

# Concavity Property of Minimal $L^2$ Integrals with Lebesgue Measurable Gain VII–Negligible Weights



In Memory of Jean-Pierre Demailly (1957–2022)

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**Abstract** In this article, we present characterizations of the concavity property of minimal  $L^2$  integrals with negligible weights degenerating to linearity on the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces. As applications, we obtain characterizations of the holding of equality in optimal jets  $L^2$  extension problem with negligible weights on the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces.

**Keywords** Plurisubharmonic function · Minimal  $L^2$  integral · Optimal  $L^2$  extension theorem

## 1 Introduction

Recall that the strong openness property of multiplier ideal sheaves [35], i.e.  $\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \bigcup_{\epsilon > 0} \mathcal{I}((1 + \epsilon)\varphi)$  (conjectured by Demailly [11]) has been widely used and discussed in several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [4, 6–9, 17, 18, 35, 38, 40, 41, 51–53]), where multiplier ideal sheaf  $\mathcal{I}(\varphi)$  is the sheaf of germs of holomorphic functions  $f$  such that  $|f|^2 e^{-\varphi}$  is locally integrable (see e.g. [11–15, 39, 42, 43, 45–47, 49]), and  $\varphi$  is a plurisubharmonic function on a complex manifold  $M$  (see [10]).

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When  $\mathcal{I}(\varphi) = O$ , the strong openness property is the openness property (conjectured by Demailly-Kollar [14]). Berndtsson [3] established an effectiveness result of the openness property, and obtained the openness property. Stimulated by Berndtsson's effectiveness result, and continuing the solution of the strong openness property [35], Guan-Zhou [37] established an effectiveness result of the strong openness property by considering the minimal  $L^2$  integral on the pseudoconvex domain  $D$ .

Considering the minimal  $L^2$  integrals on the sub-level sets of the weight  $\varphi$ , Guan [21] obtained a sharp version of Guan-Zhou's effectiveness result, and established a concavity property of the minimal  $L^2$  integrals on the sublevel sets of the weight  $\varphi$  (with constant gain), which was applied to give a proof of Saitoh's conjecture for conjugate Hardy  $H^2$  kernels [22], and the sufficient and necessary condition of the existence of decreasing equisingular approximations with analytic singularities for the multiplier ideal sheaves with weights  $\log(|z_1|^{a_1} + \dots + |z_n|^{a_n})$  [23].

For smooth gain, Guan [20] (see also [24]) obtained the concavity property on Stein manifolds (weakly pseudoconvex Kähler case was obtained by Guan-Mi [25]), which was applied by Guan-Yuan to give an optimal support function related to the strong openness property [33] and an effectiveness result of the strong openness property in  $L^p$  [29]. For Lebesgue measurable gain, Guan-Yuan [28] obtained the concavity property on Stein manifolds (weakly pseudoconvex Kähler case was obtained by Guan-Mi-Yuan [26]), which deduced a twisted  $L^p$  strong openness property [30].

Note that the linearity is a degenerate concavity. A natural problem was posed in [31]:

## ? Questions

How can one characterize the concavity property degenerating to linearity?

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For open Riemann surfaces, Guan-Yuan gave an answer to Problem 1 for single points [28] (for the case of subharmonic weights, see Guan-Mi [24]), and gave an answer to Problem 1 for finite points [31]. For products of open Riemann surfaces, Guan-Yuan [32] gave an answer to Problem 1 for products of finite points.

For fibrations over open Riemann surfaces, Bao-Guan-Yuan [1] gave an answer to Problem 1 with negligible weights pulled back from the open Riemann surfaces. For fibrations over products of open Riemann surfaces, Bao-Guan-Yuan [2] gave an answer to Problem 1 with negligible weights vanishing identically.

In this article, for the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces, we give answers to Problem 1 with negligible weights on fibrations.

We would like to recall the definition of minimal  $L^2$  integral as follows.

Let  $\Omega_j$  be an open Riemann surface, which admits a nontrivial Green function  $G_{\Omega_j}$  for any  $1 \leq j \leq n_1$ . Let  $Y$  be an  $n_2$ -dimensional weakly pseudoconvex Kähler manifold, and let  $K_Y$  be the canonical (holomorphic) line bundle on  $Y$ . Let  $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$  be an  $n$ -dimensional complex manifold, where  $n = n_1 + n_2$ .

Let  $\pi_1, \pi_{1,j}$  and  $\pi_2$  be the natural projections from  $M$  to  $\prod_{1 \leq j \leq n_1} \Omega_j$ ,  $\Omega_j$  and  $Y$  respectively. Let  $K_M$  be the canonical (holomorphic) line bundle on  $M$ .

Let  $Z_j$  be a (closed) analytic subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , and denote that  $Z_0 := \left(\prod_{1 \leq j \leq n_1} Z_j\right) \times Y \subset M$ .

Let  $\psi < 0$  be a plurisubharmonic function on  $M$  such that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$  and  $Z_0 \subset \{\psi = -\infty\}$ . Let  $\varphi_1$  be a Lebesgue measurable function on  $\left(\prod_{1 \leq j \leq n_1} \Omega_j\right)$  such that  $\pi_1^*(\varphi_1) + \psi$  is a plurisubharmonic function on  $M$ . Let  $\varphi_2$  be a plurisubharmonic function on  $Y$ . Denote  $\varphi := \pi_1^*(\varphi_1) + \pi_1^*(\varphi_2)$ . Let  $\mathcal{F}_{(z,y)} \supset \mathcal{I}(\varphi + \psi)_{(z,y)}$  be an ideal of  $\mathcal{O}_{(z,y)}$  for any  $(z, y) \in Z_0$ . Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U$  of  $Z_0$ . Let  $c(t)$  be a positive Lebesgue measurable function on  $(0, +\infty)$ . Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, (z, y)) \in (O(K_M) \otimes \mathcal{F})_{(z,y)} \right.$$

for any  $(z, y) \in Z_0$ ,

$$\left. \quad \& \tilde{f} \in H^0(\{\psi < -t\}, O(K_M)). \right\}$$

by  $G(t; c, f, \varphi, \psi, \mathcal{F})$  for any  $t \in [0, +\infty)$ . Here  $|f|^2 := (\sqrt{-1})^{n^2} f \wedge \bar{f}$  for any  $(n, 0)$  form  $f$ . We simply denote  $G(t; c, f, \varphi, \psi, \mathcal{F})$  by  $G(t)$  when there is no misunderstandings and denote  $G(t; c, \varphi, \psi, \mathcal{F})$  by  $G(t; c)$ ,  $G(t; \varphi)$ ,  $G(t; \psi)$  and  $G(t; \mathcal{F})$  when we focus on various choices of  $c(t)$ ,  $\varphi$ ,  $\psi$  and  $\mathcal{F}$  respectively.

We generally assume that  $c(t)$  is a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$ ,  $c(t)e^{-t}$  is decreasing with respect to  $t$  on  $(0, +\infty)$  and  $e^{-\varphi} c(-\psi)$  has a positive lower bound on any compact subset of  $M \setminus Z_0$  in this paper (when other assumption for  $c(t)$  is used, we introduce it explicitly). Then  $G(h^{-1}(r))$  is concave with respect to  $r \in [0, \int_0^{+\infty} c(t_1)e^{-t_1} dt_1]$  (see [26], see also Theorem 10), where  $h(t) = \int_t^{+\infty} c(s)e^{-s} ds$  for any  $t \in [0, +\infty)$ .

## 1.1 Main Results

In this section, we present characterizations of the concavity property of minimal  $L^2$  integrals with negligible weights degenerating to linearity on the fibrations over open Riemann surfaces and products of open Riemann surfaces.

### 1.1.1 Linearity of the Minimal $L^2$ Integrals on Fibrations over Open Riemann Surfaces

In this section, we present characterizations of the concavity property of minimal  $L^2$  integrals degenerating to linearity on the fibrations over open Riemann surface.

To state our results, we firstly recall the following notations (see [16], see also [26, 28, 36]).

Let  $\Omega$  be an open Riemann surface, which admits a nontrivial Green function  $G_\Omega$ . A character  $\chi$  on  $\pi_1(\Omega)$  is a homomorphism from  $\pi_1(\Omega)$  to  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  which takes values in the unit circle  $\{z \in \mathbb{C}; |z| = 1\}$ .

Let  $P : \Delta \rightarrow \Omega$  be the universal covering from unit disc  $\Delta \subset \mathbb{C}$  to  $\Omega$ . We call the holomorphic function  $f$  (resp. holomorphic  $(1, 0)$  form  $F$ ) on  $\Delta$  is a multiplicative function (resp. multiplicative differential (Prym differential)) if there is a character  $\chi$  on  $\pi_1(\Omega)$ , such that  $g^*f = \chi(g)f$  (resp.  $g^*F = \chi(g)F$ ) for every  $g \in \pi_1(\Omega)$  which naturally acts on the universal covering of  $\Omega$ . Denote the set of such kinds of  $f$  (resp.  $F$ ) by  $O^\chi(\Omega)$  (resp.  $\Gamma^\chi(\Omega)$ ).

As  $P$  is a universal covering, then for any harmonic function  $h$  on  $\Omega$ , there exists a character  $\chi_h$  associated to  $h$  and a multiplicative function  $f_h \in O^{\chi_h}(\Omega)$ , such that  $|f_h| = P^*e^h$ . And if  $g \in O(\Omega)$  and  $g$  has no zero points on  $\Omega$ , then we have  $\chi_h = \chi_{h+\log|g|}$ .

For Green function  $G_\Omega(\cdot, z_0)$ , one can find a  $\chi_{z_0}$  and a multiplicative function  $f_{z_0} \in O^{\chi_{z_0}}(\Omega)$ , such that  $|f_{z_0}| = P^*e^{G_\Omega(\cdot, z_0)}$  (see [50]).

Now we assume that  $n_1 = 1$  and then  $M = \left( \prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y = \Omega \times Y$ , where  $Y$  is an  $(n - 1)$ -dimensional weakly pseudoconvex Kähler manifold. Denote  $Z_\Omega = \{z_j : 1 \leq j < \gamma\}$  be a subset of  $\Omega$  of discrete points, where  $\gamma > 1$  is a positive integer or  $\gamma = +\infty$ . Denote  $Z_0 := Z_\Omega \times Y$ . Denote  $Z_j := \{z_j\} \times Y$  for any  $j$ .

Let  $\psi$  be a plurisubharmonic function on  $M$  such that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$  and  $Z_0 \subset \{\psi = -\infty\}$ . It follows from Siu's decomposition theorem that

$$dd^c \psi = \sum_{j \geq 1} 2p_j[Z_j] + \sum_{i \geq 1} \lambda_i[A_i] + R,$$

where  $[Z_j]$  and  $[A_i]$  are the currents of integration over an irreducible  $(n - 1)$ -dimensional analytic set, and where  $R$  is a closed positive current with the property that  $\dim E_c(R) < n - 1$  for every  $c > 0$ , where  $E_c(R) = \{x \in M : v(R, x) \geq c\}$  is the upperlevel sets of Lelong number. We assume that  $p_j > 0$  for any  $1 \leq j < \gamma$ .

Then  $N := \psi - \pi_1^*\left(\sum_{j \geq 1} 2p_j G_\Omega(z, z_j)\right)$  is a plurisubharmonic function on  $M$ , where  $\pi_1 : M \rightarrow \Omega$  be the natural projection. We assume that  $N \leq 0$  and  $N|_{Z_j}$  is not identically  $-\infty$  for any  $j$ .

Let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\psi + \pi_1^*(\varphi)$  is a plurisubharmonic function on  $M$ . By Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j \geq 1} 2\tilde{q}_j[Z_j] + \sum_{i \geq 1} \tilde{\lambda}_i[\tilde{A}_i] + \tilde{R},$$

where  $\tilde{q}_j \geq 0$  for any  $1 \leq j < \gamma$ .

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function  $g$  on  $\Omega$  such that  $ord_{z_j}(g) = q_j := [\tilde{q}_j]$  for any  $z_j \in Z_\Omega$  and  $g(z) \neq 0$  for any  $z \notin Z_\Omega$ , where  $[q]$  equals to the integral part of the nonnegative real number  $q$ . Then we know that there exists a plurisubharmonic function  $\tilde{\psi}_2 \in Psh(M)$  such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \log |g|) + \tilde{\psi}_2.$$

Let  $\varphi_2 \in Psh(Y)$ . Denote  $\pi_2 : M \rightarrow Y$  be the natural projection and  $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$ .

For any  $1 \leq j < \gamma$ , let  $\tilde{z}_j$  be a local coordinate on a neighborhood  $V_j$  of  $z_j$  satisfying  $\tilde{z}_j(z_0) = 0$  and  $V_j \cap V_k = \emptyset$ , for any  $j \neq k$ . Denote  $V_0 := \cup_{1 \leq j < \gamma} V_j$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $V_0 \times Y$  which is a neighborhood of  $Z_0$ . Denote  $\mathcal{F}_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Let  $G(t)$  be the minimal  $L^2$  integral on  $\{\psi < -t\}$  with respect to  $\varphi$ ,  $f$ ,  $\mathcal{F}$  and  $c$  for any  $t \geq 0$ .

Let  $\gamma = m + 1$  is an positive integer, i.e.,  $Z_\Omega = \{z_j : 1 \leq j < \gamma\}$  contains  $m$  points. We obtain the following characterization of the concavity of  $G(h^{-1}(r))$  degenerating to linearity on the fibrations over open Riemann surfaces.

**Theorem 1** Assume that  $G(0) \in (0, +\infty)$ . Then  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$  if and only if the following statements hold:

- (1)  $N \equiv 0$  and  $\psi = \pi_1^* \left( 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) \right)$ ;
- (2) for any  $j \in \{1, 2, \dots, m\}$ ,  $f = \pi_1^*(a_j \tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(f_Y) + f_j$  on  $V_j \times Y$ , where  $a_j \in \mathbb{C} \setminus \{0\}$  is a constant,  $k_j$  is a nonnegative integer,  $f_Y$  is a holomorphic  $(n-1, 0)$  form on  $Y$  such that  $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$ , and  $(f_j, (z_j, y)) \in (O(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$  for any  $j \in \{1, 2, \dots, m\}$  and  $y \in Y$ ;
- (3)  $\varphi_1 + 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) = 2 \log |g| + 2 \sum_{j=1}^m G_\Omega(\cdot, z_j) + 2u$ , where  $g$  is a holomorphic function on  $\Omega$  such that  $ord_{z_j}(g) = k_j$  and  $u$  is a harmonic function on  $\Omega$ ;
- (4)  $\prod_{j=1}^m \chi_{z_j} = \chi_{-u}$ , where  $\chi_{-u}$  and  $\chi_{z_j}$  are the characters associated to the functions  $-u$  and  $G_\Omega(\cdot, z_j)$  respectively;
- (5) for any  $j \in \{1, 2, \dots, m\}$ ,

$$\lim_{z \rightarrow z_j} \frac{a_j \tilde{z}_j^{k_j} d\tilde{z}_j}{g P_* \left( f_u \left( \prod_{l=1}^m f_{z_l} \right) \left( \sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)} = c_0, \quad (1)$$

where  $c_0 \in \mathbb{C} \setminus \{0\}$  is a constant independent of  $j$ ,  $f_u$  is a holomorphic function on  $\Delta$  such that  $|f_u| = P^*(e^u)$  and  $f_{z_l}$  is a holomorphic function on  $\Delta$  such that  $|f_{z_l}| = P^*(e^{G_\Omega(\cdot, z_l)})$  for any  $l \in \{1, \dots, m\}$ .

**Remark 1** When  $N \equiv \pi_1^*(N_1)$ , it follows from  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  (see Lemma 23) that Theorem 1 reduces to Theorem 1.4 in [1] (see also Theorem 11), where  $N_1 \leq 0$  is a subharmonic function on  $\Omega$  and  $N_1(z_j) > -\infty$  for any  $j$ .

Let  $\gamma = +\infty$ , i.e.,  $Z_\Omega = \{z_j : 1 \leq j < \gamma\}$  is an infinite subset of  $\Omega$  of discrete points. Assume that  $2 \sum_{j \geq 1} p_j G_\Omega(\cdot, z_j) \not\equiv -\infty$ . We present a necessary condition such that  $G(h^{-1}(r))$  is linear as follows.

**Proposition 1** Assume that  $G(0) \in (0, +\infty)$ . If  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t} dt]$ , then the following statements hold:

$$(1) N \equiv 0 \text{ and } \psi = \pi_1^* \left( 2 \sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j) \right);$$

(2) for any  $j \in \mathbb{N}_+$ ,  $f = \pi_1^*(a_j \tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(f_Y) + f_j$  on  $V_j \times Y$ , where  $a_j \in \mathbb{C} \setminus \{0\}$  is a constant,  $k_j$  is a nonnegative integer,  $f_Y$  is a holomorphic  $(n-1, 0)$  form on  $Y$  such that  $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$ , and  $(f_j, (z_j, y)) \in (O(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$  for any  $j \in \mathbb{N}_+$  and  $y \in Y$ ;

(3)  $\varphi_1 + 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) = 2 \log |g|$ , where  $g$  is a holomorphic function on  $\Omega$  such that  $\text{ord}_{z_j}(g) = k_j + 1$  for any  $j \in \mathbb{N}_+$ ;

(4) for any  $j \in \mathbb{N}_+$ ,

$$\frac{p_j}{\text{ord}_{z_j} g} \lim_{z \rightarrow z_j} \frac{dg}{a_j \tilde{z}_j^{k_j} d\tilde{z}_j} = c_0, \quad (2)$$

where  $c_0 \in \mathbb{C} \setminus \{0\}$  is a constant independent of  $j$ ;

$$(5) \sum_{j \in \mathbb{N}_+} p_j < +\infty.$$

**Remark 2** When  $N \equiv \pi_1^*(N_1)$ , Proposition 1 is Proposition 1.6 in [1] (see also Theorem 4), where  $N_1 \leq 0$  is a subharmonic function on  $\Omega$  and  $N_1(z_j) > -\infty$  for any  $j$ .

Let  $\tilde{M} \subset M$  be an  $n$ -dimensional weakly pseudoconvex submanifold satisfying that  $Z_0 \subset \tilde{M}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $\tilde{U}_0$  of  $Z_0$  in  $\tilde{M}$ .

Let  $\psi$  be a plurisubharmonic function on  $\tilde{M}$  such that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$ . It follows from Siu's decomposition theorem that

$$dd^c \psi = \sum_{j \geq 1} 2p_j [Z_j] + \sum_{i \geq 1} \lambda_i [A_i] + R,$$

where  $[Z_j]$  and  $[A_i]$  are the currents of integration over an irreducible  $(n-1)$ -dimensional analytic set, and where  $R$  is a closed positive current with the property that  $\dim E_c(R) < n-1$  for every  $c > 0$ , where  $E_c(R) = \{x \in \tilde{M} : v(R, x) \geq c\}$  is the upperlevel sets of Lelong number. We assume that  $p_j > 0$  for any  $1 \leq j < \gamma$ .

Then  $N := \psi - \pi_1^*(\sum_{j \geq 1} 2p_j G_\Omega(z, z_j))$  is a plurisubharmonic function on  $\tilde{M}$ . We assume that  $N \leq 0$  and  $N|_{Z_j}$  is not identically  $-\infty$  for any  $j$ .

Let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\psi + \pi_1^*(\varphi)$  is a plurisubharmonic function on  $\tilde{M}$ . By Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j \geq 1} 2\tilde{q}_j [Z_j] + \sum_{i \geq 1} \tilde{\lambda}_i [\tilde{A}_i] + \tilde{R},$$

where  $\tilde{q}_j \geq 0$  for any  $1 \leq j < \gamma$ .

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function  $g$  on  $\Omega$  such that  $\text{ord}_{z_j}(g) = q_j := [\tilde{q}_j]$  for any  $z_j \in Z_\Omega$  and  $g(z) \neq 0$  for any  $z \notin Z_\Omega$ , where  $[q]$  equals to the integral part of the nonnegative real number  $q$ . Then we know that there exists a plurisubharmonic function  $\tilde{\psi}_2 \in Psh(\tilde{M})$  such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \log |g|) + \tilde{\psi}_2.$$

Let  $\varphi_2 \in Psh(Y)$ . Denote  $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$ .

Let  $c(t)$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$ ,  $c(t)e^{-t}$  is decreasing with respect to  $t$  on  $(0, +\infty)$  and  $e^{-\varphi} c(-\psi)$  has a positive lower bound on any compact subset of  $\tilde{M} \setminus Z_0$ .

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, (z, y)) \in \left( O(K_{\tilde{M}}) \otimes I(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)) \right)_{(z, y)} \right. \\ \left. \text{for any } (z, y) \in Z_0, \right. \\ \left. \& \tilde{f} \in H^0(\{\psi < -t\}, O(K_{\tilde{M}})). \right\}$$

by  $\tilde{G}(t)$  for any  $t \in [0, +\infty)$ . Note that  $\tilde{G}(t)$  is a minimal  $L^2$  integrals on  $\tilde{M}$ , where  $M$  is a submanifold of  $\tilde{M}$ .

We present a necessary condition such that  $\tilde{G}(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t} dt]$  as follows.

**Proposition 2** Assume that  $\tilde{G}(0) \in (0, +\infty)$ . If  $\tilde{G}(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t} dt]$ , then  $\tilde{M} = M$ .

**Remark 3** When  $N = \pi_1^*(N_1)|_{\tilde{M}}$ , Proposition 2 is Proposition 1.7 in [1], where  $N_1 \leq 0$  is a subharmonic function on  $\Omega$  and  $N_1(z_j) > -\infty$  for any  $j$ .

### 1.1.2 Linearity of the Minimal $L^2$ Integrals on Fibrations over Products of Open Riemann Surfaces

In this section, we present characterizations of the concavity property of minimal  $L^2$  integrals degenerating to linearity on the fibrations over products of open Riemann surfaces.

When  $n_1 \geq 1$ ,  $M = \left(\prod_{1 \leq j \leq n_1} \Omega_j\right) \times Y$  is an  $n$ -dimensional complex manifold, where  $n = n_1 + n_2$ . Let  $\pi_1, \pi_{1,j}$  and  $\pi_2$  be the natural projections from  $M$  to  $\prod_{1 \leq j \leq n_1} \Omega_j$ ,  $\Omega_j$  and  $Y$  respectively. Let  $K_M$  be the canonical (holomorphic) line bundle on  $M$ . Denote  $P_j : \Delta \rightarrow \Omega_j$  be the universal covering from unit disc  $\Delta$  to  $\Omega_j$  for  $1 \leq j \leq n_1$ .

Let  $Z_j$  be a (closed) analytic subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , and denote that  $Z_0 := \left(\prod_{1 \leq j \leq n_1} Z_j\right) \times Y \subset M$ . Let  $N \leq 0$  be a plurisubharmonic function on  $M$  satisfying  $N|_{Z_0} \not\equiv -\infty$ . For any  $j \in \{1, \dots, n_1\}$ , let  $\varphi_j$  be an upper semi-continuous function on  $\Omega_j$  such that  $\varphi_j(z) > -\infty$  for any  $z \in Z_j$ . Assume that  $\sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + N$  is a plurisubharmonic function on  $M$ . Let  $\varphi_Y$  be a plurisubharmonic function on  $Y$ , and denote that  $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ .

Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ ,  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$  and  $c(-\psi)$  has a positive lower bound on any compact subset of  $M \setminus Z_0$ . Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood of  $Z_0$ .

Assume that  $Z_0 = \{z_0\} \times Y = \{(z_1, \dots, z_{n_1})\} \times Y \subset M$ . Denote

$$\hat{G} := \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\},$$

where  $p_j$  is positive real number for  $1 \leq j \leq n_1$ . Denote

$$\psi := \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\} + N.$$

We assume that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$ . Denote  $\mathcal{F}_{(z_0, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_0, y)}$  for any  $(z_0, y) \in Z_0$ . Let  $G(t)$  be the minimal  $L^2$  integral on  $\{\psi < -t\}$  with respect to  $\varphi, f, \mathcal{F}$  and  $c$  for any  $t \geq 0$ .

Let  $w_j$  be a local coordinate on a neighborhood  $V_{z_j}$  of  $z_j \in \Omega_j$  satisfying  $w_j(z_j) = 0$ . Denote that  $V_0 := \prod_{1 \leq j \leq n_1} V_{z_j}$ , and  $w := (w_1, \dots, w_{n_1})$  is a local coordinate on  $V_0$  of  $z_0 \in \prod_{1 \leq j \leq n_1} \Omega_j$ . Denote that  $E := \{(\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0}\}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $V_0 \times Y \subset M$ .

We present a characterization of the concavity of  $G(h^{-1}(r))$  degenerating to linearity for the case  $Z_0 = \{z_0\} \times Y$ .

**Theorem 2** *Assume that  $G(0) \in (0, +\infty)$ .  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$  if and only if the following statements hold:*

- (1)  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ ;

(2)  $f = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + g_0$  on  $V_0 \times Y$ , where  $g_0$  is a holomorphic  $(n, 0)$  form on  $V_0 \times Y$  satisfying  $(g_0, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$  for any point  $z \in Z_0$  and  $f_\alpha$  is a holomorphic  $(n_2, 0)$  form on  $Y$  such that  $\sum_{\alpha \in E} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \in (0, +\infty)$ ;

(3)  $\varphi_j = 2 \log |g_j| + 2u_j$ , where  $g_j$  is a holomorphic function on  $\Omega_j$  such that  $g_j(z_j) \neq 0$  and  $u_j$  is a harmonic function on  $\Omega_j$  for any  $1 \leq j \leq n_1$ ;

(4)  $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$  for any  $j \in \{1, 2, \dots, n\}$  and  $\alpha \in E$  satisfying  $f_\alpha \not\equiv 0$ , where  $\chi_{j,-u_j}$  be the character associated to  $-u_j$  on  $\Omega_j$  and  $\chi_{j,z_j}$  be the character associated to  $G_{\Omega_j}(\cdot, z_j)$  on  $\Omega_j$ .

**Remark 4** When  $N \equiv 0$  ( $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$ ), it follows from  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  (see Lemma 24) that Theorem 2 can be referred to Theorem 1.2 in [2] (see also Theorem 14).

Let  $Z_j = \{z_{j,1}, \dots, z_{j,m_j}\} \subset \Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $m_j$  is a positive integer. Denote  $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y$ . Let  $N \leq 0$  be a plurisubharmonic function on  $M$  satisfying  $N|_{Z_0} \not\equiv -\infty$ . Denote

$$\hat{G} := \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

where  $p_{j,k}$  is a positive real number. Denote

$$\psi := \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\} + N.$$

We assume that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$ .

Let  $w_{j,k}$  be a local coordinate on a neighborhood  $V_{z_{j,k}} \Subset \Omega_j$  of  $z_{j,k} \in \Omega_j$  satisfying  $w_{j,k}(z_{j,k}) = 0$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ , where  $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$  for any  $j$  and  $k \neq k'$ . Denote that  $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j \leq m_j \text{ for any } j \in \{1, \dots, n_1\}\}$ ,  $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$  for any  $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$  and  $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$  is a local coordinate on  $V_\beta$  of  $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$  satisfying  $w_\beta(z_\beta) = 0$ .

