BOUNDARY POINTS, MINIMAL L^2 INTEGRALS AND CONCAVITY PROPERTY

SHIJIE BAO, QI'AN GUAN, AND ZHENG YUAN

ABSTRACT. For the purpose of proving the strong openness conjecture of multiplier ideal sheaves, Jonsson-Mustaţă posed an enhanced conjecture and proved the two-dimensional case, which says that: the Lebesgue measure of the set $\left\{c_o^F(\psi)\psi - \log|F| < \log r\right\}$ divided by r^2 has a uniform positive lower bound independent of r, for a plurisubharmonic function ψ and a holomorphic function F near the origin o. Jonsson-Mustaţă's conjecture was proved by Guan-Zhou depending on the truth of the strong openness conjecture. However, it is still a question whether one can prove Jonsson-Mustaţă's conjecture without using the strong openness property, and obtain a sharp effectiveness result for this conjecture.

In this article, we use an L^2 method with the weight functions $\psi - \log|F|$ and firstly consider a module at at a boundary point of the sublevel sets of a plurisubharmonic function. By studying the minimal L^2 integrals on the sublevel sets of a plurisubharmonic function with respect to the module at the boundary point, we establish a concavity property of the minimal L^2 integrals. As applications, we obtain a sharp effectiveness result related to Jonsson-Mustaţă's conjecture, which completes the approach from the conjecture to the strong openness property. We also obtain a strong openness property of the module and a lower semi-continuity property with respect to the module.

CONTENTS

1. INTRODUCTION

1.1. Backgrounds and Motivations.

Let ψ be a plurisubharmonic function of a complex manifold M (see [9]). Recall that multiplier ideal sheaf $\mathcal{I}(\psi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\psi}$ is locally integrable, which was widely discussed in the study of several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [48, 42, 44, 12, 13, 11, 14, 39, 40, 45, 46, 10, 33]).

The strong openness property of multiplier ideal sheaves, i.e.

$$\mathcal{I}(\psi) = \mathcal{I}_+(\psi) := \bigcup_{\epsilon > 0} \mathcal{I}((1+\epsilon)\psi),$$

is an important feature of multiplier ideal sheaves, and "opened the door to new types of approximation techniques" (see [41]) (see e.g. [29, 37, 5, 6, 17, 7, 50, 32, 4, 51, 52, 18, 38, 8]). The strong openness property was conjectured by Demailly [11], and proved by Guan-Zhou [29] (the 2-dimensional case was proved by Jonsson-Mustaţă [35]). After that, Guan-Zhou [31] established an effectiveness result of the strong openness property by considering the minimal L^2 integral on the pseudoconvex domain D.

When $\mathcal{I}(\psi) = \mathcal{O}$, the strong openness property degenerates to the openness property, which was conjectured by Demailly-Kollár [13] and proved by Berndtsson [3] (the 2-dimensional case was proved by Favre-Jonsson in [15]).

Let D be a pseudoconvex domain in $\mathbb{C}^n$ containing the origin $o \in \mathbb{C}^n$, and let ψ be a plurisubharmonic function on D. For a holomorphic function h on a neighborhood of o, recall that

$$c_o^h(\psi) := \sup \left\{ c \geq 0 : |h|^2 e^{-2c\psi} \text{ is } L^1 \text{ on a neighborhood of } o \right\}$$

is the jumping number (see [36]). When $h \equiv 1$, $c_o^h(\psi)$ degenerates to the complex singularity exponent $c_o(\psi)$ (see [48, 13]).

In [13] (see also [36]), Demailly and Kollár conjectured the following more precise form of the openness property:

Conjecture D-K: *If $c_o(\psi) < +\infty$, $\frac{1}{r^2}\mu(\{c_o(\psi)\psi < \log r\})$ has a uniform positive lower bound independent of $r \in (0,1)$, where μ is the Lebesgue measure on $\mathbb{C}^n$.*

The 2-dimensional case of Conjecture D-K was proved by Favre-Jonsson in [15], which deduces the 2-dimensional case of the openness property. Depending on the openness property, Guan-Zhou [31] proved Conjecture D-K.

In [21], Guan gave a sharp effectiveness result related to Conjecture D-K independent of the openness property, which completed the approach from Conjecture D-K to the openness property.

In [49], Xu completed the algebraic approach to Conjecture D-K.

In order to prove the strong openness property, Jonsson and Mustaţă (see [36], see also [35]) posed the following conjecture, and proved the 2-dimensional case [35]:

Conjecture J-M: *If $c_o^F(\psi) < +\infty$ for a holomorphic function F on D, then $\frac{1}{r^2}\mu(\{c_o^F(\psi)\psi - \log|F| < \log r\})$ has a uniform positive lower bound independent of $r \in (0,1)$, where μ is the Lebesgue measure on $\mathbb{C}^n$.*

Depending on the strong openness property, Guan-Zhou [31] proved Conjecture J-M:

Theorem 1.1 ([31]). *If $c_o^F(\psi) < +\infty$ and $\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}} < +\infty$, then*

$$\liminf_{r \to 0+0} \frac{1}{r^2}\mu(\{c_o^F(\psi)\psi - \log|F| < \log r\})$$

$$\geq \sup_{\delta \in \mathbb{Z}_{>0}} \frac{C_{F^{1+\delta}, 2c_o^F(\psi)\psi + \delta\max\{2c_o^F(\psi)\psi, 2\log|F|\}}(o)}{\left(1 + \frac{1}{\delta}\right)\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}}},$$

where for a plurisubharmonic function ϕ and a holomorphic function F_0 on D,

$$C_{F_0, \phi}(o) := \inf\left\{\int_D |\tilde{F}|^2 : \tilde{F} \in \mathcal{O}(D) \And (\tilde{F} - F_0, o) \in \mathcal{I}(\phi)_o\right\}.$$

A natural question is:

Question 1.2. *Can one obtain a sharp effectiveness result related to Conjecture J-M **independent** of the strong openness property, which completes the approach from Conjecture J-M to the strong openness property?*

We give an affirmative answer to Question 1.2 in the present paper.

1.1.1. *Formulations.*

Recall that considering the minimal L^2 integrals on the sublevel sets of a given plurisubharmonic weight ψ, Guan [21] established a concavity property of the minimal L^2 integrals on the sublevel sets of ψ (see also [20, 22, 23, 26, 24, 25]), and gave a sharp effectiveness result related to Conjecture D-K independent of the openness property.

The minimal L^2 integrals on the sublevel sets of the plurisubharmonic function ψ considered in [21] (see also [20, 22, 23, 26, 24, 25]) are respect to the modules(ideals) at the inner points of the sublevel sets.

However, in Conjecture J-M, we need to deal with the weight function $\psi - \log|F|$, which is not plurisubharmonic on the entire domain D, instead of a purely plurisubharmonic function ψ. In addition, the considered singular point (which is the origin o in Conjecture J-M) can be not an inner point but a boundary point of the sublevel sets of the weight function $\psi - \log|F|$.

To cross the obstacles mentioned above, we need an (optimal) L^2 estimate result with the weight function $\psi - \log|F|$, and we also need a module at the boundary point to replace the (multiplier) ideal at the inner point. Motivated the L^2 methods in [30, 31, 21], we give an L^2 method with the weight functions $\psi - \log|F|$ in Section 2.1, where the technical part of the proof is given in the Appendix. In section 2.2,

we firstly discuss a module $I(a\Psi)$ at the boundary point and give some useful properties of the the module.

After these preparations, we consider the minimal L^2 integrals with respect to a module at a boundary point of the sublevel sets related to a function in the shape of $\psi - \log |F|$, and we obtain a concavity property of the minimal L^2 integrals. Then using the concavity property, we obtain a sharp effectiveness result of Conjecture J-M.

It is worth noting that the effectiveness result (Corollary 1.8) of Conjecture J-M in the present paper implies Theorem 1.1 (see Remark 1.9), and its proof does not depend on the truth of the strong openness property. Consequently, the present paper completes the approach from Conjecture J-M to the strong openness property moreover.

Using the concavity property of the minimal L^2 integrals with respect to a module at a boundary point, we also obtain a strong openness property of the module and a lower semi-continuity property with respect to the module.

1.2. Concavity property.

Let F be a holomorphic function on a pseudoconvex domain $D \subset \mathbb{C}^n$ containing the origin $o \in \mathbb{C}^n$, and let ψ be a plurisubharmonic function on D. Denote

$$\Psi := \min\{\psi - 2\log |F|, 0\}.$$

If $F(z) = 0$ for $z \in D$, we set $\Psi(z) = 0$. Note that $\Psi = \psi + 2\log |1/F|$ is a plurisubharmonic function on $\{\Psi < 0\}$. Note that, *if $F(o) = 0$, then o can be a boundary point of the sublevel set $\{\Psi < -t\}$ for any $t > 0$.*

Set

$$\tilde{J}(\Psi)_o := \big\{ f \in \mathcal{O}(\{\Psi < -t\} \cap V) : t \in \mathbb{R} \text{ and } V \text{ is a neighborhood of } o \big\}.$$

We define the following **module at the boundary point** o of the sublevel sets $\{\Psi < -t\} = \{\psi - 2\log |F| < -t\}$, where $t \geq 0$.

Definition 1.3. *Denote by $J(\Psi)_o$ the set of the equivalence classes f_o of $f \in \tilde{J}(\Psi)_o$ under the equivalence relation $f \sim g$ if $f = g$ on $\{\Psi < -t\} \cap V$, where $t \gg 1$ and V is some neighborhood of o.*

If $o \in \bigcap_{t>0}\{\Psi < -t\}$, then $J(\Psi)_o$ equals to $\mathcal{O}_{\mathbb{C}^n,o}$, and f_o is the germ (f, o) of a holomorphic function f.

For any $f_o \in J(\Psi)_o$, $\tilde{f}_o \in J(\Psi)_o$ and $(h, o) \in \mathcal{O}_{\mathbb{C}^n,o}$, define

$$f_o + \tilde{f}_o = (f + \tilde{f})_o, \text{ and } (h, o) \cdot f_o = (hf)_o.$$

Then it is clear that $J(\Psi)_o$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module.

Definition 1.4. *For any $h_o \in J(\Psi)_o$ and any $a \geq 0$, we call $h_o \in I(a\Psi)_o$ if and only if there exist $t \gg 1$ and a neighborhood V of o such that $\int_{\{\Psi<-t\}\cap V} |h|^2 e^{-a\Psi} < +\infty$.*

It is clear that $I(a\Psi)_o$ is an $\mathcal{O}_{\mathbb{C}^n,o}$−submodule of $J(\Psi)_o$. Denote $I_o := I(0\Psi)_o$.

Let f be a holomorphic function on $V_0 \cap \{\Psi < -t_0\}$, where V_0 is a neighborhood of o and $t_0 > 0$. Let J be an $\mathcal{O}_{\mathbb{C}^n,o}$−submodule of I_o such that $I(\Psi)_o \subset J$.

Definition 1.5. *We define the **minimal L^2 integrals on the sublevel sets** $\{\Psi < -t\} = \{\psi - 2\log |F| < -t\}$ **with respect to** J (which is a module at the*

boundary point o of the sublevel sets) as follows:

$$G(t; \Psi, J, f) := \inf \left\{ \int_{\{\Psi < -t\}} |\tilde{f}|^2 : \tilde{f} \in \mathcal{O}(\{\Psi < -t\}) \,\&\, (\tilde{f} - f)_o \in J \right\},$$

where $t \in [0, +\infty)$.

In the present article, we establish the following concavity of $G(-\log r; \Psi, J, f)$.

Theorem 1.6. *If there exists some $t_0 \geq 0$ such that $G(t_0; \Psi, J, f) < +\infty$, then $G(-\log r; \Psi, J, f)$ is concave with respect to $r \in (0, 1]$ and $\lim_{t \to +\infty} G(t; \Psi, J, f) = 0$.*

When $F \equiv 1$ and $\psi(o) = -\infty$, Theorem 1.6 can be referred to [21] (see also [20, 22, 23, 26, 24, 25]). We also give an example in Appendix (Section 7.2) for the case that F is not a constant function.

1.3. Applications.
In this section, we present some applications of Theorem 1.6.

1.3.1. *A sharp effectiveness result related to Jonsson-Mustaţă's conjecture.*
Let f be a holomorphic function on D. Denote

$$\Psi_1 := \min \left\{ 2c_o^{fF}(\psi)\psi - 2\log|F|, 0 \right\},$$

and

$$I_+(\Psi_1)_o := \bigcup_{a>1} I(a\Psi_1)_o.$$

Recall the definition of the minimal L^2 integrals G in Section 1.2. Theorem 1.6 implies the following result independent of the strong openness property.

Theorem 1.7. *If $c_o^{fF}(\psi) < +\infty$, then*

$$\frac{1}{r^2} \int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 \geq G(0; \Psi_1, I_+(\Psi_1)_o, f) > 0$$

holds for any $r \in (0, 1]$.

The constant $G(0; \Psi_1, I_+(\Psi_1)_o, f)$ is sharp. Let $D = \Delta \subset \mathbb{C}$ be the unit disc, and let $\psi = \log|z|$. Let $F \equiv 1$, and let $f \equiv 1$. It is clear that $c_o^{fF}(\psi) = 1$, $\int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 = \pi r^2$ and $G(0; \Psi_1, I_+(\Psi_1)_o, f) = \pi$, which implies the sharpness of Theorem 1.7. A boundary point example can be seen in Appendix (Section 7.2).

When $f \equiv 1$, Theorem 1.7 is the following sharp effectiveness result related to Conjecture J-M.

Corollary 1.8. *If $c_o^F(\psi) < +\infty$, then*

$$\frac{1}{r^2} \mu\{c_o^F(\psi)\psi - \log|F| < \log r\} \geq G(0; \Psi_1, I_+(\Psi_1)_o, 1) > 0$$

holds for any $r \in (0, 1]$, where $\Psi_1 = \min\{2c_o^F(\psi)\psi - 2\log|F|, 0\}$.

The proof of Corollary 1.8 is independent of the strong openness property, and Corollary 1.8 recovers Theorem 1.1 due to the following remark.

Remark 1.9. *For any $\delta \in \mathbb{Z}_{>0}$, it holds that*

$$G(0; \Psi_1, I_+(\Psi_1)_o, 1) \geq \frac{C_{F^{1+\delta}, 2c_o^F(\psi)\psi + \delta \max\{2c_o^F(\psi)\psi, 2\log|F|\}}(o)}{\left(1 + \frac{1}{\delta}\right)\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}}},$$

hence Corollary 1.8 implies that

$$\frac{1}{r^2}\mu(\{c_o^F(\psi)\psi - \log|F| < \log r\})$$
$$\geq \sup_{\delta \in \mathbb{Z}_{>0}} \frac{C_{F^{1+\delta}, 2c_o^F(\psi)\psi + \delta\max\{2c_o^F(\psi)\psi, 2\log|F|\}}(o)}{\left(1 + \frac{1}{\delta}\right)\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}}}.$$

holds for any $r \in (0, 1]$. We prove the remark in Section 4.2.

When $F \equiv 1$, Theorem 1.7 is the following sharp effectiveness result related to Conjecture D-K.

Corollary 1.10 ([21]). *If $c_o^f(\psi) < +\infty$, then*

$$\frac{1}{r^2}\int_{\{c_o^f(\psi)\psi < \log r\}} |f|^2 \geq G(0; \Psi_1, I_+(\Psi_1)_o, f) > 0$$

holds for any $r \in (0, 1]$, where $\Psi_1 = \min\{2c_o^f(\psi)\psi, 0\}$.

1.3.2. *The strong openness property of $I(a\Psi)_o$.*

In this section, we present a strong openness property of the module $I(a\Psi)_o$.

Let F be a holomorphic function on a pseudoconvex domain $D \subset \mathbb{C}^n$ containing the origin $o \in \mathbb{C}^n$, and let ψ be a plurisubharmonic function on D. Denote

$$\Psi := \min\{\psi - 2\log|F|, 0\}.$$

Let f be a holomorphic function on $\{\Psi < -t_0\}$ such that $f_o \in I_o$. Set

$$a_o^f(\Psi) := \sup\{a \geq 0 : f_o \in I(2a\Psi)_o\}.$$

Especially, when $F \equiv 1$ and $\psi(o) = -\infty$, $a_o^f(\Psi)$ is the jumping number $c_o^f(\psi)$.

Using Theorem 1.6, we obtain the following estimate for L^2 integrals on the sublevel sets of Ψ:

Theorem 1.11. *Assume that $a_o^f(\Psi) < +\infty$, then we have $a_o^f(\Psi) > 0$ and*

$$\frac{1}{r^2}\int_{\{a_o^f(\Psi)\Psi < \log r\}} |f|^2 \geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f) > 0$$

holds for any $r \in (0, e^{-a_o^f(\Psi)t_0}]$.

Theorem 1.11 implies the following strong openness property of $I(a\Psi)_o$.

Corollary 1.12. *$I(a\Psi)_o = I_+(a\Psi)_o$ holds for any $a \geq 0$.*

When $F \equiv 1$ and $\psi(o) = -\infty$, Corollary 1.12 is the strong openness property of multiplier ideal sheaves ([29]).

1.3.3. A lower semicontinuity property.

Let $\{\psi_m\}_{m\in\mathbb{Z}_{>0}}$ be a sequence of plurisubharmonic functions on a pseudoconvex domain Δ^n, and let $\{F_m\}_{m\in\mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on Δ^n. Denote $\Psi_m := \min\{\psi_m - 2\log|F_m|, 0\}$. Assume that $\{\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ converges to a Lebesgue measurable function Ψ on Δ^n in Lebesgue measure. Let $\{f_m\}_{m\in\mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on $\{\Psi_m < -t_m\}$, where $t_m > 0$.

In this section, we present a lower semicontinuity property of Ψ_m with respect to $I(\Psi_m)_o$.

Proposition 1.13. *Assume that there exist $t_0 \geq t_m$ for any $m \in \mathbb{Z}_{>0}$ and a sequence of Lebesgue functions $\{\tilde{f}_m\}_{\mathbb{Z}_{>0}}$ on Δ^n with uniform bound satisfying that $\tilde{f}_m = f_m$ on $\{\Psi_m < -t_0\}$ for any $m \in \mathbb{Z}_{>0}$ and $\{\tilde{f}_m\}_{\mathbb{Z}_{>0}}$ converges to a Lebesgue measurable function $\tilde{f}$ on Δ^n in Lebesgue measure. Assume that for any pseudoconvex domain $D \subset \Delta^n$ containing the origin o,*

$$\inf_m G(0; \Psi_m, I(\Psi_m)_o, f_m) > 0.$$

Then we have

$$|\tilde{f}|^2 e^{-\Psi} \notin L^1(U),$$

where U is any neighborhood of o.

