22, Theorem 5.7 and Remark 3.4a)], cf. [19, Theorem 4.3], that there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have

$$\left\| \frac{1}{p^{n-m}} A_p - \mathrm{Id} \right\| \leq \frac{C}{p}, \tag{2.37}$$

where the operator norm $\| \cdot \|$ is taken with respect to the L^2-norms.

In particular, from (2.37), we see that the operators A_p are invertible for p big enough. We then can rewrite (2.35) in the following form

$$\mathrm{E}_p = (\mathrm{Res}_p \circ B_p^X)^* \circ A_p^{-1}. \tag{2.38}$$

From this point, the study of the asymptotic expansion of E_p reduces to the study of the asymptotic expansion of the Bergman projection, B_p^X, done previously by Dai-Liu-Ma [12] and Ma-Marinescu [29], see [22, Proof of Lemma 5.9b)] for details. To recover the result up to the error term as in (2.15) and (2.19), we actually need to work with so-called Toeplitz operators with weak exponential decay, introduced in [22, Sect. 3.2], refining the usual Toeplitz operators.

Remark that in the symplectic reduction setting, the analogue of the operator A_p, defined through the analogue of the relation (2.36), was considered by Ma-Zhang [30, Corollary 4.13]. In this setting, they also established that this operator is a Toeplitz operator. As our work [22] suggests, it should be possible to study similarly the extension operator in this context and to get the analogue of Theorem 1 there.

The proof of the asymptotic expansion for jets instead of sections then proceeds by an inductive argument through the use of *higher order multiplicative defects* $A_{k,p}$, $k \in \mathbb{N}$, defined as follows. Using the surjectivity of (2.2), one can prove as in (2.35) that for any $k \in \mathbb{N}$, there is $p_0 \in \mathbb{N}$, such that for any $p \geq p_0$, there is a unique operator $A_{k,p}$, verifying

$$\left(\mathrm{Res}_{k,p} \circ \left(B_p^X - \sum_{i=0}^{k-1} B_p^{\perp,i}\right)\right)^* = \mathrm{E}_{k,p} \circ A_{k,p}. \tag{2.39}$$

To get now the expansions (2.15) and (2.19) for higher k, we essentially rely on the same ideas as for $k = 0$. To illustrate it better, let us explain the steps in the proof for $k = 1$.

Directly from (2.39), we obtain the formula

$$A_{1,p} = \mathrm{Res}_{1,p} \circ \Big(\mathrm{Res}_{1,p} \circ \big(B_p^X - B_p^{\perp}\big)\Big)^*. \tag{2.40}$$

The formula (2.40) allows us to study the asymptotic expansion of $A_{1,p}$ using the well-known expansion of B_p^X and the previously established asymptotic expansion of $B_p^{\perp}$, see [20, Proof of Lemma 5.9a)]. As it was previously described for the operator A_p, this analogous study would show us that for p big enough, the operators $A_{1,p}$ are invertible, see [20, Lemmas 5.9a) and 5.11]. Then, as in (2.38), we see that the following formula holds

$$\mathrm{E}_{1,p} = \Big(\mathrm{Res}_{1,p} \circ \big(B_p^X - B_p^{\perp}\big)\Big)^* \circ A_{1,p}^{-1}. \tag{2.41}$$

From the formula (2.41), it is possible to deduce the asymptotic expansion of $\mathrm{E}_{1,p}$ from the well-known expansion of B_p^X and the already established expansions for $B_p^{\perp}$ and $A_{1,p}$, see [20, Proof of Lemma 5.9b)]. Finally, we use the following simple formula, cf. [22, (5.51)],

$$B_p^{\perp,1} = \mathrm{E}_{1,p} \circ \mathrm{Res}_{1,p} \circ \big(B_p^X - B_p^{\perp}\big), \tag{2.42}$$

to deduce the asymptotic expansion of $B_p^{\perp,1}$ from the well-known expansion of B_p^X and the already established expansions of $\mathrm{E}_{1,p}$ and $B_p^{\perp}$, see [20, Proof of Lemma 5.9c)]. Then we repeat the procedure to obtain (2.15) and (2.19) for any $k \in \mathbb{N}$ by induction. Remark that along the way, we also establish (2.27) and (2.29) for any $k \in \mathbb{N}$.

3 Towers of Embeddings and Transitivity of Extensions

The main goal of this section is to a result from [19], proving that in the semiclassical setting, the optimal extension operator satisfies the asymptotic transitivity property with respect to a tower of embeddings.

More precisely, let us fix a tower of submanifolds $Y \overset{\iota_1}{\hookrightarrow} W \overset{\iota_2}{\hookrightarrow} X$, $\iota := \iota_2 \circ \iota_1$ of dimensions m, l and n respectively. To underline the dependence on the embeddings, we denote by $\mathrm{E}_p^{X|Y}$, $\mathrm{E}_p^{X|W}$ and $\mathrm{E}_p^{W|Y}$ the optimal extension operators associated for the pairs (X, Y), (X, W) and (W, Y). One of the main results of [19] goes as follows.

Theorem 2 ([19, Theorem 1.1]) *There are* $c, C > 0$, $p_1 \in \mathbb{N}$ *and an explicit constant* $C_0 \geq 0$, *depending only on the infinitesimal geometry of the embedding of* Y *inside of* W *and* X, *such that for* $p \geq p_1$, *we have*

$$\frac{C_0}{p^{\frac{n-m+3}{2}}} - \frac{C}{p^{\frac{n-m+4}{2}}} \leq \left\| \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y} \right\| \leq \frac{C_0}{p^{\frac{n-m+3}{2}}} + \frac{C}{p^{\frac{n-m+4}{2}}}. \tag{3.1}$$

Remark 2 From Theorem 1, (2.9) and (3.1), we see that the norm of the *transitivity defect*, $\mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y}$, is of three orders of magnitude less than the norm of the extension itself. We call this property the *asymptotic transitivity*.

Let us now briefly describe the main ideas behind the proof of Theorem 2. Of course, one might argue that since by (2.19) we know explicitly the asymptotic expansion of the extension operators $\mathrm{E}_p^{X|Y}$, $\mathrm{E}_p^{X|W}$ and $\mathrm{E}_p^{W|Y}$, it would be enough to apply it to get the first non-zero term of the asymptotic expansion of the transitivity defect. Remark, however, that in such approach, it would be necessary to calculate the first four terms in the asymptotic expansion of the optimal extension operator by Remark 2. The asymptotics (2.19) gives us only the first term, the second term is calculated in [18, 22], and is known to be related with the second fundamental form of the embedding. As both papers [18, 22] suggest, calculating the third and fourth terms should be a laborious exercise, refining, in particular the calculation of the corresponding asymptotic terms in the expansion of the Bergman kernel, cf. [28].

Due to this reason, the paper [19] is based on a different idea. There, we calculate the asymptotic expansion of the sequence of operators

$$T_p := \mathrm{Res}_W \circ \mathrm{E}_p^{X|Y} - \mathrm{E}_p^{W|Y}. \tag{3.2}$$

Then the needed result would follow by the use of the following basic formula

$$\mathrm{E}_p^{X|Y} - \mathrm{E}_p^{X|W} \circ \mathrm{E}_p^{W|Y} = \mathrm{E}_p^{X|W} \circ T_p, \tag{3.3}$$

and some composition laws for extension operators [19, Sect. 4.1], similar to the kernel calculus from the theory of Toeplitz operators, see [25, Sect. 7.1].

The main issue, hence, is to find an asymptotic expansion of the sequence of operators T_p. The main idea is to find an alternative expression for T_p in terms of the multiplicative defects $A_p^{X|Y}$, $A_p^{X|W}$ and $A_p^{W|Y}$, associated with our tower of embeddings, see (2.35). More precisely, our formula from [19, Lemma 5.5] says that there is $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, we have

$$\begin{aligned} T_p = &\operatorname{Res}_p^{X|W} \circ \mathrm{E}_p^{X|Y} \circ \Big[B_p^Y - \Big(\frac{1}{p^{n-m}} A_p^{X|Y} \Big) \circ \Big(\frac{1}{p^{l-m}} A_p^{W|Y} \Big)^{-1} \Big] \\ &+ \Big[B_p^W - \Big(\frac{1}{p^{n-l}} A_p^{X|W} \Big)^{-1} \Big] \circ \operatorname{Res}_p^{X|W} \circ \mathrm{E}_p^{X|Y} \\ &- \Big[B_p^W - \Big(\frac{1}{p^{n-l}} A_p^{X|W} \Big)^{-1} \Big] \circ \operatorname{Res}_p^{X|W} \circ \mathrm{E}_p^{X|Y} \circ \\ &\qquad \circ \Big[B_p^Y - \Big(\frac{1}{p^{n-m}} A_p^{X|Y} \Big) \circ \Big(\frac{1}{p^{l-m}} A_p^{W|Y} \Big)^{-1} \Big]. \end{aligned} \tag{3.4}$$

The verification of this identity is based on (2.35) and (2.38).

From (3.4), we see that the calculation of the asymptotic expansion for T_p can be essentially encapsulated in the calculations of the asymptotic expansions of $A_p^{X|Y}$, $A_p^{X|W}$ and $A_p^{W|Y}$, see [19, proof of Theorem 1.5] for details. It turns out that the calculation of the first asymptotic term of the multiplicative defect as in (2.37) is not enough, and we have to calculate the second term. To describe this term, we need to recall the definition of Toeplitz operators. For a function $f : X \to \mathbb{R}$, we denote by $T_{f,p}^X \in \operatorname{End}(L^2(X, L^p))$ the associated Berezin-Toeplitz operator, defined for $g \in L^2(X, L^p)$ as follows

$$T_{f,p}^X(g) := B_p^X(f \cdot B_p^X g). \tag{3.5}$$

We denote by r_X, r_Y the scalar curvatures of the manifolds X and Y. Assume for simplicity that the vector bundle (F, h^F) is trivial. The central formula for the proof of Theorem 2 from [19, Theorem 5.1] says that there are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have

$$\left\| \frac{1}{p^{n-m}} A_p^{X|Y} - \mathrm{Id} - \frac{1}{p} \cdot \frac{1}{8\pi} \cdot T_{r_X - r_Y, p}^Y \right\| \leq \frac{C}{p^2}. \tag{3.6}$$

From (3.4) and (3.6) we can deduce the asymptotic expansion of T_p by applying some simple composition rules for extension-like operators, see [19, Sect. 4.1]. The proof of (3.6), on its turn, relies on the precise value of the second order term in the asymptotic expansion of the Bergman kernel due to Lu [24] and Wang [42], see [19, proof of Theorem 5.1].

4 On the Metric Structure of the Section Ring

In this section, we describe the results of [20], where we propose to study metric properties of section rings associated with polarized projective manifolds.

More precisely, we fix a complex projective manifold X of dimension n and an ample line bundle L over X. We consider the *section ring* $R(X, L)$, defined as follows

$$R(X, L) := \oplus_{k=1}^{\infty} H^0(X, L^k). \tag{4.1}$$

For any $r \in \mathbb{N}^*$, $k; k_1, \ldots, k_r \in \mathbb{N}^*$, $k_1 + \cdots + k_r = k$, we define the multiplication map

$$\mathrm{Mult}_{k_1,\cdots,k_r} : H^0(X, L^{k_1}) \otimes \cdots \otimes H^0(X, L^{k_r}) \to H^0(X, L^k), \tag{4.2}$$

as $f_1 \otimes \cdots \otimes f_r \mapsto f_1 \cdots f_r$. Those multiplication maps endow $R(X, L)$ with the structure of a graded ring. It follows from the Proj-construction that the (graded) $\mathbb{C}$-ring structure of $R(X, L)$ carries the same amount of information as the pair (X, L). It is very natural then to interpret some well-known objects, defined in geometrical terms (i.e. in terms of the pair (X, L)), in algebraic terms (i.e. in terms of the section ring $R(X, L)$). In this vein, the main goal of [20] is to find an algebraic counterpart of metrics on L.

For this, let us fix a positive metric h^L on L. When restricted to the vector space of holomorphic sections, $H^0(X, L^p)$, we sometimes denote the associated L^2-norm from (2.1) by $\mathrm{Hilb}_p(h^L)$. Over $R(X, L)$, we define the induced *graded* norm

$$\mathrm{Hilb}(h^L) := \sum_{k=1}^{\infty} \mathrm{Hilb}_k(h^L). \tag{4.3}$$

The first goal of [20] is to study the metric properties of $(R(X, L), \mathrm{Hilb}(h^L))$ in their relation with the multiplication map. Roughly, our first result states that, asymptotically, the L^2-norms behave additively with respect to the multiplication operator.

To describe this result more precisely, recall that a norm $N_V = \| \cdot \|_V$ on a finitely dimensional vector space V naturally induces the norm $\| \cdot \|_Q$ on any quotient Q, $\pi : V \to Q$, of V through the following identity

$$\|f\|_Q := \inf \left\{ \|g\|_V; \quad g \in V, \pi(g) = f \right\}, \qquad f \in Q. \tag{4.4}$$

By a slight abuse of notations, we sometimes denote the quotient norm by $[N_V]$, i.e. without the reference to the quotient space. This will not cause any trouble as the space will be explicit.

Now, let us recall briefly the proof of the important fact that there is $p_0 \in \mathbb{N}$, such that for any $k_1, \cdots, k_r \geq p_0$, the map $\mathrm{Mult}_{k_1,\cdots,k_r}$ is surjective. Clearly, it is enough to prove the statement for $r = 2$. Recall first that Künneth theorem allows us to interpret the tensor product of cohomologies as the cohomology of the product manifold. In other words, we have the following isomorphism

$$H^0(X \times X, L^k \boxtimes L^l) \to H^0(X, L^k) \otimes H^0(X, L^l). \tag{4.5}$$

An easy verification then shows that under the isomorphism (4.5) and the identification of the diagonal Δ in $X \times X$ with the manifold X, the multiplication operator corresponds to the restriction operator to the diagonal. In other words, we have a commutative diagram

$$\begin{array}{ccc} H^0(X\times X, L^k\boxtimes L^l) & \xrightarrow{\mathrm{Res}_\Delta} & H^0(\Delta, L^k\boxtimes L^l|_\Delta) \\ \downarrow & & \downarrow \\ H^0(X,L^k)\otimes H^0(X,L^l) & \xrightarrow{\mathrm{Mult}_{k,l}} & H^0(X, L^{k+l}). \end{array} \tag{4.6}$$

Now, over $X\times X$, we have a short exact sequence of sheaves

$$0\to \mathscr{O}_{X\times X}(L^k\boxtimes L^l)\otimes \mathscr{J}_\Delta \to \mathscr{O}_{X\times X}(L^k\boxtimes L^l)\xrightarrow{\mathrm{Res}_\Delta} \mathscr{O}_\Delta((L^k\boxtimes L^l)|_\Delta)\to 0. \tag{4.7}$$

The associated long exact sequence in cohomology gives us

$$\begin{aligned} \cdots \to H^0(X\times X, L^k\boxtimes L^l) &\xrightarrow{\mathrm{Res}_\Delta} H^0(\Delta, (L^k\boxtimes L^l)|_\Delta) \\ &\to H^1(X\times X, L^k\boxtimes L^l\otimes \mathscr{J}_\Delta)\to \cdots. \end{aligned} \tag{4.8}$$

Clearly, from (4.6), to prove the surjectivity of $\mathrm{Mult}_{k,l}$ for k,l big enough, it is enough to establish the surjectivity of Res_Δ from (4.6). From (4.8), we see that the surjectivity of Res_Δ for k,l big enough is equivalent to the vanishing of the cohomology $H^1(X\times X, L^k\boxtimes L^l\otimes\mathscr{J}_\Delta)$. But this vanishing follows directly from Nadel vanishing theorem, cf. [14, Theorem 5.11].

From the surjectivity of the multiplication maps and the construction (4.4), we see that any Hermitian norms N_k, N_l on $H^0(X,L^k)$ and $H^0(X,L^l)$ induce the Hermitian norm $[N_k\otimes N_l]$ on $H^0(X,L^{k+l})$. The first main result of [20] states that one can asymptotically recover the L^2-norms on higher cohomology from the L^2-norms on lower cohomology as follows.

Theorem 3 ([20, Theorem 1.1]) *There are $C>0$, $p_1\in\mathbb{N}^*$, such that for any $k,l\geq p_1$, for the norms over $H^0(X,L^{k+l})$, the following relation holds*

$$1-C\Big(\frac{1}{k}+\frac{1}{l}\Big)\leq \frac{[\mathrm{Hilb}_k(h^L)\otimes \mathrm{Hilb}_l(h^L)]}{\mathrm{Hilb}_{k+l}(h^L)}\cdot\Big(\frac{k\cdot l}{k+l}\Big)^{\frac{n}{2}}\leq 1+C\Big(\frac{1}{k}+\frac{1}{l}\Big). \tag{4.9}$$

The main idea behind the proof of Theorem 3 is to apply a version of the semiclassical Ohsawa-Takegoshi extension theorem to the diagonal embedding into the product manifold $\Delta\hookrightarrow X\times X$. In fact, by the commutative diagram (4.6), the upper bound of Theorem 3 then corresponds to the following statement. For any holomorphic section $f\in H^0(X,L^{k+l})$, viewed as an element of $H^0(\Delta, L^k\boxtimes L^l|_\Delta)$, there is a holomorphic extension $\tilde f\in H^0(X\times X, L^k\boxtimes L^l)$ of it, such that in the notations of Theorem 3, the following bound is satisfied

$$\|\tilde f\|_{L^2_{k,l}(X\times X, h^L)}\leq \|f\|_{L^2_{k+l}(X,h^L)}\cdot\Big(\frac{k+l}{k\cdot l}\Big)^{\frac{n}{2}}\cdot\Big(1+C\Big(\frac{1}{k}+\frac{1}{l}\Big)\Big), \tag{4.10}$$

where $\|\cdot\|_{L^2_{k,l}(X\times X, h^L)}$ is the L^2-norm on $H^0(X \times X, L^k \boxtimes L^l)$ induced by the product metric. For $k = l$, since we have $L^k \boxtimes L^l = (L \boxtimes L)^l$, the bound (4.10) follows from Theorem 1 and (2.9), see Remark 1c. As we explain in [20, proof of Theorem 3.4], the general case, $k \neq l$, still follows from the methods of [22]. The lower bound from (4.9) can also be approached by the methods of [22], see [20, proof of Theorem 3.2] for details.

Now, the main output of [20] is the proof of the fact that the properties of the form (4.9) actually characterize the L^2-norms. It turns out that for such a characterization to hold, we need to relax the regularity assumption on h^L. Recall that a continuous metric h^L on L is called *plurisubharmonic* (or *psh* for short) if the associated Chern $(1, 1)$-current $c_1(L, h^L)$ is positive. Recall that the $(1, 1)$-current $c_1(L, h^L)$ is defined as follows

$$c_1(L, h^L) = c_1(L, h^L_0) + \frac{\partial\overline{\partial}}{2\pi\sqrt{-1}}\Big[\log\Big(\frac{h^L}{h^L_0}\Big)\Big], \tag{4.11}$$

where h^L_0 is any smooth metric on L, and by $[g]$ we mean the distribution associated with a locally integrable function g. Poincaré-Lelong equation, cf. [15, Theorem V.13.9], shows that such a definition is independent of the choice of h^L_0. Let $\mathcal{H}^L$ be the set of continuous psh metrics on L.

By Bedford-Taylor's definition of Monge-Ampère operator, see [3] or [15, Proposition I.3.2], we extend the definition of the volume form $dv_X := \frac{1}{n!}c_1(L, h^L)^n$ as a non-negative measure on X for any $h^L \in \mathcal{H}^L$. Using this measure, we extend the definition of $\mathrm{Hilb}(h^L)$ for any $h^L \in \mathcal{H}^L$.

Now, we are aiming to classify graded norms on the section ring up to the equivalence relation $(\sim)$ defined as follows. We say that graded norms $N = \sum_{k=1}^{\infty} N_k$ and $N' = \sum_{k=1}^{\infty} N'_k$ are *equivalent* if the multiplicative gap between the norms N, N' is subexponential. In other words, for any $\epsilon > 0$, we require the existence of $p_1 \in \mathbb{N}$, such that for any $p \geq p_1$, we have

$$N'_p \cdot \exp(-\epsilon p) \leq N_p \leq N'_p \cdot \exp(\epsilon p). \tag{4.12}$$

Now, one possible motivation for considering this equivalence relation is that it distinguishes elements $\mathrm{Hilb}(h^L)$ for different $h^L \in \mathcal{H}^L$, see [20, Theorem 1.7]. It was, moreover, established in [21], that for any continuous metric h^L on L, there is a unique element $P(h^L) \in \mathcal{H}^L$, such that $\mathrm{Hilb}(h^L)$ is equivalent to $\mathrm{Hilb}(P(h^L))$.

We say that a graded Hermitian norm $N := \sum_{k=1}^{\infty} N_k$ on $R(X, L)$ is *multiplicatively generated* if there is $p_0 \in \mathbb{N}$ and a function $f : \mathbb{N} \to \mathbb{R}$, verifying $f(k) = o(k)$, as $k \to \infty$, such that for any $r \in \mathbb{N}^*$, $k; k_1, \dots, k_r \geq p_0$, $k_1 + \cdots + k_r = k$, under the map (4.2), the following string of inequalities is satisfied

$$\begin{aligned} N_k \exp\Big(-\big(f(k_1) + \cdots + f(k_r) + f(k)\big)\Big) &\leq \big[N_{k_1} \otimes \cdots \otimes N_{k_r}\big] \\ &\leq N_k \exp\Big(f(k_1) + \cdots + f(k_r) + f(k)\Big). \end{aligned} \tag{4.13}$$

The condition (4.13) essentially means that the graded norm N and the norms, obtained by the inductive construction from N, differ only by a slight multiplicative margin. Clearly, this condition respects the above equivalence relation, (4.12), i.e. a norm which is equivalent to a multiplicatively generated norm is multiplicatively generated.

From Theorem 3, it is very simple to see, cf. [20, Lemma 4.9], that the norm $\mathrm{Hilb}(h^L)$ is multiplicatively generated for any smooth positive metric h^L on L. More generally, by relying on a version of Ohsawa-Takegoshi extension theorem with uniform constant due to Demailly [16, Theorem 2.8] and Ohsawa [32], we establish in [20, Theorem 1.5] that the same conclusion holds for any $h^L \in \mathcal{H}^L$. In fact, the main result of *loc. cit.*, [20, Theorem 1.6], says much more.

Theorem 4 *For any multiplicatively generated norm N, there is a unique $h^L \in \mathcal{H}^L$, such that N is equivalent to* $\mathrm{Hilb}(h^L)$*. In other words, the set of multiplicatively generated norms is a disjoint union of equivalence classes represented by* $\mathrm{Hilb}(h^L)$ *for $h^L \in \mathcal{H}^L$.*

In realms of non-Archimedean geometry, a result, similar to Theorem 4, was established by Boucksom-Jonsson [10, Theorem 6.1] and Reboulet [37, Theorem A]. As they explained, this non-Archimedean result plays an important role in the study of K-stability.

To show Theorem 4, we construct explicitly the inverse of the map Hilb using the Fubini-Study map. For this, recall that for k sufficiently large so that L^k is very ample, Fubini-Study operator associates for any norm $N_k = \| \cdot \|_k$ on $H^0(X, L^k)$, a continuous metric $FS(N_k)$ on L^k, constructed in the following way. Consider the Kodaira embedding

$$\mathrm{Kod}_k : X \hookrightarrow \mathbb{P}(H^0(X, L^k)^*), \tag{4.14}$$

defined as $x \mapsto H^0(X, L^k \otimes \mathcal{J}_x)$, where $\mathcal{J}_x$ is the ideal sheaf of holomorphic functions vanishing at x, and we implicitly used the identification between the hyperplanes in $H^0(X, L^k)$ and lines in $H^0(X, L^k)^*$. The evaluation maps, $ev_x : H^0(X, L^k) \to L^k_x$, $x \in X$, provide the isomorphism

$$L^{-k} \to \mathrm{Kod}_k^* \mathscr{O}(-1), \tag{4.15}$$

where $\mathscr{O}(-1)$ is the tautological bundle over the projective space $\mathbb{P}(H^0(X, L^k)^*)$. Let us now endow $H^0(X, L^k)^*$ with the dual norm N_k^* and induce from it a metric $h^{FS}(N_k)$ on the tautological line bundle $\mathscr{O}(-1)$ over $\mathbb{P}(H^0(X, L^k)^*)$. We define the metric $FS(N_k)$ on L^k through

$$FS(N_k) = \mathrm{Kod}_k^*(h^{FS}(N_k)^*). \tag{4.16}$$

One can see that for any $x \in X$, $l \in L^k_x$, the following identity takes place

$$|l|_{FS(N_k)} = \inf_{\substack{s \in H^0(X, L^k) \\ s(x) = l}} \|s\|_k. \tag{4.17}$$

When the norm N_k comes from a Hermitian product on $H^0(X, L^k)$, the above construction is standard and explicit evaluation shows that in this case $c_1(\mathscr{O}(-1), h^{FS}(N_k))$ coincides up to a negative constant with the Kähler form of the Fubini-Study metric on $\mathbb{P}(H^0(X, L^k)^*)$ induced by N_k. In particular, $c_1(\mathscr{O}(-1), h^{FS}(N_k))$ is a negative $(1,1)$-form. Hence, the metric $FS(N_k)$ is psh in this case. According to Kobayashi [23, Lemma on p.160], the last property holds for Fubini-Study metrics associated with any norm N_k on $H^0(X, L^k)$.

Recall also that Tian [40] proved that for any smooth positive metric h^L on L, the sequence of metrics $FS(\mathrm{Hilb}_k(h^L))^{\frac{1}{k}}$ converges uniformly, as $k \to \infty$, to h^L. A direct modification of his proof shows that the later statement holds more generally for continuous psh metrics h^L, cf. [20, Theorem 2.15]. From this, we see that if a graded norm N is equivalent to $\mathrm{Hilb}(h^L)$ for some $h^L \in \mathcal{H}^L$, then the sequence of metrics $FS(N_k)^{\frac{1}{k}}$ converges uniformly, as $k \to \infty$, to h^L. More generally, when the latter convergence holds for a graded norm N, we denote the limiting metric on L by $FS(N)$, and call it the Fubini-Study metric associated to N.

Note that the existence of the limit of the Fubini-Study metric alone doesn't determine the equivalence class of a graded Hermitian metric on $R(X, L)$, see [20, Sect. 4.4] for an explicit example. Geometrically, it is very clear. In fact, by the result of Tian, the existence of this limit basically means that over the images of the Kodaira embeddings, the multiplicative gap between the metrics on $\mathscr{O}(1)$ over $\mathbb{P}(H^0(X, L^k)^*)$ induced by N_k and $\mathrm{Hilb}_k(FS(N))$ is subexponential. The equivalence of N to $\mathrm{Hilb}(FS(N))$ means, however, that this gap is subexponential everywhere and not only over the images of the Kodaira embeddings. This distinction becomes in particular apparent since we are considering limits $k \to \infty$, and in this case the dimension of $\mathbb{P}(H^0(X, L^k)^*)$ tends to infinity, whilst the dimension of the images of Kodaira embeddings are constant.

In [20, Theorem 4.1], we prove that for any graded multiplicatively generated norm $N := \sum_{k=1}^{\infty} N_k$ on $R(X, L)$, the sequence of metrics $FS(N_k)^{\frac{1}{k}}$ converges uniformly, as $k \to \infty$, to a metric on L, which we denote by $FS(N)$. As the metrics $FS(N_k)$ are continuous for any $k \in \mathbb{N}^*$ and the convergence is uniform, we deduce that $FS(N)$ is continuous as well. Recall also that by the result of Kobayashi, the metrics $FS(N_k)$ are psh, and hence $FS(N)$ is psh as well.

The proof of the uniform convergence of $FS(N_k)^{\frac{1}{k}}$ is based on the use of Fekete's and Dini's theorems and an elementary observation that the sequence of norms $FS(N_k)$ is both nearly submultiplicative and nearly supermultiplicative. The last observation is a simple consequence of a commutative diagram relating different Kodaira embeddings through Segre embeddings, see [20, (4.31)]. The main content of Theorem 4, is then to show that when the graded Hermitian norm is multiplicatively generated, its equivalence class is determined by $FS(N)$.

The main idea of the proof of the latter statement is to establish it first for the projective space using some explicit calculations and then to deduce the general case by the semiclassical Ohsawa-Takegoshi extension theorem applied for the Kodaira embedding, see [20, proof of Theorem 4.18] for details. The explicit calculations are possible for projective spaces as any Hermitian vector space can be trivialized by a choice of an orthonormal basis.

As an application of Theorem 4, we refine in [20, Theorem 1.8] the theorem of Phong-Sturm [34, Theorem 1] and Berndtsson [6, Theorem 1.2] about quantization of Mabuchi geodesics from weaker level of Fubini-Study convergence to stronger level of norm equivalences.

More precisely, we fix $h^L_0, h^L_1 \in \mathcal{H}^L$. Consider the weak Mabuchi geodesic $h^L_t \in \mathcal{H}^L$, $t \in [0, 1]$, connecting h^L_0 and h^L_1, constructed using Bedford-Taylor theory [2], see [20, Sect. 4.2] for details. For any $k \in \mathbb{N}^*$, consider a geodesic $H_{k,t}$, $t \in [0, 1]$, connecting $\mathrm{Hilb}_k(h^L_0)$ and $\mathrm{Hilb}_k(h^L_1)$ in the space of Hermitian norms on $H^0(X, L^k)$. Explicitly, if $A_k \in \mathrm{End}(H^0(X, L^k))$ is the self-adjoint map, relating the scalar products on $H^0(X, L^k)$ associated with $\mathrm{Hilb}_k(h^L_0)$ and $\mathrm{Hilb}_k(h^L_1)$ as $\langle \cdot, \cdot \rangle_{L^2_k(X, h^L_1)} = \langle A_k \cdot, \cdot \rangle_{L^2_k(X, h^L_0)}$, then $H_{k,t}$ is the Hermitian norm associated with the scalar product $\langle A^t_k \cdot, \cdot \rangle_{L^2_k(X, h^L_0)}$. In [20, Theorem 1.8], we establish that for any $t \in [0, 1]$, the graded Hermitian norms $H_t := \sum H_{k,t}$ and $\mathrm{Hilb}(h^L_t)$ are equivalent.

The proof of this result essentially relies on two ingredients. The first ingredient is a theorem of Phong-Sturm [34, Theorem 1] and Berndtsson [6, Theorem 1.2], stating that for any $t \in [0, 1]$, the sequence of metrics $FS(H_{k,t})^{\frac{1}{k}}$, $k \in \mathbb{N}^*$ on L converge uniformly, as $k \to \infty$, to h^L_t. The second ingredient is a refinement of Theorem 4, established in the course of its proof, see [20, Theorem 4.2], stating that any norm on $R(X, L)$, for which only the lower bound from (4.13) holds, and for which the associated Fubini-Study metrics converge uniformly, is equivalent to Hilb of the associated limit of the Fubini-Study metrics. A verification of the fact that the sequence of norms $H_{k,t}$, $k \in \mathbb{N}^*$ verifies the first hypothesis above relies on a fact that $H_{k,0}$ and $H_{k,1}$ satisfy this hypothesis by [20, Theorem 1.5] and on some basic monotonicity properties from linear algebra about geodesics in the space of norms, see [20, Lemma 4.21 and Corollary 4.23], which we establish using interpolation theorem of Stein-Weiss.

5 Submultiplicative Norms on Section Rings

The main goal of this section is to review the results from [21] about the study of the set of submultiplicative norms on section rings of polarized projective manifolds.

More precisely, conserving the notations from the previous section, we say that a graded pseudonorm $N = \sum N_k$, $N_k := \| \cdot \|_k$, over the section ring $R(X, L)$ is *submultiplicative* if for any $k, l \in \mathbb{N}^*$ and $f \in H^0(X, L^k)$, $g \in H^0(X, L^l)$, we have

$$\| f \cdot g \|_{k+l} \leq \| f \|_k \cdot \| g \|_l. \tag{5.1}$$

By a *pseudonorm* over a finitely dimensional vector space V, we mean a non-negative absolutely homogeneous continuous function over V, which is equal to 0 only at $0 \in V$. In other words, it is a norm with no convexity assumption.

Directly from (4.17), cf. [21, Theorem 4.1], we see that the sequence of Fubini-Study metrics, $FS(N_k)$, $k \in \mathbb{N}^*$, see (4.16) for a definition, is submultiplicative for any submultiplicative graded pseudonorm $N = \sum N_k$. This means that for any $k, l \in \mathbb{N}^*$, the following inequalities $FS(N_{k+l}) \leq FS(N_k) \cdot FS(N_l)$ are satisfied. In particular, by Fekete's lemma, the sequence of metrics $FS(N_k)^{\frac{1}{k}}$ on L converges, as $k \to \infty$, to an upper semi-continuous metric (possibly only bounded from above and even null), which we denote by $FS(N)$. If $FS(N)$ is, moreover, lower semi-continuous and everywhere non-null, then by Dini's theorem, the convergence is uniform and $FS(N)$ is psh.

Now, for a fixed continuous metric h^L on L, we denote the induced graded L^∞-norm on $R(X, L)$ by $\mathrm{Ban}^\infty(h^L)$. Clearly, $\mathrm{Ban}^\infty(h^L)$ is an example of a submultiplicative norm. One of the main results of [21] says that essentially, these are the only possible examples up to the equivalence relation, defined in (4.12).

Theorem 5 ([21, Theorem 1.1]) *Assume that a graded norm $N = \sum N_k$ over the section ring $R(X, L)$ of an ample line bundle L is submultiplicative, and the metric $FS(N)$ on L is continuous and non-null everywhere. Then $N \sim \mathrm{Ban}^\infty(FS(N))$.*

Remark 3 There are examples of graded submultiplicative norms N with bounded metrics $FS(N)$, for which the statement doesn't hold, see [21, Proposition 3.9], which means that continuity of $FS(N)$ is a necessary assumption. As we prove, however, in [21, Theorem 1.3] a weaker equivalence still holds for norms with bounded metrics $FS(N)$.

Before describing a proof of Theorem 5, let us draw a consequence from it. We fix a compact complex manifold X of dimension n and denote by $K_X := \Lambda^n T^* X$ its canonical line bundle. Narasimhan-Simha in [31] defined pseudonorms $\mathcal{NS}_k := \| \cdot \|_k^{\mathcal{NS}}$, $k \in \mathbb{N}^*$ over the vector space of k-th *pluricanonical sections*, $f \in H^0(X, K_X^k)$, as

$$\|f\|_k^{\mathcal{NS}} := \Big(\int_X \big((-\sqrt{-1})^{k(n^2+2n)} \cdot f \wedge \overline{f} \big)^{\frac{1}{k}} \Big)^{\frac{k}{2}}. \tag{5.2}$$

Over *canonical ring*, $R(X, K_X)$, we then define the Narasimhan-Simha graded pseudonorm

$$\mathcal{NS} := \sum_{k=1}^{\infty} \mathcal{NS}_k. \tag{5.3}$$

Clearly from (5.2), the sequence of pseudonorms $\mathcal{NS}_k$, $k \in \mathbb{N}^*$, is defined without the use of any fixed metric on K_X. In particular, it depends only on the complex structure of X. Even more, it is a *birational invariant*, as birational equivalence between two complex manifolds X and Y induces the isometry with respect to the respective Narasimhan-Simha pseudonorms between $H^0(X, K_X^k)$ and $H^0(Y, K_Y^k)$ for any $k \in \mathbb{N}^*$, cf. [31].

It was, moreover, proved by Royden [38] in dimension 1 and later generalized by Chi [11, Theorem 1.4] in higher dimensions that for any canonically polarized manifolds X and Y of the same dimension, there are $k_0, k_1 \in \mathbb{N}^*$, depending only on the dimensions of X and Y, such that any isometry between vector spaces $H^0(X, K_X^k)$ and $H^0(Y, K_X^k)$, endowed with the respective Narasimhan-Simha pseudonorms, for $k \geq k_0$ and divisible by k_1, is necessarily produced from an isomorphism between X and Y. In this way, studying the isometry type of Narasimhan-Simha pseudonorms seems to be a complicated task related to birational classification of manifolds. This is why we naturally would like to understand a simpler problem of the asymptotics of $\mathcal{NS}$.

One can expect that the graded pseudonorms $\mathcal{NS}$ would be equivalent to the simplest submultiplicative norm, $\mathrm{Ban}^{\infty}(h^K)$, for a certain "canonical" metric h^K on K_X. This is, however, not the case. In fact, for any $f \in H^0(X, K_X^k)$, $k \in \mathbb{N}^*$, both the Narasimhan-Simha pseudonorms and sup-norms behave multiplicatively on the sequence f^l for $l \in \mathbb{N}^*$. Hence, if $\mathcal{NS}$ and $\mathrm{Ban}^{\infty}(h^K)$ were equivalent, it would imply that $\mathcal{NS}$ and $\mathrm{Ban}^{\infty}(h^K)$ coincide identically with the sup-norm, which is false. Nevertheless, as we shall show below, one can still understand the asymptotics of $\mathcal{NS}$, yet in a somewhat less refined way.

To explain this, recall that one can naturally associate the *convex hull* norm $\mathrm{Conv}(N_V) := \|\cdot\|_V^{\mathrm{conv}}$ for any pseudonorm $N_V := \|\cdot\|_V$ over a finitely dimensional vector space V as follows

$$\|v\|_V^{\mathrm{conv}} = \inf\Big\{\sum \|v_i\|_V : \quad \sum v_i = v\Big\}. \tag{5.4}$$

Geometrically, the unit ball of $\mathrm{Conv}(N_V)$ is the convex hull of the unit ball of N_V. We extend this definition to graded pseudonorms by considering the convex hulls of graded pieces.

Recall also that any psh metric h^K on K_X induces a volume form (possibly, with zeros) on X, denoted by dV_{h^K}. More precisely, assume first that h_0^K is a smooth metric on K_X. We define the positive volume form $dV_{h_0^K}$ for any $x \in X$ as

$$dV_{h_0^K}(x) = (-\sqrt{-1})^{n^2+2n} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n, \tag{5.5}$$

where $|dz_1 \wedge \cdots \wedge dz_n|_{h_0^K}(x) = 1$. This construction can be extended to psh metrics h^K on K_X by writing $h^K = e^{-\phi} \cdot h_0^K$ for $\phi \in L^1_{\mathrm{loc}}$ and defining $dV_{h^K} := e^{\phi} \cdot dV_{h_0^K}$. Clearly, the result doesn't depend on the choice of h_0^K. The volume form dV_{h^K} is bounded since any quasi-psh function ϕ is bounded. It might, nevertheless, vanish, as ϕ is allowed to take $-\infty$ values.

Tsuji in [41] defined the supercanonical metric h^K_{can} on K_X for manifolds X with psef K_X through the following envelope construction: for $x \in X$, we let

$$h^K_{\mathrm{can}}(x) = \inf\Big\{h^K(x) : \quad h^K \text{ is a psh metric on } K_X, \text{ with } \int_X dV_{h^K} \leq 1\Big\}. \tag{5.6}$$

In general, it seems, we cannot expect some regularity for h^K_{can}. This is due to the fact that the core argument in proving regularity of envelopes from Bedford-

Taylor's theory [2] is a possibility of constructing new contestants for the infimum (5.6) from the existent ones. This is usually done by taking a pointwise minimum of two different contesting metrics. However, if two psh metrics h_0^K, h_1^K verify the assumptions from the right-hand side of (5.6), the minimum of them, while being psh, does not necessarily verify the L^1-condition from (5.6). This contrasts a lot with the envelopes associated to L^∞-condition, for which the $\mathscr{C}^{1,1}$-regularity can be established, see Berman [4].

Berman-Demailly, however, showed in [5, Theorem 5.5] that despite the above difficulties, h^K_{can} is continuous and psh for *canonically polarized* X (i.e. when K_X is ample). Moreover, in [5, Proposition 5.19], they established that the sequence of metrics $FS(\mathcal{NS}_k)^{\frac{1}{k}}$, $k \in \mathbb{N}^*$ on K_X converges uniformly, as $k \to \infty$, to the super-canonical metric, h^K_{can}.

Note that for trivial reasons, cf. [21, (2.4)], for any vector space V with a pseudonorm N_V, which we view as a pseudonorm on $H^0(\mathbb{P}(V^*), \mathscr{O}(1))$, we have $FS(N_V) = FS(\mathrm{Conv}(N_V))$. Hence, the above result of Berman-Demailly works equally well for Fubini-Study metrics associated with $\mathrm{Conv}(\mathcal{NS}_k)$.

Remark now that by Hölder's inequality, the graded norm $\mathcal{NS}$ is submultiplicative. A trivial verification shows that in general, a convex hull of a submultiplicative norm is submultiplicative as well. In particular, from this observation, Theorem 5 and the aforementioned results of Berman-Demailly, we deduce, see [21, Theorem 1.1], that the following equivalence holds

$$\mathrm{Conv}(\mathcal{NS}) \sim \mathrm{Ban}^\infty(h^K_{\mathrm{can}}). \tag{5.7}$$

This refines the result of Berman-Demailly from weaker level of Fubini-Study convergence to stronger level of norm equivalences.

Let us say few words about the proof of Theorem 5. In fact, once the correct language of projective tensor norms is used, the proof essentially goes by the same arguments as the proof of Theorem 4 with only one serious modification.

To explain this better, recall that there is no single canonical construction of a norm on the tensor product $V_1 \otimes V_2$ of two finitely dimensional vector spaces V_i, $i = 1, 2$, endowed with norms $N_i = \|\cdot\|_i$. Instead, several definitions are widely used. The *projective tensor norm* $N_1 \otimes_\pi N_2 = \|\cdot\|_{\otimes_\pi}$ on $V_1 \otimes V_2$ is defined for $f \in V_1 \otimes V_2$ as

$$\|f\|_{\otimes_\pi} = \inf \Big\{ \sum \|x_i\|_1 \cdot \|y_i\|_2 : \quad f = \sum x_i \otimes y_i \Big\}, \tag{5.8}$$

where the infimum is taken over different ways of partitioning f into a sum of decomposable terms. The *injective tensor norm* $N_1 \otimes_\epsilon N_2 = \|\cdot\|_{\otimes_\epsilon}$ on $V_1 \otimes V_2$ is defined as

$$\|f\|_{\otimes_\epsilon} = \sup \Big\{ \big|(\phi \otimes \psi)(f)\big| : \quad \phi \in V_1^*, \psi \in V_2^*, \|\phi\|_1^* = \|\psi\|_2^* = 1 \Big\} \tag{5.9}$$

where $\|\cdot\|_i^*$, $i = 1, 2$, are the dual norms associated with $\|\cdot\|_i$.

The crucial step is then to realize that in the notations (4.4) and (5.1), the submultiplicativity condition states that the following inequality between the norms on $H^0(X, L^{k+l})$ is satisfied

$$N_{k+l} \leq [N_k \otimes_\pi N_l]. \tag{5.10}$$

By induction, we then see that for submultiplicative norms the analogue of the lower bound from the definition of multiplicatively generated norms, (4.13), is satisfied for the auxiliary function $f = 0$, once we change the Hermitian tensor product norm by the projective tensor product norm. This describes the formal analogy between Theorems 4 and 5.

As in the proof of Theorem 4, by an application of the semiclassical Ohsawa-Takegoshi extension theorem to the Kodaira embedding, we reduce the proof of the general statement of Theorem 5 to the corresponding statement on a projective space, endowed with a Fubini-Study metric associated with a normed projective space, see [21, Sect. 5] for details.

The crucial difference between the proofs of Theorems 4 and 5 is in the way we tackle projective spaces. This is due to the fact that in the setting of Theorem 5, we cannot do explicit calculations as it was done in the proof of Theorem 4, as we cannot trivialize an arbitrary *normed* vector space (in contrast with *Hermitian* vector spaces).

The proof of Theorem 5 for projective spaces, as we explain in [21, Sect. 5], essentially boils down to showing a certain functional-analytic statement about finitely dimensional normed complex vector spaces. More precisely, we show that it is essentially sufficient to establish that the multiplicative gap between two natural norms on symmetric tensor algebra of a fixed finitely dimensional normed complex vector space (V, N_V) is subexponential. The first norm we consider, $\mathrm{Sym}_\epsilon(N_V) = \sum \mathrm{Sym}^k_\epsilon(N_V)$, is induced by the injective tensor product norm. The second norm, $\mathrm{Sym}_\pi(N_V) = \sum \mathrm{Sym}^k_\pi(N_V)$, is induced by the projective tensor product norm. The statement then says that similarly to (4.12), for any $\epsilon > 0$, there is $k_1 \in \mathbb{N}$, such that for any $k \geq k_1$, we have

$$\mathrm{Sym}^k_\pi(N_V) \cdot \exp(-\epsilon k) \leq \mathrm{Sym}^k_\epsilon(N_V) \leq \mathrm{Sym}^k_\pi(N_V). \tag{5.11}$$

The upper bound corresponds to a well-known statement, saying that a projective tensor norm dominates the injective tensor norm.

Recall that from the work of Pisier [35, Théorème 3.1], the multiplicative gap between injective and projective tensor norms on the full tensor algebra of a fixed normed vector space is always exponential as long as the dimension of the vector space is strictly bigger than 1. This is why we find the above result rather unexpected as it tells that the restrictions of those non-equivalent norms to symmetric tensors suddenly become equivalent.

The proof of (5.11) is done in two steps. First, we verify it for one specific normed vector space $(U, N_U) = (\mathbb{C}^n, l_1)$, where l_1 is the L^1-norm given by the sum of the coordinates. It turns out that in this case the statement is equivalent to proving that for any $\epsilon > 0$, there is $k_1 \in \mathbb{N}$, such that for any $k \geq k_1$, and for any homogeneous polynomial P of degree k in n variables, written as

$$P(x_1, \cdots, x_n) = \sum_{|\alpha|=k} a_\alpha x^\alpha, \tag{5.12}$$

the following inequality is satisfied

$$\sum_{|\alpha|=k} |a_\alpha| \leq \exp(\epsilon k) \cdot \sup_{\substack{x_i \in \mathbb{C} \\ |x_i| \leq 1}} \big| P(x_1, \cdots, x_n) \big|. \tag{5.13}$$

As we explain in [21, Sect. 6], the bound (5.13) follows from the optimal estimates in Bohnenblust-Hille inequality, see [8] and [1, Corollary 5.3].

In the second step, we deduce the general case from the already established one for the l_1-norm. For this, let us recall that for any $\epsilon > 0$, a classical lemma from functional analysis states that there is $l \in \mathbb{N}^*$ and a surjective map $\pi : U \to V$, $U := \mathbb{C}^l$, such that N_V is related to the quotient norm associated with the l_1-norm N_U on U as follows

$$\exp(-\epsilon) \cdot [N_U] \leq N_V \leq [N_U], \tag{5.14}$$

where we used the notations from (4.4) for $[N_U]$.

Quite surprisingly, as we explain in [21, Sect. 7], for the proof of (5.11) for N_V as above, it suffices to apply the semiclassical version of Ohsawa-Takegoshi extension theorem for the embedding $\mathrm{Im}_\pi : \mathbb{P}(V^*) \to \mathbb{P}(U^*)$, where (U, N_U) and the surjective projection $\pi : U \to V$ are as in (5.14), and use the fact that $\mathrm{Sym}_\epsilon(N_U)$ and $\mathrm{Sym}_\pi(N_U)$ are equivalent by the first step of the proof.

To conclude, remark that since our arguments relied heavily on the use of the semiclassical version of Ohsawa-Takegoshi extension theorem, the above proof of (5.11) doesn't adapt to the setting of normed *real* vector spaces. This is hardly surprising since there are examples of normed real vector spaces (V, N_V) for which the induced graded norms $\mathrm{Sym}_\epsilon(N_V)$ and $\mathrm{Sym}_\pi(N_V)$ on symmetric tensors are *not* equivalent, see [21, Remark 5.2b) and 5.3]. We find it utterly remarkable that a purely functional-analytic statement like (5.11) depends somehow on the fact that the vector space V is complex.

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Balanced Metrics for Extremal Kähler Metrics and Fano Manifolds

Yoshinori Hashimoto

Abstract The first three sections of this paper are a survey of the author's work on balanced metrics and stability notions in algebraic geometry. The last section is devoted to proving the well-known result that a geodesically convex function on a complete Riemannian manifold admits a critical point if and only if its asymptotic slope at infinity is positive, where we present a proof which relies only on the Hopf–Rinow theorem and extends to locally compact complete length metric spaces.

Keywords Extremal Kähler metrics · Balanced metrics · Chow stability

1 Introduction

Let (X, L) be a pair of a compact Kähler manifold X of complex dimension n and an ample line bundle L, with a (fixed reference) Kähler metric $\omega \in c_1(L)$. We assume ω is the Kähler form associated to the (fixed reference) hermitian metric h_{ref} on L, i.e.

$$\omega = \frac{1}{2\pi\sqrt{-1}}\partial\bar{\partial}\log h_{\mathrm{ref}}.$$

Another hermitian metric, say $h := e^{-\phi}h_{\mathrm{ref}}$, defines another Kähler metric

$$\omega_h = \omega + \sqrt{-1}\partial\bar{\partial}\phi$$

assuming that ω_h is positive definite; we shall also write ω_ϕ for $\omega + \sqrt{-1}\partial\bar{\partial}\phi$.

We define $N_k := \dim_{\mathbb{C}} H^0(X, L^k)$.

Definition 1 Let h be a hermitian metric such that $\{s_i\}_{i=1}^{N_k}$ is a basis for $H^0(X, L^k)$ that is orthonormal with respect to the inner product $\int_X h^k(\cdot,\cdot)\omega_h^n/n!$. The **Bergman function** $\rho_k(\omega_h) \in C^\infty(X, \mathbb{R})$ is defined as

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$$\rho_k(\omega_h) := \sum_{i=1}^{N_k} |s_i|^2_{h^k}.$$

The right hand side of the equation above depends on h, but it turns out that it is invariant under an overall multiplicative constant and hence depends only on the associated Kähler metric ω_h. The Bergman function is the diagonal of the Bergman kernel. It is also called the density-of-states function or the distortion function.

The Bergman function (or more generally the Bergman kernel) has many properties as discussed in other chapters in this volume. An important result that we use in this chapter is the following.

Theorem 1 (Bouche [7], Tian [53], Yau [55], Zelditch [56], Catlin [11], Ruan [41], Lu [29], Ma–Marinescu [31], amongst many others) *The Bergman function admits an asymptotic expansion*

$$\rho_k(\omega_h) = k^n + \frac{k^{n-1}}{4\pi} S(\omega_h) + O(k^{n-2})$$

when $k \in \mathbb{N}$ is sufficiently large, where $S(\omega_h)$ is the scalar curvature of ω_h defined by

$$S(\omega_h) := n \frac{\mathrm{Ric}(\omega_h) \wedge \omega_h^{n-1}}{\omega_h^n}$$

where $\mathrm{Ric}(\omega_h)$ *is the Ricci curvature of ω_h, defined locally as*

$$\mathrm{Ric}(\omega_h) := -\sqrt{-1}\partial\bar{\partial} \log \omega_h^n.$$

The theorem above has many important implications and consequences, but for the moment we simply observe the appearance of the scalar curvature in the expansion above.

We recall that a Kähler metric $\omega_h \in c_1(L)$ satisfying $S(\omega_h) = \text{const}$ is said to be a **constant scalar curvature Kähler** (or **cscK**) metric. A foundational theorem of Donaldson concerning the cscK metrics states the following.

Theorem 2 (Donaldson [15]) *Suppose that* $\mathrm{Aut}_0(X, L)$ *is trivial and that X admits a cscK metric ω_{cscK} in the Kähler class $c_1(L)$. Then, there exists $k_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ with $k \geq k_0$ there exists a Kähler metric $\omega_k \in c_1(L)$ such that*

$$\rho_k(\omega_k) = \text{const}.$$

and $\omega_k \to \omega_{\mathrm{cscK}}$ in C^∞ as $k \to \infty$.

In the above, $\mathrm{Aut}_0(X, L)$ stands for the connected component of the automorphism group of (X, L), i.e. holomorphic automorphisms of X which lift to the total space of L. A Kähler metric $\omega_h \in c_1(L)$ satisfying the equation $\rho_k(\omega_h) = \text{const}$ is said to be **balanced** at the level k.

The above theorem has an important application to the stability of (X, L) in the sense of Geometric Invariant Theory, which we recall in what follows. We first recall the following important result due to Luo [30] and Zhang [57].

Theorem 3 (Luo [30], Zhang [57]) *Suppose that* $\mathrm{Aut}_0(X, L)$ *is trivial. There exists a balanced metric at level* k *in* $c_1(L)$ *if and only if* (X, L^k) *is Chow stable.*

Theorems 2 and 3 together imply that a polarised Kähler manifold (X, L) with discrete automorphism group which admits a cscK metric is asymptotically Chow stable, i.e. (X, L^k) is Chow stable for all large enough k. Chow stability that appeared in the above theorem is a classical stability notion for polarised varieties (see Remark 1), but we present a formulation that involves test configurations that is appropriate for our discussions later.

Definition 2 A **very ample test configuration** $(\mathcal{X}, \mathcal{L})$ **of exponent** k for (X, L) is a scheme $\mathcal{X}$ endowed with a flat projective morphism $\pi : \mathcal{X} \to \mathbb{C}$, which is $\mathbb{C}^*$-equivariant with respect to the natural $\mathbb{C}^*$-action on $\mathbb{C}$, with a relatively very ample Cartier divisor $\mathcal{L}$ to which the action $\mathbb{C}^* \curvearrowright \mathcal{X}$ linearises, such that $\pi^{-1}(1) \cong X$ and $\mathcal{L}|_{\pi^{-1}(1)} \cong L^k$. The preimage of $0 \in \mathbb{C}$, written $\mathcal{X}_0 := \pi^{-1}(0)$, is called the **central fibre**.

We say that $(\mathcal{X}, \mathcal{L})$ is **product** if $\mathcal{X}$ is isomorphic to $X \times \mathbb{C}$, and **trivial** if it is $\mathbb{C}^*$-equivariantly isomorphic to $X \times \mathbb{C}$ (i.e. $\mathcal{X} \xrightarrow{\sim} X \times \mathbb{C}$ and $\mathbb{C}^*$ acts trivially on X).

Suppose that we have a very ample test configuration $(\mathcal{X}, \mathcal{L})$ of exponent k. By definition $(\mathcal{X}, \mathcal{L})$ is endowed with a $\mathbb{C}^*$-action. There exists an embedding $\mathcal{X} \hookrightarrow \mathbb{P}(H^0(X, L^k)^\vee)$ by [40, Proposition 3.7] such that the generator of the $\mathbb{C}^*$-action is given by $A_k \in \mathfrak{gl}(H^0(X, L^k))$, acting linearly on $\mathbb{P}(H^0(X, L^k)^\vee)$, in the sense that $\mathcal{X}$ is equal to the flat closure of the $\mathbb{C}^*$-orbit of $X \hookrightarrow \mathbb{P}(H^0(X, L^k)^\vee))$ generated by A_k (with the $\mathbb{C}^*$-action $e^{A_k t}$). Moreover, its central fibre $\mathcal{X}_0 := \pi^{-1}(0)$ is equal to the flat limit of this $\mathbb{C}^*$-orbit (cf. [51, Sect. 6.2]).

Given $(\mathcal{X}, \mathcal{L})$, we can construct a sequence of very ample test configurations $(\mathcal{X}, \mathcal{L}^{\otimes m})$ for $m \in \mathbb{N}$, each of exponent mk. As above we can write it as a flat closure of the $\mathbb{C}^*$-orbit of $X \hookrightarrow \mathbb{P}(H^0(X, L^{mk})^\vee)$ generated by $A_{mk} \in \mathfrak{gl}(H^0(X, L^{mk}))$, say. Since the central fibre $\mathcal{X}_0$ is the flat limit of the $\mathbb{C}^*$-action generated by A_{mk}, there is a natural $\mathbb{C}^*$-action $\mathbb{C}^* \curvearrowright H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k}|_{\mathcal{X}_0})$ generated by $A_{mk} \in \mathfrak{gl}(H^0(X, L^{mk})) = \mathfrak{gl}(H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k}|_{\mathcal{X}_0}))$, by noting the isomorphism $H^0(X, L^{mk}) \xrightarrow{\sim} H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k}|_{\mathcal{X}_0})$.

By Riemann–Roch and equivariant Riemann–Roch, we write

$$\dim H^0(X, L^{mk}) = a_0(mk)^n + a_1(mk)^{n-1} + \cdots, \tag{1}$$

$$\mathrm{tr}(A_{mk}) = b_0(mk)^{n+1} + b_1(mk)^n + \cdots. \tag{2}$$

Observe that a_0 is equal to the volume $\int_X c_1(L)^n/n!$ of (X, L).

Definition 3 Let $(\mathcal{X}, \mathcal{L})$ be a very ample test configuration for (X, L) of exponent k. The **Chow weight** of $(\mathcal{X}, \mathcal{L})$ is defined by

$$\mathrm{Chow}_k(\mathcal{X}, \mathcal{L}) := kb_0 - \frac{a_0 \mathrm{tr}(A_k)}{\dim H^0(X, L^k)}.$$

Definition 4 A polarised Kähler manifold (X, L) is said to be:

1. **Chow semistable at the level** k if $\mathrm{Chow}_k(\mathcal{X}, \mathcal{L}) \geq 0$ for any very ample test configuration $(\mathcal{X}, \mathcal{L})$ of exponent k,
2. **Chow polystable at the level** k if (X, L) is Chow semistable at the level k and $\mathrm{Chow}_k(\mathcal{X}, \mathcal{L}) = 0$ holds if and only if $(\mathcal{X}, \mathcal{L})$ is product,
3. **Chow stable at the level** k if (X, L) is Chow semistable at the level k and $\mathrm{Chow}_k(\mathcal{X}, \mathcal{L}) = 0$ holds if and only if $(\mathcal{X}, \mathcal{L})$ is trivial,
4. **Chow unstable at the level** k if it is not Chow semistable at the level k.

Remark 1 It is well-known that the Chow stability as defined above is equivalent to the more conventional definition using the Chow form of the embedded manifold $X \hookrightarrow \mathbb{P}(H^0(X, L^k)^\vee)$; see e.g. [21, Sect. 2] or [38, Proposition 2.11].

A stability notion that is relevant to the cscK metrics is the K-stability introduced by Tian [54] and Donaldson [16].

Definition 5 The **Donaldson–Futaki invariant** $DF(\mathcal{X}, \mathcal{L})$ is defined as

$$DF(\mathcal{X}, \mathcal{L}) = \frac{a_1 b_0 - a_0 b_1}{a_0} = \lim_{k \to \infty} \mathrm{Chow}_{mk}(\mathcal{X}, \mathcal{L}^{\otimes m}).$$

K-stability can be defined analogously to Chow stability, using the Donaldson–Futaki invariant instead; a crucial difference is that we need to consider test configurations of *all* exponents, as opposed to a fixed one, to check K-stability.

Definition 6 A polarised Kähler manifold (X, L) is said to be K**-semistable** if $DF(\mathcal{X}, \mathcal{L}) \geq 0$ holds for any test configuration. (X, L) is K**-polystable** if it is K-semistable and $DF(\mathcal{X}, \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is product, K**-stable** if it is K-semistable and $DF(\mathcal{X}, \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is trivial.

Theorems 2 and 3 are known to imply that a polarised Kähler manifold (X, L) is K-semistable, as proved by Ross–Thomas [40]; Donaldson [17] later gave an alternative proof using the Calabi energy. This result can be further strengthened to yield K-stability, if $\mathrm{Aut}_0(X, L)$ is trivial, by the theorem of Stoppa [48] which also relies on the result of Arezzo–Pacard [1]. These results can be summarised as follows.

$$
\begin{tikzcd}
(X, L) \text{ has a cscK metric.} \arrow[rr, Rightarrow, "\text{Donaldson [15]}"] \arrow[d, Rightarrow, "\text{Donaldson [17]}"'] & & \rho_k(\omega_k) = \text{const } k \gg 1. \arrow[d, Leftrightarrow, "\text{Luo [30], Zhang [57]}"] \\
K\text{-semistability} \arrow[d, Rightarrow, "\text{Stoppa [48]}"'] & & \text{Asymptotic Chow stability} \arrow[ll, Rightarrow, "\text{Ross–Thomas [40]}"'] \\
K\text{-stability} & &
\end{tikzcd}
$$

An improvement of the result stated above, which proves the uniform K-stability by employing a variational approach, was obtained by Berman–Darvas–Lu [4]. There are many related results, e.g. [9, 14, 46, 47] amongst many others, but we omit the details.

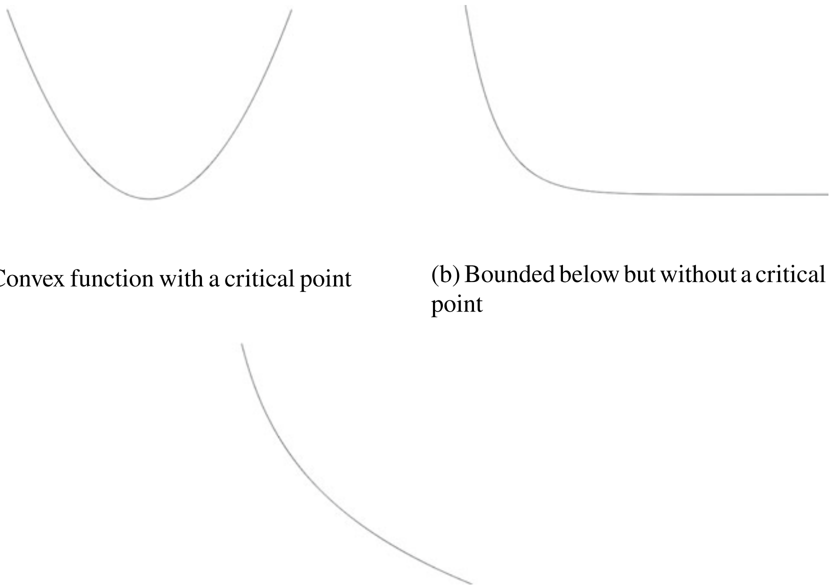
We finally recall that balanced metrics are Fubini–Study metrics, i.e. there exists a hermitian inner product on $\mathbb{C}^{N_k} = H^0(X, L^k)$ such that the Kodaira embedding

$$\iota_k : X \hookrightarrow \mathbb{P}(H^0(X, L^k)^\vee)$$

is an isometry with respect to the balanced metric on X. In other words, there exists a positive definite hermitian form H on $H^0(X, L^k)$ such that the balanced metric is equal to the pullback of the Fubini–Study metric on $\mathbb{P}(H^0(X, L^k)^\vee)$ defined by H.

The balanced metric is a critical point of an energy functional called the balancing energy, say $Z_k : Y_k \to \mathbb{R}$, where $Y_k := GL(N_k, \mathbb{C}))/U(N_k)$ is the set of all positive definite hermitian forms on $H^0(X, L^k)$. An important feature of the balancing energy is that it is convex along geodesics in the symmetric space Y_k with respect to the standard bi-invariant metric. Moreover it is well-known that the balancing energy is strictly convex along geodesics that are not contained in the $\mathrm{Aut}_0(X, L)$-orbit, which in particular shows that Z_k is strictly convex along geodesics if $\mathrm{Aut}_0(X, L)$ is trivial. Since Y_k with the bi-invariant metric is a complete Riemannian manifold, the results that we recall and prove in Sect. 4 show that (X, L) admits a balanced metric at the level k if and only if its asymptotic slope at infinity is strictly positive; see Fig. 1.

It turns out that the asymptotic slope of Z_k at infinity agrees with the Chow weight (Definition 3), for geodesic rays in Y_k generated by positive hermitian forms on $H^0(X, L^k)$ with integral eigenvalues (see [17, Proposition 3]). We can interpret Theorem 3 as a claim that the asymptotic slope of Z_k is positive along any geodesic rays in Y_k if and only if it is positive along any geodesic rays generated by "integral" generators (which correspond to algebraic weights).



(a) Convex function with a critical point

(b) Bounded below but without a critical point

(c) Unbounded below

Fig. 1 Asymptotic behaviours of convex functions

2 Balanced Metrics for Extremal Metrics

When $\mathrm{Aut}_0(X, L)$ is nontrivial, Theorem 2 still holds as long as a series of integral invariants called the higher Futaki invariant vanishes, as proved by Futaki [20] and Mabuchi [32, 34]. However, there are examples with non-vanishing higher Futaki invariants by Ono–Sano–Yotsutani [39] and Della Vedova–Zuddas [12], and for these manifolds we need to have a new definition of balanced metrics. It turns out that these results hold more widely for extremal metrics, which generalise cscK metrics, defined as follows.

Definition 7 A Kähler metric $\omega \in c_1(L)$ which satisfies

$$\bar{\partial}\mathrm{grad}^{1,0}_{\omega} S(\omega) = 0$$

is called **extremal** (i.e. (1,0)-part of the gradient vector field $\mathrm{grad}_{\omega} S(\omega)$ is a holomorphic vector field).

The extremal metric is a critical point of the Calabi energy, and can be regarded as a generalisation of the cscK metrics. When we have a non-cscK extremal metric, note that the automorphism group of X must be non-discrete.

In what follows, we assume that L is very ample and that there exists a faithful representation $\mathrm{Aut}(X, L) \hookrightarrow GL(H^0(X, L))$ such that the associated linear action

$\mathrm{Aut}(X, L) \curvearrowright \mathbb{P}(H^0(X, L)^\vee)$ induces the automorphism of the embedded submanifold $\iota(X) \subset \mathbb{P}(H^0(X, L)^\vee)$. It is well-known that this can be achieved by replacing L by a higher tensor power.

It turns out there are multiple ways of generalising Theorem 2 to the case when $\mathrm{Aut}(X, L)$ is nontrivial, as briefly recalled later. One version of them which is proved by the author is the following.

Theorem 4 ([24]) *Suppose that X admits an extremal metric $\omega_{\mathrm{ext}} \in c_1(L)$. Then, there exists $k_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ with $k \geq k_0$ there exists a Kähler metric $\omega_k \in c_1(L)$ whose Bergman function $\rho_k(\omega_k)$ satisfies*

$$\bar{\partial}\mathrm{grad}^{1,0}_{\omega_k} \rho_k(\omega_k) = 0$$

and $\omega_k \to \omega_{\mathrm{ext}}$ in C^∞ as $k \to \infty$.

As we saw in the previous section, the result above has implications to stability; here we need to consider the "relative" versions of the stability notions defined above, such as the relative K-stability defined by Székelyhidi [50], which essentially subtracts the contribution of the automorphism group. The statement is more complicated, and other generalisations of Theorem 2 give slightly different conclusions. We summarise the known results in the following diagram; see the references cited below for the precise definitions and statements.

$$\begin{tikzcd}
 & \text{Asymptotic relative Chow stability} \arrow[d, Rightarrow, "\text{Mabuchi [37]}"] \\
(X, L) \text{ has an extremal metric.} \arrow[ur, Rightarrow, "\text{Mabuchi [37]}", "\text{Seyyedali [45]}"'] \arrow[r, Rightarrow, "\text{Mabuchi [33, 34, 35, 36]}"'] \arrow[d, Rightarrow, "\text{Stoppa–Székelyhidi [49]}"'] \arrow[dr, Rightarrow, "\text{H. [24]}"'] & \text{Asymptotic weak relative Chow stability} \\
\text{Relative } K\text{-semistability} \arrow[d, Rightarrow, "\text{Stoppa–Székelyhidi [49]}"'] & \bar{\partial}\mathrm{grad}^{1,0}_{\omega_k} \rho_k(\omega_k) = 0 \ k \gg 1. \arrow[l, Rightarrow, "\text{H. [22]}"] \arrow[u, Rightarrow, "\text{H. [22]}"'] \\
\text{Relative } K\text{-stability} &
\end{tikzcd}$$

There is also a closely related result by Sano–Tipler [44] which is different from the above results. The results of Mabuchi [33–37], Seyyedali [45], Sano–Tipler [44] can all be stated in terms of the "weighted" Bergman function, but not quite the Bergman function itself (they are more natural in terms of the relative moment map). The reader is referred to the papers cited above for precise statements, and to [22, Sect. 6] and [24, Sect. 6] for the comparison of these results, which is rather technical; the only remark that we make here is that it is proved in [22] that Theorem 4 implies the asymptotic weak relative Chow stability and relative K-semistability, as indicated in the diagram above. Note also that there is another proof due to Dervan [13] for the extremal metrics implying relative K-stability, which applies more generally to compact (non-projective) Kähler manifolds.

3 Anticanonically Balanced Metrics

There is another version of balanced metrics that is adapted to Fano manifolds, which was introduced by Donaldson [18, Sect. 2.2.2] and studied further by Berman–Boucksom–Guedj–Zeriahi [2].

Let $(X, -K_X)$ be an (anticanonically polarised) Fano manifold. Recall that a hermitian metric h on $-K_X$ naturally defines a volume form $d\mu_h$ on X via the natural isomorphism $\mathcal{H}om_{C^\infty_X}((-K_X) \otimes \overline{(-K_X)}, \mathbb{C}) \xrightarrow{\sim} K_X \otimes \overline{K_X}$.

Definition 8 Let $\{\sigma_i\}_{i=1}^{N_k}$ be a basis for $H^0(X, (-K_X)^k)$ that is orthonormal with respect to the inner product $\int_X h^k(\cdot, \cdot) d\mu_h$ (note the volume form).

The **anticanonical Bergman function** $\rho_k^{\mathrm{ac}}(\omega_h) \in C^\infty(X, \mathbb{R})$ is defined to be

$$\rho_k^{\mathrm{ac}}(\omega_h) := \sum_{i=1}^{N_k} |\sigma_i|^2_{h^k}.$$

Definition 9 A Kähler metric $\omega_h \in c_1(-K_X)$ satisfying $\rho_k^{\mathrm{ac}}(\omega_h) = \mathrm{const}$ is said to be **anticanonically balanced** at level k.

The "anticanonical" version of Donaldson's quantisation was established by Berman–Witt Nyström [5].

Theorem 5 (Berman–Witt Nyström [5]) *Suppose that a Fano manifold $(X, -K_X)$ admits a Kähler–Einstein metric $\omega_{\mathrm{KE}} \in c_1(-K_X)$ and its automorphism groups is discrete. Then there exists a sequence $\{\omega_k\}_k$ of Kähler metrics in $c_1(-K_X)$ weakly converging to ω_{KE} in the sense of currents, whose anticanonical Bergman function is constant for all large enough k, i.e. $\rho_k^{\mathrm{ac}}(\omega_k) = \mathrm{const}$ for $k \gg 1$.*

While we focus on the case when $\mathrm{Aut}_0(X) = \mathrm{Aut}_0(X, -K_X)$ is trivial, Berman–Witt Nyström [5] proved an analogous result for the Kähler–Ricci g-solitons when $\mathrm{Aut}_0(X)$ is nontrivial. They proved the convergence in terms of currents, but this can be improved to the smooth convergence by Takahashi [52, Theorem 1.3 with $N = 1$] and Ioos [26, 27]. The result recently proved by the author is the following, which can be regarded as an anticanonical version of Theorem 3.

Theorem 6 ([23]) *Let $k \in \mathbb{N}$ be large enough such that $(-K_X)^k$ is very ample. A Fano manifold $(X, -K_X)$ with the discrete automorphism group admits a unique anticanonically balanced metric at level k if and only if it satisfies the following stability condition: for any very ample test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ of exponent k we have $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) + \mathrm{Chow}_k(\mathcal{X}, \mathcal{L}) \geq 0$, with equality if and only if $(\mathcal{X}, \mathcal{L})$ is trivial.*

One direction of the above result, i.e. the existence of anticanonically balanced metrics implying the stated stability condition, was proved by Saito–Takahashi [43, Theorem 1.2]. Thus the main point of the above theorem is the stability implying the anticanonically balanced metrics, although the proof in [23] easily establishes both directions.

The definition of the Ding invariant that appeared in the above statement is as follows.