Let  $\beta^* = (1, \dots, 1) \in \tilde{I}_1$ , and let  $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geq 0}^{n_1}$ . Denote that  $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{1 \leq j \leq n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}} \right\}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $\cup_{\beta \in \tilde{I}_1} V_\beta \times Y$  satisfying  $f = \pi_1^* \left( w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$  on  $V_{\beta^*} \times Y$ , where  $f_{\alpha_{\beta^*}}$  and  $f_\alpha$  are holomorphic  $(n_2, 0)$  forms on  $Y$ . Denote  $\mathcal{F}_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$  for any point

$(z, y) \in Z_0$ . Let  $G(t; c)$  be the minimal  $L^2$  integral on  $\{\psi < -t\}$  with respect to  $\varphi$ ,  $f$  and  $\mathcal{F}$  for any  $t \geq 0$ .

We present a characterization of the concavity of  $G(h^{-1}(r))$  degenerating to linearity for the case that  $Z_j$  is a set of finite points.

**Theorem 3** Assume that  $G(0) \in (0, +\infty)$ .  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$  if and only if the following statements hold:

$$(1) N \equiv 0 \text{ and } \psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\};$$

(2)  $\varphi_j = 2 \log |g_j| + 2u_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $u_j$  is a harmonic function on  $\Omega_j$  and  $g_j$  is a holomorphic function on  $\Omega_j$  satisfying  $g_j(z_{j,k}) \neq 0$  for any  $k \in \{1, \dots, m_j\}$ ;

(3) There exists a nonnegative integer  $\gamma_{j,k}$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ , which satisfies that  $\prod_{1 \leq k \leq m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$  and  $\sum_{1 \leq j \leq n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$  for any  $\beta \in \tilde{I}_1$ ;

(4)  $f = \pi_1^* \left( c_\beta \left( \prod_{1 \leq j \leq n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}} \right) dw_{1,\beta_1} \wedge \dots \wedge dw_{n,\beta_n} \right) \wedge \pi_2^*(f_0) + g_\beta \quad \text{on } V_\beta \times Y \text{ for any } \beta \in \tilde{I}_1,$  where  $c_\beta$  is a constant,  $f_0 \not\equiv 0$  is a holomorphic  $(n_2, 0)$  form on  $Y$  satisfying  $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$ , and  $g_\beta$  is a holomorphic  $(n, 0)$  form on  $V_\beta \times Y$  such that  $(g_\beta, z) \in (O(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$  for any  $z \in \{z_\beta\} \times Y$ ;

$$(4) c_\beta \prod_{1 \leq j \leq n_1} \left( \lim_{z \rightarrow z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_* \left( f_{u_j} \left( \prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left( \sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right)} \right) = c_0, \text{ for}$$

any  $\beta \in \tilde{I}_1$ , where  $c_0 \in \mathbb{C} \setminus \{0\}$  is a constant independent of  $\beta$ ,  $f_{u_j}$  is a holomorphic function  $\Delta$  such that  $|f_{u_j}| = P_j^*(e^{u_j})$  and  $f_{z_{j,k}}$  is a holomorphic function on  $\Delta$  such that  $|f_{z_{j,k}}| = P_j^* \left( e^{G_{\Omega_j}(\cdot, z_{j,k})} \right)$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ .

**Remark 5** When  $N \equiv 0$  ( $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_{j,\beta_j}) > -\infty$  for any  $1 \leq j \leq n_1$  and  $1 \leq \beta_j \leq m_j$ ), it follows from  $\mathcal{I}(\varphi + \psi)_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$  for any  $(z, y) \in Z_0$  (see Lemma 24) that Theorem 3 can be referred to Theorem 1.5 in [2] (see also Theorem 15).

Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Denote  $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y$ . Let  $p_{j,k}$  be a positive number for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$  such that  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $j$ . Let

$$\hat{G} = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

and

$$\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\} + N.$$

We assume that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$  and  $\limsup_{t \rightarrow +\infty} c(t) < +\infty$ .

Let  $w_{j,k}$  be a local coordinate on a neighborhood  $V_{z_{j,k}} \Subset \Omega_j$  of  $z_{j,k} \in \Omega_j$  satisfying  $w_{j,k}(z_{j,k}) = 0$  for any  $j \in \{1, \dots, n_1\}$  and  $1 \leq k < \tilde{m}_j$ , where  $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$  for any  $j$  and  $k \neq k'$ . Denote that  $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$ ,  $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$  for any  $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$  and  $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$  is a local coordinate on  $V_\beta$  of  $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$ .

Let  $\beta^* = (1, \dots, 1) \in \tilde{I}_1$ , and let  $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geq 0}^{n_1}$ . Denote that  $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{1 \leq j \leq n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}} \right\}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $\cup_{\beta \in \tilde{I}_1} V_\beta \times Y$  satisfying  $f = \pi_1^*(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$  on  $V_{\beta^*} \times Y$ , where  $f_{\alpha_{\beta^*}}$  and  $f_\alpha$  are holomorphic  $(n_2, 0)$  forms on  $Y$ . Denote  $\mathcal{F}_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$  for any  $(z, y) \in Z_0$ . Let  $G(t)$  be the minimal  $L^2$  integral on  $\{\psi < -t\}$  with respect to  $\varphi, f, \mathcal{F}$  and  $c$  for any  $t \geq 0$ .

When there exists  $j_0 \in \{1, \dots, n_1\}$  such that  $\tilde{m}_{j_0} = +\infty$ , we present that  $G(h^{-1}(r))$  is not linear.

**Theorem 4** If  $G(0) \in (0, +\infty)$  and there exists  $j_0 \in \{1, \dots, n_1\}$  such that  $\tilde{m}_{j_0} = +\infty$ , then  $G(h^{-1}(r))$  is not linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s} ds]$ .

**Remark 6** When  $N \equiv 0$  ( $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_{j,\beta_j}) > -\infty$  for any  $1 \leq j \leq n_1$  and  $1 \leq \beta_j \leq m_j$ ), it follows from  $\mathcal{I}(\varphi + \psi)_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$  for any  $(z, y) \in Z_0$  (see Lemma 24) that Theorem 4 can be referred to Theorem 1.7 in [2] (see also Theorem 16).

Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Denote  $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y$ .

Let  $\tilde{M} \subset M$  be an  $n$ -dimensional weakly pseudoconvex Kähler manifold satisfying that  $Z_0 \subset \tilde{M}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$ .

Let  $N \leq 0$  be a plurisubharmonic function on  $\tilde{M}$  satisfying  $N|_{Z_0} \not\equiv -\infty$ . For any  $j \in \{1, \dots, n_1\}$ , let  $\varphi_j$  be an upper semi-continuous function on  $\Omega_j$  such that  $\varphi_j(z) > -\infty$  for any  $z \in Z_j$ . Assume that  $\sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + N$  is a plurisubharmonic function on  $\tilde{M}$ . Let  $\varphi_Y$  be a plurisubharmonic function on  $Y$ , and denote that  $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ .

Let  $p_{j,k}$  be a positive number for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$  such that  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $j$ . Denote

$$\hat{G} = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

and

$$\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\} + N.$$

We assume that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$ .

Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ ,  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$  and  $c(-\psi)$  has a positive lower bound on any compact subset of  $\tilde{M} \setminus Z_0$ . Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood of  $Z_0$ .

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, (z, y)) \in \left( O(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)) \right)_{(z,y)} \right. \\ \text{for any } (z, y) \in Z_0, \\ \left. \& \tilde{f} \in H^0(\{\psi < -t\}, O(K_{\tilde{M}})). \right\}$$

by  $\tilde{G}(t)$  for any  $t \in [0, +\infty)$ . Note that  $\tilde{G}(t)$  is a minimal  $L^2$  integrals on  $\tilde{M}$ , where  $M$  is a submanifold of  $\tilde{M}$ .

We present a necessary condition such that  $\tilde{G}(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$  as follows.

**Proposition 3** *If  $\tilde{G}(0) \in (0, +\infty)$  and  $\tilde{G}(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ , we have  $\tilde{M} = M$ .*

**Remark 7** When  $N \equiv 0$  ( $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_{j,\beta_j}) > -\infty$  for any  $1 \leq j \leq n_1$  and  $1 \leq \beta_j \leq m_j$ ), it follows from  $\mathcal{I}(\varphi + \psi)_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$  for any  $(z, y) \in Z_0$  (see Lemma 24) that Proposition 3 can be referred to Proposition 1.8 in [2].

## 1.2 Applications

In this section, we present characterizations of the holding of equality in optimal jets  $L^2$  extension problem with the negligible weights on fibrations.

### 1.2.1 Background: Equality in Optimal Jets $L^2$ Extension Problem

Let  $\Omega$  be an open Riemann surface with a nontrivial Green function  $G_\Omega$ . Let  $w$  be a local coordinate on a neighborhood  $V_{z_0}$  of  $z_0 \in \Omega$  satisfying  $w(z_0) = 0$ . Let  $c_\beta(z)$  be the logarithmic capacity (see [?]) on  $\Omega$ , i.e.

$$c_\beta(z_0) := \exp \lim_{\xi \rightarrow z_0} G_\Omega(z, z_0) - \log |w(z)|.$$

Let  $B_\Omega(z_0)$  be the Bergman kernel function on  $\Omega$ . An open question was posed by Sario-Oikawa [?]: find a relation between the magnitudes of the quantities  $\sqrt{\pi} B_\Omega(z)$ ,  $c_\beta(z)$ .

In [48], Saito conjectured:  $\pi B_\Omega(z_0) \geq (c_\beta(z_0))^2$  holds, and the equality holds if and only if  $\Omega$  is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

The inequality part of Saito conjecture for bounded planar domains was proved by Błocki [5], and original form of the inequality part was proved by Guan-Zhou [34]. The equality part of Saito conjecture was proved by Guan-Zhou [36], then Saito conjecture was completed proved.

It follows from the extremal property of the Bergman kernel function that the holding of the following two equalities are equivalent

- (1)  $\pi B_\Omega(z_0) = (c_\beta(z_0))^2$ ;
- (2)  $\inf\left\{\int_\Omega |F|^2 : F \text{ is a holomorphic } (1,0) \text{ form on } \Omega \text{ such that } F(z_0) = dw\right\} = \frac{2\pi}{(c_\beta(z_0))^2}$ .

Note that (2) is equivalent to the holding of the equality in optimal 0-jet  $L^2$  extension problem for open Riemann surface  $\Omega$  and single point  $z_0$  with trivial weights  $\varphi \equiv 0$  and trivial gain  $c \equiv 1$ . Then it is natural to ask

## ? Questions

How can one characterize the holding of the equality in optimal  $k$ -jets  $L^2$  extension problem, where  $k$  is a nonnegative integer?

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For open Riemann surfaces and single points, when the weights are harmonic and gain is constant, Guan-Zhou [36] gave an answer to 0-jet version of Problem 1.2.1, i.e. a proof of the extended Saito conjecture posed by Yamada [50].

For open Riemann surfaces and single points, Guan-Yuan [28] gave an answer to 0-jet version of Problem 1.2.1 (when the weights are subharmonic and gain is smooth, Guan-Mi [24] gave an answer to 0-jet version of Problem 1.2.1), and Guan-Mi-Yuan [26] gave an answer to Problem 1.2.1.

For open Riemann surfaces and analytic sets (with finite or infinite points), Guan-Yuan [31] gave an answer to Problem 1.2.1. For products of open Riemann surfaces and products of analytic sets, Guan-Yuan [32] gave an answer to Problem 1.2.1.

For fibrations over open Riemann surfaces, Bao-Guan-Yuan [1] gave an answer to Problem 1.2.1 with negligible weights pulled back from the open Riemann surfaces. For fibrations over products of open Riemann surfaces, Bao-Guan-Yuan [2] gave an answer to Problem 1.2.1 with negligible weights pulled back from the products of open Riemann surfaces.

In the following sections, for fibrations over open Riemann surfaces and fibrations over products of open Riemann surfaces, we give answers to Problem 1.2.1 with negligible weights on fibrations.

### 1.2.2 Fibrations over Open Riemann Surfaces

In this section, we give characterizations of the holding of equality in optimal jets  $L^2$  extension problem with negligible weights from fibers over analytic subsets to fibrations over open Riemann surfaces.

Let  $\Omega$  be an open Riemann surface with nontrivial Green functions. Let  $Y$  be an  $(n-1)$ -dimensional weakly pseudoconvex Kähler manifold. Denote  $M = \Omega \times Y$ . Let  $K_M$  be the canonical line bundle on  $M$ . Let  $\pi_1$  and  $\pi_2$  be the natural projections from  $M$  to  $\Omega$  and  $Y$  respectively.

Let  $Z_\Omega = \{z_j : j \in \mathbb{N}_+, 1 \leq j < \gamma\}$  be a subset of  $\Omega$  of discrete points. Denote  $Z_0 := Z_\Omega \times Y$ . Denote  $Z_j := \{z_j\} \times Y$  for any  $j \geq 1$ .

Assume that  $\tilde{M} \subset M$  is an  $n$ -dimensional weakly pseudoconvex submanifold satisfying that  $Z_0 \subset \tilde{M}$ .

Let  $\psi$  be a plurisubharmonic function on  $\tilde{M}$  such that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$  and  $Z_0 \subset \{\psi = -\infty\}$ . It follows from Siu's decomposition theorem that

$$dd^c \psi = \sum_{j \geq 1} 2p_j [Z_j] + \sum_{i \geq 1} \lambda_i [A_i] + R,$$

where  $[Z_j]$  and  $[A_i]$  are the currents of integration over an irreducible  $n$ -dimensional analytic set, and where  $R$  is a closed positive current with the property that  $\dim E_c(R) < n$  for every  $c > 0$ , where  $E_c(R) = \{x \in M : v(R, x) \geq c\}$  is the upper-level sets of Lelong number. We assume that  $p_j \geq 0$  is a positive number for any  $1 \leq j < \gamma$  and  $2 \sum_{j \geq 1} p_j G_\Omega(\cdot, z_j) \not\equiv -\infty$ .

Then  $N := \psi - \pi_1^*(\sum_{j \geq 1} 2p_j G_\Omega(z, z_j))$  is a plurisubharmonic function on  $\tilde{M}$ . We assume that  $N \leq 0$ .

Let  $k_j$  be a nonnegative integer for any  $1 \leq j < \gamma$ . Let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\pi_1^*(\varphi_1) + \psi$  is a plurisubharmonic function on  $\tilde{M}$ . We also assume that there exists a holomorphic function  $g \in \mathcal{O}(\Omega)$  and a plurisubharmonic function  $\tilde{\psi}_2 \in Psh(\tilde{M})$  such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \log |g|) + \tilde{\psi}_2,$$

where  $ord_{z_j}(g) = k_j + 1$ , for any  $1 \leq j < \gamma$ .

Let  $\varphi_2$  be a plurisubharmonic function on  $Y$ . Denote  $\varphi := \pi_1^*(\varphi_1) + \pi_1^*(\varphi_2)$ .

For any  $1 \leq j < \gamma$ , let  $\tilde{z}_j$  be a local coordinate on a neighborhood  $V_j$  of  $z_j$  satisfying  $\tilde{z}_j(z_j) = 0$  and  $V_j \cap V_k = \emptyset$ , for any  $j \neq k$ . We assume that  $g = d_j \tilde{z}_j^{k_j+1} h_j(\tilde{z}_j)$  on each  $V_j$ , where  $h_j(z_j) = 1$ . Denote  $V_0 := \cup_{1 \leq j < \gamma} V_j$ .

Let  $\gamma = m + 1$  be a positive integer. We give an application of Theorem 1 as below.

**Theorem 5** Let  $\psi, \varphi_1$  and  $\varphi_2$  be as above. Let  $c(t)$  be a positive measurable function on  $(0, +\infty)$  satisfying that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$  and  $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$ . Let  $a_j$  be a constant for any  $j$ . Let  $F_j$  be a holomorphic  $(n, 0)$  form on  $Y$ . Assume that

$$\sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} \in (0, +\infty).$$

Let  $f$  be a holomorphic  $(n, 0)$  form on  $V_0 \times Y$  satisfying that  $f = \pi_1^*(a_j \tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(F_j)$  on  $V_j \times Y$ . Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  such that  $(F - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left( \int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (3)$$

Moreover, equality  $\left( \int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} = \inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } \tilde{M} \text{ such that } (\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}) \text{ for any } (z_j, y) \in Z_0 \right\}$  holds if and only if the following statements hold:

$$(1) N \equiv 0 \text{ and } \psi = \pi_1^*(2 \sum_{j=1}^m p_j G_{\Omega}(\cdot, z_j));$$

(2)  $\varphi_1 + 2 \sum_{j=1}^m k_j G_{\Omega}(\cdot, z_j) = 2 \log |g| + 2 \sum_{j=1}^m (k_j + 1) G_{\Omega}(\cdot, z_j) + 2u$ , where  $g$  is a holomorphic function on  $\Omega$  such that  $g(z_j) \neq 0$  for any  $j \in \{1, 2, \dots, m\}$  and  $u$  is a harmonic function on  $\Omega$ ;

(3)  $\prod_{j=1}^m \chi_{z_j}^{k_j+1} = \chi_{-u}$ , where  $\chi_{-u}$  and  $\chi_{z_j}$  are the characters associated to the functions  $-u$  and  $G_{\Omega}(\cdot, z_j)$  respectively;

$$(4) \text{ for any } j \in \{1, 2, \dots, m\},$$

$$\lim_{z \rightarrow z_j} \frac{a_j \tilde{z}_j^{k_j} d\tilde{z}_j}{g P_* \left( f_u \left( \prod_{l=1}^m f_{z_l}^{k_l+1} \right) \left( \sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)} = c_j \in \mathbb{C} \setminus \{0\}, \quad (4)$$

and there exist  $c_0 \in \mathbb{C} \setminus \{0\}$  and a holomorphic  $(n-1, 0)$  form  $F_Y$  on  $Y$  which are independent of  $j$  such that  $c_0 F_Y = c_j F_j$  for any  $j \in \{1, 2, \dots, m\}$ ;

(5)  $\tilde{M} = M$ .

**Remark 8** When  $N \equiv \pi_1^*(N_1)$ , it follows from  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  (see Lemma 23) that Theorem 5 can be referred to Theorem 1.8 in [1] (see also Theorem 12), where  $N_1 \leq 0$  is a subharmonic function on  $\Omega$ .

When  $\gamma = +\infty$ , i.e.,  $Z_\Omega = \{z_j : 1 \leq j < \gamma\}$  is an infinite subset of  $\Omega$  of discrete points. Let  $k_j$  be a nonnegative integer for any  $j \in \mathbb{N}^+$ . Assume that  $p_j = k_j + 1$ , i.e., we have  $\psi = \pi_1^*\left(\sum_{j=1}^{+\infty} 2(k_j + 1)G_\Omega(z, z_j)\right) + N$ , where  $N \leq 0$  is a plurisubharmonic function on  $\tilde{M}$ .

We give an  $L^2$  extension result from fibers over analytic subsets to fibrations over open Riemann surfaces, where the analytic subsets are infinite subsets of discrete points on open Riemann surfaces.

**Theorem 6** Let  $\psi, \varphi_1$  and  $\varphi_2$  be as above. Let  $c(t)$  be a positive measurable function on  $(0, +\infty)$  satisfying that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$  and  $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$ . Let  $a_j$  be a constant for any  $j$ . Let  $F_j$  be a holomorphic  $(n, 0)$  form on  $Y$ . Assume that

$$\sum_{j=1}^{+\infty} \frac{2\pi|a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} \in (0, +\infty).$$

Let  $f$  be a holomorphic  $(n, 0)$  form on  $V_0 \times Y$  satisfying that  $f = \pi_1^*(a_j \tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(F_j)$  on  $V_{z_j} \times Y$ . Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  such that  $(F - f, (z_j, y)) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) < \left( \int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{j=1}^{+\infty} \frac{2\pi|a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (5)$$

**Remark 9** When  $N \equiv \pi_1^*(N_1)$ , it follows from  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  (see Lemma 23) that Theorem 6 can be referred to Theorem 1.10 in [1] (see also Theorem 13), where  $N_1 \leq 0$  is a subharmonic function on  $\Omega$ .

### 1.2.3 Fibrations over Products of Open Riemann Surfaces

In this section, we present characterizations of the holding of equality in optimal jets  $L^2$  extension problem with negligible weights from fibers over analytic subsets to fibrations over products of open Riemann surfaces.

Let  $\Omega_j$  be an open Riemann surface, which admits a nontrivial Green function  $G_{\Omega_j}$  for any  $1 \leq j \leq n_1$ . Let  $Y$  be an  $n_2$ -dimensional weakly pseudoconvex Kähler manifold, and let  $K_Y$  be the canonical (holomorphic) line bundle on  $Y$ . Let  $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$  be an  $n$ -dimensional complex manifold, where  $n = n_1 + n_2$ . Let  $\pi_1, \pi_{1,j}$  and  $\pi_2$  be the natural projections from  $M$  to  $\prod_{1 \leq j \leq n_1} \Omega_j$ ,  $\Omega_j$  and  $Y$  respectively. Let  $K_M$  be the canonical (holomorphic) line bundle on  $M$ . Let  $Z_j$  be a (closed) analytic subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , and denote that  $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y$ . Let  $\tilde{M} \subset M$  be an  $n$ -dimensional complex manifold satisfying that  $Z_0 \subset \tilde{M}$ , and let  $K_{\tilde{M}}$  be the canonical (holomorphic) line bundle on  $\tilde{M}$ .

Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Let  $w_{j,k}$  be a local coordinate on a neighborhood  $V_{z_{j,k}} \Subset \Omega_j$  of  $z_{j,k} \in \Omega_j$  satisfying  $w_{j,k}(z_{j,k}) = 0$  for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$ , where  $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$  for any  $j$  and  $k \neq k'$ . Denote that  $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$ ,  $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$  and  $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$  is a local coordinate on  $V_\beta$  of  $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$  for any  $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$ . Then  $Z_0 = \{(z_\beta, y) : \beta \in \tilde{I}_1 \& y \in Y\} \subset \tilde{M}$ .

Let  $N \leq 0$  be a plurisubharmonic function on  $\tilde{M}$  and let  $\varphi_j$  be a Lebesgue measurable function on  $\Omega_j$  such that  $N + \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j)$  is a plurisubharmonic function on  $\tilde{M}$  satisfying  $(N + \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j))|_{Z_0} \not\equiv -\infty$ . Let  $\varphi_Y$  be a plurisubharmonic function on  $Y$ .

Let  $p_{j,k}$  be a positive number for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$ , which satisfies that  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $1 \leq j \leq n_1$ . Denote

$$\hat{G} := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\},$$

and denote that

$$\psi := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} + N.$$

We assume that  $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$ . Let  $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$  on  $\tilde{M}$ .

Denote that  $E_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$  and  $\tilde{E}_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \geq 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$  for any  $\beta \in \tilde{I}_1$ .

Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$  such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta}) \quad (6)$$

on  $U_0 \cap (V_\beta \times Y)$ , where  $f_{\alpha,\beta}$  is a holomorphic  $(n_2, 0)$  form on  $Y$  for any  $\alpha \in E_\beta$  and  $\beta \in \tilde{I}_1$ .

Denote that

$$c_{j,k} := \exp \lim_{z \rightarrow z_{j,k}} \left( \frac{\sum_{1 \leq k_1 < \tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any  $j \in \{1, \dots, n\}$  and  $1 \leq k < \tilde{m}_j$  (following from Lemma 3 and Lemma 4, we get that the above limit exists).

Let  $c_j(z)$  be the logarithmic capacity (see [?]) on  $\Omega_j$ , which is locally defined by

$$c_j(z_j) := \exp \lim_{z \rightarrow z_j} (G_{\Omega_j}(z, z_j) - \log |w_j(z)|).$$

For the case  $Z_0 = \{z_0\} \times Y \subset \tilde{M}$ , where  $z_0 = (z_1, \dots, z_{n_1}) \in \prod_{1 \leq j \leq n_1} \Omega_j$ , we denote that  $E := \{(\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0}\}$ . Let  $f_\alpha$  be the holomorphic  $(n_2, 0)$  form on  $Y$  which comes from formula (6). We obtain a characterization of the holding of equality in optimal jets  $L^2$  extension problem.

**Theorem 7** *Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$  and  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ . Assume that*

$$\sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \in (0, +\infty).$$

*Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in \left(O(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}\right)\right)_z$  for any  $z \in Z_0$  and*

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}. \end{aligned}$$

*Moreover, equality  $\inf \{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, O(K_{\tilde{M}})) \& (\tilde{F} - f, z) \in \left(O(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}\right)\right)_z \text{ for any } z \in Z_0\} = \left(\int_0^{+\infty} c(s) e^{-s} ds\right) \times \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}$  holds if and only if the following statements hold:*

- (1)  $\tilde{M} = \left(\prod_{1 \leq j \leq n_1} \Omega_j\right) \times Y$  and  $N \equiv 0$ ;

- (2)  $\varphi_j = 2 \log |g_j| + 2u_j$ , where  $g_j$  is a holomorphic function on  $\Omega_j$  such that  $g_j(z_j) \neq 0$  and  $u_j$  is a harmonic function on  $\Omega_j$  for any  $1 \leq j \leq n_1$ ;  
(3)  $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$  for any  $j \in \{1, 2, \dots, n\}$  and  $\alpha \in E$  satisfying  $f_\alpha \not\equiv 0$ .

**Remark 10** If  $(f_\alpha, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  for any  $y \in Y$  and  $\alpha \in \tilde{E} \setminus E$ , the above result also holds when we replace the ideal sheaf  $\mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}\right)$  by  $\mathcal{I}(\hat{G} + \pi_2^*(\varphi_2))$ . We prove the remark in Sect. 7.

Let  $Z_j = \{z_{j,1}, \dots, z_{j,m_j}\} \subset \Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $m_j$  is a positive integer. Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$  such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on  $U_0 \cap (V_\beta \times Y)$ , where  $f_{\alpha,\beta}$  is a holomorphic  $(n_2, 0)$  form on  $Y$  for any  $\alpha \in E_\beta$  and  $\beta \in \tilde{I}_1$ . Let  $\beta^* = (1, \dots, 1) \in \tilde{I}_1$ , and let  $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in E_{\beta^*}$ . Denote that  $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} > 1 \right\}$ . Assume that  $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1}\right) \wedge \pi_2^*(f_{\alpha_{\beta^*},\beta^*}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha,\beta})$  on  $U_0 \cap (V_{\beta^*} \times Y)$ .

We obtain a characterization of the holding of equality in optimal jets  $L^2$  extension problem for the case that  $Z_j$  is finite.