If $\{\psi_m\}_{m\in\mathbb{Z}_{>0}}$ and $\{\log|F_m|\}_{m\in\mathbb{Z}_{>0}}$ converge to a plurisubharmonic function ψ and $\log|F|$ in Lebesgue measure respectively, where F is a holomorphic function, and there exists a holomorphic function f on $\{\psi - 2\log|F| < -t\} \cap V$ such that $f = \tilde{f}$ on $\{\psi - 2\log|F| < -t\} \cap V$, where $t > 0$ and V is a neighborhood of o, we have

$$f_o \notin I(\Psi)_o.$$

When $F_m \equiv 1$ and $\psi_m(o) = -\infty$ for any $m \in \mathbb{Z}_{>0}$, Proposition 1.13 is Proposition 1.8 in [31].

2. Preparations

2.1. L^2 methods.

Let F be a holomorphic function on a pseudoconvex domain $D \subset \mathbb{C}^n$, and let ψ be a plurisubharmonic function on D. Let δ be a positive integer. Denote

$$\varphi := (1+\delta)\max\{\psi, 2\log|F|\},$$

and

$$\Psi := \min\{\psi - 2\log|F|, 0\}.$$

If $F(z) = 0$ for $z \in D$, we set $\Psi(z) = 0$. Then $\varphi + \Psi$ and $\varphi + (1+\delta)\Psi$ are both plurisubharmonic functions on D.

Lemma 2.1. *Let $B \in (0, +\infty)$ and $t_0 \in (0, +\infty)$ be arbitrarily given. Let f be a holomorphic function on $\{\Psi < -t_0\}$ such that*

$$\int_{\{\Psi < -t_0\}} |f|^2 < +\infty.$$

Then there exists a holomorphic function $\tilde{F}$ on D such that

$$\int_D |\tilde{F} - (1 - b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{-\varphi + v_{t_0,B}(\Psi) - \Psi}$$

$$\leq \left(\frac{1}{\delta} + 1 - e^{-t_0 - B}\right) \int_D \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f|^2 e^{-\Psi},$$

where $b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B} \mathbb{I}_{\{-t_0 - B < s < -t_0\}} ds$ and $v_{t_0,B}(t) = \int_0^t b_{t_0,B}(s) ds$.

We give the proof of Lemma 2.1 in Appendix. Lemma 2.1 implies the following lemma.

Lemma 2.2 (see [21], see also [25, 26]). *Let $B \in (0, +\infty)$ and $t_0 > t_1 \geq 0$ be arbitrarily given. Let f be a holomorphic function on $\{\Psi < -t_0\}$ such that*

$$\int_{\{\Psi < -t_0\}} |f|^2 < +\infty.$$

Then there exists a holomorphic function $\tilde{F}$ on $\{\Psi < -t_1\}$ such that

$$\int_{\{\Psi < -t_1\}} |\tilde{F} - (1 - b_{t_0,B}(\Psi))f|^2 e^{v_{t_0,B}(\Psi) - \Psi}$$

$$\leq \left(e^{-t_1} - e^{-t_0 - B}\right) \int_D \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f|^2 e^{-\Psi},$$

where $b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B} \mathbb{I}_{\{-t_0 - B < s < -t_0\}} ds$ and $v_{t_0,B}(t) = \int_0^t b_{t_0,B}(s) ds$.

Proof. Let δ be any positive integer. Set $\tilde{\varphi} := (1 + \delta) \max\{\psi + t_1, 2 \log |F|\}$ and $\tilde{\Psi} := \{\psi + t_1 - 2 \log |F|, 0\}$. Note that $\{\Psi < -t_0\} = \{\tilde{\Psi} < -(t_0 - t_1)\}$ and $\tilde{\Psi} = \Psi + t_1$ on $\{\Psi < -t_1\}$. Lemma 2.1 shows that there exists a holomorphic function $\tilde{F}_\delta$ on D such that

$$\int_D |\tilde{F}_\delta - (1 - b_{t_0 - t_1, B}(\tilde{\Psi}))fF^{1+\delta}|^2 e^{-\tilde{\varphi} + v_{t_0 - t_1, B}(\tilde{\Psi}) - \tilde{\Psi}}$$

$$\leq \left(\frac{1}{\delta} + 1 - e^{-t_0 - B + t_1}\right) \int_D \frac{1}{B} \mathbb{I}_{\{-t_0 - B + t_1 < \tilde{\Psi} < -t_0 + t_1\}} |f|^2 e^{-\tilde{\Psi}},$$

which implies that

$$\int_{\{\Psi < -t_1\}} |\tilde{F}_\delta - (1 - b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{-\varphi + v_{t_0,B}(\Psi) - \Psi}$$

$$= \int_{\{\Psi < -t_1\}} |\tilde{F}_\delta - (1 - b_{t_0 - t_1, B}(\tilde{\Psi}))fF^{1+\delta}|^2 e^{-\tilde{\varphi} + v_{t_0 - t_1, B}(\tilde{\Psi}) - \tilde{\Psi}}$$

$$\leq \int_D |\tilde{F}_\delta - (1 - b_{t_0 - t_1, B}(\tilde{\Psi}))fF^{1+\delta}|^2 e^{-\tilde{\varphi} + v_{t_0 - t_1, B}(\tilde{\Psi}) - \tilde{\Psi}} \tag{2.1}$$

$$\leq \left(\frac{1}{\delta}e^{-t_1} + e^{-t_1} - e^{-t_0 - B}\right) \int_D \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f|^2 e^{-\Psi}.$$

Let $F_\delta = \frac{\tilde{F}_\delta}{F^{1+\delta}}$ be a holomorphic function on $\{\Psi < -t_1\}$. Since $|F|^{2(1+\delta)} e^{-\varphi} = 1$ on $\{\Psi < -t_1\}$, inequality (2.1) becomes

$$\int_{\{\Psi < -t_1\}} |F_\delta - (1 - b_{t_0,B}(\Psi))f|^2 e^{v_{t_0,B}(\Psi) - \Psi}$$

$$\leq \left(\frac{1}{\delta}e^{-t_1} + e^{-t_1} - e^{-t_0 - B}\right) \int_D \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f|^2 e^{-\Psi}. \tag{2.2}$$

As $\int_{\{\Psi<-t_0\}}|f|^2<+\infty$ and $v_{t_0,B}(\Psi)-\Psi\geq 0$, it follows from inequality (2.2) that

$$\sup_\delta \int_{\{\Psi<-t_1\}}|F_\delta|^2$$

$$\leq 2\int_{\{\Psi<-t_1\}}|(1-b_{t_0,B}(\Psi))f|^2+2\sup_\delta\int_{\{\Psi<-t_1\}}|F_\delta-(1-b_{t_0,B}(\Psi))f|^2$$

$$\leq 2\int_{\{\Psi<-t_0\}}|f|^2+2\sup_\delta\left(\frac{1}{\delta}e^{-t_1}+e^{-t_1}-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}$$

$$<+\infty,$$

yielding that there exists a subsequence of $\{F_\delta\}$ (also denoted by $\{F_\delta\}$) compactly convergent to a holomorphic function $\tilde{F}$ on $\{\Psi<-t_1\}$. It follows from Fatou's Lemma and inequality (2.2) that

$$\int_{\{\Psi<-t_1\}}|F-(1-b_{t_0,B}(\Psi))f|^2e^{v_{t_0,B}(\Psi)-\Psi}$$

$$=\int_{\{\Psi<-t_1\}}\lim_{\delta\to+\infty}|F_\delta-(1-b_{t_0,B}(\Psi))f|^2e^{v_{t_0,B}(\Psi)-\Psi}$$

$$\leq\liminf_{\delta\to+\infty}\int_{\{\Psi<-t_1\}}|F_\delta-(1-b_{t_0,B}(\Psi))f|^2e^{v_{t_0,B}(\Psi)-\Psi}$$

$$\leq\liminf_{\delta\to+\infty}\left(\frac{1}{\delta}e^{-t_1}+e^{-t_1}-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}$$

$$=\left(e^{-t_1}-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}.$$

Thus, Lemma 2.2 holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2. $\mathcal{O}_{\mathbb{C}^n,o}$-module $I(a\Psi)_o$.

Let F be a holomorphic function on a pseudoconvex domain $D\subset\mathbb{C}^n$ containing the origin $o\in\mathbb{C}^n$, and let ψ be a plurisubharmonic function on D. Denote $\Psi:=\min\{\psi-2\log|F|,0\}$ and $\varphi:=2\max\{\psi,2\log|F|\}$. In this section, we discuss the $\mathcal{O}_{\mathbb{C}^n,o}$-module $I(a\Psi)_o$, where $a\geq 0$.

Let k be a positive integer. Denote $I_o:=I(0\Psi)_o$.

Lemma 2.3. *For any $f_o\in I_o$, there exists a pseudoconvex domain $D_0\subset D$ containing o and a holomorphic function $\tilde{F}$ on D_0 such that $(\tilde{F},o)\in\mathcal{I}(k\varphi)_o$ and $\int_{\{\Psi<-t\}\cap D_0}|\tilde{F}-fF^{2k}|^2e^{-k\varphi-k\Psi}<+\infty$ for some $t>0$.*

Proof. For any $f_o\in I_o$, there exists a pseudoconvex domain $D_0\subset D$ containing o and $t_0>0$ such that $f\in\mathcal{O}(\{k\Psi<-t_0\}\cap D_0)$ and $\int_{\{k\Psi<-t_0\}\cap D_0}|f|^2<+\infty$. It follows from Lemma 2.1 that there exists a holomorphic function $\tilde{F}$ on D_0 such that

$$\int_{D_0}|\tilde{F}-(1-b(k\Psi))fF^{2k}|^2e^{-k\varphi+v(k\Psi)-k\Psi}$$

$$\leq(2-e^{-t_0-1})\int_{D_0}\mathbb{I}_{\{-t_0-1<k\Psi<-t_0\}}|f|^2e^{-k\Psi},\qquad(2.3)$$

where $b(t) = \int_{-\infty}^{t} \mathbb{I}_{\{-t_0-1<s<-t_0\}} ds$ and $v(t) = \int_0^t b(s)ds$. Note that $v(t) \geq -t_0 - 1$ on $\mathbb{R}$ and $b(t) = 0$ on $(-\infty, -t_0 - 1)$. Inequality (2.3) implies that

$$\int_{D_0 \cap \{k\Psi < -t_0-1\}} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi - k\Psi}$$

$$\leq e^{t_0+1} \int_{D_0} |\tilde{F} - (1 - b(k\Psi))fF^{2k}|^2 e^{-k\varphi + v(k\Psi) - k\Psi}$$

$$\leq (2e^{t_0+1} - 1) \int_{D_0} \mathbb{I}_{\{-t_0-1<k\Psi<-t_0\}} |f|^2 e^{-k\Psi}$$

$$< +\infty.$$

Consequently, since $v(k\Psi) - k\Psi \geq 0$, $b(t) = 1$ on $[-t_0, +\infty)$, and $|F|^{4k} e^{-k\varphi} = 1$ on $\{k\Psi < -t_0\}$, we deduce from Inequality (2.3) that

$$\int_{D_0} |\tilde{F}|^2 e^{-k\varphi}$$

$$\leq 2 \int_{D_0} |(1 - b(k\Psi))fF^{2k}|^2 e^{-k\varphi} + 2 \int_{D_0} |\tilde{F} - (1 - b(k\Psi))fF^{2k}|^2 e^{-k\varphi}$$

$$\leq 2 \int_{D_0 \cap \{k\Psi < -t_0\}} |f|^2 + 2(2 - e^{-t_0-1}) \int_{D_0} \mathbb{I}_{\{-t_0-1<k\Psi<-t_0\}} |f|^2 e^{-k\Psi}$$

$$< +\infty.$$

Thus, Lemma 2.3 holds. $\qquad\square$

Let $f_o \in I_o$. Taking any $(\tilde{F}, o) \in \mathcal{I}(k\varphi)_o$ and $(\tilde{F}_1, o) \in \mathcal{I}(k\varphi)_o$, if there exist $t_1 > 0$ and a neighborhood D_1 of o such that $\int_{\{\Psi < -t_1\} \cap D_1} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi - k\Psi} < +\infty$ and $\int_{\{\Psi < -t_1\} \cap D_1} |\tilde{F}_1 - fF^{2k}|^2 e^{-k\varphi - k\Psi} < +\infty$, then we have

$$\int_{\{\Psi < -t_1\} \cap D_1} |\tilde{F}_1 - \tilde{F}|^2 e^{-k\varphi - k\Psi} < +\infty, \tag{2.4}$$

and there exists a neighborhood D_2 of o such that

$$\int_{D_2} |\tilde{F} - \tilde{F}_1|^2 e^{-k\varphi} < +\infty. \tag{2.5}$$

Combining inequality (2.4) and inequality (2.5), we obtain that $(\tilde{F} - \tilde{F}_1, o) \in \mathcal{I}(k\varphi + k\Psi)_o$. Thus, according to Lemma 2.3, there exists a map

$$\tilde{P}: I_o \to \mathcal{I}(k\varphi)_o / \mathcal{I}(k\varphi + k\Psi)_o,$$

given by

$$\tilde{P}(f_o) = [(\tilde{F}, o)],$$

for any $f_o \in I_o$, where $(\tilde{F}, o) \in \mathcal{I}(k\varphi)_o$ such that $\int_{\{\Psi < -t\} \cap D_0} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi - k\Psi} < +\infty$ for some $t > 0$ and some neighborhood D_0 of o, and $[(\tilde{F}, o)]$ is the equivalence class of $(\tilde{F}, o)$ in $\mathcal{I}(k\varphi)_o / \mathcal{I}(k\varphi + k\Psi)_o$.

Lemma 2.4. $\tilde{P}$ *is an* $\mathcal{O}_{\mathbb{C}^n, o}$*-module homomorphism, and* $\text{Ker}(\tilde{P}) = I(k\Psi)_o$.

Proof. For any $f_o \in I_o$ and $\tilde{f}_o \in I_o$, take $[(\tilde{F}, o)] = \tilde{P}(f_o)$ and $[(\tilde{F}_1, o)] = \tilde{P}(\tilde{f}_o)$. Then we have $(\tilde{F} + \tilde{F}_1, o) \in \mathcal{I}(k\varphi + k\Psi)_o$, and

$$\int_{\{\Psi < -t_1\} \cap D_1} |\tilde{F}_1 + \tilde{F} - (f + \tilde{f})F^{2k}|^2 e^{-k\varphi - k\Psi}$$

$$\leq 2 \int_{\{\Psi < -t_1\} \cap D_1} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi - k\Psi} + 2 \int_{\{\Psi < -t_1\} \cap D_1} |\tilde{F}_1 - \tilde{f}F^{2k}|^2 e^{-k\varphi - k\Psi}$$

$$< +\infty,$$

for some $t_1 > 0$ and some neighborhood D_1 of o, which shows that

$$\tilde{P}(f_o + \tilde{f}_o) = \tilde{P}((f + \tilde{f})_o) = [(\tilde{F} + \tilde{F}_1, o)] = \tilde{P}(f_o) + \tilde{P}(\tilde{f}_o).$$

For any $(h, o) \in \mathcal{O}_{\mathbb{C}^n, o}$, we have $(h\tilde{F}, o) \in \mathcal{I}(k\varphi + k\Psi)_o$ and

$$\int_{\{\Psi < -t_2\} \cap D_2} |h\tilde{F} - hfF^{2k}|^2 e^{-k\varphi - k\Psi} \leq C \int_{\{\Psi < -t_2\} \cap D_2} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi - k\Psi} < +\infty,$$

for some $C > 0$, $t_2 > 0$ and a neighborhood D_2 of o, which shows that

$$\tilde{P}((h, o) \cdot f_o) = \tilde{P}((hf)_o) = [(h\tilde{F}, o)] = (h, o) \cdot \tilde{P}(f_o).$$

Thus, $\tilde{P}$ is an $\mathcal{O}_{\mathbb{C}^n, o}$-module homomorphism from I_o to $\mathcal{I}(k\varphi)_o / \mathcal{I}(k\varphi + k\Psi)_o$.

Now, we prove $\mathrm{Ker}(\tilde{P}) = I(k\Psi)_o$. For any $f_o \in I(k\Psi)_o$, $\tilde{P}(f_o) = 0$ holds if and only if there exist a neighborhood D_3 of o and $t_3 > 0$ such that

$$\int_{\{\Psi < -t_3\} \cap D_3} |f_o F^{2k}|^2 e^{-k\varphi - k\Psi} = \int_{\{\Psi < -t_3\} \cap D_3} |f_o|^2 e^{-k\Psi} < +\infty,$$

i.e. $f_o \in I(k\Psi)_o$. $\qquad\square$

Following from Lemma 2.4, there exists an $\mathcal{O}_{\mathbb{C}^n, o}$-module homomorphism

$$P : I_o / I(k\Psi)_o \to \mathcal{I}(k\varphi)_o / \mathcal{I}(k\varphi + k\Psi)_o,$$

given by

$$P([f_o]_1) = \tilde{P}(f_o),$$

for any $f_o \in I_o$, where $[f_o]_1$ is the equivalence class of f_o in $I_o / I(k\Psi)_o$.

Note that $|F^{2k}|^2 e^{-k\varphi} = 1$ on $\{\Psi < 0\}$. For any $(\tilde{F}, o) \in \mathcal{I}(k\varphi)_o$, it is clear that $\left(\frac{\tilde{F}}{F^{2k}}\right)_o \in I_o$. Moreover, if $(\tilde{F}, o) \in \mathcal{I}(k\varphi + k\Psi)_o$, we have $\left(\frac{\tilde{F}}{F^{2k}}\right)_o \in I(k\Psi)_o$. Hence, there exists an $\mathcal{O}_{\mathbb{C}^n, o}$-module homomorphism

$$Q : \mathcal{I}(k\varphi)_o / \mathcal{I}(k\varphi + k\Psi)_o \to I_o / I(k\Psi)_o,$$

given by

$$Q([(\tilde{F}, o)]) = \left[\left(\frac{\tilde{F}}{F^{2k}}\right)_o\right]_1,$$

for any $[(\tilde{F}, o)] \in \mathcal{I}(k\varphi)_o / \mathcal{I}(k\varphi + k\Psi)_o$.