Definition 10 Let $(\mathcal{X}, \mathcal{L})$ be a very ample test configuration for $(X, -K_X)$ of exponent k, and $\nu : \mathcal{X}^\nu \to \mathcal{X}$ be its normalisation with $\mathcal{L}^\nu := \nu^* \mathcal{L}$. Let $D_{(\mathcal{X}^\nu, \mathcal{L}^\nu)}$ be a $\mathbb{Q}$-divisor on $\mathcal{X}^\nu$, whose support is contained in $\mathcal{X}^\nu_0 := (\nu \circ \pi)^{-1}(0)$, such that $-k(K_{\mathcal{X}^\nu/\mathbb{C}} + D_{(\mathcal{X}^\nu, \mathcal{L}^\nu)})$ is a Cartier divisor corresponding to $\mathcal{L}^\nu$; it is well-known that such a $\mathbb{Q}$-divisor $D_{(\mathcal{X}^\nu, \mathcal{L}^\nu)}$ exists uniquely. The **Ding invariant** of $(\mathcal{X}, \mathcal{L})$ is a real number defined by

$$\mathrm{Ding}(\mathcal{X}, \mathcal{L}) := -\frac{b_0}{a_0} - 1 + \mathrm{lct}(\mathcal{X}^\nu, D_{(\mathcal{X}^\nu, \mathcal{L}^\nu)}; \mathcal{X}^\nu_0),$$

where $\mathrm{lct}(\mathcal{X}^\nu, D_{(\mathcal{X}^\nu, \mathcal{L}^\nu)}; \mathcal{X}^\nu_0)$ is the log canonical threshold of $\mathcal{X}^\nu_0$ with respect to $(\mathcal{X}^\nu, D_{(\mathcal{X}^\nu, \mathcal{L}^\nu)})$, which is defined as

$$\mathrm{lct}(\mathcal{X}^\nu, D_{(\mathcal{X}^\nu, \mathcal{L}^\nu)}; \mathcal{X}^\nu_0) := \sup\{c \in \mathbb{R} \mid (\mathcal{X}^\nu, D_{(\mathcal{X}^\nu, \mathcal{L}^\nu)} + c\mathcal{X}^\nu_0) \text{ is sub log canonical.}\}.$$

We can define the Ding stability by using the above invariant, which is an important stability notion concerning the Kähler–Einstein metrics, but we omit the details.

The main application of Theorem 6, combined with a result by Rubinstein et al. [42], is the following.

Corollary 1 (Rubinstein et al. [42], Hashimoto [23]) *Let $(X, -K_X)$ be a Fano manifold with the discrete automorphism group. For $k \in \mathbb{N}$ large enough such that $(-K_X)^k$ is very ample, the δ_k-invariant of Fujita–Odaka satisfies $\delta_k > 1$ if and only if $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) + \mathrm{Chow}_k(\mathcal{X}, \mathcal{L}) > 0$ for any nontrivial very ample test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ of exponent k.*

The δ_k-invariant is an algebraic invariant defined by Fujita–Odaka [19] and the above statement is purely algebro-geometric, but it seems that no purely algebro-geometric proof is known at the moment of writing this paper; the proof in [23] relies on Theorem 6 and [42, Theorem 2.3] which concern the anticanonically balanced metrics in differential geometry. The summary of the results so far are as follows.

$$
\begin{tikzcd}
\mathrm{Ding}(\mathcal{X}, \mathcal{L}) + \mathrm{Chow}_k(\mathcal{X}, \mathcal{L}) > 0 \arrow[rr, Leftrightarrow, "\text{No algebraic proof known}"'] \arrow[d, Rightarrow, shift right=2, "\text{H. [23]}"'] & & \delta_k > 1 \arrow[dll, Leftrightarrow, "\text{Rubinstein–Tian–Zhang [42]}"] \\
\exists \text{ anticanonically balanced metric} \arrow[u, Rightarrow, shift right=2, "\text{Saito–Takahashi [43]}"'] & &
\end{tikzcd}
$$

Note that an analogous result for Kähler–Einstein metrics and stability is known, in which we replace δ_k by the delta invariant $\delta := \lim_{k\to\infty} \delta_k$ and $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) + \mathrm{Chow}_k(\mathcal{X}, \mathcal{L}) > 0$ by the uniform Ding stability; see Berman–Boucksom–Jonsson [3]. In this case, the equivalence between $\delta > 1$ and the uniform Ding stability can be proved by methods in algebraic geometry [6, 19].

Similarly to the usual balanced metrics, the anticanonically balanced metrics are Fubini–Study metrics. There exists an "anticanonical" version of the balancing energy, say $Z_k^{\mathrm{ac}} : Y_k \to \mathbb{R}$, where $Y_k = GL(N_k, \mathbb{C}))/U(N_k)$ is the set of all positive definite hermitian forms on $H^0(X, (-K_X)^k)$. Z_k^{ac} is convex along geodesics on Y_k and strictly convex along geodesics that are not contained in the $\mathrm{Aut}_0(X)$-orbit. Thus, just as discussed at the end of Sect. 1, the anticanonically balanced metric exists if and only if the asymptotic slope of Z_k^{ac} is strictly positive along geodesics (by the results in Sect. 4). Similarly to the case of the usual balanced metrics, the asymptotic slope of Z_k^{ac} at infinity agrees with the invariant $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) + \mathrm{Chow}_k(\mathcal{X}, \mathcal{L})$, which was first observed by Saito–Takahashi [43], for geodesic rays in Y_k generated by positive hermitian forms on $H^0(X, (-K_X)^k)$ with integral eigenvalues. Theorem 6 is proved by showing that the asymptotic slope of Z_k^{ac} is positive along any geodesic rays in Y_k if and only if it is positive along any geodesic rays generated by integral generators.

4 Geodesically Convex Functions on Complete Riemannian Manifolds

The balancing energy and its anticanonical version play a very important role in the study of (anticanonically) balanced metrics, whose important feature is the convexity along geodesics. We remarked at the end of Sects. 1 and 3 that the critical point of the (anticanonical) balancing energy exists if and only if its asymptotic slope is strictly positive along geodesics (see Fig. 1). While this fact is surely well-known to the experts, it seems that its detailed proof is not readily available in the literature. In what follows, we provide a detailed proof which relies on the Hopf–Rinow theorem. In fact, Hopf–Rinow is the only significant ingredient in the proof, and it can be generalised to any length metric space for which a generalisation of the Hopf–Rinow theorem is known to hold. Thus the proof works not only for (finite dimensional) complete Riemannian manifolds such as $Y_k = GL(N_k, \mathbb{C}))/U(N_k)$, but also for locally compact complete length spaces (see Remark 2).

Definition 11 A Riemannian manifold (Y, g_Y) is said to be **complete** if every geodesic of (Y, g_Y) can be extended indefinitely.

In what follows, d_Y stands for the distance function defined by the Riemannian metric g_Y. A foundational theorem in Riemannian geometry is the Hopf–Rinow theorem stated as follows (see e.g. [28, Chap. 4, Sect. 4]).

Theorem 7 (Hopf–Rinow) *Let (Y, g_Y) be a connected Riemannian manifold. The following are equivalent.*

1. *(Y, g_Y) is a complete Riemannian manifold.*
2. *There exists a point $y_0 \in Y$ such that any geodesic segment $\gamma(t)$ emanating from y_0 can be extended indefinitely, i.e. can be extended to a geodesic ray $\gamma : [0, +\infty) \to Y$, $\gamma(0) = y_0$;*
3. *(Y, d_Y) is a complete metric space.*
4. *A subset of Y is compact if and only if it is closed and bounded with respect to d_Y.*

Moreover, any of the above conditions implies that any two points in Y can be joined by a minimising geodesic.

Remark 2 The Hopf–Rinow theorem above, especially the fourth item (the Heine–Borel property), is the *only* property of a complete Riemannian manifold that we will use in this section. In other words, we shall provide a proof of the main results (Theorems 8 and 9) in such a way that it generalises to any length metric space for which the analogue of Hopf–Rinow holds: for example, it is well-known that Hopf–Rinow can be generalised to any locally compact length space, which is called the Hopf–Rinow–Cohn–Vossen Theorem in [10, Theorem 2.5.28].

Note the following easy consequence of the theorem above.

Lemma 1 *A continuous function $f : Y \to \mathbb{R}$ is proper, i.e. $f^{-1}(I)$ is compact in (Y, d_Y) for any closed interval $I \subset \mathbb{R}$, if and only if the sequence $\{f(y_i)\}_i \subset \mathbb{R}$ is unbounded for any d_Y-unbounded sequence $\{y_i\}_i$.*

Proof If f is proper and $\{f(y_i)\}_i$ is a bounded sequence, we find a closed interval I that contains $\{f(y_i)\}_i$, which implies that the sequence $\{y_i\}_i$ is contained in the compact set $f^{-1}(I)$ and hence must be d_Y-bounded.

Suppose conversely that $\{f(y_i)\}_i \subset \mathbb{R}$ being bounded implies that $\{y_i\}_i$ is d_Y-bounded. For a closed interval $I \subset \mathbb{R}$ pick any sequence $\{y_i\}_i \subset f^{-1}(I)$, which must be d_Y-bounded by hypothesis. On the other hand, since $f^{-1}(I)$ is closed, Theorem 7 implies that there exists a subsequence of $\{y_i\}_i$ that converges to a point in $f^{-1}(I)$, proving its compactness. □

For any $y_0 \in Y$ and $\tau > 0$, we define a subset of Y defined as

$$S(y_0, \tau) := \{y \in Y \mid d_Y(y, y_0) = \tau\} = \{\gamma(t) \mid \text{geodesic } \gamma \text{ with } \gamma(0) = y_0\}$$

which is compact by Theorem 7.

We first define a geodesically convex function, which is nothing but a function that is convex along geodesic segments.

Definition 12 A real-valued function f on a Riemannian manifold (Y, g_Y) is said to be **convex along geodesics** if for any geodesic segment $\{\gamma(\tau)\}_{\tau_0 \leq \tau \leq \tau_1}$ we have

$$f(\gamma(\tau)) \leq \frac{\tau - \tau_0}{\tau_1 - \tau_0} f(\gamma(\tau_1)) + \frac{\tau_1 - \tau}{\tau_1 - \tau_0} f(\gamma(\tau_0)).$$

f is said to be **strictly convex along geodesics** if a strict inequality holds in the above.

We first recall some elementary facts on convex functions, which can be found e.g. in [25]. First, if f is a convex function defined on a closed interval $[s_1, s_2]$, the left derivative $f_l'(\tau)$ and the right derivative $f_r'(\tau)$ are both well-defined for $\tau \in (s_1, s_2)$. These are in fact monotonically increasing in τ since for all $\tau_1, \tau_2 \in (s_1, s_2)$, $t_1 < t_2$, we have

$$f_l'(\tau_1) \leq f_r'(\tau_1) \leq \frac{f(\tau_2) - f(\tau_1)}{\tau_2 - \tau_1} \leq f_l'(\tau_2) \leq f_r'(\tau_2) \tag{3}$$

by [25, Corollary 1.1.6]. If f is continuous on $[s_1, s_2]$ and C^2 in the interior, the convexity is equivalent to $f''(\tau) \geq 0$ for $\tau \in (s_1, s_2)$ [25, Corollary 1.1.10]. The following lemma is also well-known; see e.g. [25, Theorem 1.1.5].

Lemma 2 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex. Then, for any $x \in \mathbb{R}$, $(f(x + \tau) - f(x))/\tau$ is increasing in τ.*

We observe the following elementary consequence of the above results.

Lemma 3 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex. Then*

$$\lim_{\tau \to +\infty} \frac{f(\tau)}{\tau} = \lim_{\tau \to +\infty} f_l'(\tau) = \lim_{\tau \to +\infty} f_r'(\tau),$$

where we allow the limit to be $+\infty$. The same result holds for the limit $\tau \to -\infty$, by allowing it to be $-\infty$.

All the results above carry over to geodesically convex functions in an obvious manner. Recall also the following well-known result.

Lemma 4 *Suppose that $f : Y \to \mathbb{R}$ is a C^1 function that is convex along geodesics. Then any critical point of f necessarily attains the global minimum of f over Y.*

We first prove the following technical lemma, which will be useful later.

Lemma 5 *Let (Y, g_Y) be a complete Riemannian manifold and $y_0 \in Y$ be a fixed point. Let $f : Y \to \mathbb{R}$ be a continuous function that is convex along geodesic rays in (Y, g_Y) emanating from y_0. Suppose that a sequence $\{y_i\}_i \subset Y$ satisfies the following.*

1. *$\{y_i\}_i \subset Y$ is d_Y-unbounded.*
2. *There exists a geodesic γ_* in (Y, d_Y) such that the following hold:*
   - *when we write $y_i = \gamma_i(\tau_i)$ by using a geodesic γ_i and some $\tau_i > 0$, we have $d_Y(\gamma_i(1), \gamma_*(1)) \to 0$ as $i \to \infty$;*
   - $\lim_{\tau \to +\infty} \dfrac{f(\gamma_*(\tau))}{\tau} > 0$.

Then, we have $\liminf_{i\to\infty} \dfrac{f(y_i)}{\tau_i} > 0$.

Proof Set $\lim_{\tau\to+\infty} f(\gamma_*(\tau))/\tau =: c > 0$, which we allow to be $+\infty$. This implies that, if c is finite, for any $\epsilon > 0$ there exists $\tau_0(\epsilon) > 1$ such that for all $\tau \geq \tau_0(\epsilon)$ we have $f(\gamma_*(\tau)) \geq \tau(c - \epsilon)$. The argument below carries over word by word to the case $c = +\infty$, by simply replacing $c - \epsilon$ by an arbitrarily large real number.

Suppose for contradiction that $\liminf_{i\to\infty} f(y_i)/\tau_i =: c' \leq 0$. We now take $\epsilon > 0$ to be sufficiently small and $\tau_1 := \tau_1(c, c', \epsilon) > \tau_0(\epsilon)$ to be large enough so that

$$(c - \epsilon) - \frac{f(y_0) + 1}{\tau_1} > c' + \frac{c}{2},$$

which is well-defined since $c' \leq 0$. On the other hand, by the continuity of f, there exists $\delta = \delta(\tau_1(c, c', \epsilon)) > 0$ that depends on $\tau_1(c, c', \epsilon)$ such that

$$d_Y(\gamma_i(1), \gamma_*(1)) < \delta \Rightarrow f(\gamma_i(\tau_1)) > f(\gamma_*(\tau_1)) - 1.$$

To prove the above assertion, observe that for any $\epsilon' > 0$ there exist $\epsilon, \delta > 0$ such that $d_Y(\gamma_i(1), \gamma_*(1)) < \delta \Rightarrow d_Y(\gamma_i(1+\epsilon), \gamma_*(1+\epsilon)) < \epsilon'$, and iterate using the compactness of $[1, \tau_1]$.

Note further that by convexity (see Lemma 2) we have

$$\frac{f(\gamma_i(\tau)) - f(y_0)}{\tau} \geq \frac{f(\gamma_i(\tau_1)) - f(y_0)}{\tau_1}$$

for all $\tau > \tau_1$. We take i to be large enough so that $\tau_i > \tau_1(c, c', \epsilon)$ holds and $d_Y(\gamma_i(1), \gamma_*(1)) < \delta$, and then we find

$$
\begin{aligned}
\frac{f(\gamma_i(\tau_i)) - f(y_0)}{\tau_i} &\geq \frac{f(\gamma_i(\tau_1)) - f(y_0)}{\tau_1} \\
&> \frac{f(\gamma_*(\tau_1)) - f(y_0) - 1}{\tau_1} \\
&\geq \frac{\tau_1(c - \epsilon) - f(y_0) - 1}{\tau_1} \\
&> c' + c/2.
\end{aligned}
$$

Thus, for all large enough i, we get

$$
\frac{f(y_i)}{\tau_i} > c' + \frac{c}{2} + \frac{f(y_0)}{\tau_i} > c' + \frac{c}{4},
$$

which contradicts $\liminf_{i\to\infty} f(y_i)/\tau_i =: c'$. □

We now give the proof of the main result of this section, following the approach in [8, Theorem 1.6].

Theorem 8 *Let (Y, g_Y) be a complete Riemannian manifold and $y_0 \in Y$ be a fixed point. Let $f : Y \to \mathbb{R}$ be a continuous function that is strictly convex along geodesic rays in (Y, g_Y). Then the following are equivalent.*

1. *f admits a unique global minimum over Y.*
2. *$f : Y \to \mathbb{R}$ is proper and bounded from below.*
3. *There exists $C, D > 0$ such that $f(y) \geq C d_Y(y, y_0) - D$ for all $y \in Y$.*
4. *$\lim_{\tau\to+\infty} \dfrac{f(\gamma(\tau))}{\tau} > 0$ for any geodesic γ emanating from y_0.*
5. *There exists $C > 0$ such that $\lim_{\tau\to+\infty} \dfrac{f(\gamma(\tau))}{\tau} > C$ for any geodesic γ emanating from y_0.*

Proof It suffices to prove $3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 4 \Rightarrow 3$, since $5 \Rightarrow 4$ is obvious and $3 \Rightarrow 5$ follows from

$$
\frac{f(\gamma(\tau))}{\tau} \geq C \frac{d_Y(\gamma(\tau), y_0)}{d_Y(\gamma(\tau), \gamma(0))} - \frac{D}{\tau} \geq C \frac{d_Y(\gamma(\tau), \gamma(0)) - d_Y(y_0, \gamma(0))}{d_Y(\gamma(\tau), \gamma(0))} - \frac{D}{\tau},
$$

and taking the limit of $\tau = d_Y(\gamma(\tau), \gamma(0)) \to +\infty$.

$3 \Rightarrow 2$ is an obvious consequence of Lemma 1. To prove $2 \Rightarrow 1$, take a sequence $\{y_i\}_i \subset Y$ such that $f(y_i) \to \inf_Y f =: c > -\infty$ as $i \to \infty$. Since $f^{-1}([c, c+1])$ is compact, this contains a convergent subsequence, which must attain the global minimum of f. We prove the uniqueness: suppose that there exist $y_1, y_2 \in Y$, $y_1 \neq y_2$, that satisfy $f(y_1) = f(y_2) = \inf_Y f$. We pick a geodesic segment $\{\gamma(\tau)\}_{0\leq\tau\leq 1}$ connecting $y_1 = \gamma(0)$ and $y_2 = \gamma(1)$. On the other hand, strict convexity means

$$f(\gamma(1/2)) < \frac{1}{2} f(\gamma(0)) + \frac{1}{2} f(\gamma(1)) = \inf_Y f,$$

which is a contradiction, and hence we get $y_1 = y_2$.

We prove $1 \Rightarrow 4$. Since f is strictly increasing along geodesics in a neighbourhood of the unique global minimum $y_* \in Y$, we find $f_l'(\gamma(0)) > 0$ for any geodesic that emanates from y_*. Since the left (and right) derivatives of f are monotonically increasing by (3), we get the claim for such geodesics by Lemma 3, and find in particular that $f(\gamma(\tau))$ is strictly increasing in τ for any geodesic emanating from y_*.

We now claim that for each geodesic ray $\{\gamma(\tau)\}_{\tau \geq 0}$, passing through y_0 (or indeed any point) but not necessarily through y_*, there exists τ_* such that $f(\gamma(\tau_*)) = \inf_{\tau \geq 0} f(\gamma(\tau)) > -\infty$; given such claim, we get the result $1 \Rightarrow 4$ by exactly the same argument as before, by using the strict geodesic convexity of f.

Suppose for contradiction that we have an unbounded sequence $\{\tau_i\}_i \subset \mathbb{R}_{\geq 0}$ such that $f(\gamma(\tau_i)) \to \inf_{\tau \geq 0} f(\gamma(\tau)) =: m_\gamma$ as $i \to \infty$. We then find that the d_Y-unbounded sequence $\{y_i\}_i \subset Y$, $y_i := \gamma(\tau_i)$, satisfies $f(y_i) \to m_\gamma$. Since (Y, g_Y) is complete, we can connect each y_i to y_* by a geodesic $\{\gamma_i(s)\}_{s \geq 0}$, say, with $\gamma_i(0) = y_*$ and $y_i = \gamma_i(s_i)$. By passing to a subsequence if necessary, we may assume that $\gamma_i(1) \in S(y_*, 1)$ converges to some $\gamma_*(1) \in S(y_*, 1)$ by recalling the compactness of $S(y_*, 1)$. Recalling that $\{y_i\}_i$ is d_Y-unbounded, and also that we know $\lim_{s \to +\infty} f(\gamma_*(s))/s =: c > 0$ by the argument above (we assume c is finite, but the case $c = +\infty$ can be treated similarly), we apply Lemma 5 to conclude $\liminf_{i \to \infty} f(y_i)/\tau_i > 0$. However, $f(y_i) \to m_\gamma$ $(i \to \infty)$ would imply that $\limsup_{i \to \infty} f(y_i)/\tau_i = \lim_{i \to \infty} f(y_i)/\tau_i = 0$, which is a contradiction.

We finally prove $4 \Rightarrow 3$. Suppose for contradiction that for any $C, D > 0$ there exists $y \in Y$ such that $f(y) < C d_Y(y, y_0) - D$. Pick a sequence $\{C_i\}_i, \{D_i\}_i \subset \mathbb{R}_{>0}$ such that $C_i \to 0$ and $D_i \to +\infty$ as $i \to \infty$, and also $\{y_i\}_i \subset Y$ so that

$$f(y_i) < C_i d_Y(y_i, y_0) - D_i.$$

This means that $\{y_i\}_i$ cannot be contained in a d_Y-bounded subset in Y, since otherwise we would have $f(y_i) < C' - D_i \to -\infty$ as $i \to \infty$, for some constant $C' > 0$ that does not depend on i, which is a contradiction since a d_Y-bounded subset is compact by Theorem 7 and hence $\inf_i f(y_i)$ should be finite.

We now choose a geodesic $\{\gamma_i(\tau)\}_{\tau \geq 0}$ with $y_i = \gamma_i(\tau_i)$, and $\gamma_i(0) = y_0$. Note $\tau_i = d_Y(y_i, y_0)$. Since $\{y_i\}_i$ is d_Y-unbounded we find that $\{\tau_i\}_i \subset \mathbb{R}_{\geq 0}$ is unbounded, and by taking a subsequence we further assume that it is monotonically increasing. Moreover, since $S(y_0, 1)$ is compact, we may further assume that $\gamma_i(1)$ converges to $\gamma_*(1) \in S(y_0, 1)$. We then have $\liminf_{i \to \infty} f(y_i)/\tau_i > 0$ by Lemma 5, but on the other hand

$$\frac{f(y_i)}{\tau_i} < C_i - \frac{D_i}{\tau_i}$$

implies

$$\limsup_{i\to\infty} \frac{f(y_i)}{\tau_i} \leq \limsup_{i\to\infty} C_i - \liminf_{i\to\infty} \frac{D_i}{\tau_i} \leq 0,$$

which is a contradiction. □

Remark 3 In the fourth and fifth condition of Theorem 8, the geodesic γ need not emanate from y_0. To see this, we only need to check that the first condition implies $\lim_{\tau\to+\infty} f(\gamma(\tau))/\tau > 0$ for all geodesics not necessarily emanating from y_0, which is stronger than the fourth, but this is obvious from the proof of $1 \Rightarrow 4$.

We also have a group equivariant version, which can deal with the balanced metrics when it is applied to the case where $Y = Y_k = GL(N_k, \mathbb{C}))/U(N_k)$ and $\mathrm{Aut}_0(X, L)$ is nontrivial; see [23, Sects. 2.3 and 5.1] for more details. Let H be any group, and suppose that Y admits an H-action which is a d_Y-isometry; for the sake of the exposition we assume that this is the right action but the argument is exactly the same when it is switched to the left action. We define

$$d_{Y,H}(y_1, y_2) := \inf_{h_1,h_2\in H} d_Y(y_1 \cdot h_1, y_2 \cdot h_2),$$

which is sometimes called the **reduced distance**.

We also say that a geodesic γ is **parametrised by the reduced arc-length** if $\tau = d_{Y,H}(\gamma(\tau), \gamma(0))$.

Theorem 9 *Let (Y, g_Y) be a complete Riemannian manifold with the isometric H-action, and $y_0 \in Y$ be a fixed point in Y. Let $f : Y \to \mathbb{R}$ be a H-invariant continuous function that is strictly convex along geodesic rays in (Y, g_Y) that is not contained in the H-orbit, i.e.*

$$f(\gamma(\tau)) \leq \frac{\tau - \tau_0}{\tau_1 - \tau_0} f(\gamma(\tau_1)) + \frac{\tau_1 - \tau}{\tau_1 - \tau_0} f(\gamma(\tau_0)),$$

for $\tau \in [\tau_0, \tau_1]$, with equality if and only if $\{\gamma(\tau)\}_{\tau_0\leq\tau\leq\tau_1}$ is contained in $\gamma(\tau_0)\cdot H$. The following are equivalent.

1. *There exists $y_* \in Y$ which attains the global minimum of f, which is unique modulo the H-action.*
2. *There exists $C, D > 0$ such that $f(y) \geq C d_{Y,H}(y, y_0) - D$.*
3. $\lim_{\tau\to+\infty} \frac{f(\gamma(\tau))}{\tau} > 0$ *for any geodesic γ emanating from y_0, not contained in the H-orbit.*
4. *There exists $C > 0$ such that $\lim_{\tau\to+\infty} \frac{f(\gamma(\tau))}{\tau} > C$ for any geodesic γ emanating from y_0 and not contained in the H-orbit, parametrised by the reduced arc-length.*

As pointed out in Remark 3, the geodesics in the third and fourth items need not emanate from y_0.

Proof We prove $2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 2$, as $4 \Rightarrow 3$ and $2 \Rightarrow 4$ are obvious by arguing as before in the proof of Theorem 8.

The proof of $2 \Rightarrow 1$ is just an adaptation of Lemma 1: noting that f is bounded below, we pick a minimising sequence $\{y_i\}_i$ so that $f(y_i) \to \inf_Y f$ as $i \to \infty$. We then find that $\{d_{Y,H}(y_i, y_0)\}_i \subset \mathbb{R}_{\geq 0}$ is a bounded sequence. Thus there exists $\{h'_i\}_i, \{h_i\}_i \subset H$ such that $\{d_Y(y_i \cdot h'_i, y_0 \cdot h_i)\}_i$ is also a bounded sequence. Now, since H acts on Y by isometry, we find $d_Y(y_i \cdot h'_i, y_0 \cdot h_i) = d_Y(y_i \cdot (h'_i h_i^{-1}), y_0)$. Hence defining $y'_i := y_i \cdot (h'_i h_i^{-1})$, we get a d_Y-bounded sequence $\{y'_i\}_i \subset Y$, which must contain a subsequence that converges in Y by Theorem 7. The limit of any such subsequence must attain $\inf_Y f$, since $f(y_i) = f(y'_i)$ by the H-invariance, proving the existence of the global minimiser. To show the uniqueness modulo the H-action, we argue as before: pick any $y_1, y_2 \in Y$ with $y_1 \neq y_2$ and $f(y_1) = f(y_2) = \inf_Y f$, and a geodesic γ connecting them with $y_1 = \gamma(0)$ and $y_2 = \gamma(1)$. We then find

$$f(\gamma(\tau)) \leq (1-\tau) f(\gamma(0)) + \tau f(\gamma(1)) = \inf_Y f$$

for all $0 \leq \tau \leq 1$. The above inequality must clearly be an equality, which forces $\gamma(\tau) \subset y_1 \cdot H$ as required.

The other implications are almost exactly the same as the proof of Theorem 8, but we supplement some details.

We prove $1 \Rightarrow 3$. Since f is strictly increasing along geodesics not contained in the H-orbit, we find $f'_l(\gamma(0)) > 0$ for any geodesic that emanates from the global minimum y_* which is not contained in the H-orbit. Since the left (and right) derivatives of f are monotonically increasing by (3), we get the claim for such geodesics by Lemma 3. We then prove, just as we did in Theorem 8, that for each geodesic ray $\{\gamma(\tau)\}_{\tau \geq 0}$, not necessarily passing through y_* and not contained in the H-orbit, there exists τ_* such that $f(\gamma(\tau_*)) = \inf_{\tau \geq 0} f(\gamma(\tau)) > -\infty$. The only modification required in the proof is that we need to make sure that each y_i can be connected to y_* by a geodesic that is not contained in the H-orbit; this is straightforward, since otherwise there would exist $h \in H$ such that $f(y_i) = f(y_* \cdot h) = f(y_*) = \inf_Y f$, by the H-invariance of f, which must attain $\inf_{\tau \geq 0} f(\gamma(\tau))$. This gives us the desired result.

The proof of $3 \Rightarrow 2$ is also similar. We pick a sequence $\{C_i\}_i, \{D_i\}_i \subset \mathbb{R}_{>0}$ such that $C_i \to 0$ and $D_i \to +\infty$ as $i \to \infty$, and also $\{y_i\}_i \subset Y$ so that

$$f(y_i) < C_i d_{Y,H}(y_i, y_0) - D_i,$$

which means that $\{y_i\}_i$ cannot be contained in a $d_{Y,H}$-bounded subset in Y, since as we saw in the proof of $2 \Rightarrow 1$ we can define $y'_i := y_i \cdot h_i$ for some $\{h_i\}_i \subset H$ such that for all large enough i we have

$$f(y'_i) < C_i d_Y(y'_i, y_0) - D_i + 1.$$

This means that the sequence $\{y'_i\}_i \subset Y$ must be d_Y-unbounded, as otherwise it would contradict Theorem 7. We note that we can connect each y'_i to y_0 by a geodesic that is not contained in the H-orbit (as otherwise we would have $f(y'_i) = f(y_0 \cdot h_i) = f(y_0)$), the argument for Theorem 8 carries over word by word to derive the required contradiction. $\square$

We finally summarise what we have proved so far in the form that is helpful for the study of the (anticanonical) balancing energy defined on the symmetric space $Y = Y_k = GL(N_k, \mathbb{C}))/U(N_k)$, which is a complete Riemannian manifold with respect to the bi-invariant metric.

Corollary 2 *Let (Y, g_Y) be a complete Riemannian manifold with the isometric H-action, with a fixed point $y_0 \in Y$. Suppose that an H-invariant C^1 function $f : Y \to \mathbb{R}$ is strictly convex along geodesics emanating from y_0 and not contained in the H-orbit.*

Then f admits a critical point if and only if

$$\lim_{\tau \to +\infty} \frac{f(\gamma(\tau))}{\tau} > 0$$

for any geodesic γ emanating from y_0 that is not contained in the H-orbit. Moreover, any critical point must be the global minimiser of f over Y which is unique modulo the H-action.

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Unbounded Operators on the Segal–Bargmann Space

Friedrich Haslinger

Abstract In this paper we analyse the domains of differential and multiplication operators on the Segal–Bargmann space. We consider the basic estimate for the ∂-complex, remark that this estimate is closely related to the uncertainty principle in quantum mechanics and compute the Bergman kernel of the graph norm. It is shown that the set of all functions u in the Segal–Bargmann $A^2(\mathbb{C}, e^{-|z|^2})$ such that the multiplication with a polynomial p is norm bounded gives a relatively compact subset of the Segal–Bargmann space. In the following section we give a survey of recent results on the ∂-complex on weighted Bergman spaces on Hermitian manifolds, analysing metrics which produce a similar duality between differentiation and multiplication as in the Segal–Bargmann space. Finally we study the basic estimates and the corresponding questions of compactness for the generalized ∂-complex.

Keywords Segal Bargmann space · Bergman kernel · Compactness

1 Introduction

The Segal–Bargmann space of entire functions is denoted by

$$A^2(\mathbb{C}^n, e^{-|z|^2}) = \left\{ f : \mathbb{C}^n \longrightarrow \mathbb{C} \, : \, \|f\|^2 = \int_{\mathbb{C}^n} |f(z)|^2 \, e^{-|z|^2} \, d\lambda(z) < \infty \right\}.$$

It is well known that the differentiation with respect to z_j defines an unbounded operator on $A^2(\mathbb{C}^n, e^{-|z|^2})$. We consider the operator

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} \, dz_j,$$

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which is densely defined on $A^2(\mathbb{C}^n, e^{-|z|^2})$ and maps to $A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$, the space of $(1,0)$-forms with coefficients in $A^2(\mathbb{C}^n, e^{-|z|^2})$. The domain of ∂ is given by

$$\operatorname{dom}(\partial) = \left\{ f \in A^2(\mathbb{C}^n, e^{-|z|^2}) : \sum_{j=1}^n \frac{\partial f}{\partial z_j}\, dz_j \in A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \right\}.$$

The adjoint operator ∂^* is also unbounded and densely defined and given by

$$\partial^* u = \sum_{j=1}^n z_j u_j,$$

where $u = \sum_{j=1}^n u_j\, dz_j \in A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$ belongs to the domain of ∂^* which is given by

$$\operatorname{dom}(\partial^*) = \left\{ u = \sum_{j=1}^n u_j\, dz_j \in A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) : \sum_{j=1}^n z_j u_j \in A^2(\mathbb{C}^n, e^{-|z|^2}) \right\}.$$

In general, we get the ∂-complex

$$A^2_{(p-1,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}),$$

where $1 \le p \le n-1$ and

$$\partial f = \sum_{|J|=p}{}' \sum_{j=1}^n \frac{\partial f_J}{\partial z_j}\, dz_j \wedge dz_J$$

for a $(p,0)$-form $f = \sum_{|J|=p}{}' f_J\, dz_J$ with summation over increasing multiindices $J = (j_1, \dots, j_p)$ and

$$\partial^* f = \sum_{|K|=p-1}{}' \sum_{j=1}^n z_j f_{jK}\, dz_K.$$

We will choose the domain $\operatorname{dom}(\partial)$ in such a way that ∂ becomes a closed operator on $A^2(\mathbb{C}^n, e^{-|z|^2})$. In addition we get that the corresponding complex Laplacian

$$\tilde{\Box}_p = \partial^* \partial + \partial \partial^*,$$

with $\operatorname{dom}(\tilde{\Box}_p) = \{f \in \operatorname{dom}(\partial) \cap \operatorname{dom}(\partial^*) : \partial f \in \operatorname{dom}(\partial^*) \text{ and } \partial f^* \in \operatorname{dom}(\partial)\}$ acts as an unbounded self-adjoint operator on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$. We point out that in this case the complex Laplacian is a differential operator of order one. Nevertheless, we can use the general features of a Laplacian for these differential operators of order one, see [6–8]. The canonical orthonormal basis in $A^2(\mathbb{C}^n, e^{-|z|^2})$ is given by

$$\varphi_\alpha(z) = \frac{z^\alpha}{\sqrt{\pi^n \alpha!}},$$

α is a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha! = \alpha_1! \dots \alpha_n!$ and $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$. We also write $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

An entire function with Taylor series expansion

$$f(z) = \sum_\alpha \left(\frac{\partial}{\partial z}\right)^\alpha f(0)\, \frac{z^\alpha}{\alpha!}$$

belongs to $A^2(\mathbb{C}^n, e^{-|z|^2})$ if and only if

$$\left(\left(\frac{\partial}{\partial z}\right)^\alpha f(0)/\sqrt{\alpha!}\right)_\alpha \in l^2.$$

We use Parseval's equation for the partial sums of the expansion in the canonical orthonormal basis of the Segal–Bargmann space to show that for entire functions $f \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ we have

$$\|z_j f\|^2 = \|f\|^2 + \left\|\frac{\partial f}{\partial z_j}\right\|^2. \tag{1}$$

In particular we have

$$\|z_j f\|^2 + \left\|\frac{\partial f}{\partial z_j}\right\|^2 \geq \|f\|^2, \tag{2}$$

for $f \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$.

Remark 1 The operators $a_j(f) = \frac{\partial f}{\partial z_j}$ and $a_j^*(f) = z_j f$ are densely defined unbounded operators on $A^2(\mathbb{C}^n, e^{-|z|^2})$, with closed graph.

They are adjoint to each other and $[a_j, a_j^*] = I$. Equation (1) implies that $\mathrm{dom}(a_j) = \mathrm{dom}(a_j^*)$, see [5, 7]. The operators a_j and a_j^* are the annihilation and creation operators in quantum field theory.

Now we consider the position operators $X_j(u) = x_j u$ and the momentum operators $D_j u = -i\, \frac{\partial u}{\partial x_j}$ on $L^2(\mathbb{R}^n)$ and recall that

$$\|X_j(u)\|^2 + \|D_j(u)\|^2 \geq \|u\|^2,\ u \in \mathrm{dom}(X_j) \cap \mathrm{dom}(D_j), \tag{3}$$

which is a variant of the uncertainty principle, see [5].

We indicate that (3) follows from inequality (2). To show this we use the Bargmann transform $\mathcal{B} : L^2(\mathbb{R}^n) \longrightarrow A^2(\mathbb{C}^n, e^{-|z|^2})$, which is given by

$$\mathcal{B}(u)(z) = \pi^{-n/4} \int_{\mathbb{R}^n} u(x) \exp\left[-\sum_{j=1}^{n} (z_j^2/2 - \sqrt{2} z_j x_j + x_j^2/2) \right] dx, \ u \in L^2(\mathbb{R}^n).$$

The Bargmann transform is a unitary operator between $L^2(\mathbb{R}^n)$ and $A^2(\mathbb{C}^n, e^{-|z|^2})$. Concerning the operators a_j and a_j^* we have

$$\mathcal{B}^* a_j \mathcal{B} = \frac{1}{\sqrt{2}}(X_j + iD_j) \text{ and } \mathcal{B}^* a_j^* \mathcal{B} = \frac{1}{\sqrt{2}}(X_j - iD_j),$$

see [5]. So we get

$$\|X_j(u)\|^2 = \frac{1}{2} \|\mathcal{B}^* a_j \mathcal{B}(u) + \mathcal{B}^* a_j^* \mathcal{B}(u)\|^2,$$

and

$$\|D_j(u)\|^2 = \frac{1}{2} \|\mathcal{B}^* a_j \mathcal{B}(u) - \mathcal{B}^* a_j^* \mathcal{B}(u)\|^2.$$

Hence

$$\begin{aligned} \|X_j(u)\|^2 + \|D_j(u)\|^2 &= \|\mathcal{B}^* a_j \mathcal{B}(u)\|^2 + \|\mathcal{B}^* a_j^* \mathcal{B}(u)\|^2 \\ &= \|a_j \mathcal{B}(u)\|^2 + \|a_j^* \mathcal{B}(u)\|^2 \\ &\geq \|\mathcal{B}(u)\|^2 \\ &= \|u\|^2, \end{aligned}$$

where we used (2) and that $\mathcal{B}$ is an isometry.

The last computation also shows that (3) implies (2).

2 The ∂-Complex

If $u \in A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$, we have $u \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ if and only if $\frac{\partial u_j}{\partial z_k} \in A^2(\mathbb{C}^n, e^{-|z|^2})$, for all $j, k = 1, \dots, n$. This follows from

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \sum_{j,k=1}^{n} \left\| \frac{\partial u_k}{\partial z_j} \right\|^2 + \sum_{k=1}^{n} \int_{\mathbb{C}^n} |u_k|^2 \, e^{-|z|^2} \, d\lambda, \tag{4}$$

for $u \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$, see [7]. Hence we have the basic estimate

$$\|\partial u\|^2 + \|\partial^* u\|^2 \geq \|u\|^2, \tag{5}$$

for $u \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$, which can be seen as a generalization of the uncertainty principle, see Remark 1. Equation (5) also implies that $\tilde{\Box}_1$ has a bounded inverse, see [7].

If $u = \sum'_{|J|=p} u_J\, dz_J$, then $u \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ if and only if $\frac{\partial u_J}{\partial z_k} \in A^2(\mathbb{C}^n, e^{-|z|^2})$, for all J with $|J| = p$ and all $k = 1, \dots, n$, where $1 \le p \le n-1$.

For $p = 0$ we get $u \in \mathrm{dom}(\tilde{\Box}_0)$ if and only if $\sum_{j=1}^n z_j \frac{\partial u}{\partial z_j} \in A^2(\mathbb{C}^n, e^{-|z|^2})$; for $p = n$ we get $u \in \mathrm{dom}(\tilde{\Box}_n)$ if and only if $\sum_{j=1}^n z_j \frac{\partial u}{\partial z_j} \in A^2(\mathbb{C}^n, e^{-|z|^2})$ and if $1 \le p \le n-1$ we have $u \in \mathrm{dom}(\tilde{\Box}_p)$ if and only if $\frac{\partial u_J}{\partial z_k} \in A^2(\mathbb{C}^n, e^{-|z|^2})$, for all J with $|J| = p$ and all $k = 1, \dots, n$, and $\sum_{k=1}^n z_k \frac{\partial u_J}{\partial z_k} \in A^2(\mathbb{C}^n, e^{-|z|^2})$, for all J with $|J| = p$.

Theorem 1 *Let* $u = \sum'_{|J|=p} u_J\, dz_J \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$, $0 \le p \le n$ *and let* $u_J = \sum_\alpha u_\alpha^{(J)} \varphi_\alpha$ *be the canonical expansion in* $A^2(\mathbb{C}^n, e^{-|z|^2})$, *where* $(u_\alpha^{(J)})_\alpha \in l^2$. *Then* $u \in dom(\tilde{\Box}_p)$ *if and only if* $(u_\alpha^{(J)} |\alpha|)_\alpha \in l^2$, *for all* J.

Proof The $(p, 0)$-form $u = \sum'_{|J|=p} u_J\, dz_J \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ belongs to $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ if and only if $\frac{\partial u_J}{\partial z_k} \in A^2(\mathbb{C}^n, e^{-|z|^2})$, for all J with $|J| = p$ and all $k = 1, \dots, n$. This follows from

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \sum_{|J|=p}{}' \sum_{j=1}^n \left\| \frac{\partial u_J}{\partial z_j} \right\|^2 + p \sum_{|J|=p}{}' \int_{\mathbb{C}^n} |u_J|^2 e^{-|z|^2}\, d\lambda, \qquad (6)$$

for $u \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$, see [7].

And we have $u \in \mathrm{dom}(\tilde{\Box}_p)$ if and only if $\frac{\partial u_J}{\partial z_k} \in A^2(\mathbb{C}^n, e^{-|z|^2})$, for all J with $|J| = p$ and all $k = 1, \dots, n$, and $\sum_{k=1}^n z_k \frac{\partial u_J}{\partial z_k} \in A^2(\mathbb{C}^n, e^{-|z|^2})$, for all J with $|J| = p$. So if we write

$$u_J = \sum_\alpha u_\alpha^{(J)} \varphi_\alpha, \text{ where } (u_\alpha^{(J)})_\alpha \in l^2,$$

Using the canonical orthonormal basis we obtain

$$\begin{aligned} \frac{\partial u_J}{\partial z_k} &= \frac{1}{\sqrt{\pi^n}} \sum_{\alpha_1 \ge 0 \dots \alpha_k \ge 1 \dots \alpha_n \ge 0} u_\alpha^{(J)} \alpha_k \frac{z_1^{\alpha_1} \dots z_k^{\alpha_k - 1} \dots z_n^{\alpha_n}}{\sqrt{\alpha!}} \\ &= \sum_\alpha u^{(J)}_{\alpha_1 \dots \alpha_k + 1 \dots \alpha_n} \sqrt{\alpha_k + 1}\, \varphi_\alpha, \end{aligned}$$

and

$$\begin{aligned} z_k \frac{\partial u_J}{\partial z_k} &= \frac{1}{\sqrt{\pi^n}} \sum_{\alpha_1 \ge 0 \dots \alpha_k \ge 1 \dots \alpha_n \ge 0} u_\alpha^{(J)} \alpha_k \frac{z_1^{\alpha_1} \dots z_k^{\alpha_k} \dots z_n^{\alpha_n}}{\sqrt{\alpha!}} \\ &= \sum_{\alpha_1 \ge 0 \dots \alpha_k \ge 1 \dots \alpha_n \ge 0} u_\alpha^{(J)} \alpha_k\, \varphi_\alpha. \end{aligned}$$

For $n = 2$ we have

$$z_1 \frac{\partial u_J}{\partial z_1} + z_2 \frac{\partial u_J}{\partial z_2} = \sum_{\alpha_1 \geq 1, \alpha_2 \geq 0} u_\alpha^{(J)} \alpha_1 \, \varphi_\alpha + \sum_{\alpha_1 \geq 0, \alpha_2 \geq 1} u_\alpha^{(J)} \alpha_2 \, \varphi_\alpha$$

$$= \sum_{\alpha_1 \geq 1, \alpha_2 \geq 1} u_\alpha^{(J)} (\alpha_1 + \alpha_2) \, \varphi_\alpha + \sum_{\alpha_1 \geq 1} u_{\alpha_1,0}^{(J)} \, \alpha_1 \varphi_{\alpha_1,0} + \sum_{\alpha_2 \geq 1} u_{0,\alpha_2}^{(J)} \, \alpha_2 \varphi_{0,\alpha_2}$$

$$= \sum_{\alpha_1 \geq 0, \alpha_2 \geq 0} u_\alpha^{(J)} (\alpha_1 + \alpha_2) \, \varphi_\alpha.$$

By induction, suppose that

$$\sum_{k=1}^{n} z_k \frac{\partial u_J}{\partial z_k} = \sum_{\alpha \geq 0} u_\alpha^{(J)} |\alpha| \varphi_\alpha, \tag{7}$$

then we obtain for

$$\sum_{k=1}^{n+1} z_k \frac{\partial u_J}{\partial z_k} = \sum_{\alpha_1 \geq 0 \ldots \alpha_{n+1} \geq 0} u_\alpha^{(J)} (\alpha_1 + \cdots + \alpha_n) \varphi_\alpha + \sum_{\alpha_1 \geq 0 \ldots \alpha_n \geq 0, \alpha_{n+1} \geq 1} u_\alpha^{(J)} \, \alpha_{n+1} \, \varphi_\alpha$$
$$= \sum_{\alpha_1 \geq 0 \ldots \alpha_{n+1} \geq 0} u_\alpha^{(J)} (\alpha_1 + \cdots + \alpha_{n+1}) \varphi_\alpha.$$

$\square$

3 Properties of the Graph Norm

If we endow $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ with the graph norm, we obtain Hilbert spaces of entire functions with reproducing kernels. If $1 \leq p < n$, we write for the multiindex $J = (j_1(J), \ldots j_p(J))$ and denote the missing $n - p$ indices by $j_{p+1}(J), \ldots, j_n(J)$. Then we can write (6) in the form

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \sum_{|J|=p}{}' \left(\sum_{k=1}^{p} \left(\|u_J\|^2 + \left\| \frac{\partial u_J}{\partial z_{j_k(J)}} \right\|^2 \right) + \sum_{k=p+1}^{n} \left\| \frac{\partial u_J}{\partial z_{j_k(J)}} \right\|^2 \right)$$
$$= \sum_{|J|=p}{}' \left(\sum_{k=1}^{p} \|z_{j_k(J)} u_J\|^2 + \sum_{k=p+1}^{n} \left\| \frac{\partial u_J}{\partial z_{j_k(J)}} \right\|^2 \right),$$

where we used (1). For $(p,0)$-forms $u = \sum'_{|J|=p} u_J \, dz_J \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ belonging to $\operatorname{dom}(\partial) \cap \operatorname{dom}(\partial^*)$ we have by (6)

$$\|u\|^2 = \sum_{|J|=p}{}' \|u_J\|^2 \leq \frac{1}{p}(\|\partial u\|^2 + \|\partial^* u\|^2),$$

which implies

$$\|u_J\|^2 \leq \frac{1}{p}\left(\sum_{k=1}^{p} \|z_{j_k(J)} u_J\|^2 + \sum_{k=p+1}^{n} \left\| \frac{\partial u_J}{\partial z_{j_k(J)}} \right\|^2 \right); \tag{8}$$

for each component u_J. If $p = n$ we get

$$\|u_J\|^2 \leq \frac{1}{p} \sum_{k=1}^{p} \|z_{j_k(J)} u_J\|^2. \tag{9}$$

The components $u_J dz_J \in \operatorname{dom}(\partial) \cap \operatorname{dom}(\partial^*)$ with the right hand side of (8) or (9) as norm form Hilbert spaces with reproducing kernel, since, by (8) or (9), the point evaluations are continuous linear functionals. We denote these Hilbert spaces by $\mathcal{G}_p$.

The Bergman kernel of $A^2(\mathbb{C}^n, e^{-|z|^2})$ is

$$K(z, w) = \frac{1}{\pi^n} \exp(\langle z, w \rangle),$$

where $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$, see [5].

Theorem 2 *Let* $1 \leq p \leq n$. *The Bergman kernel of* $\mathcal{G}_p$ *is of the form*

$$B_p(z, w) = \frac{p}{\pi^n} \frac{d^{p-1}}{d\zeta^{p-1}} \left(\frac{e^\zeta - \sum_{k=0}^{p-1} \frac{\zeta^k}{k!}}{\zeta} \right) \Bigg|_{\zeta = \langle z, w \rangle}. \tag{10}$$

Proof We write $p_k(u)(w) = \frac{\partial u}{\partial w_k}$ and $p_k^*(u)(w) = w_k u(w)$ for $u \in A^2(\mathbb{C}^n, e^{-|z|^2})$ and $k = 1, \ldots, n$. Without loss of generality we can assume that the graph norm is

$$\frac{1}{p}\left(\sum_{k=1}^{p} \|p_k^*(u)\|^2 + \sum_{k=p+1}^{n} \|p_k(u)\|^2 \right).$$

Since $\mathcal{B}_p$ has the reproducing property, we have

$$\begin{aligned}u(z) &= \frac{1}{p}\int_{\mathbb{C}^n}\sum_{k=1}^{p} p_k^*(u)(w)[p_k^*(v_z)(w)]^- e^{-|w|^2}\,d\lambda(w)\\ &+ \frac{1}{p}\int_{\mathbb{C}^n}\sum_{k=p+1}^{n} p_k(u)(w)[p_k(v_z)(w)]^- e^{-|w|^2}\,d\lambda(w)\\ &= \frac{1}{p}\int_{\mathbb{C}^n} u(w)\left[\left(\sum_{k=1}^{p} p_k p_k^* + \sum_{k=p+1}^{n} p_k^* p_k\right)(v_z)(w)\right]^- e^{-|w|^2}\,d\lambda(w),\end{aligned}$$

for a uniquely determined function $v_z \in \mathcal{G}_p$. Observe that

$$p\left[\sum_{k=1}^{p} p_k p_k^* + \sum_{k=p+1}^{n} p_k^* p_k\big)(v_z)(w)\right]^- = K(z,w) = \frac{1}{\pi^n}\sum_{k=0}^{\infty}\frac{\langle z,w\rangle^k}{k!}.$$

Now we claim that

$$B_p(z,w) = \frac{p}{\pi^n}\sum_{k=0}^{\infty}\frac{\langle z,w\rangle^k}{(k+p)k!}. \tag{11}$$

For this aim we compute

$$\begin{aligned}\left(\sum_{j=i}^{p} p_j p_j^* + \sum_{j=p+1}^{n} p_j^* p_j\right)(\langle w,z\rangle)^k &= p(\langle w,z\rangle)^k + k\sum_{j=1}^{p} w_j\bar{z}_j(\langle w,z\rangle)^{k-1}\\ &+ k\sum_{j=p+1}^{n} w_j\bar{z}_j(\langle w,z\rangle)^{k-1} = (k+p)(\langle w,z\rangle)^k,\end{aligned}$$

which proves (11). Comparison of the coefficients in the Taylor series yields

$$B_p(z,w) = \frac{p}{\pi^n}\sum_{k=0}^{\infty}\frac{\langle z,w\rangle^k}{(k+p)k!} = \frac{p}{\pi^n}\frac{d^{p-1}}{d\zeta^{p-1}}\left(\frac{e^\zeta - \sum_{k=0}^{p-1}\frac{\zeta^k}{k!}}{\zeta}\right)\Bigg|_{\zeta=\langle z,w\rangle}$$

and we are done. □

Examples:

$$B_1(z,w) = \frac{1}{\pi^n}\frac{e^{\langle z,w\rangle}-1}{\langle z,w\rangle} \quad\text{and}\quad B_2(z,w) = \frac{2}{\pi^n}\,\frac{\langle z,w\rangle e^{\langle z,w\rangle} - e^{\langle z,w\rangle}+1}{(\langle z,w\rangle)^2}.$$

4 Compact Subsets

Here we use the unbounded operators ∂^* and ∂ to exhibit compact subsets of $A^2(\mathbb{C}^n, e^{-|z|^2})$.

Theorem 3 *Let $n = 1$ and define*

$$\mathcal{U} = \left\{ u \in dom(\partial^*) : \|\partial^* u\|^2 \leq 1 \right\}.$$

Then $\mathcal{U}$ is a relatively compact set in $A^2(\mathbb{C}, e^{-|z|^2})$.

Proof The basic estimate implies that $\mathcal{U}$ is a bounded subset of $A^2(\mathbb{C}^n, e^{-|z|^2})$. We have to show that for each $\epsilon > 0$ there exists a constant $R > 0$ such that

$$\int_{\mathbb{C}\setminus D_R} |u(z)|^2 \, e^{-|z|^2} \, d\lambda(z) \leq \epsilon \|\partial^*(u)\|^2, \tag{12}$$

for all $u \in \mathrm{dom}(\partial^*)$, see [4]. For this purpose we take the orthonormal basis $(\varphi_k)_k$ of $A^2(\mathbb{C}, e^{-|z|^2})$, where $\varphi_k(z) = \frac{z^k}{\sqrt{\pi k!}}$, $k = 0, 1, 2, \ldots$ Then, for $u \in A^2(\mathbb{C}, e^{-|z|^2})$, we have $u(z) = \sum_{k=0}^{\infty} u_k \varphi_k(z)$, where $(u_k)_k \in l^2$.

For $u \in \mathrm{dom}(\partial^*)$ we get

$$\partial^* u(z) = z u(z) = \sum_{k=0}^{\infty} u_k \frac{z^{k+1}}{\sqrt{\pi k!}} = \sum_{k=0}^{\infty} u_k \sqrt{k+1} \frac{z^{k+1}}{\sqrt{\pi (k+1)!}}.$$

Then (12) reads as

$$\begin{aligned} \int_{\mathbb{C}\setminus D_R} \sum_{k=0}^{\infty} |u_k|^2 \frac{|z|^{2k}}{\pi k!} \, e^{-|z|^2} \, d\lambda(z) = {} & 2 \sum_{k=0}^{\infty} \frac{|u_k|^2}{k!} \int_R^{\infty} r^{2k+1} \, e^{-r^2} \, dr \\ \leq {} & \epsilon \sum_{k=0}^{\infty} |u_k|^2 (k+1), \end{aligned}$$

where $D_R = \{z \in \mathbb{C} : |z| < R\}$. Comparing the terms in the sum from above, it remains to show that

$$\frac{2}{(k+1)!} \int_R^{\infty} r^{2k+1} \, e^{-r^2} \, dr \leq \epsilon,$$

for all $k = 0, 1, 2, \ldots$. It follows that

$$2 \int_R^{\infty} r^{2k+1} \, e^{-r^2} \, dr = \int_{R^2}^{\infty} s^k \, e^{-s} \, ds.$$

Now we apply k-times integration by parts and get

$$\int_{R^2}^{\infty} s^k\, e^{-s}\, ds = e^{-R^2} k! \sum_{j=0}^{k} \frac{R^{2j}}{j!}.$$

So we still have to prove that for each $\epsilon > 0$ there exists $t > 0$ such that

$$\frac{1}{k+1} e^{-t} \sum_{j=0}^{k} \frac{t^j}{j!} \leq \epsilon, \tag{13}$$

for all $k = 0, 1, 2, \dots$. If $k \geq \frac{1}{\epsilon}$, we obtain

$$\frac{1}{k+1} e^{-t} \sum_{j=0}^{k} \frac{t^j}{j!} \leq \frac{1}{k+1} e^{-t} e^t = \frac{1}{k+1} \leq \epsilon,$$

and if $k \leq \frac{1}{\epsilon}$, we can choose $t > 0$ large enough to get

$$\frac{1}{k+1} e^{-t} \sum_{j=0}^{k} \frac{t^j}{j!} \leq \epsilon,$$

and we obtain (12). $\square$

Remark 2 (a) The operator $\tilde{\Box}_1 = \partial\partial^*$ has a bounded inverse

$$\tilde{N}_1 : A^2_{(1,0)}(\mathbb{C}, e^{-|z|^2}) \longrightarrow \mathrm{dom}(\tilde{\Box}_1).$$

Let $\tilde{j} : \mathrm{dom}(\partial^*) \longrightarrow A^2(\mathbb{C}, e^{-|z|^2})$ be the embedding, where $\mathrm{dom}(\partial^*)$ is endowed with the graph norm $\|\partial^* u\|$. It follows that $\tilde{N}_1 = \tilde{j} \circ \tilde{j}^*$. Since $\mathcal{U} = \{u \in \mathrm{dom}(\partial^*) : \|\partial^* u\|^2 \leq 1\}$ is a precompact set in $A^2(\mathbb{C}, e^{-|z|^2})$, the operator $\tilde{j}$ is compact. This implies $\tilde{N}_1$ is compact, see [7], where we showed directly that the operator $\tilde{j}$ can be approximated by finite dimensional operators. It turns out that the method in the proof of Theorem 3 can be used in more general cases.

(b) In [12] one can find a different proof of compactness using coverings in the corresponding sequence spaces of the Taylor coefficients of the entire functions.

For several variables we get for a $(1,0)$-form $u = \sum_{j=1}^n u_j\, dz_j \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial z_k} \right|^2 e^{-|z|^2}\, d\lambda + \sum_{j=1}^{n} \int_{\mathbb{C}^n} |u_j|^2\, e^{-|z|^2}\, d\lambda. \tag{14}$$

In sake of simplicity we only handle the case $n = 2$; now we use (1), and (14) reads as

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \left\|\frac{\partial u_1}{\partial z_1}\right\|^2 + \left\|\frac{\partial u_1}{\partial z_2}\right\|^2 + \left\|\frac{\partial u_2}{\partial z_1}\right\|^2 + \left\|\frac{\partial u_2}{\partial z_2}\right\|^2 + \|u_1\|^2 + \|u_2\|^2$$
$$= \frac{1}{2}(\|z_1 u_1\|^2 + \|z_2 u_1\|^2 + \|z_1 u_2\|^2 + \|z_2 u_2\|^2)$$
$$+ \frac{1}{2}\left(\left\|\frac{\partial u_1}{\partial z_1}\right\|^2 + \left\|\frac{\partial u_1}{\partial z_2}\right\|^2 + \left\|\frac{\partial u_2}{\partial z_1}\right\|^2 + \left\|\frac{\partial u_2}{\partial z_2}\right\|^2\right).$$

Theorem 4 *Let*

$$\mathcal{U} = \left\{u \in dom(\partial) \cap dom(\partial^*) : \|\partial u\|^2 + \|\partial^* u\|^2 \leq 1\right\}.$$

Then $\mathcal{U}$ is a relatively compact set in $A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$.

Proof We have to show that for each $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{C}^2 \setminus D_R \times D_R} (|u_1(z)|^2 + |u_2(z)|^2) e^{-|z|^2}\, d\lambda(z) \leq \epsilon(\|\partial u\|^2 + \|\partial^* u\|^2),$$

for all $u \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$, where $D_R = \{z \in \mathbb{C} : |z| < R\}$.

It suffices to show that for $u_j = v$, $j = 1, 2$, we have

$$\int_{\mathbb{C}^2 \setminus D_R \times D_R} |v(z)|^2 e^{-|z|^2}\, d\lambda(z) \leq \epsilon(\|z_1 v\|^2 + (\|z_2 v\|^2). \tag{15}$$

We use the orthonormal basis $\varphi_{\alpha,\beta}(z) = \frac{z_1^\alpha}{\sqrt{\pi\alpha!}} \frac{z_2^\beta}{\sqrt{\pi\beta!}}$ and write

$$v(z) = \sum_{\alpha,\beta} v_{\alpha,\beta}\varphi_{\alpha,\beta}(z),$$

where

$$\sum_{\alpha,\beta} |v_{\alpha,\beta}|^2 < \infty.$$

Then (15) reads as

$$\sum_{\alpha,\beta} |v_{\alpha,\beta}|^2 \left(\int_{\mathbb{C}\setminus D_R} \frac{|z_1|^{2\alpha}}{\pi\alpha!} e^{-|z_1|^2}\, d\lambda(z_1) \int_{\mathbb{C}\setminus D_R} \frac{|z_2|^{2\beta}}{\pi\beta!} e^{-|z_2|^2}\, d\lambda(z_2)\right)$$

$$= 4\sum_{\alpha,\beta} \frac{|v_{\alpha,\beta}|^2}{\alpha!\beta!} \left(\int_R^\infty r^{2\alpha+1} e^{-r^2}\, dr \int_R^\infty r^{2\beta+1} e^{-r^2}\, dr\right) \leq \epsilon \sum_{\alpha,\beta} |v_{\alpha,\beta}|^2(\alpha + \beta + 2).$$

Since

$$\frac{1}{\alpha!}\int_R^\infty r^{2\alpha+1}\, e^{-r^2}\, dr \leq 1,$$

for all $\alpha = 0, 1, 2, \ldots,$ we can proceed as in the proof of Theorem 3 to show that (15) is valid. □

In general, we have for $1 \leq p \leq n-1$ and for a $(p, 0)$-form $u = \sum_{|J|=p}{}' \, u_J \, dz_J \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ that

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \sum_{|J|=p}{}' \sum_{j=1}^n \left\| \frac{\partial u_J}{\partial z_j} \right\|^2 + p \sum_{|J|=p}{}' \int_{\mathbb{C}^n} |u_J|^2 \, e^{-|z|^2} \, d\lambda. \tag{16}$$

As an immediate consequence of (16) we get the basic estimates:

$$\|u\|^2 \leq \frac{1}{p}\,(\|\partial u\|^2 + \|\partial^* u\|^2). \tag{17}$$

It is now clear how to adopt the methods we developed for the case $n = 2$ for general n and $(p, 0)$-forms in $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ and we get that the set

$$\mathcal{U} = \left\{u \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*) : \|\partial u\|^2 + \|\partial^* u\|^2 \leq 1\right\}$$

is precompact in $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$, and that the corresponding complex Laplacian

$$\tilde{\Box}_p = \partial^* \partial + \partial \partial^*$$

has a compact inverse

$$\tilde{N}_p : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \longrightarrow A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}).$$

5 The ∂-Complex on Weighted Bergman Spaces on Hermitian Manifolds

The duality between differentiation and multiplication is a special feature of the Segal–Bargmann space: For general weighted spaces of holomorphic functions, we cannot expect such a duality. This poses the first question: *Under what conditions on a Hermitian metric $h_{j\bar{k}} dz^j \otimes d\bar{z}^k$ and a weight function $e^{-\psi}$ on a given complex manifold M, does the corresponding ∂-complex on the weighted Bergman spaces of $(p, 0)$-forms possess a similar duality?* To this end we exhibit conditions on the weight and the Hermitian metric under which the ∂-complex has this desired duality. Our intention here, is to give a survey of recent results together with some examples. All proofs and further examples can be found in [10, 11].

First we recall some basic facts and fix some notations. Let (M, h) be a Hermitian manifold. In holomorphic coordinates $z^1, \dots, z^n$, the metric h has the form

$$h_{j\bar{k}} dz^j \otimes dz^{\bar{k}}, \tag{18}$$

where $\left[h_{j\bar{k}}\right]$ is a positive definite Hermitian matrix with smooth coefficients. Here we use the summation convention. This metric induces a volume element which we denoted by dvol_h. If ψ is a weight function on M, then the Hilbert space of L^2 integrable functions with respect to the measure $d\mu := e^{-\psi}\mathrm{dvol}_h$ is defined by

$$L^2(M, e^{-\psi}\mathrm{dvol}_h) = \left\{ f \colon M \to \mathbb{C} \text{ measurable } : \int_M |f|^2 e^{-\psi}\mathrm{dvol}_h < +\infty \right\}. \tag{19}$$

The weighted Bergman space with weight ψ is defined to be

$$A^2(M, e^{-\psi}\mathrm{dvol}_h) = L^2\left(M, e^{-\psi}\mathrm{dvol}_h\right) \cap \mathcal{O}(M). \tag{20}$$

Here, $\mathcal{O}(M)$ denotes the space of holomorphic functions on M. Under a suitable condition on ψ, the Bergman space $A^2(M, e^{-\psi}\mathrm{dvol}_h)$ is a closed subspace of $L^2(M, e^{-\psi}\mathrm{dvol}_h)$ and thus it is a Hilbert space (although it can be trivial, finite, or infinite dimensional.)

The Hermitian metric h induces a metric on tensors of every degree. For example, if in local coordinates $u = u_j dz^j$ and $v = v_j dz^j$ are $(1, 0)$-forms, then

$$\langle u, v\rangle_h = h^{j\bar{k}} u_j v_{\bar{k}}, \quad |u|_h^2 = \langle u, u\rangle_h \tag{21}$$

where $\left[h^{j\bar{k}}\right]$ is the transpose of the inverse matrix of $\left[h_{j\bar{k}}\right]$. We define the weighted spaces of $(p, 0)$-forms

$$L^2_{(p,0)}(M, h, e^{-\psi}) = \left\{ u \text{ is a } (p, 0)\text{-form}\colon \int_M |u|_h^2\, e^{-\psi}\mathrm{dvol}_h < \infty \right\}, \quad 0 \leqslant p \leqslant n, \tag{22}$$

with inner product

$$(u, v)_{h,\psi} = \int_M \langle u, v\rangle_h\, e^{-\psi}\mathrm{dvol}_h. \tag{23}$$

We say that a $(p, 0)$-form u is holomorphic if in local holomorphic coordinates, we can write

$$u = \sum_{|J|=p}{}' u_J dz^J \tag{24}$$

with holomorphic coefficients u_J and with summation over increasing multiindices. Observe that this notion does not depend on the chosen coordinates (cf. [7]) and hence is well-defined on complex manifolds. We define the Bergman space of $(p, 0)$-forms to be

$$A^2_{(p,0)}(M,h,e^{-\psi}) = \left\{ u \text{ is a holomorphic } (p,0)\text{-form} : \int_M |u|_h^2 \, e^{-\psi} \mathrm{dvol}_h < \infty \right\}. \tag{25}$$

For smooth forms, the ∂-operator is defined in local coordinates by

$$\partial u := \sum_{|J|=p}{}' \sum_{j=1}^{n} \frac{\partial u_J}{\partial z_j} dz^j \wedge dz^J. \tag{26}$$

Thus, if u is holomorphic, then so is ∂u.

For a $(p,0)$-form u in $A^2_{(p,0)}(M,h,e^{-\psi})$, it is not necessary that the $(p+1,0)$-form ∂u is in $A^2_{(p+1,0)}(M,h,e^{-\psi})$. Therefore, we introduce the subspace

$$\mathrm{dom}(\partial_p) = \left\{ u \in A^2_{(p,0)}(M,h,e^{-\psi}) \colon \partial u \in A^2_{(p+1,0)}(M,h,e^{-\psi}) \right\}. \tag{27}$$

Clearly, $\mathrm{dom}(\partial_p)$ also depends on both the metric h and the weight function ψ.

The interesting situation is when $\mathrm{dom}(\partial_p)$ is dense in $A^2_{(p,0)}(M,h,e^{-\psi})$, for each p. In this case, ∂ is a densely defined (bounded or unbounded) operator:

$$\partial_p \colon A^2_{(p,0)}(M,h,e^{-\psi}) \to A^2_{(p+1,0)}(M,h,e^{-\psi}),$$

and the powerful theory of unbounded operators applies, see [3, 6].

Next we establish the relation between the ∂-operators on the weighted Bergman spaces and on the weighted L^2 spaces. We denote by D_p the maximal extension (in the sense of distributions) of the ∂-operator acting on $L^2_{(p,0)}(M,h,e^{-\psi}\mathrm{dvol}_h)$.

Lemma 1 *Let (M,h,ψ) be as above. Then for each $p \geq 0$, it holds that*

$$dom(\partial_p) = dom(D_p) \cap A^2_{(p,0)}(M,h,e^{-\psi}). \tag{28}$$

Let $P_{h,\psi}$ be the Bergman orthogonal projection

$$P_{h,\psi} \colon L^2_{(p,0)}(M,h,e^{-\psi}) \longrightarrow A^2_{(p,0)}(M,h,e^{-\psi}), \tag{29}$$

which is well-defined under the admissibility condition in the sense of [18]. Now we get

Lemma 2 *Suppose that $dom(\partial)$ is dense in $A^2_{(p,0)}(M,h,e^{-\psi})$ and $v \in dom(\partial^*)$. If $w \in L^2_{(p,0)}(M,h,e^{-\psi})$ such that*

$$(\partial u, v)_{h,\psi} = (u, w)_{h,\psi}, \quad \forall\, u \in dom(\partial), \tag{30}$$

then

$$\partial^* v = P_{h,\psi}(w). \tag{31}$$

In particular, if $v \in dom(D^*) \cap A^2_{(p+1,0)}(M, h, e^{-\psi})$ *then*

$$\partial^* v = P_{h,\psi}\left(D^* v\right). \tag{32}$$

For a more detailed description of the operator ∂^* we use the Chern connection of the Hermitian metric h. In local coordinates $z^1, \dots, z^n$, the nonvanishing Christoffel symbols for the Chern connection are

$$\Gamma^i_{jk} = h^{i\bar{l}} \partial_j h_{k\bar{l}}, \quad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^i_{jk}}. \tag{33}$$

Since $\partial_j \left(h^{i\bar{l}} h_{k\bar{l}}\right) = \partial_j(\delta_{ik}) = 0$, we also have $\Gamma^i_{jk} = -h_{k\bar{l}} \partial_j h^{i\bar{l}}$. The covariant derivatives can be explicitly expressed in local coordinates. For examples, if in local coordinate $u = u_k dz^k$ is a $(1, 0)$-form, then

$$\nabla_j u_k = \partial_j u_k - \Gamma^l_{jk} u_l, \quad \nabla_{\bar{j}} u_k = \partial_{\bar{j}} u_k. \tag{34}$$

Note that in the second equation, the Christoffel symbols of "mixed type" vanish and hence the covariant derivative of $(1, 0)$-forms along $(0, 1)$-direction reduces essentially to the partial derivatives of its components.

For a general Hermitian metric, the torsion tensor may be nontrivial; we define the torsion T^i_{jk} by

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad T^{\bar{i}}_{\bar{j}\bar{k}} = \overline{T^i_{jk}} \tag{35}$$

The torsion $(1, 0)$-form is then obtained by taking the trace:

$$\tau = T^i_{ji} dz^j. \tag{36}$$

In local coordinates, the volume element is given by $\mathrm{dvol}_h = \det(h_{j\bar{l}})\, d\lambda$, where $d\lambda$ is the Lebesgue measure in that coordinate patch. Thus, if ψ is an weight function, then we can write (locally) $d\mu = e^{-\psi}\mathrm{dvol} = e^{-\varphi} d\lambda$, with

$$\varphi = \psi - \log\det(h_{j\bar{l}}). \tag{37}$$

Therefore, since $\Gamma^{\bar{i}}_{\bar{k}\bar{i}} = \partial_{\bar{k}} \log\det(h_{j\bar{l}})$ we obtain

$$\varphi_{\bar{k}} = \psi_{\bar{k}} - \partial_{\bar{k}} \log\det(h_{j\bar{l}}) = \psi_{\bar{k}} - \Gamma^{\bar{i}}_{\bar{k}\bar{i}}. \tag{38}$$

If (M, h) is a complete Hermitian manifold we can use the usual cut-off procedure and get for $(1, 0)$-forms

Theorem 5 *Let* (M, h) *be a complete Hermitian manifold and* $e^{-\psi}$ *a smooth weight on* M. *Suppose that* $dom(\partial)$ *is dense in* $A^2_{(1,0)}(M, h, e^{-\psi})$ *and let* $u = u_j dz^j \in A^2_{(1,0)}(M, h, e^{-\psi})$. *Let* ∂^* *be the Hilbert space adjoint of* ∂. *If* $\langle u, \partial\psi - \tau\rangle_h \in L^2(M, h, e^{-\psi})$, *then* u *belongs to* $dom(D^*)$ *and hence* $u \in dom(\partial^*)$. *Moreover,*

$$\partial^* u = P_{h,\psi}\left(\langle u, \partial\psi - \tau\rangle_h\right). \tag{39}$$

where $\tau = \tau_j dz^j$ *is the torsion* $(1,0)$*-form.*

5.1 Holomorphic Vector Fields

Next, we give a condition under which ∂^* agrees with D^* for $u \in \mathrm{dom}(D^*)$ having holomorphic coefficients. To this end, the following notion is crucial for us: Suppose that $\xi = \xi^j \partial_j$ is a $(1,0)$-vector field expressed in a local coordinate patch U. We say that ξ is *holomorphic* if each coefficient ξ^j is holomorphic in U.