**Theorem 8** Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$  and  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ . Assume that

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta}^{2\alpha_j + 2}} \in (0, +\infty).$$

Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\right)_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta}^{2\alpha_j + 2}}. \end{aligned}$$

Moreover, equality  $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \& (\tilde{F} - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\right)_z \right\}$  for any

$$z \in Z_0 \left\{ = \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_{\beta, w})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta}^{\frac{2\alpha_j + 2}{2\alpha_j + 1}}} \right)$$

holds if and only if the following statements hold:

$$(1) \tilde{M} = \left( \prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y \text{ and } N \equiv 0;$$

(2)  $\varphi_j = 2 \log |g_j| + 2u_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $u_j$  is a harmonic function on  $\Omega_j$  and  $g_j$  is a holomorphic function on  $\Omega_j$  satisfying  $g_j(z_{j,k}) \neq 0$  for any  $k \in \{1, \dots, m_j\}$ ;

(3) There exists a nonnegative integer  $\gamma_{j,k}$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ , which satisfies that  $\prod_{1 \leq k \leq m_j} \chi_{j, z_{j,k}}^{\gamma_{j,k} + 1} = \chi_{j, -u_j}$  and  $\sum_{1 \leq j \leq n_1} \frac{\gamma_{j, \beta_j} + 1}{p_{j, \beta_j}} = 1$  for any  $\beta \in \tilde{I}_1$ ;

(4)  $f_{\alpha, \beta} = c_\beta f_0$  holds for  $\alpha = (\gamma_{1, \beta_1}, \dots, \gamma_{n_1, \beta_{n_1}})$  and  $f_{\alpha, \beta} \equiv 0$  holds for any  $\alpha \in E_\beta \setminus \{(\gamma_{1, \beta_1}, \dots, \gamma_{n_1, \beta_{n_1}})\}$ , where  $\beta \in \tilde{I}_1$ ,  $c_\beta$  is a constant and  $f_0 \not\equiv 0$  is a holomorphic  $(n_2, 0)$  form on  $Y$  satisfying  $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$ ;

$$(5) c_\beta \prod_{1 \leq j \leq n_1} \left( \lim_{z \rightarrow z_{j, \beta_j}} \frac{w_{j, \beta_j}^{\gamma_{j, \beta_j}} dw_{j, \beta_j}}{g_j(P_j)_* \left( f_{u_j} \left( \prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k} + 1} \right) \left( \sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right)} \right) = c_0 \text{ for any } \beta \in \tilde{I}_1,$$

any  $\beta \in \tilde{I}_1$ , where  $c_0 \in \mathbb{C} \setminus \{0\}$  is a constant independent of  $\beta$ ,  $f_{u_j}$  is a holomorphic function  $\Delta$  such that  $|f_{u_j}| = P_j^*(e^{u_j})$  and  $f_{z_{j,k}}$  is a holomorphic function on  $\Delta$  such that  $|f_{z_{j,k}}| = P_j^* \left( e^{G_{\Omega_j}(\cdot, z_{j,k})} \right)$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ .

**Remark 11** If  $(f_{\alpha, \beta}, y) \in (O(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  holds for any  $y \in Y$ ,  $\alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , the above result also holds when we replace the ideal sheaf  $\mathcal{I} \left( \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} \right)$  by  $\mathcal{I}(\hat{G} + \pi_2^*(\varphi_2))$ . We prove the remark in Section 7.

Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$  such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1, \beta_1} \wedge \dots \wedge dw_{n_1, \beta_{n_1}}) \wedge \pi_2^*(f_{\alpha, \beta})$$

on  $U_0 \cap (V_\beta \times Y)$ , where  $f_{\alpha, \beta}$  is a holomorphic  $(n_2, 0)$  form on  $Y$  for any  $\alpha \in E_\beta$  and  $\beta \in \tilde{I}_1$ . Let  $\beta^* = (1, \dots, 1) \in \tilde{I}_1$ , and let  $\alpha_{\beta^*} = (\alpha_{\beta^*, 1}, \dots, \alpha_{\beta^*, n_1}) \in E_{\beta^*}$ . Denote that  $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} > 1 \right\}$ . Assume that  $f = \pi_1^* \left( w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^*(f_{\alpha_{\beta^*}, \beta^*}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha, \beta})$  on  $U_0 \cap (V_{\beta^*} \times Y)$ .

When there exists  $j_0 \in \{1, \dots, n_1\}$  such that  $\tilde{m}_{j_0} = +\infty$ , we obtain that the equality in optimal jets  $L^2$  extension problem could not hold.

**Theorem 9** Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t) e^{-t} dt < +\infty$  and  $c(t) e^{-t}$  is decreasing on  $(0, +\infty)$ . Assume that

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \in (0, +\infty)$$

and there exists  $j_0 \in \{1, \dots, n_1\}$  such that  $\tilde{m}_{j_0} = +\infty$ .

Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & < \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned}$$

**Remark 12** If  $(f_{\alpha,\beta}, y) \in (O(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  holds for any  $y \in Y, \alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , the above result also holds when we replace the ideal sheaf  $\mathcal{I}(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\})$  by  $\mathcal{I}(\hat{G} + \pi_2^*(\varphi_2))$ . We prove the remark in Section 7.

**Remark 13** We note that, by using our recent progress (see [27]) on minimal  $L^2$  integrals related to boundary points, the main results in the present article hold without assuming the condition “ $\{\psi < -t\} \setminus Z_0$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$ ”.

## 2 Preparations I: Minimal $L^2$ Integrals

In this section, we recall and present some lemmas related to minimal  $L^2$  integrals.

### 2.1 Minimal $L^2$ Integrals on Weakly Pseudoconvex Kähler Manifolds

In this section, we recall some results about the concavity property of minimal  $L^2$  integrals on weakly pseudoconvex Kähler manifolds in [26].

Let  $M$  be an  $n$ -dimensional complex manifold. Let  $X$  and  $Z$  be closed subsets of  $M$ . A triple  $(M, X, Z)$  satisfies condition (A), if the following statements hold:

(1)  $X$  is a closed subset of  $M$  and  $X$  is locally negligible with respect to  $L^2$  holomorphic functions, i.e., for any coordinated neighborhood  $U \subset M$  and for any  $L^2$  holomorphic function  $f$  on  $U \setminus X$ , there exists an  $L^2$  holomorphic function  $\tilde{f}$  on  $U$  such that  $\tilde{f}|_{U \setminus X} = f$  with the same  $L^2$  norm;

(2)  $Z$  is an analytic subset of  $M$  and  $M \setminus (X \cup Z)$  is a weakly pseudoconvex Kähler manifold.

Let  $(M, X, Z)$  be a triple satisfying condition (A). Let  $K_M$  be the canonical line bundle on  $M$ . Let  $\psi$  be a plurisubharmonic function on  $M$  such that  $\{\psi < -t\} \setminus (X \cup Z)$  is a weakly pseudoconvex Kähler manifold for any  $t \in \mathbb{R}$ . Let  $\varphi$  be a Lebesgue measurable function on  $M$  such that  $\varphi + \psi$  is a plurisubharmonic function on  $M$ . Denote  $T = -\sup_M \psi$ .

We recall the concept of “gain” in [26]. A positive measurable function  $c$  on  $(T, +\infty)$  is in the class  $\mathcal{P}_{T,M}$  if the following two statements hold:

(1)  $c(t)e^{-t}$  is decreasing with respect to  $t$ ;

(2) there is a closed subset  $E$  of  $M$  such that  $E \subset Z \cap \{\psi = -\infty\}$  and for any compact subset  $K \subset M \setminus E$ ,  $e^{-\varphi}c(-\psi)$  has a positive lower bound on  $K$ .

Let  $Z_0$  be a subset of  $M$  such that  $Z_0 \cap \text{Supp}(O/\mathcal{I}(\varphi + \psi)) \neq \emptyset$ . Let  $U \supset Z_0$  be an open subset of  $M$ , and let  $f$  be a holomorphic  $(n, 0)$  form on  $U$ . Let  $\mathcal{F}_{z_0} \supset \mathcal{I}(\varphi + \psi)_{z_0}$  be an ideal of  $O_{z_0}$  for any  $z_0 \in Z_0$ .

Denote that

$$G(t; c) := \inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, O(K_M)) \right. \\ \left. \& (\tilde{f} - f) \in H^0(Z_0, (O(K_M) \otimes \mathcal{F})|_{Z_0}) \right\}, \quad (7)$$

and

$$\mathcal{H}^2(c, t) := \left\{ \tilde{f} : \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) < +\infty, \tilde{f} \in H^0(\{\psi < -t\}, O(K_M)) \right. \\ \left. \& (\tilde{f} - f) \in H^0(Z_0, (O(K_M) \otimes \mathcal{F})|_{Z_0}) \right\}, \quad (8)$$

where  $t \in [T, +\infty)$ , and  $c$  is a nonnegative measurable function on  $(T, +\infty)$ . Here  $|\tilde{f}|^2 := \sqrt{-1}^{n^2} \tilde{f} \wedge \bar{\tilde{f}}$  for any  $(n, 0)$  form  $\tilde{f}$ , and  $(\tilde{f} - f) \in H^0(Z_0, (O(K_M) \otimes \mathcal{F})|_{Z_0})$  means that  $(\tilde{f} - f, z_0) \in (O(K_M) \otimes \mathcal{F})_{z_0}$  for any  $z_0 \in Z_0$ . If there is no holomorphic  $(n, 0)$  form  $\tilde{f}$  on  $\{\psi < -t\}$  satisfying  $(\tilde{f} - f) \in H^0(Z_0, (O(K_M) \otimes \mathcal{F})|_{Z_0})$ , we set  $G(t; c) = +\infty$ .

In [26], Guan-Mi-Yuan obtained the following concavity of  $G(t; c)$ .

**Theorem 10** ([26]) *Let  $c \in \mathcal{P}_{T,M}$  such that  $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$ . If there exists  $t \in [T, +\infty)$  satisfying that  $G(t) < +\infty$ , then  $G(h^{-1}(r))$  is concave with respect to  $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt)$ ,  $\lim_{t \rightarrow T+0} G(t) = G(T)$  and  $\lim_{t \rightarrow +\infty} G(t) = 0$ , where  $h(t) = \int_t^{+\infty} c(t_1)e^{-t_1}dt_1$ .*

**Lemma 1** ([26]) *Let  $c \in \mathcal{P}_{T,M}$  satisfying  $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$ . Assume that  $G(t) < +\infty$  for some  $t \in [T, +\infty)$ . Then there exists a unique holomorphic  $(n, 0)$*

form  $F_t$  on  $\{\psi < -t\}$  satisfying  $(F_t - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$  and  $\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) = G(t)$ . Furthermore, for any holomorphic  $(n, 0)$  form  $\hat{F}$  on  $\{\psi < -t\}$  satisfying  $(\hat{F} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$  and  $\int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi) < +\infty$ , we have the following equality

$$\begin{aligned} & \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) + \int_{\{\psi < -t\}} |\hat{F} - F_t|^2 e^{-\varphi} c(-\psi) \\ &= \int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi). \end{aligned} \quad (9)$$

Guan-Mi-Yuan also obtained the following corollary of Theorem 10, which is a necessary condition for the concavity degenerating to linearity.

**Lemma 2** ([26]) Let  $c(t) \in \mathcal{P}_{T,M}$  such that  $\int_T^{+\infty} c(s) e^{-s} ds < +\infty$ . If  $G(t) \in (0, +\infty)$  for some  $t \geq T$  and  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_T^{+\infty} c(s) e^{-s} ds)$ , where  $h(t) = \int_t^{+\infty} c(l) e^{-l} dl$ , then there exists a unique holomorphic  $(n, 0)$  form  $F$  on  $M$  satisfying  $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ , and  $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$  for any  $t \geq T$ .

Furthermore, we have

$$\int_{\{-t_2 \leq \psi < -t_1\}} |F|^2 e^{-\varphi} a(-\psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(t) e^{-t} dt} \int_{t_1}^{t_2} a(t) e^{-t} dt, \quad (10)$$

for any nonnegative measurable function  $a$  on  $(T, +\infty)$ , where  $T \leq t_1 < t_2 \leq +\infty$ .

Especially, if  $\mathcal{H}^2(\tilde{c}, t_0) \subset \mathcal{H}^2(c, t_0)$  for some  $t_0 \geq T$ , where  $\tilde{c}$  is a nonnegative measurable function on  $(T, +\infty)$ , we have

$$G(t_0; \tilde{c}) = \int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} \tilde{c}(-\psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(s) e^{-s} ds} \int_{t_0}^{+\infty} \tilde{c}(s) e^{-s} ds. \quad (11)$$

**Remark 14** ([26]) Let  $c(t) \in \mathcal{P}_{T,M}$ . If  $\mathcal{H}^2(\tilde{c}, t_1) \subset \mathcal{H}^2(c, t_1)$ , then  $\mathcal{H}^2(\tilde{c}, t_2) \subset \mathcal{H}^2(c, t_2)$ , where  $t_1 > t_2 > T$ . In the following, we give some sufficient conditions of  $\mathcal{H}^2(\tilde{c}, t_0) \subset \mathcal{H}^2(c, t_0)$  for  $t_0 > T$ :

- (1)  $\tilde{c} \in \mathcal{P}_{T,M}$  and  $\lim_{t \rightarrow +\infty} \frac{\tilde{c}(t)}{c(t)} > 0$ . Especially,  $\tilde{c} \in \mathcal{P}_{T,M}$ ,  $c$  and  $\tilde{c}$  are smooth on  $(T, +\infty)$  and  $\frac{d}{dt}(\log \tilde{c}(t)) \geq \frac{d}{dt}(\log c(t))$ ;
- (2)  $\tilde{c} \in \mathcal{P}_{T,M}$ ,  $\mathcal{H}^2(c, t_0) \neq \emptyset$  and there exists  $t > t_0$  such that  $\{\psi < -t\} \Subset \{\psi < -t_0\}$ ,  $\{z \in \{\psi < -t\} : \mathcal{I}(\varphi + \psi)_z \neq O_z\} \subset Z_0$  and  $\mathcal{F}|_{\overline{\{\psi < -t\}}} = \mathcal{I}(\varphi + \psi)|_{\overline{\{\psi < -t\}}}$ .

## 2.2 The Sufficient and Necessary Conditions of the Concavity of $G(h^{-1}(r))$ Degenerating to Linearity

In this section, we recall some result on the characterizations of the concavity of  $G(h^{-1}(r))$  degenerating to linearity on the fibrations over open Riemann surfaces and products of open Riemann surfaces.

The following result can be referred to [1].

Let  $Z_0^1 := \{z_j : j \in \mathbb{N} \& 1 \leq j \leq m\}$  be a finite subset of the open Riemann surface  $\Omega$ . Let  $Y$  be an  $n - 1$  dimensional weakly pseudoconvex Kähler manifold. Let  $M = \Omega \times Y$  be a complex manifold, and  $K_M$  be the canonical line bundle on  $M$ . Let  $\pi_1, \pi_2$  be the natural projections from  $M$  to  $\Omega$  and  $Y$  and  $Z_0 := \pi_1^{-1}(Z_0^1)$ . Let  $\psi_1$  be a subharmonic function on  $\Omega$  such that  $p_j = \frac{1}{2}v(dd^c\psi_1, z_j) > 0$ , and let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\varphi_1 + \psi_1$  is subharmonic on  $\Omega$ . Let  $\varphi_2$  be a plurisubharmonic function on  $Y$ . Denote that  $\psi := \pi_1^*(\psi_1), \varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$ .

Let  $w_j$  be a local coordinate on a neighborhood  $V_{z_j} \Subset \Omega$  of  $z_j$  satisfying  $w_j(z_j) = 0$  for  $z_j \in Z_0^1$ , where  $V_{z_j} \cap V_{z_k} = \emptyset$  for any  $j, k, j \neq k$ . Denote that  $V_0 := \bigcup_{1 \leq j \leq m} V_{z_j}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $V_0 \times Y$ . Denote  $\mathcal{F}_{(z_j, y)} = \mathcal{I}(\psi + \varphi)_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Let  $G(t; c)$  be the minimal  $L^2$  integral on  $\{\psi < -t\}$  with respect to  $\varphi, f$  and  $\mathcal{F}$  for any  $t \geq 0$ .

We recall a characterization of the concavity of  $G(h^{-1}(r))$  degenerating to linearity for the fibers over sets of finite points as follows.

**Theorem 11** ([1]) *Assume that  $G(0) \in (0, +\infty)$  and  $(\psi_1 - 2p_j G_\Omega(\cdot, z_j))(z_j) > -\infty$ , where  $p_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$  for any  $j \in \{1, 2, \dots, m\}$ . Then  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t} dt]$  if and only if the following statements hold:*

- (1)  $\psi_1 = 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j);$
- (2) for any  $j \in \{1, 2, \dots, m\}$ ,  $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$  on  $V_{z_j} \times Y$ , where  $a_j \in \mathbb{C} \setminus \{0\}$  is a constant,  $k_j$  is a nonnegative integer,  $f_Y$  is a holomorphic  $(n - 1, 0)$  form on  $Y$  such that  $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$ , and  $(f_j, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$  for any  $j \in \{1, 2, \dots, m\}$  and  $y \in Y$ ;
- (3)  $\varphi_1 + \psi_1 = 2 \log |g| + 2 \sum_{j=1}^m G_\Omega(\cdot, z_j) + 2u$ , where  $g$  is a holomorphic function on  $\Omega$  such that  $\text{ord}_{z_j}(g) = k_j$  and  $u$  is a harmonic function on  $\Omega$ ;
- (4)  $\prod_{j=1}^m \chi_{z_j} = \chi_{-u}$ , where  $\chi_{-u}$  and  $\chi_{z_j}$  are the characters associated to the functions  $-u$  and  $G_\Omega(\cdot, z_j)$  respectively;
- (5) for any  $j \in \{1, 2, \dots, m\}$ ,

$$\lim_{z \rightarrow z_j} \frac{a_j w_j^{k_j} dw_j}{g P_* \left( f_u \left( \prod_{l=1}^m f_{z_l} \right) \left( \sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)} = c_0, \quad (12)$$

where  $c_0 \in \mathbb{C} \setminus \{0\}$  is a constant independent of  $j$ ,  $f_u$  is a holomorphic function  $\Delta$  such that  $|f_u| = P^*(e^u)$  and  $f_{z_{j,k}}$  is a holomorphic function on  $\Delta$  such that  $|f_{z_l}| = P^*(e^{G_\Omega(\cdot, z_l)})$  for any  $l \in \{1, \dots, m\}$ .

When  $Z_0^1 := \{z_j : j \in \mathbb{N} \& 1 \leq j < +\infty\}$  is an infinite subset of the open Riemann surface  $\Omega$  of discrete set. Assume that  $2 \sum_{j=1}^{+\infty} p_j G_\Omega(\cdot, z_j) \not\equiv -\infty$ . We recall the following necessary condition such that  $G(h^{-1}(r))$  is linear.

**Proposition 4** ([1]) *Assume that  $G(0) \in (0, +\infty)$  and  $(\psi_1 - 2p_j G_\Omega(\cdot, z_j))(z_j) > -\infty$ , where  $p_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$  for any  $j \in \mathbb{N}_+$ . Assume that  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t} dt]$ , then the following statements hold:*

$$(1) \psi_1 = 2 \sum_{j=1}^{\infty} p_j G_\Omega(\cdot, z_j);$$

(2) for any  $j \in \mathbb{N}_+$ ,  $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$  on  $V_{z_j} \times Y$ , where  $a_j \in \mathbb{C} \setminus \{0\}$  is a constant,  $k_j$  is a nonnegative integer,  $f_Y$  is a holomorphic  $(n-1, 0)$  form on  $Y$  such that  $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$ , and  $(f_j, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$  for any  $j \in \mathbb{N}_+$  and  $y \in Y$ ;

(3)  $\varphi_1 + \psi_1 = 2 \log |g|$ , where  $g$  is a holomorphic function on  $\Omega$  such that  $\text{ord}_{z_j}(g) = k_j + 1$  for any  $j \in \mathbb{N}_+$ ;

(4). for any  $j \in \mathbb{N}_+$ ,

$$\frac{p_j}{\text{ord}_{z_j} g} \lim_{z \rightarrow z_j} \frac{dg}{a_j w_j^{k_j} dw_j} = c_0, \quad (13)$$

where  $c_0 \in \mathbb{C} \setminus \{0\}$  is a constant independent of  $j$ ;

$$(5) \sum_{j \in \mathbb{N}_+} p_j < +\infty.$$

Let  $Z_0^1 := \{z_j : j \in \mathbb{N} \& 1 \leq j \leq m\}$  be a finite subset of the open Riemann surface  $\Omega$ . Let  $Y$  be an  $n-1$  dimensional weakly pseudoconvex Kähler manifold. Let  $M = \Omega \times Y$  be a complex manifold, and  $K_M$  be the canonical line bundle on  $M$ . Let  $\pi_1, \pi_2$  be the natural projections from  $M$  to  $\Omega$  and  $Y$ , and  $Z_0 := \pi_1^{-1}(Z_0^1)$ .

Let  $w_j$  be a local coordinate on a neighborhood  $V_{z_j} \Subset \Omega$  of  $z_j$  satisfying  $w_j(z_j) = 0$  for  $z_j \in Z_0^1$ , where  $V_{z_j} \cap V_{z_k} = \emptyset$  for any  $j, k, j \neq k$ . Let  $c_\beta(z)$  be the logarithmic capacity (see [?]) on  $\Omega$  which is defined by

$$c_\beta(z_j) := \exp \lim_{z \rightarrow z_0} (G_\Omega(z, z_j) - \log |w_j(z)|).$$

Denote that  $V_0 := \bigcup_{1 \leq j \leq m} V_{z_j}$ . Assume that  $\tilde{M} \subset M$  is an  $n$ -dimensional weakly pseudoconvex submanifold satisfying that  $Z_0 \subset \tilde{M}$ .

We recall the following characterization of the holding of the equality in optimal  $L^2$  extension from fibers over analytic subsets to fibrations over open Riemann surfaces (see [1]).

**Theorem 12** ([1]) Let  $k_j$  be a nonnegative integer for any  $j \in \{1, 2, \dots, m\}$ . Let  $\psi_1$  be a negative subharmonic function on  $\Omega$  satisfying that  $\frac{1}{2}v(dd^c\psi_1, z_j) = p_j > 0$  for any  $j \in \{1, 2, \dots, m\}$ . Denote  $\psi := \pi_1^*(\psi_1)$ . Let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\varphi_1 + \psi_1$  is subharmonic on  $\Omega$ ,  $\frac{1}{2}v(dd^c(\varphi_1 + \psi_1), z_j) = k_j + 1$  and  $\alpha_j := (\varphi_1 + \psi_1 - 2(k_j + 1)G_\Omega(\cdot, z_j))(z_j) > -\infty$  for any  $j$ . Let  $\varphi_2$  be a plurisubharmonic function on  $Y$ . Let  $c(t)$  be a positive measurable function on  $(0, +\infty)$  satisfying that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$  and  $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$ . Let  $a_j$  be a constant for any  $j$ . Let  $F_j$  be a holomorphic  $(n-1, 0)$  form on  $Y$  such that  $\int_Y |F_j|^2 e^{-\varphi_2} < +\infty$  for any  $j$ .

Let  $f$  be a holomorphic  $(n, 0)$  form on  $V_0 \times Y$  satisfying that  $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_j)$  on  $V_{z_j} \times Y$ . Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  such that  $(F - f, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left( \int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \quad (14)$$

Moreover, equality  $\left( \int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} = \inf \{ \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } \tilde{M} \text{ such that } (\tilde{F} - f, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)} \text{ for any } (z_j, y) \in Z_0 \}$  holds if and only if the following statements hold:

- (1)  $\psi_1 = 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j);$
- (2).  $\varphi_1 + \psi_1 = 2 \log |g| + 2 \sum_{j=1}^m (k_j + 1)G_\Omega(\cdot, z_j) + 2u$ , where  $g$  is a holomorphic function on  $\Omega$  such that  $g(z_j) \neq 0$  for any  $j \in \{1, 2, \dots, m\}$  and  $u$  is a harmonic function on  $\Omega$ ;
- (3)  $\prod_{j=1}^m \chi_{z_j}^{k_j+1} = \chi_{-u}$ , where  $\chi_{-u}$  and  $\chi_{z_j}$  are the characters associated to the functions  $-u$  and  $G_\Omega(\cdot, z_j)$  respectively;
- (4) for any  $j \in \{1, 2, \dots, m\}$ ,

$$\lim_{z \rightarrow z_j} \frac{a_j w_j^{k_j} dw_j}{g P_* \left( f_u \left( \prod_{l=1}^m f_{z_l}^{k_l+1} \right) \left( \sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)} = c_j \in \mathbb{C} \setminus \{0\}, \quad (15)$$

and there exist  $c_0 \in \mathbb{C} \setminus \{0\}$  and a holomorphic  $(n-1, 0)$  form  $F_Y$  on  $Y$  which are independent of  $j$  such that  $c_0 F_Y = c_j F_j$  for any  $j \in \{1, 2, \dots, m\}$ ;

(5)  $\tilde{M} = M$ .

Let  $Z_0^1 := \{z_j : 1 \leq j < +\infty\}$  be an infinite discrete subset of the open Riemann surface  $\Omega$ . Let  $Y$  be an  $n-1$  dimensional weakly pseudoconvex Kähler manifold.

Let  $M = \Omega \times Y$  be a complex manifold, and  $K_M$  be the canonical line bundle on  $M$ . Let  $\pi_1, \pi_2$  be the natural projections from  $M$  to  $\Omega$  and  $Y$ , and  $Z_0 := \pi_1^{-1}(Z_0^1)$ .

Let  $w_j$  be a local coordinate on a neighborhood  $V_{z_j} \Subset \Omega$  of  $z_j$  satisfying  $w_j(z_j) = 0$  for any  $z_j \in Z_0^1$ , where  $V_{z_j} \cap V_{z_k} = \emptyset$  for any  $j, k, j \neq k$ . Denote that  $V_0 := \bigcup_{j=1}^{\infty} V_{z_j}$ .

We recall the following  $L^2$  extension result from fibers over analytic subsets to fibrations over open Riemann surfaces, where the analytic subsets are infinite points on open Riemann surfaces (see [1]).

**Theorem 13** ([1]) *Let  $k_j$  be a nonnegative integer for any  $j \in \mathbb{N}_+$ . Let  $\psi_1$  be a negative subharmonic function on  $\Omega$  satisfying that  $\frac{1}{2}v(dd^c\psi_1, z_j) = k_j + 1 > 0$  for any  $j \in \mathbb{N}_+$ . Denote  $\psi := \pi_1^*(\psi_1)$ . Let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\varphi_1 + \psi_1$  is subharmonic on  $\Omega$ ,  $\frac{1}{2}v(dd^c(\varphi_1 + \psi_1), z_j) = k_j + 1$  and  $\alpha_j := (\varphi_1 + \psi_1 - 2(k_j + 1)G_{\Omega}(\cdot, z_j))(z_j) > -\infty$  for any  $j$ . Let  $\varphi_2$  be a plurisubharmonic function on  $Y$ . Let  $c(t)$  be a positive measurable function on  $(0, +\infty)$  satisfying that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$  and  $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$ . Let  $a_j$  be a constant for any  $j$ . Let  $F_j$  be a holomorphic  $(n-1, 0)$  form on  $Y$  such that  $\int_Y |F_j|^2 e^{-\varphi_2} < +\infty$  for any  $j$ .*

*Let  $f$  be a holomorphic  $(n, 0)$  form on  $V_0 \times Y$  satisfying that  $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_j)$  on  $V_{z_j} \times Y$ . If*

$$\sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j + 1)c_{\beta}(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} < +\infty,$$

*then there exists a holomorphic  $(n, 0)$  form  $F$  on  $M$  such that  $(F - f, (z_j, y)) \in (O(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and*

$$\int_M |F|^2 e^{-\varphi} c(-\psi) < \left( \int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j + 1)c_{\beta}(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \quad (16)$$

The following results can be referred to [2].