Lemma 2.5. *P is an $\mathcal{O}_{\mathbb{C}^n, o}$-module isomorphism from $I_o / I(k\Psi)_o$ to $\mathcal{I}(k\varphi)_o / \mathcal{I}(k\varphi + k\Psi)_o$ and $P^{-1} = Q$.*

Proof. According to the definitions of Q and P, we have that

$$P \circ Q([(\tilde{F}, o)]) = [(\tilde{F}, o)]$$

for any $[(\tilde{F}, o)] \in \mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi + k\Psi)_o$, which implies that P is surjective. Lemma 2.4 shows that P is injective. Thus, we get that P is an $\mathcal{O}_{\mathbb{C}^n, o}$-module isomorphism and $P^{-1} = Q$. $\qquad \square$

Let $a \in [0, k)$. Denote $P_a := P|_{I(a\Psi)_o/I(k\Psi)_o}$, where $I(a\Psi)_o/I(k\Psi)_o$ is an $\mathcal{O}_{\mathbb{C}^n, o}$-submodule of $I_o/I(k\Psi)_o$. It is clear that $k\varphi + a\Psi$ is a plurisubharmonic function on D. The the following lemma gives an isomorphism between $I(a\Psi)_o/I(k\Psi)_o$ and $\mathcal{I}(k\varphi + a\Psi)_o/\mathcal{I}(k\varphi + k\Psi)_o$.

Lemma 2.6. *P_a is an $\mathcal{O}_{\mathbb{C}^n, o}$-module isomorphism from $I(a\Psi)_o/I(k\Psi)_o$ to $\mathcal{I}(k\varphi + a\Psi)_o/\mathcal{I}(k\varphi + k\Psi)_o$.*

Proof. It suffices to prove $\text{Im}(P_a) = \mathcal{I}(k\varphi + a\Psi)_o/\mathcal{I}(k\varphi + k\Psi)_o$. For any $f_o \in I(a\Psi)_o$, taking $[(\tilde{F}, o)] = P_a([f_o]_1) = P([f_o]_1)$, we have that there exist $t > 0$ and a neighborhood D_0 of o such that

$$\begin{aligned}
&\int_{\{\Psi < -t\} \cap D_0} |\tilde{F}|^2 e^{-k\varphi - a\Psi} \\
&\leq 2 \int_{\{\psi < -t\} \cap D_0} |f|^2 e^{-a\Psi} + 2 \int_{\{\Psi < -t\} \cap D_0} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi - a\Psi} \qquad (2.6) \\
&< +\infty.
\end{aligned}$$

Combining $(\tilde{F}, o) \in \mathcal{I}(k\varphi)_o$ and inequality (2.6), we get $(\tilde{F}, o) \in \mathcal{I}(k\varphi + a\Psi)_o$, which shows $\text{Im}(P_a) \subset \mathcal{I}(k\varphi + a\Psi)_o/\mathcal{I}(k\varphi + k\Psi)_o$. For any $(\tilde{F}_1, o) \in \mathcal{I}(k\varphi + a\Psi)_o$, it is clear that $\left(\frac{\tilde{F}_1}{F^{2k}}\right)_o \in I(a\Psi)_o$. Then we obtain

$$P^{-1}([(\tilde{F}_1, o)]) = Q([(\tilde{F}_1, o)]) \in I(a\Psi)_o/I(k\Psi)_o,$$

which implies that $\text{Im}(P_a) = \mathcal{I}(k\varphi + a\Psi)_o/\mathcal{I}(k\varphi + k\Psi)_o$. $\qquad \square$

According to the definition of $I(a\Psi)_o$, we have that $I(a\Psi)_o \subset I(a'\Psi)_o$ for any $0 \leq a' < a < +\infty$. Denote

$$I_+(a\Psi)_o := \bigcup_{p > a} I(p\Psi)_o,$$

which is an $\mathcal{O}_{\mathbb{C}^n, o}$-submodule of I_o, where $a \geq 0$.

Lemma 2.7. *There exists $a' > a$ such that $I(a'\Psi)_o = I_+(a\Psi)_o$ for any $a \geq 0$.*

Proof. By the definition of $I_+(a\Psi)_o$, $I(p\Psi)_o \subset I_+(a\Psi)_o$ for any $p > a$. Then it suffices to prove that there exists $a' > a$ such that $I_+(a\Psi)_o \subset I(a'\Psi)_o$.

Let $k > a$ be an integer. It is clear that $\mathcal{I}(k\varphi + a\Psi)_o \subset \mathcal{I}(k\varphi + a'\Psi)$ for any $0 \leq a' < a \leq k$. Denote

$$\mathfrak{I} := \bigcup_{a < p < k} \mathcal{I}(k\varphi + p\Psi)_o,$$

which is an ideal of $\mathcal{O}_{\mathbb{C}^n, o}$. It follows from Lemma 2.5 that $P|_{I_+(a\Psi)_o/I(k\Psi)_o}$ is an $\mathcal{O}_{\mathbb{C}^n, o}$-module isomorphism from $I_+(a\Psi)_o/I(k\Psi)_o$ to $\mathfrak{I}/\mathcal{I}(k\varphi + k\Psi)_o$. As $\mathcal{O}_{\mathbb{C}^n, o}$ is a Noetherian ring (see [34]), we get that $\mathfrak{I}$ is finitely generated. Hence, there exists a

finite set $\{(f_1)_o, \ldots, (f_m)_o\} \subset I_+(a\Psi)_o$, which satisfies that for any $f_o \in I_+(a\Psi)_o$, there exists $(h_j, o) \in \mathcal{O}_{\mathbb{C}^n, o}$ for any $1 \leq j \leq m$, such that

$$f_o - \sum_{j=1}^m (h_j, o) \cdot (f_j)_o \in I(k\Psi)_o.$$

By the definition of $I_+(a\Psi)_o$, there exists $a' \in (a, k)$ such that $\{(f_1)_o, \ldots, (f_m)_o\} \subset I(a'\Psi)_o$. As $(h_j, o) \cdot (f_j)_o \in I(a'\Psi)_o$ for any $1 \leq j \leq m$ and $I(k\Psi)_o \subset I(a'\Psi)_o$, we obtain that $I_+(a\Psi)_o \subset I(a'\Psi)_o$.

Thus, Lemma 2.7 holds. $\qquad\square$

We recall the closedness of submodules, which can be referred to [19] and will be used in the proof of Lemma 2.9.

Lemma 2.8 (see [19]). *Let N be a submodule of $\mathcal{O}_{\mathbb{C}^n, o}^q$, $1 \leq q < +\infty$, let $f_j \in \mathcal{O}_{\mathbb{C}^n, o}(U)^q$ be a sequence of $q-$tuples holomorphic in an open neighborhood U of the origin o. Assume that the f_j converge uniformly in U towards to a $q-$tuples $f \in \mathcal{O}_{\mathbb{C}^n, o}(U)^q$, assume furthermore that all germs (f_j, o) belong to N. Then $(f, o) \in N$.*

The following lemma will be used in the proof of Theorem 1.6.

Lemma 2.9. *Let J be an $\mathcal{O}_{\mathbb{C}^n, o}$-submodule of I_o such that $I(k\Psi)_o \subset J$ for some positive integer k, and let $f \in J(\Psi)_o$. Let $D_0 \subset D$ containing o be a pseudoconvex domain, and let f_j be a sequence of holomorphic functions on $\{\Psi < -s_j\} \cap D_0$ for any $j \in \mathbb{Z}_{>0}$, where $s_j \geq 0$. Assume that $s_0 := \lim_{j \to +\infty} s_j \in [0, +\infty)$,*

$$\limsup_{j \to +\infty} \int_{\{\Psi < -s_j\} \cap D_0} |f_j|^2 \leq C < +\infty,$$

and $(f_j - f)_o \in J$. Then there exists a subsequence of $\{f_j\}_{j \in \mathbb{Z}_{>0}}$ compactly convergent to a holomorphic function f_0 on $\{\Psi < -s_0\} \cap D_0$, which satisfies that

$$\int_{\{\Psi < -s_0\} \cap D_0} |f_0|^2 \leq C,$$

and $(f_0 - f)_o \in J$.

Proof. Since $\limsup\limits_{j \to +\infty} \int_{\{\Psi < -s_j\} \cap D_0} |f_j|^2 < +\infty$ and $\lim\limits_{j \to +\infty} s_j = s_0$, we can extract a subsequence of $\{f_j\}_{j \in \mathbb{Z}_{>0}}$ (also denoted by $\{f_j\}_{j \in \mathbb{Z}_{>0}}$) compactly convergent to a holomorphic function f_0 on $\{\Psi < -s_0\} \cap D_0$ and

$$\int_{\{\Psi < -s_0\} \cap D_0} |f_0|^2 \leq C,$$

which implies $(f_0)_o \in I_o$. Thus, it suffices to prove $(f_0 - f)_o \in J$.

Denote $\varphi := 2 \max\{\psi, 2 \log |F|\}$, and $t_j := k s_j$ for $j \in \mathbb{Z}_{\geq 0}$. For any $j \in \mathbb{Z}_{>0}$, it follows from Lemma 2.1 that there exists a holomorphic function F_j on D_0 satisfying that

$$\int_{D_0} |F_j - (1 - b_{t_j, 1}(k\Psi)) f_j F^{2k}|^2 e^{-k\varphi + v_{t_j, 1}(k\Psi) - k\Psi}$$

$$\leq (2 - e^{-t_j - 1}) \int_{D_0} \mathbb{I}_{\{-t_j - 1 < k\Psi < -t_j\}} |f_j|^2 e^{-k\Psi} \tag{2.7}$$

$$\leq (2 e^{t_j + 1} - 1) \int_{\{k\Psi < -t_j\} \cap D_0} |f_j|^2,$$

where $b_{t_j,1}(t) = \int_{-\infty}^{t} \mathbb{I}_{\{-t_j-1<s<-t_j\}} ds$ and $v_{t_j,1}(t) = \int_0^t b_{t_j,1}(s) ds$. Note that $b_{t_j,1}(t) = 0$ for $t < -t_j - 1$. Then we have

$$\int_{\{k\Psi<-t_j-1\}\cap D_0} |F_j - f_j F^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty. \tag{2.8}$$

In addition, note that $b_{t_j,1}(t) = 1$ for $t \geq t_j$ and $|F|^{2k} e^{-k\varphi} = 1$ on $\{k\Psi < -t_j\} \cap D_0$. Then as $v_{t_j,1}(k\Psi) - k\Psi \geq 0$, it follows from inequality (2.7) that

$$\int_{D_0} |F_j|^2 e^{-k\varphi}$$

$$\leq 2\int_{D_0} |(1 - b_{t_j,1}(k\Psi))f_j F^{2k}|^2 e^{-k\varphi} + 2\int_{D_0} |F_j - (1 - b_{t_j,1}(k\Psi))f_j F^{2k}|^2 e^{-k\varphi}$$

$$\leq (2e^{t_j+1} + 1)\int_{\{k\Psi<-t_j\}\cap D_0} |f_j|^2. \tag{2.9}$$

Since $\lim_{j\to+\infty} t_j = t_0$ and $\limsup_{j\to+\infty} \int_{\{k\Psi<-t_j\}\cap D_0} |f_j|^2 < +\infty$, inequality (2.9) and Fatou's Lemma imply that there exists a subsequence of $\{F_j\}_{j\in\mathbb{Z}_{>0}}$ denoted by $\{F_{j_l}\}_{l\in\mathbb{Z}_{>0}}$, which compactly converges to a holomorphic function F_0 on D_0 with

$$\int_{D_0} |F_0|^2 e^{-k\varphi} \leq \liminf_{l\to+\infty} \int_{D_0} |F_{j_l}|^2 e^{-k\varphi} < +\infty. \tag{2.10}$$

As f_j converge to f_0, by inequality (2.7) and Fatou's Lemma, we obtain

$$\int_{D_0} |F_0 - (1 - b_{t_0,1}(k\Psi))f_0 F^{2k}|^2 e^{-k\varphi+v_{t_j,1}(k\Psi)-k\Psi}$$

$$= \int_{D_0} \liminf_{l\to+\infty} |F_{j_l} - (1 - b_{t_{j_l},1}(k\Psi))f_{j_l} F^{2k}|^2 e^{-k\varphi+v_{t_{j_l},1}(k\Psi)-k\Psi}$$

$$\leq \liminf_{l\to+\infty} \int_{D_0} |F_{j_l} - (1 - b_{t_{j_l},1}(k\Psi))f_{j_l} F^{2k}|^2 e^{-k\varphi+v_{t_{j_l},1}(k\Psi)-k\Psi}$$

$$< +\infty,$$

yielding that

$$\int_{\{k\Psi<-t_0-1\}\cap D_0} |F_0 - f_0 F^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty. \tag{2.11}$$

Combining inequality (2.8), inequality (2.9), inequality (2.10), inequality (2.11) and the definition of $P : I_o/I(k\Psi)_o \to \mathcal{I}(k\varphi)/\mathcal{I}(k\varphi + k\Psi)$, we get

$$P([(f_j)_o]_1) = [(F_j, o)],$$

for any $j \in \mathbb{Z}_{\geq 0}$. $(f_j - f)_o \in J$ for any $j \in \mathbb{Z}_{>0}$ implies $(f_j - f_1)_o \in J$ for any $j \in \mathbb{Z}_{>0}$. Lemma 2.5 shows that there exists an ideal $\tilde{J}$ of $\mathcal{O}_{\mathbb{C}^n,o}$ such that

$$\mathcal{I}(k\varphi + k\Psi)_o \subset \tilde{J} \subset \mathcal{I}(k\varphi)_o,$$

and

$$\tilde{J}/\mathcal{I}(k\varphi + k\Psi)_o = \text{Im}(P|_{J/I(k\Psi)_o}).$$

Then $(F_j - F_1, o) \in \tilde{J}$, for any $j \in \mathbb{Z}_{>0}$. As F_j compactly converge to F_0 on D_0, using Lemma 2.8, we obtain $(F_0 - F_1, o) \in \tilde{J}$. Recall that P is an $\mathcal{O}_{\mathbb{C}^n,o}$-module isomorphism, and $\tilde{J}/\mathcal{I}(k\varphi + k\Psi)_o = \text{Im}(P|_{J/I(k\Psi)_o})$. We deduce $(f_0 - f_1)_o \in J$, which verifies $(f_0 - f)_o \in J$.

Thus, Lemma 2.9 holds.   □

Let f be a holomorphic function on D. Denote that

$$\Psi_1 := \min\{2c_o^{fF}(\psi)\psi - 2\log|F|, 0\},$$

and

$$I_+(\Psi_1)_o := \bigcup_{a>1} I(a\Psi_1)_o.$$

Lemma 2.10. $f_o \notin I_+(\Psi_1)_o$.

Proof. We prove Lemma 2.10 by contradiction: if not, there exists $p_0 > 1$ such that $f_o \in I(p_0\Psi_1)_o$, which implies that

$$\int_{\{\Psi_1<-t\}\cap V} |f|^2|F|^{2p_0}e^{-2p_0c_o^{fF}(\psi)\psi} = \int_{\{\Psi_1<-t\}\cap V} |f|^2 e^{-p_0\Psi} < +\infty \qquad (2.12)$$

holds for some $t > 0$ and some neighborhood $V \Subset D$ of o. Note that

$$\sup_{V\cap\{\Psi_1\geq t\}} |f|^2|F|^{2p_0}e^{-2p_0c_o^{fF}(\psi)\psi} < +\infty.$$

Then inequality (2.12) indicates

$$\int_V |f|^2|F|^{2p_0}e^{-2p_0c_o^{fF}(\psi)\psi} < +\infty. \qquad (2.13)$$

As $\log|F|$ is a plurisubharmonic function on D, there exist a neighborhood $U \subset V$ of o and $p_0' \in (1, p_0)$ such that

$$\int_U |F|^{2(1-p_0')\frac{p_0}{p_0-p_0'}} = \int_U e^{-2(p_0'-1)\frac{p_0}{p_0-p_0'}\log|F|} < +\infty. \qquad (2.14)$$

It follows from inequality (2.13), inequality (2.14) and Hölder's inequality that

$$\int_U |fF|^2 e^{-2p_0'c_o^{fF}(\psi)\psi}$$
$$= \int_U |f|^2|F|^{2p_0'}e^{-2p_0'c_o^{fF}(\psi)\psi}|F|^{2-2p_0'}$$
$$\leq \left(\int_U |f|^{2\frac{p_0}{p_0'}}|F|^{2p_0}e^{-2p_0c_o^{fF}(\psi)\psi}\right)^{\frac{p_0'}{p_0}} \times \left(\int_U |F|^{2(1-p_0')\frac{p_0}{p_0-p_0'}}\right)^{\frac{p_0-p_0'}{p_0}}$$
$$< +\infty,$$

which contradicts the definition of $c_o^{fF}(\psi)$. Thus, we have $f_o \notin I_+(\Psi_1)_o$.   □

2.3. **Some properties of $G(t)$.**

Following the notations and assumptions in Section 1.2, we present some properties related to $G(t; \Psi, J, f)$ ($G(t)$ for short) in this section.

Lemma 2.11. $f_o \in J$ *if and only if* $G(t; \Psi, J, f) = 0$.

Proof. If $f_o \in J$, it follows from the definition of $G(t; \Psi, J, f)$ that $G(t; \Psi, J, f) = 0$.

If $G(t; \Psi, J, f) = 0$, there exists a sequence of holomorphic functions $\{f_j\}_{j\in\mathbb{Z}_{>0}}$ on $\{\Psi < -t\}$ such that $\lim_{j\to+\infty} \int_{\{\Psi<-t\}} |f_j|^2 = 0$ and $(f_j - f)_o \in J$. Then Lemma 2.9 implies that there exists a holomorphic function f_0 on $\{\Psi < -t\}$ such that $\int_{\{\Psi<-t\}} |f_0|^2 = 0$ and $(f_0 - f)_o \in J$, which shows $f \in J$.   □

The following lemma shows the existence and uniqueness of the minimal holomorphic function.