The notion of a holomorphic $(1,0)$-vector field does not depend on the choice of coordinate. For a $(0,1)$-form $w = w_{\bar{k}}\, d\bar{z}^k$, the "musical operator" $\sharp$ acts on w and produces an $(1,0)$-vector field $w^\sharp := h^{k\bar{j}} w_{\bar{j}}\, \partial_k$. If u and v are $(1,0)$-forms, then $\langle u, v\rangle_h = (u, \overline{v}^\sharp)$ where the right hand side is the dual pairing between vectors and covectors. Thus, if $u \in A^2_{(1,0)}(M, h, e^{-\psi})$ and $\overline{v}^\sharp$ is a holomorphic vector field, then $\langle u, v\rangle_h$ is a holomorphic function. Thus, we obtain the following

Theorem 6 *Let* (M, h) *be a complete Hermitian manifold and* $e^{-\psi}$ *a smooth weight on* M*. Suppose that* ∂ *is densely defined in the weighted Bergman space and* $(\bar{\partial}\psi - \bar{\tau})^\sharp$ *is a holomorphic vector field, then for each* $u \in dom(D^*) \cap A^2_{(1,0)}(M, h, e^{-\psi})$*, one has*

$$\partial^* u = D^* u. \tag{40}$$

5.2 Example (a)

We consider $U(n)$-invariant Kähler metrics and radial weights. To this end, we suppose that $h_{j\bar{k}}$ is a Kählerian metric induced by a radial potential $h(z) = \tilde{h}(|z|^2)$, where $\tilde{h}(r)$ is a real-valued function of a real variable. Precisely, we have

$$h_{j\bar{k}} = \partial_j \partial_{\bar{k}} \tilde{h}(|z|^2) = \tilde{h}'(|z|^2)\, \delta_{jk} + \tilde{h}''(|z|^2)\, \overline{z}_j z_k.$$

Thus, $h_{j\bar{k}}$ is a rank-one perturbation of a multiple of the identity matrix. For $h_{j\bar{k}}$ to be positive definite, we assume that $\tilde{h}'(r) > 0$ and $r\tilde{h}'' + \tilde{h}' > 0$. The Sherman-Morrison formula give the formula for the (transposed) inverse

$$h^{k\overline{j}} = \frac{1}{\tilde{h}'}\left(\delta_{jk} - \frac{\tilde{h}'' z_k \overline{z}_j}{\tilde{h}' + r\tilde{h}''}\right), \quad r = |z|^2,$$

so that $h_{\bar{l}k} h^{k\overline{j}} = \delta_{\bar{l}}^{\overline{j}}$, the Kronecker symbol.

If $\psi(z) = \tilde{\psi}(r)$, $r = |z|^2$, is a radial weight, then $\partial_{\bar{j}}\psi = \tilde{\psi}'(r)z_j$. Hence

$$\left(\overline{\partial}\psi\right)^{\sharp} = \left(\frac{\tilde{\psi}'}{\tilde{h}' + r\tilde{h}''}\right)\sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}.$$

The holomorphicity of $\left(\overline{\partial}\psi\right)^{\sharp}$ is equivalent to

$$\frac{\tilde{\psi}'}{\tilde{h}' + r\tilde{h}''} = C,$$

for some constant C. This is equivalent to

$$\tilde{\psi} = Cr\tilde{h}' + \tilde{C}, \tag{41}$$

for some constants C and $\tilde{C}$.

Now we consider the unit ball $\mathbb{B} \subset \mathbb{C}^n$ endowed with the Bergman–Kähler metric:

$$h_{j\bar{k}} = -\partial_j\partial_{\bar{k}} \log(1 - |z|^2) = (1 - |z|^2)^{-1}\delta_{jk} + (1 - |z|^2)^{-2}\bar{z}^j z^k. \tag{42}$$

From (41) with $C = \tilde{C} = \alpha$, we let the weight function be

$$\psi(z) = \frac{\alpha}{1 - |z|^2}, \quad \alpha > 0, \tag{43}$$

so that

$$d\mu = e^{-\psi}\mathrm{dvol}_h = (1 - |z|^2)^{-n-1} \exp\left(-\frac{\alpha}{1 - |z|^2}\right) d\lambda. \tag{44}$$

It turns out that this Bergman space with the so-called "exponential weight" has duality properties similar to the Segal–Bargmann space, so it can be seen as a version of the Segal–Bargmann space on a bounded domain. We will show that the adjoint of the densely defined unbounded operator ∂ is the operator multiplication by αz. But in this case we have to take care of the Hermitian metric on $\mathbb{B}$ and of the fact that ∂ maps $A^2_{(p,0)}(\mathbb{B}, h, d\mu)$ into $A^2_{(p+1,0)}(\mathbb{B}, h, d\mu)$ and ∂^* maps $A^2_{(p+1,0)}(\mathbb{B}, h, d\mu)$ into $A^2_{(p,0)}(\mathbb{B}, h, d\mu)$.

Since the weight is radial, the polynomials are dense in Bergman space $A^2_{(p,0)}$ $(\mathbb{B}, h, \psi)$. Observe that

$$h^{j\bar{k}} = (1 - |z|^2)(\delta_{jk} - z^j\bar{z}^k). \tag{45}$$

Therefore, if $u = u_j dz^j$, then

$$|u|_h^2 = u_j u_{\bar{k}} h^{j\bar{k}} = (1 - |z|^2)\left(\sum_{j=1}^{n} |u_j|^2 - \sum_{j,k} z^j\bar{z}^k u_j u_{\bar{k}}\right). \tag{46}$$

Hence, the holomorphic $(1,0)$-forms with polynomial coefficients are in $L^2_{(1,0)}(\mathbb{B}, h, e^{-\psi})$.

We also compute,

$$\psi_{\bar{k}} = \frac{\alpha z^k}{(1-|z|^2)^2}, \tag{47}$$

and find that (sum over k)

$$h^{j\bar{k}}\psi_{\bar{k}} = \alpha z^j \tag{48}$$

are holomorphic. Consequently, for $u = u_j dz^j$

$$\langle u, \partial\psi\rangle_h = \alpha z^j u_j. \tag{49}$$

Thus, if u_j's are holomorphic polynomials, then $\langle u, \partial\psi\rangle_h$ is a holomorphic polynomial and hence $u \in \operatorname{dom}(\partial^*)$. Moreover

$$\partial^*(u_j dz^j) = P_{h,\psi}(\langle u, \partial\psi\rangle_h) = \alpha z^j u_j. \tag{50}$$

5.3 *Example (b)*

Here we consider conformally flat Hermitian metrics of the form

$$h_{j\bar{k}} = e^{\varphi}\delta_{jk}$$

where $\varphi(z) = \tilde{\varphi}(|z|^2)$ and $\tilde{\varphi}(r)$ is a real-valued function of one real variable. Clearly, the torsion tensor and torsion form is

$$T^i_{jk} = \tilde{\varphi}'(|z|^2)\left(\bar{z}_j\delta^i_k - \bar{z}_k\delta^i_j\right), \quad \tau_j = (n-1)\tilde{\varphi}'(|z|^2)\bar{z}_j.$$

Assume that $\psi(z) = \tilde{\psi}(|z|^2)$ for some real-valued function of one real variable $\tilde{\psi}$, then

$$\left(\overline{\partial}\psi - \overline{\tau}\right)^\sharp = \exp\left(-\tilde{\varphi}(|z|^2)\right)\left(\tilde{\psi}'(|z|^2) - (n-1)\tilde{\varphi}'(|z|^2)\right) z^j \frac{\partial}{\partial z_j}.$$

Thus, $\left(\overline{\partial}\psi - \overline{\tau}\right)^\sharp$ is holomorphic if and only if

$$\exp\left(-\tilde{\varphi}(|z|^2)\right)\left(\tilde{\psi}'(|z|^2) - (n-1)\tilde{\varphi}'(|z|^2)\right) = C$$

for some constant C, or equivalently,

$$\tilde{\psi} = C\int e^{\tilde{\varphi}}dr + (n-1)\tilde{\varphi}. \tag{51}$$

Using (51), we can easily exhibit some examples on $\mathbb{C}^2$: Now choose $h_{j\bar{k}} = \delta_{jk}(1+|z|^2)^m$, with $m \geqslant 1$, and

$$\psi(z) = m\log(1+|z|^2) - \frac{\alpha(1+|z|^2)^{m+1}}{m+1}.$$

Then we get

$$(\bar{\partial}\psi - \bar{\tau})^\sharp = -\alpha\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right),$$

if $\alpha < 0$, we get a non-trivial Bergman space $A^2(\mathbb{C}^2, h, e^{-\psi})$. Next we take $h_{j\bar{k}} = \delta_{jk}\exp(|z|^2)$, and

$$\psi(z) = |z|^2 - \alpha\exp(|z|^2).$$

If $\alpha < 0$, we get a non-trivial Bergman space $A^2(\mathbb{C}^2, h, e^{-\psi})$ and again

$$(\bar{\partial}\psi - \bar{\tau})^\sharp = -\alpha\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right).$$

5.4 Real Holomorphic Vector Fields and Holomorphic Torsion

In order to generalize Theorem 6 for higher order forms one has to investigate the torsion of the Chern connection.

For a general Hermitian metric, the torsion tensor T^i_{jk} is defined by

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad T^{\bar{i}}_{\bar{j}\bar{k}} = \overline{T^i_{jk}}, \tag{52}$$

the torsion $(1,0)$-form is then obtained by taking the trace:

$$\tau = T^i_{ji}dz^j. \tag{53}$$

We use $h_{j\bar{k}}$ and its inverse $h^{\bar{l}}$ to lower and raise indices. For example, raising and lowering indices of the torsion, we have

$$T_q{}^{pr} := T^{\bar{i}}_{\bar{j}\bar{k}} h_{q\bar{i}} h^{p\bar{j}} h^{r\bar{k}}. \tag{54}$$

In particular, for a $(0,1)$-form $w = w_{\bar{k}}\,d\bar{z}^k$, raising indices gives the "musical" operator $\sharp$ acting on w and to produce an $(1,0)$ vector field $w^\sharp := h^{k\bar{j}}w_{\bar{j}}\,\partial_k$. Now, if $(\bar{\partial}\psi - \bar{\tau})^\sharp$ is a holomorphic vector field the adjoint operator ∂^* on $\mathrm{dom}(\partial^*) \subset A^2_{(1,0)}(M, h, e^{-\psi})$ can be expressed in the form

$$\partial^* u = \langle u, \partial\psi - \tau \rangle_h. \tag{55}$$

If, in addition, the metric h is Kählerian one has $\tau = 0$ and thus

$$\partial^* u = h^{j\bar{k}} u_j \frac{\partial \psi}{\partial \bar{z}^k}, \tag{56}$$

that means that the complex vector field

$$X := h^{j\bar{k}} \frac{\partial \psi}{\partial \bar{z}^k} \frac{\partial}{\partial z^j} \tag{57}$$

is holomorphic. In this case, the gradient field $\mathrm{grad}_h \psi$ is a *real holomorphic* vector field in the terminology of [15].

We shall see that the holomorphicity of the gradient field of a conformal factor is also related to the holomorphicity of the torsion of the conformal Kähler metric.

We consider the ∂-complex

$$A^2(M, h, e^{-\psi}) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} A^2_{(1,0)}(M, h, e^{-\psi}) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} A^2_{(2,0)}(M, h, e^{-\psi}), \tag{58}$$

and the corresponding complex Laplacian

$$\widetilde{\Box}_p = \partial\partial^* + \partial^*\partial : A^2_{(1,0)}(M, h, e^{-\psi}) \longrightarrow A^2_{(1,0)}(M, h, e^{-\psi}), \tag{59}$$

which, under suitable assumptions, will be a densely defined self-adjoint operator. For $(p, 0)$-forms with $p \geqslant 2$, the holomorphicity of $(\bar{\partial}\psi - \bar{\tau})^\sharp$ is not enough for the adjoint ∂^* to have a simple formula analogous to (55). In order to describe the formula for ∂^* on $(2, 0)$-forms we write

$$v = \frac{1}{2} \sum_{j,k} v_{jk} dz^j \wedge dz^k = \sum_{j<k} v_{jk} dz^j \wedge dz^k, \tag{60}$$

where $v_{jk} = -v_{kj}$. Define an operator $T^\sharp : \Lambda^{2,0}(M) \to \Lambda^{1,0}(M)$ by

$$T^\sharp(v) = \frac{1}{2} T_p{}^{rs} v_{rs} dz^p. \tag{61}$$

where $T_p{}^{rs}$ is given by (54). If $u = u_j dz^j$, we have

$$\partial u = \frac{1}{2} \sum_{j,k} \left(\frac{\partial u_k}{\partial z^j} - \frac{\partial u_j}{\partial z^k} \right) dz^j \wedge dz^k. \tag{62}$$

Moreover, since $v_{pq} = -v_{qp}$, we find that

$$\langle \partial u, v\rangle_h = \sum_{j,k,p,q} \overline{v_{pq}} h^{k\bar{p}} h^{j\bar{q}} \left(\frac{\partial u_k}{\partial z^j}\right). \tag{63}$$

The formula for ∂^* is then given by

$$\partial^* v = P_{h,\psi}\left(-(\psi_{\bar{j}} - \tau_{\bar{j}}) v_{pq} h^{q\bar{j}} dz^p + T^{\sharp}(v)\right). \tag{64}$$

Here, $P_{h,\psi}$ is the orthogonal projection from $L^2_{(2,0)}(M, h, e^{-\psi})$ onto $A^2_{(2,0)}(M, h, e^{-\psi})$. If h is Kähler and $(\bar{\partial}\psi)^{\sharp}$ is holomorphic, then, as in the case of $(1,0)$-forms,

$$\partial^* v = -\psi_{\bar{j}} v_{pq} h^{q\bar{j}} dz^p. \tag{65}$$

We say that h has *holomorphic torsion* if

$$\nabla_{\bar{l}} T_p{}^{rs} = 0, \tag{66}$$

where ∇ is the Chern connection.

Clearly, h has holomorphic torsion if and only if the components of the torsion $T_p{}^{rs}$ (in any holomorphic coordinate frame) are holomorphic. Moreover, it implies that $\bar{\tau}^{\sharp}$ is a holomorphic $(1,0)$-vector field.

Let D^*_p and ∂^*_p be the Hilbert space adjoints of ∂ in the Lebesgue space $L^2_{(p+1,0)}(M, h, e^{-\psi})$ and $A^2_{(p+1,0)}(M, h, e^{-\psi})$, respectively.

Theorem 7 *Let (M, h) be a complete Hermitian manifold with weight $e^{-\psi}$. Assume that the torsion $T_p{}^{rs}$ of the Chern connection is holomorphic. If $(\bar{\partial}\psi)^{\sharp}$ is holomorphic, then for $\eta \in dom(D^*_p)$, $p \geqslant 0$, that is holomorphic in an open set $U \subset M$, $D^*_p\eta$ is also holomorphic in U. In particular, if ∂_p is densely defined in the Bergman space $A^2_{(p,0)}(M, h, e^{-\psi})$, then*

$$D^*_p\eta = \partial^*_p\eta \tag{67}$$

for $\eta \in dom(\partial^*_p)$.

5.5 Conformally Kähler Metrics

Let (M, h) be a Kähler manifold and let $g = \phi^{-1}h$ be a conformal metric. In this section, we study the question when g has holomorphic torsion.

Theorem 8 *Let (M, h) be a Kähler manifold of dimension $n \geqslant 2$ and let $g = \phi^{-1}h$ be a conformally Kähler metric. Let τ^g be the torsion form of g and $\sharp_g$ the sharp "musical" operator associated to g. Then the following are equivalent.*

1. g has holomorphic torsion,
2. $(\overline{\tau}^g)^{\sharp_g}$ is holomorphic,
3. $(\bar\partial\phi)^\sharp$ is holomorphic.

Example. Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$ and let

$$h_{j\bar k} = (1-|z|^2)^{-1}\left(\delta_{jk} + \frac{\bar z_j z_k}{1-|z|^2}\right) \tag{68}$$

be the complex hyperbolic metric on $\mathbb{B}^n$. Let $g = \phi^{-1}h$ be a conformal metric on $\mathbb{B}^n$. Then g has holomorphic torsion if and only if

$$(1-|z|^2)\phi = \sum_{j,k} c_{j\bar k} z^j \bar z^k + \Re\left(\sum_k \alpha_k z^k\right) + \gamma \tag{69}$$

where $c_{j\bar k}$ is a Hermitian matrix, $\alpha_k \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

If $h_{j\bar k}$ is a Kählerian metric induced by a radial potential $h(z) = \tilde h(|z|^2)$, where $\tilde h(r)$ is a real-valued function of a real variable, we have

$$h_{j\bar k} = \partial_j \partial_{\bar k} \tilde h(|z|^2) = \tilde h'(|z|^2)\,\delta_{jk} + \tilde h''(|z|^2)\,\bar z_j z_k. \tag{70}$$

Thus, $h_{j\bar k}$ is a rank-one perturbation of a multiple of the identity matrix. For $h_{j\bar k}$ to be positive definite, we assume that $\tilde h'(r) > 0$ and $r\tilde h''(r) + \tilde h'(r) > 0$. The Sherman-Morrison formula give the formula for the (transposed) inverse

$$h^{k\bar j} = \frac{1}{\tilde h'}\left(\delta_{jk} - \frac{\tilde h'' z_k \bar z_j}{\tilde h' + r\tilde h''}\right), \quad r = |z|^2, \tag{71}$$

so that $h_{\bar l k} h^{k\bar j} = \delta_{\bar l}^{\bar j}$, the Kronecker symbol.

Let g be the conformally $U(n)$-invariant Kähler metric

$$g_{j\bar k} = e^{\tilde\sigma(|z|^2)} \partial_j \partial_{\bar k} \tilde h(|z|^2) \tag{72}$$

and $\psi(z) = \tilde\psi(|z|^2)$ is a real-valued radial weight function. The vector field $(\bar\partial\psi - \bar\tau)^\sharp$ and the torsion operator $T^\sharp$ are holomorphic if and only if

$$\tilde\sigma(r) = -\log(C_2 r\tilde h'(r) + C_3) \tag{73}$$

and

$$\tilde\psi(r) = -C_4 \log(C_2 r\tilde h'(r) + C_3) + C_5, \tag{74}$$

where $C_4 = n - 1 - (C_1/C_2)$ and the constant C_3 has to be chosen such that $C_2 r\tilde h'(r) + C_3 > 0$.

In this case we have for the vector field $(\bar\partial\psi - \bar\tau)^\sharp = C_1\sum_{j=1}^n z^j\partial_j$ and for the torsion operator

$$T^\sharp(v) = -C_2\sum_{q=1}^n z^q v_{pq}dz^p.$$

6 The Generalized ∂-Complex

We consider the following generalization of the ∂-complex. If one replaces the single derivative with respect to z_j by a differential operator of the form $p_j(\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n})$, where p_j is a complex polynomial on $\mathbb{C}^n$ (we write $p_j(u)$ for $p_j(\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n})(u)$), the duality relation is now of the form

$$(p_j(u), v) = (u, p_j^* v),$$

where $p_j^*(z_1,\dots,z_n)$ is the polynomial p_j with complex conjugate coefficients, taken as multiplication operator. Newman and Shapiro [16, 17] use this duality relation in their analysis of Fischer spaces of entire functions.

We generalize the ∂-operator by setting

$$Du = \sum_{j=1}^n p_j(u)\,dz_j, \tag{75}$$

where $u \in A^2(\mathbb{C}^n, e^{-|z|^2})$, see [6].

Operating on $(p,0)$-forms we define

$$Du = {\sum_{|J|=p}}' \sum_{k=1}^n p_k(u_J)\,dz_k \wedge dz_J, \tag{76}$$

where $u = \sum_{|J|=p}{}' u_J\,dz_J$ is a $(p,0)$-form with coefficients in $A^2(\mathbb{C}^n, e^{-|z|^2})$, here $J = (j_1,\dots,j_p)$ is a multiindex and $dz_J = dz_{j_1}\wedge\cdots\wedge dz_{j_p}$ and the summation is taken only over increasing multiindices. We get again densely defined closed operators and observe that $D^2 = 0$ and that we have

$$(Du, v) = (u, D^*v), \tag{77}$$

where $u \in \mathrm{dom}(D) = \{u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) : Du \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2})\}$ and

$$D^*v = {\sum_{|K|=p}}' \sum_{j=1}^n p_j^* v_{jK}\,dz_K$$

for $v = \sum_{|J|=p+1}{}' v_J\,dz_J$.

Replacing ∂ by D one gets a corresponding complex Laplacian $\tilde{\Box}_D = DD^* + D^*D$, for which one can use duality and the machinery of the $\bar\partial$-Neumann operator [13, 14] in order to prove existence and boundedness of the inverse to $\tilde{\Box}_D$ and to find the canonical solutions to the inhomogeneous equations $Du = \alpha$ and $D^*v = \beta$.

In the $\bar\partial$-Neumann problem the underlying Hilbert space is $L^2(\Omega)$ and the $\bar\partial$-operator is defined in the sense of distributions in order to become a densely defined unbounded operator on $L^2(\Omega)$ with closed graph. The adjoint operator $\bar\partial^*$ is again a differential operator. In our setting, the underlying Hilbert space is $A^2(\mathbb{C}^n, e^{-|z|^2})$, the operator D is a densely defined unbounded operator on $A^2(\mathbb{C}^n, e^{-|z|^2})$ with closed graph, the adjoint operator D^* is now a multiplication operator.

Similar to the classical $\bar\partial$-complex (see [6]) we consider the generalized box operator $\tilde{\Box}_{D,p} := D^*D + DD^*$ as a densely defined self-adjoint operator on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ with

$$\mathrm{dom}(\tilde{\Box}_{D,p}) = \{f \in \mathrm{dom}(D) \cap \mathrm{dom}(D^*) : Df \in \mathrm{dom}(D^*) \text{ and } D^*f \in \mathrm{dom}(D)\}.$$

see [7] for more details, see [1] for similar differential operators on the Segal Bargmann space.

The $(p, 0)$-forms with polynomial components are dense in $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$. In addition, the $(p, 0)$-forms with polynomial components are dense in $\mathrm{dom}(D) \cap \mathrm{dom}(D^*)$ endowed with the graph norm

$$u \mapsto (\|u\|^2 + \|Du\|^2 + \|D^*u\|^2)^{1/2}.$$

6.1 Basic Estimates

It is shown in [7] that the basic estimate

$$\|u\|^2 \le C(\|Du\|^2 + \|D^*u\|^2), \tag{78}$$

for any $u \in \mathrm{dom}(D) \cap \mathrm{dom}(D^*)$ implies that $\tilde{\Box}_{D,1}$ has a bounded inverse. It turns out that properties of the commutators $[p_k, p_j^*]$ give a sufficient condition for the basic estimate to hold. See Remark 1, that the basic estimate can be seen as a generalization of the uncertainty principle.

Remark 3 We give a simple example in $\mathbb{C}^2$, where the basic estimate does not hold. Let p be a polynomial differential operator without constant summand and set $D(v) = p(v)dz_1 - p(v)dz_2$ for a polynomial v. If $u = u_1dz_1 + u_2dz_2$ is a $(1, 0)$-form with polynomial components, we have $D(u) = (p(u_2) - p(u_1))dz_1 \wedge dz_2$ and $D^*u = p^*u_1 - p^*u_2$. Now take for $u = dz_1 + dz_2$. Then $D(u) = 0$ and $D^*u = p^*1 - p^*1 = 0$. As $\|u\| \neq 0$, the basic estimate does not hold.

More general, let p_2^* be a homogeneous polynomial of degree $d_2 > 1$, take a homogeneous polynomial q^* of degree $d \ge 1$ and set $p_1^* = p_2^*q^*$. Then p_1^* has degree

$d_2 + d$. Consider the $(1, 0)$-form $u = u_1 dz_1 + u_2 dz_2$ with homogeneous polynomials u_1 and u_2. Suppose that the degree of u_1 is $e_1 \geq 1$, but $e_1 < d_2$. Then $p_2(u_1) = 0$, and if we let $u_2 = -q^* u_1$, we get that the degree of u_2 equals $e_1 + d < d_2 + d$. Hence $p_1(u_2) = 0$ and

$$p_1^* u_1 + p_2^* u_2 = p_2^* q^* u_1 - p_2^* q^* u_1 = 0.$$

So, $D(u) = 0$ and $D^* u = 0$, and the basic estimate fails.

Remark 4 The basic estimates can be viewed as generalizations of the uncertainty principle, see Remark 1. Let $n = 1$ and $p = \frac{\partial^m}{\partial z^m}$. Then we have

$$\|z^m u\|^2 = \|u\|^2 + \sum_{j=0}^{m-1} \left\| \frac{\partial}{\partial z}(z^j u) \right\|^2 .$$

So we get

$$\|u\|^2 \leq \|z^m u\|^2 + \left\| \frac{\partial^m u}{\partial z^m} \right\|^2 .$$

For a general polynomial differential operator p of degree m, with leading coefficient a_m, we obtain from [8] that there exists a constant $C > 0$ such that, for each $u \in \mathrm{dom}(D^*)$, we have $\|u\| \leq C \|D^* u\| = C \|p^* u\|$, where $C = \sqrt{\frac{1}{m!|a_m|^2}}$. Hence we have again

$$\|u\|^2 \leq C(\|p^* u\|^2 + \|p(u)\|^2).$$

For several complex variables, we have only partial results. We finally discuss a general method and exhibit some examples.

6.2 Commutator Terms as a Sum of Squared Norms

In [9] it is shown that

$$([p_k, p_j^*]u_j, u_k) = \sum_{|\alpha| \geq 1} \frac{1}{\alpha!} (p_j^{(\alpha)*} p_k^{(\alpha)} u_j, u_k). \tag{79}$$

Here, the following operator theoretic method is used. Let A_j and B_j, $j = 1, \ldots, n$ be operators satisfying

$$[A_j, A_k] = [B_j, B_k] = [A_j, B_k] = 0,\ j \neq k$$

and

$$[A_j, B_j] = I,\ j = 1, \ldots, n.$$

Let P and Q be polynomials of n variables and write $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$.

The assumptions are satisfied, if one takes $A_j = \frac{\partial}{\partial z_j}$ and $B_j = z_j$ the multiplication operator. The inspiration for this comes from quantum mechanics, where the annihilation operator A_j can be represented by the differentiation with respect to z_j on $A^2(\mathbb{C}^n, e^{-|z|^2})$ and its adjoint, the creation operator B_j, by the multiplication by z_j.

Then

$$Q(A)P(B) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} P^{(\alpha)}(B) Q^{(\alpha)}(A), \tag{80}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ are multiindices and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \dots \alpha_n!$, see [19, 20].

Theorem 9 ([7, 9]) *Suppose that there exists a constant $C > 0$ such that*

$$\|u\|^2 \leq C \sum_{j,k=1}^{n} ([p_k, p_j^*]u_j, u_k), \tag{81}$$

for any $(1, 0)$-form $u = \sum_{j=1}^n u_j dz_j$ with polynomial components. Then (78) holds.

In [9] homogeneous polynomials p_1 and p_2 of arbitrary degree K in two variables are determined such that (81) holds. For this purpose (79) is used to express

$$\sum_{j,k=1}^{n} ([p_k, p_j^*]u_j, u_k)$$

as a sum of squared norms, see also [2].

6.3 Hermitian Forms

Let u and v be $(1, 0)$-forms with polynomial components and define

$$H(u, v) = \sum_{j,k=1}^{n} ([p_k, p_j^*]u_j, v_k), \tag{82}$$

Then

$$H : A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \times A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \longrightarrow \mathbb{C}$$

is a densely defined Hermitian form. Inequality (81) can be expressed by coerciveness of the Hermitian form H, see [9] for more details.

Example. We indicate it can happen that for a suitable choice of the polynomials p_1^* and p_2^* one gets

$$\sum_{j,k=1}^{2}([p_k, p_j^*]u_j, u_k) = 0,$$

for a non-zero $(1,0)$-form $u = u_1 dz_1 + z_2 dz_2$. For this purpose we take real numbers a, b, c, d and set

$$p_1^*(z_1, z_2) = az_1^2 + bz_2 \text{ and } p_2^*(z_1, z_2) = cz_1 + dz_2^2.$$

Then

$$\sum_{j,k=1}^{2}([p_k, p_j^*]u_j, u_k)$$

$$= (2a^2 + b^2)\|u_1\|^2 + (c^2 + 2d^2)\|u_2\|^2 + 4a^2 \left\| \frac{\partial u_1}{\partial z_1} \right\|^2 + 4d^2 \left\| \frac{\partial u_2}{\partial z_2} \right\|^2$$

$$+ 2ac[(u_2, z_1 u_1) + (z_1 u_1, u_2)] + 2bd[(u_1, z_2 u_2) + (z_2 u_2, u_1)].$$

Note that $\|u_1\|^2 + \|\frac{\partial u_1}{\partial z_1}\|^2 = \|z_1 u_1\|^2$ and $\|u_2\|^2 + \|\frac{\partial u_2}{\partial z_2}\|^2 = \|z_2 u_2\|^2$. Hence we get

$$4a^2\|u_1\|^2 + 4a^2 \left\| \frac{\partial u_1}{\partial z_1} \right\|^2 + 2ac[(u_2, z_1 u_1) + (z_1 u_1, u_2)] + c^2\|u_2\|^2 = \|cu_2 + 2az_1 u_1\|^2$$

and

$$4d^2\|u_2\|^2 + 4d^2 \left\| \frac{\partial u_2}{\partial z_2} \right\|^2 + 2bd[(u_1, z_2 u_2) + (z_2 u_2, u_1)] + b^2\|u_1\|^2 = \|bu_1 + 2dz_2 u_2\|^2.$$

This implies

$$-2a^2\|u_1\|^2 - 2d^2\|u_2\|^2 \le \sum_{j,k=1}^{2}([p_k, p_j^*]u_j, u_k).$$

In a similar way one obtains

$$-b^2\|u_1\|^2 - c^2\|u_2\|^2 \le \sum_{j,k=1}^{2}([p_k, p_j^*]u_j, u_k).$$

Hence the Hermitian form

$$H(u,v)=\sum_{j,k=1}^{2}([p_k,p_j^*]u_j,v_k):A^2_{(1,0)}(\mathbb{C}^2,e^{-|z|^2})\times A^2_{(1,0)}(\mathbb{C}^2,e^{-|z|^2})\longrightarrow\mathbb{C}$$

is densely defined and lower semibounded, we have

$$-A\|u\|^2\le H(u,u),$$

where $A=\max\{2a^2,b^2,c^2,2d^2\}$.

Now we choose $u_1(z_1,z_2)=-z_1z_2^2$ and $u_2(z_1,z_2)=z_1z_2$. Then $\|u_1\|^2=2\pi^2$, $\|u_2\|^2=\pi^2$, $\|\frac{\partial u_1}{\partial z_1}\|^2=2\pi^2$, $\|\frac{\partial u_2}{\partial z_2}\|^2=\pi^2$ and $(u_1,z_2u_2)=(z_2u_2,u_1)=-2\pi^2$ and $(u_2,z_1u_1)=(z_1u_1,u_2)=0$. Hence

$$\sum_{j,k=1}^{2}([p_k,p_j^*]u_j,u_k)=\pi^2(12a^2+2b^2+c^2+6d^2-8bd).$$

Now set $a=0,b=5,c=4,d=3$, then

$$\sum_{j,k=1}^{2}([p_k,p_j^*]u_j,u_k)=0.$$

It is even possible to achieve negative values for $\sum_{j,k=1}^{2}([p_k,p_j^*]u_j,u_k)$: take $a=0,b=5,c=2,d=3$, then

$$\sum_{j,k=1}^{2}([p_k,p_j^*]u_j,u_k)=25\|u_1\|^2+22\|u_2\|^2+36\left\|\frac{\partial u_2}{\partial z_2}\right\|^2+30\left(u_2,\frac{\partial u_1}{\partial z_2}\right)+30\left(\frac{\partial u_1}{\partial z_2},u_2\right).$$

Now take again $u_1(z_1,z_2)=-z_1z_2^2$ and $u_2(z_1,z_2)=z_1z_2$. Then

$$\sum_{j,k=1}^{2}([p_k,p_j^*]u_j,u_k)=-12\pi^2.$$

But taking into account that $\|u_2\|^2+\|\frac{\partial u_2}{\partial z_2}\|^2=\|z_2u_2\|^2$ we get an even better estimate

$$\begin{aligned}-8(\|u_1\|^2+\|u_2\|^2)&\le 25\|u_1\|^2+22\|u_2\|^2+36\left\|\frac{\partial u_2}{\partial z_2}\right\|^2+30\left(u_2,\frac{\partial u_1}{\partial z_2}\right)+30\left(\frac{\partial u_1}{\partial z_2},u_2\right)\\&=25\|u_1\|^2+22\|u_2\|^2+36\left\|\frac{\partial u_2}{\partial z_2}\right\|^2+30(z_2u_2,u_1)+30(u_1,z_2u_2),\end{aligned}$$

since

$$0 \leq 33\|u_1\|^2 + 30\|u_2\|^2 + 36 \left\| \frac{\partial u_2}{\partial z_2} \right\|^2 + 30(z_2 u_2, u_1) + 30(u_1, z_2 u_2)$$
$$= 3\|u_1\|^2 + 30\|u_1 + z_2 u_2\|^2 + 6 \left\| \frac{\partial u_2}{\partial z_2} \right\|^2 .$$

6.4 Further Examples

In the following we enhance the list of the D-complexes, where the basic estimate holds.

(a) Let $n = 2$ and take

$$p_1^*(z_1, z_2) = z_1^2 + 2z_1 z_2 + z_2^2 \text{ and } p_2^*(z_1, z_2) = z_1^2 - 2z_1 z_2 + z_2^2.$$

In order to show that the basic estimate holds we have to prove that there exists a constant $C > 0$ such that

$$\|u\|^2 \leq C \sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k), \tag{83}$$

for any $(1, 0)$-form $u = \sum_{j=1}^{2} u_j dz_j$ with polynomial components. Direct computations give

$$([p_1, p_1^*]u_1, u_1) = 8\|u_1\|^2 + 8\left(\frac{\partial u_1}{\partial z_1}, \frac{\partial u_1}{\partial z_1}\right) + 8\left(\frac{\partial u_1}{\partial z_2}, \frac{\partial u_1}{\partial z_2}\right)$$
$$+ 8\left(\frac{\partial u_1}{\partial z_1}, \frac{\partial u_1}{\partial z_2}\right) + 8\left(\frac{\partial u_1}{\partial z_2}, \frac{\partial u_1}{\partial z_1}\right).$$

Hence we get

$$([p_1, p_1^*]u_1, u_1) = 8\|u_1\|^2 + 8 \left\| \frac{\partial u_1}{\partial z_1} + \frac{\partial u_1}{\partial z_2} \right\|^2 \tag{84}$$

similarly

$$([p_2, p_2^*]u_2, u_2) = 8\|u_2\|^2 + 8 \left\| \frac{\partial u_2}{\partial z_1} - \frac{\partial u_2}{\partial z_2} \right\|^2 . \tag{85}$$

Finally, one observes that

$$[p_1, p_2^*] = 0\,,\ [p_2, p_1^*] = 0. \tag{86}$$

Hence

$$\|u\|^2 \le \frac{1}{8} \sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k),$$

and we get the basic estimate:

$$\|u\|^2 \le C(\|Du\|^2 + \|D^*u\|^2),$$

for $u = \sum_{j=1}^{2} u_j dz_j \in \mathrm{dom}(D) \cap \mathrm{dom}(D^*)$.

If we take

$$p_1(z_1, z_2) = az_1^2 + 2bz_1z_2 + cz_2^2, \quad p_2(z_1, z_2) = az_1^2 - 2bz_1z_2 + cz_2^2$$

it turns out, that one obtains an analogue formula as in (84), (85), (86) only if $ac = b^2$ and $a \ge b$ and $c \ge b$, so one gets two different solutions: $a = b = c$ and $a = c$, $a = -b$. In this connection it turns out that the corresponding box operator

$$\tilde{\Box}(u) = (DD^* + D^*D)(u_1dz_1 + u_2dz_2)$$

is of diagonal form:

$$\begin{aligned}\tilde{\Box}(u) &= \left[p_1p_1^*(u_1) + p_1p_2^*(u_2) + p_2^*p_2(u_1) - p_2^*p_1(u_2)\right] dz_1 \\ &\quad + \left[p_2p_1^*(u_1) + p_2p_2^*(u_2) + p_1^*p_1(u_2) - p_1^*p_2(u_1)\right] dz_2 \\ &= \left[p_1p_1^*(u_1) + p_2^*p_2(u_1)\right] dz_1 + \left[p_2p_2^*(u_2) + p_1^*p_1(u_2)\right] dz_2,\end{aligned}$$

by (86).

(b) It is easily seen that for $p_1^*(z_1, z_2) = z_1^k$ and $p_2^*(z_1, z_2) = z_2^m$, where $k, m \in \mathbb{N}$, we get (86) and

$$\|u\|^2 \le C \sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k),$$

where $C > 0$.

(c) Let

$$p_1^*(z_1, z_2) = az_1 + bz_2 \text{ and } p_2^*(z_1, z_2) = cz_1 + dz_2,$$

for $a, b, c, d \in \mathbb{R}$. A computation shows that

$$([p_1, p_1^*]u_1, u_1) = (a^2 + b^2)\|u_1\|^2 \text{ and } ([p_2, p_2^*]u_2, u_2) = (c^2 + d^2)\|u_2\|^2,$$

$$([p_1, p_2^*]u_2, u_1) = (ac + bd)(u_2, u_1) \text{ and } ([p_2, p_1^*]u_1, u_2) = (ac + bd)(u_1, u_2).$$

Given $a, b \in \mathbb{R}$, all non-zero, we suppose that $c = \mu b$, $d = -\mu a$, for $\mu \in \mathbb{R}$. Then we have $ac + bd = 0$ and we get

$$\|u\|^2 \le C \sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k),$$

where $C = \min\{(a^2+b^2), (c^2+d^2)\}^{-1}$.

d) If we take

$$p_1^*(z_1, z_2) = az_1^2 + bz_2^2 \text{ and } p_2^*(z_1, z_2) = cz_1^2 + dz_2^2,$$

for $a, b, c, d \in \mathbb{R}$, we get

$$\sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k) = 2\|au_1 + cu_2\|^2 + 2\|bu_1 + du_2\|^2 + \left\|2a\frac{\partial u_1}{\partial z_1} + 2c\frac{\partial u_2}{\partial z_1}\right\|^2 + \left\|2b\frac{\partial u_1}{\partial z_2} + 2d\frac{\partial u_2}{\partial z_2}\right\|^2.$$

Given a, b, c, all non-zero, we choose d such that $ac = -bd$. Then

$$2\|au_1 + cu_2\|^2 + 2\|bu_1 + du_2\|^2 = 2(a^2+b^2)\|u_1\|^2 + 2(c^2+d^2)\|u_2\|^2,$$

which implies that

$$\|u\|^2 \le C \sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k),$$

where $C = \min\{2(a^2+b^2), 2(c^2+d^2)\}^{-1}$.

For general $m \in \mathbb{N}$, and $p_1^*(z_1, z_2) = az_1^m + bz_2^m$ and $p_2^*(z_1, z_2) = cz_1^m + dz_2^m$ we use Leibniz' rule to show that

$$\sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k) = \sum_{k=0}^{m} \frac{m!}{k!}\binom{m}{k}\left(\left\|a\frac{\partial^k u_1}{\partial z_1^k} + c\frac{\partial^k u_2}{\partial z_1^k}\right\|^2 + \left\|b\frac{\partial^k u_1}{\partial z_2^k} + d\frac{\partial^k u_2}{\partial z_2^k}\right\|^2\right)$$

we suppose $ac = -bd$ and we have to take $C = \min\{m!(a^2+b^2), m!(c^2+d^2)\}^{-1}$.

6.5 Compactness

Let p denote the polynomial differential operator

$$p = a_0 + a_1\frac{\partial}{\partial z} + \cdots + a_m\frac{\partial^m}{\partial z^m},$$

with constant coefficients $a_0, a_1, \dots, a_m \in \mathbb{C}$, $a_m \neq 0$, $m \geq 1$ and let p_m^* denote the polynomial

$$p_m^*(z) = \overline{a}_0 + \overline{a}_1 z + \cdots + \overline{a}_m z^m,$$

with the complex conjugate coefficients $\overline{a}_0, \overline{a}_1, \dots, \overline{a}_m \in \mathbb{C}$.

First, we consider the case $m = 1$, taking $p^*(z) = \bar{a}_0 + \bar{a}_1 z$. We suppose that $a_0, a_1 \neq 0$, and choose a complex number c with

$$\frac{|a_1|^2}{1 + \frac{|a_1|^2}{|a_0|^2}} < |c|^2 < |a_1|^2. \tag{87}$$

Now let

$$d := \frac{a_0 \bar{a}_1}{\bar{c}}. \tag{88}$$

Then, using (1), we get for an arbitrary polynomial u that

$$\begin{aligned}
\|D^* u\|^2 = \; & \|\bar{a}_0 u + \bar{a}_1 z u\|^2 \\
= \; & |a_0|^2 \|u\|^2 + \bar{a}_0 a_1 (u, zu) + \bar{a}_1 a_0 (zu, u) + |a_1|^2 \|zu\|^2 \\
= \; & |a_0|^2 \|u\|^2 + \bar{a}_0 a_1 (u', u) + \bar{a}_1 a_0 (u, u') + |a_1|^2 (\|u\|^2 + \|u'\|^2) \\
\geq \; & (|a_0|^2 + |a_1|^2) \|u\|^2 + \bar{a}_0 a_1 (u', u) + \bar{a}_1 a_0 (u, u') + (|a_1|^2) \|u'\|^2 - \|du + cu'\|^2 \\
= \; & (|a_0|^2 + |a_1|^2 - |d|^2) \|u\|^2 + (|a_1|^2 - |c|^2) \|u'\|^2 \\
\geq \; & \alpha (\|u\|^2 + \|u'\|^2) \\
= \; & \alpha \|zu\|^2,
\end{aligned}$$

where

$$\alpha = \min(|a_0|^2 + |a_1|^2 - |d|^2, |a_1|^2 - |c|^2). \tag{89}$$

Note that (87) and (88) imply $\alpha > 0$.

For general p we obtain from [8] that there exists a constant $C > 0$ such that, for each $u \in \mathrm{dom}(D^*)$, we have $\|u\| \leq C \|D^* u\|$, and

$$\begin{aligned}
&\int_{\mathbb{C}} [p, p^*] \, u(z) \; \overline{u(z)} \; e^{-|z|^2} \, d\lambda(z) \\
&\qquad = \sum_{\ell=1}^{m} \ell \, ! \int_{\mathbb{C}} \left| \sum_{k=\ell}^{m} \binom{k}{\ell} a_k u^{(k-\ell)}(z) \right|^2 e^{-|z|^2} \, d\lambda(z).
\end{aligned} \tag{90}$$

Now we get from (90) taking $\ell = m$ and $\ell = m - 1$

$$\begin{aligned}
\|D^* u\|^2 = (p^* u, p^* u) = \; & ([p, p^*] u, u) + (p(u), p(u)) \\
\geq \; & m! |a_m|^2 \|u\|^2 + (m-1)! \|a_{m-1} u + m a_m u'\|^2.
\end{aligned}$$

If $a_{m-1} = 0$, we get

$$\|D^*u\|^2 \geq m!|a_m|^2(\|u\|^2 + \|u'\|^2) = m!|a_m|^2\|zu\|^2. \tag{91}$$

If $a_{m-1} \neq 0$, we set $a = \sqrt{m!}\, a_m$ and $b = \sqrt{(m-1)!}\, a_{m-1}$ and get

$$\begin{aligned} \|D^*u\|^2 \geq \; & m!|a_m|^2\|u\|^2 + (m-1)!\|a_{m-1}u + ma_m u'\|^2 \\ = \; & (|a|^2 + |b|^2)\,\|u\|^2 + b\bar{a}(u, u') + a\bar{b}(u', u) + |a|^2\,\|u'\|^2, \end{aligned}$$

and we can proceed as before to get

$$\|D^*u\|^2 \geq \alpha\|zu\|^2, \tag{92}$$

for $\alpha > 0$.

Using the orthonormal basis $(\varphi_k)_k$ we write

$$u = \sum_{k=0}^{\infty} u_k \varphi_k,$$

and get

$$\|zu\|^2 = \sum_{k=0}^{\infty} |u_k|^2(k+1).$$

Now we want to show that for each $\epsilon > 0$ there exists a constant $R > 0$ such that

$$\int_{\mathbb{C}\setminus\mathbb{B}_R} |u(z)|^2\, e^{-|z|^2}\, d\lambda(z) \leq \epsilon\|D^*u\|^2, \tag{93}$$

for all $u \in \mathrm{dom}(D^*)$. As for the operator ∂^* we get

$$\int_{\mathbb{C}\setminus\mathbb{B}_R} |u(z)|^2\, e^{-|z|^2}\, d\lambda(z) \leq \alpha\epsilon\|\partial^*u\|^2,$$

and by (91) and (92)

$$\int_{\mathbb{C}\setminus\mathbb{B}_R} |u(z)|^2\, e^{-|z|^2}\, d\lambda(z) \leq \alpha\epsilon\|D^*u\|^2,$$

and so $\{u \in \mathrm{dom}(D^*) : \|D^*u\|^2 \leq 1\}$ is a relatively compact set in $A^2(\mathbb{C}, e^{-|z|^2})$.

We can now summarize in the following

Theorem 10 *Let $Du = p_m(u)\, dz$, where*

$$p = a_0 + a_1\frac{\partial}{\partial z} + \cdots + a_m\frac{\partial^m}{\partial z^m},\ a_m \neq 0, m \geq 1.$$

Then the operator $\tilde{\Box}_{D,1} = DD^*$ *has a compact inverse*

$$\tilde{N}_{D,1} : A^2_{(1,0)}(\mathbb{C}, e^{-|z|^2}) \longrightarrow dom(\tilde{\Box}_{D,1}).$$

Finally we consider the problem of compactness for the generalized ∂-complex, where $p_1(z_1, z_2)^* = z_1^2 + z_2^2$ and $p_2(z_1, z_2)^* = 2z_1z_2$. For $(1, 0)$-forms $u = u_1dz_1 + u_2dz_2 \in \text{dom}(D) \cap \text{dom}(D^*)$ we prove the following

Theorem 11 *Let* $p_1(z_1, z_2)^* = z_1^2 + z_2^2$ *and* $p_2(z_1, z_2)^* = 2z_1z_2$. *Let*

$$\mathcal{U} = \{u \in dom(D) \cap dom(D^*) : \|Du\|^2 + \|D^*u\|^2 \leq 1\}.$$

Then $\mathcal{U}$ *is a relatively compact set in* $A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$.

Proof We have to show that for each $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{C}^2 \setminus D_R \times D_R} (|u_1(z)|^2 + |u_2(z)|^2) e^{-|z|^2} \, d\lambda(z) \leq \epsilon(\|Du\|^2 + \|D^*u\|^2), \tag{94}$$

for all $u \in \text{dom}(D) \cap \text{dom}(D^*)$.

We apply (79) and get

$$\begin{aligned} \|Du\|^2 + \|D^*u\|^2 &= \sum_{j,k=1}^{2} \|p_j(u_k)\|^2 + \sum_{j,k=1}^{2} ([p_k, p_j^*]u_j, u_k) \\ &\geq 4\left(\left\|\frac{\partial^2 u_1}{\partial z_1 \partial z_2}\right\|^2 + \left\|\frac{\partial^2 u_2}{\partial z_1 \partial z_2}\right\|^2\right) \\ &+ 4\left(\left\|\frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2}\right\|^2 + \left\|\frac{\partial u_1}{\partial z_2} + \frac{\partial u_2}{\partial z_1}\right\|^2 + \|u_1\|^2 + \|u_2\|^2\right) \\ &\geq 4\left(\left\|\frac{\partial^2 u_1}{\partial z_1 \partial z_2}\right\|^2 + \left\|\frac{\partial^2 u_2}{\partial z_1 \partial z_2}\right\|^2 + \|u_1\|^2 + \|u_2\|^2\right). \end{aligned} \tag{95}$$

We use the orthonormal basis $\varphi_{\alpha,\beta}(z) = \frac{z_1^\alpha}{\sqrt{\pi\alpha!}} \frac{z_2^\beta}{\sqrt{\pi\beta!}}$ and write

$$u_1 = \sum_{\alpha,\beta} u^{(1)}_{\alpha,\beta} \varphi_{\alpha,\beta} \text{ and } u_2 = \sum_{\alpha,\beta} u^{(2)}_{\alpha,\beta} \varphi_{\alpha,\beta},$$

where

$$\sum_{\alpha,\beta} |u^{(1)}_{\alpha,\beta}|^2 < \infty \text{ and } \sum_{\alpha,\beta} |u^{(2)}_{\alpha,\beta}|^2 < \infty.$$

Observe that

$$\left\| \frac{\partial^2 u_j}{\partial z_1 \partial z_2} \right\|^2 = \sum_{\alpha,\beta} \alpha\beta |u_{\alpha,\beta}^{(j)}|^2,$$

for $j = 1, 2$.

By (94) and (95) we have to prove that for each $\epsilon > 0$ there exists $R > 0$ such that

$$\sum_{\alpha,\beta} \frac{|u_{\alpha,\beta}^{(j)}|^2}{\alpha!\beta!} \left(\int_R^\infty r^{2\alpha+1} e^{-r^2} \, dr \int_R^\infty r^{2\beta+1} e^{-r^2} \, dr \right) \leq \epsilon \sum_{\alpha,\beta} |u_{\alpha,\beta}^{(j)}|^2 (\alpha\beta + 1). \tag{96}$$

It suffices to show that

$$\frac{1}{\alpha!\beta!(\alpha\beta+1)} e^{-R^2} \alpha! \sum_{k=0}^{\alpha} \frac{R^{2k}}{k!} e^{-R^2} \beta! \sum_{k=0}^{\beta} \frac{R^{2k}}{k!} \leq \epsilon.$$

If $\alpha = 0$ we have to show

$$e^{-2R^2} \sum_{k=0}^{\beta} \frac{R^{2k}}{k!} \leq \epsilon,$$

which is easily achieved. In all other cases we can proceed as in the proof of Theorem 4 to get the desired result. □

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Asymptotic Construction of the Optimal Degeneration for a Fano Manifold

Tomoyuki Hisamoto

Abstract Optimal degeneration is the algebraic counterpart of the prescribed geometric flow. We review some construction of the degeneration via the multiplier ideal sheaves and then consider their geometric quantization.

Keywords Kähler-Einstein metric · K-stability

1 Geometric Flow and Optimal Degeneration

1.1 Kähler-Ricci Flow

Let (X, L) be an complex n-dimensional polarized algebraic manifold. We are mainly interested in the Fano case where the polarization $L = -K_X$ is the anti-canonical line bundle. In this setting *geometric flow* means a certain time-evolution of Kähler metrics $\omega = \omega(t)$ in the first Chern class $c_1(L)$. A typical example is the normalized Kähler-Ricci flow equation:

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}\omega + \omega. \tag{1}$$

From the result of [10], the solution exists for all $0 \leqslant t < \infty$ and converges to the Kähler-Einstein metric if it exists. Using $\partial\bar{\partial}$-lemma we may rephrase the above equation in terms of the potential function. Indeed if a reference metric ω_0 is fixed one may take a function φ such that $\omega = \omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$. Similarly one has a function ρ such that $\operatorname{Ric}\omega - \omega = \sqrt{-1}\partial\bar{\partial}\rho$. We choose the normalization

$$\int_X (e^\rho - 1)\omega^n = 0 \tag{2}$$

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so that $\rho = \rho_\omega$ is uniquely determined by ω and the Eq. (1) is translated into

$$\frac{\partial}{\partial t}\varphi = -\rho. \tag{3}$$

In the analysis of the Kähler-Ricci flow it is also important to rephrase the equation in terms of the associated measures. The first measure we have in mind is the Monge-Ampère measure $\omega^n = \omega_\varphi^n$ of φ. We often divide it by the volume $V = \int_X \omega_\varphi^n$ to consider the probability measure $V^{-1}\omega_\varphi^n$. In order to encode the information of the Ricci curvature one needs another probability measure

$$d\mu_\varphi = \frac{e^{-\varphi}d\mu_0}{\int_X e^{-\varphi}d\mu_0}, \quad d\mu_0 = e^{\rho_0}\omega_0^n, \tag{4}$$

which we call *the canonical measure*. The Eq. (1) now can be written down to the second order partial differential equation

$$\frac{\partial}{\partial t}\varphi = -\log\left[\frac{d\mu_\varphi}{V^{-1}\omega_\varphi^n}\right]. \tag{5}$$

Following [30], we introduce the entropy functional

$$S(\omega) = \int_X \rho e^\rho \omega^n = \int_X \log\left[\frac{d\mu_\varphi}{V^{-1}\omega_\varphi^n}\right]\mu_\varphi \tag{6}$$

which is actually the relative entropy $\mathrm{Ent}(\mu|\nu) = \int \log\left[\frac{d\mu}{d\nu}\right]d\mu$ defined for two probability measures μ, ν. By the definition $S(\omega)$ is non-negative. One can also check that the entropy is non-increasing along the Kähler Ricci flow ([38, 40]) and that $S(\omega) = 0$ if and only if ω is Kähler-Einstein. From the standard probability theory we may express the entropy as the convex conjugate form

$$S(\omega) = \sup_f\left[\int_X f d\mu_\varphi - \log\frac{1}{V}\int_X e^f\omega_\varphi^n\right], \tag{7}$$

where f runs through arbitrary real-valued continuous functions. The above expression of the entropy is closely related to *the large deviation principle* which tells us that the rate function measuring the rarity of atypical phenomenon is given in the convex conjugate form of the moment generating function.

1.2 Optimal Degeneration

The famous Yau-Tian-Donaldson conjecture states that a polarized manifold (X, L) admits a constant scalar curvature Kähler metric in $c_1(L)$ if and only if it is K-polystable. To ensure the existence of such a standard metric many authors propose much more stronger stability condition but we do not enter the topic here. K-stability is examined by certain degenerations of (X, L).

Definition 1 ([24]. *See also* [6] *for the updated terminology.*) A flat family of polarized schemes $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1$ endowed with a torus action $\lambda : \mathbb{C}^* \to \mathrm{Aut}(\mathcal{X}, \mathcal{L})$ is called *test configuration* if it satisfies the following conditions.

(1) The projection morphism π is equivariant with respect to λ and the natural $\mathbb{C}^*$-action to $\mathbb{P}^1$.
(2) The family is equivariantly isotrivial outside $0 \in \mathbb{P}^1$. That is, over the affine line $\mathbb{A}^1$ one has $(\mathcal{X}_{\mathbb{A}^1}, \mathcal{L}_{\mathbb{A}^1}) \simeq (\mathbb{C} \times X, p_2^* L)$, where $p_2 : \mathbb{C} \times X \to X$ is the second projection. We encode the isomorphism into the datum of the test configuration.
(3) The total space $\mathcal{X}$ is normal.

In particular, for each $k \in \mathbb{N}$ the induced $\mathbb{C}^*$-action to $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$ produces the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{N_k} \in \mathbb{R}$. Distribution of these eigenvalues in the limit $k \to \infty$ defines the Duistermaat-Heckman measure

$$\mathrm{DH}(\mathcal{X}, \mathcal{L}) = \lim_{k \to \infty} \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{\frac{\lambda_i}{k}} \tag{8}$$

associated with the test configuration. As $k \to \infty$, this invariant governs the leading term coefficient of the equivariant Riemann-Roch theorem on the central fiber and hence determines the leading term of the usual Riemann-Roch theorem on the total space $\mathcal{X}$. Indeed the first moment of the Duistermaat-Heckman measure is equivalent to the self-intersection number of $\mathcal{L}$:

$$E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \frac{\mathcal{L}^{n+1}}{(n+1)V} = \hat{\lambda} := \int_{\lambda \in \mathbb{R}} \lambda \mathrm{DH}(\mathcal{X}, \mathcal{L}). \tag{9}$$

We call the second moment

$$\|(\mathcal{X}, \mathcal{L})\|_2 = \left[\int_{\lambda \in \mathbb{R}} (\lambda - \hat{\lambda})^2 \mathrm{DH}(\mathcal{X}, \mathcal{L}) \right]^{\frac{1}{2}} \tag{10}$$

as the norm of the test configuration. See [6, 31] for the detail.

A polarized manifold is called K-semistable if the essentially sub-leading term coefficient

$$M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = V^{-1} K^{\log}_{\mathcal{X}/\mathbb{P}^1} \mathcal{L}^n + \hat{S} E^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \tag{11}$$

is semipositive for all $(\mathcal{X}, \mathcal{L})$. In this article we follow the notation of [6] in order to regard the right-hand side as the non-Archimedean version of the K-energy functional

$$M(\varphi) = \mathrm{Ent}(V^{-1}\omega_\varphi^n | V^{-1}\omega_0^n) + R(\varphi) + \hat{S}E(\varphi). \tag{12}$$

Here $E\colon \mathcal{H} \to \mathbb{R}$ is the Monge-Ampère energy

$$E(\varphi) = \frac{1}{(n+1)V} \sum_{i=0}^{n} \int_X \varphi \omega_0^{n-i} \wedge \omega_\varphi^i \tag{13}$$

designed to have the derivative $(d_\varphi E)(u) = \frac{1}{V}\int_X u\omega_\varphi^n$. For more precise definition of these energies and the non-Archimedean point of view we refer [6, 7]. At any rate we do not study the K-energy in detail here. In relation with the Kähler Ricci flow we prefer *the non-Archimedean entropy*.

Definition 2 (*DS17*) Assume for simplicity that $\mathcal{X}$ has at worst Gorenstein singularities and that the central fiber $\mathcal{X}_0$ is a reduced scheme. We define the non-Archimedean entropy of the test configuration $(\mathcal{X}, \mathcal{L})$ as

$$S^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \deg \pi_*(K_{\mathcal{X}/\mathbb{P}^1} \otimes \mathcal{L}) + \log \int_{\lambda \in \mathbb{R}} e^{-\lambda} \mathrm{DH}(\mathcal{X}, \mathcal{L}). \tag{14}$$

We say (X, L) is *S-semistable* if $S^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \geqslant 0$ for all such test configurations with mild singularities.

Following [2], one can extend the first term $L^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \deg \pi_*(K_{\mathcal{X}/\mathbb{P}^1} \otimes \mathcal{L})$ to arbitrary test configurations using the log canonical threshold. On the other hand, the discussion in [34] reduces the semistability to testing the sign of $S^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ for such $(\mathcal{X}, \mathcal{L})$ with mild singularities. We see that (14) is comparable with the definition of entropy (6). In both definitions the first term encodes the information of the canonical divisor and the second term is related to the moment generating function.

We may now explain what is the optimal degeneration of a Fano manifold.

Theorem 1 ([8, 13, 21, 29]) *For an arbitrary Fano manifold one has*

$$\inf_\omega S(\omega) = \max_{(\mathcal{X},\mathcal{L})} (-S^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})). \tag{15}$$

There exists a unique test configuration $(\mathcal{X}, \mathcal{L})$ which achieves the equality and it coincides with the first one of the 2-step degenerations generated from the Gromov-Hausdorff limit of the Kähler-Ricci flow $(X, \omega(t))$.

There are various other geometric flows seeking for the standard metric. For example, in [12] the authors introduced *the inverse Monge-Ampère flow*

$$\frac{\partial}{\partial t}\varphi = 1 - e^\rho \tag{16}$$

Table 1 Geometric flows and optimal degenerations

Geometric flow	Stationary metric	Invariant	Optimal degeneration
Calabi flow	Extremal Kähler metric	$M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$	?
Inverse MA flow	Mabuchi's soliton	$D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$	[12, 43]
Kähler-Ricci flow	Kähler-Ricci soliton	$S^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$	[8, 13]
Aubin's continuity method	Twisted cscK	$M_\theta^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$	[9, 35]
Donaldon's continuity method	Conical singular KE	$M_\beta^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$	[11]

for a Fano manifold. This is the gradient flow of so-called D-energy functional

$$D(\varphi) = L(\varphi) - E(\varphi) = -\log \frac{1}{V}\int_X e^{-\varphi} d\mu_0 - E(\varphi) \tag{17}$$

and one has the non-Archimedean analogue

$$D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \deg \pi_*(K_{\mathcal{X}/\mathbb{P}^1} \otimes \mathcal{L}) - \int_{\lambda\in\mathbb{R}} \lambda \mathrm{DH}(\mathcal{X}, \mathcal{L}). \tag{18}$$

In [12] we showed that for any toric Fano manifold there exists a test configuration achieving the equality

$$\inf_{\omega} \left[\int_X (e^{\rho} - 1)^2 \omega^n\right]^{\frac{1}{2}} = \max_{(\mathcal{X},\mathcal{L})} \frac{-D^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|_2}. \tag{19}$$

Thus we obtain the degeneration $(\mathcal{X}, \mathcal{L})$ which optimizes the invariant D^{NA}. In the toric case the central fiber of the D-optimal degeneration has two irreducible component while the S-optimal degeneration is simply a product space. This is in contrast to the stable case where the Kähler-Einstein metric is characterized by both functionals S and D. In Table 1 we list the other major geometric flows and corresponding invariants for test configurations.

2 Construction via the Multiplier Ideal Sheaves

2.1 *Space of Kähler Metrics*

In [32] we gave an asymptotic construction of the optimal degeneration using multiplier ideal sheaves. This is built on the variational approach to the Kähler-Einstein metric, developed by many authors. For Fano manifolds this is accomplished by [5].

See also [44]. I refer the textbook [27] for the pluripotential theoretic ideas exploited in this area. Once a reference metric $\omega_0 \in c_1(L)$ is fixed the space of Kähler metric is identified with the collection of potentials

$$\mathcal{H} = \mathcal{H}(X, \omega_0) = \left\{ \varphi \in C^\infty(X; \mathbb{R}) : \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \right\} \tag{20}$$

and various energies like $M, D, S, \dots$ in the previous section defines functionals on $\mathcal{H}$. The tangent space at $\varphi \in \mathcal{H}$ is naturally identified with $C^\infty(X; \mathbb{R})$ and is equipped with the canonical but non-trivial Riemannian metric

$$u \mapsto \frac{1}{V} \int_X u^2 \omega_\varphi^n. \tag{21}$$

Indeed the geodesic curvature of a given curve φ_t $(a \leqslant t \leqslant b)$ is computed as

$$c(\varphi) = \ddot{\varphi} - \left|\bar{\partial}\dot{\varphi}\right|^2. \tag{22}$$

Let us take the annulus $A = \{\tau \in \mathbb{C} : e^{-b} \leqslant |\tau| \leqslant e^{-a}\}$. Now the remarkable idea of [42] is that the curve φ_t should be identified with a $\mathbb{S}^1$-invariant function Φ on the product space $A \times X$ in the manner

$$\Phi(\tau, z) = \varphi_{-\log|\tau|}(z). \tag{23}$$

In terms of Φ, geodesic equation for φ_t is translated into the degenerate Monge-Ampère equation

$$(p_2^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} \equiv 0, \tag{24}$$

where ∂-operator is taken for $(n+1)$-complex variables (τ, z).

It is a standard fact that both K-energy and D-energy are convex along any smooth geodesic on $\mathcal{H}$. In general the geodesic connecting given endpoints φ_a, φ_b is not necessarily contained in $\mathcal{H}$ and we need to consider the weak solution of (24). Singular metrics are also indispensable to construct the completion of the space $\mathcal{H}$ where one takes a convergent subsequence of the minimizing sequence of the energy functional. See [5] for the detail.

One can also consider for $p \geqslant 1$ the L^p-structure

$$u \mapsto \frac{1}{V} \int_X |u|^p \, \omega_\varphi^n \tag{25}$$

of $\mathcal{H}$. It defines a distance on $\mathcal{H}$:

$$d_p(\varphi_a, \varphi_b) = \inf_{\varphi_t} \left[\frac{1}{V} \int_a^b |\dot{\varphi}_t|^p \, \omega_{\varphi_t}^n \right]^{\frac{1}{p}}, \tag{26}$$

where φ_t runs through every smooth curve joining φ_a with φ_b. These L^p-geometry were studied in the seminal work of Darvas [16–18]. Especially $p = 1$ case plays an important role in the variational approach since the L^1-structure satisfies the following compactness result.

Theorem 2 ([4], Theorem 2.17. see also [7], Lemma 2.11) *Let $\mathcal{E}^p(X, \omega_0)$ be the L^p-completion of the space $\mathcal{H}(X, \omega_0)$. For any constant $C > 0$,*

$$\left\{\varphi \in \mathcal{E}^1(X, \omega_0) : \sup_X \varphi = 0, \ \ \mathrm{Ent}(\omega_\varphi^n | \omega_0^n) \leqslant C\right\} \tag{27}$$

is a compact subset of $\mathcal{E}^1(X, \omega_0)$.

On the other hand, it is observed by [16] that geodesics connecting given endpoints φ_a, φ_b are not unique on $\mathcal{E}^1(X, \omega_0)$. However, the weak solution of the degenerate Monge-Ampère equation (24) defines a geodesic on $\mathcal{E}^p(X, \omega_0)$ for any $p \geqslant 1$. From this reason we may call the solution of (24) *standard geodesic*.

2.2 Multiplier Ideal on the Product Space

Let us return to the Kähler-Ricci flow equation (1). If fix a sequence $0 = t_0 < t_1 < \cdots < t_j < \cdots$, we obtain a standard geodesic $\psi_{j,t}$ $(0 \leqslant t \leqslant t_j)$ joining φ_0 with φ_{t_j}. Since the K-energy is non-increasing along the flow, it follows from (12) that $\mathrm{Ent}(\omega_\varphi^n | \omega_0^n)$ is bounded on the fixed interval $[0, t_j]$. We may thus apply Theorem 2 so that obtain a geodesic ray ψ_t defined for all $0 \leqslant t < \infty$. Equivalently, a function on the product space $\Psi(\tau, z) = \psi_{-\log|\tau|}(z)$. From the linear estimate $\psi_t \leqslant ct + b$, $\Psi + c\log|\tau|$ is uniqly extended to $\mathbb{B} \times X$. We call Ψ a weak geodesic ray asymptotic to the Kähler-Ricci flow.

It is not the case for the Kähler-Ricci flow Φ but Ψ defines a plurisubharmonic function on the product space so that the multiplier ideal $\mathcal{J}(\Psi + c\log|\tau|)$ is coherent analytic thanks to the pioneering work of [37].

Theorem 3 ([32], Theorem B) *Let Ψ be the weak geodesic ray asymptotic to the Kähler-Ricci flow. Let $k \in \mathbb{N}$. The blowing up along the multiplier ideal sheaf $\mathcal{J}(k(\Psi + c\log|\tau|))$ defines $f : \mathcal{X}_k \to \mathbb{P}^1 \times X$ with the exceptional divisor E_k. Take a sufficiently large ℓ and for each k set the line bundle*

$$\mathcal{L}_k = f^* p_2^* L - \frac{1}{k+\ell} E_k + \frac{ck}{k+\ell} f^* \mathcal{X}_{k,0}. \tag{28}$$

Then $(\mathcal{X}_k, \mathcal{L}_k)$ define a sequence of test configurations and $-S^{\mathrm{NA}}(\mathcal{X}_k, \mathcal{L}_k)$ converges to the right-hand side of the equality (15).

The present construction is weaker than those of [13] or [8] but is purely complex-analytic and requires neither Cheeger-Colding theory nor the heavy results of the minimal model program. It seems difficult to analyze the singularity of the above multiplier ideal sheaf so one should ask if there exists more effective construction for each k. This problem motivates the topic in the next subsection.

3 Quantization

The detailed exposition in this subsection will appear in the forthcoming paper [33].

3.1 Geometric Quantization

Let us briefly recall the framework of the geometric quantization. For each $k \in \mathbb{N}$ we denote by $\mathcal{H}_k$ the collection of all positive definite Hermitian inner products on the vector space $H^0(X, L^{\otimes k})$. From the polar decomposition for the matrices one obtains the homogeneous space expression $\mathcal{H}_k \simeq \mathrm{GL}(N_k; \mathbb{C})/U(N_k)$. The key geometric quantization is the map

$$f_k : \mathcal{H}_k \to \mathcal{H} \tag{29}$$

which takes an orthonormal basis $s_1, s_2, \dots, s_{N_k}$ of $H \in \mathcal{H}_k$ and sends H to the metric of Fubini-Study type

$$f_k(H) = \frac{1}{k} \log \frac{1}{N_k} \sum_{i=1}^{N_k} |s_i|^2 . \tag{30}$$

One also has the converse direction map $p_k : \mathcal{H} \to \mathcal{H}_k$ taking the L^2-inner product

$$p_k(\varphi) = \int_X |\cdot|^2 e^{-k\varphi} d\mu_\varphi . \tag{31}$$

Here the reference fiber metric $|\cdot|^2$ of the line bundle L is taken such that its Chern curvature equals to ω_0. Notice that $d\mu_\varphi$ is the canonical measure defined in (4). We set $b_k = p_k \circ f_k$ and $\beta_k = f_k \circ p_k$. In general $\beta_k(\varphi) \neq \varphi$ but for any fixed φ the C^∞-convergence

$$\lim_{k \to \infty} \beta_k(\varphi) = \varphi \tag{32}$$

holds. This is very special case of the famous Bergman kernel asymptotic expansion. In this sense at lease pointwise the space $\mathcal{H}_k$ approximates $\mathcal{H}$. The homogeneous $\mathcal{H}_k$ is naturally endowed with the invariant Riemannian metric and the two spaces $\mathcal{H}_k$

and $\mathcal{H}$ shares analogous geometric structure including the expression of geodesics and the curvature tensors. See [28] for the exposition.

Lemma 1 *For any choice of the orthonormal basis s_i one has*

$$\sum \|s_i\|^2_{b_k(H)} = N_k. \tag{33}$$

Proof It is straightforward to compute

$$\sum \|s_i\|^2_{b_k(H)} = \sum \int_X |s_i|^2 \, e^{-kf_k(H)} d\mu_{f_k(H)} \tag{34}$$

$$= \int N_k e^{kf_k(H)} e^{-kf_k(H)} d\mu_{f_k(H)} \tag{35}$$

$$= N_k \int d\mu_{f_k(H)} = N_k. \tag{36}$$

□

The Hermitian inner product H is called *balanced* if $b_k(H) = H$ holds. If we take H-orthonormal basis s_i which is also $b_k(H)$-orthogonal, the balanced condition is equivalent to $\|s_i\|_{b_k(H)} = 1$. It is known from the work of Donaldson [23] that the balanced condition gives the finite-dimensional analogue of the Kähler-Einstein metric. More specifically, one has the finite-dimensional Monge-Ampère energy

$$E_k(H) = -\frac{1}{kN_k} \log \det_{H_0} H \tag{37}$$

so that D-energy is quantized to be

$$D_k = L \circ f_k - E_k. \tag{38}$$

Then the critical point of the D_k-energy is precisely the balanced Hermitian inner product defined above. Moreover, image of the k-balanced Hermitian inner product $f_k(H)$ converges to the Kähler-Einstein metric if it uniquely exists. See [3, 4] for the detailed exposition.

3.2 Quantization of the Flow

Let us first consider a curve H_t on $\mathcal{H}_k$. If H_t is geodesic one can take an orthonormal basis s_i of H_0 such that

$$e^{\frac{\lambda_i}{2}t} s_i \quad (1 \leqslant i \leqslant N_k) \tag{39}$$

gives an orthonormal basis of H_t. It implies that for a (not necessarily geodesic) curve H_t, tangent vector at t can be represented by a pair of an H_t-orthonomal basis $s_i(t)$ and weights $\lambda_i(t)$.