Let  $\Omega_j$  be an open Riemann surface, which admits a nontrivial Green function  $G_{\Omega_j}$  for any  $1 \leq j \leq n_1$ . Let  $Y$  be an  $n_2$ -dimensional weakly pseudoconvex Kähler manifold, and let  $K_Y$  be the canonical (holomorphic) line bundle on  $Y$ . Let  $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$  be an  $n$ -dimensional complex manifold, where  $n = n_1 + n_2$ . Let  $\pi_1, \pi_{1,j}$  and  $\pi_2$  be the natural projections from  $M$  to  $\prod_{1 \leq j \leq n_1} \Omega_j$ ,  $\Omega_j$  and  $Y$  respectively. Let  $K_M$  be the canonical (holomorphic) line bundle on  $M$ . Let  $Z_j$  be a (closed) analytic subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , and denote that  $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y$ .

Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Let  $w_{j,k}$  be a local coordinate on a neighborhood  $V_{z_{j,k}} \Subset \Omega_j$  of  $z_{j,k} \in Z_j$  satisfying  $w_{j,k}(z_{j,k}) = 0$  for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$ , where  $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$  for any  $j$  and  $k \neq k'$ . Denote that  $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j <$

$\tilde{m}_j$  for any  $j \in \{1, \dots, n_1\}\},  $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_j, \beta_j}$  and  $w_\beta := (w_{1, \beta_1}, \dots, w_{n_1, \beta_{n_1}})$  is a local coordinate on  $V_\beta$  of  $z_\beta := (z_{1, \beta_1}, \dots, z_{n_1, \beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$  for any  $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$ . Then  $Z_0 = \{(z_\beta, y) : \beta \in \tilde{I}_1 \text{ & } y \in Y\} \subset \tilde{M}$ .$

Let  $\varphi_j$  be a subharmonic function on  $\Omega_j$  such that  $\varphi_j(z_{j,k}) > -\infty$  for any  $1 \leq k \leq \tilde{m}_j$ . Let  $\varphi_Y$  be a plurisubharmonic function on  $Y$ .

Let  $p_{j,k}$  be a positive number for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$ , which satisfies that  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $1 \leq j \leq n_1$ . Denote that

$$\psi := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$$

and  $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$  on  $M$ .

Denote that  $E_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \text{ & } \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$  and  $\tilde{E}_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \geq 1 \text{ & } \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$  for any  $\beta \in \tilde{I}_1$ .

Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$  such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on  $U_0 \cap (V_\beta \times Y)$ , where  $f_{\alpha,\beta}$  is a holomorphic  $(n_2, 0)$  form on  $Y$  for any  $\alpha \in E_\beta$  and  $\beta \in \tilde{I}_1$ .

Let  $Z_0 = \{z_0\} \times Y = \{(z_1, \dots, z_{n_1})\} \times Y \subset M$ . Let

$$\psi = \max_{1 \leq j \leq n_1} \{2 p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\},$$

where  $p_j$  is positive real number for  $1 \leq j \leq n_1$ . Let  $w_j$  be a local coordinate on a neighborhood  $V_{z_j}$  of  $z_j \in \Omega_j$  satisfying  $w_j(z_j) = 0$ . Denote that  $V_0 := \prod_{1 \leq j \leq n_1} V_{z_j}$ , and  $w := (w_1, \dots, w_{n_1})$  is a local coordinate on  $V_0$  of  $z_0 \in \prod_{1 \leq j \leq n_1} \Omega_j$ . Denote that  $E := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j+1}{p_j} = 1 \text{ & } \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $V_0 \times Y \subset M$ .

We recall a characterization of the concavity of  $G(h^{-1}(r))$  degenerating to linearity for the case  $Z_0 = \{z_0\} \times Y$ .

**Theorem 14** ([2]) *Assume that  $G(0) \in (0, +\infty)$ .  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(t)e^{-t} dt]$  if and only if the following statements hold:*

(1)  $f = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + g_0$  on  $V_0 \times Y$ , where  $g_0$  is a holomorphic  $(n, 0)$  form on  $V_0 \times Y$  satisfying  $(g_0, z) \in (O(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$  for any  $z \in Z_0$  and  $f_\alpha$  is a holomorphic  $(n_2, 0)$  form on  $Y$  such that  $\sum_{\alpha \in E} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \in (0, +\infty)$ ;

(2)  $\varphi_j = 2 \log |g_j| + 2u_j$ , where  $g_j$  is a holomorphic function on  $\Omega_j$  such that  $g_j(z_j) \neq 0$  and  $u_j$  is a harmonic function on  $\Omega_j$  for any  $1 \leq j \leq n_1$ ;

(3)  $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$  for any  $j \in \{1, 2, \dots, n\}$  and  $\alpha \in E$  satisfying  $f_\alpha \not\equiv 0$ .

Let  $Z_j = \{z_{j,1}, \dots, z_{j,m_j}\} \subset \Omega_j$  for any  $j \in \{1, \dots, n\}$ , where  $m_j$  is a positive integer. Let

$$\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

where  $p_{j,k}$  is a positive real number. Let  $w_{j,k}$  be a local coordinate on a neighborhood  $V_{z_{j,k}} \Subset \Omega_j$  of  $z_{j,k} \in \Omega_j$  satisfying  $w_{j,k}(z_{j,k}) = 0$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ , where  $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$  for any  $j$  and  $k \neq k'$ . Denote that  $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j \leq m_j \text{ for any } j \in \{1, \dots, n_1\}\}$ ,  $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$  for any  $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$  and  $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$  is a local coordinate on  $V_\beta$  of  $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$  satisfying  $w_\beta(z_\beta) = 0$ .

Let  $\beta^* = (1, \dots, 1) \in \tilde{I}_1$ , and let  $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geq 0}^{n_1}$ . Denote that  $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} > \sum_{1 \leq j \leq n_1} \frac{\alpha_{\beta^*,j} + 1}{p_{j,1}} \right\}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $\cup_{\beta \in \tilde{I}_1} V_\beta \times Y$  satisfying  $f = \pi_1^* \left( w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^*(f_{\alpha_{\beta^*}}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$  on  $V_{\beta^*} \times Y$ , where  $f_{\alpha_{\beta^*}}$  and  $f_\alpha$  are holomorphic  $(n_2, 0)$  forms on  $Y$ .

We recall a characterization of the concavity of  $G(h^{-1}(r))$  degenerating to linearity for the case  $Z_j$  is a set of finite points.

**Theorem 15** ([2]) Assume that  $G(0) \in (0, +\infty)$ .  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s} ds]$  if and only if the following statements hold:

(1)  $\varphi_j = 2 \log |g_j| + 2u_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $u_j$  is a harmonic function on  $\Omega_j$  and  $g_j$  is a holomorphic function on  $\Omega_j$  satisfying  $g_j(z_{j,k}) \neq 0$  for any  $k \in \{1, \dots, m_j\}$ ;

(2) There exists a nonnegative integer  $\gamma_{j,k}$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ , which satisfies that  $\prod_{1 \leq k \leq m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$  and  $\sum_{1 \leq j \leq n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$  for any  $\beta \in \tilde{I}_1$ ;

(3)  $f = \pi_1^* \left( c_\beta \left( \prod_{1 \leq j \leq n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}} \right) dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}} \right) \wedge \pi_2^*(f_0) + g_\beta$  on  $V_\beta \times Y$  for any  $\beta \in \tilde{I}_1$ , where  $c_\beta$  is a constant,  $f_0 \not\equiv 0$  is a holomorphic  $(n_2, 0)$  form on  $Y$  satisfying  $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$ , and  $g_\beta$  is a holomorphic  $(n, 0)$  form on  $V_\beta \times Y$  such that  $(g_\beta, z) \in (O(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$  for any  $z \in \{z_\beta\} \times Y$ ;

$$(4) c_\beta \prod_{1 \leq j \leq n_1} \left( \lim_{z \rightarrow z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_* \left( f_{u_j} \left( \prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left( \sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right)} \right) = c_0 \text{ for}$$

any  $\beta \in \tilde{I}_1$ , where  $c_0 \in \mathbb{C} \setminus \{0\}$  is a constant independent of  $\beta$ ,  $f_{u_j}$  is a holomorphic function  $\Delta$  such that  $|f_{u_j}| = P_j^*(e^{u_j})$  and  $f_{z_{j,k}}$  is a holomorphic function on  $\Delta$  such that  $|f_{z_{j,k}}| = P_j^* \left( e^{G_{\Omega_j}(\cdot, z_{j,k})} \right)$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ .

Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Let  $p_{j,k}$  be a positive number for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$  such that  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $j$ . Let

$$\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}.$$

Assume that  $\limsup_{t \rightarrow +\infty} c(t) < +\infty$ .

Let  $w_{j,k}$  be a local coordinate on a neighborhood  $V_{z_{j,k}} \Subset \Omega_j$  of  $z_{j,k} \in \Omega_j$  satisfying  $w_{j,k}(z_{j,k}) = 0$  for any  $j \in \{1, \dots, n_1\}$  and  $1 \leq k < \tilde{m}_j$ , where  $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$  for any  $j$  and  $k \neq k'$ . Denote that  $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$ ,  $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$  for any  $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$  and  $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$  is a local coordinate on  $V_\beta$  of  $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$ .

Let  $\beta^* = (1, \dots, 1) \in I_1$ , and let  $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geq 0}^{n_1}$ . Denote that  $E' := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,1}} > \sum_{1 \leq j \leq n_1} \frac{\alpha_{\beta^*,j} + 1}{p_{j,1}} \right\}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $\cup_{\beta \in I_1} V_\beta \times Y$  satisfying  $f = \pi_1^* \left( w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^* (f_{\alpha_{\beta^*}}) + \sum_{\alpha \in E'} \pi_1^* (w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^* (f_\alpha)$  on  $V_{\beta^*} \times Y$ , where  $f_{\alpha_{\beta^*}}$  and  $f_\alpha$  are holomorphic  $(n_2, 0)$  forms on  $Y$ .

We recall that  $G(h^{-1}(r))$  is not linear when there exists  $j_0 \in \{1, \dots, n_1\}$  such that  $\tilde{m}_{j_0} = +\infty$  as follows.

**Theorem 16** ([2]) *If  $G(0) \in (0, +\infty)$  and there exists  $j_0 \in \{1, \dots, n_1\}$  such that  $\tilde{m}_{j_0} = +\infty$ , then  $G(h^{-1}(r))$  is not linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s} ds]$ .*

Let  $\tilde{M} \subset M$  be an  $n$ -dimensional complex manifold satisfying that  $Z_0 \subset \tilde{M}$ , and let  $K_{\tilde{M}}$  be the canonical (holomorphic) line bundle on  $\tilde{M}$ .

Let  $\Psi \leq 0$  be a plurisubharmonic function on  $\prod_{1 \leq j \leq n_1} \Omega_j$ , and let  $\varphi_j$  be a Lebesgue measurable function on  $\Omega_j$  such that  $\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)$  is plurisubharmonic on  $\prod_{1 \leq j \leq n_1} \Omega_j$ , where  $\tilde{\pi}_j$  is the natural projection from  $\prod_{1 \leq j \leq n_1} \Omega_j$  to  $\Omega_j$ . Let  $\varphi_Y$  be a plurisubharmonic function on  $Y$ . Let  $p_{j,k}$  be a positive number for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$ , which satisfies that  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $1 \leq j \leq n_1$ . Denote that

$$\psi := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^* (G_{\Omega_j}(\cdot, z_{j,k})) \right\} + \pi_1^*(\Psi)$$

and  $\varphi := \sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$  on  $M$ .

Denote that  $E_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$  and  $\tilde{E}_\beta := \left\{ (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} \geq 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$  for any  $\beta \in \tilde{I}_1$ . Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$  such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on  $U_0 \cap (V_\beta \times Y)$ , where  $f_{\alpha,\beta}$  is a holomorphic  $(n_2, 0)$  form on  $Y$  for any  $\alpha \in E_\beta$  and  $\beta \in \tilde{I}_1$ . Denote that

$$c_{j,k} := \exp \lim_{z \rightarrow z_{j,k}} \left( \frac{\sum_{1 \leq k_1 < \tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any  $j \in \{1, \dots, n\}$  and  $1 \leq k < \tilde{m}_j$  (following from Lemmas 3 and 4, we get that the above limit exists).

When  $Z_0 = \{z_0\} \times Y \subset \tilde{M}$ , where  $z_0 = (z_1, \dots, z_{n_1}) \in \prod_{1 \leq j \leq n_1} \Omega_j$ .

We recall a characterization of the holding of equality in optimal jets  $L^2$  extension problem for the case  $Z_0 = \{z_0\} \times Y$ .

**Theorem 17** ([2]) *Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$  and  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ . Assume that*

$$\sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \in (0, +\infty).$$

Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in \left(O(K_{\tilde{M}}) \otimes \mathcal{I} \left( \max_{1 \leq j \leq n_1} \left\{ 2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j)) \right\} \right) \right)_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}. \end{aligned}$$

Moreover, equality  $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, O(K_{\tilde{M}})) \& (\tilde{F} - f, z) \in \left(O(K_{\tilde{M}}) \otimes \mathcal{I} \left( \max_{1 \leq j \leq n_1} \left\{ 2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j)) \right\} \right) \right)_z \text{ for any } z \in Z_0 \right\} = \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \times \sum_{\alpha \in E} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_0)} \int_Y |f_\alpha|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}$  holds if and only if the following statements hold:

$$(1) \tilde{M} = \left( \prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y \text{ and } \Psi \equiv 0;$$

- (2)  $\varphi_j = 2 \log |g_j| + 2u_j$ , where  $g_j$  is a holomorphic function on  $\Omega_j$  such that  $g_j(z_j) \neq 0$  and  $u_j$  is a harmonic function on  $\Omega_j$  for any  $1 \leq j \leq n_1$ ;  
(3)  $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$  for any  $j \in \{1, 2, \dots, n\}$  and  $\alpha \in E$  satisfying  $f_\alpha \not\equiv 0$ .

**Remark 15** ([2]) If  $(f_\alpha, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  for any  $y \in Y$  and  $\alpha \in \tilde{E} \setminus E$ , the above result also holds when we replace the ideal sheaf  $\mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\})$  by  $\mathcal{I}(\varphi + \psi)$ .

Let  $Z_j = \{z_{j,1}, \dots, z_{j,m_j}\} \subset \Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $m_j$  is a positive integer. Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$  such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on  $U_0 \cap (V_\beta \times Y)$ , where  $f_{\alpha,\beta}$  is a holomorphic  $(n_2, 0)$  form on  $Y$  for any  $\alpha \in E_\beta$  and  $\beta \in \tilde{I}_1$ . Let  $\beta^* = (1, \dots, 1) \in \tilde{I}_1$ , and let  $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in E_{\beta^*}$ . Assume that

$$f = \pi_1^*(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha_{\beta^*},\beta^*}) + \sum_{\alpha \in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_{\alpha,\beta}) \text{ on } U_0 \cap (V_{\beta^*} \times Y).$$

We recall a characterization of the holding of equality in optimal jets  $L^2$  extension problem for the case that  $Z_j$  is finite.

**Theorem 18** ([2]) Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$  and  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ . Assume that

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta}^{2\alpha_j + 2}} \in (0, +\infty).$$

Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}\right)\right)_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta}^{2\alpha_j + 2}}. \end{aligned}$$

Moreover, equality  $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \& (\tilde{F} - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}\right)\right)_z \text{ for any } z \in Z_0 \right\}$

$$= \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j))(\zeta_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \quad \text{holds if}$$

and only if the following statements hold:

$$(1) \tilde{M} = \left( \prod_{1 \leq j \leq n_1} \Omega_j \right) \times Y \text{ and } \Psi \equiv 0;$$

(2)  $\varphi_j = 2 \log |g_j| + 2u_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $u_j$  is a harmonic function on  $\Omega_j$  and  $g_j$  is a holomorphic function on  $\Omega_j$  satisfying  $g_j(z_{j,k}) \neq 0$  for any  $k \in \{1, \dots, m_j\}$ ;

(3) There exists a nonnegative integer  $\gamma_{j,k}$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ , which satisfies that  $\prod_{1 \leq k \leq m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$  and  $\sum_{1 \leq j \leq n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$  for any  $\beta \in \tilde{I}_1$ ;

(4)  $f_{\alpha,\beta} = c_\beta f_0$  holds for  $\alpha = (\gamma_{1,\beta_1}, \dots, \gamma_{n_1,\beta_{n_1}})$  and  $f_{\alpha,\beta} \equiv 0$  holds for any  $\alpha \in E_\beta \setminus \{(\gamma_{1,\beta_1}, \dots, \gamma_{n_1,\beta_{n_1}})\}$ , where  $\beta \in \tilde{I}_1$ ,  $c_\beta$  is a constant and  $f_0 \not\equiv 0$  is a holomorphic  $(n_2, 0)$  form on  $Y$  satisfying  $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$ ;

$$(5) c_\beta \prod_{1 \leq j \leq n_1} \left( \lim_{z \rightarrow z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_* \left( f_{u_j} \left( \prod_{1 \leq k \leq m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left( \sum_{1 \leq k \leq m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right)} \right) = c_0 \text{ for}$$

any  $\beta \in \tilde{I}_1$ , where  $c_0 \in \mathbb{C} \setminus \{0\}$  is a constant independent of  $\beta$ ,  $f_{u_j}$  is a holomorphic function  $\Delta$  such that  $|f_{u_j}| = P_j^*(e^{u_j})$  and  $f_{z_{j,k}}$  is a holomorphic function on  $\Delta$  such that  $|f_{z_{j,k}}| = P_j^* \left( e^{G_{\Omega_j}(\cdot, z_{j,k})} \right)$  for any  $j \in \{1, \dots, n_1\}$  and  $k \in \{1, \dots, m_j\}$ .

**Remark 16** ([2]) If  $(f_{\alpha,\beta}, y) \in (O(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  holds for any  $y \in Y, \alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , the above result also holds when we replace the ideal sheaf  $\mathcal{I} \left( \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} \right)$  by  $\mathcal{I}(\varphi + \psi)$ .

Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$  such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on  $U_0 \cap (V_\beta \times Y)$ , where  $f_{\alpha,\beta}$  is a holomorphic  $(n_2, 0)$  form on  $Y$  for any  $\alpha \in E_\beta$  and  $\beta \in \tilde{I}_1$ . Let  $\beta^* = (1, \dots, 1) \in \tilde{I}_1$ , and let  $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \dots, \alpha_{\beta^*,n_1}) \in E_{\beta^*}$ . Assume that  $f = \pi_1^* \left( w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^* \left( f_{\alpha_{\beta^*},\beta^*} \right) + \sum_{\alpha \in E'} \pi_1^* \left( w^\alpha dw_{1,1} \wedge \dots \wedge dw_{n_1,1} \right) \wedge \pi_2^*(f_{\alpha,\beta})$  on  $U_0 \cap (V_{\beta^*} \times Y)$ .

We recall that the equality in optimal jets  $L^2$  extension problem could not hold when there exists  $j_0 \in \{1, \dots, n_1\}$  such that  $\tilde{m}_{j_0} = +\infty$ .

**Theorem 19** ([2]) Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t) e^{-t} dt < +\infty$  and  $c(t) e^{-t}$  is decreasing on  $(0, +\infty)$ . Assume that

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \in (0, +\infty)$$

and there exists  $j_0 \in \{1, \dots, n_1\}$  such that  $\tilde{m}_{j_0} = +\infty$ .

Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & < \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \leq j \leq n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}}. \end{aligned}$$

**Remark 17** ([2]) If  $(f_{\alpha,\beta}, y) \in (O(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  holds for any  $y \in Y, \alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , the above result also holds when we replace the ideal sheaf  $\mathcal{I}\left(\max_{1 \leq j \leq n_1} \left\{2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)$  by  $\mathcal{I}(\varphi + \psi)$ .

## 2.3 Basic Properties of the Green Functions

In this section, we recall some basic properties of the Green functions. Let  $\Omega$  be an open Riemann surface, which admits a nontrivial Green function  $G_\Omega$ , and let  $z_0 \in \Omega$ .

**Lemma 3** (see [44]) *Let  $w$  be a local coordinate on a neighborhood of  $z_0$  satisfying  $w(z_0) = 0$ .  $G_\Omega(z, z_0) = \sup_{v \in \Delta_\Omega^*(z_0)} v(z)$ , where  $\Delta_\Omega^*(z_0)$  is the set of negative subharmonic function on  $\Omega$  such that  $v - \log|w|$  has a locally finite upper bound near  $z_0$ . Moreover,  $G_\Omega(\cdot, z_0)$  is harmonic on  $\Omega \setminus \{z_0\}$  and  $G_\Omega(\cdot, z_0) - \log|w|$  is harmonic near  $z_0$ .*

**Lemma 4** (see [31]) *Let  $K = \{z_j : j \in \mathbb{Z}_{\geq 1} \& j < \gamma\}$  be a discrete subset of  $\Omega$ , where  $\gamma \in \mathbb{Z}_{>1} \cup \{+\infty\}$ . Let  $\psi$  be a negative subharmonic function on  $\Omega$  such that  $\frac{1}{2}v(dd^c\psi, z_j) \geq p_j$  for any  $j$ , where  $p_j > 0$  is a constant. Then  $2 \sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j)$  is a subharmonic function on  $\Omega$  satisfying that  $2 \sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j) \geq \psi$  and  $2 \sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j)$  is harmonic on  $\Omega \setminus K$ .*

**Lemma 5** (see [28]) *For any open neighborhood  $U$  of  $z_0$ , there exists  $t > 0$  such that  $\{G_\Omega(z, z_0) < -t\}$  is a relatively compact subset of  $U$ .*

**Lemma 6** see [31]) *There exists a sequence of open Riemann surfaces  $\{\Omega_l\}_{l \in \mathbb{Z}^+}$  such that  $z_0 \in \Omega_l \Subset \Omega_{l+1} \Subset \Omega$ ,  $\cup_{l \in \mathbb{Z}^+} \Omega_l = \Omega$ ,  $\Omega_l$  has a smooth boundary  $\partial\Omega_l$  in  $\Omega$  and  $e^{G_{\Omega_l}(\cdot, z_0)}$  can be smoothly extended to a neighborhood of  $\overline{\Omega_l}$  for any  $l \in \mathbb{Z}^+$ , where  $G_{\Omega_l}$  is the Green function of  $\Omega_l$ . Moreover,  $\{G_{\Omega_l}(\cdot, z_0) - G_\Omega(\cdot, z_0)\}$  is decreasingly convergent to 0 on  $\Omega$  with respect to  $l$ .*

Let  $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$ , where  $\Omega_j$  is an open Riemann surface and  $Y$  is an  $n_2$ -dimensional complex manifold and  $n_1 + n_2 = n$ . Let  $\pi_1, \pi_{1,j}$  and  $\pi_2$  be the natural projections from  $M$  to  $\prod_{1 \leq j \leq n_1} \Omega_j$ ,  $\Omega_j$  and  $Y$  respectively. Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $1 \leq j \leq n_1$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Denote that  $Z_0 := \left(\prod_{1 \leq j \leq n_1} Z_j\right) \times Y$ .

Let  $G = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$  be a plurisubharmonic function on  $\prod_{1 \leq j \leq n_1} \Omega_j$ , where  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $j \in \{1, \dots, n\}$  and  $\tilde{\pi}_j$  is the natural projection from  $\prod_{1 \leq j \leq n_1} \Omega_j$  to  $\Omega_j$ . Let  $\tilde{M} \subset M$  be an

$n$ -dimensional weakly pseudoconvex submanifold satisfying that  $Z_0 \subset \tilde{M}$ .

Let  $N \leq 0$  be a plurisubharmonic function on  $\tilde{M}$ . Denote  $\psi = \pi_1^*(G) + N$ .

**Lemma 7** *Let  $l(t)$  be a positive Lebesgue measurable function on  $(0, +\infty)$  satisfying that  $l$  is decreasing on  $(0, +\infty)$  and  $\int_0^{+\infty} l(t)dt < +\infty$ .*

*If  $N \not\equiv 0$ , then there exists a Lebesgue measurable subset  $\tilde{V}$  of  $\tilde{M}$  such that  $l(-\psi) < l(-\pi_1^*(G))$  on  $\tilde{V}$  and  $\mu(\tilde{V}) > 0$ , where  $\mu$  is the Lebesgue measure on  $\tilde{M}$ .*

**Proof** Let  $V_1 \Subset \prod_{1 \leq j \leq n_1} \Omega_j \setminus \{z_\beta : \exists j \in \{1, 2, \dots, n_1\} \text{ s.t. } 2 \leq \beta_j \leq \tilde{m}_j\}$  be an open neighborhood of  $z_{\beta^*}$ , where  $\beta^* = (1, 1, \dots, 1)$ . It follows from Lemma 5 that there exists  $t_0 > 0$  such that  $V_1 \cap \{G < -t_0\} \Subset V_1$ .

As  $l(t)$  is decreasing and  $\int_0^{+\infty} l(t)dt < +\infty$ , there exists  $t_1 > t_0$  such that  $l(t) < l(t_1)$  holds for any  $t > t_1$ .

For any  $w \in Y$ , let  $U \Subset Y$  be an open neighborhood of  $w$  in  $Y$  such that  $V_1 \times U \subset \tilde{M}$ .

As  $\psi$  is upper semi-continuous function and  $N \not\equiv 0$  is a plurisubharmonic function on  $\tilde{M}$ , we have

$$\sup_{(z,w) \in (V_1 \cap \{G \leq -t_1\}) \times U} \psi(z) < -t_1,$$

which implies that there exists  $t_2 \in (t_0, t_1)$  such that

$$\sup_{(z,w) \in (V_1 \cap \{G \leq -t_2\}) \times \tilde{U}} \psi(z) < -t_1,$$

where  $\tilde{U} \Subset U$  is open.

Denote  $t_3 = -\sup_{(z,w) \in (V_1 \cap \{G \leq -t_2\}) \times \tilde{U}} \psi(z)$ . Then we know  $-t_3 < -t_1$ .

Let  $V = \{-t_1 < G < -t_2\} \cap V_1$ , then  $\mu(V \times \tilde{U}) > 0$ . As  $l(t)$  is decreasing with respect to  $t$ , for any  $(z, w) \in V \times \tilde{U}$ , we have

$$l(-\psi) \leq l(t_3) < l(t_1) \leq l(\pi_1^*(-G)).$$

Lemma 7 is proved.