Lemma 2.12. *Assume that $G(t) < +\infty$. Then there exists a unique holomorphic function f_t on $\{\Psi < -t\}$ satisfying $(f_t - f)_o \in J$ and $\int_{\{\Psi<-t\}} |f_t|^2 = G(t)$. Furthermore, for any holomorphic function $\hat{f}$ on $\{\Psi < -t\}$ satisfying $(\hat{f} - f)_o \in J$ and $\int_{\{\Psi<-t\}} |\hat{f}|^2 < +\infty$, we have the following equality*

$$\int_{\{\Psi<-t\}} |f_t|^2 + \int_{\{\Psi<-t\}} |\hat{f} - f_t|^2 = \int_{\{\Psi<-t\}} |\hat{f}|^2. \tag{2.15}$$

Proof. Firstly, we prove the existence of f_t. As $G(t) < +\infty$, there exist holomorphic functions $\{F_j\}_{j\in\mathbb{Z}_{>0}}$ on $\{\Psi < -t\}$ such that $\lim_{j\to+\infty} \int_{\{\Psi<-t\}} |F_j|^2 = G(t)$ and $(F_j - f)_o \in J$. Lemma 2.9 indicates that there exists a holomorphic function f_t on $\{\Psi < -t\}$ such that $(f_t - f)_o \in J$ and $\int_{\{\Psi<-t\}} |f_t|^2 \le G(t)$. By the definition of $G(t)$, we have $\int_{\{\Psi<-t\}} |f_t|^2 = G(t)$.

Secondly, we prove the uniqueness of f_t by contradiction: if not, there exist two different holomorphic functions $\tilde{f}_1$ and $\tilde{f}_2$ on $\{\Psi < -t\}$ satisfying $\int_{\{\Psi<-t\}} |\tilde{f}_1|^2 = \int_{\{\Psi<-t\}} |\tilde{f}_2|^2 = G(t)$, $(\tilde{f}_1 - f)_o \in J$, and $(\tilde{f}_2 - f)_o \in J$. Note that

$$\int_{\{\Psi<-t\}} \left|\frac{\tilde{f}_1 + \tilde{f}_2}{2}\right|^2 + \int_{\{\Psi<-t\}} \left|\frac{\tilde{f}_1 - \tilde{f}_2}{2}\right|^2$$
$$= \frac{\int_{\{\Psi<-t\}} |\tilde{f}_1|^2 + \int_{\{\Psi<-t\}} |\tilde{f}_2|^2}{2} = G(t). \tag{2.16}$$

Then we obtain that

$$\int_{\{\psi<-t\}} \left|\frac{f_1 + f_2}{2}\right|^2 < G(t),$$

and $(\frac{\tilde{f}_1 + \tilde{f}_2}{2} - f)_o \in J$, which contradicts the definition of $G(t)$.

Finally, we prove equality (2.15). For any holomorphic function h on $\{\Psi < -t\}$ satisfying $h_o \in J$ and $\int_{\{\Psi<-t\}} |h|^2 < +\infty$, it is clear that for any complex number α, $f_t + \alpha h$ satisfies $(f_t + \alpha h - f)_o \in J$. The definition of $G(t)$ shows that

$$\int_{\{\Psi<-t\}} |f_t|^2 \le \int_{\{\Psi<-t\}} |f_t + \alpha h|^2 < +\infty,$$

yielding that

$$\mathrm{Re} \int_{\{\psi<-t\}} f_t \overline{h} = 0.$$

Then we get

$$\int_{\{\Psi<-t\}} |f_t + h|^2 = \int_{\{\Psi<-t\}} |f_t|^2 + \int_{\{\Psi<-t\}} |h|^2.$$

Choosing $h = \hat{f} - f_t$, we obtain equality (2.15).   □

We present the monotonicity and the lower semicontinuity of $G(t)$.

Lemma 2.13. *$G(t)$ is decreasing on $[0, +\infty)$, such that $\lim_{t\to t_0+0} G(t) = G(t_0)$ for $t_0 \in [0, +\infty)$. If there exists $t_1 \ge 0$ such that $G(t_1) < +\infty$, then $\lim_{t\to+\infty} G(t) = 0$. Especially, $G(t)$ is lower semicontinuous on $[0, +\infty)$ if $G(0) < +\infty$.*

Proof. By the definition of $G(t)$, it is clear that $G(t)$ is decreasing on $[0, +\infty)$. The dominated convergence theorem verifies that $\lim_{t \to +\infty} G(t) = 0$ if there exists $t_1 \geq 0$ such that $G(t_1) < +\infty$.

It suffices to prove $\lim_{t \to t_0 + 0} G(t) = G(t_0)$. We prove it by contradiction: if not, then $\lim_{t \to t_0 + 0} G(t) < G(t_0)$ for some $t_0 \in [0, +\infty)$. By Lemma 2.12, there exists a unique holomorphic function f_t on $\{\Psi < -t\}$ satisfying that $(f_t - f)_o \in J$ and $\int_{\{\Psi < -t\}} |f_t|^2 = G(t)$. Note that $G_{F, \psi, c}(t)$ is decreasing, indicating that $\lim_{t \to t_0 + 0} \int_{\{\Psi < -t\}} |f_t|^2 < G(t_0)$. It follows from Lemma 2.9 that there exists a holomorphic function $\tilde{f}_{t_0}$ on $\{\Psi < -t_0\}$ such that $(\tilde{f}_{t_0} - f)_o \in J$ and $\int_{\{\Psi < -t\}} |\tilde{f}_{t_0}|^2 \leq \lim_{t \to t_0 + 0} \int_{\{\Psi < -t\}} |f_t|^2 < G(t_0)$, which contradicts the definition of $G(t_0)$. Thus, we have $\lim_{t \to t_0 + 0} G(t) = G(t_0)$. $\qquad\square$

We consider the derivatives of $G(t)$ in the following lemma.

Lemma 2.14. *Assume that $G(t_1) < +\infty$, where $t_1 \in [0, +\infty)$. Then for any $t_0 > t_1$, we have*

$$\frac{G(t_1) - G(t_0)}{e^{-t_1} - e^{-t_0}} \leq \liminf_{B \to 0+0} \frac{G(t_0) - G(t_0 + B)}{e^{-t_0} - e^{-t_0 - B}}.$$

Proof. Lemma 2.13 tells that $G(t) < +\infty$ for any $t \geq t_1$. By Lemma 2.12, there exists a holomorphic function f_{t_0} on $\{\Psi < -t_0\}$, such that $(f_{t_0} - f)_o \in J$ and $\int_{\{\Psi < -t_0\}} |f_{t_0}|^2 = G(t_0)$.

It suffices to consider that $\liminf_{B \to 0+0} \frac{G(t_0) - G(t_0 + B)}{e^{-t_0} - e^{-t_0 - B}} \in [0, +\infty)$ because of the decreasing property of $G(t)$. Then there exists $B_j \to 0 + 0$ $(j \to +\infty)$ such that

$$\begin{aligned}
e^{t_0} \lim_{j \to +\infty} \frac{G(t_0) - G(t_0 + B_j)}{B_j} &= \lim_{j \to +\infty} \frac{G(t_0) - G(t_0 + B_j)}{e^{-t_0} - e^{-t_0 - B_j}} \\
&= \liminf_{B \to 0+0} \frac{G(t_0) - G(t_0 + B)}{e^{-t_0} - e^{-t_0 - B}},
\end{aligned} \tag{2.17}$$

and $\left\{ \frac{G(t_0) - G(t_0 + B_j)}{B_j} \right\}_{j \in \mathbb{Z}_{>0}}$ is bounded.

Since $t \leq v_{t_0, B_j}(t)$, Lemma 2.2 shows that for any B_j, there exists a holomorphic function $\tilde{F}_j$ on $\{\Psi < -t_1\}$, such that

$$\begin{aligned}
&\int_{\{\Psi < -t_1\}} |\tilde{F}_j - (1 - b_{t_0, B_j}(\Psi)) f_{t_0}|^2 \\
&\leq \int_{\{\Psi < -t_1\}} |\tilde{F}_j - (1 - b_{t_0, B_j}(\Psi)) f_{t_0}|^2 e^{-\Psi + v_{t_0, B_j}(\Psi)} \\
&\leq (e^{-t_1} - e^{-t_0 - B_j}) \int_{\{\Psi < -t_1\}} \frac{1}{B_j} (\mathbb{I}_{\{-t_0 - B_j < \Psi < -t_0\}}) |f_{t_0}|^2 e^{-\Psi} \\
&\leq (e^{t_0 - t_1 + B_j} - 1) \left(\int_{\{\Psi < -t_0\}} \frac{1}{B_j} |f_{t_0}|^2 - \int_{\{\Psi < -t_0 - B_j\}} \frac{1}{B_j} |f_{t_0}|^2 \right) \\
&\leq (e^{t_0 - t_1 + B_j} - 1) \frac{G(t_0) - G(t_0 + B_j)}{B_j}.
\end{aligned} \tag{2.18}$$

Note that $b_{t_0,B_j}(t) = 0$ for $t \le -t_0 - B_j$ and $b_{t_0,B_j}(t) = 1$ for $t \ge t_0$. According to inequality (2.18), we have $(\tilde{F}_j - f_{t_0})_o \in J$, and

$$\int_{\{\Psi < -t_1\}} |\tilde{F}_j|^2$$

$$\le 2 \int_{\{\Psi < -t_1\}} |(1 - b_{t_0,B_j}(\Psi))f_{t_0}|^2 + 2 \int_{\{\Psi < -t_1\}} |\tilde{F}_j - (1 - b_{t_0,B_j}(\Psi))f_{t_0}|^2 \quad (2.19)$$

$$\le 2 \int_{\{\Psi < -t_0\}} |f_{t_0}|^2 + 2\left(e^{t_0 - t_1 + B_j} - 1\right) \frac{G(t_0) - G(t_0 + B_j)}{B_j}.$$

Following from Lemma 2.9 and inequality (2.19), we get that there exists a subsequence of $\{\tilde{F}_j\}_{j \in \mathbb{Z}_{>0}}$ (also denoted by $\{\tilde{F}_j\}_{j \in \mathbb{Z}_{>0}}$), which compactly converges to a holomorphic function $\tilde{F}_0$ on $\{\Psi < -t_1\}$, such that $(\tilde{F}_0 - f)_o \in J$ and

$$\int_{\{\Psi < -t_1\}} |\tilde{F}_0|^2 \le \liminf_{j \to +\infty} \int_{\{\Psi < -t_1\}} |\tilde{F}_j|^2 < +\infty.$$

Note that

$$\lim_{j \to +\infty} b_{t_0,B_j}(t) = \lim_{j \to +\infty} \int_{-\infty}^{t} \frac{1}{B_j} \mathbb{I}_{\{-t_0 - B_j < s < -t_0\}} ds = \begin{cases} 0 & \text{if } x \in (-\infty, -t_0) \\ 1 & \text{if } x \in [-t_0, +\infty) \end{cases},$$

and

$$\lim_{j \to +\infty} v_{t_0,B_j}(t) = \lim_{j \to +\infty} \int_{-t_0}^{t} b_{t_0,B_j} ds - t_0 = \begin{cases} -t_0 & \text{if } x \in (-\infty, -t_0) \\ t & \text{if } x \in [-t_0, +\infty) \end{cases}.$$

According to inequality (2.18) and Fatou's Lemma, we have

$$\int_{\{\Psi < -t_0\}} |\tilde{F}_0 - f_{t_0}| + \int_{\{-t_0 \le \Psi < -t_1\}} |\tilde{F}_0|^2$$

$$= \int_{\{\Psi < -t_1\}} \lim_{j \to +\infty} |\tilde{F}_j - (1 - b_{t_0,B_j}(\Psi))f_{t_0}|^2$$

$$\le \liminf_{j \to +\infty} \int_{\{\Psi < -t_1\}} |\tilde{F}_j - (1 - b_{t_0,B_j}(\Psi))f_{t_0}|^2 \quad (2.20)$$

$$\le \left(e^{t_0 - t_1} - 1\right) \frac{G(t_0) - G(t_0 + B_j)}{B_j}.$$

Then it follows from Lemma 2.12, equality (2.17) and inequality (2.20) that

$$(e^{-t_1} - e^{-t_0}) \liminf_{B \to 0+0} \frac{G(t_0) - G(t_0 + B)}{e^{-t_0} - e^{-t_0 - B}}$$

$$= (e^{t_0 - t_1} - 1) \frac{G(t_0) - G(t_0 + B_j)}{B_j}$$

$$\ge \int_{\{\Psi < -t_0\}} |\tilde{F}_0 - f_{t_0}| + \int_{\{-t_0 \le \Psi < -t_1\}} |\tilde{F}_0|^2$$

$$= \int_{\{\Psi < -t_1\}} |\tilde{F}_0|^2 - \int_{\{\Psi < -t_0\}} |f_{t_0}|^2$$

$$\ge G(t_1) - G(t_0).$$

This proves Lemma 2.14. □

The following well-known property of concave functions will be used in the proof of Theorem 1.6.

Lemma 2.15 (see [21]). *Let $a(r)$ be a lower semicontinuous function on $(0, R]$. Then $a(r)$ is concave if and only if*

$$\frac{a(r_1) - a(r_2)}{r_1 - r_2} \geq \limsup_{r \to r_2 + 0} \frac{a(r) - a(r_2)}{r - r_2}, \tag{2.21}$$

holds for any $0 < r_2 < r_1 \leq R$.

3. Proof of Theorem 1.6

Firstly, we prove that $G(t_1) < +\infty$ for any $t_1 \in [0, t_0)$. It follows from Lemma 2.12 that there exists a holomorphic function f_{t_0} on $\{\Psi < -t_0\}$ satisfying $(f_{t_0} - f)_o \in J$ and $\int_{\{\Psi < -t_0\}} |f_{t_0}|^2 = G(t_0) < +\infty$. Using Lemma 2.2, we get a holomorphic function $\tilde{F}$ on $\{\Psi < -t_1\}$, such that

$$
\begin{aligned}
&\int_{\{\Psi < -t_1\}} |\tilde{F} - (1 - b_{t_0, B}(\Psi)) f_{t_0}|^2 \\
&\leq \int_{\{\Psi < -t_1\}} |\tilde{F} - (1 - b_{t_0, B}(\Psi)) f_{t_0}|^2 e^{-\Psi + v_{t_0, B}(\Psi)} \\
&\leq (e^{-t_1} - e^{-t_0 - B}) \int_{\{\Psi < -t_1\}} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f_{t_0}|^2 e^{-\Psi},
\end{aligned}
\tag{3.1}
$$

which implies that $(\tilde{F} - f_{t_0})_o \in J$ and

$$
\begin{aligned}
&\int_{\{\Psi < -t_1\}} |\tilde{F}|^2 \\
&\leq 2 \int_{\{\Psi < -t_1\}} |(1 - b_{t_0, B}(\Psi)) f_{t_0}|^2 + 2 \int_{\{\Psi < -t_1\}} |\tilde{F} - (1 - b_{t_0, B}(\Psi)) f_{t_0}|^2 \\
&\leq 2 \int_{\{\Psi < -t_1\}} |f_{t_0}|^2 + \frac{2(e^{t_0 + B - t_1} - 1)}{B} \int_{\{\Psi < -t_0\}} |f_{t_0}|^2 \\
&< +\infty.
\end{aligned}
$$

Then we obtain $G(t_1) \leq \int_{\{\Psi < -t_1\}} |\tilde{F}|^2 < +\infty$.

Now, as $G(h^{-1}(r))$ is lower semicontinuous (Lemma 2.13), Lemma 2.14 and Lemma 2.15 imply the concavity of $G(-\log r)$. Lemma 2.13 shows $\lim_{t \to +\infty} G(t) = 0$. Hence the proof of Theorem 1.6 is completed.

4. Proofs of Theorem 1.7 and Remark 1.9

In this section, we prove Theorem 1.7 and Remark 1.9.

4.1. **Proof of Theorem 1.7.**
As D is a pseudoconvex domain, there exist a sequence of pseudoconvex domains $D_1 \Subset \ldots D_j \Subset D_{j+1} \Subset \ldots$ such that $\bigcup_{j=1}^{+\infty} D_j = D$. Denote

$$\inf \left\{ \int_{\{p\Psi_1 < -t\} \cap D_j} |\tilde{f}|^2 : \tilde{f} \in \mathcal{O}(\{p\Psi_1 < -t\} \cap D_j) \ \& \ (\tilde{f} - f)_o \in I(p\Psi_1)_o \right\}$$

by $G_{j,p}(t)$, where $t \in [0, +\infty)$, $j \in \mathbb{Z}_{>0}$ and $p \in (1,2)$. Note that

$$p\Psi_1 = \min\left\{ \left(2pc_o^{fF}(\psi)\psi + (4-2p)\log|F|\right) - 2\log|F^2|, 0 \right\},$$

and

$$G_{j,p}(0) \le \int_{D_j} |f|^2 < +\infty.$$

Theorem 1.6 shows that $G_{j,p}(-\log r)$ is concave with respect to $r \in (0,1]$, and $\lim_{t \to +\infty} G_{j,p}(t) = 0$, yielding that

$$\frac{1}{r_1^2} \int_{\{p\Psi_1 < 2\log r_1\} \cap D_j} |f|^2 \ge \frac{1}{r_1^2} G_{j,p}(2\log r_1) \ge G_{j,p}(0), \qquad (4.1)$$

where $0 < r_1 \le 1$.

Denote

$$C_j := \inf\left\{ \int_{\{\Psi_1 < 0\} \cap D_j} |\tilde{f}|^2 : \tilde{f} \in \mathcal{O}(\{\Psi_1 < 0\} \cap D_j) \ \& \ (\tilde{f} - f)_o \in I_+(\Psi_1)_o \right\}.$$

Since $\int_{D_j} |f|^2 < +\infty$ and $G_{j,p}(0) \ge C_j$ for any $j \in \mathbb{Z}_{>0}$, according to the dominated convergence theorem and inequality (4.1), we obtain that

$$
\begin{aligned}
\frac{1}{r_1^2} & \int_{\{\Psi_1 \le 2\log r_1\} \cap D_j} |f|^2 \\
&= \lim_{p \to 1+0} \frac{1}{r_1^2} \int_{\{p\Psi_1 < 2\log r_1\} \cap D_j} |f|^2 \\
&\ge \limsup_{p \to 1+0} G_{j,p}(0) \\
&\ge C_j
\end{aligned}
\qquad (4.2)
$$

holds for any $j \in \mathbb{Z}_{>0}$ and $r_1 \in (0,1]$. Without loss of generality, assume that there exists $r_0 \in (0,1]$ such that $\frac{1}{r_0^2} \int_{\{\Psi_1 \le 2\log r_0\}} |f|^2 < +\infty$. Following from inequality (4.2), we get that $\sup_j C_j < +\infty$.