Definition 3 We call a smooth curve H_t on $\mathcal{H}_k$ is *a quantized flow* if $s_i(t)$ are chosen $b_k(H)$-orthogonal and satisfy the ODE

$$\frac{\lambda_i}{k} = -\log \|s_i\|^2_{b_k(H)}. \tag{40}$$

The equation is more naturally described as $\frac{d}{dt}\log H = k \log b_k(H) - k \log H$. It in fact defines a vector field on $\mathcal{H}_k$. By the completeness, for any given initial point $H(0)$ we obtain a unique long-time solution $H(t)$. The above definition is comparable with the expression (3). We consider the row vector $\rho_k(H) := (\log \|s_i\|^2_{b_k(H)})_{i=1}^{N_k}$ as the finite-dimensional analogue of the Ricci potential ρ. Lemma 1 now states that ρ_k automatically satisfies the normalization condition similar to (2). This is simply due to the choice of $d\mu_\varphi$ in the definition of the map p_k. If one uses unnormalized measure $e^{-\varphi}d\mu_0$ in the definition of p_k, the Eq. (40) quantizes the unnormalized Kähler-Ricci flow $\frac{\partial}{\partial t}\omega = -\operatorname{Ric}\omega$. This point was essentially observed by [1] in the context of his Bergman iteration.

As we mentioned in the previous paragraph [1] studied another quantization of the Kähler-Ricci flow. It is taking the Bergman iteration $\beta_k^j(\varphi) = (\beta_k \circ \beta_k \circ \cdots \circ \beta_k)(\varphi)$ for a given φ. We may compare the quantized flow (40) with the Bergman iteration. In particular, as an application of [1] Theorem 4.18, the quantized flow approaches the Kähler-Ricci flow in the following sense.

Theorem 4 *Let φ_t be the Kähler-Ricci flow (1) and H_t be the quantized flow (40) initiated from $H_0 = p_k(\varphi_0)$. Then for any $T > 0$ there exists a constant C_T such that*

$$\left|\varphi_{j/k} - f_k(H_{j/k})\right| \leqslant C_T \frac{j}{k^2} \tag{41}$$

holds for any $j/k \leqslant T$.

We may also quantize the entropy functional defined in (6).

Definition 4 Define the quantized entropy of $H \in \mathcal{H}_k$ by the convex conjugate form

$$S_k(H) = \sup_{\lambda_i \in \mathbb{R}} \left\{ \sum -\frac{\lambda_i}{k} \frac{\|s_i\|^2_{b_k(H)}}{N_k} - \log \frac{1}{N_k} \sum e^{-\frac{\lambda_i}{k}} \right\}, \tag{42}$$

where s_i is chosen H-orthonormal and $b_k(H)$-orthogonal.

An easy computation leads more explicit form

$$S_k(N) = \log N_k + \sum \frac{\|s_i\|^2_{b_k(H)}}{N_k} \log \frac{\|s_i\|^2_{b_k(H)}}{N_k} \tag{43}$$

and hence S_k is nonnegative. Moreover, $S_k(H) = 0$ precisely when H is balanced.

3.3 Quantized Optimal Degeneration

For terminologies about norms and filtrations, we refer [6], Sect. 1.

Recall that the non-Archimedean norm ν on the vector space V is a function $\nu \colon V \to \mathbb{R}_{\geqslant 0}$ such that

(1) $\nu(s) = 0$ if and only if $s = 0$.
(2) $\nu(a \cdot s) = \nu(s)$ holds for any $a \in \mathbb{C}^*$, $s \in V$, and
(3) $\nu(s + s') \leqslant \max\{\nu(s), \nu(s')\}$ holds for any $s, s' \in V$.

We denote by $\mathcal{H}_k^{\mathrm{NA}}$ the set of non-Archimedean norms on $H^0(X, L^{\otimes k})$. The associated filtration

$$F^\lambda H^0(X, L^{\otimes k}) = \left\{ s \in H^0(X, L^{\otimes k}) : \nu(s) \leqslant e^{-\lambda} \right\} \tag{44}$$

produces weights $\lambda_1, \lambda_2, \dots, \lambda_{N_k}$ and vectors $s_1, s_2, \dots, s_{N_k}$ describing the jump of dimension. In this way each element of $\mathcal{H}_k^{\mathrm{NA}}$ can be seen as a direction pointed to by a geodesic ray on $\mathcal{H}_k$. The space $\mathcal{H}_k^{\mathrm{NA}}$ gives the quantization of $\mathcal{H}^{\mathrm{NA}}$: the space of Kähler metrics on the Berkovich analytification X^{NA} by the non-Archimedean Fubini-Study map

$$f_k \colon \mathcal{H}_k^{\mathrm{NA}} \to \mathcal{H}^{\mathrm{NA}}. \tag{45}$$

For the background materials bout Berkovich analytifictions see [6], Sect. 6. In the non-Archimedean setting we should take the maximum in place of the square sum so that f_k is defined to be

$$f_k(\nu) = \max_s \frac{1}{k} \log \frac{|s|^2}{\nu(s)} \in \mathcal{H}^{\mathrm{NA}}. \tag{46}$$

Here $|s|$ is a function on X^{NA}, which satisfies $|s(v)|^2 = e^{-v(s)}$ for any quasi-monomial valuation $v \in X^{\mathrm{NA}}$.

Any equivalent class of test configuration defines the element of $\mathcal{H}^{\mathrm{NA}}$. In fact the filtration

$$F^\lambda_{(\mathcal{X},\mathcal{L})} H^0(X, L^{\otimes k}) = \left\{ s \in H^0(X, L^{\otimes k}) : \tau^{-\lambda} s \text{ extends holomorphically to } \mathcal{X}. \right\} \tag{47}$$

associated with the test configuration $(\mathcal{X}, \mathcal{L})$ defines $\nu_k \in \mathcal{H}_k^{\mathrm{NA}}$. Using the Gauss extension for the valuation one can see that

$$f_k(\nu_k)(v) = -\frac{1}{k} G(v)(\mathcal{L}^{\otimes k}) \tag{48}$$

holds. The right-hand side is independent of k and hence defines $\nu_{(\mathcal{X},\mathcal{L})} \in \mathcal{H}^{\mathrm{NA}}$.

Definition 5 Let $F_k^{\mathrm{NA}}(\nu) = -\log \frac{1}{N_k}\sum_i e^{-\frac{\lambda_i}{k}}$. Define the non-Archimedean version of the quantized entropy as

$$S_k^{\mathrm{NA}} = L^{\mathrm{NA}} \circ f_k - F_k^{\mathrm{NA}}. \tag{49}$$

Theorem 5 *We have the quantized version of the equality (15) so that*

$$\inf_{H \in \mathcal{H}_k} S_k(H) = \max_{\nu \in \mathcal{H}_k^{\mathrm{NA}}} (-S_k^{\mathrm{NA}}(\nu)) \tag{50}$$

holds for any fixed $k \in \mathbb{N}$.

In particular if there exists a balanced Hermitian inner product then $S_k^{\mathrm{NA}}(\nu) \geqslant 0$ for all $\nu \in \mathcal{H}_k^{\mathrm{NA}}$. This is similar to the result of [41].

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Semi-classical Spectral Asymptotics of Toeplitz Operators on Strictly Pseudodonvex Domains

Chin-Yu Hsiao and George Marinescu

Abstract On a relatively compact strictly pseudoconvex domain with smooth boundary in a complex manifold of dimension n we consider a Toeplitz operator T_R with symbol a Reeb-like vector field R near the boundary. We show that the kernel of a weighted spectral projection $\chi(k^{-1}T_R)$, where χ is a cut-off function with compact support in the positive real line, is a semi-classical Fourier integral operator with complex phase, hence admits a full asymptotic expansion as $k \to +\infty$. More precisely, the restriction to the diagonal $\chi(k^{-1}T_R)(x, x)$ decays at the rate $O(k^{-\infty})$ in the interior and has an asymptotic expansion on the boundary with leading term of order k^{n+1} expressed in terms of the Levi form and the pairing of the contact form with the vector field R.

Keywords Bergman projector · Szegö projector · Toeplitz operator · Semi-classical Fourier intergral operator

1 Introduction

Since the introduction of the Bergman kernel in [1], and the subsequent groundbreaking work by Hörmander [9], Fefferman [7], and Boutet de Monvel and Sjöstrand [4], the study of the Bergman kernel has been a central subject in several complex variables and complex geometry.

Let M be a relatively compact strictly pseudoconvex domain with smooth boundary in a complex manifold M' and let $B : L^2(M) \to H^0_{(2)}(M)$ be the Bergman projection, that is, the orthogonal projection from the space of square-integrable func-

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tions $L^2(M)$ onto the space of L^2 holomorphic functions on M. The Bergman kernel $B(x, y)$ is the Schwartz kernel of B. Fefferman [7] obtained the complete asymptotic expansion of the diagonal Bergman kernel $B(x, x)$ at the boundary. Subsequently, Boutet de Monvel-Sjöstrand [4] described the singularity of the full Bergman kernel $B(x, y)$ by showing that it is a Fourier integral operator with complex phase. They also obtained in [4] a full asymptotic expansion for the Szegő projection on a strictly pseudoconvex CR manifold. All the results mentioned above are about microlocal behavior of the Szegő and Bergman kernels. Some of these results were recently extended to weakly pseudoconvex domains of finite type in $\mathbb{C}^2$ in [16]. The structure of the Szegő projector also plays an important role in the quantization of CR manifolds [15].

On the other hand, semi-classical analysis plays an important role in modern complex geometry. For example, we can study many important problems in complex geometry by using semi-classical Bergman kernel asymptotics [5, 11, 12, 17]. Therefore, we believe that it is important to study classical several complex variables from semi-classical viewpoint. To this end, it is important to have semiclassical versions of the Boutet de Monvel-Sjöstrand's and Fefferman's results on strictly pseudoconvex CR manifolds and on complex manifolds with strictly pseudoconvex boundary. Recently, we obtained jointly with Herrmann and Shen [8] a semi-classical version of the Boutet de Monvel and Sjöstrand's result on a strictly peudoconvex CR manifold and as applications, we established Kodaira type embedding theorem and Tian type theorem on a strictly pseudoconvex CR manifold.

It is natural to establish similar results as in [8] for complex manifolds with boundary. In this paper, we consider the operator $\chi_k(T_R)$ constructed by functional calculus, where $\chi_k(\lambda) = \chi(k^{-1}\lambda)$ is a rescaled cut-off function χ with compact support in the positive real line, $k \in \mathbb{R}_+$ is a semi-classical parameter, and T_R is the Toeplitz operator on the domain M associated with a first-order differential operator given by a Reeb-like vector field from in the neighborhood of X. We show that $\chi_k(T_R)$ admits a full asymptotic expansion as $k \to +\infty$. This result can be seen as a semi-classical version of the Boutet de Monvel-Sjöstrand's and Fefferman's results on complex manifolds with boundary.

We now formulate our main result. We refer the reader to Section 2 for the notations used here. Let (M', J) be a complex manifold of dimension n with complex structure J. We fix a Hermitian metric Θ on M' and let $g^{TM'} = \Theta(\cdot, J\cdot)$ be the Riemannian metric on TM' associated to Θ and let $dv_{M'}$ be its volume form. We denote by $\langle \cdot | \cdot \rangle$ the pointwise Hermitian product induced by $g^{TM'}$ on the fibers of $\mathbb{C}TM'$ and by duality on $\mathbb{C}T^*M'$.

Let M be a relatively compact open subset in M' with smooth boundary. We set $X = \partial M$. We assume throughout the paper that M is strictly pseudoconvex. Let $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ be a defining function of M (cf. (20)), let $\mathscr{L}_x = \mathscr{L}_x(\rho)$ be the the Levi form associated to ρ at $x \in X$ (cf. (23)) and let $\det(\mathscr{L}_x)$ be the determinant of the Levi form (cf. (24)). We consider the 1-form $\omega_0 = -d\rho \circ J = i(\overline{\partial}\rho - \partial\rho)$ and we fix the contact form $\omega_0|_{TX} = 2i\overline{\partial}\rho|_{TX} = -2i\partial\rho|_{TX}$ on X (cf. (21)–(23)).

Let $(\cdot | \cdot)_M, (\cdot | \cdot)_{M'}$ be the L^2 inner products on $\mathscr{C}^\infty(\overline{M}), \mathscr{C}^\infty_c(M')$ induced by the given Hermitian metric $\langle \cdot | \cdot \rangle$ respectively (see (28)). Let $L^2(M)$ be the completion of $\mathscr{C}^\infty(\overline{M})$ with respect to $(\cdot\,|\,\cdot)_M$. Let $H^0(\overline{M}) := \big\{u \in \mathscr{C}^\infty(\overline{M});\ \overline{\partial}u = 0\big\}$, where $\overline{\partial} : \mathscr{C}^\infty(M') \to \Omega^{0,1}(M')$ denotes the standard Cauchy-Riemann operator on M'. Let $H^0_{(2)}(M)$ be the completion of $H^0(\overline{M})$ with respect to $(\cdot\,|\,\cdot,)_M$.

We denote by $\nabla\rho$ the gradient of ρ with respect to the Riemannian metric $g^{TM'}$. We consider the vector field $T = \alpha J\big(\nabla\rho\big) + Z$ on M', where $\alpha \in \mathscr{C}^\infty(M')$, $a|_X > 0$, and $Z \in \mathscr{C}^\infty(M', TM')$, $Z|_X \in \mathscr{C}^\infty(X, HX)$, cf. (25)–(26). Let R be a formally selfadjoint first order partial differential operator on M', given near X by $R = \frac{1}{2}((-iT) + (-iT)^*)$.

Since M is strictly pseudoconvex we have by [4, 7] that the Bergman projection maps the space $\mathscr{C}^\infty(\overline{M})$ of smooth functions up to the boundary into itself, $B : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M})$. Let

$$T_R := BRB : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M}).$$

We extend T_R to $L^2(M)$:

$$\begin{aligned} &T_R : \mathrm{Dom}(T_R) \subset L^2(M) \to L^2(M), \\ &\mathrm{Dom}(T_R) = \big\{u \in L^2(M);\ BRBu \in L^2(M)\big\}, \end{aligned} \tag{1}$$

where $BRBu$ is defined in the sense of distributions on M. In Theorem 2, we will show that T_R is self-adjoint. We consider a function

$$\chi \in \mathscr{C}^\infty_c(\mathbb{R}_+, \mathbb{R}), \tag{2}$$

and set for $k > 0$,

$$\chi_k \in \mathscr{C}^\infty_c(\mathbb{R}_+, \mathbb{R}), \quad \chi_k(\lambda) := \chi(k^{-1}\lambda). \tag{3}$$

We let

$$\chi_k(T_R) : L^2(M) \to L^2(M), \tag{4}$$

be obtained by functional calculus of T_R and let $\chi_k(T_R)(\cdot\,,\cdot) \in \mathscr{D}'(M \times M)$ be the distribution kernel of $\chi_k(T_R)$. We will show that $\chi_k(T_R)(\cdot\,,\cdot) \in \mathscr{C}^\infty(\overline{M} \times \overline{M})$ (cf. Corollary 2). We consider a function χ with support in $(0, +\infty)$ in order to avoid that the spectral operator $\chi_k(T_R)$ takes into account the zero eigenvalue of T_R, whose eigenspace contains the kernel of B. With this choice the image of $\chi_k(T_R)$ is contained in $H^0_{(2)}(M)$.

The main result of this paper is the following.

Theorem 1 *Let M be a relatively compact strictly pseudoconvex domain with smooth boundary X of a complex manifold M' of dimension n. Let $T_R : \mathrm{Dom}(T_R) \subset L^2(M) \to L^2(M)$ be the Toeplitz operator (1) and let $\chi_k(T_R)$ be as in (4). Then the following assertion hold:*

(i) For any $\tau, \hat{\tau} \in \mathscr{C}^\infty(\overline{M})$ with $\operatorname{supp}\tau \cap \operatorname{supp}\hat{\tau} = \emptyset$ we have

$$\tau\chi_k(T_R)\hat{\tau} = O(k^{-\infty}) \quad \text{on } \overline{M} \times \overline{M}. \tag{5}$$

(ii) For any $\tau \in \mathscr{C}_c^\infty(M)$ we have

$$\tau\chi_k(T_R) = O(k^{-\infty}) \quad on\ \overline{M} \times \overline{M}. \tag{6}$$

(iii) For any $p \in X$ and any open local coordinate patch U around p in M' we have

$$\chi_k(T_R)(x, y) = \int_0^{+\infty} e^{ikt\Psi(x,y)} b(x, y, t, k) dt + O(k^{-\infty}) \quad on\ (U \times U) \cap (\overline{M} \times \overline{M}), \tag{7}$$

where $b(x, y, t, k) \in S^{n+1}_{\mathrm{loc}}(1; ((U \times U) \cap (\overline{M} \times \overline{M})) \times \mathbb{R}_+)$,

$$\begin{aligned}
&b(x, y, t, k) \sim \sum_{j=0}^{\infty} b_j(x, y, t) k^{n+1-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1; ((U \times U) \cap (\overline{M} \times \overline{M})) \times \mathbb{R}_+),\\
&b_j(x, y, t) \in \mathscr{C}^\infty(((U \times U) \cap (\overline{M} \times \overline{M})) \times \mathbb{R}_+), \ j = 0, 1, 2, \ldots,\\
&b_0(x, x, t) = \frac{1}{\pi^n} \det(\mathscr{L}_x)\, \chi(t\omega_0(T(x)))t^n \not\equiv 0, \quad x \in U \cap X,
\end{aligned} \tag{8}$$

and for some compact interval $I \Subset \mathbb{R}_+$,

$$\mathrm{supp}_t\, b(x, y, t, k), \quad \mathrm{supp}_t\, b_j(x, y, t) \subset I, \quad j = 0, 1, 2, \ldots, \tag{9}$$

and

$$\begin{aligned}
&\Psi(z, w) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \quad \mathrm{Im}\,\Psi \geq 0,\\
&\Psi(z, z) = 0, \ z \in U \cap X,\\
&\mathrm{Im}\,\Psi(z, w) > 0 \textit{ if } (z, w) \notin (U \times U) \cap (X \times X),\\
&d_x\Psi(x, x) = -d_y\Psi(x, x) = -2i\partial\rho(x), \quad x \in U \cap X,\\
&\Psi|_{(U\times U)\cap(X\times X)} = \varphi_-,
\end{aligned} \tag{10}$$

where φ_- is a phase function as in (39), cf. [13, Theorem 4.1]. Moreover, using the local coordinates $z = (x_1, \ldots, x_{2n-1}, \rho)$ on M' near p, where $x = (x_1, \ldots, x_{2n-1})$ are local coordinates on X near p with $x(p) = 0$, we have

$$\Psi(z, w) = \Psi(x, y) - i\rho(z)(1 + f(z)) - i\rho(w)(1 + \overline{f(w)}) + O(|(z, w)|^3) \ \ near\ (p, p), \tag{11}$$

where f is smooth near p and $f = O(|z|)$.

The representation (7) shows that near the boundary $\chi_k(T_R)$ is a semi-classical Fourier integral operator with complex phase and canonical relation generated by the phase $\Psi(x, y)t$. The integral in (7) is a smooth kernel, since t runs in the bounded interval I. The term $O(k^{-\infty})$ denotes a k-negligible smooth kernel (cf. (17)–(18)).

The idea of the proof of Theorem 1 follows the strategy of [4, 10]. We express the Bergman projection in terms of the Poisson operator and a projector $\mathcal{S}$ on a subspace of functions annihilated by a system of pseudo-differential operators simulating $\overline{\partial}_b$.

We can express in the same way a Toeplitz operator T_R in terms of a Toeplitz operator on the boundary X, given by $\mathcal{T}_{\mathcal{R}} = \mathcal{S}\mathcal{R}\mathcal{S}$. The operator $\mathcal{S}$ is a Fourier integral operator having a structure similar to the Szegő projector cf. [3, 4, 8, 10] and we can apply the results obtained in [8] for the asymptotics of $\chi_k(\mathcal{T}_{\mathcal{R}})$.

As a consequence we have the following asymptotics of the kernel of $\chi_k(T_R)$ on the diagonal.

Corollary 1 *In the situation of Theorem 1 we have:*

$$\chi_k(T_R)(z,z) = O(k^{-\infty}), \quad \text{as } k \to \infty \text{ on } M. \tag{12}$$

$$\chi_k(T_R)(x,x) = \sum_{j=0}^{\infty} b_j(x) k^{n+1-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1; X) \text{ on } X, \tag{13}$$

where for $x \in X$ and with $b_j(x,x,t)$ as in (8),

$$b_j(x) = \int_0^{+\infty} b_j(x,x,t)dt, \quad j \in \mathbb{N}_0, \tag{14}$$

with

$$b_0(x) = \frac{1}{\pi^n} \det(\mathscr{L}_x) \int_0^{+\infty} \chi(t\omega_0(T(x)))t^n dt, \tag{15}$$

Moreover, there exist $C_1, C_2 > 0$ such that for k large enough $C_1 k^n \leq \operatorname{Tr} \chi_k(T_R) \leq C_2 k^n$.

2 Preliminaries

2.1 Notions from Microlocal and Semi-classical Analysis

We shall use the following notations: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R};\ x > 0\}$.

Let W be a smooth paracompact manifold. We let TW and T^*W denote the tangent bundle of W and the cotangent bundle of W respectively. The complexified tangent bundle of W and the complexified cotangent bundle of W are be denoted by $\mathbb{C}TW$ and $\mathbb{C}T^*W$, respectively. Write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TW and T^*W. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}TW \times \mathbb{C}T^*W$. Let G be a smooth vector bundle over W. The fiber of G at $x \in W$ will be denoted by G_x. Let $Y \subset W$ be an open set. From now on, the spaces of distributions of Y and smooth functions of Y will be denoted by $\mathscr{D}'(Y)$ and $\mathscr{C}^\infty(Y)$ respectively. Let $\mathscr{E}'(Y)$ be the subspace of $\mathscr{D}'(Y)$ whose elements have compact support in Y and let $\mathscr{C}^\infty_c(Y)$ be the subspace of $\mathscr{C}^\infty(Y)$ whose elements have compact support in Y. For $m \in \mathbb{R}$, let $H^m(Y)$ denote the Sobolev space of order m of Y. Put

$$H^m_{\rm loc}(Y) = \left\{u \in \mathscr{D}'(Y);\ \varphi u \in H^m(Y),\ \text{for every } \varphi \in \mathscr{C}^\infty_c(Y)\right\},$$
$$H^m_{\rm comp}(Y) = H^m_{\rm loc}(Y) \cap \mathscr{E}'(Y)\,.$$

If $A : \mathscr{C}^\infty_c(W) \to \mathscr{D}'(W)$ is continuous, we write $A(x, y)$ to denote the distribution kernel of A. The following two statements are equivalent

1. A is continuous: $\mathscr{E}'(W) \to \mathscr{C}^\infty(W)$,
2. $A(x, y) \in \mathscr{C}^\infty(W \times W)$.

If A satisfies (1) or (2), we say that A is smoothing on W. Let $A, B : \mathscr{C}^\infty_c(W) \to \mathscr{D}'(W)$ be continuous operators. We write

$$A \equiv B \text{ (on } W \times W) \tag{16}$$

if $A - B$ is a smoothing operator on W.

Let $H(x, y) \in \mathscr{D}'(W \times W)$. We write H to denote the unique continuous operator $\mathscr{C}^\infty_c(W) \to \mathscr{D}'(W)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

Let D be an open set of a smooth manifold X. For $0 \leq \rho, \delta \leq 1$, $m \in \mathbb{R}$, let

$$L^m_{\rho,\delta}(D)\,, \quad L^m_{\rm cl}(D),$$

denote the space of pseudodifferential operators on D of order m type (ρ, δ) and the space of classical pseudodifferential operators on D of order m respectively. Let $W \subset \mathbb{R}^N$ be an open set. For $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$, let $S^m_{\rho,\delta}(W \times \mathbb{R}^{N_1})$ be the Hörmander symbol space on $W \times \mathbb{R}^{N_1}$ of order m and type (ρ, δ). Let $S^m_{\rm cl}(W \times \mathbb{R}^{N_1})$ be the classical symbol space on $W \times \mathbb{R}^{N_1}$ of order m.

Let W_1 be an open set in $\mathbb{R}^{N_1}$ and let W_2 be an open set in $\mathbb{R}^{N_2}$. A k-dependent continuous operator $F_k : \mathscr{C}^\infty_c(W_2) \to \mathscr{D}'(W_1)$ is called k-negligible on $W_1 \times W_2$ if, for k large enough, F_k is smoothing and, for any $K \Subset W_1 \times W_2$, any multi-indices α, β and any $N \in \mathbb{N}$, there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$\left|\partial^\alpha_x \partial^\beta_y F_k(x, y)\right| \leq C_{K,\alpha,\beta,N} k^{-N} \text{ on } K, \quad \text{for } k \gg 1. \tag{17}$$

In that case we write

$$F_k(x, y) = O(k^{-\infty}) \text{ or } F_k = O(k^{-\infty}) \text{ on } W_1 \times W_2. \tag{18}$$

If $F_k, G_k : \mathscr{C}^\infty_c(W_2) \to \mathscr{D}'(W_1)$ are k-dependent continuous operators, we write $F_k = G_k + O(k^{-\infty})$ on $W_1 \times W_2$ or $F_k(x, y) = G_k(x, y) + O(k^{-\infty})$ on $W_1 \times W_2$ if $F_k - G_k = O(k^{-\infty})$ on $W_1 \times W_2$.

Let Ω_1 and Ω_2 be smooth manifolds. Let $F_k, G_k : \mathscr{C}^\infty(\Omega_2) \to \mathscr{C}^\infty(\Omega_1)$ be k-dependent smoothing operators. We write $F_k = G_k + O(k^{-\infty})$ on $\Omega_1 \times \Omega_2$ if on every local coordinate patch D of Ω_1 and local coordinate patch D_1 of Ω_2, $F_k = G_k + O(k^{-\infty})$ on $D \times D_1$.

We recall the definition of the semi-classical symbol spaces.

Definition 1 Let W be an open set in $\mathbb{R}^N$. Let

$$S(1; W) := \Big\{a \in \mathscr{C}^\infty(W);\ \text{for every } \alpha \in \mathbb{N}_0^N : \sup_{x\in W} \big|\partial^\alpha a(x)\big| < \infty\Big\},$$

$$S^0_{\mathrm{loc}}(1; W) := \Big\{(a(\cdot, k))_{k\in\mathbb{R}};\ \text{for all } \alpha \in \mathbb{N}_0^N,\ \chi \in \mathscr{C}^\infty_c(W) : \sup_{k\geq 1}\sup_{x\in W} \big|\partial^\alpha(\chi a(x, k))\big| < \infty\Big\}.$$

For $m \in \mathbb{R}$, let

$$S^m_{\mathrm{loc}}(1) := S^m_{\mathrm{loc}}(1; W) = \Big\{(a(\cdot, k))_{k\in\mathbb{R}};\ (k^{-m}a(\cdot, k)) \in S^0_{\mathrm{loc}}(1; W)\Big\}.$$

Hence $a(\cdot, k) \in S^m_{\mathrm{loc}}(1; W)$ if for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in \mathscr{C}^\infty_c(W)$, there exists $C_\alpha > 0$ independent of k, such that $|\partial^\alpha(\chi a(\cdot, k))| \leq C_\alpha k^m$ holds on W.

Consider a sequence $a_j \in S^{m_j}_{\mathrm{loc}}(1)$, $j \in \mathbb{N}_0$, where $m_j \searrow -\infty$, and let $a \in S^{m_0}_{\mathrm{loc}}(1)$. We say that

$$a(\cdot, k) \sim \sum_{j=0}^{\infty} a_j(\cdot, k) \ \text{ in } S^{k_0}_{\mathrm{loc}}(1),$$

if for every $\ell \in \mathbb{N}_0$, we have $a - \sum_{j=0}^{\ell} a_j \in S^{m_{\ell+1}}_{\mathrm{loc}}(1)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\mathrm{loc}}(1) = S^{-\infty}_{\mathrm{loc}}(1; W) := \cap_m S^m_{\mathrm{loc}}(1)$.

Similarly, we can define $S^m_{\mathrm{loc}}(1; Y)$ in the standard way, where Y is a smooth manifold.

2.2 Set Up of Complex Manifolds with Smooth Boundary

Let (M', J) be a complex manifold of dimension n, where $J : TM' \to TM'$ is the complex structure of M'. We fix a Hermitian metric Θ on M' and let $g^{TM'} = \Theta(\cdot, J\cdot)$ be the Riemannian metric on TM' associated to Θ and let $dv_{M'}$ be its volume form. We denote by $\langle \cdot \,|\, \cdot \rangle$ the pointwise Hermitian product induced by $g^{TM'}$ on the fibers of the bundle $\Lambda^q(T^{*(0,1)}M')$ of $(0, q)$-forms for every $q \in \{0, \ldots, n\}$. Let $\Omega^{0,q}(M')$ be the space of smooth $(0, q)$-forms on M' and let $\Omega^{0,q}_c(M')$ be the subspace of $\Omega^{0,q}(M')$ whose elements have compact support in M'. The L^2 inner product on $\Omega^{0,q}_c(M')$ is given by

$$(\alpha \,|\, \beta)_{M'} = \int_{M'} \langle \alpha \,|\, \beta \rangle dv_{M'}. \tag{19}$$

The corresponding L^2 space is denoted by $L^2_{0,q}(M')$, and we set $L^2(M') = L^2_{0,0}(M')$.

Let M be a relatively compact open subset of M' with smooth boundary. Hence $X := \partial M$ is a submanifold of M' of real dimension $2n - 1$. We denote by $HX =$

$TX \cap J(TX)$ the complex tangent bundle of X. The triple (X, HX, J) forms a CR structure on X and we set $T^{1,0}X := T^{1,0}M' \cap \mathbb{C}TX$, $T^{0,1}X := \overline{T^{1,0}X}$. Let $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ be a defining function of X, that is,

$$M = \{x \in M' : \rho(x) < 0\}, \quad X = \partial M = \{x \in M' : \rho(x) = 0\}, \quad \text{and } d\rho \neq 0 \text{ on } X. \tag{20}$$

From now on, we fix a defining function ρ so that $|d\rho| = 1$ on X. Define a real 1-form ω_0 on M' by

$$\omega_0 = -d\rho \circ J. \tag{21}$$

Hence

$$\omega_0 = i(\overline{\partial}\rho - \partial\rho), \quad d\omega_0 = 2i\partial\overline{\partial}\rho. \tag{22}$$

The Levi form of ρ is Hermitian symmetric map $\mathscr{L}_x = \mathscr{L}_x(\rho)$ given by

$$\mathscr{L}_x : T^{1,0}_x X \times T^{1,0}_x X \to \mathbb{C}, \quad \mathscr{L}_x(U, \overline{V}) = \frac{1}{2i} d\omega_0(U, \overline{V}) = \partial\overline{\partial}\rho(U, \overline{V}), \quad U, V \in T^{1,0}_x X. \tag{23}$$

We assume that M is a strictly pseudoconvex domain, that is, the Levi form $\mathscr{L}_x$ is positive definite for every $x \in X$. In this case the hyperplane field HX is a contact structure on X. Indeed, $HX = \ker(\omega_0|_{TX})$ and for every $u \in HX \setminus \{0\}$ we have $d\omega_0(u, Ju) = 4\mathscr{L}(U, \overline{U}) > 0$, where $U = \frac{1}{2}(u - iJu) \in T^{1,0}X$. So $d\omega_0|_{HX}$ is symplectic, and hence HX is a contact structure, with $\omega_0|_{TX} = 2i\overline{\partial}\rho|_{TX} = -2i\partial\rho|_{TX}$ a contact form.

We denote by $\lambda_j(x)$, $j = 1, \ldots, n-1$, the eigenvalues of $\mathscr{L}_x$ with respect to $\langle \cdot | \cdot \rangle$ (note that $T^{0,1}X$ has rank $n-1$). The determinant of the Levi form is defined by

$$\det(\mathscr{L})_x := \lambda_1(x) \ldots \lambda_{n-1}(x). \tag{24}$$

Let $\nabla\rho$ be the gradient of ρ with respect to the Riemannian metric $g^{TM'}$. We define the vector field T on M' by

$$T = \alpha J\big(\nabla\rho\big) + Z \in \mathscr{C}^\infty(M', TM'), \tag{25}$$

where

$$\alpha \in \mathscr{C}^\infty(M'), \;\; \alpha|_X > 0, \;\; Z \in \mathscr{C}^\infty(M', TM'), \;\; Z|_X \in \mathscr{C}^\infty(X, HX). \tag{26}$$

The vector field T does not vanish on a neighborhood of X. Indeed, we have on X that $\langle J(\nabla\rho), Z\rangle = -\langle \nabla\rho, JZ\rangle = 0$ hence $|T|^2 = a^2 + |Z|^2 > 0$. Note also that

$$\omega_0(T) = -(d\rho \circ J)(\alpha J(\nabla\rho) + Z) = \alpha d\rho(\nabla\rho) = \alpha|\nabla\rho|^2 = \alpha \quad \text{on } X, \tag{27}$$

since $|\nabla\rho| = |d\rho| = 1$ on X.

Let U be an open set in M'. Let

$$\mathscr{C}^\infty(U\cap\overline{M}),\ \ \mathscr{D}'(U\cap\overline{M}),\ \ \mathscr{C}^\infty_c(U\cap\overline{M}),\ \ \mathscr{E}'(U\cap\overline{M}),$$
$$H^s(U\cap\overline{M}),\ \ H^s_{\mathrm{comp}}(U\cap\overline{M}),\ \ H^s_{\mathrm{loc}}(U\cap\overline{M}),$$

(where $s\in\mathbb{R}$) denote the spaces of restrictions to $U\cap\overline{M}$ of elements in

$$\mathscr{C}^\infty(U\cap M'),\ \ \mathscr{D}'(U\cap M'),\ \ \mathscr{C}^\infty(U\cap M'),\ \ \mathscr{E}'(U\cap M'),$$
$$H^s(M'),\ \ H^s_{\mathrm{comp}}(M'),\ \ H^s_{\mathrm{loc}}(M'),$$

respectively. Write

$$L^2(U\cap\overline{M}) := H^0(U\cap\overline{M}),\ \ L^2_{\mathrm{comp}}(U\cap\overline{M}) := H^0_{\mathrm{comp}}(U\cap\overline{M}),$$
$$L^2_{\mathrm{loc}}(U\cap\overline{M}) := H^0_{\mathrm{loc}}(U\cap\overline{M}).$$

Let $dv_{M'}$ be the volume form on M' induced by the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TM'$ and and let $(\,\cdot\,|\,\cdot\,)_M$ and $(\,\cdot\,|\,\cdot\,)_{M'}$ be the inner products on $\mathscr{C}^\infty(\overline{M})$ and $\mathscr{C}^\infty_c(M')$ defined by

$$\begin{aligned}(\,f\,|\,h\,)_M &= \int_M f\overline{h}dv_{M'},\quad f,h\in\mathscr{C}^\infty(\overline{M}),\\ (\,f\,|\,h\,)_{M'} &= \int_{M'} f\overline{h}dv_{M'},\quad f,h\in\mathscr{C}^\infty_c(M').\end{aligned}\tag{28}$$

Let $\|\cdot\|_M$ and $\|\cdot\|_{M'}$ be the corresponding norms with respect to $(\,\cdot\,|\,\cdot\,)_M$ and $(\,\cdot\,|\,\cdot\,)_{M'}$ respectively. Let $L^2(M)$ be the completion of $\mathscr{C}^\infty(\overline{M})$ with respect to $(\,\cdot\,|\,\cdot\,)_M$. We extend $(\,\cdot\,|\,\cdot\,)_M$ to $L^2(M)$ in the standard way. For $q=1,2,\ldots,n$, let $\Omega^{0,q}(M')$ be the space of smooth $(0,q)$ forms on M' and let $\Omega^{0,q}_c(M')$ be the subspace of $\Omega^{0,q}(M')$ whose elements have compact support in M'. As in (28), let $(\,\cdot\,|\,\cdot\,)_{M'}$ be the L^2 inner product on $\Omega^{0,q}_c(M')$ induced by $dv_{M'}$ and $\langle\cdot\,|\,\cdot\rangle$.

The boundary $X=\partial M$ is a compact CR manifold of dimension $2n-1$ with natural CR structure $T^{1,0}X := T^{1,0}M'\cap\mathbb{C}TX$. Let $T^{0,1}X := \overline{T^{1,0}X}$. The Hermitian metric on $\mathbb{C}TM'$ induces a Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TX$ and let $(\,\cdot\,|\,\cdot\,)_X$ be the L^2 inner product on $\mathscr{C}^\infty(X)$ induced by $\langle\cdot\,|\,\cdot\rangle$.

Let U be an open set in M'. Let

$$F_1, F_2 : \mathscr{C}^\infty_c(U\cap M)\to\mathscr{D}'(U\cap M)$$

be continuous operators. Let $F_1(x,y), F_2(x,y)\in\mathscr{D}'((U\times U)\cap(M\times M))$ be the distribution kernels of F_1 and F_2 respectively. We write

$$F_1\equiv F_2\mod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$$

or $F_1(x,y) \equiv F_2(x,y) \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ if $F_1(x,y)=F_2(x,y)+r(x,y)$, where $r(x,y)\in\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$.

Let $F_k, G_k : \mathscr{C}^\infty_c(U\cap M)\to\mathscr{D}'(U\cap M)$ be k-dependent continuous operators. Let $F_k(x,y), G_k(x,y)\in\mathscr{D}'((U\times U)\cap(M\times M))$ be the distribution kernels of F_k and G_k respectively. We write

$$F_k(x,y)\equiv G_k(x,y) \mod O(k^{-\infty}) \quad \text{on } (U\times U)\cap(\overline{M}\times\overline{M}) \tag{29}$$

or $F_k\equiv G_k \mod O(k^{-\infty})$ on $(U\times U)\cap(\overline{M}\times\overline{M})$ if there is a $r_k(x,y)\in\mathscr{C}^\infty(U\times U)$ with $r_k(x,y)=O(k^{-\infty})$ on $U\times U$ such that

$$r_k(x,y)|_{(U\times U)\cap(\overline{M}\times\overline{M})}=F_k(x,y)-G_k(x,y),\ \text{for } k\gg 1.$$

Let $m\in\mathbb{R}$. Let U be an open set in M'. Let

$$S^m_{\mathrm{loc}}(1,(U\times U)\cap(\overline{M}\times\overline{M})) \tag{30}$$

denote the space of restrictions to $(U\times U)\cap(\overline{M}\times\overline{M})$ of elements in $S^m_{\mathrm{loc}}(1,U\times U)$. Let

$$a_j\in S^{m_j}_{\mathrm{loc}}(1,(U\times U)\cap(\overline{M}\times\overline{M})),\quad j=0,1,2,\ldots,$$

with $m_j\searrow-\infty$, $j\to\infty$. Then there exists $a\in S^{m_0}_{\mathrm{loc}}(1,(U\times U)\cap(\overline{M}\times\overline{M}))$ such that for every $\ell\in\mathbb{N}$,

$$a-\sum_{j=0}^{\ell-1}a_j\in S^{m_\ell}_{\mathrm{loc}}(1,(U\times U)\cap(\overline{M}\times\overline{M})).$$

If a and a_j have the properties above, we write

$$a\sim\sum_{j=0}^{\infty}a_j \text{ in } S^{m_0}_{\mathrm{loc}}(1,(U\times U)\cap(\overline{M}\times\overline{M})).$$

3 The Toeplitz Operator T_R

Let R be a first order partial differential operator on M' such that R is formally self-adjoint with respect to $(\cdot\,|\,\cdot)_{M'}$ and near X,

$$R=\frac{1}{2}((-iT)+(-iT)^*), \tag{31}$$

where T is given by (25) and $(-iT)^*$ is the formal adjoint of $-iT$ with respect to $(\cdot\,|\,\cdot)_{M'}$. Let T_R the Toeplitz operator introduced in (1). The goal of this section is to prove the following.

Theorem 2 *The operator* $T_R : \mathrm{Dom}(T_R) \subset L^2(M) \to L^2(M)$ *is self-adjoint.*

For the proof of the theorem 2 we need some preparation. Let

$$\overline{\partial}^*_f : \Omega^{0,1}(M') \to \mathscr{C}^\infty(M')$$

be the formal adjoint of $\overline{\partial}$ with respect to $(\cdot\,|\,\cdot)_{M'}$, that is, $(\,\overline{\partial} f \,|\, h\,)_{M'} = (\, f \,|\, \overline{\partial}^*_f h\,)_{M'}$, for any $f \in \mathscr{C}^\infty_c(M')$, $h \in \Omega^{0,1}(M')$. Let

$$\Box_f = \overline{\partial}^*_f\,\overline{\partial} : \mathscr{C}^\infty(M') \to \mathscr{C}^\infty(M')$$

denote the complex Laplace-Beltrami operator on functions. Let

$$P : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(\overline{M}) \tag{32}$$

be the Poisson operator associated to $\Box_f$. The Poisson operator P satisfies

$$\begin{aligned} &\Box_f Pu = 0, \quad u \in \mathscr{C}^\infty(X), \\ &\gamma Pu = u, \quad u \in \mathscr{C}^\infty(X), \end{aligned} \tag{33}$$

where γ denotes the operator of restriction to the boundary X. It is known that P extends continuously

$$P : H^s(X) \to H^{s+\frac{1}{2}}(\overline{M}), \quad \forall s \in \mathbb{R}$$

(see [2, Page 29]). Let

$$P^* : \hat{\mathscr{D}}'(\overline{M}) \to \mathscr{D}'(X)$$

be the operator defined by

$$(\,P^*u \,|\, v\,)_X = (\,u \,|\, Pv\,)_M, \quad u \in \hat{\mathscr{D}}'(\overline{M}), \quad v \in \mathscr{C}^\infty(X),$$

where $\hat{\mathscr{D}}'(\overline{M})$ denotes the space of continuous linear maps from $\mathscr{C}^\infty(\overline{M})$ to $\mathbb{C}$ with respect to $(\cdot\,|\,\cdot)_M$. It is well-known (see [2, page 30]) that P^* is continuous $P^* : H^s(\overline{M}) \to H^{s+\frac{1}{2}}(X)$ for every $s \in \mathbb{R}$ and

$$P^* : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(X).$$

It is well-known that the operator

$$P^*P : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$$

is a classical elliptic pseudodifferential operator of order -1 and invertible since P is injective (see [2]). Moreover, the operator

$$(P^*P)^{-1} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$$

is a classical elliptic pseudodifferential operator of order one. We define a new inner product on $H^{-\frac{1}{2}}(X)$ as follows:

$$[\, u \mid v\,]_X := (\, Pu \mid Pv)_M, \quad u, v \in H^{-\frac{1}{2}}(X). \tag{34}$$

For an operator A on $H^{-\frac{1}{2}}(X)$ we denote by $A^\dagger$ the formal adjoint of A with respect to the inner product $[\,\cdot \mid \cdot\,]_X$.

The next result shows that we can link the Bergman projection with a certain approximate projector on the boundary X, which is a Fourier integral operator.

Theorem 3 ([10]) *There exists a continuous operator*

$$\mathcal{S} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X), \quad \mathcal{S} \in L^0_{\frac{1}{2},\frac{1}{2}}(X) \tag{35}$$

such that

$$B = P\mathcal{S}(P^*P)^{-1}P^*, \quad on\, \mathscr{C}^\infty(\overline{M}), \tag{36}$$

with the following properties,

$$\mathcal{S}^\dagger = \mathcal{S}, \quad \mathcal{S}^2 = \mathcal{S} \text{ on } \mathscr{D}'(X), \tag{37}$$

and for any local coordinate patch (D, x), we have

$$\mathcal{S}(x, y) \equiv \int_0^\infty e^{it\varphi(x,y)} s(x, y, t)dt \quad on\ D \times D, \tag{38}$$

where $\varphi = \varphi_- \in \mathscr{C}^\infty(D \times D)$ is the phase function φ_- as in [13, Theorem 4.1] satisfying

$$\begin{aligned}
&\varphi \in \mathscr{C}^\infty(D \times D), \ \ \mathrm{Im}\,\varphi(x, y) \ge 0,\\
&\varphi(x, x) = 0, \ \ \varphi(x, y) \ne 0 \ \ if\ \ x \ne y,\\
&d_x\varphi(x, y)\big|_{x=y} = -d_y\varphi(x, y)\big|_{x=y} = \omega_0(x),\\
&\varphi(x, y) = -\overline{\varphi}(y, x).
\end{aligned} \tag{39}$$

and

$$
\begin{aligned}
& s(x,y,t) \sim \sum_{j=0}^{+\infty} s_j(x,y) t^{n-1-j} \ \textit{in}\ S^{n-1}_{1,0}(D \times D \times \mathbb{R}_+), \\
& s_j(x,y) \in \mathscr{C}^\infty(D \times D),\ j = 0,1,\ldots, \\
& s_0(x,x) = \frac{1}{2\pi^n} \det(\mathscr{L}_x),\ \ \textit{for all}\ x \in D_0.
\end{aligned} \tag{40}
$$

Proof We recall here the construction from [10] for the convenience of the reader. In order to link the Bergman projection to a boundary operator we consider a version of the tangential Cauchy-Riemann operator $\overline{\partial}_b$ on X, denoted $\overline{\partial}_\beta$, expressed in terms of the Poisson extension operator, the $\overline{\partial}$ operator and the restriction to the boundary, namely

$$
\overline{\partial}_\beta : \Omega^{0,\star}(X) \to \Omega^{0,\star+1}(X), \quad \overline{\partial}_\beta = Q\gamma\overline{\partial}P,
$$

where P is the Poisson operator, $\gamma : \Omega^{0,\star}(\overline{M}) \to \Omega^{0,\star}(X)$ is the restriction operator and $Q : H^{-1/2}(X, \Lambda^{0,\star}TM') \to \ker(\overline{\partial}\rho \wedge \cdot)^* \subset H^{-1/2}(X, \Lambda^{0,\star}TM')$ is the orthogonal projection, cf. [10, (5.1), p. 103]. Note that Q is the operator T in [10, (3.6), p. 96] and Q is the identity in degree zero. The operator $\overline{\partial}_\beta$ is a classical pseudo-differential operator of order one on X, such that $\overline{\partial}_\beta = \overline{\partial}_b +$ l.o.t. and $\overline{\partial}_\beta^2 = 0$. The corresponding Laplace operator (cf. [10, (5.6), p. 104]),

$$
\Box^{(\star)}_\beta = \overline{\partial}_\beta\,\overline{\partial}^\dagger_\beta + \overline{\partial}^\dagger_\beta\,\overline{\partial}_\beta : \Omega^{0,\star}(X) \to \Omega^{0,\star}(X)
$$

is a classical pseudo-differential operator of order two on X, with the same principal symbol and the same characteristic manifold as the Kohn Laplacian $\Box_b = \overline{\partial}_b\,\overline{\partial}^*_b + \overline{\partial}^*_b\,\overline{\partial}_b$,

$$
\Sigma = \left\{(x, t\omega_0(x)) \in T^*X : x \in X,\ t \in \mathbb{R} \setminus \{0\}\right\}. \tag{41}
$$

We have

$$
\Sigma = \Sigma^+ \cup \Sigma^-,\ \ \Sigma^+ := \{(x, t\omega_0(x)) \in T^*X : x \in X,\ t > 0\},\ \ \Sigma^- := \Sigma \setminus \Sigma^+. \tag{42}
$$

Note that we use here a different sign convention than in [10], where ω_0 equals $d\rho \circ J$ (compare [10, (1.9), p. 84], (21)), thus we swap here the roles of Σ^+ and Σ^- compared to [10].

By Theorem [10, Theorem 6.15, p. 114] the operator $\Box^{(0)}_\beta$ acting in degree $q = 0$ (that is on functions) has a parametrix A and an approximate projector S (denoted B_- in [10]) such that

$$
\begin{aligned}
A \in L^{-1}_{\frac12,\frac12}(X), &\quad S \in L^{0}_{\frac12,\frac12}(X), \\
A\Box^{(0)}_\beta + S \equiv I, &\quad \Box^{(0)}_\beta A + S \equiv I, \\
S^2 \equiv S, &\quad S^\dagger \equiv S, \\
\overline{\partial}_\beta\, S \equiv 0, &\quad \overline{\partial}^\dagger_\beta\, S \equiv 0.
\end{aligned} \tag{43}
$$

Morevover the wavefront set of the distribution kernel $S(\cdot,\cdot)$ of S is given by

$$\mathrm{WF}(S(\cdot,\cdot)) = \{(x,\xi,x,-\xi) : (x,\xi) \in \Sigma^+\}. \tag{44}$$

In [10, (7.4), p. 120] the operator $P\mathcal{S}Q(P^*P)^{-1}P^*$ is defined on $\Omega^{0,q}(\overline{M})$ and it is shown in [10, Proposition 7.5] that its kernel equals the Bergman kernel on $(0,q)$-forms up to a smooth form on $\overline{M}\times\overline{M}$. For $q=0$ the operator Q is the identity so we have $B = PS(P^*P)^{-1}P^* + F$, where F is smoothing. We set

$$\mathcal{S} = S + (P^*P)^{-1}P^*FP. \tag{45}$$

Then $\mathcal{S}^2 = \mathcal{S}$, $\mathcal{S}^\dagger = \mathcal{S}$, and $B = P\mathcal{S}(P^*P)^{-1}P^*$. We have thus obtained (36) and (37). The prpperties (38), (39) and (38) follow from the corresponding properties of S. □

Remark 1 The operator $\mathcal{S}$ is a Toeplitz structure on Σ^+ in the sense of [3, Definition 2.10].

Lemma 1 *For any* $u \in \mathrm{Dom}(T_R)$ *there exists* $u_j \in \mathscr{C}^\infty(\overline{M})$*,* $j=1,2,\ldots$*, such that* $\lim_{j\to+\infty} u_j = Bu$ *in* $L^2(M)$ *and* $\lim_{j\to+\infty} BRBu_j = BRBu$ *in* $L^2(M)$*.*

Proof Let $u \in \mathrm{Dom}(T_R)$. We may assume that $u = Bu$. Then,

$$u = P(P^*P)^{-1}P^*u = P\mathcal{S}(P^*P)^{-1}P^*u.$$

From (36), we have

$$BRBu = P\mathcal{S}(P^*P)^{-1}P^*RP\mathcal{S}(P^*P)^{-1}P^*u = P\mathcal{S}L\mathcal{S}(P^*P)^{-1}P^*u, \tag{46}$$

where $L = (P^*P)^{-1}P^*RP$. It is straightforward to check that $L \in L^1_{\mathrm{cl}}(X)$ and

$$\sigma^0_L(x,\omega_0(x)) \neq 0$$

at every $x \in X$, where σ^0_L denotes the principal symbol of L. Since $\mathcal{S}L\mathcal{S}(P^*P)^{-1}P^*u \in H^{-\frac{1}{2}}(X)$, we can repeat the proof of [8, Theorem 3.3] and deduce that $(P^*P)^{-1}P^*u \in H^{\frac{1}{2}}(X)$. Let $v_j \in \mathscr{C}^\infty(X)$, $j=1,2,\ldots$, $v_j \to (P^*P)^{-1}P^*u$ in $H^{\frac{1}{2}}(X)$ as $j\to+\infty$. Then, $u_j := Pv_j \to P(P^*P)^{-1}P^*u = u$ in $H^1(\overline{M})$ as $j\to+\infty$ and $BRBu_j \to BRBu$ in $L^2(\overline{M})$ as $j\to+\infty$. □

Proof (*Proof of Theorem* 2) Let $T_R^* : \mathrm{Dom}(T_R^*) \subset L^2(X) \to L^2(X)$ be the L^2 adjoint of T_R. Let $u \in \mathrm{Dom}(T_R)$. From Lemma 1, for every $v \in \mathrm{Dom}(T_R)$, we have

$$(\,u\,|\,Av\,)_M = (\,Bu\,|\,Av\,)_M = \lim_{j\to+\infty}(\,Bu_j\,|\,BRBv_j\,)_M, \tag{47}$$

where $u_j, v_j \in \mathscr{C}^\infty(\overline{M})$, $j=1,2,\ldots$, such that $\lim_{j\to+\infty} u_j = Bu$ in $L^2(M)$, $\lim_{j\to+\infty} v_j = Bv$ in $L^2(M)$, $\lim_{j\to+\infty} BRBu_j = BRBu$ in $L^2(M)$ and $\lim_{j\to+\infty}$

$BRBv_j = BRBv$ in $L^2(M)$. From (47) and since $R\rho = 0$ on X, we can integrate by parts and deduce that

$$(u \,|\, T_R v)_M = (T_R u \,|\, v)_M, \quad \text{for every } v \in \mathrm{Dom}(T_R).$$

Thus, $u \in \mathrm{Dom}(T_R^*)$ and $T_R^* u = T_R u$. Let $u \in \mathrm{Dom}(T_R^*)$. Since $\mathscr{C}^\infty(\overline{M}) \subset \mathrm{Dom}(T_R)$, we deduce that there is a constant $C > 0$ such that

$$|(u \,|\, BRBv)_M| \leq C \, \|v\|_M \,, \quad \text{for every } v \in \mathscr{C}^\infty(\overline{M}).$$

Thus, $BRBu \in L^2(M)$ and hence $u \in \mathrm{Dom}(T_R)$. □

4 Asymptotic Expansion of $\chi_k(T_R)$

In this Section we will reduce the study of the Toeplitz operator T_R to the study of a Toeplitz operator $\mathcal{T}_\mathcal{R}$ on the boundary X and apply results from [8] in order to prove Theorem 1. The Toeplitz operator on the boundary is defined by

$$\mathcal{T}_\mathcal{R} := \mathcal{S}\mathcal{R}\mathcal{S} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X), \tag{48}$$

where $\mathcal{S}$ is as in Theorem 3 and

$$\mathcal{R} := (P^* P)^{-1} P^* R P \in L^1_{\mathrm{cl}}(X). \tag{49}$$

Note that by (46) we have

$$T_R = P(\mathcal{S}\mathcal{R}\mathcal{S})(P^* P)^{-1} P^* = P\mathcal{T}_\mathcal{R}(P^* P)^{-1} P^*. \tag{50}$$

We extend $\mathcal{T}_\mathcal{R}$ to $H^{-\frac{1}{2}}(X)$:

$$\begin{aligned} &\mathcal{T}_\mathcal{R} : \mathrm{Dom}(\mathcal{T}_\mathcal{R}) \subset H^{-\frac{1}{2}}(X) \to H^{-\frac{1}{2}}(X), \\ &\mathrm{Dom}(\mathcal{T}_\mathcal{R}) = \left\{ u \in H^{-\frac{1}{2}}(X);\ \mathcal{S}\mathcal{R}\mathcal{S}u \in H^{-\frac{1}{2}}(X) \right\}. \end{aligned} \tag{51}$$

The operator $\mathcal{S}$ is a Toeplitz structure (generalized Szegő projector) in the sense of [3, Definition 2.10]. Let $\mathrm{Im}(\mathcal{S})$ be the image of $\mathcal{S}$ in $L^2(X)$. By [3, Proposition 2.14] the spectrum of the operator $\mathcal{T}_\mathcal{R}|_{\mathrm{Im}(\mathcal{S})} : \mathrm{Im}(\mathcal{S}) \to \mathrm{Im}(\mathcal{S})$ consists only of isolated eigenvalues of finite multiplicity, is bounded from below and has only $+\infty$ as a point of accumulation. We have $\mathrm{Spec}(\mathcal{T}_\mathcal{R}) \setminus \{0\} = \mathrm{Spec}(\mathcal{T}_\mathcal{R}|_{\mathrm{Im}(\mathcal{S})}) \setminus \{0\}$ and the restrictions to $\mathbb{R} \setminus \{0\}$ of spectral measures of these operators coincide. We conclude that the operator $\mathcal{T}_\mathcal{R}$ in (51) is self-adjoint with respect to $[\,\cdot\,|\,\cdot\,]_X$ and its spectrum consists only of isolated eigenvalues, is bounded from below and has only $+\infty$ as a point of accumulation. Moreover, for every $\lambda \in \mathrm{Spec}(\mathcal{T}_\mathcal{R})$, $\lambda \neq 0$, the eigenspace

$$E_\lambda = \{u \in \mathrm{Dom}(\mathcal{T}_\mathcal{R}) : \mathcal{T}_\mathcal{R} u = \lambda u\}$$

is a finite dimensional subspace of $\mathscr{C}^\infty(X)$.

Remark 2 The kernel of $\mathcal{T}_\mathcal{R}$ contains the kernel of $\mathcal{S}$, so in order to avoid the zero eigenvalue we consider the operator $\chi_k(\mathcal{T}_\mathcal{R})$ associated to a function χ with support in $(0, +\infty)$. In this way the image of $\chi_k(\mathcal{T}_\mathcal{R})$ is contained in $\mathrm{Im}(\mathcal{S})$.

Lemma 2 *For every $z \in \mathbb{C}$, $z \notin \mathbb{R}$, we have*

$$(z - T_R)^{-1} B = P(z - \mathcal{T}_\mathcal{R})^{-1} \mathcal{S}(P^* P)^{-1} P^*. \tag{52}$$

Proof From (36) and (46) we have

$$(z - T_R)B = P(z - \mathcal{S}\mathcal{R}\mathcal{S})(P^* P)^{-1} P^* B = P(z - \mathcal{T}_\mathcal{R})\mathcal{S}(P^* P)^{-1} P^*. \tag{53}$$

From (53), we have

$$P(z - \mathcal{T}_\mathcal{R})^{-1}(P^* P)^{-1} P^*(z - T_R)B = B.$$

Thus,

$$\begin{aligned}(z - T_R)^{-1} B &= P(z - \mathcal{T}_\mathcal{R})^{-1}(P^* P)^{-1} P^* B \\ &= P(z - \mathcal{T}_\mathcal{R})^{-1}(P^* P)^{-1} P^* P \mathcal{S}(P^* P)^{-1} P^* \\ &= P(z - \mathcal{T}_\mathcal{R})^{-1} \mathcal{S}(P^* P)^{-1} P^*.\end{aligned}$$

The lemma follows. □

Lemma 3 *We have*

$$\chi_k(T_R) = P \chi_k(\mathcal{T}_\mathcal{R})(P^* P)^{-1} P^*. \tag{54}$$

Proof From the Helffer-Sjöstrand formula [6, §8] and (52), we have

$$\begin{aligned}\chi_k(T_R) &= \frac{1}{2\pi i} \int_\mathbb{C} \frac{\partial \tilde{\chi}_k}{\partial \overline{z}}(z)(z - T_R)^{-1} dz d\overline{z} \\ &= \frac{1}{2\pi i} \int_\mathbb{C} \frac{\partial \tilde{\chi}_k}{\partial \overline{z}}(z) P(z - \mathcal{T}_\mathcal{R})^{-1} \mathcal{S}(P^* P)^{-1} P^* dz d\overline{z} \\ &= P \chi_k(\mathcal{T}_\mathcal{R})(P^* P)^{-1} P^*,\end{aligned}$$

where $\tilde{\chi}_k$ denotes an almost analytic extension of χ_k. The lemma follows. □

Corollary 2 *We have*

$$\chi_k(T_R)(x, y) \in \mathscr{C}^\infty(\overline{M} \times \overline{M}). \tag{55}$$

Proof This follows from (54) and from the fact that $\chi_k(\mathcal{T}_\mathcal{R}) \in \mathscr{C}^\infty(X \times X)$. □

We need the following variant of [8, Theorem 1.1].

Theorem 4 *Let (X, HX, J) be an orientable compact strictly pseudoconvex Cauchy-Riemann manifold of dimension $2n-1$, $n \geq 2$. We consider:*

(a) A Riemannian metric g^{TX} compatible with J, with volume form dv_X and the associated L^2-space $L^2(X) = L^2(X, dv_X)$.

(b) A contact form ω_0 on X such that the Levi form $\mathcal{L} = \frac{1}{2}d\omega_0(\cdot, J\cdot)$ is positive definite. We denote by $dv_{\omega_0} = \omega_0 \wedge (d\omega_0)^{n-1}$.

(c) An operator $\mathcal{S} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$, satisfying (35) and (37)- (40).

(d) For a formally self-adjoint first order pseudodifferential operator $Q \in L^1_{\mathrm{cl}}(X)$ we consider the Toeplitz operator $\mathcal{T}_Q = \mathcal{S}Q\mathcal{S} : L^2(X) \to L^2(X)$.

Let (D, x) be any coordinates patch and let $\varphi : D \times D \to \mathbb{C}$ be the phase function satisfying (38) and (39). Then for any formally self-adjoint first order pseudodifferential operator $Q \in L^1_{\mathrm{cl}}(X)$ whose symbol σ_Q satisfies $\sigma_Q(\omega_0) > 0$ on X, and for any $\chi \in \mathscr{C}^\infty_c((0, +\infty))$, $\chi \not\equiv 0$, the Schwartz kernel of $\chi_k(T_Q)$, $\chi_k(\lambda) := \chi\left(k^{-1}\lambda\right)$, can be represented for k large by

$$\chi_k(\mathcal{T}_Q)(x, y) = \int_0^{+\infty} e^{ikt\varphi(x,y)} A(x, y, t, k) dt + O\left(k^{-\infty}\right) \text{ on } D \times D, \tag{56}$$

where $A(x, y, t, k) \in S^n_{\mathrm{loc}}(1; D \times D \times \mathbb{R}_+)$,

$$\begin{aligned} &A(x, y, t, k) \sim \sum_{j=0}^{+\infty} A_j(x, y, t) k^{n-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1; D \times D \times \mathbb{R}_+),\\ &A_j(x, y, t) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}_+),\ \ j = 0, 1, 2, \dots,\\ &A_0(x, x, t) = \frac{1}{2\pi^n} \frac{dv_{\omega_0}}{dv_X}(x)\, \chi(t\sigma_Q(\omega_0(x)))\, t^{n-1} \not\equiv 0, \end{aligned} \tag{57}$$

and for some compact interval $I \Subset \mathbb{R}_+$,

$$\operatorname{supp}_t A(x, y, t, k),\ \ \operatorname{supp}_t A_j(x, y, t) \subset I, \quad j = 0, 1, 2, \dots. \tag{58}$$

Moreover, for any $\tau_1, \tau_2 \in \mathscr{C}^\infty(X)$ such that $\operatorname{supp}(\tau_1) \cap \operatorname{supp}(\tau_2) = \emptyset$, *we have*

$$\tau_1 \chi_k(T_P) \tau_2 = O\left(k^{-\infty}\right). \tag{59}$$

The proof of Theorem 4 is completely analogous to the proof of [8, Theorem 1.1] on account of the structure of $\mathcal{S}$ as a Fourier integral operator given in Theorem 3.

Proof (*Proof of Theorem* 1) We will apply Theorem 4 for $X = \partial M$ as in Theorem 1. The metric g^{TX} in (a) is induced by the metric $g^{TM'}$ and the contact form ω_0 in (b) is given by (21)–(23). The operator $\mathcal{S}$ in (c) is the operator constructed in Theorem 3, which in particular fulfills (36). Moreover, we apply Theorem 4 for $Q = \mathcal{R}$ given by (49). In this situation, we have

$$\frac{dv_{\omega_0}}{dv_X}(x) = \det(\mathscr{L}_x), \quad \sigma_{\mathcal{R}}(\omega_0) = \omega_0(T). \tag{60}$$

By (26) and (27) we have $\sigma_{\mathcal{R}}(\omega_0) = \omega_0(T) > 0$ on X.

We first prove (i). Let $\tau, \hat{\tau} \in \mathscr{C}^\infty(\overline{M})$, $\operatorname{supp}\tau \cap \operatorname{supp}\hat{\tau} = \emptyset$. We have

$$\begin{aligned}
&\tau\chi_k(T_R)\hat{\tau}\\
&= \tau P\chi_k(\mathcal{T}_{\mathcal{R}})(P^*P)^{-1}P^*\hat{\tau}\\
&= \tau P\tau_1\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1(P^*P)^{-1}P^*\hat{\tau} + \tau P(1-\tau_1)\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1(P^*P)^{-1}P^*\hat{\tau}\\
&\quad + \tau P\chi_k(\mathcal{T}_{\mathcal{R}})(1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau},
\end{aligned} \tag{61}$$

where $\tau_1, \hat{\tau}_1 \in \mathscr{C}^\infty(X)$, $\operatorname{supp}\tau_1 \cap \operatorname{supp}\hat{\tau}_1 = \emptyset$, $\operatorname{supp}\tau \cap \operatorname{supp}(1-\tau_1) = \emptyset$,

$$\operatorname{supp}(1-\hat{\tau}_1) \cap \operatorname{supp}\hat{\tau} = \emptyset.$$

We apply now Theorem 4 for the operator $Q = \mathcal{R}$ and we see that $\tau_1\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1 = O(k^{-\infty})$ and hence

$$\tau P\tau_1\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1(P^*P)^{-1}P^*\hat{\tau} = O(k^{-\infty}) \text{ on } \overline{M}\times\overline{M}. \tag{62}$$

From [14, Lemma 4.1], we see that

$$(\tau P(1-\tau_1))(x,y) \in \mathscr{C}^\infty(\overline{M}\times X), \tag{63}$$

where $(\tau P(1-\tau_1))(x,y)$ denotes the distribution kernel of $\tau P(1-\tau_1)$. From (63), we can repeat the proof of [8, Theorem 4.6] with minor changes and deduce that

$$\tau P(1-\tau_1)\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1(P^*P)^{-1}P^*\hat{\tau} = O(k^{-\infty}) \text{ on } \overline{M}\times\overline{M}. \tag{64}$$

Similarly, from [14, Lemma 4.2], we see that

$$((1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau})(x,y) \in \mathscr{C}^\infty(\overline{X}\times\overline{M}), \tag{65}$$

where $((1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau})(x,y)$ denotes the distribution kernel of $(1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau}$. From (65), we can repeat the proof of [8, Theorem 4.6] with minor changes and deduce that

$$\tau P\chi_k(\mathcal{T}_{\mathcal{R}})(1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau} = O(k^{-\infty}) \text{ on } \overline{M}\times\overline{M}. \tag{66}$$

From (61), (62), (64) and (66), we get (5).

We prove now (ii) and (iii). Fix $p \in \overline{M}$. We first assume that $p \notin X$ and let U be an open set of p with $U \cap X = \emptyset$. Let $\tau \in \mathscr{C}^\infty_c(U)$. Since $(\tau P)(x,y) \in \mathscr{C}^\infty(\overline{M}\times X)$, we can repeat the proof of [8, Theorem 4.6] with minor changes and get

$$\tau P \chi_k(\mathcal{T}_{\mathcal{R}})(P^*P)^{-1}P^* = O(k^{-\infty}) \text{ on } \overline{M} \times \overline{M}. \tag{67}$$

From (54) and (67), we get (6).

Now, assume that $p \in X$ and let U be an open local coordinate patch of p in M'. Let $D := U \cap X$. We can repeat the proof of [8, Theorem 1.1] (in the situation of Theorem 4) and deduce

$$\chi_k(\mathcal{T}_{\mathcal{R}})(x, y) = \int_0^{+\infty} e^{ikt\varphi(x,y)} a(x, y, t, k) dt + O\left(k^{-\infty}\right) \quad \text{on } D \times D, \tag{68}$$

where $a(x, y, t, k) \in S^n_{\mathrm{loc}}(1; D \times D \times \mathbb{R}_+)$,

$$\begin{aligned}
&a(x, y, t, k) \sim \sum_{j=0}^{\infty} a_j(x, y, t) k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1; D \times D \times \mathbb{R}_+),\\
&a_j(x, y, t) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}_+), \ j = 0, 1, 2, \ldots,\\
&a_0(x, x, t) = \frac{1}{2\pi^n} \det(\mathscr{L}_x)\, \chi(t\omega_0(T(x))) t^{n-1} \not\equiv 0,
\end{aligned}$$

and for some compact interval $I \Subset \mathbb{R}_+$,

$$\mathrm{supp}_t\, a(x, y, t, k), \ \ \mathrm{supp}_t\, a_j(x, y, t) \subset I, \quad j = 0, 1, 2, \ldots.$$

From (68), we can repeat the WKB procedure in [10, Part II,Proposition 7.8, Theorem 7.9] and get (7). □

Proof (*Proof of Corollary* 1) The asymptotics $\chi_k(T_R)(x, x) = O(k^{-\infty})$, $k \to \infty$, on M from (12) follow immediately from (6). Let $p \in X$ be fixed and consider local coordinates near p on M' of the form $z = (x_1, \ldots, x_{2n-1}, \rho)$, where $x = (x_1, \ldots, x_{2n-1})$ are local coordinates on X near p with $x(p) = 0$ and the phase function ψ in (7) has the form (11). In this local chart we have near (p, p),

$$i\Psi(z, z) = 2\rho(z)\big(1 + O(|z|\big) + O(|z|^3). \tag{69}$$

By (7) we have

$$\chi_k(T_P)(z, z) = \sum_{j=0}^{\infty} k^{n+1-j} \int_0^{+\infty} e^{ikt\psi(z,z)} b_j(z, z, t)\, dt + O(k^{-\infty}) \tag{70}$$

Since $\Psi(x, x) = 0$ for $x \in X$ this yields the asymptotic expansion (13) with the coefficients (14). The expression (15) of $b_0(x)$ follows from (8). We have $b_0(x) > 0$ for every $x \in X$. Note also the exponential decay of the integrands in (70) for $z \in M$ near p due to (69) and on account of $\rho(z) < 0$.

The trace of the operator $\chi_k(T_P)$ is given by

$$\begin{aligned}\operatorname{Tr}\chi_k(T_P) &= \int_M \chi_k(T_P)(z,z)\,dv_{M'}(z)\\ &= \int_{\{\rho<\varepsilon\}} \chi_k(T_P)(z,z)\,dv_{M'}(z) + \int_{\{\varepsilon\le\rho<0\}} \chi_k(T_P)(z,z)\,dv_{M'}(z)\\ &=: I_1(k) + I_2(k). \end{aligned} \tag{71}$$

where $\varepsilon < 0$ is chosen small enough. We have $I_1(k) = O(k^{-\infty})$ by (12). By using (69) and (70) and the fact that $2k\int_\varepsilon^0 e^{2k\rho}d\rho \to 1$ as $k \to \infty$, we obtain that there exist $C_1, C_2 > 0$ such that $C_1k^n \le I_2(k) \le C_2k^n$ for k large enough. □

Remark 3 It is interesting to compare the result of Corollary 1 to the corresponding result regarding Toeplitz operators on the boundary X (see also [8, Corollary 1.2]). By Theorem 4 we have for the operator $\mathcal{T}_\mathcal{R}$ from (48),

$$\chi_k(\mathcal{T}_\mathcal{R})(x,x) = \sum_{j=0}^{\infty} A_j(x)k^{n-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1;X) \text{ on } X, \tag{72}$$

where

$$A_j(x) = \int_0^{+\infty} A_j(x,x,t)dt, \quad j \in \mathbb{N}_0, \tag{73}$$

with $A_j(x,x,t)$ as in (57), and

$$A_0(x) = \frac{1}{2\pi^n}\det(\mathscr{L}_x)\int_0^{+\infty}\chi(t\omega_0(T(x)))t^{n-1}dt. \tag{74}$$

Moreover,

$$\operatorname{Tr}\chi_k(\mathcal{T}_\mathcal{R}) = \frac{k^n}{2\pi^n}\int_X\int_0^{+\infty}\det(\mathscr{L}_x)\chi(t\omega_0(x))\,t^{n-1}dt + O(k^{n-1})\,. \tag{75}$$

We see that $\chi_k(T_R)(x,x)$ and $\chi_k(\mathcal{T}_\mathcal{R})(x,x)$ have an asymptotic expansion on the boundary X, the former with leading term of order k^{n+1}, the latter of order k^n. On the other hand both traces $\operatorname{Tr}\chi_k(T_R)$ and $\operatorname{Tr}\chi_k(\mathcal{T}_\mathcal{R})$ have growth of order k^n as $k \to +\infty$.

Remark 4 If we do not normalize the definition function ρ such that $|d\rho| = 1$, Theorem 1 holds with the same proof, but we need to take $\omega_0 = -J \circ d(\rho/|d\rho|)$. With this ω_0 the leading term $b_0(x,x,t)$ has the same formula as in (8).

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On a Concrete Realization of Simply Connected Complex Domains Admitting Homogeneous Kähler Metrics

Hideyuki Ishi

Abstract We show that any simply connected complex domain admitting homogeneous Kähler metrics is realized as a complex orbit in the Siegel-Jacobi domain under the action of a solvable Lie subgroup of the Jacobi group.