## 2.4 Other Lemmas

We call a positive measurable function  $c$  on  $(S, +\infty)$  in class  $\tilde{\mathcal{P}}_S$  if  $\int_S^s c(l)e^{-l}dl < +\infty$  for some  $s > S$  and  $c(t)e^{-t}$  is decreasing with respect to  $t$ .

**Lemma 8** (see [26]) *Let  $B \in (0, +\infty)$  and  $t_0 \geq S$  be arbitrarily given. Let  $(M, \omega)$  be an  $n$ -dimensional weakly pseudoconvex Kähler manifold. Let  $\psi < -S$  be a plurisubharmonic function on  $M$ . Let  $\varphi$  be a plurisubharmonic function on  $M$ . Let  $F$  be a holomorphic  $(n, 0)$  form on  $\{\psi < -t_0\}$  such that*

$$\int_{K \cap \{\psi < -t_0\}} |F|^2 < +\infty,$$

for any compact subset  $K$  of  $M$ , and

$$\int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B < \psi < -t_0\}} |F|^2 e^{-\varphi} \leq C < +\infty.$$

Then there exists a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $X$ , such that

$$\int_M |\tilde{F} - (1 - b_{t_0, B}(\psi))F|^2 e^{-\varphi + v_{t_0, B}(\psi)} c(-v_{t_0, B}(\psi)) \leq C \int_S^{t_0+B} c(t)e^{-t} dt. \quad (17)$$

where  $b_{t_0, B}(t) = \int_{-\infty}^t \frac{1}{B} \mathbb{I}_{\{-t_0-B < s < -t_0\}} ds$ ,  $v_{t_0, B}(t) = \int_{-t_0}^t b_{t_0, B}(s)ds - t_0$  and  $c(t) \in \tilde{\mathcal{P}}_S$ .

**Lemma 9** (see [26]) *Let  $M$  be a complex manifold. Let  $S$  be an analytic subset of  $M$ . Let  $\{g_j\}_{j=1,2,\dots}$  be a sequence of nonnegative Lebesgue measurable functions on  $M$ , which satisfies that  $g_j$  are almost everywhere convergent to  $g$  on  $M$  when  $j \rightarrow +\infty$ , where  $g$  is a nonnegative Lebesgue measurable function on  $M$ . Assume that for any compact subset  $K$  of  $M \setminus S$ , there exist  $s_K \in (0, +\infty)$  and  $C_K \in (0, +\infty)$  such that*

$$\int_K g_j^{-s_K} dV_M \leq C_K$$

for any  $j$ , where  $dV_M$  is a continuous volume form on  $M$ .

Let  $\{F_j\}_{j=1,2,\dots}$  be a sequence of holomorphic  $(n, 0)$  form on  $M$ . Assume that  $\liminf_{j \rightarrow +\infty} \int_M |F_j|^2 g_j \leq C$ , where  $C$  is a positive constant. Then there exists a subsequence  $\{F_{j_l}\}_{l=1,2,\dots}$  which satisfies that  $\{F_{j_l}\}$  is uniformly convergent to a holomorphic  $(n, 0)$  form  $F$  on  $M$  on any compact subset of  $M$  when  $l \rightarrow +\infty$ , such that

$$\int_M |F|^2 g \leq C.$$

**Lemma 10** (see [19]) *Let  $N$  be a submodule of  $\mathcal{O}_{\mathbb{C}^n, o}^q$ ,  $1 \leq q < \infty$ , let  $f_j \in \mathcal{O}_{\mathbb{C}^n}(U)^q$  be a sequence of  $q$ -tuples holomorphic function in an open neighborhood  $U$  of the*

origin  $o$ . Assume that the  $f_j$  converges uniformly in  $U$  towards a  $q$ -tuples  $f \in \mathcal{O}_{\mathbb{C}^n}(U)^q$ , assume furthermore that all germs  $(f_j, o)$  belong to  $N$ . Then  $(f, o) \in N$ .

**Lemma 11** (see [32]) Let  $\psi = \max_{1 \leq j \leq n} \{2p_j \log |w_j|\}$  be a plurisubharmonic function on  $\mathbb{C}^n$ , where  $p_j > 0$ . Let  $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} b_\alpha w^\alpha$  (Taylor expansion) be a holomorphic function on  $\{\psi < -t_0\}$ , where  $t_0 > 0$ . Denote that  $q_\alpha := \sum_{1 \leq j \leq n} \frac{\alpha_j + 1}{p_j} - 1$  for any  $\alpha \in \mathbb{Z}_{\geq 0}^n$  and  $E_1 := \{\alpha \in \mathbb{Z}_{\geq 0}^n : q_\alpha = 0\}$ . Let  $k_1$  be a constant satisfying  $k_1 + 1 > 0$ . Then

$$\int_{\{-t-1-k_1 < \psi < -t\}} |f|^2 e^{-\psi} d\lambda_n = \sum_{\alpha \in E_1} \frac{|b_\alpha|^2 \pi^n (1 + k_1)}{\prod_{1 \leq j \leq n} (\alpha_j + 1)} + \sum_{\alpha \notin E_1} \frac{|b_\alpha|^2 \pi^n (q_\alpha + 1) (e^{-q_\alpha t} - e^{-q_\alpha(t+1+k_1)})}{q_\alpha \prod_{1 \leq j \leq n} (\alpha_j + 1)}$$

for any  $t > t_0$ .

**Remark 18** Lemma 11 is stated in [32] in the case  $k_1 = 0$ , the same proof as in [32] shows that Lemma 11 holds for  $k_1$  which satisfying  $k_1 + 1 > 0$ .

**Lemma 12** Let  $\psi = \max_{1 \leq j \leq n} \{2p_j \log |w_j|\}$  be a plurisubharmonic function on  $\mathbb{C}^n$ , where  $p_j > 0$ . Let  $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} b_\alpha w^\alpha$  (Taylor expansion) be a holomorphic function on  $\{\psi < -t_0\}$ , where  $t_0 > 0$ . Let  $c(t)$  be a nonnegative measurable function on  $(t_0, +\infty)$ . Denote that  $q_\alpha := \sum_{1 \leq j \leq n} \frac{\alpha_j + 1}{p_j} - 1$  for any  $\alpha \in \mathbb{Z}_{\geq 0}^n$ . Let  $A$  be a real constant such that  $t_0 + A > 0$ . Then

$$\int_{\{\psi+A < -t\}} |f|^2 c(-\psi - A) d\lambda_n = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} \left( \int_t^{+\infty} c(s) e^{-(q_\alpha+1)s} ds \right) e^{-(q_\alpha+1)A} \frac{(q_\alpha + 1)|b_\alpha|^2 \pi^n}{\prod_{1 \leq j \leq n} (\alpha_j + 1)}$$

holds for any  $t \geq t_0$ .

**Proof** For any  $t \geq t_0$ , by direct calculations, we obtain that (note that  $t + A > 0$ )

$$\begin{aligned} & \int_{\{\psi+A < -t\}} |w^\alpha|^2 c(-\psi - A) d\lambda_n \\ &= (2\pi)^n \int_{\left\{ \max_{1 \leq j \leq n} \left\{ s_j^{p_j} \right\} < e^{-\frac{t+A}{2}} \text{ & } s_j > 0 \right\}} \prod_{1 \leq j \leq n} s_j^{2\alpha_j + 1} \cdot c \left( -\log \max_{1 \leq j \leq n} \left\{ s_j^{2p_j} \right\} - A \right) ds_1 ds_2 \dots ds_n \\ &= (2\pi)^n \frac{1}{\prod_{1 \leq j \leq n} p_j} \\ & \quad \times \int_{\left\{ \max_{1 \leq j \leq n} \{r_j\} < e^{-\frac{t+A}{2}} \text{ & } r_j > 0 \right\}} \prod_{1 \leq j \leq n} r_j^{\frac{2\alpha_j + 2}{p_j} - 1} \cdot c \left( -\log \max_{1 \leq j \leq n} \{r_j^2\} - A \right) dr_1 dr_2 \dots dr_n. \end{aligned} \tag{18}$$

By the Fubini's theorem, we have

$$\begin{aligned}
& \int_{\left\{\max_{1 \leq j \leq n} \{r_j\} < e^{-\frac{t+A}{2}} \text{ and } r_j > 0\right\}} \prod_{1 \leq j \leq n} r_j^{\frac{2\alpha_j+2}{p_j}-1} \cdot c \left( -\log \max_{1 \leq j \leq n} \{r_j^2\} - A \right) dr_1 dr_2 \dots dr_n \\
&= \sum_{j'=1}^n \int_0^{e^{-\frac{t+A}{2}}} \left( \int_{\{0 \leq r_j < r_{j'}, j \neq j'\}} \prod_{j \neq j'} r_j^{\frac{2\alpha_j+2}{p_j}-1} \cdot \wedge_{j \neq j'} dr_j \right) r_{j'}^{\frac{2\alpha_{j'}+2}{p_{j'}}-1} c(-2 \log r_{j'} - A) dr_{j'} \\
&= \sum_{j'=1}^n \left( \prod_{j \neq j'} \frac{p_j}{2\alpha_j + 2} \right) \int_0^{e^{-\frac{t+A}{2}}} r_{j'}^{\sum_{1 \leq k \leq n} \frac{2\alpha_k+2}{p_k}-1} c(-2 \log r_{j'} - A) dr_{j'} \\
&= (q_\alpha + 1) e^{-(q_\alpha+1)A} \left( \int_t^{+\infty} c(s) e^{-(q_\alpha+1)s} ds \right) \prod_{1 \leq j \leq n} \frac{p_j}{2\alpha_j + 2}. \tag{19}
\end{aligned}$$

Following from  $\int_{\{\psi < -t\}} |f|^2 c(-\psi) d\lambda_n = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} |b_\alpha|^2 \int_{\{\psi < -t\}} |w^\alpha|^2 c(-\psi) d\lambda_n$ , equality (18) and equality (19), we obtain that

$$\int_{\{\psi < -t\}} |f|^2 d\lambda_n = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} \left( \int_t^{+\infty} c(s) e^{-(q_\alpha+1)s} ds \right) e^{-(q_\alpha+1)A} \frac{(q_\alpha + 1) |b_\alpha|^2 \pi^n}{\prod_{1 \leq j \leq n} (\alpha_j + 1)}.$$

**Lemma 13** Let  $\Delta^n \subset \mathbb{C}^n$  be a polydisc. Let  $\varphi$  be a bounded subharmonic function on  $\Delta^n$ . Assume that  $v$  is a nonnegative continuous real function on  $\Delta^n$ . Denote

$$\begin{aligned}
I_t &:= \int_{\{z \in \Delta^n : -t-1-k_2 < \max_{1 \leq j \leq n} \{2p_j \log |z_j|\} < -t+k_1\}} v(z) \frac{\prod_{j=1}^n |z_j|^{2\alpha_j}}{\max_{1 \leq j \leq n} \{|z_j|^{2p_j}\}} e^{-\varphi(z_1, \dots, z_n)} dz \wedge d\bar{z}, \\
J_t &:= \int_{\{z \in \Delta^n : -t-1-k_2 < \max_{1 \leq j \leq n} \{2p_j \log |z_j|\} < -t+k_1\}} v(z) \frac{\prod_{j=1}^n |z_j|^{2\beta_j}}{\max_{1 \leq j \leq n} \{|z_j|^{2p_j}\}} e^{-\varphi(z_1, \dots, z_n)} dz \wedge d\bar{z},
\end{aligned}$$

where  $p_j, \alpha_j, \beta_j > 0$  are constants for any  $j$  satisfying  $\sum_{1 \leq j \leq n} \frac{\alpha_j+1}{p_j} = 1$  and  $\sum_{1 \leq j \leq n} \frac{\beta_j+1}{p_j} > 1$ ,  $k_1, k_2$  are constants satisfying  $k_1 + k_2 + 1 > 0$ .

Then

$$\limsup_{t \rightarrow +\infty} I_t \leq \frac{(2\pi)^n (1 + k_1 + k_2)}{\prod_{1 \leq j \leq n} (\alpha_j + 1)} v(0) e^{-\varphi(0)}. \tag{20}$$

and

$$\lim_{t \rightarrow +\infty} J_t = 0.$$

**Proof** The idea of the proof can be referred to Lemma 3.3 in [52]. For the convenience of the readers, we give a proof below.

Denote

$$I_{t+k_1} := \int_{\{z \in \Delta^n : -t-1-k_1-k_2 < \max_{1 \leq j \leq n} \{2p_j \log |z_j|\} < -t\}} v(z) \frac{\prod_{j=1}^n |z_j|^{2\alpha_j}}{\max_{1 \leq j \leq n} \{|z_j|^{2p_j}\}} e^{-\varphi(z_1, \dots, z_n)} dz \wedge d\bar{z},$$

Note that

$$\limsup_{t \rightarrow +\infty} I_t = \limsup_{t \rightarrow +\infty} I_{t+k_1},$$

then we only need to prove (20) for  $I_{t+k_1}$ .

Denote

$$B_{\delta,t} := \{z \in \Delta^n : \varphi(e^{\frac{-t}{2p_1}} z_1, e^{\frac{-t}{2p_2}} z_2, \dots, e^{\frac{-t}{2p_n}} z_n) < (1+\delta)\varphi(0)\},$$

where  $\delta \in (0, +\infty)$  and  $t \in (0, +\infty)$ .

Let  $\lambda(B_{\delta,t})$  be the  $2n$ -dimensional Lebesgue measure of  $B_{\delta,t}$ .

Since the computation is local, we may assume that  $\varphi$  is a negative upper semi-continuous function on  $\Delta^n$ . Note that  $\varphi(0) > -\infty$ . For any  $\epsilon \in (0, 1)$ , there exists  $t_\epsilon > 0$  such that

$$\varphi(e^{\frac{-t}{2p_1}} z_1, e^{\frac{-t}{2p_2}} z_2, \dots, e^{\frac{-t}{2p_n}} z_n) \leq (1-\epsilon)\varphi(0)$$

for any  $z \in \Delta^n$ , when  $t \geq t_\epsilon$ . We denote  $\varphi(e^{\frac{-t}{2p_1}} z_1, e^{\frac{-t}{2p_2}} z_2, \dots, e^{\frac{-t}{2p_n}} z_n)$  by  $\varphi(e^{\frac{-t}{2p}} z)$  when there is no misunderstanding.

Note that for fixed  $t$ ,  $\varphi(e^{\frac{-t}{2p}} z)$  is subharmonic on  $\Delta^n$  with respect to  $z$ . It follows from mean value inequality that, for all  $t \geq t_\epsilon$ , we have

$$\begin{aligned} \varphi(0) &\leq \frac{1}{\pi} \int_{z_1 \in \Delta} \varphi(e^{\frac{-t}{2p_1}} z_1, 0 \dots, 0) d\lambda_{z_1} \\ &\leq \frac{1}{\pi^n} \int_{z \in \Delta^n} \varphi(e^{\frac{-t}{2p_1}} z_1, e^{\frac{-t}{2p_2}} z_2, \dots, e^{\frac{-t}{2p_n}} z_n) d\lambda_z \\ &= \frac{1}{\pi^n} \int_{\Delta^n \setminus B_{\delta,t}} \varphi(e^{\frac{-t}{2p}} z) d\lambda_z + \frac{1}{\pi} \int_{B_{\delta,t}} \varphi(e^{\frac{-t}{2p}} z) d\lambda_z \\ &\leq \frac{(1-\epsilon)\varphi(0)}{\pi^n} (\pi^n - \lambda(B_{\delta,t})) + \frac{(1+\delta)\varphi(0)}{\pi^n} \lambda(B_{\delta,t}) \\ &= \varphi(0)(1-\epsilon + \frac{\delta+\epsilon}{\pi^n} \lambda(B_{\delta,t})). \end{aligned}$$

As  $\varphi(0) < 0$ , we have

$$\lambda(B_{\delta,t}) \leq \frac{\pi^n \epsilon}{\delta + \epsilon} \leq \frac{\pi^n \epsilon}{\delta}$$

for any  $t \geq t_\epsilon$ . Hence

$$\lim_{t \rightarrow +\infty} \lambda(B_{\delta,t}) = 0.$$

Since  $\varphi$  is bounded, we have  $e^{-\varphi} \leq C_1$  for some  $C_1 > 0$ . As  $v(t)$  is continuous, when  $t$  is large enough, we have

$$\sup_{\{z \in \Delta^n : -t - 1 - k_1 - k_2 < \max_{1 \leq j \leq n} \{2p_j \log |z_j|\} < -t\}} v(z) \leq C_2,$$

where  $C_2 > 0$  is a constant independent of  $t$ . We denote  $v(e^{\frac{-t}{2p_1}} z_1, e^{\frac{-t}{2p_2}} z_2, \dots, e^{\frac{-t}{2p_n}} z_n)$  by  $v(e^{\frac{-t}{2p}} z)$  when there is no misunderstandings.

Let  $z_i = e^{\frac{-t}{2p_i}} w_i$ . We denote  $v(e^{\frac{-t}{2p_1}} w_1, e^{\frac{-t}{2p_2}} w_2, \dots, e^{\frac{-t}{2p_n}} w_n)$  by  $v(e^{\frac{-t}{2p}} w)$  and denote  $\varphi(e^{\frac{-t}{2p_1}} w_1, e^{\frac{-t}{2p_2}} w_2, \dots, e^{\frac{-t}{2p_n}} w_n)$  by  $\varphi(e^{\frac{-t}{2p}} w)$  for simplicity. By Lemma 11, we have

$$\begin{aligned} I_{t+k_1} &= \int_{\{z \in \Delta^n : -t - 1 - k_1 - k_2 < \max_{1 \leq j \leq n} \{2p_j \log |z_j|\} < -t\}} v(z) \frac{\prod_{j=1}^n |z_j|^{2\alpha_j}}{\max_{1 \leq j \leq n} \{|z_j|^{2p_j}\}} e^{-\varphi(z)} dz \wedge d\bar{z} \\ &= \int_{\{w \in \Delta^n : e^{-1-k_1-k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\}} v(e^{\frac{-t}{2p}} w) \frac{\prod_{j=1}^n |w_j|^{2\alpha_j}}{\max_{1 \leq j \leq n} \{|w_j|^{2p_j}\}} e^{-\varphi(e^{\frac{-t}{2p}} w)} dw \wedge d\bar{w} \\ &= \int_{\{w \in \Delta^n : e^{-1-k_1-k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\} \cap B_{\delta,t}} v(e^{\frac{-t}{2p}} w) \frac{\prod_{j=1}^n |w_j|^{2\alpha_j}}{\max_{1 \leq j \leq n} \{|w_j|^{2p_j}\}} e^{-\varphi(e^{\frac{-t}{2p}} w)} dw \wedge d\bar{w} + \\ &\quad \int_{\{w \in \Delta^n : e^{-1-k_1-k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\} \setminus B_{\delta,t}} v(e^{\frac{-t}{2p}} w) \frac{\prod_{j=1}^n |w_j|^{2\alpha_j}}{\max_{1 \leq j \leq n} \{|w_j|^{2p_j}\}} e^{-\varphi(e^{\frac{-t}{2p}} w)} dw \wedge d\bar{w} \\ &\leq CC_1C_2\lambda(B_{\delta,t}) + \left( \sup_{\{e^{-1-k_1-k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\}} v(e^{\frac{-t}{2p}} w) \right) e^{-(1+\delta)\varphi(0)} \frac{(2\pi)^n(1+k_1+k_2)}{\prod_{1 \leq j \leq n} (\alpha_j + 1)}. \end{aligned} \tag{21}$$

Hence we know

$$\limsup_{t \rightarrow +\infty} I_{t+k_1} = \frac{(2\pi)^n(1+k_1+k_2)}{\prod_{1 \leq j \leq n} (\alpha_j + 1)} e^{-(1+\delta)\varphi(0)} v(0).$$

By the arbitrariness of  $\delta$ , we know (20) holds for  $I_{t+k_1}$ .

For  $J_t$ , we know when  $t$  is large enough, the function  $v(z)$  and  $e^{-\varphi(z)}$  are uniformly bounded by some constant  $M > 0$  with respect to  $t$ . Then it follows from Lemma 11 that

$$\begin{aligned} \limsup_{t \rightarrow +\infty} J_{t+k_1} &\leq M \limsup_{t \rightarrow +\infty} \int_{\{z \in \Delta^n : -t - 1 - k_1 - k_2 < \max_{1 \leq j \leq n} \{2p_j \log |z_j|\} < -t\}} \frac{\prod_{j=1}^n |z_j|^{2\beta_j}}{\max_{1 \leq j \leq n} \{|z_j|^{2p_j}\}} dz \wedge d\bar{z} \\ &= M \lim_{t \rightarrow +\infty} \frac{\pi^n (q_\alpha + 1) (e^{-q_\alpha t} - e^{-q_\alpha(t+1+k_1+k_2)})}{q_\alpha \prod_{1 \leq j \leq n} (\alpha_j + 1)}, \end{aligned}$$

where  $q_\alpha = \frac{\beta_j+1}{p_j} - 1$ . As  $k_1 + k_2 + 1 > 0$ , we have  $\lim_{t \rightarrow +\infty} J_t = 0$ .

Lemma 13 is proved.

**Lemma 14** Let  $\Delta^n \subset \mathbb{C}^n$  be a polydisc. Let  $f = \sum_{\alpha \in E} b_\alpha w^\alpha$  (Taylor expansion) be a holomorphic function on  $\Delta^n$ , where  $E := \left\{ \alpha = (\alpha_1, \dots, \alpha_n) : \sum_{1 \leq j \leq n} \frac{\alpha_j + 1}{p_j} = 1 \right\}$ . Assume that  $v$  is a nonnegative continuous real function on  $\Delta^n$ . Denote

$$S_t := \int_{\{z \in \Delta^n : -t - 1 - k_2 < \max_{1 \leq j \leq n} \{2p_j \log |z_j|\} < -t + k_1\}} v(z) \frac{|f|^2}{\max_{1 \leq j \leq n} \{|z_j|^{2p_j}\}} e^{-\varphi(z_1, \dots, z_n)} dz \wedge d\bar{z}.$$

Then we have

$$\limsup_{t \rightarrow +\infty} S_t \leq \sum_{\alpha \in E} \frac{|b_\alpha|^2 (2\pi)^n (1 + k_1 + k_2)}{\prod_{1 \leq j \leq n} (\alpha_j + 1)} v(0) e^{-\varphi(0)}.$$

**Proof** In the proof of Lemma 14, we follow the notations we used in the proof of Lemma 13. Let  $z_i = e^{\frac{-t}{2p_i}} w_i$ , by Lemma 11, we have

$$\begin{aligned} S_t &= \int_{\{z \in \Delta^n : -t - 1 - k_2 < \max_{1 \leq j \leq n} \{2p_j \log |z_j|\} < -t + k_1\}} v(z) \frac{|f|^2}{\max_{1 \leq j \leq n} \{|z_j|^{2p_j}\}} e^{-\varphi(z_1, \dots, z_n)} dz \wedge d\bar{z} \\ &= \int_{\{w \in \Delta^n : e^{-1 - k_1 - k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\} \cap B_{\delta, t}} v(e^{\frac{-t}{2p}} w) \frac{|f(w)|^2}{\max_{1 \leq j \leq n} \{|w_j|^{2p_j}\}} e^{-\varphi(e^{\frac{-t}{2p}} w)} dw \wedge d\bar{w} + \\ &\quad \int_{\{w \in \Delta^n : e^{-1 - k_1 - k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\} \setminus B_{\delta, t}} v(e^{\frac{-t}{2p}} w) \frac{|f(w)|^2}{\max_{1 \leq j \leq n} \{|w_j|^{2p_j}\}} e^{-\varphi(e^{\frac{-t}{2p}} w)} dw \wedge d\bar{w} \\ &\leq CC_1 C_2 \lambda(B_{\delta, t}) + \left( \sup_{\{e^{-1 - k_1 - k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\}} v(e^{\frac{-t}{2p}} w) \right) e^{-(1+\delta)\varphi(0)} \times \\ &\quad \int_{\{e^{-1 - k_1 - k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\}} \frac{|f(w)|^2}{\max_{1 \leq j \leq n} \{|w_j|^{2p_j}\}} dw \wedge d\bar{w} \\ &= CC_1 C_2 \lambda(B_{\delta, t}) + \left( \sup_{\{e^{-1 - k_1 - k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\}} v(e^{\frac{-t}{2p}} w) \right) e^{-(1+\delta)\varphi(0)} \times \\ &\quad \sum_{\alpha \in E} \int_{\{e^{-1 - k_1 - k_2} < \max_{1 \leq j \leq n} \{|w_j|^{2p_j}\} < 1\}} \frac{|b_\alpha|^2 \prod_{j=1}^{n_1} |w_j|^{2\alpha_j}}{\max_{1 \leq j \leq n} \{|w_j|^{2p_j}\}} dw \wedge d\bar{w}. \end{aligned}$$

Hence we know

$$\limsup_{t \rightarrow +\infty} S_t \leq \sum_{\alpha \in E} \frac{|b_\alpha|^2 (2\pi)^n (1 + k_1 + k_2)}{\prod_{1 \leq j \leq n} (\alpha_j + 1)} v(0) e^{-\varphi(0)}.$$

**Lemma 15** (see [32]) Let  $c(t)$  be a positive measurable function on  $(0, +\infty)$ , and let  $a \in \mathbb{R}$ . Assume that  $\int_t^{+\infty} c(s) e^{-as} ds \in (0, +\infty)$  when  $t$  near  $+\infty$ . Then we have

$$(1) \quad \lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s) e^{-as} ds}{\int_t^{+\infty} c(s) e^{-s} ds} = 1 \text{ if and only if } a = 1,$$

- $$(2) \lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as} ds}{\int_t^{+\infty} c(s)e^{-s} ds} = 0 \text{ if and only if } a > 1,$$
- $$(3) \lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} c(s)e^{-as} ds}{\int_t^{+\infty} c(s)e^{-s} ds} = +\infty \text{ if and only if } a < 1.$$

**Lemma 16** (see [31]) If  $c(t)$  is a positive measurable function on  $(T, +\infty)$  such that  $c(t)e^{-t}$  is decreasing on  $(T, +\infty)$  and  $\int_{T_1}^{+\infty} c(s)e^{-s} ds < +\infty$  for some  $T_1 > T$ , then there exists a positive measurable function  $\tilde{c}$  on  $(T, +\infty)$  satisfying the following statements:

- (1)  $\tilde{c} \geq c$  on  $(T, +\infty)$ ;
- (2)  $\tilde{c}(t)e^{-t}$  is strictly decreasing on  $(T, +\infty)$  and  $\tilde{c}$  is increasing on  $(a, +\infty)$ , where  $a > T$  is a real number;
- (3)  $\int_{T_1}^{+\infty} \tilde{c}(s)e^{-s} ds < +\infty$ .

Moreover, if  $\int_T^{+\infty} c(s)e^{-s} ds < +\infty$  and  $c \in \mathcal{P}_T$ , we can choose  $\tilde{c}$  satisfying the above conditions,  $\int_T^{+\infty} \tilde{c}(s)e^{-s} ds < +\infty$  and  $\tilde{c} \in \mathcal{P}_T$ .