Lemma 2.7 tells us that there exists $p_0 \in (1,2)$ such that $I_+(\Psi_1)_o = I(p_0\Psi_1)_o$. Then it follows from Lemma 2.12 that there exists a holomorphic function $\tilde{f}_j$ on $\{\Psi_1 < 0\} \cap D_j$ such that $C_j = \int_{\{\Psi_1 < 0\} \cap D_j} |\tilde{f}_j|^2$ and $(\tilde{f}_j - f)_o \in I(p_0\Psi_1)_o$ for any $j \in \mathbb{Z}_{>0}$. Note that $\sup_j \int_{\{\Psi_1 < 0\} \cap D_j} |\tilde{f}_j|^2 < +\infty$. Then we can apply Lemma 2.9 and the diagonal method to extract a subsequence of $\{\tilde{f}_j\}_{j \in \mathbb{Z}_{>0}}$ (also denoted by $\{\tilde{f}_j\}_{j \in \mathbb{Z}_{>0}}$) compactly convergent to a holomorphic function $\tilde{f}_0$ on $\{\Psi_1 < 0\}$, which satisfies $(\tilde{f}_0 - f)_o \in I(p_0\Psi)_o$. As C_j is increasing with respect to j, using Fatou's Lemma and the definition of $G(0; \Psi_1, I_+(\Psi_1)_o, f)$, we obtain that

$$G(0; \Psi_1, I_+(\Psi_1)_o, f) \le \int_{\{\Psi < 0\}} |\tilde{f}_0|^2 \le \liminf_{j \to +\infty} \int_{\{\Psi_1 < 0\} \cap D_j} |\tilde{f}_j|^2 = \lim_{j \to +\infty} C_j. \quad (4.3)$$

Combining inequality (4.2) and inequality (4.3), we get

$$
\begin{aligned}
\frac{1}{r_1^2} &\int_{\{\Psi_1 \leq 2\log r_1\}} |f|^2 \\
&\geq \lim_{j\to+\infty} \frac{1}{r_1^2} \int_{\{\Psi_1 \leq 2\log r_1\}\cap D_j} |f|^2 \\
&\geq \lim_{j\to+\infty} C_j \\
&\geq G(0;\Psi_1, I_+(\Psi_1)_o, f).
\end{aligned}
\tag{4.4}
$$

Note that

$$
\left\{2c_o^{fF}(\psi)\psi - 2\log|F| < 2\log r\right\} = \bigcup_{0<r_1<r} \{\Psi_1 \leq 2\log r_1\}
$$

where $r \in (0,1]$. Then inequality (4.4) implies

$$
\begin{aligned}
\int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 &\geq \sup_{r_1\in(0,r)} r_1^2 G(0;\Psi_1, I_+(\Psi_1)_o, f) \\
&= r^2 G(0;\Psi_1, I_+(\Psi_1)_o, f).
\end{aligned}
$$

Lemma 2.10 shows $f_o \notin I_+(\Psi_1)_o$. In addition, $I_+(\Psi_1)_o = I(p_0\Psi_1)_o$ and Lemma 2.11 verify $G(0;\Psi_1, I_+(\Psi_1)_o, f) > 0$. Thus, Theorem 1.7 holds.

4.2. **Proof of Remark 1.9.**

Note that $G(0;\Psi_1, I_+(\Psi_1)_o, 1) \geq G(0;\Psi_1, I(\Psi_1)_o, 1)$. Then it suffices to prove

$$
G(0;\Psi_1, I(\Psi_1)_o, 1) \geq \frac{C_{F^{1+\delta}, 2c_o^F(\psi)\psi+\delta\max\{2c_o^F(\psi)\psi, 2\log|F|\}}(o)}{\left(1+\frac{1}{\delta}\right)\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}}},
$$

for any $\delta \in \mathbb{Z}_{>0}$. Without loss of generality, assume that $G(0;\Psi_1, I(\Psi_1)_o, 1) < +\infty$. Denote $G(t;\Psi_1, I(\Psi_1)_o, 1)$ by $G(t)$ for short.

Lemma 2.12 indicates that there exists a holomorphic function f_0 on $\{\Psi_1 < 0\}$ such that $(f_0 - 1)_o \in I(\Psi_1)_o$ and $G(0) = \int_{\{\Psi_1<0\}} |f_0|^2$. Denote

$$
\varphi := (1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}.
$$

For any $B > 0$, it follows from Lemma 2.1 that there exists a holomorphic function $\tilde{F}_B$ on D such that

$$
\begin{aligned}
\int_D &|\tilde{F}_B - (1-b_B(\Psi_1))f_0 F^{1+\delta}|^2 e^{-\varphi+v_B(\Psi_1)-\Psi_1} \\
&\leq \left(\frac{1}{\delta}+1-e^{-B}\right)\int_D \frac{1}{B}\mathbb{I}_{\{-B<\Psi_1<0\}}|f_0|^2 e^{-\Psi_1} \\
&\leq \left(\left(\frac{1}{\delta}+1\right)e^B - 1\right)\int_D \frac{1}{B}\mathbb{I}_{\{-B<\Psi_1<0\}}|f_0|^2 \\
&\leq \left(\left(\frac{1}{\delta}+1\right)e^B - 1\right)\frac{G(0)-G(B)}{B},
\end{aligned}
\tag{4.5}
$$

where $b_B(t) = \int_{-\infty}^t \frac{1}{B}\mathbb{I}_{\{-B<s<0\}}ds$ and $v_B(t) = \int_0^t b_B(s)ds$. Theorem 1.6 shows that $G(-\log r)$ is concave with respect to r and $\lim_{t\to+\infty} G(t) = 0$, which implies

that

$$\lim_{B \to 0+0} \left(\left(\frac{1}{\delta} + 1 \right) e^B - 1 \right) \frac{G(0) - G(B)}{B}$$

$$= \lim_{B \to 0+0} \left(\left(\frac{1}{\delta} + 1 \right) e^B - 1 \right) \frac{G(-\log 1) - G(-\log e^{-B})}{1 - e^{-B}} \cdot \frac{1 - e^{-B}}{B} \qquad (4.6)$$

$$\leq \frac{1}{\delta} G(0).$$

Combining inequality (4.5), inequality (4.6) and the fact $v_B(\Psi_1) - \Psi_1 \geq 0$, we have

$$\lim_{B \to 0+0} \int_D |\tilde{F}_B|^2 e^{-\varphi}$$

$$\leq \lim_{B \to 0+0} 2 \left(\int_D |\tilde{F}_B - (1 - b_B(\Psi_1))f_0 F^{1+\delta}|^2 e^{-\varphi} + \int_D |(1 - b_B(\Psi_1))f_0 F^{1+\delta}|^2 e^{-\varphi} \right)$$

$$\leq 2 \int_{\{\Psi_1 < 0\}} |f_0|^2 + 2 \lim_{B \to 0+0} \left(\left(\frac{1}{\delta} + 1 \right) e^B - 1 \right) \frac{G(0) - G(B)}{B}$$

$$\leq 2 \left(1 + \frac{1}{\delta} \right) G(0)$$

$$< + \infty.$$

$$\qquad (4.7)$$

This implies that there exists a subsequence of $\{\tilde{F}_B\}_{B>0}$, denoted by $\{\tilde{F}_{B_j}\}_{j \in \mathbb{Z}_{>0}}$, satisfying that $\lim_{j \to +\infty} B_j = 0$ and $\{\tilde{F}_{B_j}\}_{j \in \mathbb{Z}_{>0}}$ converges to a holomorphic function $\tilde{F}$ on D. It follows from Fatou's Lemma and inequality (4.7) that

$$\int_D |\tilde{F}|^2 e^{-\varphi} \leq \liminf_{j \to +\infty} \int_D |\tilde{F}_{B_j}|^2 e^{-\varphi} \leq \left(1 + \frac{1}{\delta} \right) G(0) < +\infty. \qquad (4.8)$$

Using Fatou's Lemma, inequality (4.5) and inequality (4.6), we obtain

$$\int_{\{\Psi_1 < 0\}} |\tilde{F} - f_0 F^{1+\delta}|^2 e^{-\varphi - \Psi_1} + \int_{\{\Psi_1 \geq 0\}} |\tilde{F}|^2 e^{-\varphi}$$

$$\leq \liminf_{j \to +\infty} \int_D |\tilde{F}_{B_j} - (1 - b_{B_j}(\Psi_1))f_0 F^{1+\delta}|^2 e^{-\varphi + v_{B_j}(\Psi_1) - \Psi_1}$$

$$\leq \liminf_{j \to +\infty} \left(\left(\frac{1}{\delta} + 1 \right) e^{B_j} - 1 \right) \frac{G(0) - G(B_j)}{B_j} \qquad (4.9)$$

$$\leq \frac{1}{\delta} G(0).$$

Note that

$$\int_{\{\Psi_1 < 0\}} \left| \frac{\tilde{F}}{F^{1+\delta}} \right|^2 = \int_{\{\Psi_1 < 0\}} |\tilde{F}|^2 e^{-\varphi} < +\infty,$$

and

$$\int_{\{\Psi_1 < 0\}} \left| \frac{\tilde{F}}{F^{1+\delta}} - f_0 \right|^2 e^{-\Psi_1} = \int_{\{\Psi_1 < 0\}} |\tilde{F} - f_0 F^{1+\delta}|^2 e^{-\varphi - \Psi_1} < +\infty.$$

Then Lemma 2.12 indicates that

$$
\int_{\{\Psi_1<0\}} |\tilde{F} - f_0 F^{1+\delta}|^2 e^{-\varphi} = \int_{\{\Psi_1<0\}} \left| \frac{\tilde{F}}{F^{1+\delta}} - f_0 \right|^2
$$

$$
= \int_{\{\Psi_1<0\}} \left| \frac{\tilde{F}}{F^{1+\delta}} \right|^2 - \int_{\{\Psi_1<0\}} |f_0|^2. \tag{4.10}
$$

Combining inequality (4.9) and equality (4.10), we get

$$
\int_D |\tilde{F}|^2 e^{-\varphi}
$$

$$
= \int_{\{\Psi_1<0\}} |\tilde{F}|^2 e^{-\varphi} - \int_{\{\Psi_1<0\}} |f_0|^2 + \int_{\{\Psi_1\geq0\}} |\tilde{F}|^2 e^{-\varphi} + \int_{\{\Psi_1<0\}} |f_0|^2
$$

$$
= \int_{\{\Psi_1<0\}} |\tilde{F} - f_0 F^{1+\delta}|^2 e^{-\varphi} + \int_{\{\Psi_1\geq0\}} |\tilde{F}|^2 e^{-\varphi} + G(0) \tag{4.11}
$$

$$
\leq \left(1 + \frac{1}{\delta}\right) G(0).
$$

As $(f-1)_o \in I(\Psi_1)_o$ and

$$
\int_{\{\Psi_1<0\}} \left| \frac{\tilde{F}}{F^{1+\delta}} - f_0 \right|^2 e^{-\Psi_1} = \int_{\{\Psi_1<0\}} |\tilde{F} - f_0 F^{1+\delta}|^2 e^{-\varphi-\Psi_1} < +\infty,
$$

we have $(\frac{\tilde{F}}{F^{1+\delta}} - 1)_o \in I(\Psi_1)_o$, yielding that there exist a neighborhood $U \Subset D$ of o and $t > 0$ such that

$$
\int_{\{\Psi_1<-t\}\cap U} |\tilde{F} - F^{1+\delta}|^2 e^{-\varphi-\Psi_1} = \int_{\{\Psi_1<-t\}\cap U} \left| \frac{\tilde{F}}{F^{1+\delta}} - 1 \right|^2 e^{-\Psi_1} < +\infty. \tag{4.12}
$$

Sicne

$$
\int_{\{\Psi_1>-t\}\cap U} |\tilde{F}|^2 e^{-\varphi-\Psi_1} \leq e^t \int_D |\tilde{F}|^2 e^{-\varphi} < +\infty,
$$

and

$$
\int_{\{\Psi_1>-t\}\cap U} |F^{1+\delta}|^2 e^{-\varphi-\Psi_1} \leq \int_{U\cap\{\Psi_1\geq-t\}} e^{-\Psi_1} < +\infty,
$$

inequality (4.12) verifies that

$$
(\tilde{F} - F^{1+\delta}, o) \in \mathcal{I}(\varphi + \Psi_1)_o
$$

$$
= \mathcal{I}\left(2c_o^F(\psi)\psi + \delta \max\left\{2c_o^F(\psi)\psi, 2\log|F|\right\}\right)_o.
$$

Recall the definition of $C_{F^{1+\delta}, 2c_o^F(\psi)\psi+\delta\max\{2c_o^F(\psi)\psi, 2\log|F|\}}(o)$. According to inequality (4.11), we obtain that

$$
\frac{C_{F^{1+\delta}, 2c_o^F(\psi)\psi+\delta\max\{2c_o^F(\psi)\psi, 2\log|F|\}}(o)}{\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}}} \leq \int_D |\tilde{F}|^2 e^{-\varphi} \leq \left(1 + \frac{1}{\delta}\right) G(0).
$$

Thus, Remark 1.9 holds.

5. Proofs of Theorem 1.11 and Corollary 1.12

In this section, we prove Theorem 1.11 and Corollary 1.12.

5.1. **Proof of Theorem 1.11.**

Lemma 2.7 shows that there exists some $p_0 > 2a_o^f(\Psi)$ such that $I(p_0\Psi)_o = I_+(2a_o^f(\Psi)\Psi)_o$. According to the definition of $a_o^f(\Psi)$ and Lemma 2.11, we obtain that

$$G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f) > 0. \tag{5.1}$$

Without loss of generality, assume that there exists $t > t_0$ such that $\int_{\{\Psi < -t\}} |f|^2 < +\infty$. Denote

$$t_1 := \inf\left\{ t \geq t_0 : \int_{\{\Psi < -t\}} |f|^2 < +\infty \right\},$$

and

$$G_p(t) := \inf\left\{ \int_{\{p\Psi < -t\}} |\tilde{f}|^2 : \tilde{f} \in \mathcal{O}(\{p\Psi < -t\}) \ \& \ (\tilde{f} - f)_o \in I(p\Psi)_o \right\},$$

where $t \in [0, +\infty)$ and $p > 2a_o^f(\Psi)$. By the definition of $G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f)$, we have that $G_p(0) \geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f)$ for any $p > 2a_o^f(\Psi)$. Note that

$$p\Psi = \min\left\{ 2\psi + (2\lceil p \rceil - 2p)\log|F| - 2\log\left|F^{\lceil p \rceil}\right|, 0 \right\},$$

where $\lceil p \rceil = \min\{n \in \mathbb{Z} : n \geq p\}$, and

$$G_p(pt) \leq \int_{\{\Psi < -t\}} |f|^2 < +\infty,$$

for any $t > t_1$. Theorem 1.6 indicates that $G_{j,p}(-\log r)$ is concave with respect to $r \in (0, 1]$ and $\lim_{t \to +\infty} G_{j,p}(t) = 0$, which verifies that

$$\frac{1}{r_1^2} \int_{\{p\Psi < 2\log r_1\}} |f|^2 \geq \frac{1}{r_1^2} G_p(-2\log r_1) \geq G_p(0) \tag{5.2}$$
$$\geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f),$$

where $0 < r_1 \leq e^{-\frac{pt_0}{2}}$.

We prove $a_o^f(\Psi) > 0$ by contradiction: if $a_o^f(\Psi) = 0$, as $\int_{\{\Psi < -t_1 - 1\}} |f|^2 < +\infty$, the dominated convergence theorem and inequality (5.2) show that

$$\frac{1}{r_1^2} \int_{\{\Psi = -\infty\}} |f|^2 = \lim_{p \to 0+0} \frac{1}{r_1^2} \int_{\{p\Psi < 2\log r_1\}} |f|^2 \tag{5.3}$$
$$\geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f).$$

Note that $\mu(\{\Psi = -\infty\}) = \mu(\{\psi = -\infty\}) = 0$, where μ is the Lebesgue measure on $\mathbb{C}^n$. Then inequality (5.3) implies that $G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f) = 0$, which contradicts inequality (5.1). Thus, we get $a_o^f(\Psi) > 0$.

For any $r_2 \in (0, e^{-a_o^f(\Psi)t_1})$ and $p \in (2a_0^f(\Psi), -\frac{2\log r_2}{t_1})$, since $\frac{2\log r_2}{p} < -t_1$, we have $\int_{\{p\Psi < 2\log r_2\}} |f|^2 < +\infty$. Then it follows from the dominated convergence theorem and inequality (5.2) that

$$\frac{1}{r_2^2} \int_{\{2a_0^f(\Psi)\Psi \leq 2\log r_2\}} |f|^2 = \lim_{p \to 2a_0^f(\Psi)+0} \frac{1}{r_2^2} \int_{\{p\Psi < 2\log r_2\}} |f|^2 \tag{5.4}$$
$$\geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f).$$

Set $r \in (0, e^{-a_o^f(\Psi)t_0}]$. If $r > e^{-a_o^f(\Psi)t_1}$, we have

$$\int_{\{a_0^f(\Psi)\Psi < \log r\}} |f|^2 = +\infty > G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f).$$

If $r \in (0, e^{-a_o^f(\Psi)t_1}]$, it follows from

$$\{a_o^f(\Psi)\Psi < \log r\} = \bigcup_{0 < r_2 < r} \{a_o^f(\Psi)\Psi < \log r_2\}$$

and inequality (5.4) that

$$\begin{aligned}
\int_{\{a_0^f(\Psi)\Psi < \log r\}} |f|^2 &= \sup_{r_2 \in (0, r)} \int_{\{2a_0^f(\Psi)\Psi \le 2 \log r_2\}} |f|^2 \\
&\ge \sup_{r_2 \in (0, r)} r_2^2 G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f) \\
&= r^2 G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f).
\end{aligned}$$

Thus, Theorem 1.11 holds.

5.2. Proof of Corollary 1.12.

It is clear that $I_+(a\Psi)_o \subset I(a\Psi)_o$. Hence, it suffices to prove $I(a\Psi) \subset I_+(a\Psi)_o$.