Keywords homogeneous Kähler metric · Bounded homogeneous domain · Siegel-Jacobi domain · Solvable lie group

1 Introduction

A Kähler metric ds^2 on a complex manifold M is said to be *homogeneous* if the Kähler automorphism group $\mathrm{Aut}(M, ds^2)$ of the holomorphic automorphisms preserving the metric ds^2 acts on M transitively. In this case, (M, ds^2) is called *a homogeneous Kähler manifold*. The fundamental theorem of homogeneous Kähler manifolds, conjectured in [7] and proved in [2], states that every homogeneous Kähler manifold M is a holomorphic fiber bundle over a bounded homogeneous domain in which the fiber is (with the induced Kähler metric) the product of a fiat homogeneous Kähler manifold and a compact simply connected homogeneous Kähler manifold. This fundamental theorem tells us that, if M is a simply connected complex domain admitting a homogeneous Kähler metric, M is biholomprphic to a product of a bounded homogeneous domain and a complex vector space as a complex manifold because there is no compact homogenoeus Kähler submanifold of M in this case. In this paper, we give a concrete realization of such M together with homogeneous Kähler metrics.

A typical example of a homogeneous Kähler manifold is a bounded homogeneous domain $\mathcal{U} \subset \mathbb{C}^n$ with its Bergman metric. Note that there are many homogeneous Kähler metrics on $\mathcal{U}$ other than the Bergman metric, and all of such metrics are thoroughly classified by Dorfmeister [1]. On the other hand, it is shown in [4] that every bounded homogeneous domain $\mathcal{U}$ is realized as a complex orbit $\mathcal{O}$ in the Siegel upper half plane $\mathcal{S}_N$ under the action of a solvable Lie subgroup of the real symplectic

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group. Based on these results, we claim in Theorem 1 that homogeneous Kähler metrics on the domain $\mathcal{U}$ are obtained as a restriction of some specific homogeneous Kähler metrics on $\mathcal{S}_N$ to the complex orbit $\mathcal{O}$.

Another typical example of a homogeneous Kähler manifold is the Siegel-Jacobi domain $\mathcal{D}_N$ with a Kähler metric that is invariant under the action of the Jacobi group. The Siegel-Jacobi domain $\mathcal{D}_N$ is biholomorphic to a product $\mathcal{S}_N \times \mathbb{C}^N$, whereas the Kähler metric is not factorized along the product. In Theorem 2, we claim that every simply connected complex domain $M \subset \mathbb{C}^n$ admitting a homogeneous Kähler metric is realized as a complex orbit $\tilde{\mathcal{O}}$ in the Siegel-Jacobi domain $\mathcal{D}_N$ under the action of a solvable Lie subgroup of the Jacobi group, where the homogenoeus Kähler metric on M is isometric to the restriction of a certain specific homogeneous Kähler metric on $\mathcal{D}_N$ to the orbit $\tilde{\mathcal{O}}$. The key ingredient of our argument is a nice equivariant projection $\pi : \mathcal{S}_{N+1} \to \mathcal{D}_N$. We give a concise description (11) of the orbit $\tilde{\mathcal{O}}$ as a product of a Siegel domain and a complex vector space.

Let us fix notaion used in this paper. The space of $n \times m$ matrices with $\mathbb{K}$-entries is denoted by $\mathrm{Mat}(n, m, \mathbb{K})$. The space $\mathrm{Mat}(n, n, \mathbb{K})$ is also denoted by $\mathrm{Mat}(n, \mathbb{K})$. We write $\mathrm{Sym}(n, \mathbb{K})$ for the space of symmetric matrices with $\mathbb{K}$-entries. When a real symmetric matrix $S \in \mathrm{Sym}(n, \mathbb{R})$ is positive defiite, we write $S > 0$. We denote by $\mathbb{K}^n$ the space of n-dimensional column vectors of $\mathbb{K}$-entries, that is to say, $\mathbb{K}^n = \mathrm{Mat}(n, 1, \mathbb{K})$. The transpose of a matrix A is denoted by ${}^{\mathrm{t}}A$. The identity matrix of size n is denoted by I_n.

2 The Siegel Upper Half Plane

We write $Sp(N, \mathbb{R})$ for the real symplectic group of rank N, that is, the group of matrices $g \in GL(2N, \mathbb{R})$ satisfying ${}^{\mathrm{t}}g \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$. It is well known that $Sp(N, \mathbb{R})$ acts transitively on the Siegel upper plane

$$\mathcal{S}_N := \{\, \zeta \in \mathrm{Sym}(N, \mathbb{C})\,;\ \mathrm{Im}\, \zeta > 0\}$$

by

$$g \cdot \zeta := (A\zeta + B)(C\zeta + D)^{-1} \qquad (\zeta \in \mathcal{S}_N),$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(N, \mathbb{R})$ with $A, B, C, D \in \mathrm{Mat}(N, \mathbb{R})$. Let $K_N \subset Sp(N, \mathbb{R})$ be the isotropy subgroup at $p_N := iI_N \in \mathcal{S}_N$, that is, $\{\, g \in Sp(N, \mathbb{R})\,;\ g \cdot p_N = p_N\}$. Then

$$K_N = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\,;\ A + iB \in U(N),\ A, B \in \mathrm{Mat}(N, \mathbb{R}) \right\} \simeq U(N),$$

and we have a bijection $Sp(N, \mathbb{R})/K_N \ni gK_N \mapsto g \cdot p_N \in \mathcal{S}_N$. On the other hand, we have

$$\mathrm{Im}\,(g \cdot \zeta) = {}^{\mathrm{t}}\overline{(C\zeta + D)}^{-1}(\mathrm{Im}\,\zeta)(C\zeta + D)^{-1}, \tag{1}$$

so that

$$-\log\det\mathrm{Im}\,(g\cdot\zeta) = -\log\det\mathrm{Im}\,\zeta + \log\det(C\zeta + D) + \overline{\log\det(C\zeta + D)}. \tag{2}$$

Let ds^2_{det} be the Kähler metric whose potential function ϕ on $\mathcal{S}_N$ is given by $\phi(\zeta) := -\log\det\mathrm{Im}\,\zeta \quad (\zeta \in \mathcal{S}_N)$. For tangent vectors $V, V' \in T_\zeta\mathcal{S}_N = \mathrm{Sym}(N, \mathbb{C})$, we have $ds^2_{\mathrm{det}}(V, V')_\zeta = \mathrm{tr}\,(\mathrm{Im}\,\zeta)^{-1}V(\mathrm{Im}\,\zeta)^{-1}\overline{V'}/4$. We see from (2) that ds^2_{det} is invariant under the action of $Sp(N, \mathbb{R})$, so that ds^2_{det} is a homogeneous Kähler metric on $\mathcal{S}_N$.

Let us consider other homogeneous Kähler metrics on $\mathcal{S}_N$. Let H_N be the group of real lower triangular matrices of size N with positive diagonal entries. For $T \in H_N$ and $X \in \mathrm{Sym}(N, \mathbb{R})$. define

$$b(X, T) := \begin{pmatrix} I_N & X \\ 0 & I_N \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & {}^{\mathrm{t}}T^{-1} \end{pmatrix} \in Sp(N, \mathbb{R}).$$

Then $b(X, T) \cdot \zeta = X + T\zeta{}^{\mathrm{t}}T$ for $\zeta \in \mathcal{S}_N$. Put

$$B_N := \{\, b(X, T)\,;\ X \in \mathrm{Sym}(N, \mathbb{R}),\ T \in H_N \,\},$$

which is an Iwasawa subgroup (a maximal connected split solvable subgroup) of $Sp(N, \mathbb{R})$ acting simply transitively on $\mathcal{S}_N$. For $\underline{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N$, we introduce a one-dimensional representation $\chi_{\underline{\alpha}} : B_N \ni b(X, T) \mapsto \prod_{k=1}^N (T_{kk})^{2\alpha_k} \in \mathbb{C}$. Keeping in mind a diffeomorphism $B_N \ni b \mapsto b \cdot p_N \in \mathcal{S}_N$, define $\Phi_{\underline{\alpha}} : \mathcal{S}_N \to \mathbb{C}$ by $\Phi_{\underline{\alpha}}(b \cdot p_N) := \chi_{\underline{\alpha}}(b) \quad (b \in B_N)$. For $\underline{\alpha} \in \mathbb{R}^N_{>0}$, let $ds^2_{\underline{\alpha}}$ be the Kähler metric whose potential function on $\mathcal{S}_N$ is $\log\Phi_{-\underline{\alpha}}$. The positivity of $ds^2_{\underline{\alpha}}$ is verified as follows. In general, for $A = (a_{ij}) \in \mathrm{Mat}(N, \mathbb{C})$ and $k = 1, \ldots, N$, we denote by $A^{[k]}$ the matrix $(a_{ij})_{1\le i\le k,\,1\le j\le k}$ of size k. Then we can show that $\Phi_{\underline{\alpha}}(\zeta) = \prod_{k=1}^N (\det\mathrm{Im}\,\zeta^{[k]})^{\alpha_k - \alpha_{k+1}}$ for $\zeta \in \mathcal{S}_N$, where $\alpha_{N+1} := 0$. Thus, for $V, V' \in T_\zeta\mathcal{S}_N = \mathrm{Sym}(N, \mathbb{C})$, we have

$$ds^2_{\underline{\alpha}}(V, V')_\zeta = \sum_{k=1}^N (\alpha_k - \alpha_{k+1})\mathrm{tr}\,\Big((\mathrm{Im}\,\zeta^{[k]})^{-1}V^{[k]}(\mathrm{Im}\,\zeta^{[k]})^{-1}\overline{(V')^{[k]}}\Big)/4.$$

Especially, when $\zeta = p_N$ and $V = V' \neq 0$, we obtain

$$ds^2_{\underline{\alpha}}(V, V)_{p_N} = \sum_{k=1}^N \alpha_k\Big(|V_{kk}|^2 + 2\sum_{j<k}|V_{kj}|^2\Big)/4 > 0.$$

On the other hand, since

$$\log \Phi_{-\underline{\alpha}}(b \cdot \zeta) = \log \Phi_{-\underline{\alpha}}(\zeta) - \log \chi_{\underline{\alpha}}(b) \qquad (\zeta \in \mathcal{S}_N,\ b \in B_N),$$

the tensor $ds^2_{\underline{\alpha}}$ is invariant under the transitive action of B_N on $\mathcal{S}_N$. Therefore we get the positivity of $ds^2_{\underline{\alpha}}$, and we see also that $ds^2_{\underline{\alpha}}$ is a homogeneous Kähler metric on $\mathcal{S}_N$. In fact, the metrics $ds^2_{\underline{\alpha}}$ ($\underline{\alpha} \in \mathbb{R}^N_{>0}$) essentially exhaust all the homogeneous Kähler metrics on $\mathcal{S}_N$ in the following sense.

Proposition 1 *For any homogeneous Kähler metric ds^2 on the Siegel upper half plane, there exist $\underline{\alpha} \in \mathbb{R}^N_{>0}$ and $g \in Sp(N, \mathbb{R})$ such that $ds^2 = g^* ds^2_{\underline{\alpha}}$, where g is regarded as a holomorphic automorphism $\zeta \mapsto g \cdot \zeta$ on $\mathcal{S}_N$.*

Proof The Kähler automorphism group $\mathrm{Aut}(\mathcal{S}_N,\ ds^2)$ is a closed subgroup of $Sp(N, \mathbb{R})$. By [1, Sect. 3.3, Theorem 2 (c)], there exists a split solvable Lie group $B \subset \mathrm{Aut}(\mathcal{S}_N,\ ds^2)$. Then we have two Iwasawa decomposition $Sp(N, \mathbb{R}) = B_N K_N = B K_N$. It follows from the uniqueness of the Iwasawa subgroup up to conjugation (see [6]) that there exists $g \in Sp(N, \mathbb{R})$ for which $B_N = g B g^{-1}$. Then the pull-back metric $(g^{-1})^* ds^2$ is B_N-invariant. Therefore, thanks to [1, Sect. 3.4, Theorem 1], there exists $\underline{\alpha} \in \mathbb{R}^N_{>0}$ for which $(g^{-1})^* ds^2 = ds^2_{\underline{\alpha}}$.

3 Realization of Bounded Homogeneous Domains

Let $\mathcal{U} \subset \mathbb{C}^n$ be a bounded homogeneous domain. Then there exists a split solvable group B acting holomorphically and simply transitively on $\mathcal{U}$ ([5, Chap. 2]). Taking a point $p_0 \in \mathcal{U}$, we have a diffeomorphism $B \ni b \mapsto b \cdot p_0 \in \mathcal{U}$. Let $\mathfrak{b}$ be the Lie algebra of B. Then we have a linear isomorphism $\mathfrak{b} \ni Y \mapsto Y \cdot p_0 \in T_{p_0}\mathcal{U} = \mathbb{C}^n$. Let $j : \mathfrak{b} \to \mathfrak{b}$ be the linear map defined in such a way that $(jY) \cdot p_0 = \sqrt{-1} Y \cdot p_0 \in \mathbb{C}^n$ $(Y \in \mathfrak{b})$. Then $(\mathfrak{b}, j)$ is a normal j-algebra, which means that the following conditions are satisfied:
(i) $[Y_1, Y_2] + j[jY_1, Y_2] + j[Y_1, jY_2] - [jY_1, jY_2] = 0$ for all $Y_1, Y_2 \in \mathfrak{b}$,
(ii) there exists a linear form $\xi : \mathfrak{b} \to \mathbb{R}$ such that the bilinear form $B_\xi(Y_1, Y_2) := \xi([jY_1, Y_2])$ $(Y_1, Y_2 \in \mathfrak{b})$ defines a j-invariant inner product on $\mathfrak{b}$.

Let us consider a normal j-algebra associated to the simply transitive action of B_N on the Siegel upper half plane $\mathcal{S}_N$. Let $\mathfrak{h}_N$ and $\mathfrak{b}_N$ be the Lie algebras of H_N and B_N respectively. Then $\mathfrak{h}_N$ is the set of lower triangular matrices, and $\mathfrak{b}_N$ is the set of matrices $Y \in \mathrm{Mat}(2N, \mathbb{R})$ of the form

$$Y = \begin{pmatrix} T & X \\ 0 & -{}^{\mathrm{t}}T \end{pmatrix} \qquad (T \in \mathfrak{h}_N,\ \ X \in \mathrm{Sym}(N, \mathbb{R})).$$

For $X \in \mathrm{Sym}(N, \mathbb{R})$, there exists a unique $\underset{\vee}{X} \in \mathfrak{h}_N$ for which $X = \underset{\vee}{X} + {}^{\mathrm{t}}(\underset{\vee}{X})$. Indeed, we have $(\underset{\vee}{X})_{ij} = \begin{cases} X_{ij} & (i > j) \\ X_{ii}/2 & (i = j) \\ 0 & (i < j) \end{cases}$. Let $J_N : \mathfrak{b}_N \to \mathfrak{b}_N$ be the linear map given by

$$J_N Y := \begin{pmatrix} \underset{\vee}{X} & -(T + {}^{\mathrm{t}}T) \\ 0 & -{}^{\mathrm{t}}(\underset{\vee}{X}) \end{pmatrix} \quad \left(Y = \begin{pmatrix} T & X \\ 0 & -{}^{\mathrm{t}}T \end{pmatrix} \in \mathfrak{b}_N,\ T \in \mathfrak{h}_N,\ X \in \mathrm{Sym}(N, \mathbb{R})\right).$$

Then we have $(J_N Y) \cdot p_N = \sqrt{-1}(Y \cdot p_N)$ for $Y \in \mathfrak{b}_N$, and $(\mathfrak{b}_N,\ J_N)$ is a normal j-algebra. It is shown in [4] that every normal j-algebra is realized as a j-subalgebra of $(\mathfrak{b}_N,\ J_N)$ constructed in the following way. Let n_k $(k = 0, 1, \ldots, r)$ be positive integers such that $N = n_0 + n_1 + \cdots + n_r$. Let $\mathcal{V}_{lk} \subset \mathrm{Mat}(n_l, n_k, \mathbb{R})$ $(1 \le k < l \le r)$ and $\mathcal{W}_l \subset \mathrm{Mat}(n_l, n_0, \mathbb{C})$ $(1 \le l \le r)$ be respectively real and complex vector spaces satisfying

(V1) $A \in \mathcal{V}_{lk},\ B \in \mathcal{V}_{ki} \Rightarrow AB \in \mathcal{V}_{li}$ $(1 \le i < k < l \le r)$,
(V2) $A \in \mathcal{V}_{li},\ B \in \mathcal{V}_{ki} \Rightarrow A\,{}^{\mathrm{t}}B \in \mathcal{V}_{lk}$ $(1 \le i < k < l \le r)$,
(V3) $A \in \mathcal{V}_{lk} \Rightarrow A\,{}^{\mathrm{t}}A \in \mathbb{R}I_{n_l}$ $(1 \le k < l \le r)$,
(W1) $A \in \mathcal{V}_{lk},\ C \in \mathcal{W}_k \Rightarrow AC \in \mathcal{W}_l$ $(1 \le k < l \le r)$,
(W2) $C \in \mathcal{W}_l,\ C' \in \mathcal{W}_k \Rightarrow C\,{}^{\mathrm{t}}\overline{C'} \in (\mathcal{V}_{lk})_{\mathbb{C}}$ $(1 \le k < l \le r)$,
(W3) $C \in \mathcal{W}_l \Rightarrow C\,{}^{\mathrm{t}}\overline{C} + \overline{C}\,{}^{\mathrm{t}}C \in \mathbb{R}I_{n_l}$.

Let $\mathcal{V}$ be the set of real symmetric matrices $X \in \mathrm{Sym}(N - n_0, \mathbb{R})$ of the form

$$X = \begin{pmatrix} X_{11} & {}^{\mathrm{t}}X_{21} & \cdots & {}^{\mathrm{t}}X_{r1} \\ X_{21} & X_{22} & & {}^{\mathrm{t}}X_{r2} \\ \vdots & & \ddots & \vdots \\ X_{r1} & X_{r2} & \cdots & X_{rr} \end{pmatrix} \in \mathrm{Sym}(N - n_0, \mathbb{R}) \quad \begin{pmatrix} X_{kk} = x_{kk} I_{n_k},\ x_{kk} \in \mathbb{R}\,(k = 1, \ldots, r) \\ X_{lk} \in \mathcal{V}_{lk} \ \ (1 \le k < l \le r) \end{pmatrix},$$

and $\mathcal{W}$ the set of complex matrices $U \in \mathrm{Mat}(N - n_0, n_0, \mathbb{C})$ of the form

$$U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_r \end{pmatrix} \quad (U_k \in \mathcal{W}_k,\ k = 1, \ldots, r).$$

Finally, let $\mathfrak{b}_{\mathcal{V},\mathcal{W}}$ be the set of $Y \in \mathfrak{b}_N$ of the form

$$Y = \begin{pmatrix} 0 & 0 & 0 & \mathrm{Re}\,{}^{\mathrm{t}}U \\ \mathrm{Im}\,U & \underset{\vee}{X} & \mathrm{Re}\,U & X' \\ 0 & 0 & 0 & -\mathrm{Im}\,{}^{\mathrm{t}}U \\ 0 & 0 & 0 & -{}^{\mathrm{t}}(\underset{\vee}{X}) \end{pmatrix} \quad (X, X' \in \mathcal{V},\ \ U \in \mathcal{W}).$$

Then $\mathfrak{b}_{\mathcal{V},\mathcal{W}}$ is a Lie subalgebra of $\mathfrak{b}_N$ and $J_N \mathfrak{b}_{\mathcal{V},\mathcal{W}} = \mathfrak{b}_{\mathcal{V},\mathcal{W}}$. In other words, $(\mathfrak{b}_{\mathcal{V},\mathcal{W}},\ J_N|_{\mathfrak{b}_{\mathcal{V},\mathcal{W}}})$ is a normal j-algebra. The integer r is called *the rank* of the normal j-algebra $\mathfrak{b}_{\mathcal{V},\mathcal{W}}$, which is equal to the dimension of $\mathfrak{b}_{\mathcal{V},\mathcal{W}}/[\mathfrak{b}_{\mathcal{V},\mathcal{W}},\ \mathfrak{b}_{\mathcal{V},\mathcal{W}}]$.

For a normal j-algebra $(\mathfrak{b},\ j)$ associated to a bounded homogeneous domain $\mathcal{U}$, there exist an appropriate $\mathfrak{b}_{\mathcal{V},\mathcal{W}} \subset \mathfrak{b}_N$ and a Lie algebra isomorphism $f : \mathfrak{b} \to \mathfrak{b}_{\mathcal{V},\mathcal{W}}$ such that $f(jY) = J_N f(Y)$ for $Y \in \mathfrak{b}$ ([4]). Then we have an equivariant holomorphic injective map $F : \mathcal{U} \ni (\exp Y) \cdot p_0 \mapsto (\exp f(Y)) \cdot p_N \in \mathcal{S}_N$. Let $B_{\mathcal{V},\mathcal{W}}$

denote the Lie group $\exp \mathfrak{b}_{\mathcal{V},\mathcal{W}} \subset B_N$. The image of F is exactly the $B_{\mathcal{V},\mathcal{W}}$-orbit $\mathcal{O}_{\mathcal{V},\mathcal{W}}$ through $p_N \in \mathcal{S}_N$, which is described as

$$\mathcal{O}_{\mathcal{V},\mathcal{W}} = \left\{ \begin{pmatrix} iI_{n_0} & {}^{\mathrm{t}}U \\ U & Z - \frac{i}{2}U{}^{\mathrm{t}}U \end{pmatrix} ; \begin{array}{l} Z \in \mathcal{V}_{\mathbb{C}},\ U \in \mathcal{W} \\ \operatorname{Im} Z - (U{}^{\mathrm{t}}\overline{U} + \overline{U}{}^{\mathrm{t}}U)/4 > 0 \end{array} \right\}.$$

Here we note that the condition $\operatorname{Im} Z - (U{}^{\mathrm{t}}\overline{U} + \overline{U}{}^{\mathrm{t}}U)/4 > 0$ is equivalent to the positive definiteness of the imaginary part of the matrix $\begin{pmatrix} iI_{n_0} & {}^{\mathrm{t}}U \\ U & Z - \frac{i}{2}U{}^{\mathrm{t}}U \end{pmatrix}$. This expression of $\mathcal{O}_{\mathcal{V},\mathcal{W}}$ leads us to the notion of Siegel domain naturally. Let $\Omega_{\mathcal{V}}$ be the set $\{ X \in \mathcal{V};\ X > 0\}$, which is an open convex cone in the vector space $\mathcal{V}$. For $U, U' \in \mathcal{W}$, the complex symmetric matrix $U\,{}^{\mathrm{t}}\overline{U'} + \overline{U'}\,{}^{\mathrm{t}}U$ belongs to $\mathcal{V}_{\mathbb{C}}$ thanks to (W2) and (W3). We define a vector-valued Hermitian form $Q_{\mathcal{W}} : \mathcal{W} \times \mathcal{W} \to (\mathcal{V})_{\mathbb{C}}$ by $Q_{\mathcal{W}}(U, U') := (U\,{}^{\mathrm{t}}\overline{U'} + \overline{U'}\,{}^{\mathrm{t}}U)/4$. We can check easily the $\Omega_{\mathcal{V}}$-positivity of $Q_{\mathcal{W}}$ which means that $Q_{\mathcal{W}}(U, U)$ belongs to the closure of the cone $\Omega_{\mathcal{V}}$ for all $U \in \mathcal{W}$, and that $Q_{\mathcal{W}}(U, U) \neq 0$ if $U \neq 0$. The Siegel domain $\mathcal{S}(\Omega_{\mathcal{V}}, Q_{\mathcal{W}})$ associated to the cone $\Omega_{\mathcal{V}} \subset \mathcal{V}$ and the Hermitian form $Q_{\mathcal{W}}$ is the set $\{ (Z, U) \in \mathcal{V}_{\mathbb{C}} \times \mathcal{W};\ \operatorname{Im} Z - Q_{\mathcal{W}}(U, U) \in \Omega_{\mathcal{V}}\}$ (see [5, Chap. 1]). Then we have a biholomorphic map

$$\mathcal{S}(\Omega_{\mathcal{V}}, Q_{\mathcal{W}}) \ni (Z, U) \mapsto \begin{pmatrix} iI_{n_0} & {}^{\mathrm{t}}U \\ U & Z - \frac{i}{2}U{}^{\mathrm{t}}U \end{pmatrix} \in \mathcal{O}_{\mathcal{V},\mathcal{W}}. \tag{3}$$

Here we remark that some of $\mathcal{V}_{lk}$ and $\mathcal{W}_k$ can be the zero spaces. If $\mathcal{W} = \{0\}$, then $\mathcal{O}_{\mathcal{V},\mathcal{W}} (\simeq \mathcal{S}(\Omega_{\mathcal{V}}, Q_{\mathcal{W}}))$ is biholomorphic to the tube domain $\mathcal{V} + i\Omega_{\mathcal{V}}$ in $\mathcal{V}_{\mathbb{C}}$.

Recall the B_N-invariat Kähler metrics $ds^2_{\underline{\alpha}}$ ($\underline{\alpha} \in \mathbb{R}^N_{>0}$) on $\mathcal{S}_N$. The restriction of $ds^2_{\underline{\alpha}}$ to $\mathcal{O}_{\mathcal{V},\mathcal{W}}$ is a homogeneous Kähler metric on $\mathcal{O}_{\mathcal{V},\mathcal{W}}$. We shall claim in Theorem 1 below that any homogeneous Kähler metric on a homogeneous bounded domain is obtained this way. By (V3) and (W3), we can define Euclidean norms on $\mathcal{V}_{lk}$ $(1 \le k < l \le r)$ and $\mathcal{W}_k$ $(1 \le k \le r)$ in such a way that $A{}^{\mathrm{t}}A = \|A\|^2 I_{n_l}$ $(A \in \mathcal{V}_{lk})$ and $C{}^{\mathrm{t}}\overline{C} + \overline{C}{}^{\mathrm{t}}C = 2\|C\|^2 I_{n_k}$ $(C \in \mathcal{W}_k)$. Furthermore, define a Hermitian norm on $(\mathcal{V}_{lk})_{\mathbb{C}}$ by $\|C\|^2 := \|\operatorname{Re} C\|^2 + \|\operatorname{Im} C\|^2$ for $C \in (\mathcal{V}_{lk})_{\mathbb{C}}$. A tangent vector $V \in T_{p_N}\mathcal{O}_{\mathcal{V},\mathcal{W}}$ is of the form $V = \begin{pmatrix} 0 & {}^{\mathrm{t}}U \\ U & Z \end{pmatrix}$ with $Z \in \mathcal{V}_{\mathbb{C}}$ and $U \in \mathcal{W}$. Then we have

$$ds^2_{\det}(V, V)_{p_N} = \operatorname{tr} V\overline{V}/4 = \sum_{l=1}^{r} n_l \Big(|z_{ll}|^2 + 2\|U_l\|^2 + 2\sum_{k<l} \|Z_{lk}\|^2\Big)/4.$$

For $\underline{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^r_{>0}$, put

$$\underline{\alpha}(\underline{\sigma}) := (\underbrace{1, \ldots, 1}_{n_0}, \underbrace{\frac{\sigma_1}{n_1}, \ldots, \frac{\sigma_1}{n_1}}_{n_1}, \underbrace{\frac{\sigma_2}{n_2}, \ldots, \frac{\sigma_2}{n_2}}_{n_2}, \ \ldots, \underbrace{\frac{\sigma_r}{n_r}, \ldots, \frac{\sigma_r}{n_r}}_{n_r}) \in \mathbb{R}^N_{>0}.$$

Then

$$ds^2_{\underline{\alpha}(\underline{\sigma})}(V, V)_{p_N} = \sum_{l=1}^{r} \sigma_l \Big(|z_{ll}|^2 + 2\|U_l\|^2 + 2\sum_{k<l} \|Z_{lk}\|^2 \Big)/4.$$

Comparing this formula with the parametrization of homogeneous Kähler metrics on a homogeneous Siegel domain [1, Sect. 3.4, Theorem 1], we obtain the following theorem.

Theorem 1 *For any homogeneous Kähler metric ds^2 on a bounded homogeneous domain $\mathcal{U}$, there exists a Kähler isomorphism $F : (\mathcal{U}, ds^2) \to (\mathcal{O}_{\mathcal{V},\mathcal{W}}, ds^2_{\underline{\alpha}(\underline{\sigma})}|_{\mathcal{O}_{\mathcal{V},\mathcal{W}}})$ with appropriate $\mathcal{V}, \mathcal{W}$ and $\underline{\sigma} \in \mathbb{R}^r_{>0}$.*

In particular, if $F : \mathcal{U} \to \mathcal{O}_{\mathcal{V},\mathcal{W}}$ is a biholomorphic map, then the Bergman metric of $\mathcal{U}$ corresponds to the parameter $\underline{\sigma} \in \mathbb{R}^r_{>0}$ with

$$\sigma_k = 2 + \sum_{i<k} \dim_{\mathbb{R}} \mathcal{V}_{ki} + \sum_{l>k} \dim_{\mathbb{R}} \mathcal{V}_{lk} + \dim_{\mathbb{C}} \mathcal{W}_k \quad (k = 1, \dots, r)$$

because the Jacobian of the action of $b \in B_{\mathcal{V},\mathcal{W}}$ on the Siegel domain $\mathcal{S}(\mathcal{V}, \mathcal{W}) \simeq \mathcal{O}_{\mathcal{V},\mathcal{W}}$ equals $\chi_{\underline{\alpha}(\underline{\sigma})/2}(b)$ (cf. [3, Lemma 5.1]).

4 The Siegel-Jacobi Domain

The Lie algebra $\mathfrak{sp}(N, \mathbb{R})$ of $Sp(N, \mathbb{R})$ is the space of matrices of the form

$$\begin{pmatrix} P & Q \\ R & -{}^{\mathrm{t}}P \end{pmatrix} \quad (P \in \mathrm{Mat}(N, \mathbb{R}),\ \ Q, R \in \mathrm{Sym}(N, \mathbb{R})).$$

Let $\mathfrak{g}_N \subset \mathfrak{sp}(N+1, \mathbb{R})$ be the centralizer of the element $\begin{pmatrix} 0 & E_{N+1,N+1} \\ 0 & 0 \end{pmatrix} \in$ sp$(N+1, \mathbb{R})$. Then we can describe $\mathfrak{g}_N$ as

$$\mathfrak{g}_N = \left\{ \begin{pmatrix} P & 0 & Q & u \\ {}^{\mathrm{t}}v & 0 & {}^{\mathrm{t}}u & \theta \\ R & 0 & -{}^{\mathrm{t}}P & -v \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \begin{array}{l} P \in \mathrm{Mat}(N, \mathbb{R}),\ Q, R \in \mathrm{Sym}(N, \mathbb{R}) \\ u, v \in \mathbb{R}^N,\ \theta \in \mathbb{R} \end{array} \right\}.$$

Clearly, we have a Lie algebra homomorphism

$$d\iota : \mathfrak{sp}(N, \mathbb{R}) \ni \begin{pmatrix} P & Q \\ R & -{}^{\mathrm{t}}P \end{pmatrix} \mapsto \begin{pmatrix} P & 0 & Q & 0 \\ 0 & 0 & 0 & \theta \\ R & 0 & -{}^{\mathrm{t}}P & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_N \subset \mathfrak{sp}(N+1, \mathbb{R}),$$

which is a differential of the Lie group homomorphism

$$\iota : Sp(N,\mathbb{R}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp(N+1,\mathbb{R}).$$

On the other hand, putting $N(u,v,\theta) := \begin{pmatrix} 0 & 0 & 0 & u \\ {}^{\mathrm{t}}v & 0 & {}^{\mathrm{t}}u & \theta \\ 0 & 0 & 0 & -v \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_N$, we have

$$[N(u,v,\theta), N(u',v',\theta')] = N(0,0,-2({}^{\mathrm{t}}uv' - {}^{\mathrm{t}}vu')),$$
$$\left[d\iota \begin{pmatrix} P & Q \\ R & -{}^{\mathrm{t}}P \end{pmatrix}, N(u,v,\theta)\right] = N(Pu - Qv,\ -Ru - {}^{\mathrm{t}}Pv, 0),$$

so that $\mathfrak{heis}_N := \left\{ N(u,v,\theta)\,;\ u,v \in \mathbb{R}^N,\ \theta \in \mathbb{R} \right\}$ forms an ideal of $\mathfrak{g}_N$. Indeed, $\mathfrak{heis}_N$ is isomorphic to the so-called Heisenberg Lie algebra. The Lie group $\mathrm{Heis}_N := \exp \mathfrak{heis}_N \subset Sp(N+1,\mathbb{R})$ consists of the matrices

$$n(u,v,\theta) := \exp N(u,v,\theta) = \begin{pmatrix} I_N & 0 & 0 & u \\ {}^{\mathrm{t}}v & 1 & {}^{\mathrm{t}}u & \theta \\ 0 & 0 & I_N & -v \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (u,v \in \mathbb{R}^N,\ \theta \in \mathbb{R}).$$

The group $G_N := \exp \mathfrak{g}_N \subset Sp(N+1,\mathbb{R})$ equals $\{n(u,v,\theta)\iota(g)\,;\ u,v \in \mathbb{R}^N,\ \theta \in \mathbb{R},\ g \in Sp(N,\mathbb{R})\}$, which is called *the Jacobi group*. It is easy to check that G_N is a semidirect product of Heis_N and $\iota(Sp(N,\mathbb{R}))$.

Let us observe the action of G_N on the Siegel upper half plane $\mathcal{S}_{N+1}$. Note that a complex symmetric matrix $\tilde{\zeta} = \begin{pmatrix} \zeta & w \\ {}^{\mathrm{t}}w & c \end{pmatrix}$ $(\zeta \in \mathrm{Sym}(N,\mathbb{C}),\ w \in \mathbb{C}^N,\ c \in \mathbb{C})$ belongs to $\mathcal{S}_{N+1}$ if and only if $\zeta \in \mathcal{S}_N$ and $\mathrm{Im}\, c - (\mathrm{Im}\, {}^{\mathrm{t}}w)(\mathrm{Im}\, \zeta)^{-1}(\mathrm{Im}\, w) > 0$. In this case, we see that

$$\det \mathrm{Im}\, \tilde{\zeta} = \det \mathrm{Im}\, \zeta \cdot (\mathrm{Im}\, c - (\mathrm{Im}\, {}^{\mathrm{t}}w)(\mathrm{Im}\, \zeta)^{-1}(\mathrm{Im}\, w)). \tag{4}$$

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(N,\mathbb{R})$, we have $\iota(g) \cdot \tilde{\zeta} = \begin{pmatrix} \zeta' & w' \\ {}^{\mathrm{t}}w' & c' \end{pmatrix}$ with

$$\zeta' = (A\zeta + B)(C\zeta + D)^{-1}, \quad w' = {}^{\mathrm{t}}(C\zeta + D)^{-1}w, \quad c' = c - {}^{\mathrm{t}}w(C\zeta + D)^{-1}Cw. \tag{5}$$

Then we see from (1), (4) and (5) that

$$\det \operatorname{Im} \zeta' = |\det(C\zeta + D)|^{-2} \det \operatorname{Im} \zeta, \tag{6}$$

$$\operatorname{Im} c' - (\operatorname{Im} {}^{\mathrm{t}}w')(\operatorname{Im} \zeta')^{-1}(\operatorname{Im} w') = \operatorname{Im} c - (\operatorname{Im} {}^{\mathrm{t}}w)(\operatorname{Im} \zeta)^{-1}(\operatorname{Im} w), \tag{7}$$

and (7) implies

$$\begin{aligned}
&(\operatorname{Im} {}^{\mathrm{t}}w')(\operatorname{Im} \zeta')^{-1}(\operatorname{Im} w') = (\operatorname{Im} {}^{\mathrm{t}}w)(\operatorname{Im} \zeta)^{-1}(\operatorname{Im} w) - \operatorname{Im} c + \operatorname{Im} c' \\
&= (\operatorname{Im} {}^{\mathrm{t}}w)(\operatorname{Im} \zeta)^{-1}(\operatorname{Im} w) + \operatorname{Im} ({}^{\mathrm{t}}w(C\zeta + D)^{-1}Cw) \\
&= (\operatorname{Im} {}^{\mathrm{t}}w)(\operatorname{Im} \zeta)^{-1}(\operatorname{Im} w) - i\, {}^{\mathrm{t}}w(C\zeta + D)^{-1}Cw/2 + i\, \overline{{}^{\mathrm{t}}w(C\zeta + D)^{-1}Cw}/2.
\end{aligned} \tag{8}$$

For $u, v \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$, we have $n(u, v, \theta) \cdot \tilde{\zeta} = \begin{pmatrix} \zeta'' & w'' \\ {}^{\mathrm{t}}w'' & c'' \end{pmatrix}$ with

$$\zeta'' = \zeta, \quad w'' = w + u + \zeta v, \quad c'' = c + \theta + {}^{\mathrm{t}}uv + 2{}^{\mathrm{t}}vw + {}^{\mathrm{t}}w\zeta w. \tag{9}$$

By a direct calculation, we get

$$\operatorname{Im} c'' - (\operatorname{Im} {}^{\mathrm{t}}w'')(\operatorname{Im} \zeta'')^{-1}(\operatorname{Im} w'') = \operatorname{Im} c - (\operatorname{Im} {}^{\mathrm{t}}w)(\operatorname{Im} \zeta)^{-1}(\operatorname{Im} w),$$

which yields that

$$\begin{aligned}
&(\operatorname{Im} {}^{\mathrm{t}}w'')(\operatorname{Im} \zeta'')^{-1}(\operatorname{Im} w'') = (\operatorname{Im} {}^{\mathrm{t}}w)(\operatorname{Im} \zeta)^{-1}(\operatorname{Im} w) - \operatorname{Im} c + \operatorname{Im} c'' \\
&= (\operatorname{Im} {}^{\mathrm{t}}w)(\operatorname{Im} \zeta)^{-1}(\operatorname{Im} w) + \operatorname{Im} (\theta + {}^{\mathrm{t}}uv + 2{}^{\mathrm{t}}vw + {}^{\mathrm{t}}w\zeta w) \\
&= (\operatorname{Im} {}^{\mathrm{t}}w)(\operatorname{Im} \zeta)^{-1}(\operatorname{Im} w) - i({}^{\mathrm{t}}vw + {}^{\mathrm{t}}w\zeta w/2) + i(\overline{{}^{\mathrm{t}}vw + {}^{\mathrm{t}}w\zeta w/2}).
\end{aligned} \tag{10}$$

Let π denote the canonical projection $\mathrm{Sym}(N+1, \mathbb{C}) \ni \tilde{\zeta} \mapsto \tilde{\zeta} + \mathbb{C}E_{N+1,N+1} \in \mathrm{Sym}(N+1, \mathbb{C})/\mathbb{C}E_{N+1,N+1}$. We call the image $\mathcal{D}_N := \pi(\mathcal{S}_{N+1})$ of the Siegel upper half plane $\mathcal{S}_{N+1}$ by the projection π *the Siegel-Jacobi domain*. When $\tilde{\zeta} = \begin{pmatrix} \zeta & w \\ {}^{\mathrm{t}}w & c \end{pmatrix}$, we denote by $\begin{pmatrix} \zeta & w \\ {}^{\mathrm{t}}w & * \end{pmatrix}$ the image $\pi(\tilde{\zeta})$. Then $\mathcal{D}_N = \left\{ \begin{pmatrix} \zeta & w \\ {}^{\mathrm{t}}w & * \end{pmatrix} ; \ \zeta \in \mathcal{S}_N, \ w \in \mathbb{C}^N \right\}$, which means that $\mathcal{D}_N$ is biholomprphic to the product of the Siegel upper half plane $\mathcal{S}_N$ and the complex vector space $\mathbb{C}^N$. By (5) and (9), we see that the action of G_N on $\mathcal{S}_{N+1}$ induces naturally an action on $\mathcal{D}_N = \pi(\mathcal{S}_{N+1})$ by

$$\iota(g) \cdot \begin{pmatrix} \zeta & w \\ {}^{\mathrm{t}}w & * \end{pmatrix} := \begin{pmatrix} \zeta' & w' \\ {}^{\mathrm{t}}w' & * \end{pmatrix}, \quad n(u, v, \theta) \cdot \begin{pmatrix} \zeta & w \\ {}^{\mathrm{t}}w & * \end{pmatrix} := \begin{pmatrix} \zeta'' & w'' \\ {}^{\mathrm{t}}w'' & * \end{pmatrix}.$$

In other words, $\pi : \mathcal{S}_{N+1} \to \mathcal{D}_N$ is equivariant under the actions of the Jacobi group. Moreover, the action of G_N is transitive on $\mathcal{D}_N$, while it is not effective. Indeed, the action of the center $\{n(0, 0, \theta) ; \theta \in \mathbb{R}\}$ of G_N is trivial. Let $d\tilde{s}^2_{\det}$ be the Kähler metric whose potential function $\tilde{\phi}$ on $\mathcal{D}_N$ is defined by $\tilde{\phi}(\tilde{\zeta}) := -\log \det \operatorname{Im} \zeta +$

$(\mathrm{Im}\,{}^{\mathrm{t}}w)(\mathrm{Im}\,\zeta)^{-1}(\mathrm{Im}\,w)$ for $\tilde{\zeta} = \begin{pmatrix} \zeta & w \\ {}^{\mathrm{t}}w & * \end{pmatrix} \in \mathcal{D}_N$. In fact, for a tangent vector $\tilde{V} = \pi(V) \in T_{\tilde{p}_N}\mathcal{D}_N$, $V = (V_{ij}) \in \mathrm{Sym}(N+1, \mathbb{R})$ at $\tilde{p}_N := \pi(p_{N+1}) \in \mathcal{D}_N$, we have $d\tilde{s}^2_{\det}(\tilde{V}, \tilde{V})_{\tilde{p}_N} = (\sum_{k=1}^N |V_{kk}|^2 + 2\sum_{1\le k<l\le N+1} |V_{lk}|^2)/4$, while the tensor $d\tilde{s}^2_{\det}$ is G_N-invariant thanks to (6), (8) and (10). Therefore the positivity of $d\tilde{s}^2_{\det}$ is verified, and we conclude that $d\tilde{s}^2_{\det}$ is a homogeneous Kähler metric on $\mathcal{D}_N$. This metric $ds^2_{\det}$ equals the one given in [8, Theorem 1.1] with $A = B = 1/4$.

For $\underline{\alpha} \in \mathbb{R}^N_{>0}$, let $d\tilde{s}^2_{\underline{\alpha}}$ be the Kähler metric whose potential function $\tilde{\phi}_{\underline{\alpha}}$ on $\mathcal{D}_N$ is defined by $\tilde{\phi}_{\underline{\alpha}}(\tilde{\zeta}) := \log \Phi_{-\underline{\alpha}}(\mathrm{Im}\,\zeta) + (\mathrm{Im}\,{}^{\mathrm{t}}w)(\mathrm{Im}\,\zeta)^{-1}(\mathrm{Im}\,w)$ for $\tilde{\zeta} = \begin{pmatrix} \zeta & w \\ {}^{\mathrm{t}}w & * \end{pmatrix} \in \mathcal{D}_N$. Indeed, for a tangent vector $\tilde{V} = \pi(V) \in T_{\tilde{p}_N}\mathcal{D}_N$, $V = (V_{ij}) \in \mathrm{Sym}(N+1, \mathbb{R})$ at $\tilde{p}_N$ we have $d\tilde{s}^2_{\underline{\alpha}}(\tilde{V}, \tilde{V})_{\tilde{p}_N} = \sum_{k=1}^N \alpha_k(|V_{kk}|^2 + 2\sum_{j<k} |V_{kj}|^2)/4 + \sum_{j=1}^N |V_{N+1,j}|^2/2$. Let $\tilde{B}_N$ be the subgroup $\{n(u, v, \theta)\iota(b);\, u, v \in \mathbb{R}^N, \theta \in \mathbb{R}, b \in B_N\}$ of G_N. In other words, $\tilde{B}_N$ is a semidirect product $\mathrm{Heis}_N \rtimes \iota(B_N)$. Owing to the relative invariance of $\Phi_{-\alpha}$ under the transitive action of B_N on $\mathcal{S}_N$, the tensor $d\tilde{s}^2_{\underline{\alpha}}$ on the Siegel-Jacobi domain $\mathcal{D}_N$ is invariant under the transitive action of $\tilde{B}_N$ on $\mathcal{D}_N$. Therefore the positivity of $d\tilde{s}^2_{\underline{\alpha}}$ is checked, and $d\tilde{s}^2_{\underline{\alpha}}$ is a homogeneous Kähler metric on $\mathcal{D}_N$.

5 Realization of Simply Connected Complex Domains with Homogeneous Kähler Metrics

In this section, just as in Sect. 3, we show that every simply connected complex domain M with a homogeneous Kähler metrit is realized as an orbit of a subgroup of $\tilde{B}_N$ in the Siegel-Jacobi domain $\mathcal{D}_N$, while the metric is obtained as a restriction of $d\tilde{s}^2_{\underline{\alpha}}$.

First, we observe that the Heis_N-orbit through $\tilde{p}_N$ is exactly $\left\{ \begin{pmatrix} iI_N & w \\ {}^{\mathrm{t}}w & * \end{pmatrix};\ w \in \mathbb{C}^N \right\}$, which is naturaly isomorphic to $\mathbb{C}^N$. For a tangent vector $\tilde{V} = \begin{pmatrix} 0 & v \\ {}^{\mathrm{t}}v & * \end{pmatrix}$ $(v \in \mathbb{C}^N)$ at any point p in the orbit, we have $ds^2_{\underline{\alpha}}(V, V)_p = \sum_{k=1}^N |v_k|^2/2$, which is independent of $\underline{\alpha} \in \mathbb{R}^N_{>0}$. In this way, the flat Kähler manifold $\mathbb{C}^N$ is realized in $\mathcal{D}_N$.

Recall the family of vector spaces $\mathcal{V}_{lk} \subset \mathrm{Mat}(n_l, n_k, \mathbb{R})$ $(1 \le k < l \le r)$ and $\mathcal{W}_k \subset \mathrm{Mat}(n_k, n_0, \mathbb{C})$ $(k = 1, \dots, r)$ in Sect. 3. Let $\mathcal{L}_k \subset \mathbb{R}^{n_k}$ $(k = 1, \dots, r)$ and $\mathcal{L}_0 \subset \mathbb{C}^{n_0}$ be respectively real and complex vector spaces satisfying
(L1) $A \in \mathcal{V}_{lk}, x \in \mathcal{L}_k \Rightarrow Ax \in \mathcal{L}_l$ $(1 \le k < l \le r)$,
(L2) $A \in \mathcal{V}_{lk}, x \in \mathcal{L}_l \Rightarrow {}^{\mathrm{t}}Ax \in \mathcal{L}_k$ $(1 \le k < l \le r)$,
(L3) $C \in \mathcal{W}_k, x \in \mathcal{L}_k \Rightarrow {}^{\mathrm{t}}Cx \in \mathcal{L}_0$ $(k = 1, \dots, r)$,
(L4) $C \in \mathcal{W}_k, z \in \mathcal{L}_0 \Rightarrow \overline{C}z \in (\mathcal{L}_k)_{\mathbb{C}}$ $(k = 1, \dots, r)$,
and define

$$\mathcal{L} := \left\{ \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_r \end{pmatrix} \in \mathbb{C}^N \,;\; w_0 \in \mathcal{L}_0,\; w_k \in (\mathcal{L}_k)_{\mathbb{C}} \;\, (k = 1, \ldots, r) \right\}.$$

Let $\tilde{B}_{\mathcal{V},\mathcal{W},\mathcal{L}}$ be the subgroup of $\tilde{B}_N$ given by

$$\tilde{B}_{\mathcal{V},\mathcal{W},\mathcal{L}} := \left\{ n(\operatorname{Re} w,\; \operatorname{Im} w, \theta)\iota(b) \,;\; w \in \mathcal{L},\; \theta \in \mathbb{R},\; b \in B_{\mathcal{V},\mathcal{W}} \right\},$$

and put $\tilde{O}_{\mathcal{V},\mathcal{W},\mathcal{L}} := \tilde{B}_{\mathcal{V},\mathcal{W},\mathcal{L}} \cdot \tilde{p}_N \subset \mathcal{D}_N$.

Theorem 2 *Let $M \subsetneqq \mathbb{C}^n$ be a simply connected complex domain, and ds^2 be a homogeneous Kähler metric on M. Then there exists a Kähler isomorphism $F : (M, ds^2) \to (\tilde{O}_{\mathcal{V},\mathcal{W},\mathcal{L}},\; d\tilde{s}^2_{\underline{\alpha}}|_{\tilde{O}_{\mathcal{V},\mathcal{W},\mathcal{L}}})$ with appropriate $\mathcal{V}, \mathcal{W}, \mathcal{L}$ and $\underline{\alpha} \in \mathbb{R}^N_{>0}$.*

In order to describe ideas for the proof of Theorem 2, we introduce some notations. Setting $n_{r+1} := 1$, $\mathcal{V}_{r+1,k} := \left\{ {}^{\mathrm{t}}x \,;\; x \in \mathcal{L}_k \right\} \subset \mathrm{Mat}(n_{r+1}, n_k, \mathbb{R})$ $(k = 1, \ldots, r)$ and $\mathcal{W}_{r+1} := \left\{ {}^{\mathrm{t}}z \,;\; z \in \mathcal{L}_0 \right\} \subset \mathrm{Mat}(n_{r+1}, n_0, \mathbb{C})$, we see that the conditions (V1)–(V3) and (W1)–(W3) in Sect. 3 are satisfied thanks to (L1)–(L4), where r and N are respectively replaced by $r + 1$ and $N + 1$ in an obvious way. Putting

$$\tilde{\mathcal{V}} := \left\{ \begin{pmatrix} X_{11} & & {}^{\mathrm{t}}X_{r1} & {}^{\mathrm{t}}X_{r+1,1} \\ \vdots & \ddots & & \vdots \\ X_{r1} & \cdots & X_{rr} & {}^{\mathrm{t}}X_{r+1,r} \\ X_{r+1,1} & \cdots & X_{r+1,r} & X_{r+1,r+1} \end{pmatrix} ; \begin{array}{l} X_{kk} \in \mathbb{R}I_{n_k} \;\, (k = 1, \ldots, r+1) \\ X_{lk} \in \mathcal{V}_{lk} \;\, (1 \le k < l \le r+1) \end{array} \right\}$$

and

$$\tilde{\mathcal{W}} := \left\{ \begin{pmatrix} U_1 \\ \vdots \\ U_r \\ U_{r+1} \end{pmatrix} ;\; U_k \in \mathcal{W}_k \;\, (k = 1, \ldots, r+1) \right\},$$

we obtain a normal j-algebra $\mathfrak{b}_{\tilde{\mathcal{V}},\tilde{\mathcal{W}}} \subset \mathfrak{b}_{N+1}$. Then $\tilde{\mathfrak{b}}_{\mathcal{V},\mathcal{W},\mathcal{L}}$ is a codimension-one subalgebra of $\mathfrak{b}_{\tilde{\mathcal{V}},\tilde{\mathcal{W}}}$. In view of the equivariance of $\pi : \mathcal{S}_{N+1} \to \mathcal{D}_N$, we see that the $B_{\tilde{\mathcal{V}},\tilde{\mathcal{W}}}$-orbit $O_{\tilde{\mathcal{V}},\tilde{\mathcal{W}}} \subset \mathcal{S}_{N+1}$ is mapped onto $\tilde{O}_{\mathcal{V},\mathcal{W},\mathcal{L}} \subset \mathcal{D}_N$.

Theorem 2 is proved in the following way. Take an Iwasawa subgroup B_M of the Kähler automorphism group $\mathrm{Aut}(M, ds^2)$. Then B_M acts simply transitively on M. We can show that a central extension $\tilde{B}$ of B_M is obtained as a codimension-one subgroup of $B = \exp \mathfrak{b}$, where $\mathfrak{b}$ is a normal j-algebra. Then $\mathfrak{b}$ and $\tilde{\mathfrak{b}} = \mathrm{Lie}(\tilde{B})$ are realized as $\mathfrak{b}_{\tilde{\mathcal{V}},\tilde{\mathcal{W}}}$ and $\tilde{\mathfrak{b}}_{\mathcal{V},\mathcal{W},\mathcal{L}}$ respectively, so that M is realized as $\tilde{O}_{\mathcal{V},\mathcal{W},\mathcal{L}}$.

We shall conclude this paper by showing that the simply connected complex domain $M \simeq \tilde{O}_{\mathcal{V},\mathcal{W},\mathcal{L}}$ is biholomorphic to the product of a Siegel domain and a complex vector space. Recall the Siegel domain $\mathcal{S}(\Omega_{\mathcal{V}}, Q_{\mathcal{W}})$ and the biholomorphic map

(3). For $(Z, U) \in \mathcal{S}(\Omega_{\mathcal{V}}, Q_{\mathcal{W}}) \subset \mathcal{V}_{\mathbb{C}} \times \mathcal{W}$, $w = \begin{pmatrix} w_0 \\ w' \end{pmatrix} \in \mathcal{L}$ with $w' = \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix}$,

and $c \in \mathbb{C}$, we put

$$\tilde{Z} := \begin{pmatrix} Z & w' \\ {}^{\mathrm{t}}w' & c \end{pmatrix} \in \tilde{\mathcal{V}}_{\mathbb{C}}, \quad \tilde{U} := \begin{pmatrix} U \\ {}^{\mathrm{t}}w_0 \end{pmatrix} \in \tilde{\mathcal{W}}.$$

We note that, if $\operatorname{Im} c$ is sufficiently large, then $(\tilde{Z}, \tilde{U})$ belongs to the Siegel domain $\mathcal{S}(\Omega_{\tilde{\mathcal{V}}}, Q_{\tilde{\mathcal{W}}})$, that is to say,

$$\operatorname{Im} \tilde{Z} - (\tilde{U}\,{}^{\mathrm{t}}\overline{\tilde{U}} + \overline{\tilde{U}}\,{}^{\mathrm{t}}\tilde{U})/4 > 0$$

because the $N \times N$ upper-left principal submatrix of the left-hand side is $\operatorname{Im} Z - (U\,{}^{\mathrm{t}}\overline{U} + \overline{U}\,{}^{\mathrm{t}}U)/4$, which is positive definite. On the other hand, the projection $\pi \begin{pmatrix} i I_{n_0} & {}^{\mathrm{t}}\tilde{U} \\ \tilde{U} & \tilde{Z} - \frac{i}{2}\tilde{U}\,{}^{\mathrm{t}}\tilde{U} \end{pmatrix}$ is independent of such c. Thus we have a holomorphic map

$$\mathcal{S}(\Omega_{\mathcal{V}}, Q_{\mathcal{W}}) \times \mathcal{L} \ni (Z, U, w) \mapsto \pi \begin{pmatrix} i I_{n_0} & {}^{\mathrm{t}}\tilde{U} \\ \tilde{U} & \tilde{Z} - \frac{i}{2}\tilde{U}\,{}^{\mathrm{t}}\tilde{U} \end{pmatrix} \in \tilde{\mathcal{O}}_{\mathcal{V},\mathcal{W},\mathcal{L}} = \pi(\mathcal{O}_{\tilde{\mathcal{V}},\tilde{\mathcal{W}}}), \tag{11}$$

which is easily seen to be biholomorphic.

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The Asymptotic Behavior of the Bergman Kernel on Pseudoconvex Model Domains

Joe Kamimoto

Abstract In this paper, we investigate the asymptotic behavior of the Bergman kernel at the boundary for some pseudoconvex model domains. This behavior can be described by the geometrical information of the Newton polyhedron of the defining function of the respective domains. We deal with not only the finite type cases but also some infinite type cases.

Keywords Bergman kernel · Newton data · Pseudoconvex model domain

1 Introduction

Let Ω be a domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ be the Hilbert space of the L^2-holomorphic functions on Ω. The *Bergman kernel* $B_\Omega(z)$ of Ω (on the diagonal) is defined by $B_\Omega(z) = \sum_\alpha |\phi_\alpha(z)|^2$, where $\{\phi_\alpha\}$ is a complete orthonormal basis of $A^2(\Omega)$. Throughout this paper, we assume that the boundary $\partial\Omega$ of Ω is always C^∞-smooth.

Since the behavior of the Bergman kernel at the boundary plays essentially important roles in the study of several complex variables and complex geometry, many interesting results about its behavior have been obtained.

In the case of strictly pseudoconvex domains, a beautiful asymptotic expansion of the Bergman kernel was given by Fefferman [11] and Boutet de Monvel and Sjöstrand [4]. In the case of weakly pseudoconvex domains, many kinds of important results have been obtained (see the references in [5, 15], etc.). In particular, in the two-dimensional finite type case, an asymptotic expansion analogous to that of Fefferman was recently given by Hsiao and Savale [14]. On the other hand, in the higher dimensional case, there does not seem to be such strong and general results. In [15], the author investigated a special case of pseudoconvex model domains and computed some asymptotic expansion of the Bergman kernel. The purpose of this paper is to generalize the results in [15].

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In [15], only the finite type case was dealt with, while more general cases will be considered in this paper; for example, some infinite type cases can be also dealt with. Some two-dimensional infinite type cases have been precisely investigated in [2, 3]. We will consider higher dimensional cases, which are more complicated.

From the results in [5–8, 15], it might be recognized that the information of the boundary from the viewpoint of singularity theory is valuable for the exact analysis of the Bergman kernel in the higher dimensional weakly pseudoconvex case. In particular, the *Newton polyhedron* determined from the boundary contains fruitful information for the singularity of the Bergman kernel at the boundary.

One of the difficulty of the analysis in the infinite type case is caused by the existence of non-zero *flat functions* (see Sect. 2.1). Notice that flat functions do not affect the geometry of the Newton polyhedron and the influence of flat functions is subtle in the singularity of the Bergman kernel. However, this practical influence cannot always be negligible. In the main theorem (Theorem 2), we will give a certain condition on the geometry of the Newton polyhedron to determine the case where the above influence of flat functions can be negligible in some sense.

This paper is organized as follows. In Sect. 2, we state a main theorem and explain its significance. In Sect. 3, we exhibit an integral formula of the Bergman kernel given by Haslinger [12, 13], on which our analysis is based. In Sect. 4, we show that the singularity of the Bergman kernel can be completely determined by the local geometry of the boundary. In our analysis, it is necessary to consider various kinds of C^∞ functions, but the C^∞ class contains many troublesome functions. In [18, 19], a certain class of C^∞ functions, called the $\hat{\mathcal{E}}$-class, is introduced by the use of Newton polyhedra, which is easy to deal with. Moreover, we also introduce a class analogous to the $\hat{\mathcal{E}}$-class in the complex variables case in Sect. 5. In Sect. 6, we investigate the singularity of the Bergman kernel in the $\hat{\mathcal{E}}$-case. In Sect. 7, a main theorem can be shown by the use of the results in the $\hat{\mathcal{E}}$-case. By the way, the behavior of some Laplace type integrals is a key in the analysis in Sect. 6. The work of Varchenko in [23] (see also [1]) concerning local zeta functions and oscillatory integrals plays crucial roles in the investigation of the above behavior. In Sect. 8, we explain some results in [9, 18, 19], which generalize the above Varchenko's results, and apply these results to the analysis of the Bergman kernel. In the last section, we will explain some important words and concepts.

Notation and symbols

- We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ the set consisting of all natural numbers, integers, real numbers, complex numbers, respectively. Moreover, we denote by $\mathbb{Z}_+$, $\mathbb{R}_+$ the set consisting of all nonnegative integers, real numbers, respectively. For $s \in \mathbb{C}$, $\mathrm{Im}(s)$ expresses the imaginary part of s.
- Let $\alpha := (\alpha_1, \ldots, \alpha_n)$, $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$. The multi-indices will be used as follows. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \|x\|_{\mathbb{R}} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

For $z = (z_1, \ldots, z_n)$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in \mathbb{C}^n$, define

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \ \bar{z}^\beta := \bar{z}_1^{\beta_1} \ldots \bar{z}_n^{\beta_n}, \ |z|^{2\alpha} := |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n},$$
$$\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

- For a positive number R, we denote

$$D_{\mathbb{R}}(R) = \{x \in \mathbb{R}^n : \|x\|_{\mathbb{R}} < R\}, \quad D(R) = \{z \in \mathbb{C}^n : \|z\| < R\}.$$

2 Main Results

2.1 Newton Data

In this paper, many concepts in *convex geometry* play useful roles. We will explain the exact meanings of necessary words in convex geometry in Sect. 9.2 (see also [24]).

Let F be a real-valued C^∞ function defined near the origin in $\mathbb{C}^n$. Let

$$\sum_{\alpha,\beta \in \mathbb{Z}_+^n} C_{\alpha\beta} z^\alpha \bar{z}^\beta = \sum_{\alpha,\beta \in \mathbb{Z}_+^n} C_{\alpha\beta} z_1^{\alpha_1} \cdots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \cdots \bar{z}_n^{\beta_n}$$

be the Taylor series of F at the origin. The *support of* F is the set $S_F = \{\alpha + \beta \in \mathbb{Z}_+^n : C_{\alpha\beta} \neq 0\}$ and the *Newton polyhedron of* F is the integral polyhedron

$$\mathcal{N}_+(F) = \text{ the convex hull of the set } \bigcup \{\alpha + \beta + \mathbb{R}_+^n : \alpha + \beta \in S_F\} \text{ in } \mathbb{R}_+^n.$$

We say that F is *flat* if $\mathcal{N}_+(F) = \emptyset$ and that F is *convenient* if $\mathcal{N}_+(F)$ intersects all the axes. For a compact face γ of $\mathcal{N}_+(F)$, the γ-part of F is defined by

$$F_\gamma(z) = \sum_{\alpha+\beta \in \gamma \cap \mathbb{Z}_+^n} C_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

We define a quantity ρ_F ($\in \mathbb{Z}_+$) as follows. When F is convenient, let

$$\rho_F := \max\{\rho_j(F) : j = 1, \ldots, n\},$$

where

$$\rho_j(F) := \min\{t \geq 0 : (0, \ldots, \overset{(j)}{t}, \ldots, 0) \in \mathcal{N}_+(F)\}.$$

When F is not convenient, let $\rho_F := \infty$.

Hereafter, we assume that F is not flat. The *Newton distance of F* is the nonnegative number

$$d_F := \min\{t \geq 0 : (t, \ldots, t) \in \mathcal{N}_+(F)\}.$$

Since F is not flat, there exists the minimum proper face of the Newton polyhedron $\mathcal{N}_+(F)$ containing the point $P_F := (d_F, \ldots, d_F)$, which is called the *principal face* of $\mathcal{N}_+(F)$ and is denoted by γ_*. The codimension of γ_* is called the *Newton multiplicity* of F, which is denoted by m_F (i.e., $m_F = n - \dim(\gamma_*)$). In particular, when P_F is a vertex of $\mathcal{N}_+(F)$, γ_* is the point P_F and $m_F = n$. When γ_* is compact, the *principal part* of F is defined by

$$F_*(z) = \sum_{\alpha,\beta \in \gamma_* \cap \mathbb{Z}_+^n} C_{\alpha\beta} z^\alpha \bar{z}^\beta$$

(i.e., the principal part of F is the γ_*-part of F).

2.2 Main Results

Let U be a complete pseudoconvex Reinhardt domain in $\mathbb{C}^n$ (possibly $U = \mathbb{C}^n$). Let F be a real-valued C^∞ function on U satisfying the following conditions

(A) $F(z) = 0$ if and only if $z = 0$ and F is not flat at the origin;
(B) F is a plurisubharmonic function on U;
(C) $F(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) = F(z_1, \ldots, z_n)$ for any $\theta_j \in \mathbb{R}$ and $z \in U$;
(D) If U is unbounded, then there are some positive numbers c, β, L such that $F(z) \geq c\|z\|^\beta$ for $z \in U \setminus D(L)$.

We will mainly deal with pseudoconvex model domains in $\mathbb{C}^{n+1}$ of the form

$$\Omega_F = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C} \times U : \mathrm{Im}(z_0) > F(z_1, \ldots, z_n)\}.$$

Note that the condition (D) implies that the dimension of $A^2(\Omega_F)$ is infinity.

In the case of the domain Ω_F, the finite type condition can be easily seen in the information of the Newton polyhedron of F. Let $\Delta_1(\partial\Omega_F, 0)$ be the D'Angelo type of $\partial\Omega_F$ at the origin.

Lemma 1 ([16]) $\Delta_1(\partial\Omega_F, 0) = \rho_F$. *In particular, the following two conditions are equivalent.*

1. *$\partial\Omega_F$ is of finite type at* 0 *(i.e., $\Delta_1(\partial\Omega_F, 0) < \infty$);*
2. *F is convenient (i.e., $\mathcal{N}_+(F)$ intersects all axes).*

Let $B_{\Omega_F}(z_0, z) = B_{\Omega_F}(z_0, z_1, \ldots, z_n)$ be the Bergman kernel (on the diagonal) of Ω_F. Since we are interested in the behavior of the restriction of the Bergman kernel of Ω_F to the vertical line, we define

$$\mathcal{B}_F(\rho) := B_{\Omega_F}(0 + i\rho, 0, \ldots, 0) \quad \text{for } \rho > 0.$$

The behavior of $\mathcal{B}_F(\rho)$, as ρ tends to 0, can be exactly expressed by the use of the geometry of the Newton polyhedron $\mathcal{N}_+(F)$ of F.

Theorem 1 ([15]) *If $\partial\Omega_F$ is of finite type at 0, then*

$$\mathcal{B}_F(\rho) = \frac{\Psi(\rho)}{\rho^{2/d_F+2}(\log\rho)^{m_F-1}}, \tag{1}$$

where $\Psi(\rho)$ admits the asymptotic expansion:

$$\Psi(\rho) \sim \sum_{j=0}^{\infty}\sum_{k=a_j}^{\infty} C_{jk}\frac{\rho^{j/m}}{(\log(1/\rho))^k} \quad \text{as } \rho \to 0, \tag{2}$$

where m is a positive integer, a_j are integers and C_{jk} are real numbers (the exact meaning of the above asymptotic expansion is explained in Sect. 9.1, below).

Furthermore, the first coefficient of the above expansion can be determined by the use of the Newton data of F as follows:

$$\mathcal{B}_F(\rho) = \frac{C(F_*)}{\rho^{2/d_F+2}(\log\rho)^{m_F-1}} \cdot (1 + o(\rho^{\varepsilon})) \quad \text{as } \rho \to 0, \tag{3}$$

where ε is a positive constant and $C(F_*)$ is a positive constant depending only on the principal part F_* of F.

In this paper, we improve the above theorem in some sense as follows.

Theorem 2 *Suppose that there exists a C^∞ function F_0 defined near the origin in $\mathbb{C}^n$ such that F_0 satisfies the condition (C) and the $\hat{\mathcal{E}}$-condition at the origin (see Sect. 5) and $F(z) - F_0(z)$ is a nonnegative flat function. If the principal face γ_* of the Newton polyhedron of F is compact, then*

$$\mathcal{B}_F(\rho) = \frac{C(F_*)}{\rho^{2/d_F+2}(\log\rho)^{m_F-1}} \cdot (1 + o(\rho^{\varepsilon})) \quad \text{as } \rho \to 0, \tag{4}$$

where ε is a positive constant and $C(F_*)$ is a positive constant depending only on the principal part F_* of F.

Remark 1 (1) Note that the finite type condition is equivalent to the convenience condition of F from Lemma 1. When F is convenient, F itself satisfies the $\hat{\mathcal{E}}$-condition (see [18, 19]) and the principal face of $\mathcal{N}_+(F)$ is always compact. Therefore, the assumption of Theorem 2 is weaker than that of Theorem 1. Indeed, Theorem 2 can be applied to some infinite type cases.

(2) In the two-dimensional case (i.e. $\Omega_F \subset \mathbb{C}^2$), Lemma 1 implies that the finite type condition is equivalent to the nonflatness of F. Since we assume the nonflatness

of F in (A), the advantage of Theorem 2 can be only seen in the case where the dimension is higher than two.

(3) Theorem 2 can be applied to the following examples, which are in the infinite type case.

- When $F(z_1, z_2) = |z_1|^6 + |z_1|^2|z_2|^4 + e^{-1/|z_2|^2}$ near the origin,

$$\mathcal{B}_F(\rho) = \frac{C(F_*)}{\rho^{8/3}} \cdot (1 + o(\rho^\varepsilon)) \quad \text{as } \rho \to 0.$$

(Note that $F_*(z_1, z_2) = |z_1|^6 + |z_1|^2|z_2|^4$.)

- When $F(z_1, z_2) = |z_1|^6 + |z_1|^2|z_2|^2 + e^{-1/|z_2|^2}$ near the origin,

$$\mathcal{B}_F(\rho) = \frac{C(F_*)}{\rho^3 \log \rho} \cdot (1 + o(\rho^\varepsilon)) \quad \text{as } \rho \to 0.$$

(Note that $F_*(z_1, z_2) = |z_1|^2|z_2|^2$.)

- When $F(z_1, z_2) = |z_1|^2|z_2|^2 + e^{-1/|z_1|^2} + e^{-1/|z_2|^2}$ near the origin,

$$\mathcal{B}_F(\rho) = \frac{C(F_*)}{\rho^3 \log \rho} \cdot (1 + o(\rho^\varepsilon)) \quad \text{as } \rho \to 0.$$

(Note that $F_*(z_1, z_2) = |z_1|^2|z_2|^2$.)

- When $F(z_1, z_2, z_3) = |z_1|^8 + |z_2|^8 + |z_1|^2|z_2|^2|z_3|^2 + e^{-1/|z_3|^2}$ near the origin,

$$\mathcal{B}_F(\rho) = \frac{C(F_*)}{\rho^3 (\log \rho)^2} \cdot (1 + o(\rho^\varepsilon)) \quad \text{as } \rho \to 0.$$

(Note that $F_*(z_1, z_2) = |z_1|^2|z_2|^2|z_3|^2$.)

We remark that $e^{-1/|z_j|^2}$ $(j = 1, 2, 3)$ in the above examples can be replaced by any flat functions which are positive away from the origin.

(4) Our method in the proof of Theorem 2 cannot generally show the existence of an asymptotic expansion of the form (2). We guess that the pattern of the asymptotic expansion might not be expressed as in the form (2) in general from some observation of the strange phenomena in [20–22], which are seen in the analytic continuation of local zeta functions.

(5) In the case where the principal face is noncompact, there exist many examples in which the behavior of the Bergman kernel $\mathcal{B}_F(\rho)$ as $\rho \to 0$ is different from (4). For example, in the case where $F(z_1, z_2) = |z_1|^2 + e^{-1/|z_2|^p}$ near the origin, $\mathcal{B}_F(\rho)$ locally satisfies

$$\frac{c_1 |\log \rho|^{1/p}}{\rho^3} \le \mathcal{B}_F(\rho) \le \frac{c_2 |\log \rho|^{1/p}}{\rho^3}$$

near $\rho = 0$, where c_1, c_2 are positive constants ([17]). Notice that the logarithmic functions appear in the numerators in the above estimates. Observing the above estimates, we can see that the behavior of $\mathcal{B}_F(\rho)$ depends on p, which is an information of the flat term $e^{-1/|z_2|^p}$. In the noncompact principal face case, the information of the Newton polyhedron of F cannot always determine the singularity of the Bergman kernel completely.

(6) Of course, the constant ε in (3), (4) does not depend on ρ. The equations of similar type to (3), (4) will be often seen in this paper. In these equations, the constant ε can be chosen by the use of the geometry of the respective Newton polyhedron.

Since $\rho^\varepsilon = o((\log(1/\rho)^{-1})$ holds for any $\varepsilon > 0$, $o(\rho^\varepsilon)$ in the equations in (3), (4) and in the examples in Remark 1 (3) can be replaced by $o((\log(1/\rho)^{-1})$.

(7) It is desirable to show that the conditions (A–C) of F imply the existence of F_0 in the assumption of the theorem (in other words, the existence of F_0 might be removed in the assumption). Even if the existence of F_0 is not known, we can give the following estimate:

$$\mathcal{B}_F(\rho) \leq \frac{C(F_*)}{\rho^{2/d_F+2}(\log \rho)^{m_F-1}} \cdot (1 + C\rho^\varepsilon) \quad \text{for } \rho \in (0, \delta),$$

where $C(F_*)$ is as in the theorem and C, ε, δ are positive constants. This can be easily seen from the proof of Theorem 2.