### 3 Preparations II: Multiplier Ideal Sheaves and Optimal $L^2$ Extensions

In this section, we recall and present some lemmas related to multiplier ideal sheaves and optimal  $L^2$  extensions.

#### 3.1 Multiplier Ideal Sheaves

We need the following lemmas in the local cases.

Let  $\Delta \subset \mathbb{C}$  be the unit disc. Let  $X = \Delta^m$  and let  $Y = \Delta^{n-m}$ . Denote  $M = X \times Y$ . Let  $\pi_1$  and  $\pi_2$  be the natural projections from  $M$  to  $X$  and  $Y$  respectively.

Let  $\psi_1$  be a plurisubharmonic function on  $X$ . Let  $\psi = \pi_1^*(\psi_1)$ . Let  $\varphi_1$  be a Lebesgue measurable function on  $X$  and  $\varphi_2 \in Psh(Y)$ . Denote  $\varphi = \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$ .

**Lemma 17** Assume that

$$\int_M |f|^2 e^{-\varphi} c(-\psi) < +\infty.$$

Then for any  $w \in \Delta^{n-m}$ ,

$$\int_{z \in \Delta^m} |f(z, w)|^2 e^{-\varphi_1} c(-\psi_1) < +\infty.$$

**Proof** According to the Fubini's Theorem, we have

$$\begin{aligned} & \int_{z \in \Delta^m} |f(z, w)|^2 e^{-\varphi_1} c(-\psi_1) \\ & \leq \frac{1}{(\pi r^2)^{n-m}} \int_{w' \in \Delta^{n-m}(w, r)} \left( \int_{z \in \Delta^m} |f(z, w')|^2 e^{-\varphi_1} c(-\psi_1) \right) \\ & \leq \frac{e^T}{(\pi r^2)^{n-m}} \int_{w' \in \Delta^{n-m}(w, r)} \left( \int_{z \in \Delta^m} |f(z, w')|^2 e^{-\varphi_1} c(-\psi_1) \right) e^{-\varphi_2} \\ & \leq \frac{e^T}{(\pi r^2)^{n-m}} \int_{\Delta^n} |f|^2 e^{-\varphi} c(-\psi) < +\infty, \end{aligned}$$

where  $r > 0$  such that  $\Delta^{n-m}(w, r) \Subset \Delta^{n-m}$ , and  $T := -\sup_{w' \in \Delta^{n-m}(w, r)} \varphi_2(w')$ .

**Lemma 18** Let  $f_1(z)$  be a holomorphic function on  $X$  such that  $(f_1, o) \in \mathcal{I}(\varphi_1 + \psi_1)_o$ , and  $f_2(w)$  be a holomorphic function on  $Y$  such that  $(f_2, w) \in \mathcal{I}(\varphi_2)_w$  for any  $w \in Y$ . Let  $\tilde{f}(z, w) = f_1(z)f_2(w)$  on  $M$ , then  $(\tilde{f}, (o, w)) \in \mathcal{I}(\varphi + \psi)_{(o, w)}$  for any  $(o, w) \in Y$ .

**Proof** According to  $(f_1, o) \in \mathcal{I}(\varphi_1 + \psi_1)_o$  and  $(f_2, w) \in \mathcal{I}(\varphi_2)_w$ , we can find some  $r > 0$  such that

$$\int_{\Delta^m(o, r)} |f_1|^2 e^{-\varphi_1 - \psi_1} < +\infty,$$

and

$$\int_{\Delta^{n-m}(w, r)} |f_2|^2 e^{-\varphi_2} < +\infty.$$

Then using Fubini's Theorem, we get

$$\int_{\Delta^n((o, w), r)} |\tilde{f}|^2 e^{-\varphi - \psi} = \int_{\Delta^m(o, r)} |f_1|^2 e^{-\varphi_1 - \psi_1} \int_{\Delta^{n-m}(w, r)} |f_2|^2 e^{-\varphi_2} < +\infty,$$

which means that  $(\tilde{f}, (o, w)) \in \mathcal{I}(\varphi + \psi)_{(o, w)}$ .

Let  $\Delta^{n_1} = \{w \in \mathbb{C}^{n_1} : |w_j| < 1 \text{ for any } j \in \{1, \dots, n_1\}\}$  be product of the unit disks. Let  $Y$  be an  $n_2$ -dimensional complex manifold, and let  $M = \Delta^{n_1} \times Y$ . Denote  $n = n_1 + n_2$ . Let  $\pi_1$  and  $\pi_2$  be the natural projections from  $M$  to  $\Delta^{n_1}$  and  $Y$  respectively. Let  $\rho_1$  be a nonnegative Lebesgue measurable function on  $\Delta^{n_1}$  satisfying that  $\rho_1(w) = \rho_1(|w_1|, \dots, |w_{n_1}|)$  for any  $w \in \Delta^{n_1}$  and the Lebesgue measure of  $\{w \in \Delta^{n_1} : \rho_1(w) > 0\}$  is positive. Let  $\rho_2$  be a nonnegative Lebesgue measurable function on  $Y$ , and denote that  $\rho = \pi_1^*(\rho_1) \times \pi_2^*(\rho_2)$  on  $M$ .

**Lemma 19** (see [2]) For any holomorphic  $(n, 0)$  form  $F$  on  $M$ , there exists a unique sequence of holomorphic  $(n_2, 0)$  forms  $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$  on  $Y$  such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha), \quad (22)$$

where the right term of the above equality is uniformly convergent on any compact subset of  $M$ . Moreover, if  $\int_M |F|^2 \rho < +\infty$ , we have

$$\int_Y |F_\alpha|^2 \rho_2 < +\infty \quad (23)$$

for any  $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$ .

Let  $\tilde{M} \subset M$  be an  $n$ -dimensional complex manifold satisfying that  $\{o\} \times Y \subset \tilde{M}$ , where  $o$  is the origin in  $\Delta^{n_1}$ .

**Lemma 20** (see [2]) For any holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$ , there exist a unique sequence of holomorphic  $(n_2, 0)$  forms  $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$  on  $Y$  and a neighborhood  $M_2 \subset \tilde{M}$  of  $\{o\} \times Y$ , such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha)$$

on  $M_2$ , where the right term of the above equality is uniformly convergent on any compact subset of  $M_2$ . Moreover, if  $\int_{\tilde{M}} |F|^2 \rho < +\infty$ , we have

$$\int_K |F_\alpha|^2 \rho_2 < +\infty$$

for any compact subset  $K$  of  $Y$  and  $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$ .

Let  $f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} b_\alpha w^\alpha$  (Taylor expansion) be a holomorphic function on  $D = \{w \in \mathbb{C}^n : |w_j| < r_0 \text{ for any } j \in \{1, \dots, n\}\}$ , where  $r_0 > 0$ . Let

$$\psi = \max_{1 \leq j \leq n_1} \{2p_j \log |w_j|\}$$

be a plurisubharmonic function on  $\mathbb{C}^n$ , where  $n_1 \leq n$  and  $p_j > 0$  is a constant for any  $j \in \{1, \dots, n_1\}$ . We recall a characterization of  $\mathcal{I}(\psi)_o$ , where  $o$  is the origin in  $\mathbb{C}^n$ .

**Lemma 21** (see [2])  $(f, o) \in \mathcal{I}(\psi)_o$  if and only if  $\sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} > 1$  for any  $\alpha \in \mathbb{Z}_{\geq 0}^n$  satisfying  $b_\alpha \neq 0$ .

Let  $\Omega = \Delta \subset \mathbb{C}$  be an unit disk. Let  $Y = \Delta^{n_1}$ . Denote  $M = \Omega \times Y$ . Let  $\pi_1$  and  $\pi_2$  be the natural projections from  $M$  to  $\Omega$  and  $Y$  respectively.

Let  $\psi = \pi_1^*(2p \log |z|) + N$  be a plurisubharmonic function on  $M$ , where  $N \leq 0$  is a plurisubharmonic function on  $M$  and  $N|_{\{0\} \times \Delta^n} \not\equiv -\infty$ . Assume that there exist a holomorphic function  $g$  on  $\Delta$  and a function  $\tilde{\psi}_2 \in Psh(M)$  such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \log |g|) + \tilde{\psi}_2,$$

where  $ord_0(g) = q$ . We assume that  $g = dz^q h(z)$  on  $\Delta$ , where  $d$  is a constant,  $h(z)$  is a holomorphic function on  $\Delta$  and  $h(0) = 1$ .

Let  $\varphi_2 \in Psh(Y)$ . Denote  $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$ .

Let  $F$  be a holomorphic  $(n, 0)$  form on  $M$ , where

$$F = \sum_{j=k}^{\infty} \pi_1^*(z^j dz) \wedge \pi_2^*(F_j)$$

according to Lemma 19. Here  $k \in \mathbb{N}$  and  $F_j$  is a holomorphic  $(n, 0)$  form on  $Y$  for any  $j \geq k$ .

Assume that  $\int_M |F|^2 e^{-\varphi} c(-\psi) < +\infty$  and  $c(t)$  is increasing near  $+\infty$ . As  $\psi = \pi_1^*(2p \log |z|) + N \leq \pi_1^*(2p \log |z|)$ , when  $t$  is large enough, we have

$$S := \int_{\{\pi_1^*(2p \log |z|) < -t\}} |F|^2 e^{-\varphi} c(-\pi_1^*(2p \log |z|)) \leq \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty.$$

**Lemma 22** *Let  $c$  be a positive measurable function on  $(0, +\infty)$  such that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ ,  $c$  is increasing near  $+\infty$ , and  $\int_0^{+\infty} c(s)e^{-s} ds < +\infty$ . Assume that  $k \geq q$ , and*

$$S = \int_{\{\pi_1^*(2p \log |z|) < -t\}} |F|^2 e^{-\varphi} c(-\pi_1^*(2p \log |z|)) < +\infty.$$

Then

$$(F, (0, y)) \in (O(K_M)) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0, y)}$$

for any  $y \in Y$ .

**Proof** It follows from  $M$  is a Stein manifold that there exist smooth plurisubharmonic functions  $\tilde{\psi}_{2,l}$  on  $M$  such that  $\tilde{\psi}_{2,l}$  are decreasingly convergent to  $\tilde{\psi}_2$ . Since the computation is local, we assume that  $|\tilde{\psi}_{2,l}(z, w) - \tilde{\psi}_{2,l}(0, w)| \leq \epsilon$  for any  $(z, w) \in M = \Delta \times \Delta^n$ .

We also assume that  $F = z^k \tilde{h}(z, w) dz \wedge dw$  on  $M$  and  $|h(z) - h(0)| < \epsilon$  for any  $z \in \Delta$ .

$$\begin{aligned}
S &= \int_{\{\pi_1^*(2p \log |z|) < -t\}} |F|^2 e^{-\pi_1^*(2 \log |g|) - \tilde{\psi}_2 + \pi_1^*(2p \log |z|) + N - \pi_2^*(\varphi_2)} c(-\pi_1^*(2p \log |z|)) \\
&\geq \int_{\{\pi_1^*(2p \log |z|) < -t\}} \frac{|z|^{2k+2p} |\tilde{h}(z, w)|^2}{d|z|^{2q} |h(z)|^2} e^{-\tilde{\psi}_{2,l} + N - \pi_2^*(\varphi_2)} c(-\pi_1^*(2p \log |z|)) \\
&\geq \int_{w \in \Delta^n} \int_{\{2p \log |z| < -t\}} |z|^{2k+2p-2q} e^N \frac{|\tilde{h}(z, w)|^2}{d|1+\epsilon|^2} e^{-\tilde{\psi}_{2,l}(0, w) - \epsilon - \pi_2^*(\varphi_2)} c(-\pi_1^*(2p \log |z|)) \\
&= \int_{w \in \Delta^n} \left( 2 \int_{\{2p \log |r| < -t\}} \int_0^{2\pi} |r|^{2k+2p-2q+1} e^{N(re^{i\theta}, w)} |\tilde{h}(re^{i\theta}, w)|^2 c(-2p \log r) d\theta dr \right) \times \\
&\quad \frac{e^{-\tilde{\psi}_{2,l}(0, w) - \epsilon - \varphi_2(w)}}{d|1+\epsilon|^2} \\
&\geq \frac{4\pi}{d|1+\epsilon|^2} e^{-\epsilon} \int_{\{2p \log |r| < -t\}} r^{2k+2p-2q+1} c(-2p \log r) dr \int_{w \in \Delta^n} |\tilde{h}(0, w)|^2 e^{N(0, w) - \tilde{\psi}_{2,l}(0, w)} e^{-\varphi_2(w)} \\
&= \frac{2\pi}{pd|1+\epsilon|^2} e^{-\epsilon} \int_t^{+\infty} c(s) e^{-\left(\frac{k-q+1}{p}+1\right)s} ds \int_{w \in \Delta^n} |\tilde{h}(0, w)|^2 e^{N(0, w) - \tilde{\psi}_{2,l}(0, w)} e^{-\varphi_2(w)}. \tag{24}
\end{aligned}$$

Since  $k \geq q$  and  $\int_0^{+\infty} c(s) e^{-s} ds < +\infty$ , we have

$$\int_t^{+\infty} c(s) e^{-\left(\frac{k-q+1}{p}+1\right)s} ds < +\infty.$$

Letting  $t \rightarrow +\infty$  in (24), it follows from  $S < +\infty$  and Levi's Theorem that we have

$$\frac{2\pi}{pd|1+\epsilon|^2} e^{-\epsilon} \int_t^{+\infty} c(s) e^{-\left(\frac{k-q+1}{p}+1\right)s} ds \int_{w \in \Delta^n} |\tilde{h}(0, w)|^2 e^{N(0, w) - \tilde{\psi}_2(0, w)} e^{-\varphi_2(w)} < +\infty. \tag{25}$$

It follows from inequality (25) that we have

$$\int_{w \in \Delta^n} |\tilde{h}(0, w)|^2 e^{N(0, w) - \tilde{\psi}_2(0, w)} e^{-\varphi_2(w)} < +\infty. \tag{26}$$

Note that  $N|_{\{0\} \times \Delta^n} \not\equiv -\infty$ . It follows from (26) that there must exist  $w_1 \in Y$  such that  $\tilde{\psi}_2(0, w_1) > -\infty$ .

Since  $k \geq q$ , we know  $z^k \in \mathcal{I}(2 \log |g|)_0$ . It follows from Lemma 18 that

$$(\pi_1^*(z^k dz) \wedge \pi_2^*(F_k), (0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0, y)}, \tag{27}$$

for any  $y \in Y$ .

Let  $0 < \delta < 1$  be a constant such that  $w_1 \in \Delta_\delta^n$ . It follows from Fubini's Theorem and  $c(t)e^{-t}$  is decreasing with respect to  $t$  that

$$\begin{aligned}
& \int_{\Delta_\delta \times \Delta_\delta^n} |\pi_1^*(z^k dz) \wedge \pi_2^*(F_k)|^2 e^{-\varphi} c(-\pi_1^*(2p \log |z|)) \\
&= \int_{\Delta_\delta} |z^k|^2 e^{-\varphi_1} c(-2p \log |z|) |dz|^2 \cdot \int_{\Delta_\delta^n} |F_k|^2 e^{-\varphi_2} \\
&\leq C \int_{\Delta_\delta} |z^k|^2 e^{-\varphi_1 - 2p \log |z|} |dz|^2 \cdot \int_{\Delta_\delta^n} |F_k|^2 e^{-\varphi_2},
\end{aligned} \tag{28}$$

where  $C$  is a positive constant independent of  $F$ .

Consider

$$\begin{aligned}
I &:= \int_{\Delta_\delta} |z|^{2k} e^{-\varphi_1 - 2p \log |z|} |dz|^2 \\
&= \int_{\Delta_\delta} |z|^{2k} e^{-2 \log |g| - u(z)} |dz|^2 \\
&= \int_{\Delta_\delta} |z|^{2k} e^{-2 \log |g| + N(z, w) - \tilde{\psi}_2(z, w)} |dz|^2.
\end{aligned} \tag{29}$$

As  $N$  is a plurisubharmonic function on  $M$ ,  $e^N$  has an upper bound  $C_1$  on  $\Delta_\delta \times \Delta_\delta^n$  (especially,  $C_1$  is independent of  $w$ ). Hence

$$I \leq C_1 \int_{\Delta_\delta} |z|^{2k} e^{-2 \log |g| - \tilde{\psi}_2(z, w)} |dz|^2 = C_1 \int_{\Delta_\delta} |z|^{2k-2q} |h(z)|^2 e^{-\tilde{\psi}_2(z, w)} |dz|^2.$$

Denote  $M(w) = \int_{\Delta_\delta} |z|^{2k-2q} |h(z)|^2 e^{-\tilde{\psi}_2(z, w)} |dz|^2$ . We have  $I \leq M(w)$  for any  $w \in \Delta_\delta^n$ , especially  $I \leq M(w_1)$ .

Next we prove  $M(w_1) < +\infty$ . Note that  $M(w_1) = \int_{\Delta_\delta} |z|^{2k-2q} |h(z)|^2 e^{-\tilde{\psi}_2(z, w_1)} |dz|^2$ . As  $e^{-\tilde{\psi}_2(0, w_1)} > -\infty$ ,  $k \geq q$  and  $h(z)$  is a holomorphic function on  $\Delta_\delta$ , by Hölder inequality, we have

$$\begin{aligned}
M(w_1) &= \int_{\Delta_\delta} |z|^{2k-2q} |h(z)|^2 e^{-\tilde{\psi}_2(z, w_1)} |dz|^2 \\
&\leq \left( \int_{\Delta_\delta} |z|^{s(2k-2q)} |h(z)|^{2s} |dz|^2 \right)^{\frac{1}{s}} \left( \int_{\Delta_\delta} e^{-\frac{s}{s-1} \tilde{\psi}_2(z, w_1)} |dz|^2 \right)^{\frac{s-1}{s}} \\
&< +\infty,
\end{aligned}$$

where  $s > 1$  is a real number. Hence we know  $I < +\infty$ . Then

$$\int_{\Delta_\delta \times \Delta_\delta^n} |\pi_1^*(z^k dz) \wedge \pi_2^*(F_k)|^2 e^{-\varphi} c(-\pi_1^*(2p \log |z|)) < +\infty.$$

As  $S = \int_{\Delta_\delta \times \Delta_\delta^n} |F|^2 e^{-\varphi} c(-\pi_1^*(2p \log |z|)) < +\infty$ , we have

$$\int_{\Delta_\delta \times \Delta_\delta^n} |F - \pi_1^*(z^k dz) \wedge \pi_2^*(F_k)|^2 e^{-\varphi} c(-\pi_1^*(2p \log |z|)) < +\infty.$$

Note that

$$F - \pi_1^*(z^k dz) \wedge \pi_2^*(F_k) = \sum_{j=k+1}^{\infty} \pi_1^*(z^j dz) \wedge \pi_2^*(F_j)$$

and  $k+1 > q$ . Using the same method as above, we can get that

$$(\pi_1^*(z^{k+1} dz) \wedge \pi_2^*(F_{k+1}), (0, y)) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(0, y)}$$

for any  $y \in Y$  and

$$\int_{\Delta_\delta \times \Delta_\delta^n} \left| F - \pi_1^*(z^k dz) \wedge \pi_2^*(F_k) - \pi_1^*(z^{k+1} dz) \wedge \pi_2^*(F_{k+1}) \right|^2 e^{-\varphi} c(-\pi_1^*(2p \log |z|)) < +\infty.$$

By induction, we know that

$$(\pi_1^*(z^j dz) \wedge \pi_2^*(F_j), (0, y)) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(0, y)}$$

for any  $j \geq k$ ,  $y \in Y$ . Then it follows from Lemma 10 that

$$(F, (0, y)) = \left( \sum_{j=k}^{\infty} \pi_1^*(z^j dz) \wedge \pi_2^*(F_j), (0, y) \right) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(0, y)}$$

for any  $y \in Y$ .

Let  $\Omega = \Delta$  be the unit disk in  $\mathbb{C}$ , where the coordinate is  $z$ . Let  $Y = \Delta^n$  be the unit polydisc in  $\mathbb{C}^n$ , where the coordinate is  $w = (w_1, \dots, w_n)$ . Let  $M = \Omega \times Y$ . Let  $\pi_1, \pi_2$  be the natural projections from  $M$  to  $\Omega$  and  $Y$ .

Let  $\psi_1 = 2p \log |z| + \psi_0$  on  $\Omega$ , where  $p > 0$  and  $\psi_0$  is a negative subharmonic function on  $\Omega$  with  $\psi_0(0) > -\infty$ . Let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\varphi_1 + \psi$  is a subharmonic function on  $\Omega$ . It follows from the Weierstrass Theorem on open Riemann surfaces (see [16]) and the Siu's Decomposition Theorem, that  $\varphi_1 + \psi_1 = 2 \log |g| + 2u$ , where  $g$  is a holomorphic function on  $\Omega$  with  $\text{ord}(g)_0 = q \in \mathbb{N}$ ,  $u$  is a subharmonic function on  $\Omega$  such that  $v(dd^c u, z) \in [0, 1)$  for any  $z \in \Omega$ . Let  $\varphi_2$  be a plurisubharmonic function on  $Y$ . Denote  $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$  on  $M$ .

**Lemma 23**  $\mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0, y)} = \mathcal{I}(\pi_1^*(\psi_1) + \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2))_{(0, y)}$  for any  $y \in Y$ .

**Proof** It is easy to see that  $\mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0, y)} \supset \mathcal{I}(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2))_{(0, y)}$  for any  $y \in Y$ .

Now we prove  $\mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0, y)} \subset \mathcal{I}(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2))_{(0, y)}$  for any  $y \in Y$ .

Let  $F \in \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0,y)}$ . Then by Lemma 19 (although Lemma 19 is stated for the holomorphic  $(n, 0)$  forms, since our case is local, the decomposition still holds for holomorphic functions), we know  $F = \sum_{j=k}^{+\infty} z^j F_j$  on  $M$ , where the right hand side is uniformly convergent to  $F$  on  $M$ ,  $F_j(w)$  is a holomorphic function on  $Y$  for any  $j \geq k$  and  $F_k \not\equiv 0$ .

Since the case is local, we also assume that  $F = z^k h_1(z, w)$  on  $M$ , where  $h_1(z, w)$  is a holomorphic function on  $M$  satisfying  $h_1(0, w) \neq 0$ .  $F \in \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0,y)}$  implies that  $k \geq q$  and

$$\int_{\Delta \times \Delta^n} |z|^{2k-2q} |h_1(z, w)| e^{-\pi_2^*(\varphi_2(w))} < +\infty.$$

By Fubini's theorem and sub-mean value inequality of subharmonic functions, we have

$$\begin{aligned} \int_{\Delta \times \Delta^n} |z|^{2k-2q} |h_1(z, w)| e^{-\pi_2^*(\varphi_2(w))} &= \int_{w \in \Delta^n} \int_{z \in \Delta} |z|^{2k-2q} |h_1(z, w)| e^{-\pi_2^*(\varphi_2(w))} \\ &\geq C \int_{w \in \Delta^n} |h_1(0, w)| e^{-\varphi_2(w)} \\ &= C \int_{w \in \Delta^n} |F_k|^2 e^{-\varphi_2(w)}, \end{aligned}$$

where  $C > 0$  is a constant. Hence we have  $(F_k, y) \in \mathcal{I}(\varphi_2)_y$  for any  $y \in Y$ .

As  $v(dd^c u, 0) \in [0, 1)$ , we have  $(z^k, 0) \in \mathcal{I}(2 \log |g|)_0 = \mathcal{I}(2 \log |g| + 2u)_0 = \mathcal{I}(\psi_1 + \varphi_1)_0$ . It follows from Lemma 18 that we know  $(z^k F_k, (0, y)) \in \mathcal{I}(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2))_{(0,y)}$ . Then we know

$$\int_{\Delta \times \Delta^n} |z^k F_k|^2 e^{\pi_1^*(-2 \log |g| + 2u) - \pi_2^*(\varphi_2)} < +\infty,$$

which implies

$$\int_{\Delta \times \Delta^n} |z^k F_k|^2 e^{\pi_1^*(-2 \log |g|) - \pi_2^*(\varphi_2)} < +\infty.$$

Hence we have  $(F - z^k F_k, (0, y)) \in \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0,y)}$ .

Denote  $\tilde{F}_{k+1} = F - z^k F_k$  on  $\Delta \times \Delta^n$ . Note that  $\tilde{F}_{k+1} = \sum_{j=k+1}^{+\infty} z^j F_j$  on  $M$  and  $(\tilde{F}_{k+1}, (0, y)) \in \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0,y)}$ .

By using similar discussion as above, we know

$$(z^{k+1} \tilde{F}_{k+1}, (0, y)) \in \mathcal{I}(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2))_{(0,y)},$$

and

$$\int_{\Delta \times \Delta^n} |z^{k+1} \tilde{F}_{k+1}|^2 e^{\pi_1^*(-2 \log |g|) - \pi_2^*(\varphi_2)} < +\infty.$$

Hence  $(\tilde{F}_{k+1} - z^{k+1} F_{k+1}, (0, y)) \in \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0,y)}$ . Denote  $\tilde{F}_{k+2} = \tilde{F}_{k+1} - z^{k+1} F_{k+1}$  on  $\Delta \times \Delta^n$ . Note that  $\tilde{F}_{k+2} = \sum_{j=k+2}^{+\infty} z^j F_j$  on  $M$  and  $(\tilde{F}_{k+2}, (0, y)) \in \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(0,y)}$ .

By induction, we know that for any  $j \geq k$ ,

$$(z^j F_j, (0, y)) \in \mathcal{I}(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2))_{(0,y)}$$

holds. Then it follows from Lemma 10 that we know

$$(F, (0, y)) \in \mathcal{I}(\pi_1^*(\psi_1 + \varphi_1) + \pi_2^*(\varphi_2))_{(0,y)}$$

for any  $y \in Y = \Delta^n$ .

We recall a well known result about multiplier ideal sheaves.

**Lemma 24** (see [2]) *Let  $\Phi_1$  and  $\Phi_2$  be plurisubharmonic functions on  $\Delta^n$  satisfying  $\Phi_2(o) > -\infty$ , where  $n \in \mathbb{Z}_{>0}$  and  $o$  is the origin in  $\Delta^n$ . Then  $\mathcal{I}(\Phi_1)_o = \mathcal{I}(\Phi_1 + \Phi_2)_o$ .*

### 3.2 Optimal Jet $L^2$ Extensions

Let  $\Omega$  be an open Riemann surface with nontrivial Green function. Let  $Z_\Omega = \{z_j : j \in \mathbb{N}_+ \& j < \gamma\}$  be a subset of  $\Omega$  of discrete points, where  $\gamma \in \mathbb{Z}_{\geq 2}^+$  or  $\gamma = +\infty$ . Let  $Y$  be an  $n$ -dimensional weakly pseudoconvex Kähler manifold. Denote  $M = \Omega \times Y$ . Let  $\pi_1$  and  $\pi_2$  be the natural projections from  $M$  to  $\Omega$  and  $Y$  respectively. Denote  $Z_0 := Z_\Omega \times Y$ . Denote  $Z_j := \{z_j\} \times Y$ .