If there exists a holomorphic function f on $\{\Psi < -t_0\} \cap D_0$ such that $f_o \in I(a\Psi)_o$ and $f_o \notin I_+(a\Psi)_o$, where $t_0 > 0$ and $D_0 \subset D$ is a neighborhood of o, then $a_o^f(\Psi)_o = a/2 < +\infty$. Theorem 1.11 shows that $a > 0$. For any neighborhood $U \subset D_0$ of o, it follows from Theorem 1.11 that there exists $C_U > 0$ such that

$$\frac{1}{r^2} \int_{\{a\Psi < 2 \log r\} \cap U} |f|^2 \ge C_U, \tag{5.5}$$

for any $r \in (0, e^{-\frac{at_0}{2}}]$. Given any $t > at_0$, Fubini's Theorem and inequality (5.5) imply that

$$\begin{aligned}
\int_{\{a\Psi < -t\} \cap U} |f|^2 e^{-a\Psi} &= \int_{\{a\Psi < -t\} \cap U} \left(|f|^2 \int_0^{e^{-a\Psi}} dl \right) \\
&= \int_0^{+\infty} \left(\int_{\{l < e^{-a\Psi}\} \cap \{a\Psi < -t\} \cap U} |f|^2 \right) dl \\
&\ge \int_{e^t}^{+\infty} \left(\int_{\{a\Psi < -\log l\} \cap U} |f|^2 \right) dl \\
&\ge C_U \int_{e^t}^{+\infty} \frac{1}{l} dl \\
&= +\infty,
\end{aligned}$$

which contradicts $f_o \in I(a\Psi)_o$. Thus, there does not exist $f_o \in I(a\Psi)_o$ such that $f_o \notin I_+(a\Psi)_o$, i.e. $I(a\Psi) = I_+(a\Psi)_o$ for any $a \ge 0$.

5.2.1. *Another proof of Corollary 1.12.*

We give another proof of Corollary 1.12 by using Lemma 2.6 and the strong openness property of multiplier ideal sheaves.

According to Lemma 2.6, we only need to prove that

$$\mathcal{I}(k\varphi + a\Psi)_o = \bigcup_{p \in (a, k)} \mathcal{I}(k\varphi + p\Psi)_o,$$

where $\varphi = 2\max\{\psi, 2\log|F|\}$, and $k > a$ is an integer. It is easy to see

$$\bigcup_{p \in (a,k)} \mathcal{I}(k\varphi + p\Psi)_o \subset \mathcal{I}(k\varphi + a\Psi)_o.$$

In the following, we prove that

$$\mathcal{I}(k\varphi + a\Psi)_o \subset \bigcup_{p \in (a,k)} \mathcal{I}(k\varphi + p\Psi)_o.$$

Note that $k\varphi + p\Psi$ is a plurisubharmonic function on D for any $p \in [0, k]$.

When $a > 0$, for any $(f, o) \in \mathcal{I}(k\varphi + a\Psi)_o$, it follows from the strong openness property of multiplier ideal sheaves ([29]) that there exists $r > 1$ such that $(f, o) \in \mathcal{I}(rk\varphi + ar\Psi)_o$. As

$$\mathcal{I}(rk\varphi + ar\Psi)_o \subset \bigcup_{p \in (a,k)} \mathcal{I}(k\varphi + p\Psi)_o,$$

we have

$$(f, o) \in \bigcup_{p \in (a,k)} \mathcal{I}(k\varphi + p\Psi)_o.$$

When $a = 0$, for any $(f, o) \in \mathcal{I}(k\varphi + a\Psi)_o$, as ψ is plurisubharmonic, it follows from the strong openness property of multiplier ideal sheaves ([29]) and Hölder's inequality that there exist $r_1 > 1$ and a neighborhood $U \Subset D$ of o satisfying that

$$\int_U |f|^2 e^{-k\varphi - r_1\psi} \le \left(\int_U |f|^{2q_1} e^{-q_1 k\varphi}\right)^{\frac{1}{q_1}} \cdot \left(\int_U e^{-q_2 r_1\psi}\right)^{\frac{1}{q_2}} < +\infty, \qquad (5.6)$$

where $q_1 > 1$ and $q_2 > 1$ satisfy $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Since $\varphi/2 + \Psi = \psi$, inequality (5.6) implies

$$(f, o) \in \bigcup_{p \in (a,k)} \mathcal{I}(k\varphi + p\Psi)_o.$$

Thus, Corollary 1.12 holds.

6. Proof of Proposition 1.13

Denote

$$C := \inf_m G(0; \Psi_m, I(\Psi_m)_o, f_m) > 0.$$

It follows from Theorem 1.6 that $G(-\log r; \Psi_m, I(\Psi_m)_o, f_m)$ is concave with respect to $r \in (0, 1]$. Then for some $t_0 > 0$, it holds that

$$\int_{\{\Psi_m < -t\}} |f_m|^2 \ge e^{-t} G(0; \Psi_m, I(\Psi_m)_o, f_m) \ge e^{-t} C, \ \forall t \ge t_0. \qquad (6.1)$$

For any $\epsilon > 0$, as $\{\Psi_m\}_{m \in \mathbb{Z}_{>0}}$ converges to Ψ, there exists $m_0 > 0$ such that

$$\mu\{|\Psi_m - \Psi| \ge 1\} < \epsilon, \ \forall m \ge m_0, \qquad (6.2)$$

where μ is the Lebesgue measure on $\mathbb{C}^n$. Denote

$$M := \sup_m \sup_D |\tilde{f}_m|^2 < +\infty.$$

Note that $\tilde{f}_m = f_m$ on $\{\Psi_m < -t_0\}$. Then inequality (6.1) and inequality (6.2) indicate that

$$\int_{\{\Psi < -t+1\}} |\tilde{f}_m|^2$$
$$\geq \int_{\{\Psi_m < -t\}} |\tilde{f}_m|^2 - \int_{\{|\Psi_m - \Psi| \geq 1\}} |\tilde{f}_m|^2$$
$$\geq e^{-t}C - M\epsilon,$$

for any $t \geq t_0$ and $m \geq m_0$, yielding that

$$\liminf_{m \to +\infty} \int_{\{\Psi < -t+1\}} |\tilde{f}_m|^2 \geq e^{-t}C, \ \forall t \geq t_0. \tag{6.3}$$

As $\{\tilde{f}_m\}_{m \in \mathbb{Z}_{>0}}$ converges to $\tilde{f}$ in Lebesgue measure, and $\sup_m \sup_D |\tilde{f}_m| < +\infty$, the dominated convergence theorem and inequality (6.3) verify that

$$\int_{\{\Psi < -t+1\}} |\tilde{f}|^2 \geq e^{-t}C, \ \forall t \geq t_0.$$

Using Fubini's Theorem, we get that

$$\int_D |\tilde{f}|^2 e^{-\Psi} \geq \int_{\{\Psi < -t_0\}} |\tilde{f}|^2 e^{-\Psi} = \int_{\{\Psi < -t_0\}} \left(|\tilde{f}|^2 \int_0^{e^{-\Psi}} dl \right)$$
$$= \int_0^{+\infty} \left(\int_{\{l < e^{-\Psi}\} \cap \{\Psi < -t_0\}} |\tilde{f}|^2 \right) dl$$
$$\geq \int_{e^{t_0}}^{+\infty} \left(\int_{\{\Psi < -\log l\}} |\tilde{f}|^2 \right) dl$$
$$\geq e^{-1}C \int_{e^{t_0}}^{+\infty} \frac{1}{l} dl$$
$$= +\infty,$$

which holds for any pseudoconvex domain $D \subset \Delta^n$ containing o. Consequently, we conclude that

$$|\tilde{f}|^2 e^{-\Psi} \notin L^1(U),$$

where U is any neighborhood of o.

7. Appendix

7.1. Proof of Lemma 2.1.

In the first part of the appendix, we give the proof of Lemma 2.1.

Firstly, we do some preparations. We recall some lemmas on L^2 estimates for $\bar{\partial}$-equations. In the following, $\bar{\partial}^*$ denotes the Hilbert adjoint operator of $\bar{\partial}$.

Lemma 7.1. *(see* [44], *see also* [2]*) Let $\Omega \Subset \mathbb{C}^n$ be a domain with C^∞ boundary $b\Omega$, $\Phi \in C^\infty(\bar{\Omega})$. Let ρ be a C^∞ defining function for Ω such that $|d\rho| = 1$ on $b\Omega$. Let η be a smooth function on $\bar{\Omega}$. For any $(0,1)$-form $\alpha = \sum_{j=1}^n \alpha_{\bar{j}} d\bar{z}^j \in$*

$Dom_\Omega(\bar{\partial}^*) \cap C^\infty_{(0,1)}(\bar{\Omega})$,

$$\int_\Omega \eta|\bar{\partial}^*_\Phi \alpha|^2 e^{-\Phi} + \int_\Omega \eta|\bar{\partial}\alpha|^2 e^{-\Phi} = \sum_{i,j=1}^n \int_\Omega \eta|\bar{\partial}_i \alpha_{\bar{j}}|^2 e^{-\Phi}$$

$$+ \sum_{i,j=1}^n \int_{b\Omega} \eta(\partial_i \bar{\partial}_j \rho)\alpha_{\bar{i}}\bar{\alpha}_{\bar{j}} e^{-\Phi} + \sum_{i,j=1}^n \int_\Omega \eta(\partial_i \bar{\partial}_j \Phi)\alpha_{\bar{i}}\bar{\alpha}_{\bar{j}} e^{-\Phi} \qquad (7.1)$$

$$+ \sum_{i,j=1}^n \int_\Omega -(\partial_i \bar{\partial}_j \eta)\alpha_{\bar{i}}\bar{\alpha}_{\bar{j}} e^{-\Phi} + 2\mathrm{Re}(\bar{\partial}^*_\Phi \alpha, \alpha \llcorner(\bar{\partial}\eta)^\sharp)_{\Omega,\Phi},$$

where $\alpha \llcorner(\bar{\partial}\eta)^\sharp = \sum_j \alpha_{\bar{j}} \partial_j \eta$.

The symbols and notations can be referred to [30]. See also [44] or [47].

Lemma 7.2. (see [2], see also [30]) Let $\Omega \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain with C^∞ boundary $b\Omega$ and $\Phi \in C(\bar{\Omega})$. Let λ be a $\bar{\partial}$ closed smooth form of bidegree $(n,1)$ on $\bar{\Omega}$. Assume the inequality

$$|(\lambda,\alpha)_{\Omega,\Phi}|^2 \le C \int_\Omega |\bar{\partial}^*_\Phi \alpha|^2 \frac{e^{-\Phi}}{\mu} < +\infty,$$

where $\frac{1}{\mu}$ is an integrable positive function on Ω and C is a constant, holds for all $(n,1)$-form $\alpha \in Dom_\Omega(\bar{\partial}^*) \cap Ker(\bar{\partial}) \cap C^\infty_{(n,1)}(\bar{\Omega})$. Then there is a solution u to the equation $\bar{\partial}u = \lambda$ such that

$$\int_\Omega |u|^2 \mu e^{-\Phi} \le C.$$

The following lemma will be used in the proof of Lemma 7.4.

Lemma 7.3 (see [9]). *For arbitrary $\eta = (\eta_1,\ldots,\eta_p) \in (0,+\infty)^p$, the function*

$$M_\eta(t_1,\ldots,t_p) = \int_{\mathbb{R}^p} \max\{t_1 + h_1,\ldots,t_p + h_p\} \prod_{1\le j\le p} \theta\left(\frac{h_j}{\eta_j}\right) dh_1 \ldots dh_p$$

possesses the following properties:

(1) $M_\eta(t_1,\ldots,t_p)$ *is non decreasing in all variables, smooth and convex on $\mathbb{R}^p$;*

(2) $\max\{t_1,\ldots,t_p\} \le M_\eta(t_1,\ldots,t_p) \le \max\{t_1 + \eta_1,\ldots,t_p + \eta_p\}$;

(3) $M_\eta(t_1,\ldots,t_p) = M_{(\eta_1,\ldots,\hat{\eta}_j,\ldots,\eta_p)}(t_1,\ldots,\hat{t}_j,\ldots,t_p)$ *if $t_j + \eta_j \le \max_{k\ne j}\{t_k - \eta_k\}$;*

(4) $M_\eta(t_1 + a,\ldots,t_p + a) = M_\eta(t_1,\ldots,t_p) + a$ *for any $a \in \mathbb{R}$;*

(5) *if $u_1,\ldots,u_p$ are plurisubharmonic functions, then $u = M_\eta(u_1,\ldots,u_p)$ is plurisubharmonic.*

Following the notations and assumptions in Lemma 2.1, we present the following approximation property of φ and Ψ.

Lemma 7.4. *For any $t_0 > 0$, there exist smooth plurisubharmonic functions $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ and smooth functions $\{\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ on $D\backslash\{F = 0\}$, such that*

(1) $\{\varphi_m + \Psi_m\}_{m\in\mathbb{Z}_{>0}}$ *and $\{\varphi_m + (1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ are smooth plurisubharmonic functions;*

(2) *the sequence $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ is convergent to φ, and there exists a smooth plurisubharmonic function φ_0 on $D\backslash\{F = 0\}$ such that $\varphi_0 \ge \varphi_m \ge \varphi$ on $D\backslash\{F = 0\}$ for any $m \in \mathbb{Z}_{>0}$;*

(3) *the sequence $\{\varphi_m + (1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ is decreasingly convergent to $\varphi + (1+\delta)\Psi$;*

(4) $\{\Psi_m < -t_0\} \subset \{\Psi < -t_0\}$ *and $\Psi_m \leq 0$ on $D\backslash\{F=0\}$ for any $m \in \mathbb{Z}_{>0}$.*

Proof. As D is a pseudoconvex domain, there exists a sequence of plurisubharmonic functions $\{\psi_m\}_{m\in\mathbb{Z}_{>0}}$, which is decreasingly convergent to ψ. Let $\eta_m = (\frac{t_0}{3m}, \frac{t_0}{3m})$. Set

$$\varphi_m = (1+\delta)M_{\eta_m}(\psi_m, 2\log|F|),$$

and

$$\Psi_m = \psi_m - \frac{\varphi_m}{1+\delta},$$

which are smooth functions on $D\backslash\{F=0\}$.

Now, we prove that (φ_m, Ψ_m) satisfies the four conditions. Lemma 7.3 shows that φ_m is plurisubharmonic. Note that

$$\varphi_m + \Psi_m = \psi_m + \frac{\delta}{1+\delta}\varphi_m$$

and

$$\varphi_m + (1+\delta)\Psi_m = (1+\delta)\psi_m$$

are smooth plurisubharmonic functions, and $\{\varphi_m + (1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ decreasingly converges to $\varphi + (1+\delta)\Psi = (1+\delta)\psi$. Set

$$\varphi_0 = (1+\delta)\left(\max\{\psi_1, 2\log|F|\} + \frac{t_0}{3}\right).$$

It follows from Lemma 7.3 that

$$\varphi \leq (1+\delta)\max\{\psi_m, 2\log|F|\} \leq \varphi_m \leq (1+\delta)\left(\max\{\psi_m, 2\log|F|\} + \frac{t_0}{3m}\right),$$

which implies that $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ is convergent to φ and $\varphi \leq \varphi_m \leq \varphi_0$ on $D\backslash\{F=0\}$. It can be verified by Lemma 7.3 that:

(1) if $\psi_m \leq 2\log|F| - \frac{2t_0}{3m}$ holds, then $\varphi_m = 2(1+\delta)\log|F|$ and $\Psi_m = \psi_m - 2\log|F| \leq 0$;

(2) if $\psi_m \geq 2\log|F| + \frac{2t_0}{3m}$ holds, then $\varphi_m = (1+\delta)\psi_m$ and $\Psi_m = 0$;

(3) if $2\log|F| - \frac{2t_0}{3m} < \psi_m < 2\log|F| + \frac{2t_0}{3m}$ holds, then $-\frac{t_0}{m} \leq \Psi_m \leq 0$, and

$$(1+\delta)\max\{\psi_m, 2\log|F|\} \leq \varphi_m \leq (1+\delta)\left(\psi_m + \frac{t_0}{m}\right).$$

Thus, we have

$$\{\Psi_m < -t_0\} \subset \{\psi - 2\log|F| < -t_0\} \subset \{\Psi < -t_0\},$$

and $\Psi_m \leq 0$ on $D\backslash\{F=0\}$. $\qquad\square$

Now we begin to prove Lemma 2.1.

Note that $D\backslash\{F=0\}$ is a pseudoconvex domain. The following remark shows that it suffices to consider the case of Lemma 2.1 that $F(z) \neq 0$ for any $z \in D$.

Remark 7.5. *Assume that there exists a holomorphic function F_1 on $D\backslash\{F=0\}$ such that*

$$\int_{D\backslash\{F=0\}} |F_1 - (1 - b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{-\varphi + v_{t_0,B}(\Psi) - \Psi}$$

$$\leq \left(\frac{1}{\delta} + 1 - e^{-t_0 - B}\right)\int_D \frac{1}{B}\mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}}|f|^2 e^{-\Psi}. \tag{7.2}$$

Note that $v_{t_0,B}(\Psi) - \Psi \geq 0$ on D and $|F|^{2(1+\delta)}e^{-\varphi} = 1$ on $\{\Psi < -t_0\}$. Then it follows from $\int_{\{\Psi < -t_0\}} |f|^2 < +\infty$ and inequality (7.2) that

$$\int_{D\backslash\{F=0\}} |F_1|^2 e^{-\varphi}$$

$$\leq 2\int_{D\backslash\{F=0\}} |(1-b_{t_0,B}(\Psi))f|^2 + 2\int_{D\backslash\{F=0\}} |F_1 - (1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{-\varphi}$$

$$\leq 2\int_{\{\Psi<-t_0\}} |f|^2 + \left(\frac{1}{\delta} + 1 - e^{-t_0-B}\right)\int_D \frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi}$$

$$< +\infty,$$

which implies that there exists a holomorphic function $\tilde{F}$ on D such that $\tilde{F} = F_1$ on $D\backslash\{F=0\}$. Thus, we have

$$\int_D |\tilde{F} - (1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{-\varphi+v_{t_0,B}(\Psi)-\Psi}$$

$$\leq \left(\frac{1}{\delta} + 1 - e^{-t_0-B}\right)\int_D \frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi}.$$

Let $D_1 \Subset \ldots \Subset D_j \Subset D_{j+1} \Subset \ldots$ be a sequence of strongly pseudoconvex domains such that $\bigcup_{j=1}^{+\infty} D_j = D$. As discussed above, there exist smooth plurisubharmonic functions $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ and smooth functions $\{\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ on D, such that:

(1) $\{\varphi_m + \Psi_m\}_{m\in\mathbb{Z}_{>0}}$ and $\{\varphi_m + (1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ are smooth plurisubharmonic functions;

(2) the sequence $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ is convergent to φ, and there exists a smooth plurisubharmonic function φ_0 on D such that $\varphi_0 \geq \varphi_m \geq \varphi$ on D for any $m \in \mathbb{Z}_{>0}$;

(3) the sequence $\{\varphi_m + (1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ is decreasingly convergent to $\varphi + (1+\delta)\Psi$;

(4) $\{\Psi_m < -t_0\} \subset \{\Psi < -t_0\}$ and $\Psi_m \leq 0$ on D for any $m \in \mathbb{Z}_{>0}$.