3 Integral Formula of the Bergman Kernel

Let U be a domain in $\mathbb{C}^n$ and let $F : U \to \mathbb{R}_+$ be a C^∞-smooth plurisubharmonic function. The weighted Hilbert space $H_\tau(U)$ ($\tau > 0$) consists of all entire functions $\psi : U \to \mathbb{C}$ such that

$$\int_U |\psi(z)|^2 e^{-2\tau F(z)} dV(z) < \infty,$$

where $dV(z)$ denotes the Lebesgue measure on $\mathbb{C}^n$. We only consider the case where $H_\tau(U)$ is a nontrivial Hilbert space with the reproducing kernel. (When U is bounded, this is obvious. When U is unbounded, the condition (D) in Sect. 2.2 can imply the above nontriviality.) We denote the above kernel (on the diagonal) by $K_F(z;\tau)$. We remark that the function $\tau \mapsto K_F(z;\tau)$ is continuous for fixed $z \in U$ from the result in [10]. Haslinger [12, 13] shows that the Bergman kernel $B_{\Omega_F}(z_0, z)$ of the domain Ω_F can be expressed by the use of $K_F(z;\tau)$ as follows.

$$B_{\Omega_F}(z_0, z) = \frac{1}{2\pi} \int_0^\infty e^{-2\rho\tau} K_F(z;\tau)\tau d\tau, \tag{5}$$

where ρ is the imaginary part of z_0.

In the case where F satisfies the conditions (C), (D) in Sect. 2.2, we can take a complete orthonormal system for $H_\tau(U)$ as

$$\left\{ \frac{z^\alpha}{c_\alpha(\tau)} : \alpha \in \mathbb{Z}_+^n \right\}, \quad \text{with } c_\alpha(\tau)^2 = \int_U |z|^{2\alpha} e^{-2\tau F(z)} dV(z).$$

Therefore, $K_F(z;\tau)$ can be expressed as

$$K_F(z;\tau) = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{|z|^{2\alpha}}{c_\alpha(\tau)^2}. \tag{6}$$

From (5), (6), $\mathcal{B}_F(\rho)$ can be expressed as

$$\mathcal{B}_F(\rho) = \frac{1}{2\pi} \int_0^\infty e^{-2\rho\tau} \frac{\tau}{c_0(\tau)^2} d\tau. \tag{7}$$

In order to see the behavior of $\mathcal{B}_F(\rho)$ as $\rho \to 0$, it suffices to investigate that of $c_0(\tau)^2$ as $\tau \to \infty$.

4 Localization Lemma

Let U be an open neighborhood of the origin in $\mathbb{C}^n$ and let $F : U \to \mathbb{R}$ be a nonnegative C^∞ function with $F(0) = 0$.

Let R be a positive number such that $D(R) \subset U$. Let $\varphi : \mathbb{R}^n \to [0,1]$ be a C^∞ function such that φ identically equals 1 on $D_{\mathbb{R}}(R/2)$ and its support is contained in $D_{\mathbb{R}}(R)$. Let $\Phi : \mathbb{C}^n \to [0,1]$ be a C^∞ function defined by $\Phi(z_1,\ldots,z_n) = \varphi(|z_1|,\ldots,|z_n|)$.

We define an integral of the form

$$\tilde{\mathcal{B}}_F(\rho) = \frac{1}{2\pi} \int_0^\infty e^{-2\rho\tau} \frac{\tau}{\tilde{c}_0(\tau)^2} d\tau \quad \text{for } \rho > 0, \tag{8}$$

where

$$\tilde{c}_0(\tau)^2 = \int_U e^{-2\tau F(z)} \Phi(z) dV(z). \tag{9}$$

Since there exists a positive number a such that $F(z) \le a\|z\|$ on $D(R)$, we see

$$\begin{aligned} \tilde{c}_0(\tau)^2 &\ge \int_{D(R/2)} e^{-2\tau F(z)} dV(z) \\ &\ge c \int_0^{R/2} e^{-2a\tau x} x^{2n-1} dx \ge \frac{c}{(2a\tau)^{2n}} \int_0^1 e^{-s} s^{2n-1} ds = \frac{C}{\tau^{2n}}, \end{aligned} \tag{10}$$

for $\tau \ge 1$, where c, C are positive constants independent of τ.

By the use of the estimate (10), the integral $\tilde{\mathcal{B}}_F(\rho)$ in (8) can be considered as a C^∞ function defined on $(0,\infty)$, which is samely denoted by $\tilde{\mathcal{B}}_F(\rho)$.

Since $\tilde{c}_0(\tau)^2 \le c_0(\tau)^2$, the relationship between $\tilde{\mathcal{B}}_F(\rho)$ and the Bergman kernel $\mathcal{B}_F(\rho)$ can be seen: $\tilde{\mathcal{B}}_F(\rho) \ge \mathcal{B}_F(\rho)$. The following proposition shows that the singularities of $\mathcal{B}_F(\rho)$ and $\tilde{\mathcal{B}}_F(\rho)$ at the origin are essentially the same.

Proposition 1 *If F satisfies the conditions (A–D) in Sect. 2.2, then $\tilde{\mathcal{B}}_F(\rho) - \mathcal{B}_F(\rho)$ can be real-analytically extended to an open neighborhood of $\rho = 0$.*

Proof From the integral expressions in (7) and (8), we have

$$\mathcal{B}_F(\rho) - \tilde{\mathcal{B}}_F(\rho) = \frac{1}{2\pi}\int_0^\infty e^{-2\rho\tau}\left(\frac{1}{c_0(\tau)^2} - \frac{1}{\tilde{c}_0(\tau)^2}\right)\tau d\tau.$$

Therefore, the proposition can be easily seen by the use of the following lemma. □

Lemma 2 *There exist positive numbers L, C, q, ϵ such that*

$$\left|\frac{1}{\tilde{c}_0(\tau)^2} - \frac{1}{c_0(\tau)^2}\right| \le C\tau^q e^{-2\epsilon\tau} \quad \textit{for } \tau \ge L.$$

Proof Let

$$e(\tau) := c_0(\tau)^2 - \tilde{c}_0(\tau)^2 = \int_U e^{-2\tau F(z)}(1 - \Phi(z))dV(z).$$

From the conditions (A), (C) in Sect. 2.2, there exist positive numbers c, ϵ such that

$$F(z) \ge \epsilon + c(|z_1|^\beta + \cdots + |z_n|^\beta) \quad \text{for } z \in U \setminus D(R/2).$$

By the use of the above inequality, we can

$$\begin{aligned} |e(\tau)| &\le \int_{U\setminus D(R/2)} e^{-2\tau F(z)} dV(z) \\ &\le (2\pi)^n e^{-2\epsilon\tau}\int_{\mathbb{R}^n_+} e^{-2c\tau(x_1^\beta + \cdots + x_n^\beta)}\left(\prod_{j=1}^n x_j\right) dx \le C\tau^{-2n/\beta}e^{-2\epsilon\tau}. \end{aligned} \tag{11}$$

Applying (10), (11) to the right hand side of the equation

$$\frac{1}{\tilde{c}_0(\tau)^2} - \frac{1}{c_0(\tau)^2} = \frac{e(\tau)}{\tilde{c}_0(\tau)^4(1 + e(\tau)/\tilde{c}_0(\tau)^2)},$$

we can get the estimate in the lemma. □

5 The $\hat{\mathcal{E}}$-Condition

In this section, we introduce some classes of C^∞ functions, which are defined by the use of Newton polyhedra. When F belongs to this class, the behavior of $\tilde{\mathcal{B}}_F(\rho)$ is relatively easy to be understood (see Theorem 3, below).

5.1 *Newton Polyhedra in the Real Case*

Let f be a real-valued C^∞ function defined near the origin in $\mathbb{R}^n$. Let

$$\sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x^\alpha = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \tag{12}$$

be the Taylor series of f at the origin. The Newton polyhedron of f is the integral polyhedron

$$\mathcal{N}_+(f) = \text{the convex hull of the set } \bigcup \{\alpha + \mathbb{R}_+^n : \alpha \in S(f)\} \text{ in } \mathbb{R}_+^n,$$

where $S(f) := \{\alpha \in \mathbb{Z}_+^n : c_\alpha \neq 0\}$. In the real case, we also say that f is *flat* if $\mathcal{N}_+(f) = \emptyset$ and that f is *convenient* if $\mathcal{N}_+(f)$ intersects all the axes.

5.2 *The $\hat{\mathcal{E}}$-Condition in the Real Case*

Let f be a C^∞ function defined near the origin in $\mathbb{R}^n$. We say that f *admits the* γ*-part* on an open neighborhood V of the origin in $\mathbb{R}^n$ if for any $x \in V$ the limit

$$\lim_{t \to 0} \frac{f(t^{a_1} x_1, \ldots, t^{a_n} x_n)}{t^l}$$

exists for *all* pairs $(a, l) = (a_1, \ldots, a_n, l) \in \mathbb{Z}_+^n \times \mathbb{Z}_+$ defining γ (i.e., $\{x \in P : \sum_{j=1}^n a_j x_j = l\} = \gamma$). It is known in [18] that when f admits the γ-part, the above limits take the same value for any (a, l), which is denoted by $f_\gamma(x)$. We consider f_γ as the function on V, which is called the γ*-part* of f. For a compact face γ of $\mathcal{N}_+(f)$, f always admits the γ-part near the origin and $f_\gamma(x)$ equals the polynomial $\sum_{\alpha \in \gamma \cap \mathbb{Z}_+^n} c_\alpha x^\alpha$, where c_α are as in (12).

Definition 1 ([18, 19]) We say that f satisfies the $\hat{\mathcal{E}}$*-condition at the origin* if f admits the γ-part for every proper face γ of $\mathcal{N}_+(f)$.

Remark 2 (1) If f is real analytic near the origin or f is convenient, then f satisfies the $\hat{\mathcal{E}}$-condition (see [18, 19]). In particular, in the one-dimensional case, nonflat functions satisfy the $\hat{\mathcal{E}}$-condition.

(2) For example, $f(x_1, x_2) = x_1^2 + e^{-1/x_2^2}$ does not satisfy the $\hat{\mathcal{E}}$-condition.

Every C^∞ function f can be decomposed into an $\hat{\mathcal{E}}$-function and a flat function.

Proposition 2 *For any C^∞ function f defined near the origin in $\mathbb{R}^n$, there exists a C^∞ function f_0 satisfying the $\hat{\mathcal{E}}$-condition at the origin such that $f - f_0$ is flat at the origin.*

Proof Since the proposition is obvious when f does not vanish at the origin or f is flat at the origin, we will only consider the other cases.

Let $p = (p_1, \ldots, p_n) \in \mathbb{Z}_+^n$ be a vertex of the Newton polyhedron of f. We inductively define C^∞ functions $R_0, R_1, \ldots, R_n$ defined near the origin as follows.

Let $R_0(x) = f(x)$. Let k is an integer with $1 \le k \le n$. If $p_k \ge 1$, then there exists a C^∞ function R_k defined near the origin such that

$$R_{k-1}(x) = \sum_{j=0}^{p_k-1} c_{kj}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) x_k^j + x_k^{p_k} R_k(x), \tag{13}$$

where c_{jk} are C^∞ functions of $(n-1)$-variables. Note that (13) is the Taylor expansion of R_{k-1} with respect to the variable x_k. On the other hand, if $p_k = 0$, then set $R_k(x) = R_{k-1}(x)$.

By the use of the equations in (13) for $k = 1, \ldots, n$, f can be expressed as

$$f(x) = P(x) + x^p R_n(x), \tag{14}$$

where P is written in the form of the sum of a monomial times a C^∞ function of $(n-1)$-variables, and $x^p = x_1^{p_1} \cdots x_n^{p_n}$. It is easy to see that $R_n(0) = c_p$, where c_p is as in (12) (i.e., c_p is the coefficient of the term containing x^p of the Taylor series of f). Since p is the vertex of the Newton polyhedron of f, c_p does not vanish, which implies $R_n(0) \neq 0$. It follows from Proposition 6.3 in [18], we can see that $x^p R_n(x)$ satisfies the $\hat{\mathcal{E}}$-condition.

Now, let us construct a C^∞ function f_0 in the proposition by induction on n. Note that every non-flat C^∞ function of one variable satisfies the $\hat{\mathcal{E}}$-condition. Let f be a non-flat C^∞ function of n-variables with $f(0) = 0$. Then f can be expressed as in (14) and P takes the form of the sum of a monomial times a C^∞ function of $(n-1)$-variables. Now we assume that every C^∞ function of $(n-1)$-variables can be decomposed into an $\hat{\mathcal{E}}$-function and a flat function. Then P has the same type decomposition and so a desirable $\hat{\mathcal{E}}$-function f_0 can be obtained.

5.3 The $\hat{\mathcal{E}}$-Condition in the Complex Case

Now, let us consider a nonnegative C^∞ function F defined near the origin in $\mathbb{C}^n$ with $F(0) = 0$. If F satisfies the condition (C), then there exists a C^∞ function f defined near the origin in $\mathbb{R}^n$ such that $f(|z_1|, \ldots, |z_n|) = F(z_1, \ldots, z_n)$. We say that F satisfies the $\hat{\mathcal{E}}$-condition at the origin, if f satisfies the $\hat{\mathcal{E}}$-condition at the origin. It is easy to see that the $\hat{\mathcal{E}}$-condition of F is independent of the choice of the function f.

Proposition 3 *For any C^∞ function F defined near the origin in $\mathbb{C}^n$ satisfying the condition (C), there exists a C^∞ function F_0 satisfying the condition (C) and the $\hat{\mathcal{E}}$-condition at the origin such that $F - F_0$ is flat at the origin.*

Remark 3 It is desirable to show that the additional conditions (A), (B) imply the existence of F_0 such that $F - F_0$ is a *nonnegative* flat function. If this implication can be shown, then the existence of F_0 can be removed in the assumption in Theorem 2.

6 Behavior of $\tilde{\mathcal{B}}_F(\rho)$ in the $\hat{\mathcal{E}}$-Case

In order to prove Theorem 2, it suffices to consider the case where F satisfies the $\hat{\mathcal{E}}$-condition.

Theorem 3 *Suppose that F is a C^∞ function with $F(0) = 0$ satisfying the condition (C) in Sect. 2.2 and the $\hat{\mathcal{E}}$-condition at the origin. If the principal face of $\mathcal{N}_+(F)$ is compact, then*

$$\tilde{\mathcal{B}}_F(\rho) = \frac{C(F_*)}{\rho^{2+2/d_F}(\log \rho)^{m_F-1}} \cdot (1 + o(\rho^\varepsilon)) \quad \text{as } \rho \to 0,$$

where d_F and m_F are as in Sect. 2.1, ε is a positive constant, and $C(F_*)$ is a positive constant depending only on the principal part F_* of F.

Proof Recall the expression of $\tilde{\mathcal{B}}_F(\rho)$:

$$\tilde{\mathcal{B}}_F(\rho) = \frac{1}{2\pi}\int_0^\infty e^{-2\rho\tau}\frac{\tau}{\tilde{c}_0(\tau)^2}d\tau$$
$$\text{with } \tilde{c}_0(\tau)^2 = \int_{\mathbb{C}^n} e^{-2\tau F(z)}\Phi(z)dV(z).$$

This theorem can be directly shown by applying the following lemma to the above integral formulas. □

Lemma 3 *Under the same assumptions on F as those of Theorem 3, $\tilde{c}_0(\tau)^2$ satisfies*

$$\tilde{c}_0(\tau)^2 = c(F_*)\tau^{-2/d_F}(\log \tau)^{m_F-1} \cdot (1 + o(\tau^{-\varepsilon})) \quad \text{as } \tau \to \infty, \tag{15}$$

where ε is a positive constant and $c(F_*)$ is a positive constant depending only on the principal part F_* of F.

The proof of the above lemma will be given in Sect. 8.

7 Proof of Theorem 2

Let us give a proof of Theorem 2. We assume that F is a C^∞ function satisfying the conditions (A–D) in Sect. 2.2. Moreover, let F_0 be as in Theorem 2. It is easy to see that F_0 also satisfies the conditions (B), (C).

For a positive number M, we define

$$F_1(z) = F_0(z) + \sum_{j=1}^{n} |z_j|^{2M}.$$

Since the principal face of $\mathcal{N}_+(F_0)$ is compact, there exists a positive constant M such that the principal face of $\mathcal{N}_+(F_1)$ is the same as that of $\mathcal{N}_+(F_0)$. Moreover, since F_1 is convenient, F_1 also satisfies the $\hat{\mathcal{E}}$-condition at the origin (see Remark 2 (1)). It is easy to see that there exists a positive number R such that

$$F_0(z) \le F(z) \le F_1(z) \quad \text{for } z \in D(R). \tag{16}$$

For $j = 0, 1$, we define

$$\tilde{\mathcal{B}}_{F_j}(\rho) = \frac{1}{2\pi}\int_0^\infty e^{-2\rho\tau}\frac{\tau}{g_j(\tau)}d\tau \quad \text{for } \rho > 0,$$

with

$$g_j(\tau) = \int_{\mathbb{C}^n} e^{-2\tau F_j(z)}\Phi(z)dV(z),$$

where Φ is as in Sect. 4. From the relationship in (16), we see

$$\tilde{\mathcal{B}}_{F_0}(\rho) \le \tilde{\mathcal{B}}_{F}(\rho) \le \tilde{\mathcal{B}}_{F_1}(\rho). \tag{17}$$

Since it is easy to see that F_0, F_1 satisfy the assumptions in Theorem 3, we have

$$\tilde{\mathcal{B}}_{F_j}(\rho) = \frac{C(F_*)}{\rho^{2+2/d_F}(\log\rho)^{m_F-1}}\cdot(1+o(\rho^{\varepsilon_j})) \quad \text{as } \rho \to 0, \tag{18}$$

for $j = 0, 1$, where ε_0, ε_1 are positive numbers. Therefore, (17), (18) imply that

$$\tilde{\mathcal{B}}_{F}(\rho) = \frac{C(F_*)}{\rho^{2+2/d_F}(\log\rho)^{m_F-1}}\cdot(1+o(\rho^{\varepsilon})) \quad \text{as } \rho \to 0,$$

where ε is the minimum of $\varepsilon_1, \varepsilon_2$. From the localization lemma in Proposition 1, we can obtain the Eq. (4) in the theorem.

8 Asymptotic Analysis of Some Laplace Integrals

The purpose of this section is to give a proof of Lemma 3. For this purpose, we will investigate the behavior of a Lapace integral of the form

$$L_f(\tau) = \int_{\mathbb{R}^n_+} e^{-2\tau f(x)} \varphi(x) \left(\prod_{j=1}^{n} x_j \right) dx, \tag{19}$$

for large $\tau > 0$, where $f, \varphi : V \to \mathbb{R}_+$ are nonnegative and nonflat C^∞ functions defined on a small open neighborhood V of the origin with $f(0) = 0$ and $\varphi(0) = 1$ and the support of φ is contained in V of the origin. Since the compact support of φ implies the convergence of the above integral, L_f can be considered as a C^∞ function defined on $(0, \infty)$.

The analysis in this section is based on the studies in [8, 18, 19], which deal with oscillatory integrals and local zeta functions.

8.1 Newton Data in the Real Case

Let f be a nonflat real-valued C^∞ function defined near the origin in $\mathbb{R}^n$. Let f admit the Taylor series $\sum_{\alpha \in \mathbb{Z}^n_+} c_\alpha x^\alpha$ at the origin and let $\mathcal{N}_+(f)$ be the Newton polyhedron of f defined in Sect. 5.1. The *Newton distance* of f is a nonnegative number

$$d_f = \min\{t \geq 0 : (t, \ldots, t) \in \mathcal{N}_+(f)\}.$$

The minimum proper face of $\mathcal{N}_+(f)$ containing the point $P_f := (d_f, \ldots, d_f)$ is called the *principal face* of $\mathcal{N}_+(f)$, and is denoted by γ_*. The codimension of γ_* is called the *Newton multiplicity*, which is denoted by m_f. When γ_* is compact, the *principal part* of f is defined by $f_*(x) = \sum_{\alpha \in \gamma_* \cap \mathbb{Z}^n_+} c_\alpha x^\alpha$.

8.2 Meromorphic Extension of Some Local Zeta Functions

Let us consider an integral of the form

$$Z_f(s) = \int_{\mathbb{R}^n_+} f(x)^s \varphi(x) \left(\prod_{j=1}^{n} x_j \right) dx \quad \text{for } s \in \mathbb{C}, \tag{20}$$

where f, φ are the same as in (19). The convergence of the integral easily implies that Z_f can be regarded as a holomorphic function on the right half-plane, which is called a *local zeta function* and is samely denoted by Z_f.

The situation of analytic extension of the above local zeta function plays crucial roles in the investigation of the behavior of Laplace integrals as $\tau \to \infty$.

Theorem 4 *Suppose that*

1. *f satisfies the $\hat{\mathcal{E}}$-condition (see Sect. 5);*
2. *f is Newton nondegenerate (see Sect. 9.3);*
3. *The principal face of $\mathcal{N}_+(f)$ is compact.*

Then Z_f can be analytically continued as a meromorphic function to the whole complex plane, which will be samely denoted by Z_f. More precisely, the set of the poles of Z_f are contained in the set $\{-j/m : j \in \mathbb{N}\}$ where m is a positive integer.

Furthermore, the most right pole of Z_f exists at $s = -2/d_f$ and its order is m_f, where d_f, m_f are as in Sect. 8.1. The coefficient of the leading term of the Laurent expansion of Z_f at $s = -2/d_f$

$$c_Z(f_*) = \lim_{s \to -2/d_f} \left(s + \frac{2}{d_f}\right)^{m_f} \cdot Z_f(s)$$

is a positive constant depending only on the principal part f_* of f.

Proof The above theorem is a special case of Theorems 10.1 and 10.8 in [19]. Indeed, f satisfies the conditions (b) and (c) in Theorem 10.8. □

Remark 4 The idea of the proof is based on the method of Varchenko in [1, 23] (see also [8, 18, 19]). In his method, the toric resolution of singularities based on the geometry of Newton polyhedra plays an essential role.

8.3 Asymptotic Behavior of Some Laplace Integrals

From an information about the analytic continuation of Z_f in Theorem 4, we can see the behavior of the Laplace integral in (19) as $\tau \to \infty$.

Theorem 5 *Suppose that f, φ satisfy the same conditions as in Theorem 4. Then L_f admits the asymptotic expansion: for any $N \in \mathbb{N}$, there exists a positive constant C_N such that*

$$\left| L_f(\tau) - \sum_{j=0}^{N} \sum_{k=1}^{n} c_{jk} \tau^{-j/m} (\log \tau)^{k-1} \right| < C_N \tau^{-N/m-\varepsilon}, \tag{21}$$

for $\tau \geq 2$*, where* m *is a positive integer determined by* $\mathcal{N}_+(f)$*,* ε *is a positive number and* c_{jk} *are constants. In particular, the first term of the expansion can be expressed as*

$$L_f(\tau) = c_L(f_*)\tau^{-2/d_f}(\log \tau)^{m_f-1} \cdot (1 + o(\tau^{-\varepsilon})) \quad \text{as } \tau \to \infty,$$

where d_f *and* m_f *are as in Sect. 8.1 and* $c_L(f_*)$ *is a positive number depending only on the principal part* f_* *of* f*.*

Proof Define the fiber integral $H_f : \mathbb{R}_+ \to \mathbb{R}$ as

$$H_f(u) = \int_{W_u} \varphi(x) \left(\prod_{j=1}^{n} x_j \right) \omega,$$

where $W_u := \{x \in \mathbb{R}^n : f(x) = u\}$ and ω is the surface element on W_u, which is determined by $df \wedge \omega = dx_1 \wedge \cdots \wedge dx_n$.

It is easy to see that the Laplace integral L_f and the local zeta function Z_f can be represented by the use of H_f as follows.

$$L_f(\tau) = \int_0^\infty e^{-2\tau u} H_f(u)du, \tag{22}$$

$$Z_f(s) = \int_0^\infty u^s H_f(u)du. \tag{23}$$

Applying the inverse formula of the Mellin transform to (23), we have

$$H_f(u) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} Z_f(s) u^{-s-1} ds,$$

where $r > 0$ and the integral contour follows the line $\mathrm{Re}(s) = r$ upwards. The meromorphic extension of Z_f in Theorem 4 implies that the deformation of the integral contour as r tends to $-\infty$ gives an asymptotic expansion of $H_f(u)$ as $u \to 0$ by the residue formula. For any $N \in \mathbb{N}$, there exists a positive constant B_N such that

$$\left| H_f(u) - \sum_{j=0}^{N} \sum_{k=1}^{n} b_{jk} u^{j/m} (\log u)^{k-1} \right| < B_N u^{N/m+\varepsilon}, \tag{24}$$

for $u \in (0, 1/2)$, where m is a positive integer determined by $\mathcal{N}_+(f)$, ε is a positive number and b_{jk} are constants. We remark that the above deformation of the contour can be done. By applying the asymptotic expansion (24) to (22), we can obtain (21).

In particular, we have

$$H_f(u) = c_H(f_*)u^{2/d_f-1}(\log u)^{m_f-1} \cdot (1 + o(u^{\varepsilon})) \quad \text{as } u \to 0. \tag{25}$$

Substituting (25) into (22), we have

$$L_f(\tau) = c_L(f_*)\tau^{-2/d_f}(\log \tau)^{m_f-1} \cdot (1 + o(\tau^{-\varepsilon})) \quad \text{as } \tau \to \infty. \qquad \square$$

8.4 *Proof of Lemma 3*

By the use of the polar coordinates $z_j = x_j e^{i\theta_j}$, with $x_j \geq 0, \theta_j \in \mathbb{R}$, for $j = 1, \ldots, n$, then we have

$$\begin{aligned}\tilde{c}_0(\tau)^2 &= \int_U e^{-2\tau F(z)}\Phi(z)dV(z) \\ &= (2\pi)^n \int_{\mathbb{R}^n_+} e^{-2\tau f(x)} \left(\prod_{j=1}^n x_j\right) \varphi(x)dV(x),\end{aligned}$$

where φ is as in Sect. 4 and f is a C^∞ function defined near the origin satisfying $f(|z_1|, \ldots, |z_n|) = F(z_1, \ldots, z_n)$.

From [16], the conditions (A–C) implies that F_γ is positive on $(\mathbb{R} \setminus \{0\})^n$ for every compact face of $\mathcal{N}_+(F)$, which implies the Newton nondegeneracy of f (see Sect. 9.3). Moreover, it follows from the assumptions of F that the the principal face of f is compact and f satisfies the $\hat{\mathcal{E}}$-condition. Therefore, since f satisfies all the assumptions in Theorem 5, we can obtain

$$\tilde{c}_0(\tau)^2 = C(f_*)\tau^{-2/d_f}(\log \tau)^{m_f-1} \cdot (1 + o(\tau^{-\varepsilon})),$$

where $C(f_*)$ is a positive constant depending only on f_* and ε is a positive number. Since $d_f = d_F$ and $m_f = m_F$, we can obtain (15) in the lemma.

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9 Appendix

9.1 Asymptotic Expansion in (2)

Theorem 1 implies that the singularity of the Bergman kernel can be expressed by the following complicated asymptotic expansion

$$\Psi(\rho) \sim \sum_{j=0}^{\infty} \sum_{k=a_j}^{\infty} C_{jk} \frac{\rho^{j/m}}{(\log(1/\rho))^k} \quad \text{as } \rho \to 0,$$

where m is a positive integer, a_j are integers and C_{jk} are real numbers.

Let us explain the exact meaning of the above asymptotic expansion. For any $N \in \mathbb{N}$, there exists a positive number C_N such that

$$\left| \Psi(\rho) - \sum_{j=0}^{N} C_j(\zeta) \rho^{j/m} \right| < C_N \rho^{N/m+\varepsilon} \quad \text{for } \rho \in (0, \delta),$$

where $\zeta = \log(1/\rho)^{-1}$ and every $C_j(\zeta)$ satisfies that, for any $M \in \mathbb{N}$ satisfying $M \geq a_j$ there exists a positive number $\tilde{C}_M$ such that

$$\left| C_j(\zeta) - \sum_{k=a_j}^{M} C_{jk} \zeta^k \right| < \tilde{C}_M \zeta^{M+\tilde{\varepsilon}} \quad \text{for } \zeta \in (0, \delta).$$

Here $\varepsilon, \tilde{\varepsilon}, \delta$ are some positive numbers. Note that $\zeta \to 0$ if and only if $\rho \to 0$.

9.2 Convex Geometry

Let us explain fundamental notions in the theory of convex polyhedra which are necessary for our investigation. Refer to [24] for a general theory of convex polyhedra.

For $(a, l) \in \mathbb{R}^n \times \mathbb{R}$, let $H(a, l)$ and $H_+(a, l)$ be a hyperplane and a closed half-space in $\mathbb{R}^n$ defied by

$$H(a, l) = \{x \in \mathbb{R}^n; \langle a, x \rangle = l\},$$
$$H_+(a, l) = \{x \in \mathbb{R}^n; \langle a, x \rangle \geq l\},$$

respectively. Here $\langle a, x \rangle := \sum_{j=1}^{n} a_j x_j$.

A (*convex rational*) *polyhedron* is an intersection of closed half-spaces: a set $P \subset \mathbb{R}^n$ presented in the form $P = \bigcap_{j=1}^{N} H_+(a^j, l_j)$ for some $a^1, \ldots, a^N \in \mathbb{Z}^n$ and $l_1 \ldots, l_N \in \mathbb{Z}$.

Let P be a polyhedron in $\mathbb{R}^n$. A pair $(a, l) \in \mathbb{Z} \times \mathbb{Z}$ is said to be valid for P if P is contained in $H_+(a, l)$. A face of P is any set of the form $F = P \cap H(a, l)$, where (a, l) is valid for P. Since (0, 0) is always valid, we consider P itself as a trivial face of P; the other faces are called *proper faces*. Conversely, it is easy to see that any face is a polyhedron. Considering the valid pair $(0, -1)$, we see that the empty set is always a face of P. Indeed, $H_+(0, -1) = \mathbb{R}^n$, but $H(0, -1) = \emptyset$.

The dimension of a face F is the dimension of its affine hull (i.e., the intersection of all affine flats that contain F), which is denoted by $\dim(F)$. The faces of dimensions 0, 1 and $\dim(P) - 1$ are called vertices, edges and facets, respectively.

When P is the Newton polyhedron of some non-flat C^∞ function, γ is a compact face if and only if every valid pair $(a, l) = (a_1, \ldots, a_n, l)$ defining γ satisfies $a_j > 0$ for any j.

9.3 Newton Nondegeneracy Condition

We say that f is *Newton nondegenerate* if the gradient of the γ-part of f has no zero in $(\mathbb{R} \setminus \{0\})^n$ for every compact face γ of the Newton polyhedron $\mathcal{N}_+(f)$. This concept is very important in the study of singularity theory.

When γ is a compact face of $\mathcal{N}_+(f)$ with the valid pair $(a_1, \ldots, a_n, l)$ defining γ, the following Euler identity holds:

$$l f_\gamma(x) = a_1 x_1 \frac{\partial f_\gamma}{\partial x_1}(x) + \cdots + a_n x_n \frac{\partial f_\gamma}{\partial x_n}(x).$$

It follows from this identity that if f_γ has no zero in $(\mathbb{R} \setminus \{0\})^n$ for every compact face γ, then f is Newton nondegenerate.

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Bundle-Convexity and Kernel Asymptotics on a Class of Locally Pseudoconvex Domains

Takeo Ohsawa

Abstract A generalization of Grauert's solution of the Levi problem for strongly pseudoconvex domains will be given. On a class of bounded locally pseudoconvex domains in complex manifolds, spaces of holomorphic sections of vector bundles are analyzed by the L^2 method. Under appropriate positivity conditions on the curvature, a bundle-convexity theorem holds as a generalization of Grauert's theorem. Based on it, function-theoretic properties of such domains will be discussed.

Keywords Pseudoconvex · Bergman kernel · Bundle-convexity

1 Introduction

Analytic sheaves are of basic importance in several complex variables. As topological spaces, locally free analytic sheaves over complex manifolds consist of uncountably many connected components which are locally pseudoconvex over the base manifolds. It is known conversely that locally pseudoconvex domains over some manifolds are naturally identified with the components of the sheaves. For instance, if a compact complex manifold M admits a Kähler metric with positive holomorphic bisectional curvature, every locally pseudoconvex domain over M is equivalent to a connected component of the structure sheaf $\mathscr{O}_M \to M$ (cf. [10, 34]. See also [24]). A decisive existence result supporting this assertion is Grauert's theorem in [13] which says in particular that, for any complex manifold X, every strongly pseudoconvex domain $\Omega \Subset X$ is holomorphically convex and turns out to be a nonsingular model of a Stein space with isolated singularities. It is remarkable that Grauert's theorem is a direct consequence of the finite-dimensionality of the sheaf cohomology groups $H^1(\Omega, \mathscr{F})$ for coherent analytic sheaves $\mathscr{F} \to X$. Finite-dimensionality works here to produce global sections with singularities on $\partial\Omega$ from the vanishing of nontrivial linear combinations of cohomology classes. As a result, it turns out that the function

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space $H^0(\Omega, \mathscr{O})$ contains sufficiently many elements to make Ω holomorphically convex. To enlarge the scope from this viewpoint, generalizations and refinements of Grauert's theorem have been obtained (cf. [1, 16, 20, 22]). The present paper is a continuation of [25] whose motivation was to understand the L^2 $\bar{\partial}$ cohomology of complex surfaces which arose as the complements of complex curves with topologically trivial normal bundles in Ueda's work [35]. As a result, we could describe an asymptotic behavior of the weighted L^2 cohomology of a class of locally pseudoconvex domains of any dimension (cf. [25, Theorem 0.2]). This was done by establishing a variant of [20] which amounts to a generalization of [13]. The motivation of the present work is to explore function-theoretic applications of what we found in [25]. For that we shall study the space $H^{n,0}(\Omega, E)(\cong H^0(\Omega, \mathscr{O}(K_X \otimes E)), n = \dim X)$ for holomorphic vector bundles $E \to X$, where K_X denotes the canonical line bundle of X. Assuming that Ω is a locally pseudoconvex bounded domain in X, we want to derive for $H^{n,0}(\Omega, E)$ analogous results as Grauert's by imposing a positivity condition on the curvature of E.

Before stating the results of the present article, let us recall what has been proved in [20] at first. (See also [19].)

Theorem 1 (Grauert-type approximation theorem) *Let X be an n-dimensional complex manifold admitting a C^∞ plurisubharmonic exhaustion function $\varphi : X \to [0, \infty)$ and let $L \to X$ be a holomorphic Hermitian line bundle whose curvature form is positive on the complement of $X_c := \{x; \varphi(x) < c\}$ for some $c < \infty$. Then*

$$\dim H^{n,q}(X, L) < \infty$$

holds for $q \geq 1$ and the restriction homomorphism

$$H^{n,q}(X, L) \to H^{n,q}(X_c, L)$$

has dense image for $q \geq 0$, where $H^{n,q}(X, L)$ denotes the L-valued $\bar{\partial}$-cohomology group of X of type (n, q).

In the situation of Theorem 1, X is not necessarily holomorphically convex, but it does not mean that Theorem 1 is an empty generalization of Grauert's theorem. In fact, it is easy to see that the above X_c is $(K_X \otimes L^\mu)$-convex in the sense of Grauert [14] for sufficiently large μ. Here, given a holomorphic vector bundle $E \to X$, a domain Ω in X is said to be E-convex in the sense of Grauert if, for any compact subset K of E, one can find a compact set $\hat{K} \subset E$ such that, for any point $x \in X \setminus \hat{K}$ and for any $v \in E_x$, there exists a holomorphic section s of E satisfying $s(K \cap X) \subset \hat{K}$ and $s(x) = v$. (X is identified with the zero section of E.)

Theorem 1 was obtained by an application of a method of Hörmander (Proposition 3.4.5 in [16]).

The purpose of the present article is to prove a similar theorem for a wider class of domains by replacing the bundle-convexity by the following weaker one.

Definition 1 Given a Hermitian holomorphic vector bundle (E, h) over X, a domain Ω in X is said to be (E, h)-convex if, for any $\gamma \in \Omega^{\mathbb{N}}$ such that $\overline{\gamma(\mathbb{N})} \cap \Omega$ is noncompact, one can find a holomorphic section s of E over Ω satisfying $\sup_{\mu \in \mathbb{N}} |s(\gamma(\mu))|_h = \infty$.

In [26, Theorem 0.3] a special case of the following was proved.

Theorem 2 (Bundle-convexity theorem) *Let $\Omega \Subset X$ be a bounded locally pseudoconvex domain and let (L, h) be a holomorphic Hermitian line bundle over X whose curvature form is positive on $\partial\Omega$. Assume that every arcwise connected component of $\partial\Omega$ is either a C^2 real hypersurface or the support of a divisor with semipositive normal bundle. Then, for any Hermitian holomorphic vector bundle (F, h_F) over X, Ω is $(F \otimes L^\mu, h_F h^\mu)$-convex for sufficiently large μ.*

The case $F = X \times \mathbb{C}$ was proved in [26] and applied in [27, 28].

Note that not every locally pseudoconvex domain with C^2 smooth boundary is of the form X_c as in Theorem 1 (cf. [6, 8]).

When X is compact and (L, h) is a positive line bundle over X, (L^μ, h^μ)-convexity of Ω for $\mu \gg 1$ was proved by Pinney [29] when $\partial\Omega$ is C^2-smooth. So Theorem 2 amounts to a generalization of [29]. Asserda [5] has generalized [29] when Ω is an arbitrary locally pseudoconvex domain, but assuming also that X is compact and L is positive. Note that Theorem 2 does not hold if Ω is only assumed to be locally pseudoconvex, as one can see when X is the one point blow-up of $\mathbb{CP}^2$, $\Omega = X \setminus D$ for the exceptional set D and $L = [D]^{-1}$.

Anyway, in the absence of plurisubharmonic exhaustion functions, one cannot directly apply Hörmander's method. Pinney's method is an extension of [21] and Asserda applied Skoda's L^2 division theorem [31, 32]. Extensions of Cartan's theorems A and B to smoothly bounded domains are given also by Pinney in [30]. Theorem 2 will be proved by combining the following variant of Theorem 1, which is essentially equivalent to Theorem 0.2 in [25], with a technique in [9] for estimating the Bergman distance.

Theorem 3 (Stability theorem) *Let (X, g) be a complete Hermitian manifold of dimension n equipped with a (not necessarily exhaustive) C^∞ function $\varphi : X \to [0, \infty)$ such that*

$$X \setminus \{x; \varepsilon\omega_g + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ at } x\}$$

is compact for all $\varepsilon > 0$, where ω_g denotes the fundamental form of g. Then, for any holomorphic Hermitian vector bundle (F, h_F) over X and for any holomorphic Hermitian line bundle (L, h) over X whose curvature form Θ_h satisfies $\sqrt{-1}\Theta_h = \omega_g$ outside a compact subset of X, there exist positive integers m and μ_0 such that the L^2 $\bar{\partial}$-cohomology groups $H^{n,q}_{(2)}(X, F \otimes L^m, g, h_F \otimes h^m e^{-\mu\varphi})$ $(q \geq 1)$ are isomorphic to each other for $\mu \geq \mu_0$ by the homomorphisms

$$H^{n,q}_{(2)}(X, F \otimes L^m, g, h_F \otimes h^m e^{-\mu\varphi}) \to H^{n,q}_{(2)}(X, F \otimes L^m, g, h_F \otimes h^m e^{-\nu\varphi}) \ \ (\nu \geq \mu)$$

induced by inclusions.

We shall prove Theorem 3 at first and give a proof of Theorem 2 by exploring the technique of [9] more explicitly than in [26]. Then, we shall derive a function-theoretic property of locally pseudoconvex bounded domains in Theorem 2 by applying the bundle-convexity. The problem will be reduced to the case where the bundle is positive not only near the boundary but also on the whole interior of the domain.

2 Preliminaries and the Proof of Theorem 3

After recalling basic results on the L^2 $\bar{\partial}$-cohomology groups, we shall prove Theorem 3 at first.

Let (X, g) be a complete Hermitian manifold of dimension n and let (E, h) be a holomorphic Hermitian vector bundle over X. By $L^{p,q}_{(2)}(X, E, g, h)$ we denote the space of measurable E-valued (p, q)-forms on X which are square integrable with respect to (g, h). For any $u \in L^{p,q}_{(2)}(X, E, g, h)$, the pointwise norm of u with respect to (g, h) is denoted by $|u|$. Let ω_g denote the fundamental form of g and set $dV_g = \dfrac{1}{n!}\omega_g^n$. For any measurable set $A \subset X$ and $u \in L^{p,q}_{(2)}(X, E, g, h)$ we put

$$\|u\|^2 = \int_A |u|^2 dV_g.$$

Let

$$\bar{\partial} : \bigoplus_{p,q} L^{p,q}_{(2)}(X, E, g, h) \to \bigoplus_{p,q} L^{p,q+1}_{(2)}(X, E, g, h)$$

be the maximal closed extension of the complex exterior differentiation

$$\bar{\partial} : \bigoplus_{p,q} C_0^{p,q}(X, E, g, h) \to \bigoplus_{p,q} C_0^{p,q+1}(X, E, g, h)$$

of type (0,1), where $C_0^{p,q}(X, E)$ denotes the set of compactly supported E-valued C^∞ forms on X, equipped with the structure of pre-Hilbert space with respect to the metrics g and h. $\mathrm{Dom}\bar{\partial}$, $\mathrm{Im}\bar{\partial}$ and $\mathrm{Ker}\bar{\partial}$ will stand for the domain, the image and the kernel of $\bar{\partial}$ on $\oplus L^{p,q}_{(2)}(X, E, g, h)$, respectively.

Let

$$\bar{\partial}^* : \bigoplus_{p,q} L^{p,q}_{(2)}(X, E, g, h) \to \bigoplus_{p,q} L^{p,q-1}_{(2)}(X, E, g, h)$$

be the adjoint of $\bar{\partial}$. We put $\mathscr{H}(X, E, g, h) = \mathrm{Ker}\bar{\partial} \cap \mathrm{Ker}\bar{\partial}^*$ and

$$H^{p,q}_{(2)}(X, E, g, h) = \frac{\mathrm{Ker}\bar{\partial} \cap L^{p,q}_{(2)}(X, E, g, h)}{\mathrm{Im}\bar{\partial} \cap L^{p,q}_{(2)}(X, E, g, h)}.$$

It is elementary that

$$\mathscr{H}(X, E, g, h) \cap L^{p,q}_{(2)}(X, E, g, h) \cong \frac{\mathrm{Ker}\bar{\partial} \cap L^{p,q}_{(2)}(X, E, g, h)}{[\mathrm{Im}\bar{\partial}] \cap L^{p,q}_{(2)}(X, E, g, h)},$$

where $[\mathrm{Im}\bar{\partial}]$ denotes the closure of $\mathrm{Im}\bar{\partial}$.

The following is the most basic for our purpose.

Proposition 1 $\dim H^{p,q}_{(2)}(X, E, g, h) < \infty$ *if there exist a compact set* $K \subset X$ *and a constant* $C > 0$ *such that*

$$\|u\|^2 \leq C \left(\int_K |u|^2 dV_g + \|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 \right) \tag{2.1}$$

holds for all $u \in \mathrm{Dom}\bar{\partial} \cap \mathrm{Dom}\bar{\partial}^* \cap L^{p,q}_{(2)}(X, E, g, h)$.

Recall that the proof of Proposition 1 is done by combining Rellich's lemma and the finite-dimensionality of Banach spaces whose bounded sets are relatively compact. Note that no estimate for $\dim H^{p,q}_{(2)}(X, E, g, h)$ is obtained from Proposition 1.

Recall also that (2.1) follows immediately if there exists a compact set K_1 in the interior of K such that $d\omega_g = 0$ on $X \setminus K_1$ and the curvature form of h say Θ_h satisfies $\sqrt{-1}\Theta_h \geq Id_E \otimes \omega_g$ on $X \setminus K_1$. In fact, that (2.1) holds for $u \in C^{n,q}_0(X, E)$ ($q \geq 1$) follows from the equality

$$((\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} - \partial_h(\partial_h)^* - (\partial_h)^*\partial_h)u, u) = \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 - \|\partial_h u\|^2 - \|(\partial_h)^*u\|^2$$

applied to $u \in C^{n,q}_0(X \setminus K_1, E)$ through integration by parts, combined with Nakano's formula

$$\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} - \partial_h(\partial_h)^* - (\partial_h)^*\partial_h = [\sqrt{-1}\Theta_h, \omega_g^*] \tag{2.2}$$

with $([\sqrt{-1}\Theta_h, \omega_g^*]u, u) \geq \|u\|^2$, where $(\,,\,)$ denotes the inner product, ∂_h denotes the (1,0)-part of the Chern connection for h, $(\partial_h)^*$ the adjoint of ∂_h, Θ_h is identified with its exterior multiplication from the left-hand side and ω_g^* denotes the pointwise adjoint of $u \mapsto \omega_g \wedge u$.

The estimate (2.1) for $C^{n,q}_0(X, E)$ extends to $\mathrm{Dom}\bar{\partial} \cap \mathrm{Dom}\bar{\partial}^* \cap L^{n,q}_{(2)}(X, E)$ by the completeness of g. (cf. [11]. See also [2–4].)

For the proof of Theorem 3 we need the following variant of Proposition 3.4.5 in [16].

Proposition 2 *Let* φ_μ ($\mu = 1, 2, \dots$) *be an increasing sequence of* C^∞ *functions on* X *converging to a* C^∞ *function* φ *on* X *such that* (2.1) *for fixed* p, q, K *and* C *holds for all* $u \in \mathrm{Dom}\bar{\partial} \cap \mathrm{Dom}\bar{\partial}^* \cap L^{p,q}_{(2)}(X, E, g, he^{-\varphi_\mu})$. *Then there exist* $\mu_0 > 0$ *and* $C_1 > 0$ *such that*

$$\|u\|^2 \leq C_1(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) \tag{2.3}$$

holds for those $u \in \mathrm{Dom}\bar\partial \cap \mathrm{Dom}\bar\partial^* \cap L^{p,q}_{(2)}(X, E, g, he^{-\varphi_\mu})$ *which are orthogonal to* $\mathscr{H}(X, E, g, he^{-\varphi})$ *in* $L^{p,q}_{(2)}(X, E, g, he^{-\varphi})$, *if* $\mu \geq \mu_0$.

Proof The proof is done by contradiction. Assume on the contrary that there exists no such C_1. Then one can find a sequence $u_\mu \in \mathrm{Dom}\bar\partial \cap \mathrm{Dom}\bar\partial^* \cap L^{p,q}_{(2)}(X, E, g, he^{-\varphi_\mu})$ $(\mu = 1, 2, \dots)$ with $\|u_\mu\| = 1$ which has a subsequence u_{μ_k} $(k = 1, 2, \dots)$ such that $\lim_{k\to\infty} \|\bar\partial u_{\mu_k}\| = 0$, $\lim_{k\to\infty} \|\bar\partial^* u_{\mu_k}\| = 0$ and $u_{\mu_k} \perp \mathscr{H}(X, E, g, he^{-\varphi})$. By Rellich's lemma (or by Sobolev's embedding theorem) one has a strongly locally convergent subsequence of u_{μ_k} whose limit, say u_∞ is an element of $L^{p,q}_{(2)}(X, E, g, he^{-\varphi})$ satisfying $\bar\partial u_\infty = 0$, $\bar\partial^* u_\infty = 0$ and $u_\infty \perp \mathscr{H}(X, E, g, he^{-\varphi})$. But (2.1) for fixed K and C for all u_{μ_k} implies that $u_\infty \neq 0$, which is clearly a contradiction. □

Hence we obtain the following by virtue of a fundamental theorem of Hörmander (cf. [16, Theorem 1.1.4]).

Proposition 3 *In the situation of Proposition 2,*

$$\dim H^{p,q}_{(2)}(X, E, g, he^{-\varphi_\mu}) < \infty$$

for all μ *and the homomorphisms* $H^{p,q}_{(2)}(X, E, g, he^{-\varphi_\mu}) \to H^{p,q}_{(2)}(X, E, g, he^{-\varphi_\nu})$ $(\nu \geq \mu)$ *induced by the inclusions are bijective for sufficiently large* μ.

Proof (***Proof of Theorem*** 3) Let (X, L, g, h, φ) be as in Theorem 3. By the assumption, one can find an increasing sequence of C^∞ convex increasing functions say $\lambda_\mu : \mathbb{R} \to \mathbb{R}$ which converges to a C^∞ function λ such that

$$\sqrt{-1}(\Theta_h + Id_E \otimes \partial\bar\partial\lambda_\mu(\varphi)) \geq \frac{1}{2} Id_E \otimes \omega_g \quad (\mu = 1, 2, \dots)$$

hold on $X \setminus \{x; \varphi(x) < c\}$ for some $c < \infty$. Therefore, by applying Propositions 2 and 3 to $(E, h) = (F \otimes L^m, h_F \otimes h^m)$ for sufficiently large m, we obtain Theorem 3. □

Remark 1 The completeness assumption on g in Theorem 3 is actually not necessary. In fact, one may choose λ_μ as above in such a way that $g + \varepsilon\partial\bar\partial\lambda_\mu(\varphi)$ and $g + \varepsilon\partial\bar\partial\lambda(\varphi)$ are complete metrics for all $\varepsilon \in (0, 1)$. (Here $\partial\bar\partial$ stands for the complex Hessian for simplicity.) Then one can apply Theorem 3 and the corresponding solvability of the $\bar\partial$-equations for $(g + \varepsilon\lambda_\mu(\varphi), he^{-\lambda_\mu(\varphi)})$ with L^2 estimates following from (2.3), and let $\varepsilon \to 0$.

3 Complete Kähler Metrics on Punctured Domains with a Control of Potentials

It was observed by Grauert [12] that, for any Stein manifold X and any point $x \in X$, $X \setminus \{x\}$ admits a complete Kähler metric of the form $\partial\bar\partial\varphi + \partial\bar\partial\psi$, where φ is a C^∞ plurisubharmonic function on X and ψ is a C^∞ function on $X \setminus \{x\}$ such that $\mathrm{supp}\psi$

is relatively compact in X. For the proof of Theorem 2, we need some additional information on the metric for suitable choices of φ and ψ, which can be formulated as follows.

Proposition 4 *Let Ω be a bounded domain in $\mathbb{C}^n$ which admits a C^∞ plurisubharmonic exhaustion function $\varphi : \Omega \to [1, \infty)$ satisfying $\partial\bar{\partial}\varphi \geq \partial\varphi\bar{\partial}\varphi$ and*

$$\lim_{c\to\infty} \sup\left\{R^{\varphi}_{z,w};\, z, w \in \Omega \text{ and } |\varphi(z) - \varphi(w)| > c\right\} = 0, \tag{3.1}$$

where $R^{\varphi}_{z,w} := \dfrac{\log(|\log\|z-w\|| + 1)}{\varphi(w)}$.

Then there exists a C^∞ strictly plurisubharmonic function Φ on a neighborhood of $\overline{\Omega}$ and constants $A > 0$ and $B > 0$ such that, for any sufficiently large c and any $z_0 \in \Omega$ satisfying $2c < \varphi(z_0) < 3c$, one can find a C^∞ function $\psi : \Omega \setminus \{z_0\} \to (-\infty, 0]$ satisfying the following conditions.

(1) *$\partial\bar{\partial}(\Phi + \psi + A\varphi)$ is a complete Kähler metric on $\Omega \setminus \{z_0\}$ satisfying*

$$\partial\bar{\partial}(\Phi + \psi + A\varphi) \geq \partial\varphi\bar{\partial}\varphi + \frac{1}{A}\partial\psi\bar{\partial}\psi.$$

(2) $e^{-\psi(z)} = 2n \log \dfrac{1}{\|z - z_0\|} + B$ *on* $\{z \in \Omega \setminus \{z_0\};\, 2c < \varphi(z) < 3c\}$.

(3) $\mathrm{supp}\psi \Subset \{z \in \Omega;\, c < \varphi(z) < 4c\}$.

Proof Let $\chi : \mathbb{R} \to [0, 1]$ be a C^∞ function satisfying $\chi|_{(-\infty,1)\cup(4,\infty)} = 0$ and $\chi|_{(2,3)} = 1$. Let $c > 1$ and let $z_0 \in \Omega$ with $2c < \varphi(z_0) < 3c$.

We put

$$\psi(z) = \chi\left(\frac{\varphi(z)}{c}\right)\log(-2n\log\|z - z_0\| + B),$$

where B is a constant to make $\log(-2n\log\|z - z_0\| + B)$ positive on $\overline{\Omega} \setminus \{z_0\}$. Then

$$\begin{aligned}\partial\bar{\partial}\psi(z) =& \frac{1}{c}\chi'\left(\frac{\varphi}{c}\right) 2\mathrm{Re}\,\partial\varphi\bar{\partial}\log(-2n\log\|z - z_0\| + B)\\ &+ \frac{1}{c^2}\chi''\left(\frac{\varphi}{c}\right)\log(-2n\log\|z - z_0\| + B)\partial\varphi\bar{\partial}\varphi\\ &+ \frac{1}{c}\chi'\left(\frac{\varphi}{c}\right)\log(-2n\log\|z - z_0\| + B)\partial\bar{\partial}\varphi\\ &+ \chi\left(\frac{\varphi}{c}\right)\partial\bar{\partial}\log(-2n\log\|z - z_0\| + B).\end{aligned}$$

For simplicity we put $\eta = \log(-2n\log\|z - z_0\| + B)$.

By $\partial\bar{\partial}\varphi \geq \partial\varphi\bar{\partial}\varphi$ and (2), we have

$$\frac{1}{c^2}\left|\chi''\left(\frac{\varphi}{c}\right)\eta\right|\partial\varphi\bar{\partial}\varphi + \frac{1}{c}\left|\chi'\left(\frac{\varphi}{c}\right)\eta\right|\partial\bar{\partial}\varphi \leq \partial\bar{\partial}\varphi$$

for sufficiently large c.

Since

$$\left|\frac{2}{c}\chi'\left(\frac{\varphi}{c}\right)\partial\varphi\bar{\partial}\eta\right| \leq \frac{2\left|\chi'\left(\frac{\varphi}{c}\right)\right|^2}{c^2\chi\left(\frac{\varphi}{c}\right)}\partial\varphi\bar{\partial}\varphi + \frac{1}{2}\chi\left(\frac{\varphi}{c}\right)\partial\eta\bar{\partial}\eta,$$

$$\partial\bar{\partial}\eta \geq \partial\eta\bar{\partial}\eta$$

and χ can be chosen in such a way that

$$\sup_{\chi(t)\neq 0}\frac{(\chi'(t))^2}{\chi(t)} < \infty,$$

one can find a constant $A > 0$ such that

$$\partial\bar{\partial}\psi + A\partial\bar{\partial}\varphi \geq \partial\varphi\bar{\partial}\varphi + \frac{1}{2}\chi\left(\frac{\varphi}{c}\right)\partial\eta\bar{\partial}\eta$$

holds for sufficiently large c.

Hence, by replacing A by $A+1$ if necessary, for any C^∞ strictly plurisubharmonic function Φ on $\overline{\Omega}$, $\partial\bar{\partial}\Phi + \partial\bar{\partial}\psi + A\partial\bar{\partial}\varphi$ becomes a complete Kähler metric on $\Omega \setminus \{z_0\}$ such that ψ satisfies (1), (2) and (3). □

Similarly as above one can deduce the following proposition which is crucial for the proof of Theorem 2.

Proposition 5 *Let Ω be a locally pseudoconvex bounded domain with C^2-smooth boundary in a Hermitian manifold (X, g) and let ρ be C^2 function on X such that $\Omega = \{x; \rho(x) < 0\}$ and $d\rho$ has no zeros on $\partial\Omega$. Then there exist constants $d_0, d_1 > 0$ and a C^∞ exhaustion function $\varphi : \Omega \to [0, \infty)$ satisfying*

$$\sup_x |\varphi(x) + \log(-\rho(x))| < \infty$$

and $g + d_0\partial\bar{\partial}\varphi \geq d_1\partial\varphi\bar{\partial}\varphi$. Moreover, there exists a constant $C > 0$ such that, for any such φ, for any sufficiently large c and for any $x_0 \in \Omega$ satisfying $4c < \varphi(x_0) < 5c$, one can find a C^∞ function $\psi : \Omega \setminus \{x_0\} \to (-\infty, 0]$ satisfying the following conditions.

(i) *$C^2 g + C(\partial\bar{\partial}\psi + \partial\bar{\partial}\varphi)$ is a complete metric on $\Omega \setminus \{x_0\}$ satisfying*

$$C^2 g + C(\partial\bar{\partial}\psi + \partial\bar{\partial}\varphi) \geq \partial\varphi\bar{\partial}\varphi + \partial\psi\bar{\partial}\psi.$$

(ii) $e^{e^{-\psi}}$ *is not integrable on any neighborhood of* x_0.

(iii) $\operatorname{supp}\psi \Subset \{x \in \Omega; c < \varphi(x) < 8c\}$.

(iv) $\limsup_{c\to\infty} \sup \left\{ \left| \dfrac{e^{-\psi(x)}}{\varphi(x)} \right| ; |\varphi(x) - \varphi(x_0)| > 2c \right\} < \infty.$

4 Proof of Theorem 2

Let X, Ω and $L \to X$ be as in Theorem 2 and let h be a fiber metric of L whose curvature form Θ_h is positive. Let us fix a fiber metric κ of the canonical bundle K_X of X. First we assume that $\partial\Omega$ is a real hypersurface of class C^2 and fix a Hermitian metric g on X whose fundamental form ω_g is $\sqrt{-1}\Theta_h$ on a neighborhood of $\partial\Omega$. Let φ be as in Proposition 5.

Let x_ν $(\nu = 1, 2, \dots)$ be a sequence of points in Ω which does not have any accumulation point in Ω. Then, by Proposition 5, for any holomorphic Hermitian vector bundle (F, h_F) on X, one can find a subsequence x_{ν_k} of x_ν, an increasing sequence $c_k \in (1, \infty)$ $(k = 1, 2, \dots)$ satisfying $8c_k + 1 < c_{k+1}$, a complete Hermitian metric $\tilde{g}$ on $\Omega \setminus \{x_{\nu_k}; k = 1, 2, \dots\}$ and a C^∞ function $\tilde{\psi} : \Omega \setminus \{x_{\nu_k}; k = 1, 2, \dots\} \to (-\infty, 0)$ such that

$$\operatorname{supp}\tilde{\psi} \cap \bigcup_{k=1}^{\infty} \{x; 8c_k < \varphi(x) < c_{k+1}\} = \phi,$$

$e^{e^{-\tilde{\psi}}}$ is not integrable around each x_{ν_k}, and

$$\sqrt{-1}(\Theta_F + Id_F \otimes (-\Theta_\kappa - \partial\bar{\partial}e^{-\tilde{\psi}} + \mu\partial\bar{\partial}(\tilde{\psi} + \varphi) + \mu^2\Theta_h)) \geq Id_F \otimes \omega_{\tilde{g}}$$

holds outside a compact subset of $\Omega \setminus \{x_{\nu_k}; k = 1, 2, \dots\}$ for sufficiently large μ.

Therefore, by Nakano's formula and Proposition 1,

$$\dim H^{n,1}_{(2)}(\Omega', K_X^{-1} \otimes F \otimes L^{\mu^2}, \tilde{g}, \kappa^{-1} \otimes h_F \otimes e^{e^{-\tilde{\psi}}} e^{-\mu(\tilde{\psi}+\varphi)} h^{\mu^2}) < \infty$$

holds for sufficiently large μ. Here $\Omega' = \Omega \setminus \{x_{\nu_k}; k = 1, 2, \dots\}$ and $n = \dim\Omega$.

On the other hand, it is clear in this situation that one can find C^∞ sections s_ℓ $(\ell = 1, 2, \dots)$ of $F \otimes L^{\mu^2}$ on Ω satisfying

$$\lim_{k\to\infty} \left|s_\ell(x_{\nu_k})\right|_{h_F\otimes h^{\mu^2}} = \infty,$$

$$\lim_{k\to\infty} \left| \frac{s_\ell(x_{\nu_k})}{s_{\ell+1}(x_{\nu_k})} \right| = 0$$

and
$$\bar\partial s_\ell \in L^{n,1}_{(2)}(\Omega', K_X^{-1}\otimes F\otimes L^{\mu^2}, \tilde g, \kappa^{-1}\otimes h_F\otimes e^{e^{-\tilde\psi}} e^{-\mu(\tilde\psi+\varphi)}h^{\mu^2})$$

for all ℓ.

Hence one can find a nontrivial linear combination of s_ℓ, say $\sigma = \sum_{\ell=1}^N a_\ell s_\ell$ such that there exists a solution to $\bar\partial s = \bar\partial\sigma$ with $s \in L^{n,0}_{(2)}(\Omega', K_X^{-1}\otimes F\otimes L^{\mu^2}, \tilde g, h_F\otimes e^{e^{-\tilde\psi}} e^{-\mu(\tilde\psi+\varphi)}h^{\mu^2})$.

Clearly $\sigma - s$ extends to a holomorphic section $\tilde\sigma$ of $F\otimes L^{\mu^2}$ which satisfies

$$\lim_{k\to\infty} |\tilde\sigma(x_{\nu_k})| = \infty.$$

If $\partial\Omega$ is the support of an effective divisor whose normal bundle is semipositive, namely if $\partial\Omega = |D|$ for an effective divisor D with $[D]|_{|D|} \geq 0$, X is compact and it has been shown in [23, Theorem 0.3] that the multiplication by a canonical section of $[D]$ induces bijections between $H^{n,1}(X, K_X^{-1}\otimes F\otimes L^\mu\otimes[D]^m)$ and $H^{n,1}(X, K_X^{-1}\otimes F\otimes L^\mu\otimes[D]^{m+1})$ for sufficiently large μ and $m \gg \mu$, so that the restriction homomorphisms

$$H^{0,0}(X, F\otimes L^\mu\otimes[D]^{m+1}) \to H^{0,0}(|D|, F\otimes L^\mu\otimes[D]^{m+1})$$

are surjective for $m \gg \mu \gg 1$, which follows from Theorem 3 in particular. Since $L|_{|D|}$ is ample, $X\setminus|D|$ is $F\otimes L^\mu$-convex for sufficiently large μ. $\square$

5 Application to the Kernel Asymptotics

In the situation of Theorem 2, it is natural to ask whether one can see the asymptotic behavior of the weighted Bergman kernels of the Bergman spaces $H^{n,0}_{(2)}(\Omega, F\otimes L^\nu, h_F\otimes e^{-\mu\varphi}h^\nu)$ as $z\to\partial\Omega$, $\mu\to\infty$ or $\nu\to\infty$. Here, by the weighted Bergman kernels we mean the functions $B_{\Omega,F\otimes L^\nu,\mu}(x)$ on Ω defined by

$$B_{\Omega,F\otimes L^\nu,\mu}(x) := \sup\left\{\frac{|s(x)|^2}{\|s\|^2}; s\in H^{n,0}_{(2)}(\Omega, F\otimes L^\nu, h_F\otimes e^{-\mu\varphi}h^\nu)\setminus\{0\}\right\}.$$

From the proof of Theorem 2, one cannot directly see it. Nevertheless, after reducing the question to the case where $\sqrt{-1}\Theta_{h^\nu e^{-\mu\varphi}} > 0$ on Ω, such asymptotics can be analyzed by solving the $\bar\partial$ equations with L^2 norm estimates by a standard technique (cf. [7, 16] or [21, 23]). The reduction is done by the following.

Lemma 1 *Let X be a complex manifold and let $L\to X$ be a holomorphic line bundle. If the meromorphic map*

$$\sigma : X \cdots\to \mathrm{Proj}(H^{0,0}(X, L)) := H^{0,0}(X, L)^*/(\mathbb{C}\setminus\{0\})$$

induced from the correspondence $x \mapsto \{s; s(x) = 0\}$ *is proper onto its image, there exists a complex manifold* $\tilde{X}$ *with a proper and bimeromorphic holomorphic map* $\pi : \tilde{X} \to X$ *and a divisor* A *on* $\tilde{X}$ *with compact support such that* $\sigma \circ \pi$ *is holomorphic and*

$$\pi^* L \otimes [A] = (\sigma \circ \pi)^* \mathscr{O}(1),$$

where $\mathscr{O}(1)$ *denotes the hyperplane section bundle over* $\mathrm{Proj}(H^{0,0}(X, L))$.

Proof Given such σ as above, by virtue of a theorem of Hironaka, there exists a succession of blow-ups $X_N \to X_{N-1} \to \cdots X_1 \to X_0 = X$ which ends up with a map $\pi : \tilde{X} \to X$ with the required property. □

Hence it is not difficult to deduce the following from the above-mentioned proof of Theorem 2.

Theorem 4 *In the situation of Theorem 2, assume moreover that* $\partial\Omega$ *is a real hypersurface and let* (Ω, g, ρ) *be as in Proposition 5. Then, for any* $\varepsilon > 0$ *one can find* $\nu \in \mathbb{N}$ *such that*

$$\liminf_{x \to \partial\Omega} B_{\Omega, F \otimes L^\nu, \varepsilon}(x) \cdot \rho(z)^2 > 0. \tag{5.1}$$

It is very likely that (5.1) holds also for $\varepsilon = 0$.

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The $\overline{\partial}$-Equation on the Hartogs Triangles in $\mathbb{C}^2$ and $\mathbb{CP}^2$

Mei-Chi Shaw

Abstract This is a survey paper of the recent progress of the Cauchy-Riemann equation on the Hartogs triangles in $\mathbb{C}^2$ and $\mathbb{CP}^2$. In particular, L^2 and W^1 Dolbeault cohomology groups on the Hartogs triangles in $\mathbb{CP}^2$ are investigated.

Keywords Cauchy-Riemann operator · Hartogs triangles · Complex projective spaces

1 Introduction

The Hartogs triangle in $\mathbb{C}^2$ is an important example in several complex variables. In this paper, we survey some results on estimates for $\overline{\partial}$ on the Hartogs triangles in $\mathbb{C}^2$ and $\mathbb{CP}^2$. The Hartogs triangle is a bounded pseudoconvex domain in $\mathbb{C}^2$. The L^2 existence theorem for $\overline{\partial}$ of Hörmander holds. On the other hand, the L^2 theory for $\overline{\partial}$ on the Hartogs triangles in $\mathbb{CP}^2$ is quite different from that in $\mathbb{C}^2$. In fact, the L^2 Dolbeault cohomology groups do not vanish for (2.1)-forms.

The plan of the paper is as follows: In Sect. 2 we review some known results for $\overline{\partial}$ on the Hartogs triangle in $\mathbb{C}^2$, including some recent work in [3]. Next we discuss some results on the L^2 theory for $\overline{\partial}$ on domains in $\mathbb{CP}^n$ in Sect. 3. In Sect. 4 we discuss the L^2 theory for $\overline{\partial}$ on the Hartogs triangles in $\mathbb{CP}^2$ and show that it is very different from $\mathbb{C}^2$.

The boundary of the Hartogs triangles in $\mathbb{CP}^2$ can be viewed as a non-Lipschitz Levi-flat hypersurface in the sense that it divides $\mathbb{CP}^2$ into two pseudoconvex domains. Thus the Hartogs triangles in $\mathbb{CP}^2$ are especially important not only in complex analysis, but also in foliation theory and complex dynamics. The non-existence of Levi-flat hypersurfaces have been an active area of research in recent years and we raise some open questions at the end.

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2 Boundary Regularity for $\bar\partial$ on the Hartogs Triangle in $\mathbb{C}^2$

Let T be the Hartogs triangle in $\mathbb{C}^2$ defined by

$$T = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < |z_2| < 1\}.$$

We first recall some well-known results for the Hartogs triangle in $\mathbb{C}^2$. First note that T is pseudoconvex and is biholomorphic to $\triangle \times \triangle_*$ where $\triangle$ is the unit disc and $\triangle_* = \triangle \setminus \{0\}$ is the punctured disc.

Recall that a domain Ω is called Lipschitz if its boundary $b\Omega$ is locally the graph of a Lipschitz function. The T is not Lipschitz near the point (0, 0) since it is not a graph.

Definition 1 A domain $\Omega \subset \mathbb{R}^N$ is called a *Sobolev extension domain*, if for each $k \in \mathbf{N}$ and $1 \le p \le \infty$, there exists a bounded linear operator

$$\eta_k : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)$$

such that $\eta_k f\big|_\Omega = f$ for all $f \in W^{k,p}(\Omega)$.

It is well known by a theorem of Calderon–Stein [45] that every bounded Lipschitz domain in $\mathbb{R}^N$ is a Sobolev extension domain. One of the basic questions is to know if T is a Sobolev extension domain. The following lemma is proved in Theorem 2.12 in the recent paper [3].

Lemma 1 *The Hartogs triangle T is a Sobolev extension domain.*

Let $H^{p,q}(\Omega)$ (or $H^{p,q}(\overline{\Omega})$) denote the Dolbeault cohomology group with $C^\infty(\Omega)$ coefficients (or $C^\infty(\overline{\Omega})$ coefficients). Let $H^{p,q}_{W^s}(\Omega)$ be the Dolbeault cohomology with Sobolev W^s coefficients defined by

$$H^{p,q}_{W^s}(\Omega) = \frac{\{f \in W^s_{p,q}(\Omega) \mid \bar\partial f = 0\}}{\{f \in W^s_{p,q}(\Omega) \mid f = \bar\partial u, u \in W^s_{p,q-1}(\Omega)\}}. \tag{1}$$

When $s = 0$, we use the notation $H^{p,q}_{L^2}(\Omega)$. Similarly, we denote the cohomology group $H^{p,q}_{W^{k,r}}(\Omega)$ with forms with $W^{k,r}(\Omega)$ coefficients with $k \in \mathbb{N}$ and $1 < r < \infty$.

Since T is a pseudoconvex domain, we have

$$H^{p,1}(T) = 0, \qquad p = 0, 1, 2.$$

Moreover, using Hörmander's theorem, we have

$$H^{p,1}_{L^2}(T) = 0, \qquad p = 0, 1, 2.$$

The Sobolev estimates and boundary regularity for $\bar\partial$ on bounded smooth pseudoconvex domains in $\mathbb{C}^n$ are well known from Kohn's work (see [27] or [9]). However, when the domain is not smooth, it is still unknown if similar results hold for Lipschitz pseudoconvex domains (see Problem 2 at the end of this section). However, if the domain is not Lipschitz, the following results is proved by Sibony [41, 42] earlier (see also [10]). We repeat their arguments for the benefit of the reader.

Theorem 1 *For any ζ in the bidisc $P = \Delta \times \Delta$ and $\zeta \in P \setminus \overline{T}$, their exists a C^∞-smooth, $\overline{\partial}$-closed $(0, 1)$-form α_ζ defined in $\mathbb{C}^2 \setminus \{\zeta\}$ such that there does not exist any C^∞-smooth function β on $\overline{T}$ such that $\overline{\partial}\beta = \alpha_\zeta$.*

Proof Let $B(z) = B(z_1, z_2)$ the $(0, 1)$-form

$$B(z) = \frac{\overline{z}_1\, d\overline{z}_2 - \overline{z}_2\, d\overline{z}_1}{|z|^4}$$

derived from the Bochner–Martinelli kernel in $\mathbb{C}^2$. For any $\zeta \in P \setminus \overline{T}$, consider the $(0, 1)$-form $B(z - \zeta)$. It is a $\overline{\partial}$-closed form on $\mathbb{C}^2 \setminus \{\zeta\}$. Also we have

$$-\overline{\partial}\left(\frac{\overline{(z_2 - \zeta_2)}}{|z - \zeta|^2}\right) = (z_1 - \zeta_1)B(z - \zeta)$$

on $\mathbb{C}^2 \setminus \{\zeta\}$. On the other hand, if $H^{0,1}(\overline{T}) = 0$, there exists a $\alpha \in C^\infty(\overline{T})$ such that $\overline{\partial}\alpha = B$ on T. Set

$$F(z_1, z_2) = (z_1 - \zeta_1)\alpha + \frac{\overline{z_2 - \zeta_2}}{|z - \zeta|^2}.$$

Then F is a holomorphic function on T and $F \in C^\infty(\overline{T})$. Thus it extends holomorphically to P. But we have

$$F(\zeta_1, z_2) = \frac{1}{(z_2 - \zeta_2)}$$

on $T \cap \{z_1 = \zeta_1\}$, which is holomorphic and singular at $z_2 = \zeta_2$. This gives the contradiction since $\zeta \in P \setminus \overline{T}$. $\square$

One has a strengthened version (see [31]).