Let  $\tilde{M} \subset M$  be an  $n$ -dimensional weakly pseudoconvex Kähler manifold satisfying that  $Z_0 \subset \tilde{M}$ . Let  $F$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$ .

Let  $\psi$  be a plurisubharmonic function on  $\tilde{M}$ . It follows from Siu's decomposition theorem that

$$dd^c \psi = \sum_{j \geq 1} 2p_j[Z_j] + \sum_{i \geq 1} \lambda_i[A_i] + R,$$

where  $[Z_j]$  and  $[A_i]$  are the currents of integration over an irreducible  $(n-1)$ -dimensional analytic set, and where  $R$  is a closed positive current with the property that  $\dim E_c(R) < n-1$  for every  $c > 0$ , where  $E_c(R) = \{x \in \tilde{M} : v(R, x) \geq c\}$  is the upperlevel sets of Lelong number. We assume that  $p_j > 0$  for any  $1 \leq j < \gamma$ .

Then  $N := \psi - \pi_1^*\left(\sum_{j \geq 1} 2p_j G_\Omega(z, z_j)\right)$  is a plurisubharmonic function on  $\tilde{M}$ . We assume that  $N \leq 0$ .

Let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\psi + \pi_1^*(\varphi)$  is a plurisubharmonic function on  $\tilde{M}$ . With similar discussion as above, by Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j \geq 1} 2\tilde{q}_j[Z_j] + \sum_{i \geq 1} \tilde{\lambda}_i[\tilde{A}_i] + \tilde{R},$$

where  $\tilde{q}_j \geq 0$  for any  $1 \leq j < \gamma$ .

By Weierstrass theorem on open Riemann surface, there exists a holomorphic function  $g$  on  $\Omega$  such that  $ord_{z_j}(g) = q_j := [\tilde{q}_j]$  for any  $z_j \in Z_\Omega$  and  $g(z) \neq 0$  for any  $z \notin Z_\Omega$ , where  $[q]$  equals to the integer part of the nonnegative real number  $q$ . Then we know that there exists a plurisubharmonic function  $\tilde{\psi}_2 \in Psh(\tilde{M})$  such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \log |g|) + \tilde{\psi}_2.$$

Let  $\varphi_2 \in Psh(Y)$ . Denote  $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$ .

For  $1 \leq j < \gamma$ , let  $(V_j, \tilde{z}_j)$  be a local coordinate open neighborhood of  $z_j$  in  $\Omega$  satisfying  $V_j \Subset \Omega$ ,  $\tilde{z}_j(z_j) = 0$  under the local coordinate and  $V_j \cap V_k = \emptyset$  for any  $j \neq k$ . Denote  $V_0 := \cup_{1 \leq j < \gamma} V_j$ . We assume that  $g = d_j \tilde{z}_j^{q_j} h_j(z)$  on  $V_j$ , where  $d_j$  is a constant,  $h_j(z)$  is a holomorphic function on  $V_j$  and  $h(z_j) = 1$ .

Let  $c(t)$  be a positive measurable function on  $(0, +\infty)$  satisfying that  $c(t)e^{-t}$  is decreasing and  $\int_0^{+\infty} c(s)e^{-s} ds < +\infty$ .

We have the following lemma.

**Lemma 25** *Let  $F$  be a holomorphic  $(n, 0)$  form on  $U_0$  such that for  $1 \leq j < \gamma$ ,  $F = \pi_1^*(\tilde{z}_j^{k_j} f_j dz_j) \wedge \pi_2^*(F_j)$  on  $U_j \Subset U_0 \cap (V_j \times Y)$ , where  $U_j$  is an open neighborhood of  $Z_j$  in  $\tilde{M}$ ,  $k_j$  is a nonnegative integer,  $f_j$  is a holomorphic function on  $V_j$  satisfying  $f_j(z_j) = a_j \neq 0$  and  $F_j$  is a holomorphic  $(n-1, 0)$  form on  $Y$ .*

*Denote  $I_F := \{j : 1 \leq j < \gamma \& k_j + 1 - q_j \leq 0\}$ . Assume that  $k_j + 1 = q_j$  for any  $j \in I_F$  and  $\tilde{\psi}_2|_{Z_j}$  is not identically  $-\infty$  on  $Z_j$ . If*

$$\sum_{j \in I_F} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty \quad (30)$$

and

$$\int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty, \quad (31)$$

for any  $j \notin I_F$ . Then there exists a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $\tilde{M}$  such that  $(\tilde{F} - F, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $1 \leq j < \gamma$  and  $y \in Y$  and

$$\int_{\tilde{M}} |\tilde{F}|^2 c(-\psi) e^{-\varphi} \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty. \quad (32)$$

**Proof** Denote  $G = \sum_{j \geq 1} 2p_j G_\Omega(z, z_j)$ . Note that  $\psi \leq \pi_1^*(G)$  and  $c(t)e^{-t}$  is decreasing. We have

$$c(-\psi) e^{-\varphi} \leq c(-\pi_1^*(G)) e^{-N - \pi_1^*(\varphi_1) - \pi_2^*(\varphi_2)}.$$

To prove Lemma 25, it suffice to prove

$$\begin{aligned} & \int_{\tilde{M}} |\tilde{F}|^2 c(-\pi_1^*(G)) e^{-N - \pi_1^*(\varphi_1) - \pi_2^*(\varphi_2)} \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty. \end{aligned} \quad (33)$$

The following remark shows that we only need to prove formula (33) when  $Z_\Omega$  is a finite set.

**Remark 19** It follows from Lemma 6 that there exists a sequence of open Riemann surfaces  $\{\Omega_l\}_{l \in \mathbb{Z}^+}$  such that  $\Omega_l \Subset \Omega_{l+1} \Subset \Omega$ ,  $\cup_{l \in \mathbb{Z}^+} \Omega_l = \Omega$  and  $\{G_{\Omega_l}(\cdot, z_0) - G_\Omega(\cdot, z_0)\}$  is decreasingly convergent to 0 on  $\Omega$  with respect to  $l$  for any  $z_0 \in \Omega$ .

Denote  $Z_l := \Omega_l \cap Z_0$ . As  $Z_\Omega$  is a subset of  $\Omega$  of discrete points,  $Z_l$  is a set of finite points.

Denote

$$G_l := \sum_{z_j \in Z_l} 2p_j G_\Omega(z, z_j)$$

and

$$\varphi_{1,l} := \varphi_1 + G - G_l.$$

Then we have  $N + \pi_1^*(G_l) + \pi_1^*(\varphi_{1,l}) = N + \pi_1^*(G) + \pi_1^*(\varphi_1) = \psi + \pi_1^*(\varphi_1)$ .

Let  $I_l := I_F \cap \{j : z_j \in Z_l\}$ . Denote  $\tilde{M}_l := (\Omega_l \times Y) \cap \tilde{M}$ . We note that  $\tilde{M}_l$  is weakly pseudoconvex Kähler. Now we assume that the formula (33) holds on  $\tilde{M}_l$ , i.e. we have

$$\begin{aligned} & \int_{\tilde{M}_l} |\tilde{F}_l|^2 c(-\pi_1^*(G_l)) e^{-N - \pi_1^*(\varphi_{1,l}) - \pi_2^*(\varphi_2)} \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_l} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty, \end{aligned}$$

where  $F_l$  is a holomorphic  $(n, 0)$  form on  $M_l$  satisfying  $(F_l - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g_0|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $l, z_j \in Z_l$ , and  $y \in Y$ .

As  $G \leq G_l$  and  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ , we have

$$\begin{aligned} & \int_{\tilde{M}_l} |\tilde{F}_l|^2 c(-\pi_1^*(G)) e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)} \\ & \leq \int_{\tilde{M}_l} |\tilde{F}_l|^2 c(-\pi_1^*(G_l)) e^{-N-\pi_1^*(\varphi_{1,l})-\pi_2^*(\varphi_2)} \\ & \leq (\int_0^{+\infty} c(s) e^{-s} ds) \sum_{j \in I_l} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty. \end{aligned} \quad (34)$$

Note that  $\pi_1^*(G)$  is continuous on  $\tilde{M} \setminus Z_0$ , where  $Z_0$  is a closed analytic subset of  $\tilde{M}$  and  $N + \pi_1^*(G) + \pi_1^*(\varphi_1) = \psi + \pi_1^*(\varphi_1)$  is a plurisubharmonic function on  $M$ . For any compact subset  $K$  of  $\tilde{M} \setminus Z_0$ , there exist  $\hat{l}$  (depending on the choice of  $K$ ) such that  $K \subset \subset \tilde{M}_{\hat{l}}$  and  $C_K > 0$  such that  $\frac{e^{N+\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)}}{c(-\pi_1^*(G))} = \frac{e^{N+\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)+\pi_1^*(G)}}{c(-\pi_1^*(G))e^{\pi_1^*(G)}} \leq C_K$  on  $K$ . It follows from Lemma 9 and diagonal method that there exists a subsequence of  $\{F_l\}$  (also denoted by  $\{F_l\}$ ), which is compactly convergent to a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $\tilde{M}$ . Combining formula (34) and Fatou's lemma, we have

$$\begin{aligned} & \int_{\tilde{M}_l} |\tilde{F}|^2 c(-\pi_1^*(G)) e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)} \\ & \leq \liminf_{l \rightarrow +\infty} \int_{\tilde{M}_l} |\tilde{F}_l|^2 c(-\pi_1^*(G)) e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)} \\ & \leq \liminf_{l \rightarrow +\infty} \int_{\tilde{M}_l} |\tilde{F}_l|^2 c(-\pi_1^*(G_l)) e^{-N-\pi_1^*(\varphi_{1,l})-\pi_2^*(\varphi_2)} \\ & \leq \liminf_{l \rightarrow +\infty} (\int_0^{+\infty} c(s) e^{-s} ds) \sum_{j \in I_l} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} \\ & \leq (\int_0^{+\infty} c(s) e^{-s} ds) \sum_{j \in I_F} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty. \end{aligned}$$

As  $\{F_l\}$  is compactly convergent to  $\tilde{F}$  on  $\tilde{M}$  and  $(F_l - F, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g_0|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $l$ ,  $z_j \in Z_l$ , and  $y \in Y$ . It follows from Lemma 10 that  $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g_0|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $z_j \in Z_{\Omega}$ , and  $y \in Y$ .

We continue to prove Lemma 25. Now we assume that  $\gamma = m + 1$  i.e.  $Z_{\Omega} = \{z_j : 1 \leq j \leq m\}$  and  $I_F = \{1, 2, \dots, m_1\}$ , where  $m_1 \leq m$ .

Denote  $\tilde{\psi}_{2,l} = \max\{\psi_2, -l\}$ , where  $l$  is a positive integer. As  $\tilde{\psi}_2$  is plurisubharmonic, we know  $\{\tilde{\psi}_{2,l}\}_{l=1}^{+\infty}$  is a sequence of plurisubharmonic functions on  $\tilde{M}$  decreasingly convergent to  $\tilde{\psi}_2$ . We also note that every  $\tilde{\psi}_{2,l}$  is lower bounded.

When  $t_0$  is large enough, we know that  $\{G < -t\} \Subset V_0$ , for any  $t > t_0$ . As  $\tilde{M}$  is a weakly pseudoconvex Kähler manifold, there exists a sequence of weakly

pseudoconvex Kähler manifolds  $\tilde{M}_s$  satisfying  $\tilde{M}_1 \Subset \tilde{M}_2 \cdots \Subset \tilde{M}_s \Subset \cdots \tilde{M}$  and  $\cup_{s \in \mathbb{N}^+} \tilde{M}_s = \tilde{M}$ .

It is easy to verify that  $\int_{\{\pi_1^*(G) < -t\} \cap \tilde{M}_s} |F|^2 < +\infty$ , and it follows from formula (31) and Fubini's theorem that

$$\int_{\tilde{M}_s} \mathbb{I}_{\{-t-1 < \pi_1^*(G) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} < +\infty.$$

It follows from Lemma 8 that there exists a holomorphic  $(n, 0)$  form  $F_{t,l,s}$  on  $M$  such that

$$\begin{aligned} & \int_{\tilde{M}_s} |F_{t,l,s} - (1 - b_{t,1}(\pi_1^*(G)))F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + v_{t,1}(\pi_1^*(G))} c(-v_{t,1}(\pi_1^*(G))) \\ & \leq (\int_0^{t+1} c(s) e^{-s} ds) \int_{\tilde{M}_s} \mathbb{I}_{\{-t-1 < \pi_1^*(G) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} < +\infty, \end{aligned} \quad (35)$$

for any  $t \geq t_0$ .

As  $b_{t,1}(\hat{t}) = 0$  for  $\hat{t}$  largely enough, then we know  $(F_{t,l,s} - F, (z_0, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g_0|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0 \cap \tilde{M}_s$ .

For any  $\epsilon > 0$ , there exists  $t_1 > t_0$  such that

(1) for any  $j \in \{1, 2, \dots, m\}$

$$\sup_{\tilde{z}_j \in \{G < -t_1\} \cap V_j} |g_1(\tilde{z}) - g_1(z_j)| < \epsilon,$$

where  $g_1(z)$  is a smooth function on  $V_0$  satisfying  $g_1(\tilde{z})|_{V_j} := G - 2p_j \log |\tilde{z}_j|$ .

(2) for any  $j \in \{1, 2, \dots, m\}$

$$\sup_{\tilde{z}_j \in \{G < -t_1\} \cap V_j} |f_j(\tilde{z}_j) - a_j| < \epsilon.$$

(3) for any  $j \in \{1, 2, \dots, m\}$

$$\sup_{\tilde{z}_j \in \{G < -t_1\} \cap V_j} |h_j(\tilde{z}_j) - 1| < \epsilon.$$

For any  $(z_j, y) \in Z_0 \cap \tilde{M}_s$ , letting  $(U_y, w)$  be a small local coordinated open neighborhood of  $y$  and shrinking  $V_j$  if necessary, we have  $V_j \times U_y \Subset U_j$  for any  $j$ . Recall that  $V_0 := \cup_{j \geq 1} V_j$ . Assume that  $F_j = \tilde{h}_j(w)dw$  on  $U_y$ , where  $\tilde{h}_j$  is a holomorphic function on  $U_y$  and  $dw = dw_1 \wedge dw_2 \wedge \dots \wedge dw_n$ . There exists  $t_2 > t_1$  such that when  $t > t_2$ ,  $(\{G < -t\} \times U_y) \cap (V_j \times U_y) \Subset V_j \times U_y$ , for any  $j$ .

When  $t > t_2$ , direct calculation shows

$$\begin{aligned} & \int_{V_0 \times U_y} \mathbb{I}_{\{-t-1 < \pi_1^*(G) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} \\ &= \sum_{j=1}^m \int_{\{-t-1 < G < -t\} \times U_y} \frac{|\tilde{z}_j|^{2k_j} |f_j|^2 |F_j|^2}{|d_j|^2 |\tilde{z}|^{2q_j} |h_j|^2} e^{-\tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} |d\tilde{z}_j|^2 |dw|^2 \\ &\leq \sum_{j=1}^m \int_{y \in U_y} \left( \int_{\{-t-1-\epsilon-g_1(z_0) < 2p_j \log |\tilde{z}| < -t+\epsilon-g_1(z_0)\}} |\tilde{z}_j|^{2k_j-2q_j} \frac{|f_j|^2}{|d_j|^2 |h_j|^2} e^{-\tilde{\psi}_{2,l}(z,w)} |d\tilde{z}_j|^2 \right) \times \\ &\quad |\tilde{h}_j|^2 e^{-\varphi_2} |dw|^2. \end{aligned}$$

Note that  $2k_j - 2q_j = -2$  for any  $1 \leq j \leq m_1$  and  $2k_j - 2q_j \geq 0$  for  $m_1 < j \leq m$ . When  $t$  is large enough, for any  $j$ , the integral

$$\int_{\{-t-1-\epsilon-g_1(z_0) < 2p_j \log |\tilde{z}| < -t+\epsilon-g_1(z_0)\}} |\tilde{z}_j|^{2k_j-2q_j} \frac{|f_j|^2}{|d_j|^2|h_j|^2} e^{-\tilde{\psi}_{2,l}(z,w)} |d\tilde{z}_j|^2$$

is uniformly bounded with respect to  $t$ . It follows from (31) that  $\int_{U_y} |\tilde{h}_j|^2 e^{-\varphi_2} |dw|^2 < +\infty$  for any  $j$ . Then, by Fatou's lemma and Lemma 13 (we use Lemma 13 for the case  $n = 1$ ), we have

$$\begin{aligned} & \limsup_{t \rightarrow +\infty} \int_{V_0 \times U_y} \mathbb{I}_{\{-t-1 < \pi_1^*(G) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} \\ & \leq \limsup_{t \rightarrow +\infty} \sum_{j=1}^m \int_{y \in U_y} \left( \int_{\{-t-1-\epsilon-g_1(z_0) < 2p_j \log |\tilde{z}| < -t+\epsilon-g_1(z_0)\}} |\tilde{z}_j|^{2k_j-2q_j} \times \right. \\ & \quad \left. \frac{|f_j|^2}{|d_j|^2|h_j|^2} e^{-\tilde{\psi}_{2,l}(z,w)} |d\tilde{z}_j|^2 \right) |\tilde{h}_j|^2 e^{-\varphi_2} |dw|^2 \\ & \leq \sum_{j=1}^m \int_{y \in U_y} \limsup_{t \rightarrow +\infty} \left( \int_{\{-t-1-\epsilon-g_1(z_0) < 2p_j \log |\tilde{z}| < -t+\epsilon-g_1(z_0)\}} |\tilde{z}_j|^{2k_j-2q_j} \times \right. \\ & \quad \left. \frac{|f_j|^2}{|d_j|^2|h_j|^2} e^{-\tilde{\psi}_{2,l}(z,w)} |d\tilde{z}_j|^2 \right) |\tilde{h}_j|^2 e^{-\varphi_2} |dw|^2 \\ & \leq \sum_{j=1}^{m_1} \int_{y \in U_y} \frac{2\pi(1+2\epsilon)|a_j|^2}{p_j|d_j|^2} e^{-\tilde{\psi}_{2,l}(z_0,w)} |\tilde{h}_j|^2 e^{-\varphi_2} |dw|^2. \end{aligned}$$

Let  $\epsilon \rightarrow +\infty$ , we have

$$\begin{aligned} & \limsup_{t \rightarrow +\infty} \int_{V_0 \times U_y} \mathbb{I}_{\{-t-1 < \pi_1^*(G) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} \\ & \leq \sum_{j=1}^{m_1} \int_{y \in U_y} \frac{2\pi|a_j|^2}{p_j|d_j|^2} e^{-\tilde{\psi}_{2,l}(z_0,w)} |\tilde{h}_j|^2 e^{-\varphi_2} |dw|^2. \end{aligned}$$

As  $y$  and  $U_y$  are arbitrarily chosen, we have

$$\begin{aligned} & \limsup_{t \rightarrow +\infty} \int_{\tilde{M}_s} \mathbb{I}_{\{-t-1 < \pi_1^*(G) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} \\ & \leq \sum_{j=1}^{m_1} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_{\{(z_j) \times Y\} \cap \tilde{M}_s} |F_j|^2 e^{-\tilde{\psi}_{2,l}(z_0,w) - \varphi_2}. \end{aligned} \tag{36}$$

Since  $v_{t,1}(\pi_1^*(G)) \geq \pi_1^*(G)$  and  $c(t)e^{-t}$  is decreasing with respect to  $t$ , it follows from (35) and (36) that

$$\begin{aligned}
& \limsup_{t \rightarrow +\infty} \int_{\tilde{M}_s} |F_{t,l,s} - (1 - b_{t,1}(\pi_1^*(G)))F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) \\
& \leq \limsup_{t \rightarrow +\infty} \int_{\tilde{M}_s} |F_{t,l,s} - (1 - b_{t,1}(\pi_1^*(G)))F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + v_{t,1}(\pi_1^*(G))} c(-v_{t,1}(\pi_1^*(G))) \\
& \leq \limsup_{t \rightarrow +\infty} \left( \int_0^{t+1} c(t_1) e^{-t_1} dt_1 \right) \int_{\tilde{M}_s} \mathbb{I}_{\{-t-1 < \pi_1^*(G) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} \\
& \leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{j=1}^{m_1} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_{\{z_j\} \times Y \cap \tilde{M}_s} |F_j|^2 e^{-\tilde{\psi}_{2,l}(z_0, w) - \varphi_2}.
\end{aligned} \tag{37}$$

Note that  $k_j - q_j = -1$  for  $1 \leq j \leq m_1$ ,  $k_j - q_j \geq 0$  for  $m_1 < j \leq m$ , when  $t$  is large enough, we have

$$\begin{aligned}
& \int_{\{\pi_1^*(G) < -t\} \times Y \cap \tilde{M}_s} |F|^2 e^{\pi_1^*(-2 \log |g| + G) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2)} c(-\pi_1^*(G)) \\
& \leq \sum_{j=1}^m C_1 \int_{\{G < -t\}} |z_j|^{2k_j} |f_j|^2 e^{-2 \log |g_0|} e^{-G} c(-G) \int_{\{z_j\} \times Y} |F_j|^2 e^{-\varphi_2} \\
& \leq C_2 \sum_{j=1}^m \int_t^{+\infty} c(t_1) e^{-t_1} dt_1 \int_{\{z_j\} \times Y \cap \tilde{M}_s} |F_j|^2 e^{-\varphi_2} < +\infty,
\end{aligned}$$

where  $C_1$  and  $C_2$  are constants. Hence we have

$$\limsup_{t \rightarrow +\infty} \int_{\tilde{M}_s} |(1 - b_{t,1}(\pi_1^*(G)))F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) < +\infty.$$

Combining with (37), we know

$$\limsup_{t \rightarrow +\infty} \int_{\tilde{M}_s} |F_{t,l,s}|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) < +\infty.$$

By Lemma 9, we know there exists a subsequence of  $\{F_{t,l,s}\}_{t \rightarrow +\infty}$  (still denoted by  $\{F_{t,l,s}\}_{t \rightarrow +\infty}$ ) compactly convergent to a holomorphic  $(n, 0)$  form  $F_{l,s}$  on  $M_s$ . It follows from (37) and Fatou's lemma that

$$\begin{aligned}
& \int_{\tilde{M}_s} |F_{l,s}|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) \\
& \leq \liminf_{l \rightarrow +\infty} \int_{\tilde{M}_s} |F_{t,l,s} - (1 - b_{t,1}(\pi_1^*(G)))F|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) \\
& \leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{j=1}^{m_1} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_{\{z_j\} \times Y \cap \tilde{M}_s} |F_j|^2 e^{-\tilde{\psi}_{2,l}(z_0, w) - \varphi_2} < +\infty.
\end{aligned} \tag{38}$$

As  $\tilde{\psi}_{2,l}$  is decreasingly convergent to  $\tilde{\psi}_2$ , when  $l \rightarrow +\infty$ , and  $\int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \pi_2^*(\varphi_2)} < +\infty$  for any  $j$ ,

$$\begin{aligned} & \int_{\tilde{M}_s} |F_{l,s}|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) \\ & \leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{j=1}^{m_1} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_{(\{z_j\} \times Y) \cap \tilde{M}_s} |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2} < +\infty. \end{aligned} \quad (39)$$

As  $\tilde{\psi}_{2,l}$  is decreasingly convergent to  $\tilde{\psi}_2$ , for any compact subset  $K \subset \tilde{M}_s$ , we have

$$\inf_l \inf_K e^{-\tilde{\psi}_{2,l}} \geq \inf_K e^{-\tilde{\psi}_{2,1}} > 0,$$

then it follows from (39) that

$$\sup_l \int_K |F_{l,s}|^2 e^{\pi_1^*(-2 \log |g|) - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) < +\infty.$$

Hence it follows from Lemma 9 and diagonal method that there exists a subsequence of  $\{F_{l,s}\}_{l \rightarrow +\infty}$  (still denoted by  $\{F_{l,s}\}_{l \rightarrow +\infty}$ ) compactly convergent to a holomorphic  $(n, 0)$  form  $F_s$  on  $\tilde{M}_s$ . It follows from (31), (39) and Fatou's lemma that

$$\begin{aligned} & \int_{\tilde{M}_s} |F_s|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_2 - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) \\ & \leq \liminf_{l \rightarrow +\infty} \int_{\tilde{M}_s} |F_{l,s}|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) \\ & \leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{j=1}^{m_1} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_{(\{z_j\} \times Y) \cap \tilde{M}_s} |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2} \\ & \leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{j=1}^{m_1} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2} < +\infty. \end{aligned} \quad (40)$$

Again using Lemma 9 and diagonal method, we know that there exists a subsequence of  $\{F_s\}_{s \rightarrow +\infty}$  (still denoted by  $\{F_s\}_{s \rightarrow +\infty}$ ) compactly convergent to a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $\tilde{M}$ . It follows from (40) and Fatou's lemma that

$$\begin{aligned} & \int_{\tilde{M}} |\tilde{F}|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_2 - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) \\ & \leq \liminf_{s \rightarrow +\infty} \int_{M_s} |F_{l,s}|^2 e^{\pi_1^*(-2 \log |g|) - \tilde{\psi}_{2,l} - \pi_2^*(\varphi_2) + \pi_1^*(G)} c(-\pi_1^*(G)) \\ & \leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{j=1}^{m_1} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2} < +\infty. \end{aligned} \quad (41)$$

Note that  $N + \pi_1^*(G) + \pi_1^*(\varphi_1) = 2 \log |g| + \tilde{\psi}_2$ . We have

$$\begin{aligned} & \int_{\tilde{M}} |\tilde{F}|^2 e^{-\pi_1^*(\varphi_1) - N - \pi_2^*(\varphi_2)} c(-\pi_1^*(G)) \\ & \leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{j=1}^{m_1} \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2} < +\infty. \end{aligned} \quad (42)$$

It follows from  $(F_{t,l,s} - F, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0 \cap \tilde{M}_s$ , Lemma 10 and the compactly convergence of all the sequences, we know that

$$(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$$

for any  $y \in Y$ .

Lemma 25 is proved.

Let  $\Omega_j$  be an open Riemann surface, which admits a nontrivial Green function  $G_{\Omega_j}$  for any  $1 \leq j \leq n_1$ . Let  $Y$  be an  $n_2$ -dimensional weakly pseudoconvex Kähler manifold, and let  $K_Y$  be the canonical (holomorphic) line bundle on  $Y$ . Let  $M = (\prod_{1 \leq j \leq n_1} \Omega_j) \times Y$  be an  $n$ -dimensional complex manifold, where  $n = n_1 + n_2$ , and let  $K_M$  be the canonical (holomorphic) line bundle on  $M$ . Let  $\pi_1, \pi_{1,j}$  and  $\pi_2$  be the natural projections from  $M$  to  $\prod_{1 \leq j \leq n_1} \Omega_j$ ,  $\Omega_j$  and  $Y$  respectively.