In the following, we prove Lemma 2.1 in 6 steps.

Step 1: Some Notations

Let $\epsilon \in (0, \frac{1}{8}B)$. Let $\{v_\epsilon\}_{\epsilon\in(0,\frac{1}{8}B)}$ be a family of smooth increasing convex functions on $\mathbb{R}$, which are continuous functions on $\mathbb{R} \cup \{-\infty\}$, such that:

1) $v_\epsilon(t) = t$ for $t \geq -t_0 - \epsilon$, $v_\epsilon(t) = constant$ for $t < -t_0 - B + \epsilon$ and are pointwise convergent to $v_{t_0,B}$, when $\epsilon \to 0$;

2) $v_\epsilon''(t)$ are pointwise convergent to $\frac{1}{B}\mathbb{I}_{(-t_0-B,-t_0)}$, when $\epsilon \to 0$, and $0 \leq v_\epsilon''(t) \leq \frac{2}{B}\mathbb{I}_{(-t_0-B+\epsilon,-t_0-\epsilon)}$ for any $t \in \mathbb{R}$;

3) $v_\epsilon'(t)$ are pointwise convergent to $b_{t_0,B}(t)$ which is a continuous function on $\mathbb{R} \cup \{-\infty\}$, when $\epsilon \to 0$, and $0 \leq v_\epsilon'(t) \leq 1$ for any $t \in \mathbb{R}$.

One can construct the family $\{v_\epsilon\}_{\epsilon\in(0,\frac{1}{8}B)}$ by the setting

$$v_\epsilon(t) := \int_0^t \left(\int_{-\infty}^{t_1} \left(\frac{1}{B-4\epsilon}\mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon}\right)(s)ds\right)dt_1, \qquad (7.3)$$

where $\rho_{\frac{1}{4}\epsilon}$ is the kernel of convolution satisfying $\mathrm{supp}(\rho_{\frac{1}{4}\epsilon}) \subset (-\frac{1}{4}\epsilon, \frac{1}{4}\epsilon)$. Then it follows that

$$v_\epsilon''(t) = \frac{1}{B-4\epsilon}\mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon}(t),$$

and
$$v'_\epsilon(t) = \int_{-\infty}^t \left(\frac{1}{B - 4\epsilon} \mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon} \right)(s)ds.$$

It is clear that $\lim_{\epsilon \to 0} v_\epsilon(t) = v_{t_0,B}(t)$, and $\lim_{\epsilon \to 0} v'_\epsilon(t) = b_{t_0,B}(t)$.

Let $\eta = s(-v_\epsilon(\Psi_m))$ and $\phi = u(-v_\epsilon(\Psi_m))$, where $s \in C^\infty((0,+\infty))$ satisfies $s \geq \frac{1}{\delta}$ and $s' > 0$, and $u \in C^\infty([0,+\infty))$, such that $u''s - s'' > 0$ and $s' - u's = 1$ on $(0,+\infty)$. Let $\Phi = \phi + \varphi_m + \Psi_m$.

Step 2: Solving the $\bar{\partial}$-equation

Now let $\alpha = \sum_{j=1}^n \alpha_{\bar{j}} d\bar{z}^j \in \mathrm{Dom}_{D_j}(\bar{\partial}^*) \cap \mathrm{Ker}(\bar{\partial}) \cap C_{(0,1)}^\infty(\overline{D_j})$. It follows from Cauchy-Schwarz's inequality that

$$2\mathrm{Re}(\bar{\partial}_\Phi^* \alpha, \alpha \llcorner (\bar{\partial}\eta)^\sharp)_{D_j,\Phi} \geq -\int_{D_j} g^{-1}|\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} + \sum_{j,k=1}^n \int_{D_j} -g(\partial_j \eta)(\bar{\partial}_k \eta)\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}} e^{-\Phi},$$
$$(7.4)$$

where g is a positive continuous function on $\overline{D_j}$. Using Lemma 7.1 and inequality (7.4), since D_j is a strongly pseudoconvex domain, we get

$$\int_{D_j} (\eta + g^{-1})|\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi}$$
$$\geq \sum_{j,k=1}^n \int_{D_j} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \Phi - g(\partial_j \eta)(\bar{\partial}_k \eta))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}} e^{-\Phi} \qquad (7.5)$$
$$= \sum_{j,k=1}^n \int_{D_j} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k(\phi + \varphi_m + \Psi_m) - g(\partial_j \eta)(\bar{\partial}_k \eta))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}} e^{-\Phi}.$$

We need some calculations to determine g.

Since
$$\partial_j \bar{\partial}_k \eta = -s'(-v_\epsilon(\Psi_m))\partial_j \bar{\partial}_k(v_\epsilon(\Psi_m)) + s''(-v_\epsilon(\Psi_m))\partial_j v_\epsilon(\Psi_m)\bar{\partial}_k v_\epsilon(\Psi_m),$$

and
$$\partial_j \bar{\partial}_k \phi = -u'(-v_\epsilon(\Psi_m))\partial_j \bar{\partial}_k(v_\epsilon(\Psi_m)) + u''(-v_\epsilon(\psi_m))\partial_j v_\epsilon(\Psi_m)\bar{\partial}_k v_\epsilon(\Psi_m),$$

for any j, k $(1 \leq j, k \leq n)$, we have

$$\sum_{j,k=1}^n (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \phi - g(\partial_j \eta)(\bar{\partial}_k \eta))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}$$
$$=(s' - su') \sum_{j,k=1}^n \partial_j \bar{\partial}_k v_\epsilon(\Psi_m)\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}$$
$$+ ((u''s - s'') - gs'^2) \sum_{j,k=1}^n \partial_j(-v_\epsilon(\Psi_m))\bar{\partial}_k(-v_\epsilon(\Psi_m))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}} \qquad (7.6)$$
$$=(s' - su') \sum_{j,k=1}^n (v'_\epsilon(\Psi_m)\partial_j\bar{\partial}_k\Psi_m + v''_\epsilon(\Psi_m)\partial_j(\Psi_m)\bar{\partial}_k(\Psi_m))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}$$
$$+ ((u''s - s'') - gs'^2) \sum_{j,k=1}^n \partial_j(-v_\epsilon(\Psi_m))\bar{\partial}_k(-v_\epsilon(\Psi_m))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}.$$

We omit the composite item $-v_\epsilon(\Psi_m)$ after $s' - su'$ and $(u''s - s'') - gs'^2$ in the above equalities. As $\varphi_m + \Psi_m$ and $\varphi_m + (1+\delta)\Psi_m$ are plurisubharmonic functions and $v'_\epsilon \in [0, 1]$, we have

$$(1 - v'_\epsilon(\Psi_m))\sqrt{-1}\partial\bar{\partial}(\varphi_m + \Psi_m) + v'_\epsilon(\Psi_m)\sqrt{-1}\partial\bar{\partial}(\varphi_m + (1+\delta)\Psi_m) \geq 0$$

on $\overline{D_j}$, which implies that

$$\eta\sqrt{-1}\partial\bar{\partial}(\varphi_m + \Psi_m) + v'_\epsilon(\Psi_m)\sqrt{-1}\partial\bar{\partial}\Psi_m$$
$$\geq \frac{1}{\delta}\sqrt{-1}\partial\bar{\partial}(\varphi_m + \Psi_m) + v'_\epsilon(\Psi_m)\sqrt{-1}\partial\bar{\partial}\Psi_m \geq 0 \tag{7.7}$$

on $\overline{D_j}$.

Let $g = \frac{u''s - s''}{s'^2}(-v_\epsilon(\Psi_m))$. It follows that $\eta + g^{-1} = \left(s + \frac{s'^2}{u''s - s''}\right)(-v_\epsilon(\Psi_m))$. As $s' - su' = 1$, using inequality (7.5), inequality (7.6) and inequality (7.7), we obtain that

$$\int_{D_j}(\eta + g^{-1})|\bar{\partial}^*_\Phi\alpha|^2 e^{-\Phi} \geq \int_{D_j} v''_\epsilon(\Psi_m)|\alpha_{\llcorner}(\bar{\partial}\Psi_m)^\sharp|^2 e^{-\Phi}. \tag{7.8}$$

As $fF^{1+\delta}$ is holomorphic on $\{\Psi < -t_0\}$ and $\overline{\{\Psi_m < -t_0 - \epsilon\}} \subset \{\Psi_m < -t_0\} \subset \{\Psi < -t_0\}$,

$$\lambda := \bar{\partial}\left(\left(1 - v'_\epsilon(\Psi_m)\right)fF^{1+\delta}\right)$$

is well-defined and smooth on D. By the definition of contraction, Cauchy-Schwarz's inequality and inequality (7.8), it follows that

$$\begin{aligned}
|(\lambda, \alpha)_{D_j,\Phi}|^2 &= |(v''_\epsilon(\Psi_m)fF^{1+\delta}\bar{\partial}\Psi_m, \alpha)_{D_j,\Phi}|^2 \\
&= |(v''_\epsilon(\Psi_m)fF^{1+\delta}, \alpha_{\llcorner}(\bar{\partial}\Psi_m)^\sharp)_{D_j,\Phi}|^2 \\
&\leq \left(\int_{D_j} v''_\epsilon(\Psi_m)|fF^{1+\delta}|^2 e^{-\Phi}\right)\left(\int_{D_j} v''_\epsilon(\Psi_m)|\alpha_{\llcorner}(\bar{\partial}\Psi_m)^\sharp|^2 e^{-\Phi}\right) \quad (7.9) \\
&\leq \left(\int_{D_j} v''_\epsilon(\Psi_m)|fF^{1+\delta}|^2 e^{-\Phi}\right)\left(\int_{D_j}(\eta + g^{-1})|\bar{\partial}^*_\Phi\alpha|^2 e^{-\Phi}\right).
\end{aligned}$$

Set $\mu := (\eta + g^{-1})^{-1}$. By Lemma 7.2, there exists a locally L^1 function $u_{m,\epsilon,j}$ on D_j such that $\bar{\partial}u_{m,\epsilon,j} = \lambda$, and

$$\int_{D_j}|u_{m,\epsilon,j}|^2(\eta + g^{-1})^{-1}e^{-\Phi} \leq \int_{D_j} v''_\epsilon(\Psi_m)|fF^{1+\delta}|^2 e^{-\Phi}. \tag{7.10}$$

Assume that we can choose η and ϕ such that $e^{v_\epsilon(\Psi_m)}e^\phi = (\eta + g^{-1})^{-1}$. Then inequality (7.10) becomes

$$\int_{D_j}|u_{m,\epsilon,j}|^2 e^{v_\epsilon(\Psi_m)-\varphi_m-\Psi_m} \leq \int_{D_j} v''_\epsilon(\Psi_m)|F|^2 e^{-\phi-\varphi_m-\Psi_m}. \tag{7.11}$$

Let

$$F_{m,\epsilon,j} := -u_{m,\epsilon,j} + \left(1 - v'_\epsilon(\Psi_m)\right)fF^{1+\delta}.$$

Then $F_{m,\epsilon,j}$ is holomorphic on D_j. Now inequality (7.11) becomes

$$\int_{D_j} |F_{m,\epsilon,j} - (1 - v'_\epsilon(\Psi_m))fF^{1+\delta}|^2 e^{v_\epsilon(\Psi_m) - \varphi_m - \Psi_m}$$

$$\leq \int_{D_j} (v''_\epsilon(\Psi_m))|fF^{1+\delta}|^2 e^{-\phi - \varphi_m - \Psi_m}. \tag{7.12}$$

Step 3: $m \to +\infty$

Note that

$$\sup_m \sup_{D_j} e^{-\phi} = \sup_m \sup_{D_j} e^{-u(-v_\epsilon(\Psi_m))} < +\infty.$$

As $\{\Psi_m < -t_0 - \epsilon\} \subset \{\Psi < -t_0\}$ and $|F^{1+\delta}|^2 e^{-\varphi} = 1$ on $\{\Psi < -t_0\}$, we have

$$(v''_\epsilon(\Psi_m))|fF^{1+\delta}|^2 e^{-\phi - \varphi_m - \Psi_m} \leq e^{t_0 + B}(v''_\epsilon(\Psi_m))|fF^{1+\delta}|^2 e^{-\phi - \varphi}$$

$$\leq \left(\sup_m \sup_{D_j} e^{-\phi}\right) e^{t_0 + B}\mathbb{I}_{\{\Psi < -t_0\}}|f|^2$$

on D_j. Then it follows from $\int_{\{\Psi < -t_0\}} |f|^2 < +\infty$ and the dominated convergence theorem that

$$\lim_{m \to +\infty} \int_{D_j} (v''_\epsilon(\Psi_m))|fF^{1+\delta}|^2 e^{-\phi - \varphi_m - \Psi_m}$$

$$= \int_{D_j} (v''_\epsilon(\Psi))|fF^{1+\delta}|^2 e^{-u(-v_\epsilon(\Psi)) - \varphi - \Psi} < +\infty. \tag{7.13}$$

Since $\inf_m \inf_{D_j} e^{v_\epsilon(\Psi_m) - \varphi_m - \Psi_m} \geq \inf_{D_j} e^{-\varphi_0} > 0$, by inequality (7.12), we have

$$\sup_m \int_{D_j} |F_{m,\epsilon,j} - (1 - v'_\epsilon(\Psi_m))fF^{1+\delta}|^2 < +\infty.$$

According to

$$\left|(1 - v'_\epsilon(\Psi_m))fF^{1+\delta}\right| \leq \left(\sup_{D_j} |F|^{1+\delta}\right) \mathbb{I}_{\{\Psi < -t_0\}}|f|,$$

and $\int_{\{\Psi < -t_0\}} |f|^2 < +\infty$, we get $\sup_m \int_{D_j} |F_{m,\epsilon,j}|^2 < +\infty$, which implies that there exists a subsequence of $\{F_{m,\epsilon,j}\}_{m \in \mathbb{Z}_{>0}}$ (also denoted by $F_{m,\epsilon,j}$) compactly convergent to a holomorphic function $F_{\epsilon,j}$ on D_j. Now Fatou's Lemma, inequality (7.12) and inequality (7.13) indicate that

$$\int_{D_j} |F_{\epsilon,j} - (1 - v'_\epsilon(\Psi))fF^{1+\delta}|^2 e^{v_\epsilon(\Psi) - \varphi - \Psi}$$

$$\leq \liminf_{m \to +\infty} \int_{D_j} \left|F_{m,\epsilon,j} - (1 - v'_\epsilon(\Psi_m))fF^{1+\delta}\right|^2 e^{v_\epsilon(\Psi_m) - \varphi_m - \Psi_m}$$

$$\leq \lim_{m \to +\infty} \int_{D_j} v''_\epsilon(\Psi_m)|fF^{1+\delta}|^2 e^{-\phi - \varphi_m - \Psi_m} \tag{7.14}$$

$$= \int_{D_j} v''_\epsilon(\Psi)|fF^{1+\delta}|^2 e^{-u(-v_\epsilon(\Psi)) - \varphi - \Psi}$$

$$= \int_{D_j} v''_\epsilon(\Psi)|f|^2 e^{-u(-v_\epsilon(\Psi)) - \Psi}.$$

Step 4: $\epsilon \to 0 + 0$

Note that $\sup_\epsilon \sup_{D_j} e^{-u(-v_\epsilon(\Psi))} < +\infty$, and

$$(v_\epsilon''(\Psi))|f|^2 e^{-u(-v_\epsilon(\Psi))-\Psi} \le \left(\sup_\epsilon \sup_{D_j} e^{-u(-v_\epsilon(\Psi))} \right) e^{t_0+B} \mathbb{I}_{\{\Psi < -t_0\}} |f|^2$$

on D_j. Then it follows from $\int_{\{\Psi < -t_0\}} |f|^2 < +\infty$ and the dominated convergence theorem that

$$
\begin{aligned}
&\lim_{\epsilon \to 0} \int_{D_j} (v_\epsilon''(\Psi))|f|^2 e^{-u(-v_\epsilon(\Psi))-\Psi} \\
&= \int_{D_j} \frac{1}{B} \mathbb{I}_{\{-t_0-B < \Psi < -t_0\}} |f|^2 e^{-u(-v_{t_0,B}(\Psi))-\Psi} \\
&\le \left(\sup_{D_j} e^{-u(-v_{t_0,B}(\Psi))} \right) \int_{D_j} \frac{1}{B} \mathbb{I}_{\{-t_0-B < \Psi < -t_0\}} |f|^2 e^{-\Psi} \\
&< +\infty.
\end{aligned}
\tag{7.15}
$$

Since $\inf_\epsilon \inf_{D_j} e^{v_\epsilon(\Psi)-\varphi-\Psi} > 0$, according to inequality (7.14) and (7.15), we have

$$\int_{D_j} \left| F_{\epsilon,j} - (1 - v_\epsilon'(\Psi)) f F^{1+\delta} \right|^2 < +\infty.$$