Corollary 1 *The cohomology group $H^{0,1}(\overline{T})$ is non-Hausdorff.*

On the other hand, let $C^{k,\alpha}$ be the Hölder space of order k and $0 < \alpha < 1$. The following results are obtained by Chaumat–Chollet [10].

Theorem 2 *For each nonnegative integer k and $0 < \alpha < \infty$, we have $H^{0,1}_{C^{k,\alpha}}(T) = 0$.*

Remark 1 1. Theorem 2 is in sharp contrast with Theorem 1. Notice that

$$\cap_k\, C^{k,\alpha}(T) = C^\infty(\overline{T}).$$

2. There are also results for $\overline{\partial}$ on the Sobolev spaces $W^{k,p}(T)$ with $p \geq 4$ (see the recent paper [36]).

There are many open questions related to the Hartogs triangle in $\mathbb{C}^2$ (see e.g. [3, 40]). We only mention one here.

Problem 1 (*Dollar Bill Question*) Determine if

1. $H^{0,1}_{W^1}(T) = 0$;
2. $H^{0,1}(\Omega)$ is Hausdorff where $\Omega = B \setminus \overline{T}$, B is the ball of radius 2 centered at 0.

Remark 2 We remark that (1) and (2) in Problem 1 are equivalent. The proof is essentially the same as the proof in [3, 30].

When $\Omega = B \setminus \overline{P}$ is the annulus between the ball B and the the bidisk $P = \Delta \times \Delta$, it is proved in [7] that $H^{0,1}(B \setminus \overline{P})$ is Hausdorff. Thus $H^{0,1}_{W^1}(P) = 0$. This question was referred to as the *Chinese Coin Problem* (see [7]). The proof in [7] is completely different from the proof of Kohn's theorem for smooth domains (see [27]).

However, the following result is proved recently.

Theorem 3 *Let $\Omega = B \setminus \overline{T}$ where B is the ball of radius 2 centered at 0. Then $H^{0,1}_{W^1}(\Omega)$ is Hausdorff. Furthermore, we have*

$$H^{0,1}_{W^1}(\Omega) \cong \mathcal{H}(T)$$

where $\mathcal{H}(T)$ is the Bergman space on T.

The theorem follows from L^2 Serre duality [8] and that T is a Sobolev extension domain (see Theorem 4.3 in [3]).

A more general question arises naturally.

Problem 2 Let D be a bounded pseudoonvex domain $\mathbb{C}^n$ with Lipschitz boundary. Determine if

1. $H^{0,1}_{W^1}(D) = 0$;
2. $H^{0,1}(\overline{D}) = 0$.

We remark that if D has C^2 boundary, we have $H^{0,1}_{W^1}(D) = 0$. This is proved by Kohn [27] for domains with C^4 boundary and by Harrington [18]) for domains with C^2 boundary.

3 L^2 Theory for $\overline{\partial}$ on Lipschitz Pseudoconvex Domains in $\mathbb{CP}^n$

Next we discuss the function theory for domains in the complex projective space $\mathbb{CP}^n$. We equip $\mathbb{CP}^n$ with the Fubini–Study metric ω. Let Ω be a proper domain in the complex projective space $\mathbb{CP}^n$ with C^2-smooth boundary $b\Omega$. Let $\rho(z)$ be the signed distance function from z to $b\Omega$ such that $\rho(z) = -d(z, b\Omega)$ for $z \in \Omega$ and $\rho(z) = d(z, b\Omega)$ when $z \in \mathbb{CP}^n \setminus \Omega$. Let φ be a real-valued C^2 function on $\overline{\Omega}$. Let $L^2_{p,q}(\Omega, e^{-\varphi})$ be the space of (p, q)-forms u on Ω such that

$$\|u\|^2_\varphi = \int_\Omega |u|^2_\omega e^{-\varphi} dV_\omega < \infty.$$

We will also use $(\cdot,\ \cdot)_\varphi$ to denote the associated inner product. Let $\overline{\partial}^*_\varphi$ be the adjoint of the maximally defined $\overline{\partial}\colon L^2_{p,q}(\Omega, e^{-\varphi}) \to L^2_{p,q}(\Omega, e^{-\varphi})$.

The Kähler form ω associated with the Fubini–Study metric h is given by

$$\omega = i\partial\overline{\partial}\log(1 + |z|^2) \tag{2}$$

$$= i \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}(z)\, dz_\alpha \wedge d\bar{z}_\beta \tag{3}$$

in local inhomogeneous coordinates, where

$$g_{\alpha\bar{\beta}}(z) = \frac{\partial^2 \log(1 + |z|^2)}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{(1 + |z|^2)\delta_{\alpha\bar{\beta}} - \bar{z}_\alpha z_\beta}{(1 + |z|^2)^2}. \tag{4}$$

The volume form is then

$$dV_\omega = \det(g_{\alpha\bar{\beta}}(z)) dV_{\mathrm{E}} = \frac{1}{(1 + |z|^2)^{n+1}} dV_{\mathrm{E}} \tag{5}$$

where dV_{E} is the Euclidean volume form.

Let ∇ be the Levi–Civita connection for the associated Riemannian metric $g = \mathrm{Re}\ h$, which is identical to the Chern connection on the holomorphic tangent bundle $T^{1,0}X$ due to the Kähler condition. Let

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

be the curvature tensor, extended to be $\mathbb{C}$-linear and to act on tensors of any type. The curvature tensor is then given by

$$
\begin{aligned}
R_{\alpha\bar\beta\gamma\bar\delta} &= h\left(R\left(\frac{\partial}{\partial z_\gamma},\ \frac{\partial}{\partial \bar z_\delta}\right)\frac{\partial}{\partial \bar z_\beta},\ \frac{\partial}{\partial z_\alpha}\right)\\
&= \frac{\partial^2 h_{\alpha\bar\beta}}{\partial z^\gamma \partial \bar z^\delta} - \sum_{\varepsilon,\tau} h^{\bar\varepsilon\tau}\frac{\partial h_{\alpha\bar\varepsilon}}{\partial z^\gamma}\frac{\partial h_{\tau\bar\beta}}{\partial \bar z^\delta}
\end{aligned} \tag{6}
$$

where $h^{\bar\varepsilon\tau}$ denotes the inverse of $h_{\tau\bar\varepsilon}$.

Let $L_1, \ldots, L_n$ be a local orthonormal frame field of type $(1,0)$ and $\omega^1, \ldots, \omega^n$ be the coframe field. For a (p,q)-form u, we set

$$\langle \Theta u, u\rangle = \sum_{j,k=1}^{n} \langle \bar\omega^j \wedge \big(\bar L_k \lrcorner R(L_j, \bar L_k) u\big),\ u\rangle,$$

where $\lrcorner$ is the usual contraction operator. For a C^2-smooth function φ, we set

$$\langle (\partial\overline\partial\varphi)u,\ u\rangle = \sum_{j,k=1}^{n} \partial\overline\partial\varphi(L_j, \bar L_k)\langle \bar L_j \lrcorner u, \bar L_k \lrcorner u\rangle.$$

It follows that the complex projective space $\mathbb{CP}^n$ with the Fubini–Study metric has constant holomorphic sectional curvature 2 and its holomorphic bisectional curvature is bounded between 1 and 2. Furthermore, we have that if u is a (p,q)-form on $\mathbb{CP}^n$ with $q \ge 1$, then

$$\langle \Theta u, u\rangle = 0, \quad \text{if} \quad p = n; \qquad \langle \Theta u, u\rangle \ge 0, \quad \text{if} \quad p \ge 1; \tag{7}$$

and

$$\langle \Theta u, u\rangle = q(2n+1)|u|^2 \quad \text{if} \quad p = 0. \tag{8}$$

With the above notations, we can now state the following *Basic Identity* (see [5, 47]).

Theorem 4 (Bochner–Kodaira–Morrey–Kohn–Hörmander) *Let Ω be a proper domain in $\mathbb{CP}^n$ with C^2-smooth boundary $b\Omega$. For any $u \in C^1_{p,q}(\overline\Omega) \cap \mathrm{dom}\,(\overline\partial^*_\varphi)$, we have*

$$\|\overline\partial u\|^2_\varphi + \|\overline\partial^*_\varphi u\|^2_\varphi = \|\overline\nabla u\|^2_\varphi + (\Theta u,\ u)_\varphi + ((i\partial\overline\partial\varphi)u,\ u)_\varphi + \int_{b\Omega}\langle (i\partial\overline\partial\rho)u,\ u\rangle e^{-\varphi}dS \tag{9}$$

where dS is the induced surface element on $b\Omega$, Θ is the curvature term with respect to the Fubini–Study metric and $|\overline\nabla u|^2 = \sum_{j=1}^{n} |\nabla_{\bar L_j} u|^2$.

For a proof of these results, see [47] or Proposition A.5 in the Appendix in [5].

Proposition 1 *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with C^2 boundary and $1 \le q \le n-1$. Then*

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 \geq q(2n+1)\|u\|^2 \tag{10}$$

for any $(0, q)$*-form* $u \in \mathrm{dom}\,(\overline{\partial}) \cap \mathrm{dom}\,(\overline{\partial}^*)$.

Proof This is a direct consequence of the curvature property (8) and (9) with $\varphi = 0$.

$$\begin{aligned}\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 &= \|\overline{\nabla}u\|^2 + (\Theta u,\ u) + \int_{b\Omega} \langle(\partial\overline{\partial}\rho)u,\ u\rangle dS \\ &\geq (\Theta u,\ u) \geq q(2n+1)\|u\|^2.\end{aligned} \tag{11}$$

□

3.1 L^2 Theory of $\overline{\partial}$ for $(0, q)$-Forms

Theorem 5 *Let* Ω *be a pseudoconvex domain in* $\mathbb{CP}^n$ *such that* $\overline{\Omega} \neq \mathbb{CP}^n$ *and* $1 \leq q \leq n-1$. *For any* $\overline{\partial}$*-closed* $(0, q)$*-form* $f \in L^2_{0,q}(\Omega)$*, there exists a* $(0, q-1)$*-form* $u \in L^2_{0,q-1}(\Omega)$ *such that* $\overline{\partial}u = f$ *with*

$$\|u\|^2 \leq \frac{1}{q(2n+1)}\|f\|^2. \tag{12}$$

Proof If Ω has C^2 boundary, estimate (12) is then a consequence of (11). The general case is then proved by exhausting Ω from inside by pseudoconvex domains with smooth boundaries. □

Corollary 2 *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n$ *with diameter* R*, where* $R = \sup_{z,z' \in \Omega} |z - z'|$. *Then for any* $f \in L^2_{p,q}(\Omega)$ *with* $\overline{\partial} f = 0$*, there is* $u \in L^2_{(p,q-1)}(\Omega)$ *such that* $\overline{\partial}u = f$ *with*

$$\|u\|^2 \leq C_{n,q} R^2 \|f\|^2 \tag{13}$$

where $C_{n,q}$ *is a constant depending only on n and q, but is independent of* Ω.

Proof We may assume that $p = 0$. First we assume that Ω has C^2 boundary and its diameter $R < 1$. The estimate (13) follows from (11). The general case follows by exhausting Ω by smooth subdomains with C^2 boundary.

For general bounded domain Ω with radius R, the estimate (13) follows from scaling argument. □

Remark 3 Corollary 2 gives an alternative proof of the Hörmander's L^2 theorem where the constant $C_{n,q} = e/q$ is independent of n. The best constant in (13) is still unknown. It would be interesting to compare the constants.

3.2 Bounded Plurisubharmonic Exhaustion Functions

When the complex manifold is Kähler with positive curvature, we have also the following result of Ohsawa–Sibony [35].

Theorem 6 *Let $\Omega \subset\subset \mathbb{CP}^n$ be a pseudoconvex domain with C^2 boundary $b\Omega$ and let $\delta(x) = d(x, b\Omega)$ be the distance function to $b\Omega$ with the Fubini–Study metric ω. Then there exists $t_0 = t_0(\Omega)$ with $0 < t_0 \le 1$ such that*

$$i\partial\overline{\partial}(-\delta^{t_0}) \ge 0. \tag{14}$$

Remark 4 (1) Theorem 6 also holds for pseudoconvex domains with Lipschitz boundary in $\mathbb{CP}^n$ (see Harrington [19]).

(2) If Ω is a Lipschitz bounded pseudoconvex domain in $\mathbb{C}^n$ or a Stein manifold, Diederich–Forneass [12], Demailly [11] prove earlier that there exists a bounded strictly plurisubharmonic function in Ω (see also Kerzman–Rosay [28] for the C^1 case).

(3) The exponent t_0 in Theorem 6 is closely related to the Levi-flat hypersurfaces in $\mathbb{CP}^n$, see the articles [1, 16].

Based on the existence of the bounded plurisubharmonic function (14), we have the following results (see [4, Theorem 2.3] and [5, Theorems 2.6, 3.5]).

Theorem 7 *Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. Then the $\overline{\partial}$-Neumann Laplacian $\Box$ has a bounded inverse N on $L^2_{p,q}(\Omega)$, where $0 \le p \le n$ and $1 \le q \le n-1$ and for $u \in Dom(\overline{\partial}) \cap Dom(\overline{\partial}^*)$,*

$$C_\eta \|u\|^2 \le \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2. \tag{15}$$

Furthermore, the operator N is bounded from $W^s_{p,q}(\Omega) \to W^s_{p,q}(\Omega)$ with

$$\|\overline{\partial}^* Nu\|^2_s \le C_\eta \|u\|^2_s; \qquad \|\overline{\partial}Nu\|^2_s \le C_\eta \|u\|^2_s \tag{16}$$

for any $u \in W^s_{p,q}(\Omega)$ with $0 < s < t_0/2$.

Problem 3 Let Ω be a bounded pseudoconvex domain in $\mathbb{CP}^n$ with smooth boundary. For every $0 \le p \le n$, $0 < q < n$, prove or disprove

$$H^{p,q}_{W^1}(\Omega) = 0. \tag{17}$$

Remark 5 Suppose Ω is a domain in $\mathbb{C}^n$. We have the following theorem (see [27]).

Theorem 8 (Kohn) *If $\Omega \subset\subset \mathbb{C}^n$ is a bounded pseudoconvex domain with smooth boundary, then*

$$H^{p,q}_{W^s}(\Omega) = 0 \quad \text{for } s \in \mathbb{N} \tag{18}$$

and

$$H^{p,q}(\overline{\Omega}) = 0. \tag{19}$$

Not much is known for Sobolev estimates for $\overline{\partial}$ if we consider Ω to be a smooth pseudoconvex domain in $\mathbb{CP}^n$ except for small $s < \frac{1}{2}$ (see Theorem 7). The lack of Sobolev estimates for $\overline{\partial}$ in the complex projective space $\mathbb{CP}^n$ is one of the most challenging obstacles in many applications (see Problem 5 and the Remark after that).

4 Hartogs Triangles in $\mathbb{CP}^2$

Theorem 7 might not hold if we drop the Lipschitz condition. Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$, not necessarily with Lipschitz boundary.

Consider the Hartogs triangle in $\mathbb{CP}^2$. We denote the homogeneous coordinates by $[z_0, z_1, z_2]$. Let $\mathbb{H}^+$ and $\mathbb{H}^-$ be defined by

$$\mathbb{H}^+ = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| < |z_2|\}$$
$$\mathbb{H}^- = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| > |z_2|\}$$

then $\mathbb{H}^+ \cap \mathbb{H}^- = \emptyset$ and $\overline{\mathbb{H}}^+ \cup \overline{\mathbb{H}}^- = \mathbb{CP}^2$.

Lemma 2 $H^{0,1}_{L^2}(\mathbb{H}^\pm) = 0$.

Proof Let $U_j = \{[z_0, z_1, z_2] \mid z_j \neq 0\}$, $j = 0, 1, 2$. Then $\mathbb{H}^+ \subset U_2$. In local coordinates, $\mathbb{H}^+$ is the product $\mathbb{C}$ and the unit disc. Hence $\mathbb{H}^+$ is pseudoconvex. Similarly, $\mathbb{H}^-$ is also pseudoconvex. The lemma follows from Theorem 5. $\square$

However, The situation is different for (2, 1)-forms. We have the following results.

Theorem 9 *The cohomology group $H^{2,1}_{L^2}(\mathbb{H}^\pm)$ is infinite dimensional.*

The proof of Theorem 9 essentially follows from [32] combining with the recent results in [3]. We shall give a simpler proof here.

Lemma 3 *The Hartogs triangles $\mathbb{H}^+$ and $\mathbb{H}^-$ are Sobolev extension domains.*

Proof The Hartogs triangles are smooth except at the point $[1, 0, 0]$. If we set $z = z_1/z_0$ and $w = z_2/z_0$, then the domain $\mathbb{H}^+$ is defined by the inhomolgeneous coordinates (z, w) by

$$\mathbb{H}^+ = \{z, w) \in \mathbb{C}^2 \mid |z| < |w|\}.$$

The Hartogs triangle $\mathbb{H}^+$ and $\mathbb{H}^-$ are not Lipschitz at $(0, 0)$. At (0,0), the singularity of $\mathbb{H}^+$ and $\mathbb{H}^-$ are the same as the Hartogs triangle T in $\mathbb{C}^2$. Thus the lemma follows from Lemma 1. $\square$

Definition 2 1. Let $\overline{\partial}_s : L^2_{p,q-1}(\mathbb{H}^-) \to L^2_{p,q}(\mathbb{H}^-)$ be the *strong maximal closure* of $\overline{\partial}$ in the sense that $f \in \mathrm{Dom}(\overline{\partial}_s)$ if there exists a sequence $f_\nu \in C^\infty_{p,q-1}(\overline{\mathbb{H}}^-)$ such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ strongly in $L^2(\mathbb{H}^-)$.

2. Let $\overline{\partial}_c : L^2_{p,q-1}(\mathbb{H}^-) \to L^2_{p,q}(\mathbb{H}^-)$ be the *strong minimal closure* of $\overline{\partial}$ in the sense that $f \in \mathrm{Dom}(\overline{\partial}_s)$ if there exists a sequence $f_\nu \in \mathcal{D}_{p,q-1}(\overline{\mathbb{H}}^-)$ such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ strongly in $L^2(\mathbb{H}^-)$.
3. Let $\overline{\partial}_{\tilde{c}} : L^2_{p,q-1}(\mathbb{H}^-) \to L^2_{p,q}(\mathbb{H}^-)$ be the *weak minimal closure* of $\overline{\partial}$ in the sense that $f \in \mathrm{Dom}(\overline{\partial}_c)$ if $\overline{\partial} f \in L^2_{p,q}(\mathbb{H}^-)$ where we have set $f = 0$ on $\mathbb{CP}^2 \setminus \mathbb{H}^-$.

The operators $\overline{\partial}$ and $\overline{\partial}_c$ are dual to each other. And $\overline{\partial}_s$ and $\overline{\partial}_{\tilde{c}}$ are dual to each other for rectifiable domains (see [30]).

Lemma 4 (Weak equals Strong) *On* $\mathbb{H}^-$, *we have*

1. $\overline{\partial} = \overline{\partial}_s$;
2. $\overline{\partial}_c = \overline{\partial}_{\tilde{c}}$.

Proof It is proved in Theorem 3.13 in [3] that $\overline{\partial}$ and $\overline{\partial}_s$ are the same on the Hartogs triangle T. The proof of (1) is the same. Thus $\overline{\partial}_c = \overline{\partial}_{\tilde{c}}$ by duality. □

From Lemma 4, there is no need to distinguish $\overline{\partial}_c$ and $\overline{\partial}_{\tilde{c}}$. From now on, we will only use $\overline{\partial}$ and $\overline{\partial}_c$. Define the L^2 Dolbeault cohomolgy group with respect to $\overline{\partial}_c$ as follows:

$$H^{p,q}_{\overline{\partial}_c, L^2}(\mathbb{H}^-) = \frac{\mathrm{Ker}(\overline{\partial}_c)}{\mathrm{Range}(\overline{\partial}_c)}.$$

The following lemma is proved in Proposition 6 in [8].

Lemma 5 *Let* $\mathbb{H}^+ \subset \mathbb{CP}^2$ *be the Hartogs' triangle. Then we have the following:*

1. The Bergman space of L^2 *holomorphic functions* $L^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ *on the domain* $\mathbb{H}^+$ *separates points in* $\mathbb{H}^+$.
2. There exist nonconstant functions in the space $W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$. *However, this space does not separate points in* $\mathbb{H}^+$ *and is not dense in the Bergman space* $L^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$.
3. Let $f \in W^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ *be a holomorphic function on* $\mathbb{H}^+$ *which is in the Sobolev space* $W^2(\mathbb{H}^+)$. *Then* f *is a constant.*

Theorem 10 *Let* $\mathbb{H}^- \subset \mathbb{CP}^2$ *be the Hartogs' triangle. Then the cohomology group* $H^{0,1}_{\overline{\partial}_c, L^2}(\mathbb{H}^-) \neq 0$ *and is infinite dimensional.*

Proof From Lemma 5, the space of holomorphic functions in $W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ is infinite dimensional. In the non-homogeneous coordinates $\mathbb{H}^+ = \{(z, w) \in \mathbb{C} \mid |z| < |w|\}$. Let

$$f_k = \left(\frac{z}{w}\right)^k, \qquad k \in \mathbb{N}.$$

Then $f_k \in W^1(\mathbb{H}^+)$.

Using Lemma 1, each holomorphic function f_k can be extended to a function $\tilde{f}_k \in W^1(\mathbb{CP}^2)$. Suppose that $H^{0,1}_{\overline{\partial}_c, L^2}(\overline{\mathbb{H}}^-) = 0$. Then we can solve $\bar{\partial}_{\tilde{c}} u_k = \overline{\partial} \tilde{f}_k$ in $\mathbb{CP}^2$

with prescribed support for u_k in $\overline{\mathbb{H}^-}$. Let $H_k = \tilde{f}_k - u_k$. Then H_k is a holomorphic function in $\mathbb{CP}^2$, hence a constant. But $H_k = f_k$ on $\mathbb{H}^+$, a contradiction. This implies that the space $H^{0,1}_{\overline{\partial}_c, L^2}(\overline{\mathbb{H}^-})$ is non-trivial.

Next we prove that $H^{0,1}_{\overline{\partial}_c, L^2}(\overline{\mathbb{H}^-})$ is infinite dimensional. Each function $\tilde{f}_k$ corresponds to a (0,1)-form $\overline{\partial}\tilde{f}_k$. We set $g_k = \overline{\partial}\tilde{f}_k$. Then g_k is in $\mathrm{Dom}(\overline{\partial}_{\tilde{c}})$ and satisfies $\overline{\partial}_c g_k = 0$. Thus it induces an element $[g_k]$ in $H^{0,1}_{\overline{\partial}_c, L^2}(\overline{\mathbb{H}^-})$. To see that $[g_k]$'s are linearly independent, let $N > 1$ be a positive integer and $F_N = \sum_{k=1}^N c_k f_k$, where c_k are constants. Set $G_N = \sum_{k=1}^N c_k g_k$. Suppose that $[G_N] = 0$, then we can solve $\overline{\partial}_{\tilde{c}} u = G_N$ and the function F_N holomorphic in $\mathbb{H}^+$ extends holomorphically to $\mathbb{CP}^2$. Thus F_N must be a constant and $c_1 = \cdots = c_N = 0$. Thus $[g_k]$'s are linearly independent. This proves that $H^{0,1}_{\overline{\mathbb{H}^-}, L^2}(\mathbb{CP}^2)$ is infinite dimensional. □

Lemma 6 *The range of*

$$\overline{\partial} : L^2_{2,1}(\mathbb{H}^-) \to L^2_{2,2}(\mathbb{H}^-) \tag{20}$$

is closed and equal to $L^2_{2,2}(\mathbb{H}^-)$.

Proof It is clear that $\overline{\partial}$ has closed range in the top degree and the range is $L^2_{2,2}(\mathbb{H}^-)$. Let $f \in L^2_{2,2}(\mathbb{H}^-)$. We extend f to be zero outside $\mathbb{H}^-$. Let U be an open neighbourhood of $\overline{\mathbb{H}^-}$, then f is in $L^2_{2,2}(U)$. We can choose U such that $\overline{U}$ is a proper subset of $\mathbb{CP}^2$ and U has Lipschitz boundary. Since one can solve the $\overline{\partial}$ equation for top degree forms on U, there exists $u \in L^2_{2,1}(U)$ such that $\overline{\partial} u = f$ in the weak sense. □

Corollary 3 *The cohomology group* $H^{0,1}_{\overline{\partial}_c, L^2}(\overline{\mathbb{H}^-})$ *is Hausdorff and infinite dimensional.*

Corollary 4 *The cohomology group* $H^{2,1}_{L^2}(\mathbb{H}^-)$ *is infinite dimensional.*

Proof Suppose that $\overline{\partial} : L^2_{2,0}(\mathbb{H}^-) \to L^2_{2,1}(\mathbb{H}^-)$ does not have closed range. Then $H^{2,1}_{\overline{\partial}, L^2}(\mathbb{H}^-)$ is non-hausdorff, hence infinite dimensional.

Suppose that $\overline{\partial} : L^2_{2,0}(\mathbb{H}^-) \to L^2_{2,1}(\mathbb{H}^-)$ has closed range. Using Lemma 6, $\overline{\partial} : L^2_{2,1}(\mathbb{H}^-) \to L^2_{2,2}(\mathbb{H}^-)$ has closed range. From Lemma 4 and the L^2 Serre duality, we have $\overline{\partial}_c : L^2(\mathbb{H}^-) \to L^2_{0,1}(\mathbb{H}^-)$ and $\overline{\partial}_c : L^2_{0,1}(\mathbb{H}^-) \to L^2_{0,2}(\mathbb{H}^-)$ both have closed range.

Furthermore, from L^2 Serre duality (see [8]),

$$H^{2,1}_{\overline{\partial}, L^2}(\mathbb{H}^-) \cong H^{0,1}_{\overline{\partial}_c, L^2}(\overline{\mathbb{H}^-}). \tag{21}$$

Thus from Theorem 10, it is infinite dimensional. □

Since $\mathbb{H}^-$ and $\mathbb{H}^+$ have the same properties, we have proved Theorem 9.

Remark 6 From Lemma 2, $H^{0,1}_{L^2}(\mathbb{H}^+) = 0$. Thus p plays important role for the L^2 cohomology groups on Hartogs triangles in $\mathbb{CP}^2$. The reason is that when $p > 0$, the curvature term Θ is only nonnegative, in contrast to the case of (0, 1)-forms.

Problem 4 The following questions remain open:

1. Determine if $H^{2,1}(\mathbb{H}^-)$ is Hausdorff.
2. Determine if $H^{1,1}_{L^2}(\mathbb{H}^-)$ is Hausdorff or nonzero.

5 Levi-Flat Hypersurfaces in $\mathbb{CP}^n$

A C^2-smooth hypersurface M in a complex manifold of dimension n is called *Levi-flat* if its Levi form vanishes on M. A Levi-flat hypersurface is foliated by complex manifolds of dimension $n-1$ We can also define a Lipschitz (or C^1) hypersurface M to be *Levi-flat* if it is locally foliated by complex manifolds of complex dimension $n-1$ (see [2, 6]). More generally, we can define Levi-flat hypersurfaces in $\mathbb{CP}^n$ as follows.

Definition 3 A compact connected hypersurface M in $\mathbb{CP}^n$ is said to be *Levi-flat* if $\mathbb{CP}^n \setminus M$ consists of two pseudoconvex domains.

Definition 3 is a natural generalization of Levi-flatness to non-smooth hypersurfaces. We refer the reader to an extensive survey paper of Levi-flat hypersurfaces by Ohsawa [34].

Theorem 11 *There exist no Lipschitz Levi-flat hypersurfaces in $\mathbb{CP}^n$ for $n \geq 3$.*

Lins–Neto [29] first proved the nonexistence of real-analytic Levi-flat hypersurfaces in $\mathbb{CP}^n$ with $n \geq 3$. The nonexistence of smooth Levi-flat hypersurfaces in $\mathbb{CP}^n$ with $n \geq 3$ was established by Siu [44]. The nonexistence of Lipschitz Levi-flat hypersurfaces in $\mathbb{CP}^n$ with $n \geq 3$ was proved in Cao–Shaw [6].

Problem 5 Does there exist smooth (or Lipschitz) Levi-flat hypersurface in $\mathbb{CP}^2$?

In [5], it was stated that there exist no C^2 Levi-flat hypersurfaces in $\mathbb{CP}^n$ for all $n \geq 2$, but the proof only works for $n \geq 3$. An affirmative answer to Problem 3 will bridge the gap of our paper.

Remark 7 Let Ω be a bounded pseudoconvex domain in $\mathbb{CP}^n$ with Lipschitz boundary. If we consider the complement $\Omega^+ = \mathbb{CP}^n \setminus \overline{\Omega}$, then

$$\overline{\partial} : W^1_{p,q-1}(\Omega^+) \to W^1_{p,q}(\Omega^+)$$

has closed range for all $1 \leq q \leq n-1$. For the domain Ω^+ with pseudoconcave boundary, we only need that $b\Omega^+$ to be Lipschitz (see [17]). When $n \geq 3$ and $1 \leq q < n-1$, one even has $H^{p,q}_{W^1}(\Omega^+) = 0$ if $p \neq q$. This is why one can prove Theorem 11 in [6] since one can solve $\overline{\partial}$ in Sobolev spaces on pseudoconcave domains with Lipschitz boundary when $n \geq 3$.

There exist many non-Lipschitz Levi-flat hypersurfaces in $\mathbb{CP}^2$. The boundary of the Hartogs triangles is such an example (see Henkin–Iordan [20]).

Example The boundary $M = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| = |z_2|\}$ of the Hartogs triangles $\mathbb{H}^{\pm}$ provides an example of a singular Levi-flat hypersurface in $\mathbb{CP}^2$ since $\mathbb{CP}^2 \setminus M$ are two pseudoconvex domains. We write M by

$$M = \cup_{\theta \in \mathbb{R}} S_\theta$$

where

$$S_\theta = \{[z_0, z_1, z_2] \mid z_1 = e^{i\theta} z_2\}, \qquad \theta \in \mathbb{R}.$$

Each S_θ is a compact Riemann surface. Thus M is foliated by the Riemann surface S_θ except at $[1, 0, 0]$. It is not a foliation near $[1, 0, 0]$ since each leaf S_θ passes through that point.

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Dynamical Systems of p-Bergman Kernels

Hajime Tsuji

Abstract In this paper, we consider the dynamical system of p-Bergman kernels. This is useful when we consider a projective manifold with intermediate Kodaira dimension.

Keywords Dynamical systems of Bertman kernels · Kähler-Einstein metrics · L^2-estimates · Canonical measures

1 Introduction

In complex geometry, there has been introduced several intrinsic (pseudo) volume forms on complex manifolds (cf. [2, 6, 11, 28]).

In [21, 25], we considered the approximation of Kähler-Einstein forms by Bergman kernels.

The Bergman kernel is the reproduction kernel of the Hilbert space of L^2-holomorphic functions. The diagonal part of the Bergman kernel is characterized as the pointwise supremum of the norm square of the L^2-holomorphic functions whose L^2-norm is less than or equal to 1.

Similarly the p-Bergman kernel is defined as the supremum of the norm square of the L^p-holomorphic functions whose L^p-norm is less than or equal to 1. The main difference of p-Bergman kernels and usual Bergman kernels is that p-Bergman kernel is not related to reproduction kernels. Since L^p-spaces are difficult to handle in comparison with L^2-sapces, p-Bergman kernels are considered to be less interesting than usual Bergman kernels.

However we see that to handle pluricanonical forms there is several advantages to use p-Bergman kernels. For example for a compact complex manifold X with ample canonical bundle, the space of L^2-canonical forms is nothing but $H^0(X, K_X)$ and it does not contain enough information, unless K_X is very ample. Hence to study a canonically polarized manifold X we need to consider the graded ring

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$\oplus_{m=0}^{\infty} H^0(X, mK_X)$ instead of $H^0(X, K_X)$. We note that $H^0(X, mK_X)$ has the natural $L^{\frac{2}{m}}$-structure with respect to the pseudonorm:

$$\| \sigma \| = \left| \int_X (\sigma \wedge \overline{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}.$$

Hence it is natural to consider the $L^{\frac{2}{m}}$-space of pluricanonical forms to handle compact Kähler manifolds with pseudoeffective canonical bundles as in [16].

In this paper, we shall consider the approximation of Kähler-Einstein forms and twisted Kähler-Einstein forms by p-Bergman kernels.

2 Some Invariant Volume Forms

In this section, we shall summarize the definitions of invariant volume forms on a bounded pseudoconvex domain and their generalization to compact complex manifolds. These invariant volume forms are used in this paper.

2.1 Bergman Volume Forms

First we shall review Bergman volume forms.

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let K_Ω denote the canonical bundle of Ω. Then the space of L^2-canonical forms on Ω:

$$A^2(\Omega, K_\Omega) = \left\{ \sigma \in \Gamma(\Omega, \mathcal{O}_\Omega(K_\Omega)) \,\middle|\, \int_\Omega |\sigma|^2 < +\infty \right\} \quad (|\sigma|^2 = (\sqrt{-1})^{n^2} \sigma \wedge \overline{\sigma})$$

is a Hilbert space with respect to the inner product:

$$(\sigma, \tau) := (\sqrt{-1})^{n^2} \int_\Omega \sigma \wedge \overline{\tau}.$$

We set

$$K(\Omega)(x) := \sup\{|\sigma(x)|^2 \mid \sigma \in A^2(\Omega, K_\Omega),\ \|\sigma\| = 1\}(x \in \Omega), \tag{2.1}$$

where $\| \sigma \|$ denotes the L^2-norm of σ. We call $K(\Omega)$ *the Bergman volume form* on Ω. This definition is a little bit different from the usual one. But this definition coincides with the usual definition:

$$K(\Omega) := \sum_i |\sigma_i|^2,$$

where $\{\sigma_i\}$ is a complete orthonomal basis of $A^2(\Omega, K_\Omega)$ (This definition is independent of the choice of the complete orthonormal basis $\{\sigma_i\}$). Then by [12],

$$\omega_B := -\mathrm{Ric}\, K(\Omega) \tag{2.2}$$

is the pull back of the Fubini-Study Kähler form by the embedding

$$\Phi : \Omega \longrightarrow \mathbb{P}^\infty$$

defined by

$$\Phi(x) := [\sigma_1(x) : \sigma_2(x) : \cdots : \sigma_k(x) : \cdots],$$

where

$$\mathrm{Ric}\, K(\Omega) = -\sqrt{-1}\partial\bar{\partial}\log K(\Omega).$$

Hence ω_B is C^∞ and

$$\omega_B > 0 \tag{2.3}$$

holds, i.e., ω_B is a C^∞ Kähler form on Ω.

The Bergman volume form can be generalized to the case of compact complex manifolds as follows. Let X be a compact complex manifold of dimension n and let (L, h_L) be a singular hermitian line bundle (cf. Definition 1 below) on X. Let $\{\sigma_j\}$ be a complete orthonormal basis of $H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(h_L))$ with respect to the L^2-inner product:

$$(\sigma, \tau) := (\sqrt{-1})^{n^2} \int_X \sigma \wedge \overline{\tau} \cdot h_L,$$

where $\mathcal{J}(h_L)$ denotes the multiplier ideal sheaf with respect to the singular hermitian metric h_L (cf, Definition 1 and [15]) and set

$$K(X, K_X + L, h_L) := \sum_j |\sigma_j|^2, \tag{2.4}$$

where $|\sigma|^2 = (\sqrt{-1})^{n^2}\sigma \wedge \overline{\sigma} (n = \dim X)$. Then $K(X, K_X + L, h_L)$ is a semipositive $|L|^2 = L \otimes \bar{L}$-valued (n, n)-form on X such that $K(X, K_X + L, h_L)^{-1}$ is a singular hermitian metric on $K_X + L$ with semipositive curvature current, unless it is not identically $+\infty$. As in the usual Bergman kernel, the extremal property:

$$\begin{aligned} K(X, &K_X + L, h_L)(x) \\ &:= \sup\{|\sigma(x)|^2 \mid \sigma \in H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(h_L)), \|\sigma\| = 1\} \end{aligned} \tag{2.5}$$

holds.

2.2 Kähler-Einstein Volume Forms

On the other hand, there exists another invariant volume form on a bounded pseudoconvex domain in $\mathbb{C}^n$. For a Kähler form $\omega = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$, we define

$$\mathrm{Ric}_\omega := -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$$

and call it the **Ricci form** of ω. More generally for a volume form $dV := \phi \cdot 2^{-n}|dz_1 \wedge \cdots dz_n|^2$, we define

$$\mathrm{Ric}\, dV := -\sqrt{-1}\partial\bar{\partial}\log\phi$$

and call it the Ricci form of dV. We note that $-(2\pi)^{-1}\mathrm{Ric}_\omega$ is the 1-st Chern form on the canonical bundle with respect to ω.

Theorem 1 ([6, 14]) *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then there exists a unique complete Kähler form ω_E on Ω such that*

$$-\,Ric(\omega_E) = \omega_E$$

holds on Ω.

By definition, we see that **the Kähler-Einstein volume form**:

$$dV_E := \frac{1}{n!}\omega_E^n$$

satisfies

$$-\,\mathrm{Ric}(dV_E) = \omega_E > 0. \tag{2.6}$$

In the compact case, we have the following well known existence theorem,

Theorem 2 ([1, 28]) *Let X be a smooth projective variety with ample K_X. Then there exists a unique C^∞-Kähler form ω_E such that $-Ric(\omega_E) = \omega_E$ holds on X.*

2.3 Supercanonical Volume Form

First we shall recall several definitions.

Definition 1 Let L be a holomorphic line bundle on a complex manifold M. A **singular hermitian metric** h_L on L is given by

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{\mathrm{loc}}(M)$ is an arbitrary function on M. We call φ the weight function of h_L with respect to h_0. Let Θ_{h_0} be the curvature of h_0. We define the curvature current Θ_h of h by

$$\Theta_h := \Theta_{h_0} + \partial\bar{\partial}\varphi,$$

where $\partial\bar{\partial}$ is taken in the sense of current. We define the **multiplier ideal sheaf** of (L, h_L) by

$$\mathcal{J}(h_L)(U) = \{f \in \mathcal{O}(U) \mid |f|^2 \cdot e^{-\varphi} \in L^1_{\mathrm{loc}}(U)\},$$

where φ is the weight function as above.

Definition 2 Let L be a line bundle on a compact complex manifold M. L is said to be **pseudoeffective**, if there exists a singular hermitian metric h_L on L such that $\sqrt{-1}\Theta_{h_L}$ is a closed (semi)positive $(1,1)$-current. A singular hermitian line bundle (F, h_F) is said to be pseudoeffective, if $\sqrt{-1}\Theta_{h_F}$ is a closed (semi) positive $(1,1)$ current.

The following notion is useful.

Definition 3 Let M be a compact complex manifold and let L be a line bundle on X. A singular hermitian metric h_L is said to be an **AZD (Analytic Zariski decomposition)** of L, if the followings are satisfied:

(1) $\sqrt{-1}\Theta(h_L) \geqq 0$ holds on M, where $\Theta(h_L)$ denotes the curvature current of h_L,
(2) $H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h_L^m)) \simeq H^0(M, \mathcal{O}_M(mL))$ holds for every $m \geqq 0$.

We note that since the uppersemicontinuous envelope of the supremum of a family of plurisubharmonic functions uniformly bounded from above is again plurisubharmonic by the theorem of P. Lelong (cf. [13, p. 26, Theorem 5]), we see that an AZD exists for every pseudoeffective line bundle on a compact complex manifold.

In [23], we have introduced the canonical volume form on a smooth projective variety with pseudoeffective canonical bundle.

Let X be a smooth projective n-fold such that the canonical bundle K_X is pseudoeffective. Let A be a sufficiently ample line bundle such that for every pseudoeffective singular hermitian line bundle (L, h_L) (cf, Definition 2) on X, $\mathcal{O}_X(A+L) \otimes \mathcal{I}(h_L)$ and $\mathcal{O}_X(K_X + A + L) \otimes \mathcal{I}(h_L)$ are globally generated. The existence of such an ample line bundle A follows from Nadel's vanishing theorem ([15, p.561]).

For every $x \in X$ we set

$$\hat{K}^A_m(x) := \sup\left\{ |\sigma(x)|^{\frac{2}{m}} \,\middle|\, \sigma \in \Gamma(X, \mathcal{O}_X(mK_X + A)), \|\sigma\|^{\frac{1}{m}} = 1 \right\}, \tag{2.7}$$

where

$$\| \sigma \|_{\frac{1}{m}} := \left| \int_X h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}. \tag{2.8}$$

Here $\mid \sigma \mid^{\frac{2}{m}}$ is not a function on X, but the supremum is takan as a section of the real line bundle $\mid A \mid^{\frac{2}{m}} \otimes \mid K_X \mid^2$ in the obvious manner.[1] Then $h_A^{\frac{1}{m}} \cdot \hat{K}_m^A$ is a continuous semipositive (n,n)-form on X. This construction is similar to the [16], where there is no twist by (A, h_A), Under the above notations, we have the following theorem.

Theorem 3 ([23]) *We set*

$$\hat{K}_\infty^A := \limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot \hat{K}_m^A \tag{2.9}$$

and

$$\hat{h}_{can,A} := \textit{the lower envelope of } (\hat{K}_\infty^A)^{-1}. \tag{2.10}$$

Then $\hat{h}_{can,A}$ is an AZD of K_X. And we define

$$\hat{h}_{\rm can} := \textit{the lower envelope of } \inf_A \hat{h}_{{\rm can},A}, \tag{2.11}$$

where inf *denotes the pointwise infimum and A runs all the ample line bundles on X. Then $\hat{h}_{can}$ is a well defined AZD (see Definition 3) on K_X with minimal singularities depending only on X.*

Remark 1 As one sees later (see the proof of Theorem 1.11), $\hat{h}_{can,A} = \hat{h}_{can}$ holds for every sufficiently ample A.

Definition 4 Let X be a smooth projective variety with pseudoeffective canonical bundle. Let $\hat{h}_{can}$ be as in Theorem 3. We set

$$d\hat{\mu}_{\rm can} := \hat{h}_{\rm can}^{-1}$$

and call it the **supercanonical measure**.

2.4 *Extremal Measures*

In [22] we have introduced a new intrinsic volume form on a compact Kähler manifold. This construction also works for a bounded pseudoconvex domain.

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. We shall introduce the invariant volume form on Ω by

$$d\mu_{\rm ext}(\Omega)(x) := \left(\sup\left\{dV(x) \middle| -{\rm Ric}\, dV \geqq 0, \int_\Omega dV = 1\right\}\right)^* (x \in \Omega), \tag{2.12}$$

where dV runs all the uppersemicontinuous semipostive (n,n)-forms such that dV^{-1} is a singular hermitian metric on K_Ω and $-{\rm Ric}\, dV = \sqrt{-1}\partial\bar{\partial}\log dV \geqq 0$

[1] We have abused the notations $|A|$, $|K_X|$ here. These notations are similar to the notations of corresponding linear systems. But we shall use the notation if without fear of confusion.

in the sense of current and $(\ )^*$ denotes the uppersemicontinuous envelope. We call $d\mu_{\text{ext}}(\Omega)$ **the extremal measure** of Ω. Then we see that

$$-\operatorname{Ric} d\mu_{\text{ext}}(\Omega) \geqq 0 \tag{2.13}$$

holds in the sense of current.

This definition can be generalized to the case of compact complex manifolds with pseudoeffective canonical bundles [22].

2.5 Equivalence of the Extremal Maasure and the Supercanonical Measure

The following theorem is in [27].

Theorem 4 *Let X be a smooth projective variety with pseudoeffective canonical bundle. Then $d\hat{\mu}_{\text{can}}(X) = d\mu_{\text{ext}}(X)$ holds.*

Remark 2 If X is of general type, then Theorem 4 has already been proven in [3, Sect. 5].

Theorem 4 means that the extremal measure can be approximated by the supremum of the norm square of the mth root of m-ple canonical forms.

3 Dynamical System of p-Bergman Kernels (The Case of Canonically Polarized Manifolds)

In this section we shall approximate the Kähler-Einstein volume form on a canonically polarized manifold by a dynamical system of p-Bergman kernels.

3.1 p-Bergman Kernels

Let D be a bounded domain in $\mathbb{C}^n$ and $p > 0$. By using the extremal property of the Bergman kernel we set

$$K_p(z) := \left(\sup\left\{|f(z)|^2 \mid f \in \mathcal{O}(\Omega), \int_\Omega |f(w)|^p d\mu = 1\right\}\right)^*,$$

where $d\mu$ is the standard Lebesgue measure on $\mathbb{C}^n$ and $(\ )^*$ denotes the uppersemicontinuous envelope. We call K_p the p-Bergman kernel of Ω and $(\ )^*$ denotes the uppersemicontinuous envelope. Since K_p is the uppersemicontinuous envelope of the supremum of plurisubharmonic functions, K_p is plurisubharmonic on Ω ([13, p. 26, Theorem 5]).

Let X be compact complex manifold and let (L, h_L) be a singular hermitian line bundle on X. Let p be a positive integer. We set

$$K_{\frac{2}{p}}(X, K_X + L, h_L)(x) := \left(\sup\left\{|\sigma(x)|^{\frac{2}{p}} \mid \sigma \in H^0(X, \mathcal{O}_X(p(K_X + L)) \otimes \mathcal{J}(h_L^p)), \int_X |\sigma|^{\frac{2}{p}} h_L = 1\right\}\right)^*$$

and call it the $\frac{2}{p}$-Bergman kernel of the adjoint bundle $K_X + L$ with respect to h_L.

3.2 Dynamical System of p-Bergman Kernels

Let X be a smooth projective manifold of dimension n with ample K_X. Then by [1, 28], we see that there exists a C^∞ Kähler form ω_E such that

$$-\operatorname{Ric}(\omega_E) = \omega_E$$

We set

$$dV_E := \frac{\omega_E^n}{n!}$$

and call it the Kähler-Einstein volume form on X.

And let A be a sufficiently ample line bundle such that for every pseudoeffective singular hermitian line bundle (L, h_L), the sheaf $\mathcal{O}_X(A + L) \otimes \mathcal{J}(h_L)$ is globally generated on X. Let $p \geqq 2$ be a positive integer, We set

$$K_1 := K_{\frac{2}{p}}(X, p(A + K_X), h_A^p)$$

and inductively

$$K_m^{(p)} := K_{\frac{2}{p}}(X, p(A + mK_X), K_{m-1}^{-1})$$

for $m \geqq 2$. Then we have the following theorem.

Theorem 5 *Let X, $\{K_m^{(p)}\}_{m=1}^\infty$ be as above. Then*

$$\lim_{m\to\infty} h_A^{-\frac{1}{pm}} \cdot ((m!)^{-1} K_m^{(p)})^{\frac{1}{m}} = \frac{1}{(2\pi)^n} dV_E$$

where $n = \dim X$ and dV_E is the Kähler-Einstein volume form on X with constant negative Ricci curvature -1.

Proof By definition, we see that

$$K_m^{(1)} \leqq K_m^{(p)}$$

holds for every $m \geqq 1$. This is simply because the test functions to define $K_m^{(p)}$ is of the form $|(\sigma \wedge \overline{\sigma})^{\frac{1}{p}}|(\sigma \in H^0(X, \mathcal{O}_X(p(A + mK_X)))$ than that of $K_m^{(1)}$ of the form $|\tau \wedge \overline{\tau}|(\tau \in H^0(X.\mathcal{O}_X(A + mK_X))$ (τ^p is a test function of $K_m^{(p)}$). □

Now let us consider another dynamical system

$$d\mu_1 = d\mu_{ext}(X, A + K_X, h_A)$$

and inductively

$$d\mu_m = d\mu_{ext}(X, A + mK_X, \left(\prod_{i=1}^{m-1} d\mu_i)^{-1} \cdot h_A\right)$$

for $m \geqq 2$. Then by definition we see that

$$K_1^{(p)} \leqq h_A^{-1} \cdot d\mu_1$$

and by induction on m, we see that

$$K_m^{(p)} \leqq h_A^{-1} \cdot \prod_{i=1}^{m} d\mu_i$$

holds for every $m \geqq 1$. We set

$$K_m^{(\infty)} := h_A^{-1} \cdot \prod_{i=1}^{m} d\mu_i.$$

Hence we have the following lemma.

Lemma 1 *The inequalities*

$$K_m^{(1)} \leqq K_m^{(p)} \leqq K_m^{(\infty)}$$

hold on X.

Now we quote the following theorems:

Theorem 6 ([21])

$$\lim_{m\to\infty} h_A^{\frac{1}{m}} \cdot \sqrt[m]{(m!)^{-1} K_m^{(1)}} = \frac{1}{(2\pi)^n} dV_E$$

holds on X.

Theorem 7 ([27])

$$\lim_{m\to\infty} h_A^{\frac{1}{m}} \cdot \sqrt[m]{(m!)^{-1} K_m^{(\infty)}} = \frac{1}{(2\pi)^n} dV_E$$

Combining Theorems 6 and 7, by Lemma 1. we see that

$$\lim_{m\to\infty} h_A^{-\frac{1}{pm}} \cdot ((m!)^{-1} K_m^{(p)})^{\frac{1}{m}} = \frac{1}{(2\pi)^n} dV_E$$

holds. This completes the proof of Theorem 5.

4 Canonical Measures and p-Bergman Kernels

In this section, we consider the twisted version of Theorem 5. Let X be a compact complex manifold. We set

$$\kappa(X) := \limsup_{m\to\infty} \frac{\log \dim H^0(X, mK_X)}{\log m}$$

and call it the **Kodaira dimension** of X. $\kappa(X) = -\infty$ or integer between 0 and $\dim X$. If $\kappa(X) \geqq 0$, for sufficiently large m $|m!K_X|$ gives a rational fibration $f : X \dashrightarrow Y$. For every sufficiently large m, this fibration is birationally stable. We call this fibration the Iitaka fibration of X. It is easy to see $\dim Y = \kappa(X)$.

4.1 Hodge Bundles

Let X be a smooth projective manifold with $\kappa(X) \geqq 0$ and let $f : X \to Y$ be an Iitaka fibration of X such that f is a morphism. Let $K_{X/Y} := K_X \otimes f^* K_Y^{-1}$ be the relative canonical bundle of $f : X \to Y$. We say that

$$L_{X/Y} = \frac{1}{m!} \left(f_* \mathcal{O}_X(m! K_{X/Y}) \right)^{**} \in \mathrm{Pic}(Y) \otimes \mathbb{Q}$$

the **Hodge $\mathbb{Q}$-line bundle**, where $**$ denotes the double dual.

We define the singular hermitian metric on $L_{X/Y}$ by

$$h_{X/Y}^{m!}(\sigma, \sigma) = \left| \int_{X/Y} (\sigma \wedge \bar{\sigma})^{\frac{1}{m!}} \right|$$

We call $h_{X/Y}$ the **Hodge metric** on $L_{X/Y}$. By the curvature computation by P.A. Griffith we see that

$$\sqrt{-1}\Theta_{h_{X/Y}} \geqq 0$$

holds in the sense of current.

4.2 Canonical Measure

The canonical measure is a substitute of Kähler-Einstein metrics for projective manifolds with nonnegative Kodaira dimension. Let $f : X \to Y$ be an Iitaka fibration such that f is a morphism.

Definition 5 A closed positive (1, 1)-current ω_Y satisfying:

(1) ω_Y is a C^∞ Kähler form on a nonempty Zariski open subset U on Y,
(2) On U, ω_Y satisfies the equation:

$$-\operatorname{Ric}(\omega_Y) + \sqrt{-1}\Theta_{h_{X/Y}} = \omega_Y,$$

where $\Theta_{h_{X/Y}}$ denotes the curvature current of $h_{X/Y}$,
(3) Let dV_Y be the volume form associated with ω_Y. Then $dV_Y^{-1} \otimes h_{X/Y}$ is a singular hermitian metric on $K_Y + L_{X/Y}$ with minimal singularities

is said to be the **twisted Kähler-Einstein form** on Y.

Theorem 8 ([17, 26]) *Let X be a smooth projective variety with $\kappa(X) \geqq 0$ and let $f : X \to Y$ There exists a unique twisted Kähler-Einstein current on Y.*

Definition 6 Let $f : X \to Y$ be an Iitaka fibration and let ω_Y be the twisted Kähler-Einstein current on Y. Let dV_Y denote the volume form of ω_Y. We identify $h_{X/Y}^{-1}$ as an relative volume form on $f : X \to Y$ and we set

$$d\mu_X := f^*(dV_Y \otimes h_{X/Y}^{-1}).$$

We call $d\mu_X$ the **canonical measure** on X. $d\mu_X$ is a degenerate volume form on X.

4.3 Dynamical System of p-Bergman Kernels (The Case of Compact Kähler Manifolds with Kodaira Dimension

Let X be a smooth projective variety with $\kappa(X) \geqq 0$ and let $f : X \to Y$ an Iitaka fibration of X such that f is a morphism. By taking a suitable modification, we may and do assume that f is a morphism and Y is smooth. Let $(L_{X/Y}, h_{X/Y})$ be the Hodge $\mathbb{Q}$-line bundle associated with $f : X \to Y$. Let p be a positive integer such that $pL_{X/Y} \in \mathrm{Pic}(Y)$. Hereafter we shall assume $p \geqq 2$ (if $p = 1$, then the dynamical system is essentially same as in the previous section). Let A be a sufficiently ample line bundle on Y such that $\mathcal{O}_Y(A + F) \otimes \mathcal{J}(h_F)$ is globally generated for every pseudoeffective singular hermitian line bundle (F, h_F) (such an ample line bundle A exists by Nadel's vanishing theorem [15, p.561]). We define the dynamical system of p-Bergman kernels by

$$K_1^{(p)} := K_{\frac{2}{p}}(X, p(K_Y + L_{X/Y} + A), h_{X/Y} \cdot h_A)$$

and inductively for $m \geqq 2$

$$K_m^{(p)} := K_{\frac{2}{p}}(X, mp(K_Y + L_{X/Y}) + pA, h_{X/Y} \cdot (K_{m-1}^{(p)})^{-1}). \tag{4.1}$$

The following theorem is the main result in this paper.

Theorem 9 *Let $f : X \to Y$ be the Iitaka fibration and $\{K_m\}$ be as above. Let κ denote the Kodaira dimension $\kappa(X)(= \dim Y)$ of X. And let Y° denote the locus where the twisted Kähler-Einstein current ω_Y is C^∞ and strictly positive.*

Then

$$\lim_{m\to\infty} \left((m!)^{-\kappa} \cdot h_A^{\frac{1}{m}} K_m^{(p)}\right)^{\frac{1}{m}} = \frac{1}{(2\pi)^\kappa} dV_Y \cdot h_{X/Y}^{-1}$$

holds. Here the convergence is globally L^1 and uniform on every compact subset of Y°.

4.4 Dynamical System of Bergman Kernels for Twisted Kähler-Einstein Volume Forms

To prove Theorem 9, we first consider a dynamical system of usual Bergman kernels.

Since the Hodge bundle $L_{X/Y}$ is not a genuine line bundle, we cannot take the Bergman kernel of $K_Y + L_{X/Y}$ in general. This is the main difficulty to approximate the twisted Kähler-Einstein volume forms. To overcome this difficulty we shall modify the dynamical systems.

Let A be a sufficiently ample line bundle on Y and let h_A be a C^∞-hermitian metric on A with strictly positive curvature. Let p be a positive integer such that $pL_{X/Y}$ is a genuine line bundle on Y. We set

$$K_1 = K(Y, K_Y + A, h_A)$$

and for $m \geqq 2$ inductively we define

$$K_m = \begin{cases} K(Y, mK_Y + \lfloor \frac{m}{p} \rfloor pL_{X/Y}, K_{m-1}^{-1} \cdot h_{X/Y}^p), & \text{if } p|m, \\ \\ K(Y, mK_Y + \lfloor \frac{m}{p} \rfloor pL_{X/Y}, K_{m-1}^{-1}), & \text{if } p \nmid m, \end{cases} \tag{4.2}$$

where $\lfloor a \rfloor$ denotes the greatest integer which do not exceed a.

Theorem 10

$$\lim_{m\to\infty} \left((m!)^{-\kappa} \cdot h_A^{\frac{1}{m}} \cdot K_m\right)^{\frac{1}{m}} = \frac{1}{(2\pi)^\kappa} dV_Y \cdot h_{X/Y}^{-1}$$

holds.

Proof The proof of Theorem 10 is essentially the same as the argument in [20], i.e., the lower estimate comes from the L^2-estimate of $\bar\partial$-operator and the upper estimate comes from the dimension counting. □

Let $y_0 \in Y$ be an arbitrary point on Y such that the twisted Kähler-Einstein form ω_Y is a C^∞-Kähler form on a neighbourhood of y_0. Then we have the following lemma.

Lemma 2 *There exists a normal coordinate neighbourhood* $(U, , z_1, \cdots, z_\kappa)$ *around* y_0 *such that*

(1) Let g_{ij} *be the function defined by:*

$$\omega_Y = \frac{\sqrt{-1}}{2} \sum_{i=1}^{\kappa} g_{ij} dz_i \wedge d\bar z_j.$$

Then $g_{i\bar j}(O) = \delta_{ij}$,

(2) There exists a local holomorphic frame $\mathbf{e}$ *of* $pL_{X/Y}$ *on* U *such that*

$$\det(g_{i\bar j}) \cdot (h_{X/Y}^p(\mathbf{e}, \mathbf{e}))^{-\frac{1}{p}} = \prod_{j=1}^{\kappa} \left(1 - \frac{1}{2}|z_j|^2\right)^{-1} + O(\| z \|^3), \tag{4.3}$$

(The formula follows from the Taylor expansion of $\det(g_{i\bar j})$ *and the twisted Kähler-Einstein condition directly),*

(3) U is biholomorphic to $\Delta^\kappa(O,R)$ for some $0 < R < 1$ via $(z_1, \cdots, z_\kappa)$, where $\Delta^\kappa(O,R) = \{z \in \mathbb{C}^n \mid |z_i| < R, 1 \leqq i \leqq \kappa\}$.

Let us estimate $\{K_m\}$ from below. To get a uniform estimate we use the following lemma.

Lemma 3 *Taking a suitable modification of $f : X \to Y$, we may assume that there exists an effective $\mathbb{Q}$-divisor E and a C^∞ hermitian metric h_E on the $\mathbb{Q}$-line bundle $\mathcal{O}_Y(E)$ such that $dV_Y^{-1} \cdot h_{X/Y} \cdot h_E^{-\delta}$ has strictly positive curvature on Y for every sufficiently small $\delta > 0$.*

This lemma is the direct consequence of Kodaira's lemma. Let σ_E is a multivalued holomorphic section on $\mathcal{O}_Y(E)$ with divisor E and $\parallel \sigma_E \parallel$ denote the norm of σ_E with respect to h_E. We shall take σ_E so that $\parallel \sigma_E \parallel \leqq 1$ holds on Y.

We note that

$$\frac{\sqrt{-1}}{2}\int_{\Delta(0,r)} \left(1 - \frac{1}{2}|t|^2\right)^m dt \wedge d\bar{t} = \frac{2\pi}{m+1}\left(1 - \left(1 - \frac{1}{2}r^2\right)^{m+1}\right) \tag{4.4}$$

holds, where $\Delta(0,r)$ denotes the unit open disk in $\mathbb{C}$ with center 0 and radius r.

Suppose that there exists a positive number $C_{m,\delta}$ such that

$$K_m(y_0) \geqq C_{m,\delta} \cdot h_A^{-1} \cdot dV_Y^m \cdot h_{X/Y}^{-p\lfloor\frac{m}{p}\rfloor} \cdot \parallel \sigma_E \parallel^{2\delta m}$$

holds on Y.

To get a lower estimate of $C_{m+1,\delta}$ we shall apply the same strategy as in [21],

Let $\mathbf{e}_A$ be a local holomorphic frame of A on U (shrinking U if necessary). By the L^2-estimate for $\bar{\partial}$-operator, we shall extend a (twisted) m-ple canonical form

$$(\mathbf{e}_A \cdot (dz_1 \wedge \cdots dz_\kappa)^{m+1} \otimes \mathbf{e}^{\lfloor\frac{m*1}{p}\rfloor})(y_0) \tag{4.5}$$

at y_0 to a global holomorphic section of $A + (m+1)K_Y + p\lfloor\frac{m+1}{p}\rfloor$. We note that by (4.4) and Lemma 2, the L^2-norm of the local section

$$\mathbf{e}_A \cdot (dz_1 \wedge \cdots dz_\kappa)^{m+1} \otimes \mathbf{e}^{\lfloor\frac{m+1}{p}\rfloor}$$

on U with respect to the singular hermitian metric

$$h_{m,\delta} := dV_Y^{-m} \cdot h_{X/Y}^{p\lfloor\frac{m+1}{p}\rfloor} \cdot \parallel \sigma_E \parallel^{-2\delta(m+1)} \tag{4.6}$$

concentrates near the origin O of the coordinate $(z_1, \cdots, z_\kappa)$ as m tends to infinity and satisfies the inequality:

$$\int_U \left|\mathbf{e}_A \cdot (dz_1 \wedge \cdots dz_\kappa)^{m+1} \otimes \mathbf{e}^{\lfloor\frac{m+1}{p}\rfloor}\right|^2_{h_{m,\delta}} \leqq \left(\frac{m+1}{2\pi}\right)^\kappa (1 - \varepsilon(m)), \tag{4.7}$$

where $\varepsilon(m) < 1$ is a positive constant depending only on m such that $\lim_{m\to\infty} \varepsilon(m) = 0$ (we note that the fluctuation coming from the fractional part m/p $(= m/p - \lfloor m/p \rfloor)$ is getting small as m tends to inifnity, because of the concentration of the L^2-norm near y_0). Let ρ be a cut off function on U which is identically 1 on a neighbourhood of y_0, i.e.,

(1) $\rho \in C^\infty(Y)$ and $f \equiv 0$ on $Y \backslash U$ holds,
(2) There exists a neighbourhood $V \subset U$ of y_0 such that $\rho \equiv 1$ on V,
(3) $0 \leqq \rho \leqq 1$ hold on Y.

Now we shall solve the $\bar{\partial}$-equation:

$$\bar{\partial} u = \bar{\partial}(\rho \cdot (\mathbf{e}_A \cdot (dz_1 \wedge \cdots dz_\kappa)^m \otimes \mathbf{e}^{\lfloor \frac{m}{p} \rfloor})). \tag{4.8}$$

Here we note that

$$\bar{\partial}(\rho \cdot (\mathbf{e}_A \cdot (dz_1 \wedge \cdots dz_\kappa)^m \otimes \mathbf{e}^{\lfloor \frac{m}{p} \rfloor})) = \bar{\partial}\rho \wedge (\mathbf{e}_A \cdot (dz_1 \wedge \cdots dz_\kappa)^m \otimes \mathbf{e}^{\lfloor \frac{m}{p} \rfloor}))$$

and $\bar{\partial}\rho \equiv 0$ on V. And we also note that taking Θ_{h_A} sufficiently positive, we may assume that there exists a positive constant c such that

$$\sqrt{-1}\Theta_{h_{m,\delta}} > c \cdot m\Theta_{h_A}$$

holds on Y. Then by the standard L^2-estimate for $\bar{\partial}$ operator, we see that there 1exists a C^∞ solution u of (4.8) such that

$$\begin{aligned}\int_Y e^{-2\kappa\rho \log\|z\|} |u|^2_{h_{m,\delta}} &\leqq \int_Y e^{-2\kappa\rho \log\|z\|} \left| \bar{\partial}(\rho \cdot (\mathbf{e}_A \cdot (dz_1 \wedge \cdots dz_\kappa)^m \otimes \mathbf{e}^{\lfloor \frac{m}{p} \rfloor})) \right|^2_{h_{m,\delta}} \\ &\leqq \frac{C'}{m} \int_U \left| \mathbf{e}_A \cdot (dz_1 \wedge \cdots dz_\kappa)^{m+1} \otimes \mathbf{e}^{\lfloor \frac{m+1}{p} \rfloor} \right|^2_{h_{m,\delta}}\end{aligned}$$

where $|\ \ |_{h_{m,\delta}}$ denotes the hermitian norm with respect to $h_{m,\delta}$ (see (4.6)) and the Kähler form $\sqrt{-1}\Theta_{h_{m,\delta}}$ and C' is a positive constant independent of m. Then we have that

$$\| u \| \leqq \frac{C}{\sqrt{m}} \left(\frac{m+1}{2\pi} \right)^\kappa \cdot C_{m,\delta}$$

holds, where C is a positive constant independent of m. We note that since $e^{-2\kappa\rho \log\|z\|}$ is not locally integrable around y_0, $u(y_0) = 0$ holds. Thus by the extremal property of the Bergman kernel and (4.7), we may take $C_{m+1,\delta}$ so that

$$C_{m+1,\delta} \geqq \left(1 - \frac{C}{\sqrt{m}}\right) \left(\frac{m+1}{2\pi} \right)^\kappa \cdot C_{m,\delta}$$

holds where C is a positive constant independent of m. This implies that

$$\lim_{m\to\infty}\left((m!)^{-\kappa}\cdot h_A^{\frac{1}{m}} K_m\right)^{\frac{1}{m}} \geqq \frac{1}{(2\pi)^{\kappa}} dV_Y \cdot h_{X/Y}^{-1} \parallel \sigma_E \parallel^{2\delta} .$$

holds. Letting δ tend to 0, we have the following lemma:

Lemma 4

$$\lim_{m\to\infty}\left((m!)^{-\kappa}\cdot h_A^{\frac{1}{m}} \cdot K_m\right)^{\frac{1}{m}} \geqq \frac{1}{(2\pi)^{\kappa}} dV_Y \cdot h_{X/Y}^{-1}$$

holds.

Now we shall give the upper estimate for K_m. By Hölder's inequality

$$\int_Y (K_m \cdot h_A \cdot h_{X/Y}^{p\lfloor\frac{m}{p}\rfloor})^{\frac{1}{m}} \leqq \left(\int_Y \frac{K_m \cdot h_{X/Y}^{p\lfloor\frac{m}{p}\rfloor}}{K_{m-1}\cdot h_{X/Y}^{p\lfloor\frac{m-1}{p}\rfloor}}\right)^{\frac{1}{m}} \cdot \left(\int_Y (K_{m-1}\cdot h_A \cdot h_{X/Y}^{p\lfloor\frac{m-1}{p}\rfloor})^{\frac{1}{m-1}}\right)^{\frac{m-1}{m}} \tag{4.9}$$

holds. We note that

$$\int_Y \frac{K_m \cdot h_{X/Y}^{p\lfloor\frac{m}{p}\rfloor}}{K_{m-1}\cdot h_{X/Y}^{p\lfloor\frac{m-1}{p}\rfloor}} = \dim H^0(Y, \mathcal{O}_Y(mK_Y + p\lfloor\frac{m}{p}\rfloor \cdot L_{X/Y}) \otimes \mathcal{J}(h_{X/Y}^{p\lfloor\frac{m}{p}\rfloor}))$$

holds and again by Hölder's inequality we have

$$\begin{aligned}&\left(\int_Y (K_{m-1}\cdot h_A \cdot h_{X/Y}^{p\lfloor\frac{m-1}{p}\rfloor})^{\frac{1}{m-1}}\right)^{\frac{m-1}{m}} \\ &\qquad \leqq \left(\int_Y \frac{K_{m-1} \cdot h_{X/Y}^{p\lfloor\frac{m-1}{p}\rfloor}}{K_{m-2}\cdot h_{X/Y}^{p\lfloor\frac{m-2}{p}\rfloor}}\right)^{\frac{1}{m}} \cdot \left(\int_Y (K_{m-2}\cdot h_A \cdot h_{X/Y}^{p\lfloor\frac{m-2}{p}\rfloor})^{\frac{1}{m-2}}\right)^{\frac{m-2}{m-1}} .\end{aligned}$$

Hence continuing this process inductively, by the asymptotic Riemann-Roch theorem, we have that

$$\int_Y \left(\lim_{m\to\infty}\left((m!)^{-\kappa}\cdot h_A^{\frac{1}{m}} K_m\right)^{\frac{1}{m}} \cdot h_{X/Y}\right) \leqq \frac{1}{(2\pi)^{\kappa}} \int_Y dV_Y \tag{4.10}$$

holds. Combining Lemma 4 and (4.10), we have that

$$\lim_{m\to\infty}\left((m!)^{-\kappa}\cdot h_A^{\frac{1}{m}} \cdot K_m\right)^{\frac{1}{m}} \cdot h_{X/Y} = dV_Y$$

holds on Y. This completes the proof of Theorem 10. We note that by the proof, the convergence in Theorem 10 is globally L^1 and uniform on every compact subset of Y°, where Y° denotes the locus where ω_Y is C^∞ and strictly positive. □

The following corollary is a direct consequence of the plurisubharmonic variation property of Bergman kernels ([2, 5]) and Theorem 10.

Corollary 1 *Let $f : X \to S$ be smooth projective family of projective variety of nonnegative Kodaira dimension. Let $d\mu_{X/S,can}$ be the relative canonical measure on X defined by*

$$d\mu_{X/S}|_{X_s} = d\mu_s$$

where $d\mu_s$ denote the canonical measure on $X_s := f^{-1}(s)$. Then $d\mu_{X/S}^{-1}$ is a singular hermitian metric on $K_{X/S}$ with semipositive curvature current.

4.5 Proof of Theorem 9

Let $\{K_m\}_{m=1}^{\infty}$ be as (4.2). We define the dynamical system of p-Bergman kernels by

$$\tilde{K}_1^{(p)} := K_{\frac{2}{p}}(Y, p(K_Y + L + A), h_A \cdot h_L)$$

and inductively for $m \geqq 2$

$$\tilde{K}_m^{(p)} = \begin{cases} K_{\frac{2}{p}}(Y, mK_Y + \lfloor \frac{m}{p} \rfloor pL_{X/Y}, K_{m-1}^{-1} \cdot h_{X/Y}^{p}) & \text{, if } p|m, \\ \\ K_{\frac{2}{p}}(Y, mK_Y + \lfloor \frac{m}{p} \rfloor pL_{X/Y}, (K_{m-1}^{(p)})^{-1}) & \text{, if } p\nmid m. \end{cases}, \tag{4.11}$$

Then by definitions of $\{K_m\}$ and $\{\tilde{K}_m^{(p)}\}$ we see that

$$K_m \leqq \tilde{K}_m^{(p)}$$

holds for every $m \geqq 1$. On the other hand for a positive integer ℓ, we define the dynamical system of p-Bergman kernels by

$$\hat{K}_1^{(p)} := K_{\frac{2}{p}}(Y, p(K_Y + L_{X/Y} + \ell A), h_{X/Y} \cdot h_A^{\ell}) \tag{4.12}$$

and inductively for $m \geqq 2$

$$\hat{K}_m^{(p)} := K_{\frac{2}{p}}(Y, mp(K_Y + L_{X/Y}) + p\ell A, h_{X/Y} \cdot (\hat{K}_{m-1}^{(p)})^{-1}). \tag{4.13}$$

Then taking ℓ sufficiently large to absorb the fluctuation of $\lfloor \frac{m}{p} \rfloor$, we see that

$$h_A^{\ell} \cdot h_{X/Y}^{m} \cdot \hat{K}_m^{(p)} \geqq h_A \cdot h_{X/Y}^{p\lfloor \frac{m}{p} \rfloor} \cdot \tilde{K}_m^{(p)} \geqq h_A \cdot h_{X/Y}^{m} \cdot K_m \tag{4.14}$$

hold. Hence we have the following lemma.

Lemma 5

$$\lim_{m\to\infty}\left((m!)^{-\kappa}\cdot h_A^{\frac{\ell}{m}}\cdot\hat{K}_m^{(p)}\right)^{\frac{1}{m}} \geqq \frac{1}{(2\pi)^\kappa}dV_Y\cdot h_{X/Y}^{-1}$$

holds.

4.6 Dynamical Systems of Extremal Measures

Now we shall estimate

$$\lim_{m\to\infty}\left((m!)^{-\kappa}\cdot h_A^{\frac{\ell}{m}}\cdot\hat{K}_m^{(p)}\right)^{\frac{1}{m}}$$

from above.

To get the upper estimate, we shall consider dynamical systems of extremal measures.