Combining with

$$\sup_\epsilon \int_{D_j} |(1 - v_\epsilon'(\Psi)) f F^{1+\delta}|^2 \le \left(\sup_{D_j} |F|^{2(1+\delta)} \right) \int_{D_j} \mathbb{I}_{\{\Psi < -t_0\}} |f|^2 < +\infty,$$

one can obtain $\sup_\epsilon \int_{D_j} |F_{\epsilon,j}|^2 < +\infty$, which implies that there exists a subsequence of $\{F_{\epsilon,j}\}_{\epsilon \to 0+0}$ (denoted by $\{F_{\epsilon_l,j}\}_{l \in \mathbb{Z}_{>0}}$) compactly convergent to a holomorphic function F_j on D_j. Now Fatou's Lemma, inequality (7.14) and inequality (7.15) verify that

$$
\begin{aligned}
&\int_{D_j} |F_j - (1 - b_{t_0,B}(\Psi)) f F^{1+\delta}|^2 e^{v_{t_0,B}(\Psi)-\varphi-\Psi} \\
&\le \liminf_{l \to +\infty} \int_{D_j} |F_{\epsilon_l,j} - (1 - v_{\epsilon_l}'(\Psi)) f F^{1+\delta}|^2 e^{v_{\epsilon_l}(\Psi)-\varphi-\Psi} \\
&\le \liminf_{l \to +\infty} \int_{D_j} (v_{\epsilon_l}''(\Psi))|f|^2 e^{-u(-v_{\epsilon_l}(\Psi))-\Psi} \\
&\le \left(\sup_{D_j} e^{-u(-v_{t_0,B}(\Psi))} \right) \int_{D_j} \frac{1}{B} \mathbb{I}_{\{-t_0-B < \Psi < -t_0\}} |f|^2 e^{-\Psi}.
\end{aligned}
\tag{7.16}
$$

Step 5: $j \to +\infty$

Note that

$$\lim_{j\to+\infty}\left(\sup_{D_j} e^{-u(-v_{t_0,B}(\Psi))}\right)\int_{D_j}\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi}$$
$$\leq\left(\sup_D e^{-u(-v_{t_0,B}(\Psi))}\right)\int_D \frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi}<+\infty,$$

(7.17)

and $v_{t_0,B}(\Psi)-\Psi\geq 0$. Then it follows from inequality (7.16) that

$$\limsup_{j\to+\infty}\int_{D_j}|F_j-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{-\varphi}<+\infty.$$

Combining with

$$\int_{D_j}|(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{-\varphi}\leq\int_{\{\Psi<-t_0\}}|f|^2<+\infty,$$

we obtain

$$\limsup_{j\to+\infty}\int_{D_j}|F_j|^2 e^{-\varphi}<+\infty.$$

Then we can extract a compactly convergent subsequence of $\{F_j\}$ (also denoted by $\{F_j\}$), which is convergent to a holomorphic function $\tilde{F}$ on D. It follows from Fatou's Lemma, inequality (7.16) and inequality (7.17) that

$$\int_D|\tilde{F}-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{v_{t_0,B}(\Psi)-\varphi-\Psi}$$
$$\leq\liminf_{j\to+\infty}\int_{D_j}|F_j-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{v_{t_0,B}(\Psi)-\varphi-\Psi}$$
$$\leq\lim_{j\to+\infty}\left(\sup_{D_j} e^{-u(-v_{t_0,B}(\Psi))}\right)\int_{D_j}\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi}$$
$$\leq\left(\sup_D e^{-u(-v_{t_0,B}(\Psi))}\right)\int_D \frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi}.$$

(7.18)

Step 6: ODE system

Finally, it suffices to find η and ϕ such that $\eta+g^{-1}=e^{-v_\epsilon(\Psi_m)}e^{-\phi}$ on D_j and $s'-u's=1$. As $\eta=s(-v_\epsilon(\Psi_m))$ and $\phi=u(-v_\epsilon(\Psi_m))$, we have

$$(\eta+g^{-1})e^{v_\epsilon(\Psi_m)}e^{\phi}=\left(\left(s+\frac{s'^2}{u''s-s''}\right)e^{-t}e^u\right)\circ\left(-v_\epsilon(\Psi_m)\right).$$

Summarizing the above discussions about s and u, we are naturally led to a system of ODEs (see [27, 28, 30, 31]):

$$(1)\ \left(s+\frac{s'^2}{u''s-s''}\right)e^{u-t}=1,$$
$$(2)\ s'-su'=1,$$

(7.19)

where $t\in(0,+\infty)$. Solving the ODE system (7.19), we get

$$u(t)=-\log\left(\frac{1}{\delta}+1-e^{-t}\right),$$

and
$$s(t) = \frac{\left(1 + \frac{1}{\delta}\right) t + \frac{1}{\delta}\left(1 + \frac{1}{\delta}\right)}{\frac{1}{\delta} + 1 - e^{-t}} - 1,$$

(see [31]). It follows that $s \in C^\infty((0, +\infty))$ satisfying $s \geq \frac{1}{\delta}$ and $s' > 0$, and $u \in C^\infty([0, +\infty))$ satisfying $u''s - s'' > 0$.

As $u(t) = -\log\left(\frac{1}{\delta} + 1 - e^{-t}\right)$ is decreasing with respect to t, and $0 \geq v(t) \geq \max\{t, -t_0 - B_0\} \geq -t_0 - B_0$ for any $t \leq 0$, we can see that

$$\sup_D e^{-u(-v_{t_0,B}(\Psi))} \leq \sup_{t \in [0, t_0+B]} e^{-u(t)} = \frac{1}{\delta} + 1 - e^{-t_0-B}.$$

Therefore, we are done.

Eventually, we complete the proof of Lemma 2.1.

7.2. An example on the Hartogs triangle.

In the second part of the appendix, for the sake of the completeness, we drag an explicit example for the objects described in the present paper. In this example, the function F is not trivial (i.e. not a constant function). A similar example can also be seen in [1].

Let $D = \Delta^2$ be the unit polydisc in $\mathbb{C}^2$ with the coordinate (z, w), $\psi = 2\log|z|$ a plurisubharmonic function on Δ^2, and $F = w^k$ a holomorphic function on Δ^2 for a positive integer k. Then we have

$$\Psi(z, w) := \min\{\psi - 2\log|F|, 0\} = \min\left\{2\log\left|\frac{z}{w^k}\right|, 0\right\},$$

where $\Psi(o, w) := 0$ for any $w \in \Delta$. It follows that

$$D_t := \{\Psi < -t\} = \left\{(z, w) \in \Delta \times \Delta : |z| < e^{-t/2}|w|^k\right\}, \ \forall t \in [0, +\infty).$$

Especially,

$$D_0 = \left\{(z, w) \in \mathbb{C}^2 : |z| < |w|^k < 1\right\},$$

which is also called the (generalized) *Hartogs triangle* (see [16, 43]). One can see that o is a boundary point of D_t for any $t \geq 0$.

Fix $t \geq 0$. Note that D_t is a Reinhardt domain (which means that D_t is invariant under the $\mathbb{T}^n$−action). Then any holomorphic function on D_t can be written as its Laurent expansion. Let

$$f(z, w) = \sum_{(j,l) \in \mathbb{Z}^2} c_{j,l} z^j w^l \in \mathcal{O}(D_t).$$

One can verify that

$$\int_{D_t} |f|^2 = \sum_{(j,l) \in \mathbb{Z}^2} |c_{j,l}|^2 \int_{D_t} |z|^{2j}|w|^{2l},$$

where

$$\int_{D_t} |z|^{2j}|w|^{2l} = \begin{cases} \dfrac{\pi^2 e^{-(j+1)t}}{(j+1)(jk+k+l+1)}, & \text{if } j \geq 0, \ jk+k+l \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Similarly,

$$\int_{D_t} |f|^2 e^{-\Psi} = \sum_{(j,l) \in \mathbb{Z}^2} |c_{j,l}|^2 \int_{D_t} |z|^{2j-2}|w|^{2l+2k},$$

where

$$\int_{D_t} |z|^{2j-2}|w|^{2l+2k} = \begin{cases} \dfrac{\pi^2 e^{-jt}}{j(jk+k+l+1)}, & \text{if } j \geq 1, \ jk+k+l \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then it can be summarized that

$$I(\Psi)_o = \left\{ h(z,w) = \sum_{(j,l)\in\mathbb{Z}^2} a_{j,l} z^j w^l \text{ near } o : a_{j,l} = 0, \ \forall (j,l) \in \mathcal{S} \right\},$$

where

$$\mathcal{S} = \left\{ (j,l) \in \mathbb{Z}^2 : j \leq 0 \text{ or } l \leq -(jk+k+1) \right\}.$$

Set $J = I(\Psi)_o$, and $f = w^{-k}$. It is not hard to calculate that

$$G(t; \Psi, J, f) = \int_{D_t} |f|^2 = \pi^2 e^{-t},$$

which shows that $G(-\log r; \Psi, J, f)$ is concave with respect to $r \in (0,1]$ in this case.

Under the same assumptions above, one can also verify that $c_o^{fF}(\psi) = 1/2$, and

$$\frac{1}{r^2} \int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 = \frac{1}{r^2} \int_{D_{-2\log r}} |w|^{-2k} = \pi^2.$$

On the other hand, since

$$\Psi_1 := \min\{2c_o^{fF}(\psi)\psi - 2\log|F|, 0\} = \Psi,$$

with some similar computations for $a\Psi$ ($a > 1$), one can see that

$$I_+(\Psi)_o = \bigcup_{a>1} I(a\Psi)_o = I(\Psi)_o.$$

Now according to the value of $G(t; \Psi, J, f)$, it can be obtained that

$$\frac{1}{r^2} \int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 = \pi^2 = G(0; \Psi_1, I_+(\Psi)_o, f),$$

for any $r \in (0,1]$. This also shows that the effectiveness result in Theorem 1.7 is sharp.

It is easy to check that $a_o^f(\Psi) = 1/2$, thus one can verify that the lower bound in Theorem 1.11 is also sharp. The strong openness property for Ψ has been checked.

References

[1] S.J. Bao and Q.A. Guan, Modules at boundary points, fiberwise Bergman kernels, and log-subharmonicity II – on Stein manifolds, Preprint, arXiv:2205.08044v2.

[2] B. Berndtsson, The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman, Ann. L'Inst. Fourier (Grenoble) 46 (1996), no.4, 1083–1094.

[3] B. Berndtsson, The openness conjecture for plurisubharmonic functions, arXiv:1305.5781.

[4] B. Berndtsson, Lelong numbers and vector bundles, J. Geom. Anal. 30 (2020), no. 3, 2361-2376.

[5] J.Y. Cao, Ohsawa-Takegoshi extension theorem for compact Kähler manifolds and applications, Complex and symplectic geometry, 19-38, Springer INdAM Ser., 21, Springer, Cham, 2017.

[6] J.Y. Cao, J.-P. Demailly and S. Matsumura, A general extension theorem for cohomology classes on non reduced analytic subspaces, Sci. China Math. 60 (2017), no. 6, 949-962, DOI 10.1007/s11425-017-9066-0.

[7] T. Darvas, E. Di Nezza and H.C. Lu, Monotonicity of nonpluripolar products and complex Monge-Ampére equations with prescribed singularity, Anal. PDE 11 (2018), no. 8, 2049-2087.

[8] T. Darvas, E. Di Nezza and H.C. Lu, The metric geometry of singularity types, J. Reine Angew. Math. 771 (2021), 137-170.

[9] J.-P. Demailly, Complex analytic and differential geometry, electronically accessible at https://www-fourier.ujf-grenoble.fr/~demailly/documents.html.

[10] J.-P. Demailly, Analytic Methods in Algebraic Geometry, Higher Education Press, Beijing, 2010.

[11] J.-P. Demailly, Multiplier ideal sheaves and analytic methods in algebraic geometry, School on Vanishing Theorems and Effective Result in Algebraic Geometry (Trieste,2000),1-148,ICTP lECT.Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[12] J.-P. Demailly, L.Ein and R.Lazarsfeld, A subadditivity property of multiplier ideals, Michigan Math. J. 48 (2000) 137-156.

[13] J.-P. Demailly and J. Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, Ann. Sci. Éc. Norm. Supér. (4) 34 (4) (2001) 525-556.

[14] J.-P. Demailly and T. Peternell, A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds, J. Differential Geom. 63 (2) (2003) 231-277.

[15] C. Favre and M. Jonsson, Valuations and multiplier ideals, J. Amer. Math. Soc. 18 (2005), no. 3, 655-684.

[16] L.D. Edholm and J.D. McNeal, The Bergman projection on fat Hartogs triangles: L^p boundedness, Proc. Amer. Math. Soc., 144(5):2185-2196, 2016.

[17] J.E. Fornæss and J.J. Wu, A global approximation result by Bert Alan Taylor and the strong openness conjecture in $\mathbb{C}^n$, J. Geom. Anal. 28 (2018), no. 1, 1-12.

[18] J.E. Fornæss and J.J. Wu, Weighted approximation in $\mathbb{C}$, Math. Z. 294 (2020), no. 3-4, 1051-1064.

[19] H. Grauert and R. Remmert, Coherent analytic sheaves, Grundlehren der mathematischen Wissenchaften, 265, Springer-Verlag, Berlin, 1984.

[20] Q.A. Guan, Genral concavity of minimal L^2 integrals related to multiplier sheaves, arXiv:1811.03261.v4 [math.CV].

[21] Q.A. Guan, A sharp effectiveness result of Demailly's strong openness conjecture, Adv. Math. 348 (2019): 51-80.

[22] Q.A. Guan and Z.T. Mi, Concavity of minimal L^2 integrals related to multiplier ideal sheaves, Peking Math. J. 6, 393-457 (2023). https://doi.org/10.1007/s42543-021-00047-5.

[23] Q.A. Guan and Z.T. Mi, Concavity of minimal L^2 integrals related to multiplier ideal sheaves on weakly pseudoconvex Kähler manifolds, Sci. China Math. 65 (2022), no. 5, 887-932.

[24] Q.A. Guan, Z.T. Mi and Z. Yuan, Concavity property of minimal L^2 integrals with Lebesgue measurable gain II, https://www.researchgate.net/publication/354464147.

[25] Q.A. Guan and Z. Yuan, An optimal support function related to the strong openness property, J. Math. Soc. Japan 74 (2022), no. 4, 1269-1293.

[26] Q.A. Guan and Z. Yuan, Concavity property of minimal L^2 integrals with Lebesgue measurable gain, Nagoya Math. J. (2023) 252, 842-905, doi: 10.1017/nmj.2023.12.

[27] Q.A. Guan and X.Y. Zhou, Optimal constant problem in the L^2 extension theorem, C. R. Acad. Sci. Paris. Ser. I. 350 (2012), no. 15–16, 753–756.

[28] Q.A. Guan and X.Y. Zhou, Optimal constant in an L^2 extension problem and a proof of a conjecture of Ohsawa, Sci. China Math., 2015, 58(1): 35–59.

[29] Q.A. Guan and X.Y. Zhou, A proof of Demailly's strong openness conjecture, Ann. of Math. (2) 182 (2015), no. 2, 605-616.

[30] Q.A. Guan and X.Y. Zhou, A solution of an L^2 extension problem with an optimal estimate and applications, Ann. of Math. (2) 181 (2015), no. 3, 1139–1208.

[31] Q.A. Guan and X.Y Zhou, Effectiveness of Demailly's strong openness conjecture and related problems, Invent. Math. 202 (2015), no. 2, 635-676.

[32] Q.A. Guan and X.Y Zhou, Restriction formula and subadditivity property related to multiplier ideal sheaves, J. Reine Angew. Math. 769, 1-33 (2020).

[33] H. Guenancia, Toric plurisubharmonic functions and analytic adjoint ideal sheaves, Math. Z. 271 (3-4) (2012) 1011-1035.

[34] L. Hörmander, An introduction to complex analysis in several variables, 3rd edition (North-Holland, Amsterdam, 1990).

[35] M. Jonsson and M. Mustaţă, Valuations and asymptotic invariants for sequences of ideals, Annales de l'Institut Fourier A. 2012, vol. 62, no.6, pp. 2145–2209.

[36] M. Jonsson and M. Mustaţă, An algebraic approach to the openness conjecture of Demailly and Kollár, J. Inst. Math. Jussieu (2013), 1–26.

[37] D. Kim, Skoda division of line bundle sections and pseudo-division, Internat. J. Math. 27 (2016), no. 5, 1650042, 12 pp.

[38] D. Kim and H. Seo, Jumping numbers of analytic multiplier ideals (with an appendix by Sebastien Boucksom), Ann. Polon. Math., 124 (2020), 257-280.

[39] R. Lazarsfeld, Positivity in Algebraic Geometry. I. Classical Setting: Line Bundles and Linear Series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004.

[40] R. Lazarsfeld, Positivity in Algebraic Geometry. II. Positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 49. Springer-Verlag, Berlin, 2004.

[41] J. D. McNeal, D. Varolin, L^2 estimates for the $\bar{\partial}$ operator. Bull. Math. Sci. 5 (2015), no. 2, 179–249.

[42] A. Nadel, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. (2) 132 (3) (1990) 549-596.

[43] M.C. Shaw, The Hartogs triangle in complex analysis, In Geometry and topology of submanifolds and currents, volume 646 of Contemp. Math., pages 105-115, Amer. Math. Soc., Providence, RI, 2015.

[44] Y.T. Siu, The Fujita conjecture and the extension theorem of Ohsawa-Takegoshi, Geometric Complex Analysis, World Scientific, Hayama, 1996, pp.223-277.

[45] Y.T. Siu, Multiplier ideal sheaves in complex and algebraic geometry, Sci. China Ser. A 48 (suppl.) (2005) 1-31.

[46] Y.T. Siu, Dynamic multiplier ideal sheaves and the construction of rational curves in Fano manifolds, Complex Analysis and Digtial Geometry, in: Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., vol.86, Uppsala Universitet, Uppsala, 2009, pp.323-360.

[47] Y.T. Siu, Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type. Complex geometry (Göttingen, 2000), 223–277, Springer, Berlin, (2002).

[48] G. Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, Invent. Math. 89 (2) (1987) 225-246.

[49] C.Y. Xu, A minimizing valuation is quasi-monomial. Ann. of Math. (2) 191 (2020), no. 3, 1003–1030.

[50] X.Y Zhou and L.F. Zhu, An optimal L^2 extension theorem on weakly pseudoconvex Kähler manifolds, J. Differential Geom.110(2018), no.1, 135-186.

[51] X.Y Zhou and L.F. Zhu, Optimal L^2 extension of sections from subvarieties in weakly pseudoconvex manifolds. Pacific J. Math. 309 (2020), no. 2, 475-510.

[52] X.Y Zhou and L.F. Zhu, Siu's lemma, optimal L^2 extension and applications to twisted pluricanonical sheaves, Math. Ann. 377 (2020), no. 1-2, 675-722.

Shijie Bao: Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email address: bsjie@amss.ac.cn

Qi'an Guan: School of Mathematical Sciences, Peking University, Beijing 100871, China.
Email address: guanqian@math.pku.edu.cn

Zheng Yuan: Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.
Email address: yuanzheng@amss.ac.cn