First we set

$$d\nu_1 = \sup\{dV \mid -\mathrm{Ric}(dV) + \sqrt{-1}\Theta_{h_{X/Y}} + \sqrt{-1}\ell\cdot\Theta_{h_A} \geqq 0, \int_Y dV = 1\},$$

where dV runs semipositive volume form such that dV^{-1} is a singular hermitian metric on K_Y. And we set

$$h_1 = (d\nu_1)^{-1}\cdot h_{X/Y}\cdot h_A^{\ell}.$$

For $m \geqq 2$, we define inductively

$$d\nu_m = \sup\{dV \mid -\mathrm{Ric}(dV) + \sqrt{-1}\Theta_{h_{m-1}} + \sqrt{-1}\Theta_{h_{X/Y}} \geqq 0, \int_Y dV = 1\}$$

and

$$K_m^{(\infty)} = d\nu_m\cdot h_{X/Y}^{-1}\cdot K_{m-1}^{\infty}.$$

Next we shall estimate $K_m^{(\infty)}$ from above. Let g be the twisted Kähler-Einstein metric on Y associated with ω_Y. Let $y_0 \in Y$ be an arbitrary point such that ω_Y is a C^∞ Kähler form on a neighbourhood of y_0. Then by Lemma 2, there exists a holomorphic normal coordinate $(U, (z_1, \cdots, z_\kappa))$ with center y_0 such that

(1) Let g_{ij} be the function defined by:

$$\omega_Y = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{\kappa} g_{ij}dz_i\wedge d\bar{z}_j,$$

Then $g_{i\bar{j}}(O) = \delta_{ij}$,

(2) There exists a local holomorphic frame $\mathbf{e}$ of $pL_{X/Y}$ on U such that

$$\det(g_{i\bar{j}}) \cdot (h^{p}_{X/Y}(\mathbf{e}, \mathbf{e}))^{-\frac{1}{p}} = \prod_{j=1}^{\kappa}\left(1 - \frac{1}{2}|z_j|^2\right)^{-1} + O(\| z \|^3) \tag{4.15}$$

(The formula follows from the Taylor expansion of $\det(g_{i\bar{j}})$ and the twisted Kähler-Einstein condition directly),

(3) U is biholomorphic to $\Delta^{\kappa}(O, R)$ for some $0 < R < 1$ via $(z_1, \cdots, z_\kappa)$, where $\Delta^{\kappa}(O, R) = \{z \in \mathbb{C}^{\kappa} \,|\, |z_i| < R, 1 \leqq i \leqq \kappa\}$.

We shall identify U with $\Delta^{\kappa}(O, R)$ and we set $U(r) := \Delta^{\kappa}(O, r)$ for $0 < r \leqq R$,

For $m = 1$, there exists a positive constant C_1 such that

$$K_1^{(\infty)} \leqq C_1 \cdot h_A^{-\ell} \cdot dV_Y \cdot h^{-1}_{X/Y}$$

holds on U. Suppose that for some $m \geqq 1$,there exists a constant C_m such that

$$K_m^{(\infty)} \leqq C_m \cdot h_A^{-1} \cdot (dV_Y)^m \cdot h^{-m}_{X/Y}$$

holds on U. We note that

$$K_m^{(\infty)}(y_0) \leqq K^{\infty}(U, C_m^{-1} \cdot h_A \cdot (dV_Y)^{-m} \cdot h^m_{X/Y})(y_0) \tag{4.16}$$

holds, where

$$\begin{aligned} K^{(\infty)}(U, &C_m^{-1} \cdot h_A \cdot dV_Y^{-m})(y) \\ &= \sup\{h(y)^{-1} \mid \sqrt{-1}\Theta(h) \geqq 0, C_m^{-1}\int_X h^{-1} \cdot h_A \cdot dV_Y^{-(m-1)} \cdot h^m_{X/Y} = 1\}. \end{aligned}$$

Here h runs lowersemicontinuous singular hermitian metrics on $m(K_Y + K_{X/Y}) + A$ on U.

By Lemma 2, there exist a function $\varepsilon(r)$ of $0 < r < R$ such that

$$dV_Y^{-1} \cdot (2^{-\kappa}|dz_1 \wedge \cdots \wedge dz_\kappa|^2) \cdot (h^p_{X/Y}(\mathbf{e}, \mathbf{e}))^{-\frac{1}{p}} \geqq \prod_{j=1}^{\kappa}\left(1 - \frac{1}{2}(1 + \varepsilon(r))|z_j|^2\right) \tag{4.17}$$

holds on $U(r)$ and

$$\lim_{r \downarrow 0} \varepsilon(r) = 0. \tag{4.18}$$

We set

$$K_{m,\mathrm{loc}}(y_0) := \sup\Big\{e^{\varphi(y_0)} \mid \varphi \in \mathrm{PSH}(U),$$
$$\int_U e^{\varphi}\cdot\left(\prod_{j=1}^{\kappa}\left(1-2^{-1}(1+\varepsilon(R))|z_j|^2\right)\right)^m 2^{-\kappa}|dz_1\wedge\cdots dz_\kappa|^2 = 1\Big\}. \tag{4.19}$$

We note that

$$\frac{\sqrt{-1}}{2}\int_{\Delta(0,r)}\left(1-\frac{1}{2}|t|^2\right)^m dt\wedge d\bar{t} = \frac{2\pi}{m+1}\left(1-\left(1-\frac{1}{2}r^2\right)^{m+1}\right) \tag{4.20}$$

holds, where $\Delta(0,r)$ denotes the unit open disk in $\mathbb{C}$ with center 0 and radius r. By the symmetry of the diagonal $(S^1)^\kappa$-action on $\Delta^\kappa(O,r)$, we have that

$$K_{m,\mathrm{loc}}(y_0) \leqq (1+\varepsilon(r))^\kappa\left(\frac{2\pi}{m+1}\right)^\kappa \tag{4.21}$$

holds. By (4.17), there exists a positive constant $c<1$ and and a positive function $\varepsilon(r)$ of r so that we may take C_{m+1} so that

$$C_{m+1} \leqq C_m\cdot\left(\frac{m+1}{2\pi}\right)^\kappa(1+\varepsilon(r))^\kappa(1-c^{m+1})^{-\kappa}$$

holds. This implies that

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}}\cdot((m!)^{-\kappa}K_m^{(\infty)})^{\frac{1}{m}}(y_0) \leqq \frac{1}{(2\pi)^\kappa}(1+\varepsilon(r))^n dV_Y(y_0)\cdot h_{X/Y}^{-1}(y_0)$$

holds. Letting $r\downarrow 0$, by (4.18), we have that

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}}\cdot((m!)^{-\kappa}K_m^{(\infty)})^{\frac{1}{m}}(y_0) \leqq \frac{1}{(2\pi)^\kappa}dV_Y(y_0)\cdot h_{X/Y}^{-1}(y_0) \tag{4.22}$$

holds. Since y_0 is arbitrary, we have the following lemma.

Lemma 6

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}}\cdot((m!)^{-\kappa}K_m^{(\infty)})^{\frac{1}{m}}(y_0) \leqq \frac{1}{(2\pi)^\kappa}dV_Y(y_0)\cdot h_{X/Y}^{-1}(y_0) \tag{4.23}$$

holds on Y.

Let $\{K_m\}$ be as in Theorem 10, Then by definitions, $K_m \leqq K_m^{(\infty)}$ holds on Y for every $m \geqq 1$. By Theorems 10 and 6, we have the following theorem.

Theorem 11

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot ((m!)^{-\kappa} K_m^{(\infty)})^{\frac{1}{m}} = \frac{1}{(2\pi)^\kappa} dV_Y \cdot h_{X/Y}^{-1}$$

holds.

We note that by the proof, the convergence in Theorem 11 is uniform on every compact subset of Y°, where Y° denotes the locus where ω_Y is C^∞ and strictly positive.

Now we shall complete the proof of Theorem 9. Let $\{\hat{K}_m^{(\infty)}\}$ be the dynamical system obtained by replacing (A, h_A) by $(\ell A, h_A^\ell)$. Then by definitions (cf. (4.12)),

$$K_m \leqq \hat{K}_m^{(p)} \leqq \hat{K}_m^{(\infty)}$$

hold on Y. By Lemma 4.4, Theorem 10 and Theorem 11 we see that

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot ((m!)^{-\kappa} K_m^{(p)})^{\frac{1}{m}} = \frac{1}{(2\pi)^\kappa} dV_Y \cdot h_{X/Y}^{-1}$$

holds. Since

$$K_m^{(p)} \leqq \hat{K}_m^{(p)} \leqq \hat{K}_m^{(\infty)}$$

hold on Y, we see that

$$\limsup_{m\to\infty} h_A^{\frac{1}{m}} \cdot ((m!)^{-\kappa} K_m^{(p)})^{\frac{1}{m}} \leqq \frac{1}{(2\pi)^\kappa} dV_Y \cdot h_{X/Y}^{-1} \tag{4.24}$$

holds. Combining (4.24) and Lemma 5, we see that

$$\lim_{m\to\infty} h_A^{\frac{1}{m}} \cdot ((m!)^{-\kappa} K_m^{(p)})^{\frac{1}{m}} = \frac{1}{(2\pi)^\kappa} dV_Y \cdot h_{X/Y}^{-1}$$

holds. This completes the proof of Theorem 9.

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Wehrl-Type Inequalities for Bergman Spaces on Domains in $\mathbb{C}^d$ and Completely Positive Maps

Genkai Zhang

Abstract We prove certain Wehrl type L^p-inequalities for the Bergman spaces on bounded domains D in $\mathbb{C}^d$ and apply the result to bounded symmetric domains $D = G/K$. We introduce also G-invariant completely positive trace preserving map $A \to \mathcal{T}(A)$ from operators A on a weighted Bergman space H_μ on $D = G/K$ to operators $\mathcal{T}(A)$ on another weighted Bergman space $H_{\mu+\nu}$ and prove that the L^p-norm of Bergman functions $f \in H_\mu$ can be obtained as limit of trace of $\mathcal{T}(f \otimes f^*)$ as the weight $\nu \to \infty$.

Keywords Bergman spaces · Bounded domains · Bounded symmetric domains · Wehrl inequality · Reproducing kernels

1 Introduction

In a series of papers [10–12] Lieb and Solovej proved certain Wehrl $L^2 - L^p$ inequality for matrix coefficients of representations $\odot^m \mathbb{C}^d$ of $SU(d)$ for all $p \geq 2$ and for the Bergman spaces on the unit disc as representations of $SU(1, 1)$ for even integer $p = 2k$. The L^p-inequality for Bergman spaces is recently proved in [6, 8] for all $p \geq 2$ using classical complex analysis. The proof in [11, 12] for finite dimensional representations $\odot^m \mathbb{C}^d$ of $SU(d)$ is rather intriguing; Lieb and Solovej introduced certain completely positive trace preserving map $A \to \mathcal{T}_{m,n}(A)$ for operators $A \in End(\odot^m \mathbb{C}^d)$ to operators $\mathcal{T}(A) \in End(\odot^{m+n} \mathbb{C}^d)$ obtained by taking the leading component in tensor product decomposition of $\odot^m \mathbb{C}^d \otimes \odot^n \mathbb{C}^d$. They proved that the L^p-norm of the matrix coefficients $\langle \tau(g)v, v_0 \rangle, g \in SU(d), v \in \odot^m \mathbb{C}^d, \|v\| = 1$, can be obtained from the functional calculus of $\mathcal{T}_{m,n}(A)$ for rank one projection $A = P_v = v \otimes v^* \in End(\odot^m \mathbb{C}^d)$ by taking the limit $n \to \infty$, and that the eigenvalue distribution of $\mathcal{T}(A)$ is dominated by $\mathcal{T}(A_0)$ for $A = v_0 \otimes v_0^*$, where v_0 is a

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unit highest weight vector. The Wehrl inequality then follows by a general abstract result about functional calculus $A \to f(A)$ for convex functions f and self-adjoint operators A.

In the present paper we shall study two kinds of Wehrl type L^p-inequalities related to the tensor product of weighted Bergman spaces and one limit formula related to the L^p-norm of holomorphic functions in terms of functional calculus. The Wehrl type inequality in [6, 8, 11, 12] is to find the $L^2 - L^p$ norm of the identity operator on Bergman spaces. Firstly we shall find a general $L^2 - L^p$ optimal estimate for even integers p for a reproducing kernel space H and apply it to bounded symmetric domains $D = G/K$. This is proved by considering the tensor product $\otimes^s H$ of H. Next we consider the tensor product $H \otimes \bar{H}$ of H with its conjugate and related Toeplitz and Berezin calculus, we find the optimal bound of $R \to R(A)$ for the dual operator of the Toeplitz calculus $f \to T_f$ from $L^2(G/K) \to S^2(H) = H \otimes \overline{H}$. The Wehrl inequality is related to this map applied to the rank one projection $A = f \otimes f^*$ for a unit element f in the Bergman space. Finally we consider the tensor product decomposition of $H \otimes H$ for weighted Bergman space H on the unit ball $B = SU(d, 1)/U(d)$, and we introduce a large class of completely positive maps $A \to \mathcal{T}_m(A)$ for operators on Bergman spaces on the unit ball B in $\mathbb{C}^d$. We prove a limit formula expressing the L^p-norm of a Bergman space function f using the trace of the functional calculus of the operator $\mathcal{T}_0(A)$. We pose some open questions related to these three results.

2 Wehrl-Type Inequality for Bounded Domains

2.1 Bergman Spaces on Bounded Domains D

We prove below certain Wehrl-type L^p-inequalities for general reproducing kernel spaces. The proof is based on some simple considerations of polarization and restriction of holomorphic functions and Toeplitz operators.

Let $D \subset \mathbb{C}^d$ be a bounded domain and let $d\sigma$ be a probability measure on D. For simplicity we assume that $0 \in D$ and that $d\sigma$ is a weighted measure with the weight $\frac{d\sigma(z)}{dm(z)}$ being positive bounded continuous on D, $dm(z)$ being the Lebesgue measure. Let $H = L^2_h(D, d\sigma) \subset L^2(D, d\sigma)$ be the L^2-Bergman space of holomorphic functions and $K(z, w)$ its reproducing kernel. We recall first some simple $L^2 - L^p$ inequalities as a motivation to the Wehrl type inequalities.

It follows by Cauchy-Schwartz inequality that for any $f \in H = L^2_h(D, d\sigma)$,

$$|g(z)| \leq \|f\|,\ g(z) := |f(z)K(z, z)^{-\frac{1}{2}}|.$$

In other words we have

$$\|g\|_{L^2(D,K(z,z)d\sigma(z))} = \|f\|_H,\ \|g\|_{L^\infty(D,K(z,z)d\sigma(z))} \leq \|f\|_H,$$

and further by interpolation that

$$\|g\|_{L^p(D,K(z,z)d\sigma(z))} \leq \|f\|_H,$$

for $p \geq 2$. We may formulate the Wehrl type inequality as finding the best constant of this inequality, i.e. the operator norm of the inclusion operator from H to $L^p(D, K(z,z)d\sigma(z))$.

We fix a positive integer s. The power $K(z,w)^s$ is then a positive definite kernel and is thus the reproducing kernel of a Hilbert space which we denote by H_s. Namely H_s is the closure of holomorphic functions f of the form

$$f(x) = \sum_\alpha c_\alpha K(x, x_\alpha)^s, \quad x \in D, \tag{1}$$

with norm square

$$\|f\|_{H_s}^2 := \sum_{\alpha,\beta} c_\alpha \overline{c_\beta} K(x_\beta, x_\alpha)^s.$$

Here $\{x_\alpha\} \subset D$ is any finite subset.

For each function $f \in H$ we consider the H_s-norm of f^s.

Theorem 1 *The following inequality holds,*

$$\|f^s\|_{H_s}^2 \leq \|f\|_H^{2s}, \quad f \in H. \tag{2}$$

Furthermore if the set of bounded point evaluations of H is D then the equality of (2) holds when f is a scalar multiple of the reproducing kernel $K_w(z) = K(z,w)$ for some $w \in D$.

The tensor product $\otimes^s H$ is realized as the space of holomorphic functions $F(z_1, \cdots, z_s)$ such that

$$\int_{D^s} |F(z_1, \cdots, z_s)|^2 d\sigma(z_1) \cdots d\sigma(z_s) < \infty.$$

It has the reproducing kernel

$$L(z,w) = L_s(z,w) = \prod_{j=1}^{s} K(z_j, w_j), \quad z = (z_1, \cdots, z_s), w = (w_1, \cdots, w_s) \in D^s.$$

Write the diagonal imbedding $D \to D^s$ and $z \to \widetilde{z} = (z, \cdots, z)$.

The following lemma can be proved using definitions.

Lemma 1 *The natural extension operator*

$$Qf(z_1, \cdots, z_s) = \sum_\alpha c_\alpha L((z_1, \cdots, z_s), (x_\alpha, \cdots, x_\alpha)) = \sum_\alpha c_\alpha \prod_{j=1}^{s} K(z_j, x_\alpha)$$

for $f \in H_s$, $f(x) = \sum_\alpha c_\alpha K(x, x_\alpha)^s$, $x \in D$, *extends to an isometry* $H_s \to \otimes^s H$ *and its adjoint is the restriction to the diagonal* $Q^* F(z) = RF(z) = F(z, \cdots, z)$, $z \in D$.

Indeed the isometric property of Q is a direct computation: For f as above we have

$$\begin{aligned}\|Qf\|^2_{\otimes^s H} &= \|\sum_\alpha c_\alpha L(\cdot, \widetilde{x_\alpha})\|^2_{\otimes^s H} = \sum_{\alpha,\beta} c_\alpha \overline{c_\beta} L(\widetilde{x_\beta}, \widetilde{x_\alpha})\\ &= \sum_{\alpha,\beta} c_\alpha \overline{c_\beta} K(x_\beta, x_\alpha)^s = \|f\|^{2s}_{H_s},\end{aligned}$$

by the reproducing kernel properties. To find Q^* we take first $g \in \otimes^s H$ of the form $g = K(z_1, w_1) \cdots K(z_s, w_s)$ and compute $\langle Qf, g\rangle$ where $f \in H_s$ is as above by using the reproducing kernel property in $\otimes^s H$ and in H_s:

$$\langle Qf, g\rangle_{\otimes^s H} = \sum_\alpha c_\alpha K(w_1, x_\alpha) \cdots K(w_s, x_\alpha) = \langle f, Rg\rangle_{H_s}$$

with $Rg(z) = K(z, w_1) \cdots K(z, w_s)$. This prove $Rg = Q^* g$ and then $R = Q^*$ on $\otimes^s H$ since the span of such g's is dense in $\otimes^s H$.

We observe that if $F \in \otimes^s H$ and $F(z_1, \cdots, z_s)$ has no constant term along the diagonal $(z_1, \cdots, z_s) = (z_1, \cdots, z_1)$,

$$F(z_1, \cdots, z_s) = f_2(z_1, \cdots, z_s; w)(z_2 - z_1) + \cdots + f_s(z_1, \cdots, z_s; w)(z_s - z_1) + \cdots$$

then $RF = 0$; in other words if $F(z_1, \cdots, z_1) = 0$, $z_1 \in D$ then F is the ideal generated by $z_i - z_j$, $1 \le i < j \le s$.

We recall also some elementary facts about Toeplitz operators. Let H be the Hilbert space above realized as $H \subset L^2(D, d\sigma)$. Let T_f be the Toeplitz operator on H, $T_f = PM_f P$ where $P : L^2(D, d\sigma) \to H$ is the orthogonal projection and M_f is the multiplication operator on $L^2(D, d\sigma)$ by a bounded function $f \in L^\infty(\sigma)$. Write a point in D as $x = (x^1, \cdots, x^d) \in D \subset \mathbb{C}^d$ (to differ the previous notation $(z_1, \cdots, z_s) \in D^s$.) The Toeplitz operators $\{T_{x^j}\}$ form a commuting family. The following lemma on the joint eigenvectors of $\{T^*_{x^j}\}$ is elementary and follows directly from the definition.

Lemma 2 *Suppose that the set of polynomials is dense in* H. *The joint eigenvectors of* $\{T^*_{x^j}\} = \{T_{\bar{x}^j}\}$ *on* H *are the scalar multiples of Riesz kernels of bounded point* *evaluations* $f \in H \to f(z)$, $z \in \mathbb{C}^d$. *In particular if the set of bounded point evaluations of* H *is* D *then they are the reproducing kernels* K_x, $x \in D$ *with eigenvalues* $\{\overline{x^j}\}$.

We prove Theorem 1.

Proof The inequality is immediate by applying the partial isometry $R : \otimes^s H \to H_s$ to $\otimes^s f \in \otimes^s H$ with $\| \otimes^s f\|^2 = \|f\|^{2s}$. Now suppose $f \in H$ is a maximizer. This is equivalent to $\otimes^s f \in \otimes^s H$ is in the image of R^*. But then $\otimes^s f \perp \operatorname{Ker} R$ since $\operatorname{Im} R^* \perp \operatorname{Ker} R$. Let $\mathbf{T}_a$ be the Toeplitz operator on $\otimes^s H$ for $a = a(z_1, \cdots, z_s)$ on D^s and T_b the Toeplitz operator on H for b on D. Let α be any linear functional on $\mathbb{C}^d$, the ambient vector space. Consider the Toeplitz operators $\mathbf{T}_{\alpha(z_i - z_j)}$, $z = (z_1, \cdots, z_s) \in D^s$ and $T_{\alpha(z)}$, $1 \le i \ne j \le s$. We have $\mathbf{T}_{\alpha(z_i - z_j)} p \in \operatorname{Ker} R$ for any polynomial $p(z_1, \cdots, z_s)$. Thus $\otimes^s f \perp \mathbf{T}_{\alpha(z_i - z_j)} p$, and this becomes

$$0 = \langle \otimes^s f, \mathbf{T}_{\alpha(z_i - z_j)} p\rangle = \langle \mathbf{T}^*_{\alpha(z_i - z_j)} \otimes^s f, p\rangle.$$

But the polynomials are dense in $\otimes^s H$ and we get $\mathbf{T}^*_{\alpha(z_i - z_j)} \otimes^s f = 0$. Observe that for any function $f(z_1, \cdots, z_s)$ depending only on one variable, say $f(z_1, \cdots, z_s) = f(z_1)$, the Toeplitz operator $\mathbf{T}_f$ on $\otimes^s H$ is $\mathbf{T}_f = T^{(1)}_f \otimes^{s-1} I$ where $T^{(1)}_f$ is the Toeplitz operator on the first factor and I is the identity operator on the remaining factors. In particular $\mathbf{T}^*_{\alpha(z_i)} = \otimes^{i-1} I \otimes (T^{(i)}_{\alpha(z_i)})^* \otimes^{s-i} I$. The condition $0 = \mathbf{T}^*_{\alpha(z_i - z_j)}(\otimes^s f) = \mathbf{T}^*_{\alpha(z_i)}(\otimes^s f) - \mathbf{T}^*_{\alpha(z_j)}(\otimes^s f)$ is then

$$\otimes^{i-1} f \otimes (T^{(i)}_{\alpha(z_i)})^* f \otimes^{s-i} f - \otimes^{j-1} f \otimes (T^{(j)}_{\alpha(z_j)})^* f \otimes^{s-j} f = 0, \quad \forall 1 \le i \ne j \le s.$$

Observe that if two simple tensor $f_1 \otimes \cdots \otimes f_s$ and $g_1 \otimes \cdots \otimes g_s$ are the same, then f_i and g_i are proportional. Thus $(T^{(i)}_{\alpha(z_i)})^* f$ and f are proportional, i.e. f is an eigenvector of $(T^{(i)}_{\alpha(z_i)})^*$; in other words, f is a joint eigenvector of $T^*_{\alpha(x)}$ on H for all linear functionals α on $\mathbb{C}^d$. This is the case if f is a scalar multiple of a Riesz kernel of some bounded point evaluation at $x \in \mathbb{C}^d$ and furthermore a reproducing kernel K_x, $x \in D$ if the point evaluations are in $D \subset \mathbb{C}^d$, by Lemma 2. This completes the proof. □

2.2 Bounded Symmetric Domains

We apply the above result to Bergman spaces on bounded symmetric domains D. They are bounded realization of non-compact Hermitian symmetric spaces G/K and we refer [5, 7, 13] for general results on the geometry and analysis of symmetric spaces. We start with an irreducible bounded symmetric domain $D = G/K$, in $\mathbb{C}^d$ with $0 \in D$, G being the connected component of biholomorphic mappings of D where $K = \{k \in G; k \cdot 0 = 0\}$ acting linearly and unitarily on $\mathbb{C}^d$. Each irreducible bounded symmetric domain is characterized by a triple of non-negative integers (r, a, b) where $r \geq 1$ is the rank, a, b are certain root multiplicities [5]. Consider the Bergman space of holomorphic functions in $L^2(D)$ with respect to the Lebesgue measure $dm(z)$. Up to a constant factor its reproducing kernel is $h(z, w)^{-p_D}$ where $p_D = 2 + (r-1)a + b$ is called the *genus* of D, where $h(z, w)$ is an irreducible polynomial holomorphic in z and anti-holomorphic in w, $h(z, 0) = 1$. Write $h(z) = h(z, z)$. The Bergman metric is given by $\langle B(z, z)^{-1}v, v\rangle = \partial_v \partial_{\bar{v}} \log h(z, z)^{-1}$, where $B(z, z)$ is a $\mathrm{End}(V)$-valued polynomial in $(z, \bar{z})$; see Sect. 4.1 below for the case of the unit ball. The G-invariant (Kähler) measure is given by $d\iota(z) = h(z)^{-p_D} dm(z)$.

Let $\nu > p_D - 1 = 1 + (r-1)a + b$ and consider the weighted probability measure $d\iota_\nu(z) = C_\nu h(z)^{\nu - p_D} dz$, where

$$C_\nu^{-1} := \int_D h(z)^\nu d\iota(z) = \frac{\pi^d \Gamma_a(\nu - \frac{d}{r})}{\Gamma_a(\nu)},$$

and Γ_a the so-called Gindikin Gamma function; see [5]. Let H_ν be the weighted Bergman space of holomorphic functions

$$\int_D |f(z)|^2 d\iota_\nu(z) < \infty.$$

Note that under our normalization the constant function 1 is a unit vector in H_ν. The reproducing kernel of H_ν is

$$K_\nu(z, w) = h(z, w)^{-\nu}.$$

Now let $D = D_1 \times \cdots \times D_k$ be a product of irreducible bounded symmetric domains $D_1, \cdots, D_k$. Write points z in D as $z = (z_1, \cdots, z_k)$, $z_j \in D_j$. Consider the Bergman space $H_{(\nu_1, \cdots, \nu_k)}$ of holomorphic functions on D with the measure $d\iota_{\nu_1}(z_1) \cdots d\iota_{\nu_k}(z_k)$ with each $\nu_j > p_{D_j} - 1$. The reproducing kernel of $H = H_{(\nu_1, \cdots, \nu_k)}$ is then $K_w(z) = h_1(z_1, w_1)^{-\nu_1} \cdots h_k(z_k, w_k)^{-\nu_k}$. In this case the reproducing kernel space H_s introduced in the previous section with reproducing kernel $K_w(z)^s$ is precisely the Bergman space with measure $d\iota_{s\nu_1}(z_1) \cdots d\iota_{s\nu_k}(z_k)$. The set of bounded point evaluations for $H = H_{(\nu_1, \cdots, \nu_k)}$ is exactly D, which can be proved by direct computations. Applying Theorem 1 to this case we get the following

Theorem 2 *Let $D = D_1 \times \cdots \times D_k$ be a bounded symmetric domains with D_j being irreducible with genus p_j and Bergman reproducing kernel $h_j^{-p_j}$ respectively. Let $\nu_1 > p_1 - 1, \cdots, \nu_k > p_k - 1$ and consider the Bergman space $H_{(\nu_1, \cdots, \nu_k)}$ on D with weighted measure $\prod_{j=1}^k h_j(z_j, z_j)^{\nu_j} d\iota_j(z_j)$. Then*

$$C_{(s\nu_1, \cdots, s\nu_k)} \int_D |f(z) \prod_{j=1}^k h_j(z_j)^{\frac{\nu_j}{2}}|^{2s} \prod_{j=1}^k d\iota_j(z_j) \leq \|f\|^{2s}_{H_{(\nu_1, \cdots, \nu_k)}}, \quad f \in H. \tag{3}$$

The equality holds if f is a scalar multiple of the reproducing kernel $K_w(z) = h_1(z_1, w_1)^{-\nu_1} \cdots h_k(z_1, w_k)^{-\nu_k}$ *for some* $w = (w_1, \cdots, w_k) \in \prod_{j=1}^k D_j$.

Note that the constant $C_{(s\nu_1,\cdots,s\nu_k)}$ appears in LHS, and this is due to the fact that $K(z, w)^s = h_1(z_1, w_1)^{-s\nu_1} \cdots h_k(z_1, w_k)^{-s\nu_k}$ is the reproducing for the normalized measure $C_{(s\nu_1,\cdots,s\nu_k)} \prod_{j=1}^k h_j(z_j)^{s\nu_j} d\iota_j(z_j)$.

Remark 1 It seems that even when $D = \Delta^k = \Delta \times \cdots \times \Delta$ is a polydisc, the above result can not be deduced from the known results [6, 8, 10]. The above result can also be generalized [3] to a large class of weighted Bergman spaces of vector-valued holomorphic functions on bounded symmetric domains $D = G/K$ as unitary highest weight representations of G by using the similar idea as in [15] for representations of compact groups.

3 Wehrl-Type Inequalities Related to Berezin Transform

We prove another Wehrl-type inequality related to restriction of holomorphic functions on a real form and Berezin transform. This is closely related to Theorem 2; see Remark 2 below. Let D be an *irreducible* bounded symmetric domain $D = G/K$ with characteristic (r, a, b) as above. Let $S^p(H_\nu) \subset B(H_\nu)$ be the Schatten - von Neumann ideal in the space $B(H_\nu)$ of all bounded operators. The space $S^2(H_\nu)$ will be realized as the tensor product $H_\nu \otimes \overline{H_\nu} \subset L^2(D \times D, d\iota_\nu(z)d\iota_\nu(w))$, a general operator F in $B(H_\nu)$ is identified with its kernel $F(z, w)$, and accordingly the rank-one tensor $f \otimes f^*$ is identified with $f(z)\overline{f(w)}$.

Let $R : S^2(H_\nu) = H_\nu \otimes \overline{H_\nu} \to C^\infty(D)$ the operator

$$Rf(z) = f(z, z)h^\nu(z, z).$$

It is easy to prove by interpolation that $R : S^2(H_\nu) \to L^2(D, d\iota)$ is bounded and G-invariant. Its adjoint R^* is $R : L^2(D, d\iota) \to S^2(H_\nu)$ is the more familiar Toeplitz operator; see e.g. [2] and references therein. In [16] the spectral decomposition of RR^* was found in terms of Harish-Chandra Plancherel formula on $L^2(D, d\iota)$. We can deduce the following result from [16, Theorem 3.43] and we omit the proof here.

Proposition 1 *The Berezin transform RR^* is bounded on $L^2(D, \iota)$ and its norm is*

$$\|RR^*\| = b_\nu^2 := \frac{\Gamma_a^2(\nu - \frac{1+b}{2})}{\Gamma_a(\nu)\Gamma_a(\nu - \frac{d}{r})}.$$

We note that the norm of $\|RR^*\| = \|R\|^2 = \|R^*\|^2$ can not be achieved since RR^* has only continuous spectrum. However the norm can be approximated by spectral cut-off functions near the infimum of the spectrum of (positive) Laplace-Beltrami operator, i.e. it is achieved approximately by the Harish-Chandra spherical function ϕ_0 [7, 16, 19] at $\lambda = 0$.

Theorem 3 *1. Let $p \geq 2$. The following inequalities hold*

$$\|Rf\|_{L^p(D,d\iota)} \leq b_\nu^{\frac{2}{p}} \|f\|_{S_2(H_\nu)}, \tag{4}$$

for $f \in S_2(H_\nu)$, and

$$\|Rf\|_{L^p(D,d\iota)} \leq b_\nu^{\frac{2}{p}} \|f\|_{S_p(H_\nu)}, \tag{5}$$

$f \in S_p(H_\nu)$.

2. Let $p = 2s$ be an even integer. Then

$$\|Rf\|_{L^p(D,d\iota)} \leq \frac{b_{s\nu}^{\frac{1}{s}}}{C_{s\nu}^{\frac{1}{s}}} \|f\|_{S_2(H_\nu)}. \tag{6}$$

Proof The first part is obtained by interpolation, since $\|Rf\|_{L^\infty} \leq \|f\|_{B(H_\nu)} \leq \|f\|_{S^2(H_\nu)}$. The second part is combination of Proposition 1 applied to $H_{s\nu} \otimes H_{s\nu}$ and Theorem 2 for the Hilbert space $H_\nu \otimes \overline{H_\nu}$ as the Bergman space of holomorphic functions on $D \times \bar{D}$ ($\bar{D}$ being D with the opposite complex structure), namely

$$\begin{aligned}\|Rf\|^{2s}_{L^{2s}(D,d\iota)} &= \int_D |f(z,z)^s h(z)^{s\nu}|^2 d\iota(z) \leq b_{s\nu}^2 \|f^s\|^2_{H_{s\nu}\otimes\overline{H_{s\nu}}} \\ &\leq b_{s\nu}^2 \frac{1}{C_{s\nu}^2} \|f\|^{2s}_{H_\nu \otimes \overline{H_\nu}}.\end{aligned}$$

This completes the proof. □

It is possible that the estimate (6) is sharper than (5) for $p = 2s$, and that both are not best estimates. Thus it is a natural question to find the norm of $\|R\|_{S^2\to L^p}$ and $\|R\|_{S^p\to L^p}, 2 < p < \infty$. We conjecture that the norm $\|R\|_{S^p\to L^p}$ can be approximated by the polarization $\phi_0(z,w)h(z,w)^{-\nu}$ of the function $\phi_0(z)h(z)^{-\nu}$. An orthogonal expansion of $\phi_0(z,w)h(z,w)^{-\nu}$ has been found in [18, Theorem 10.1]; indeed the Harish-Chandra spherical function ϕ_0 is in L^p for $p > 2$.

Remark 2 Observe that each unit vector $f \in H_\nu$ defines a rank-one projection $f \otimes f^*$ with kernel $f(z)\overline{f(w)}$, and the map $R(f \otimes f^*)$ is then

$$R(f \otimes f^*)(z) = |f(z)|^2 h(z)^\nu,$$

and its L^p norm is

$$\|R(f \otimes f^*)\|^p_{L^p(D,d\iota)} = \int_D (|f(z)|^2 h(z)^\nu)^p d\iota(z).$$

Thus Theorem 1 gives the optimal bound for (2) acting on the subset of rank one projections in the space of all Hilbert-Schmidt operators. In particular when D is the unit disc the result of [6, 8] gives the optimal bound for (2) on rank one projections for any $p \geq 1$. Thus our question here is a natural generalization of the Wehrl $L^2 - L^p$ inequality [6, 8, 10].

Remark 3 The above question about various norms can be also formulated in the flat space $\mathbb{C}^d$. In this case the corresponding $R^*(f)$ is the Toeplitz operator T_f on the Fock space and the polarization of R^* produces the Weyl transform W, and W is bounded from $L^2(\mathbb{C}^d)$ to S^p on the Fock space. The best constant for W was found in [9] for $p \in (1, 2)$ being of the form $p = \frac{m}{m-1}$ for an even integer m. In view of the Wehrl inequality it might be interesting to find the norm of the dual map R, which might be more accessible than the Weyl transform.

4 Tensor Product of Bergman Spaces on the Unit Ball and Construction of Completely Positive Maps

As we have seen above tensor products of weighted Bergman spaces provide a tool to study relevant $L^2 - L^p$ inequalities, at least for p even integers. Indeed Lieb and Solovej [11] proved the Wehrl inequality for $SU(2)$-representations for general $p \geq 2$ using some construction of so-called quantum channeling involving tensor product decompositions of representations of $SU(2)$. We give now a construction of a large family of quantum channeling operators $A \mapsto \mathcal{T}_m(A)$ for the Bergman spaces on the unit ball $D = B^d$ as representations of the non-compact group $SU(d, 1)$. The operators $\mathcal{T}_m(A)$ are now on Bergman spaces of vector-valued holomorphic functions. This generalizes the construction of Lieb-Solovej, which corresponds to the case $m = 0$ for the projective space $\mathbb{P}^d$. The same construction can be made for general bounded symmetric domains, however it would require much more preparations.

4.1 Bergman Spaces of Symmetric Tensor-Valued Holomorphic Functions

Let $D = B^d$ be now the unit ball in $\mathbb{C}^d$ as symmetric space of $SU(d, 1)$. The genus is now $p_{B^d} = d + 1$ and the function $h(z, w) = 1 - \langle z, w\rangle$. The weighted Bergman space H_ν has the reproducing kernel $(1 - \langle z, w\rangle)^{-\nu}$ for $\nu > d$. It defines an irreducible (projective) unitary representation of G by

$$\pi_\nu(g) : f(z) \mapsto J_{g^{-1}}(z)^{\frac{\nu}{d+1}} f(g^{-1}z).$$

The unit disc is equipped with the Bergman metric $\langle B(z, z)^{-1}v, v\rangle$, for $v \in \mathbb{C}^d = T_z^{(1,0)}(B^d)$, $B(z, z) = (1 - |z|^2)(1 - zz^*)$. The dual space $(\mathbb{C}^d)' = \mathbb{C}^d$ is equipped with the dual metric $\langle B(z,z)^t v, v\rangle$. Now we define the weighted Bergman space $H_{\mu,m}$ of $\odot^m \mathbb{C}^d$-valued holomorphic functions f such that

$$\|f\|^2_{\mu,m} = c_{m,\mu} \int_{B^d} \langle \otimes^m B(z,z)^t f(z), f(z)\rangle (1-|z|^2)^\mu d\iota(z) < \infty,$$

where $\mu > d$, and

$$c_{m,\mu} = \frac{\Gamma(m+\mu-1)(2m+\mu-1)}{\pi^d \Gamma(m+\mu-d)\Gamma(d+1)}.$$

The reproducing kernel for the space $H_{\mu,m}$ is

$$K_{\mu,m}(z,w) = \otimes^m B^{-t}(z,w)(1-\langle z,w\rangle)^{-\mu},$$

in the sense that

$$\langle f(z), v\rangle = \langle f, K_{\mu,m}(\cdot,z)v\rangle_{\mu,m}.$$

The group G acts on $H_{\mu,m}$ by the defining geometric action

$$\pi_{\mu,m}(g) : f(z) \mapsto J_{g^{-1}}(z)^{\frac{\mu}{d+1}} \otimes^m dg^{-t}(z) f(g^{-1}z)$$

where dg^{-t} is the pull-back of the $dg^{-1} : T_z^{(1,0)} \to T_{g^{-1}z}^{(1,0)}$ of $g^{-1} \in G$.

4.2 Irreducible Decomposition of Tensor Product $H_\mu \otimes H_\nu$ and $SU(d,1)$-Invariant Completely Positive Maps

Consider the tensor product $H_\mu \otimes H_\nu$ of weighted Bergman spaces H_μ and H_ν, $\mu, \nu > d$, as representation of G. It is decomposed [14] under $G = SU(d,1)$ as

$$H_\mu \otimes H_\nu = \bigoplus_{m=0}^{\infty} H_{\mu+\nu,m},$$

where $H_{\mu+\nu,m}$ is the above weighted Bergman space of holomorphic functions taking values in the symmetric tensor of the co-tangent space.

The explicit decomposition is obtained by an integral $SU(d,1)$-invariant operator $Q_m : H_{\mu+\nu,m} \to H_\mu \otimes H_\nu$. Let $q_\zeta(\bar z)$ be the vector-valued function $q_\zeta(\bar z) = \frac{\bar z}{1-\langle \zeta, z\rangle}$ considered as a $(1,0)$-form at $\zeta \in B^d$ and anti-holomorphic function in z,

$$q_\zeta(z) = -\partial_\zeta \log(1-\langle \zeta, z\rangle) = \sum_{j=1}^d \frac{\bar z_j}{1-\langle \zeta, z\rangle} d\zeta_j = \frac{\bar z}{1-\langle \zeta, z\rangle}.$$

It is a polarization of the $\bar{\partial}$-primitive of the Bergman metric $-\partial_\zeta \log(1-\langle \zeta, \zeta\rangle)$. Then Q_m is given by

$$\begin{aligned} &Q_m f(z, w) \\ &= c_{\mu,\nu,m} \int_{B^d} (1-\langle z, \zeta\rangle)^{-\mu} (1-\langle w, \zeta\rangle)^{-\nu} \left(\otimes^m B^t(\zeta, \zeta) f(\zeta), (q_z(\zeta) - q_w(\zeta))^m\right) d\iota_{\mu+\nu}(\zeta), \end{aligned}$$

where $q_z(\zeta) = \frac{z}{1-\langle z, \zeta\rangle}$ and $(q_z(\zeta) - q_w(\zeta))^m$ is viewed as an element in $\odot^m T_\zeta^{(0,1)}$, $\otimes^m B^t(\zeta, \zeta) f(\zeta)$ as an element in $\odot^m T_\zeta^{(0,1)}$ as the Riesz representative of $f(\zeta) \in \odot^m (T_\zeta^{(1,0)})'$, and $(\cdot, \cdot)$ the natural bilinear pairing. Here $c_{\mu,\nu,m}$ is the constant so that Q_m is an isometry, and for $m = 0$ we have $c_{\mu,\nu,0} = 1$ since $Q_0 : 1 \in H_{\mu+\nu} \mapsto 1 \otimes 1 \in H_\mu \otimes H_\nu$. The dual operator $Q_m^* : H_\mu \otimes H_\nu \to H_{\mu+\nu,m}$ is a bilinear operator onto $H_{\mu+\nu,m}$ and is a higher dimensional analogue of the Rankin-Cohen bracket,

$$Q_m^* F(\zeta) = c \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{1}{(\mu)_j (\nu)_{m-j}} \partial_z^j \partial_w^{m-j} F(z, w)|_{z=w=\zeta},$$

where c is a constant depending on μ, ν, m. Observe also that for $\nu = \mu$ the operator Q_0 here is precisely the extension operator Q in Lemma 1 for $s = 2$.

For each component $H_{\mu+\nu,m}$ in the sum we construct completely positive maps from bounded operators on H_μ to operators on the component. Let

$$\mathcal{T}_m : A \in B(H_\mu) \mapsto \mathcal{T}_m(A) = Q_m^*(A \otimes I) Q_m \in B(H_{\mu+\nu,m})$$

where I is the identity operator on the space H_ν. The following result is obtained by direct computation.

Proposition 2 *The map $\frac{C_\mu}{C_{\mu+\nu}} \mathcal{T}_0$ is a completely positive trace preserving map from $S_1(H_\mu)$ to $S_1(H_{\mu+\nu})$. The operator $\mathcal{T}_0(f \otimes f^*)$ is given by*

$$\mathcal{T}_0(f \otimes f^*) g(z) = \int_{B^d} (1-\langle z, w\rangle)^{-\nu} f(z) \overline{f(w)} g(w) d\iota_{\mu+\nu}(w)$$

It is interesting to note that by construction we have for any orthonormal basis $\{f_\alpha\}$ of H_μ

$$\sum_\alpha \mathcal{T}_0(f_\alpha \otimes f_\alpha^*) = I$$

is the identity operator on $H_{\mu+\nu}$. Moreover when $f = 1$ and μ is an integer the operator $\mathcal{T}_0(f \otimes f^*)$ is

$$\mathcal{T}_0(f \otimes f^*) h(z) = \int_{B^d} \frac{(1-\langle z, w\rangle)^\mu}{(1-\langle z, w\rangle)^{\mu+\nu}} h(w) d\iota_{\mu+\nu}(w)$$

and is thus the formal "quantization" of the symbol $(1-\langle z, w\rangle)^{\mu}$. For example for $d=1$ it is

$$\sum_{j=0}^{\mu}(-1)^j\binom{\mu}{j}T_{z^j}T_{z^j}^*$$

corresponding to the expansion $(1-z\bar{w})^{\mu}=\sum_{j=0}^{\mu}(-1)^j\binom{\mu}{j}z^j\overline{w^j}$.

4.3 Limit of the Trace $Tr\,\mathcal{T}_0^p(f\otimes f^*)$

We find an asymptotic formula for the trace $\mathrm{Tr}\,\psi(\mathcal{T}_0(f\otimes f^*))$ of the functional calculus $\psi(\mathcal{T}_0(f\otimes f^*))$ of $\mathcal{T}_0(f\otimes f^*)$ for a continuous function ψ on $[0,1]$; in particular by taking $\psi(x)=x^p$, $p\geq 1$ we prove that the trace $\mathrm{Tr}\mathcal{T}_0^p(f\otimes f^*)$ is approximately the L^p-norm of $|f|_{\mu,z}^2=:|f(z)|^2(1-|z|^2)^{\mu}$. This is proved in [11, 12] in the setup of finite dimensional representations of $SU(N)$ using approximation of integral as Riemann sum and is one important step in proving the Wehrl inequality using quantum channeling operators.

Theorem 4 *Let $f\in H_{\mu}$ be a unit vector. For any polynomial ψ without constant term we have*

$$\lim_{\nu\to\infty}\frac{1}{C_{\mu+\nu}}Tr\,\psi(\mathcal{T}(f\otimes f^*))=\int_{B^d}\psi(|f|_{\mu,z}^2)d\iota(z).$$

This result can be proved for any bounded symmetric domain and strongly pseudo-convex domains and we shall not present the details here.

In the case of finite dimensional representations $\odot^s\mathbb{C}^d$ of $SU(d)$ Lieb and Solovej [11, 12] have proved that the operators $\mathcal{T}_0(f\otimes f^*)$ are dominated by $\mathcal{T}_0(f_0\otimes f_0^*)$ in the sense of eigenvalue distribution for f_0 being the reproducing kernel. The final and presumably very challenging question is whether similar property holds for our operators $\mathcal{T}_0(f\otimes f^*)$, even for the unit disc. Some preliminary results have been obtained in [3] where we express the eigenvalues of certain operators $\mathcal{T}_0(f\otimes f^*)$ as the summand in certain hypergeometric series. It might be also interesting to study the trace class property of $\mathcal{T}_m(f\otimes f^*)$ for $m>0$ and for general bounded symmetric domains.

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Converse of L^2 Existence and Extension of Cohomology Classes

Xiangyu Zhou

Abstract The present paper grew out of my talk at Hayama conference in 2022. We give a survey of recent progress on a converse of L^2 existence theorem and a criterion of Nakano positivity which is used to answer a problem of Lempert affirmatively. We also mention a recent result on the extension of cohomology classes which answers a problem of Cao–Demailly–Matsumura affirmatively, and an injectivity theorem as an application.

Keywords Multiplier ideal sheaves · Optimal L^2 extension · Converse L^2 theory · Nakano positivity

1 Background and Motivation

In this section, we recall some facts on the plurisubharmonic functions, multiplier ideal sheaves, holomorphic vector bundles.

1.1 Plurisubharmonic Functions

In several complex variables and complex geometry, a fundamental question is to construct or obtain specified holomorphic functions or sections. A basic philosophy to approach such a question is to reduce the question to the problem to construct or obtain specified plurisubharmonic functions. We list an example.

There were two fundamental questions in several complex variables and complex algebraic geometry:

Which kind of complex manifolds is Stein? (Levi problem).

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Which kind of complex manifolds is projective algebraic? (Kodaira embedding theorem).

The first was solved by Oka, Bremermann, Norguet, and Grauert: A complex manifold is Stein if and only if it's Kähler exhaustion (admitting a Kähler metric with exhaustion and global potential, which means that there is a smooth strictly plurisubharmonic exhaustion function). The later was solved by Kodaira: A compact complex manifold is projective if and only if there is a positive holomorphic line bundles. We know that a smooth Hermitian metric h on a holomorphic line bundle L can be locally written as $h = e^{-\varphi}$, where φ is a smooth function. And (L, h) is called positive if φ is smooth strictly plurisubharmonic.

The plurisubharmonic (*psh for short*) function, introduced by Oka and Lelong, plays a fundamental role in several complex variables and complex geometry, and has vast applications in various areas, as we already see from the above questions and their solutions. A psh function is upper semi-continuous and subharmonic when the function restricts to any complex line. Here, it's important that the psh function may take the value $-\infty$. The singularity of a psh function φ is the set $\{z : \varphi(z) = -\infty\}$. A typical example is that let $f_1, \ldots, f_k$ be holomorphic functions and $c > 0$, then $\varphi = c \log(|f_1|^2 + \cdots + |f_k|^2)$ is plurisubharmonic and the singularity $\varphi^{-1}(-\infty) = f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$ is a complex analytic subset.

A psh function with (neat) analytic singularities if locally $\varphi = c \log(|f_1|^2 + \cdots + |f_k|^2)$ up to a bounded function (smooth function). We say that φ is more singular than ψ, denoted by $\varphi \preccurlyeq \psi$, if $\varphi \le \psi + O(1)$, and that φ and ψ have equivalent singularities if $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \varphi$.

A singular hermitian metric h on a holomorphic line bundle L over a complex manifold (X, ω) is locally given by $e^{-\varphi}$, $\varphi \in L^1_{loc}$. Thus the curvature current is well-defined as $\Theta = i\partial\bar{\partial}\varphi$. A holomorphic line bundle is called pseudo-effective(*resp.* big) if $\Theta \ge 0$, i.e., φ is plurisubharmonic (*resp.* strictly but not necessarily smooth plurisubharmonic, $\Theta \ge \epsilon\omega$ for some $\epsilon > 0$).

1.2 Hörmander's L^2 Existence Theorem

Hörmander [31] obtained an L^2 estimate of $\bar{\partial}-$equation by using psh weights, which is an important technique in constructing and obtaining the holomorphic functions or sections. This also reflects the above philosophy mentioned at the beginning.

Theorem 1 (Hörmander's L^2 existence theorem) *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, φ be a plurisubharmonic function on D, then one can solve $\bar{\partial}u = v$ (where $\bar{\partial}v = 0$) with L^2 estimate $||u||_\varphi := \int_D |u|^2 e^{-\varphi} \le C||v||_\varphi$, for some constant $C > 0$.*

One can regard the psh weight in the above theorem as a singular metric on the (trivial) line bundle such that it's pseudoeffective. It is worthy to mention that there is an interesting application if we choose appropriate singularities for φ, which gives a solution of Levi problem.

Consider equation: $\bar{\partial} u = \bar{\partial}(\rho f)$, choose singular φ such that $e^{-\varphi}$ is not locally integrable at each z_i and $||v||_\varphi < +\infty$, therefore $u(z_i) = 0$. Take $F = \rho f - u$, so one can get a solution of the interpolation problem.

Corollary 1 *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $A = \{z_i\}$ be a discrete set in D, c_i be prescribed. Let $f(z_i) = c_i$, then there exists $F \in \mathcal{O}(D)$ such that $F|_A = f$.*

1.3 Multiplier Ideal Sheaves

The multiplier ideal sheaf associated to a psh function φ is an ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}$ whose germs consist of holomorphic functions $f \in \mathcal{O}_x$ such that $|f|^2 e^{-\varphi}$ is locally integrable at x. Similarly we can define L^p multiplier ideal sheaf if we replace 2 by p, i.e., $|f|^p e^{-\varphi}$ is locally integrable. Actually the origin of this notion goes back to Hörmander, Bombieri, Skoda as follows, which is actually an L^2 extension from a point.

Theorem 2 (Hörmander–Bombieri–Skoda theorem) *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and $\varphi \in \mathrm{psh}(\Omega)$, if $e^{-\varphi}$ is integrable on $\mathbb{B}^n(z_0; r) \subset\subset \Omega$, then there exists a holomorphic function $f \in \mathcal{O}(\Omega)$ so that $f(z_0) = 1$ and $\int_\Omega |f(z)|^2 e^{-\varphi}(r^2 + |z|^2)^{-n-\varepsilon} d\lambda < +\infty$ for any $\varepsilon > 0$.*

It is well-known that there are several classical invariants for the singularities of plurisubharmonic functions in addition to multiplier ideal sheaves:

1. the zero variety $V(\mathcal{I}(\varphi)) := \mathrm{Supp}\, \mathcal{O}/\mathcal{I}(\varphi) = \{x | e^{-\varphi} \text{ not locally integrable at } x\}$;
2. the Lelong number $v(\varphi, x) := \liminf\limits_{z \to x} \frac{\varphi(z)}{\log|z-x|}$;
3. the complex singularity exponent (log canonical threshold)

$$c_x(\varphi) := \sup\{c \geq 0 : e^{-2c\varphi} \text{ is locally integrable at } x\}.$$

Let us recall some basic properties of multiplier ideal sheaves.

Theorem 3 (Nadel's coherence theorem [43]) *The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is coherent.*

Notice that the quotient sheaf of coherent analytic sheaf is also coherent, as a corollary, the zero variety $V(\mathcal{I}(\varphi))$ is an analytic set.

Theorem 4 ([35]) *The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is integrally closed, i.e., the integral closure of $\mathcal{I}(\varphi)$ is itself.*

Theorem 5 (Nadel's vanishing theorem [11, 43]) *Let* $(L, e^{-\varphi})$ *be a big line bundle on a weakly pseudoconvex Kähler manifold* X. *Then*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0, \tag{1}$$

for any $q \geq 1$.

By Nadel's vanishing theorem which generalized Kawamata–Viehweg's vanishing theorem and Kodaira's vanishing theorem, multiplier ideal sheaves could be used to give a unified solution of both Levi problem and Kodaira embedding theorem (see [15]).

1.4 Holomorphic Vector Bundles

Let (X, ω) be a complex manifold of complex dimension n, equipped with a Hermitian metric ω, and (E, h) be a Hermitian holomorphic vector bundle of rank r over X. Let $D = D' + \bar{\partial}$ be the Chern connection of (E, h), and $\Theta_{E,h} = [D', \bar{\partial}] = D'\bar{\partial} + \bar{\partial}D'$ be the Chern curvature tensor. Denote by $(e_1, \ldots, e_r)$ an orthonormal frame of E over a coordinate patch $\Omega \subset X$ with complex coordinates $(z_1, \ldots, z_n)$, and

$$i\Theta_{E,h} = \sum_{1\leq j,k\leq n, 1\leq \lambda,\mu\leq r} ic_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu, \quad \bar{c}_{jk\lambda\mu} = c_{kj\mu\lambda}. \tag{2}$$

Associated to $i\Theta_{E,h}$ a natural Hermitian form $\theta_{E,h}$ on $TX \otimes E$ is given by

$$\theta_{E,h}(u, u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}(x) u_{j\lambda} \bar{u}_{k\mu}, \quad u = \sum_{j,\lambda} u_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_x X \otimes E_x. \tag{3}$$

Definition 1 (1) E is said to be Nakano positive (*resp.* Nakano semi-positive) if $\theta_{E,h}$ is positive (*resp.* semi-positive) definite as a Hermitian form on $TX \otimes E$, i.e., for every $u \in TX \otimes E$, $u \neq 0$, we have

$$\theta(u, u) > 0 \quad (\textit{resp.} \geq 0). \tag{4}$$

(2) E is said to be Griffiths positive (resp. Griffiths semi-positive) if for any $x \in X$, all $\xi \in T_x X$ with $\xi \neq 0$, and $s \in E_x$ with $s \neq 0$, we have

$$\theta(\xi \otimes s, \xi \otimes s) > 0 \quad (\textit{resp.} \geq 0). \tag{5}$$

(3) Nakano negative (semi-negative) and Griffiths negative (semi-negative) are similarly defined by replacing > 0 (*resp.* ≥ 0) with < 0 (*resp.* ≤ 0) in the above.

Let (X, ω) be a Kähler manifold, (E, h) be a Hermitian holomorphic vector bundle over X. A direct computation shows that

$$\langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda}\bar{u}_{k\mu} \tag{6}$$

where $u = \sum u_{j\lambda} dz \wedge d\bar{z}_j \otimes e_\lambda \in \Lambda^{n,1}T_X^* \otimes E$, and $dz = dz_1 \wedge \cdots \wedge dz_n$. Hence (E, h) is Nakano positive (resp. semi-positive) if and only if the Hermitian operator $[i\Theta_{E,h}, \Lambda_\omega]$ is positive definite (resp. semi-positive definite) on $\Lambda^{n,1}T_X^* \otimes E$.

Then Hörmander's L^2 existence Theorem 1 can be generalized to vector bundles as follows.

Theorem 6 ([12]) *Let X be a complete Kähler manifold, with a Kähler metric ω which is not necessarily complete. Let (E, h) be a Nakano semi-positive Hermitian vector bundle of rank r over X and $B := [i\Theta_{E,h}, \Lambda_\omega]$. Then for any form $g \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ satisfying $\bar\partial g = 0$ and $\int_X \langle B^{-1}g, g\rangle dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{n,q-1}T_X^* \otimes E)$ such that $\bar\partial f = g$ and*

$$\int_X |f|^2 dV_\omega \le \int_X \langle B^{-1}g, g\rangle dV_\omega. \tag{7}$$

2 Some Known Results

2.1 An Optimal L^2 Extension Theorem

An L^2 extension theorem is important. The above Theorem 2 addresses an L^2 extension from a point. The famous Ohsawa–Takegoshi L^2 extension theorem addresses an L^2 extension from a complex submanifold. There is a rich theory on L^2 extension (see [39, 44–46]), now we list a general optimal L^2 extension theorem, note that the paper [57] is a turning point toward solving the optimal L^2 extension problem.

Assume the pair (M, S) is almost Stein. Let (E, h) be a Hermitian holomorphic vector bundle on M with rank r which is Nakano semi-positive . Let $A \in (-\infty, +\infty)$ and $\Psi - A$ be a negative psh polar function on M, which is smooth on $M \setminus S$, $c_A(t)$ belongs to a class of smooth real functions satisfying some suitable conditions.

Theorem 7 ([24]) *There exists a uniform constant $\mathbf{C} = 1$ such that, for any holomorphic section f of $K_M \otimes E|_S$ on S of pure codimension k satisfying*

$$\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi] < \infty, \tag{8}$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M c_A(-\Psi)|F|_h^2 dV_M \le \mathbf{C}\left(\int_{-A}^{\infty} c_A(t)e^{-t}dt\right)\left(\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi]\right). \tag{9}$$

There are some applications of optimal L^2 extension theorem, here we list a few:

1. a solution of the Suita conjecture (Blocki [5, 6] for inequality part, [22] for the inequality part and its reformulation conjecture posed by Pommerenke–Suita [49], and [22, 24] for the full version of the Suita conjecture);
2. solutions of several questions of Ohsawa [23, 24, 53];
3. geometric property of the optimal L^2 extension theorem found in [24], a new understanding of Berndtsson's log-plurisubharmonicity theorem ([24], Berndtsson–Lempert [2]);
4. a use of the above geometric property to study the Iitaka conjecture by Hacon–Popa–Schnell [29] and the subadditivity of generalized Kodaira dimensions in [55];
5. the pseudo-effectivity of twisted relative pluricanonical bundles by combining the optimal $L^{\frac{2}{m}}$ extension theorem and a generalized Siu's lemma (in [54]), which was firstly established by Berndtsson–Paun [3], Paun–Takayama [48]); and an answer of a comparison question about two singular metrics of twisted pluricanonical bundles over exceptional fibers [54], which was asked by Berndtsson–Paun–Takayama.

2.2 Demailly's Strong Openness Conjecture

Denote by

$$\mathcal{I}_+(\varphi) := \bigcup_{\varepsilon>0} \mathcal{I}((1+\varepsilon)\varphi). \tag{10}$$

Demailly posed the following beautiful conjecture in [12].
Strong openness conjecture (SOC):

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi).$$

The conjecture was also posed by Siu [51], Demailly–Kollár [14], and many others.

It is easy to see that for any fixed $x \in X$, there exists $\varepsilon_0 > 0$ such that

$$\mathcal{I}_+(\varphi)_x = \mathcal{I}((1+\varepsilon_0)\varphi)_x. \tag{11}$$

Then the conjecture means that there exists an $\varepsilon_0 > 0$ such that $|f|^2 e^{-(1+\varepsilon_0)\varphi}$ is also integrable near x if $|f|^2 e^{-\varphi}$ is integrable near x. In other words, it means that $\{p \in \mathbb{R} : |f|^2 e^{-p\varphi}$ is integrable near $x\}$ is open.

Its origin is from calculus: $\{p \in \mathbb{R} : 1/|x|^{pc} = e^{-p\varphi}$ is locally integrable at the origin$\}$ is open, where $\varphi = c \log |x|$, $c > 0$.

The Openness conjecture asserts that: $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ under the assumption that $\mathcal{I}(\varphi)$ is trivial, which was proved by Berndtsson [1].

For the strong openness conjecture, the case of $\dim X = 1$ was well known; the case of $\dim X = 2$ was proved by Favre–Jonsson [19], Jonsson–Mustaţă [34]. Their methods are both algebraic. The conjecture "was thought to be rather inaccessible for $\dim X > 2$" (Math. Reviews, MR3418526).

In [25], Guan–Zhou used a different method to obtain the following result:

Theorem 8 *Demailly's strong openness conjecture holds.*

As noted in [25], Guan–Zhou's proof is suitable for the other two forms of SOC:

1. if the psh functions φ_j increasingly converges to a psh function φ, then

$$\bigcup_j \mathcal{I}(\varphi_j) = \mathcal{I}(\varphi). \tag{12}$$

2. for two psh functions φ, ψ,

$$\bigcup_{\varepsilon>0} \mathcal{I}((\varphi + \varepsilon\psi)) = \mathcal{I}(\varphi). \tag{13}$$

Under the assumption that the strong openness conjecture holds, Cao [8], Boucksom–Favre–Jonsson [7] and Demailly–Ein–Lazarsfeld [13] obtained important results.

In addition, the following stability of the multiplier ideal sheaves is established, based on [26].

Theorem 9 ([28]) *Let $(\varphi_j)_{j\in\mathbb{N}^+}$ be a sequence of negative psh functions on D, which converges to $\varphi \in Psh(D)$ in Lebesgue measure, and $\mathcal{I}(\varphi_j)_o \subset \mathcal{I}(\varphi)_o$. Let $F_j \in \mathcal{O}(D)$, $j \in \mathbb{N}^+$ such that $(F_j, o) \in \mathscr{I}(\varphi)_o$, which compactly converges to $F \in \mathcal{O}(D)$. Then, $|F_j|^2 e^{-\varphi_j}$ converges to $|F|^2 e^{-\varphi}$ in the L^1 norm near o. In particular, there exists $\varepsilon_0 > 0$ such that $\mathcal{I}(\varphi_j)_o = \mathcal{I}((1+\varepsilon_0)\varphi_j)_o = \mathcal{I}(\varphi)_o$ for any large enough j.*

Corollary 2 ([14, 30]) *Semi-continuity of complex singularity exponents and the weighted log canonical threshold holds.*

There is a Skoda's famous result about Lelong number.

Theorem 10 ([52]) *If $v(\varphi, 0) < 2$, then $\mathcal{I}(\varphi)_0 = \mathcal{O}_0$.*

Using the solution of the strong openness conjecture, we obtain the following result.

Theorem 11 ([27]) *If $v(\varphi, 0) = 2$, then either $\mathcal{I}(\varphi)_0 = \mathcal{O}_0$ or $\mathcal{I}(\varphi)_0 = \mathcal{I}(\log|h|)_0$ where h is the defining function of a germ of a regular analytic hypersurface through 0.*

For $n = 2$, the result is due to Blel–Mimouni [4], Favre and Jonsson [19].

3 Some Recent Progress

3.1 Converse of L^2 Existence Theorem

Let (X, ω) be a weakly pseudoconvex Kähler or almost Stein manifold, and (E, h) be a Hermitian holomorphic vector bundle over X.

If (E, h) is Nakano semi-positive, then (E, h) satisfies the optimal L^2-estimate condition and the multiple coarse L^2-estimate condition by works of Hörmander and Demailly.

Recently, a systematic study of the converse aspects of L^2 existence and L^2 extension in several complex variables has been started [16–18, 32].

Definition 2 ([18]) We say (E, h) satisfies the optimal L^2-estimate condition, if for any Stein coordinate U such that $E|_U$ is trivial, any Kähler form ω_U on U and any $\psi \in \mathrm{Spsh}(U) \cap C^\infty(U)$ such that for any $\bar\partial$-closed f $\in L^2_{(n,1)}(U, \omega_U, E|_U, he^{-\psi})$, there exists $u \in L^2_{(n,0)}(U, \omega_U, E|_U, he^{-\psi})$ such that $\bar\partial u = f$ and

$$\int_U |u|^2_{\omega_U,h} e^{-\psi} dV_{\omega_U} \leq \int_U \langle B^{-1}_{\omega_U,\psi} f, f\rangle_{\omega_U,h} e^{-\psi} dV_{\omega_U} \tag{14}$$

provided that the right hand is finite, where $B_{\omega_U,\psi} = [\sqrt{-1}\partial\bar\partial\psi \otimes \mathrm{Id}_E, \Lambda_{\omega_U}]$.

For the converse of L^2 existence theorem, one has the following converse proposition.

Theorem 12 (Characterization of Nakano positivity [18]) *If h is C^2 smooth, then (E, h) satisfies the optimal L^2-estimate condition if and only if h is Nakano semi-positive.*

As for the converse aspect of Ohasawa–Takegoshi L^2 extension theorem or optimal L^2 extension theorem, one can obtain the following result.

Theorem 13 ([16]) *If (E, h)* (h maybe Finsler) *satisfies the optimal or the multiple coarse L^2-extension condition, then (E, h) is Griffiths semi-positive.*

In particular, in the case of line bundles, $h = e^{-\varphi}$ locally, then φ is psh.

The above Theorem 12 can be used to answer the following basic problem on the complex differential geometry of holomorphic vector bundles affirmatively.

Lempert's problem ([36]): Let a C^2 Hermitian metric (E, h) be a convergence of an increasing sequence h_j of C^2-smooth Nakano semi-positive Hermitian metrics. Then is h Nakano semi-positive in the usual sense?

It is well-known that h_j satisfies the optimal L^2-estimate condition; consequently the limit metric h also satisfies the condition (see [33]). Therefore, as observed in [37], by Theorem 12, h is Nakano semi-positive in the usual sense, which affirmatively answers Lempert's problem.

3.2 Multiplier Submodule Sheaves

Let (L, h) be a pseudoeffective line bundle, i.e., the curvature current of h is semi-positive, then by Theorem 8 the associated multiplier ideal sheaf $\mathcal{I}(h)$ satisfies the strong openness, i.e., let a sequence of metrics h_j decreasingly converge to h, then $\bigcup_j \mathcal{I}(h_j) = \mathcal{I}(h)$. It's natural to ask if the multiplier submodule sheaves for the vector bundles satisfy the analogue properties.

Let X be a complex manifold, E be a holomorphic vector bundle of rank r over X. The following fact is well-known:

Let h be a smooth Hermitian metric on E. Then the following are equivalent:
(1) h is smooth Griffiths semi-negative.
(2) $|u|_h$ is plurisubharmonic for any local holomorphic section u of E.
(3) $\log |u|_h$ is plurisubharmonic for any local holomorphic section u of E.
(4) the dual metric h^* on E^* is smooth Griffiths semi-positive.

Now, we consider singular Hermitian metrics. A singular hermitian metric on E is a measurable map h from the base manifold X to the set of hermitian forms on the fibers of E, which assigns to almost every point $x \in X$ a hermitian form h_x on E_x satisfying $0 < \det h_x < +\infty$.

Definition 3 ([3, 48, 50]) Let E be a holomorphic vector bundle over a complex manifold X.
(1) A singular Hermitian metric h on E is called Griffiths semi-negative if $|u|_h$ is plurisubharmonic for any local holomorphic section u of E.
(2) A singular Hermitian metric h is Griffiths semi-positive if h^* is Griffiths semi-negative on E^*.

Inayama used the above Deng–Ning–Wang–Zhou's Theorem 12 on characterization of Nakano semipositivity to define Nakano semi-positivity for singular Hermitian metrics on holomorphic vector bundles.

Definition 4 ([33]) We say that a singular Hermitian metric h is Nakano semi-positive if
(1) h is Griffiths semi-positive;
(2) h satisfies the optimal L^2-estimate condition.

A standard limit argument in L^2-method gives the closeness of Nakano positivity under increasing approximation. Actually one can get the following singular version of Lempert's problem:

Theorem 14 ([38]) *Let h be a Griffiths semi-positive singular Hermitian metric. Let $\{h_j\}$ be a sequence of Nakano semi-positive singular Hermitian metrics. Assume that $\{h_j\}$ is bounded below by a continuous Hermitian metric and $h_j \le h$ converges to h. Then h is also Nakano semi-positive.*

Definition 5 ([10]) The multiplier submodule sheaf $\mathcal{E}(h)$ of $\mathcal{O}(E)$ associated to a singular Hermitian metric h on E is the sheaf of the germs of $s_x \in \mathcal{O}(E)_x$ such that $|s_x|_h^2$ is integrable in some neighborhood of x.

Theorem 15 ([32]) *The multiplier submodule sheaf $\mathcal{E}(h)$ associated to a Nakano semi-positive singular Hermitian metric h is coherent.*

Furthermore, Liu–Yang–Zhou proves the strong openness and stability for the multiplier submodule sheaf.

Theorem 16 ([38]) *Let E be a holomorphic vector bundle over a complex manifold X with Nakano semi-positive singular Hermitian metrics h and h_j, $j \geq 1$. Assume that $h_j \geq h$ and $-\log\det h_j$ converges to $-\log\det h$ locally in measure. Then $\bigcup_j \mathcal{E}(h_j) = \mathcal{E}(h)$.*

Moreover, assume that $(F_j, o) \in \mathcal{E}(h)_o$ and F_j compactly converges to F, then $|F_j|^2_{h_j}$ converges to $|F|^2_h$ in L^p in some neighborhood of o for some $p > 1$.

3.3 Injectivity Theorem and Extension of Cohomology Classes

Let (X, ω) be a holomorphically convex Kähler manifold, ψ be an L^1_{loc} function on X which is locally bounded above, and (L, h) be a singular Hermitian line bundle over X.

Assume that $\alpha > 0$ is a positive continuous function on X, and that the following two inequalities hold on X in the sense of currents:
(i) $\sqrt{-1}\Theta_{L,h} + \sqrt{-1}\partial\bar{\partial}\psi \geq 0$,
(ii) $\sqrt{-1}\Theta_{L,h} + (1+\alpha)\sqrt{-1}\partial\bar{\partial}\psi \geq 0$.

Theorem 17 ([9, 56]) *Keep the above setting, then the homomorphism induced by the natural inclusion $\mathcal{I}(he^{-\psi}) \longrightarrow \mathcal{I}(h)$,*

$$H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi})\big) \longrightarrow H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

is injective for every $q \geq 0$.

In other words, the homomorphism induced by the natural sheaf surjection $\mathcal{I}(h) \longrightarrow \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$,

$$H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big) \longrightarrow H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})\big)$$

is surjective for every $q \geq 0$.

The above was proved by Cao–Demailly–Matsumura [9] when ψ is a quasi-psh function with neat analytic singularities. The general case when ψ is a quasi-psh function with arbitrary singularities was posed as a question in the same paper of Cao–Demailly–Matsumura. As an application of the above extension theorem of cohomology classes, an injectivity theorem on holomorphically convex Kähler manifolds was obtained.

Theorem 18 ([56]) *Let X be a holomorphically convex Kähler manifold. Let (F, h_F) and (G, h_G) be two singular Hermitian line bundles over X.*

Assume that the following two inequalities hold on X in the sense of currents:
(i) $\sqrt{-1}\Theta_{F,h_F} \geq 0$,
(ii) $\sqrt{-1}\Theta_{F,h_F} \geq b\sqrt{-1}\Theta_{G,h_G}$ for some $b \in (0, +\infty)$.

Then, for a non-zero global holomorphic section s of G satisfying $\sup_{\Omega} |s|_{h_G} < +\infty$ for every $\Omega \subset\subset X$, the following map β induced by the tensor product with s

$$H^q\big(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h_F)\big) \xrightarrow{\beta} H^q\big(X, \mathcal{O}_X(K_X \otimes F \otimes G) \otimes \mathcal{I}(h_F h_G)\big)$$

is injective for every $q \geq 0$.

The above injectivity theorem unifies several recent injectivity theorems obtained by Matsumura [40–42], Fujino–Matsumura [20], Gongyo–Matsumura [21].

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