Let  $\tilde{M} \subset M$  be an  $n$ -dimensional weakly pseudoconvex Kähler manifold satisfying that  $Z_0 \subset \tilde{M}$ . Let  $F$  be a holomorphic  $(n, 0)$  form on a neighborhood  $U_0 \subset \tilde{M}$  of  $Z_0$ .

Let  $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$  be a discrete subset of  $\Omega_j$  for any  $j \in \{1, \dots, n_1\}$ , where  $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$ . Denote that  $Z_0 := (\prod_{1 \leq j \leq n_1} Z_j) \times Y$ .

Let  $p_{j,k}$  be a positive number for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$ , which satisfies that  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $1 \leq j \leq n_1$ . Denote that

$$G := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}.$$

Let  $N \leq 0$  be a plurisubharmonic function on  $\tilde{M}$ . Denote  $\psi := G + N$ .

Let  $\varphi_X$  be a Lebesgue measurable function on  $\prod_{1 \leq j \leq n_1} \Omega_j$ . Assume that  $\pi_1^*(\varphi_X) + N$  is a plurisubharmonic function on  $\tilde{M}$  and  $(\pi_1^*(\varphi_X) + N)|_{Z_0} \not\equiv -\infty$ . Denote  $\Phi = \pi_1^*(\varphi_X) + N$ . Let  $\varphi_Y$  be a plurisubharmonic function on  $Y$ , and

$$\varphi := \pi_1^*(\varphi_X) + \pi_2^*(\varphi_Y)$$

on  $M$ .

Let  $w_{j,k}$  be a local coordinate on a neighborhood  $V_{z_{j,k}} \Subset \Omega_j$  of  $z_{j,k} \in \Omega_j$  satisfying  $w_{j,k}(z_{j,k}) = 0$  for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$ , where  $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$  for any  $j$  and  $k \neq k'$ .

Denote that  $\tilde{I}_1 := \{\beta = (\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$ ,  $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$  and  $w_\beta := (w_{1,\beta_1}, \dots, w_{n_1,\beta_{n_1}})$  is a local coordinate on  $V_\beta$  of  $z_\beta := (z_{1,\beta_1}, \dots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \leq j \leq n_1} \Omega_j$  for any  $\beta = (\beta_1, \dots, \beta_{n_1}) \in \tilde{I}_1$ . Denote that  $E_\beta := \left\{ \alpha = (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$  and  $\tilde{E}_\beta := \left\{ \alpha = (\alpha_1, \dots, \alpha_{n_1}) : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} \geq 1 \& \alpha_j \in \mathbb{Z}_{\geq 0} \right\}$  for any  $\beta \in \tilde{I}_1$ . Let  $U_0$  be an open neighborhood of  $Z_0$  in  $\tilde{M}$ . Let  $f$  be a holomorphic  $(n, 0)$  form on  $U_0$  such that

$$f = \sum_{\alpha \in \tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on  $U_0 \cap (V_\beta \times Y)$ , where  $f_{\alpha,\beta}$  is a holomorphic  $(n_2, 0)$  form on  $Y$  for any  $\alpha \in \tilde{E}_\beta$  and  $\beta \in \tilde{I}_1$ . Denote that

$$c_{j,k} := \exp \lim_{z \rightarrow z_{j,k}} \left( \frac{\sum_{1 \leq k_1 < \tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any  $j \in \{1, \dots, n\}$  and  $1 \leq k < \tilde{m}_j$  (following from Lemmas 3 and 4, we get that the above limit exists).

**Lemma 26** *Let  $c$  be a positive function on  $(0, +\infty)$  such that  $\int_0^{+\infty} c(t)e^{-t} dt < +\infty$  and  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ . Assume that  $f_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)_y$  for any  $y \in Y$ , where  $\alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , and*

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} < +\infty. \quad (43)$$

Then there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, (z, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z,y)}$  for any  $(z, y) \in Z_0$  and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}.$$

**Remark 20** If we don't assume that  $f_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)_y$  for any  $y \in Y$ , where  $\alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , we can still find a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, (z, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(G))_{(z,y)}$  for any  $(z, y) \in Z_0$  and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}$$

as long as (43) holds.

**Proof (Proof of Lemma 26)** As  $\psi = G + N \leq G$ , we know that  $c(-\psi)e^{-\varphi} \leq c(-G)e^{-\varphi-N}$ . To prove Lemma 26, it suffice to prove that there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_z$  for any  $z \in Z_0$  and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi-N} c(-G) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}.$$

The following Remark shows that it suffices to prove Proposition 26 for the case  $\tilde{m}_j < +\infty$  for any  $j \in \{1, \dots, n_1\}$ .

**Remark 21** Assume that Proposition 26 holds for the case  $\tilde{m}_j < +\infty$  for any  $j \in \{1, \dots, n_1\}$ . For any  $j \in \{1, \dots, n_1\}$ , it follows from Lemma 6 that there exists a sequence of Riemann surfaces  $\{\Omega_{j,l}\}_{l \in \mathbb{Z}_{\geq 1}}$ , which satisfies that  $\Omega_{j,l} \Subset \Omega_{j,l+1} \Subset \Omega_j$  for any  $l$ ,  $\cup_{l \in \mathbb{Z}_{\geq 1}} \Omega_{j,l} = \Omega_j$  and  $\{G_{\Omega_{j,l}}(\cdot, z) - G_{\Omega_j}(\cdot, z)\}_{l \in \mathbb{Z}_{\geq 1}}$  is decreasingly convergent to 0 with respect to  $l$  for any  $z \in \Omega_j$ . As  $Z_j$  is a discrete subset of  $\Omega_j$ ,  $Z_{j,l} := \Omega_{j,l} \cap Z_j$  is a set of finite points. Denote that  $\tilde{M}_l := ((\prod_{1 \leq j \leq n_1} \Omega_{j,l}) \times Y) \cap \tilde{M}$  and  $G_l := \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( \sum_{z_{j,k} \in Z_{j,l}} 2p_{j,k} G_{\Omega_{j,l}}(\cdot, z_{j,k}) \right) \right\}$  on  $\tilde{M}_l$ . Note that  $\tilde{M}_l$  is weakly pseudoconvex Kähler manifold. Denote that

$$c_{j,k,l} = \exp \lim_{z \rightarrow z_{j,k}} \left( \frac{\sum_{z_{j,k_1} \in Z_{j,l}} p_{j,k_1} G_{\Omega_{j,l}}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any  $1 \leq j \leq n_1$ ,  $l \in \mathbb{Z}_{\geq 1}$  and  $1 \leq k < \tilde{m}_j$  satisfying  $z_{j,k} \in Z_{j,l}$ . Hence  $c_{j,k,l}$  is decreasingly convergent to  $c_{j,k}$  with respect to  $l$ ,  $G_l$  is decreasingly convergent to  $G$  with respect to  $l$  and  $\cup_{l \in \mathbb{Z}_{\geq 1}} \tilde{M}_l = \tilde{M}$ .

Then there exists a holomorphic  $(n, 0)$  form  $F_l$  on  $\tilde{M}_l$  such that  $(F_l - f, (z_\beta, y)) \in (O(K_{\tilde{M}_l}) \otimes (\mathcal{I}(G_l) + \pi_2^*(\varphi_Y)))_{(z_\beta, y)} = (O(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z_\beta, y)}$  for any  $\beta \in \{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1 \leq j \leq n_1} \Omega_{j,l}\}$  and  $y \in Y$ , and  $F_l$  satisfies

$$\begin{aligned} & \int_{\tilde{M}_l} |F_l|^2 e^{-\varphi-N} c(-G_l) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1 \leq j \leq n_1} \Omega_{j,l}\}} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \\ & < +\infty. \end{aligned}$$

Since  $\psi \leq \psi_l$  and  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ , we have

$$\begin{aligned}
& \int_{\tilde{M}_l} |F_l|^2 e^{-\varphi - N - G_l + G} c(-G) \\
& \leq \int_{\tilde{M}_l} |F_l|^2 e^{-\varphi - N} c(-G_l) \\
& \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j + 2}}.
\end{aligned} \tag{44}$$

Note that  $\psi$  is continuous on  $M \setminus Z_0$ ,  $\psi_l$  is continuous on  $M_l \setminus Z_0$  and  $Z_0$  is a closed complex submanifold of  $M$ . For any compact subset  $K$  of  $M \setminus Z_0$ , there exist  $l_K > 0$  such that  $K \subseteq M_{l_K}$  and  $C_K > 0$  such that  $\frac{e^{\varphi + N + G_l - G}}{c(-G)} \leq C_K$  for any  $l \geq l_K$ . It follows from Lemma 9 and the diagonal method that there exists a subsequence of  $\{F_l\}$ , denoted still by  $\{F_l\}$ , which is uniformly convergent to a holomorphic  $(n, 0)$  form  $F$  on any compact subset of  $M$ . It follows from Fatou's Lemma and inequality (44) that

$$\begin{aligned}
\int_{\tilde{M}} |F|^2 e^{-\varphi - N} c(-G) &= \int_{\tilde{M}} \lim_{l \rightarrow +\infty} |F_l|^2 e^{-\varphi - N - G_l + G} c(-G) \\
&\leq \liminf_{l \rightarrow +\infty} \int_{\tilde{M}_l} |F_l|^2 e^{-\varphi - N - G_l + G} c(-G) \\
&\leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j + 2}}.
\end{aligned}$$

Since  $\{F_l\}$  is uniformly convergent to  $F$  on any compact subset of  $\tilde{M}$  and  $(F_l - f, (z_\beta, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z_\beta, y)}$  for any  $\beta \in \{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1 \leq j \leq n_1} \Omega_{j,l}\}$  and  $y \in Y$ , it follows from Lemma 10 that  $(F - f, (z_\beta, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z_\beta, y)}$  for any  $\beta \in \tilde{I}_1$  and  $y \in Y$ .

In the following, we assume that  $\tilde{m}_j < +\infty$  for any  $1 \leq j \leq n_1$ . Denote that  $m_j = \tilde{m}_j - 1$ .

As  $\tilde{M}$  is a weakly pseudoconvex Kähler manifold, there exists a sequence of weakly pseudoconvex Kähler manifolds  $\tilde{M}_s$  satisfying  $\tilde{M}_1 \Subset \tilde{M}_2 \dots \Subset \tilde{M}_s \Subset \dots \tilde{M}$  and  $\cup_{s \in \mathbb{N}^+} \tilde{M}_s = \tilde{M}$ .

Recall that  $\Phi = \pi_1^*(\varphi_X) + N \in Psh(M)$ . Denote  $\Phi_l = \max\{\Phi, -l\}$ , where  $l$  is a positive integer. We note that  $\Phi_l$  is a bounded plurisubharmonic function on  $\tilde{M}$ .

It follows from Lemmas 3 and 4 that there exists a local coordinate  $\tilde{w}_{j,k}$  on a neighborhood  $\tilde{V}_{z_{j,k}} \Subset V_{z_{j,k}}$  of  $z_{j,k}$  satisfying  $\tilde{w}_{j,k}(z_{j,k}) = 0$  and

$$|\tilde{w}_{j,k}| = \exp \left( \frac{\sum_{1 \leq k_1 \leq m_j} p_{j,k_1} G_{\Omega_j}(\cdot, z_{j,k_1})}{p_{j,k}} \right)$$

on  $\tilde{V}_{z_{j,k}}$ .

Denote that  $\tilde{V}_\beta := \prod_{1 \leq j \leq n_1} \tilde{V}_{j,\beta_j}$  for any  $\beta \in \tilde{I}_1$ . Let  $\tilde{f}$  be a holomorphic  $(n, 0)$  form on  $(\cup_{\beta \in \tilde{I}_1} \tilde{V}_\beta \times Y) \cap U_0$  satisfying

$$\tilde{f} = \sum_{\alpha \in E_\beta} c_{\alpha,\beta} \pi_1^*(\tilde{w}_\beta^\alpha d\tilde{w}_{1,\beta_1} \wedge d\tilde{w}_{2,\beta_2} \wedge \dots \wedge d\tilde{w}_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on  $(\tilde{V}_\beta \times Y) \cap U_0$ , where  $c_{\alpha,\beta} = \prod_{1 \leq j \leq n_1} \left( \lim_{z \rightarrow z_{j,\beta_j}} \frac{w_{j,\beta_j}(z)}{\tilde{w}_{j,\beta_j}(z)} \right)^{\alpha_j+1}$ . It follows from  $f_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)_y$  for any  $y \in Y$ , where  $\alpha \in \tilde{E}_\beta$  and  $\beta \in \tilde{I}_1$ , and Lemma 21 that

$$(f - \tilde{f}, (z, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z,y)} \quad (45)$$

for any  $(z, y) \in Z_0$ .

Denote that  $G_1 := \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$  on  $\prod_{1 \leq j \leq n_1} \Omega_j$ , where  $\tilde{\pi}_j$  is the natural projection from  $\prod_{1 \leq j \leq n_1} \Omega_j$  to  $\Omega_j$  and  $G = \pi_1^*(G_1)$ . It follows from Lemmas 4 and 5 that there exists  $t_0 > 0$  such that  $\{G_1 < -t_0\} \Subset \cup_{\beta \in \tilde{I}_1} \tilde{V}_\beta$ , which implies that  $\int_{\{G_1 < -t\} \cap \tilde{M}_s} |\tilde{f}|^2 < +\infty$ . It follows from (43),  $\Phi_l$  is bounded and Fubini's theorem that we know

$$\int_{\tilde{M}_s} \mathbb{I}_{\{-t-1 < G < -t\}} |\tilde{f}|^2 e^{-G - \pi_2^*(\varphi_Y) - \Phi_l} < +\infty.$$

Using Lemma 8, there exists a holomorphic  $(n, 0)$  form  $F_{l,s,t}$  on  $\tilde{M}_s$  such that

$$\begin{aligned} & \int_{\tilde{M}_s} |F_{l,s,t} - (1 - b_{t,1}(\psi)) \tilde{f}|^2 e^{-G - \pi_2^*(\varphi_Y) - \Phi_l + v_{t,1}(G)} c(-v_{t,1}(G)) \\ & \leq \left( \int_0^{t+1} c(t) e^{-t} dt \right) \int_{\tilde{M}_s} \mathbb{I}_{\{-t-1 < G < -t\}} |\tilde{f}|^2 e^{-G - \pi_2^*(\varphi_Y) - \Phi_l}, \end{aligned} \quad (46)$$

where  $t \geq t_0$ . Note that  $b_{t,1}(t_1) = 0$  for large enough  $t_1$ , then  $(F_{l,s,t} - \tilde{f}, (z, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z,y)}$  for any  $(z, y) \in Z_0 \cap \tilde{M}_s$ .

For any  $(z_\beta, y) \in Z_0 \cap \tilde{M}_s$ , letting  $(U_y, w)$  be a small local coordinated open neighborhood of  $y$  and shrinking  $\tilde{V}_\beta$  if necessary, we have  $\tilde{V}_\beta \times U_y \Subset U_0 \cap (V_\beta \times Y)$  for any  $\beta \in \tilde{I}_1$ . Assume that  $F_j = \tilde{h}_j(w) dw$  on  $U_y$ , where  $\tilde{h}_j$  is a holomorphic function on  $U_y$  and  $dw = dw_1 \wedge dw_2 \wedge \dots \wedge dw_n$ . There exists  $t_2 > t_1$  such that when  $t > t_2$ ,  $(\{G_1 < -t\} \times U_y) \cap (\tilde{V}_\beta \times U_y) \Subset (V_\beta \times Y) \cap \tilde{M}$ , for any  $\beta \in \tilde{I}_1$ .

Now we consider

$$\begin{aligned} & \int_{(\cup_{\beta \in \tilde{I}_1} \tilde{V}_\beta) \times U_y} \mathbb{I}_{\{-t-1 < G < -t\}} |\tilde{f}|^2 e^{-G - \pi_2^*(\varphi_Y) - \Phi_l} \\ & = \sum_{\beta \in \tilde{I}_1} \int_{w' \in U_y} \left( \int_{\{-t-1 < G_1 < -t\}} |f(\tilde{w}, w')|^2 \frac{e^{-\Phi_l(\tilde{w}, w')}}{\max_{1 \leq j \leq n} \{|\tilde{w}_{j,\beta_j}|^{2p_{j,\beta_j}}\}} \right) e^{-\varphi_Y}. \end{aligned} \quad (47)$$

Note that  $|c_{\alpha, \beta}| = \frac{1}{\prod_{1 \leq j \leq n_1} c_{j, \beta_j}^{\alpha_j+1}}$  and  $\int_{w' \in Y_s} |f_{\alpha, \beta}|^2 e^{-\varphi_Y} < +\infty$ . It follows from (47), Fatou's Lemma and Lemma 14 that we have

$$\begin{aligned}
& \limsup_{t \rightarrow +\infty} \int_{U_{\beta \in \tilde{I}_1} \tilde{V}_{\beta} \times U_y} \mathbb{I}_{\{-t-1 < G < -t\}} |\tilde{f}|^2 e^{-G - \pi_2^*(\varphi_Y) - \Phi_l} \\
&= \limsup_{t \rightarrow +\infty} \sum_{\beta \in \tilde{I}_1} \int_{w' \in U_y} \left( \int_{\{-t-1 < G_1 < -t\}} |f(\tilde{w}, w')|^2 \frac{e^{-\Phi_l(\tilde{w}, w')}}{\max_{1 \leq j \leq n} \{|\tilde{w}_{j, \beta_j}|^{2p_{j, \beta_j}}\}} \right) e^{-\varphi_Y} \\
&\leq \sum_{\beta \in \tilde{I}_1} \int_{w' \in U_y} \limsup_{t \rightarrow +\infty} \left( \int_{\{-t-1 < G_1 < -t\}} |f(\tilde{w}, w')|^2 \frac{e^{-\Phi_l(\tilde{w}, w')}}{\max_{1 \leq j \leq n} \{|\tilde{w}_{j, \beta_j}|^{2p_{j, \beta_j}}\}} \right) e^{-\varphi_Y} \\
&\leq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_{\beta}} \frac{(2\pi)^{n_1} \int_{U_y} |f_{\alpha, \beta}|^2 e^{-\varphi_Y - \Phi_l(z_{\beta}, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j+2}}. \tag{48}
\end{aligned}$$

Note that  $y$  and  $U_y$  are arbitrarily chosen,  $v_{t,1}(\psi) \geq \psi$  and  $c(t)e^{-t}$  is decreasing. Combining inequalities (46) and (48), then we have

$$\begin{aligned}
& \int_{\tilde{M}_s} |F_{l,s,t} - (1 - b_{t,1}(\psi)) \tilde{f}|^2 e^{-\pi_2^*(\varphi_Y) - \Phi_l} c(-G) \\
&\leq \int_{\tilde{M}_s} |F_{l,s,t} - (1 - b_{t,1}(\psi)) \tilde{f}|^2 e^{-G - \pi_2^*(\varphi_Y) - \Phi_l + v_{t,1}(G)} c(-v_{t,1}(G)) \\
&\leq \left( \int_0^{t+1} c(t_1) e^{-t_1} dt_1 \right) \int_{\tilde{M}_s} \mathbb{I}_{\{-t-1 < G < -t\}} |\tilde{f}|^2 e^{-G - \pi_2^*(\varphi_Y) - \Phi_l} \\
&\leq \left( \int_0^{t+1} c(t_1) e^{-t_1} dt_1 \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_{\beta}} \frac{(2\pi)^{n_1} \int_{\{(z_{\beta}) \times Y\} \cap \tilde{M}_s} |f_{\alpha, \beta}|^2 e^{-\varphi_Y - \Phi_l(z_{\beta}, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j+2}}. \tag{49}
\end{aligned}$$

Note that  $G$  is continuous on  $\tilde{M}_s \setminus Z_0$ . For any open set  $K \Subset \tilde{M}_s \setminus Z_0$ , as  $b_{t,1}(t_1) = 1$  for any  $t_1$  large enough and  $c(t_2)e^{-t_2}$  is decreasing with respect to  $t_2$ , we get that there exists a constant  $C_K > 0$  such that

$$\int_K |(1 - b_{t,1}(\psi)) \tilde{f}|^2 e^{-\pi_2^*(\varphi_Y) - \Phi_l} c(-G) \leq C_K \int_{\{G < -t_1\} \cap K} |\tilde{f}|^2 < +\infty$$

for any  $t > t_1$ , which implies that

$$\limsup_{t \rightarrow +\infty} \int_K |F_{l,s,t}|^2 e^{-\pi_2^*(\varphi_Y) - \Phi_l} c(-G) < +\infty.$$

Using Lemma 9 and the diagonal method, we obtain that there exists a subsequence of  $\{F_{l,s,t}\}_{t \rightarrow +\infty}$  denoted by  $\{F_{l,s,t_m}\}_{m \rightarrow +\infty}$  uniformly convergent on any compact subset of  $\tilde{M}_s \setminus Z_0$ . As  $Z_0$  is a closed complex submanifold of  $\tilde{M}$ , we obtain that  $\{F_{l,s,t_m}\}_{m \rightarrow +\infty}$  is uniformly convergent to a holomorphic  $(n, 0)$  form  $F_{l,s}$  on  $\tilde{M}_s$  on any compact subset of  $\tilde{M}_s$ . Then it follows from inequality (49) and Fatou's Lemma that

$$\begin{aligned} & \int_{\tilde{M}_s} |F_{l,s}|^2 e^{-\pi_2^*(\varphi_Y) - \Phi_l} c(-G) \\ &= \int_{\tilde{M}_s} \liminf_{m \rightarrow +\infty} |F_{l,s,t_m} - (1 - b_{t_m,1}(\psi)) \tilde{f}|^2 e^{-\pi_2^*(\varphi_Y) - \Phi_l} c(-G) \\ &\leq \liminf_{m \rightarrow +\infty} \int_{\tilde{M}_s} |F_{l,s,t_m} - (1 - b_{t_m,1}(\psi)) \tilde{f}|^2 e^{-\pi_2^*(\varphi_Y) - \Phi_l} c(-G) \\ &\leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_{\{z_\beta\} \times Y} |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \Phi_l(z_\beta, w')} }{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \\ &< +\infty. \end{aligned}$$

As  $\Phi_l(z_\beta, w')$  is decreasingly convergent to  $\Phi(z_\beta, w')$  for any  $\beta \in \tilde{I}_1$ , then we have

$$\begin{aligned} & \limsup_{l \rightarrow +\infty} \int_{\tilde{M}_s} |F_{l,s}|^2 e^{-\pi_2^*(\varphi_Y) - \Phi_l} c(-G) \\ &\leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_{\{z_\beta\} \times Y} |f_{\alpha,\beta}|^2 e^{-\varphi_Y - \Phi(z_\beta, w')} }{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \\ &= \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_{\{z_\beta\} \times Y} |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')} }{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \\ &< +\infty. \end{aligned} \tag{50}$$

Note that  $G$  is continuous on  $\tilde{M} \setminus Z_0$  and  $Z_0$  is a closed complex submanifold of  $\tilde{M}$ . Using Lemma 9, we obtain that there exists a subsequence of  $\{F_{l,s}\}_{l \rightarrow +\infty}$  (also denoted by  $\{F_{l,s}\}_{l \rightarrow +\infty}$ ) uniformly convergent to a holomorphic  $(n, 0)$  form  $F_s$  on  $\tilde{M}_s$  on any compact subset of  $\tilde{M}_s$ , which satisfies that

$$\begin{aligned} & \int_{\tilde{M}_s} |F_s|^2 e^{-\pi_2^*(\varphi_Y) - N - \varphi_X} c(-G) \\ &\leq \left( \int_0^{+\infty} c(t_1) e^{-t_1} dt_1 \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_{\{z_\beta\} \times Y} |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')} }{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}}. \end{aligned}$$

As  $\cup_{s \in \mathbb{Z}_{\geq 1}} \tilde{M}_s = \tilde{M}$ , we have

$$\begin{aligned}
& \limsup_{s \rightarrow +\infty} \int_{\tilde{M}_s} |F_s|^2 e^{-\varphi - N} c(-\psi) \\
& \leq \lim_{s \rightarrow +\infty} \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_{(\{z_\beta\} \times Y) \cap \tilde{M}_s} |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \varphi_X)(z_\beta, w')} }{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \\
& = \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \\
& < +\infty.
\end{aligned} \tag{51}$$

Note that  $\psi$  is continuous on  $\tilde{M} \setminus Z_0$ ,  $Z_0$  is a closed complex submanifold of  $\tilde{M}$  and  $\cup_{s \in \mathbb{Z}_{\geq 1}} \tilde{M}_s = \tilde{M}$ . Using Lemma 9 and the diagonal method, we get that there exists a subsequence of  $\{F_s\}$  (also denoted by  $\{F_s\}$ ) uniformly convergent to a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  on any compact subset of  $\tilde{M}$ . Then it follows from inequality (51) and Fatou's Lemma that

$$\begin{aligned}
\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) &= \int_{\tilde{M}} \liminf_{s \rightarrow +\infty} \mathbb{I}_{\tilde{M}_s} |F_{s'}|^2 e^{-\varphi} c(-\psi) \\
&\leq \liminf_{s \rightarrow +\infty} \int_{\tilde{M}_s} |F_{s'}|^2 e^{-\varphi} c(-\psi) \\
&\leq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}.
\end{aligned}$$

Following from Lemma 10, we have  $(F - f, (z, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z,y)}$  for any  $(z, y) \in Z_0$ .

Thus, Proposition 26 holds.

**Proof (Proof of Remark 20)** If we don't assume that  $f_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)_y$  for any  $y \in Y$ , where  $\alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , it follows from Lemma 21 that for any  $(z, y) \in Z_0$ , we will have

$$(f - \tilde{f}, (z, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(G))_{(z,y)}. \tag{52}$$

Replace the formula (45) by  $(f - \tilde{f}, (z, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(G))_{(z,y)}$ . Then if (43) holds, by the same proof as Proposition 26, we know Remark 20 holds.

### 3.3 Other Calculations

Let  $\Omega$  be an open Riemann surface with nontrivial Green functions. Let  $Z_\Omega = \{z_j : j \in \mathbb{N}_+ \& j < \gamma\}$  be a subset of  $\Omega$  of discrete points. Let  $Y$  be an  $(n - 1)$ -dimensional weakly pseudoconvex Kähler manifold. Denote  $M = \Omega \times Y$ . Let  $\pi_1$  and  $\pi_2$  be the natural projections from  $M$  to  $\Omega$  and  $Y$  respectively. Denote  $Z_0 := Z_\Omega \times Y$ . Denote  $Z_j := \{z_j\} \times Y$ .

Let  $\psi$  be a plurisubharmonic function on  $M$ . It follows from Siu's decomposition theorem that

$$dd^c \psi = \sum_{j \geq 1} 2p_j [Z_j] + \sum_{i \geq 1} \lambda_i [A_i] + R,$$

where  $[Z_j]$  and  $[A_i]$  are the currents of integration over an irreducible  $(n - 1)$ -dimensional analytic set, and where  $R$  is a closed positive current with the property that  $\dim E_c(R) < n - 1$  for every  $c > 0$ . We assume that  $p_j > 0$  for any  $1 \leq j < \gamma$ .

Then  $N := \psi - \pi_1^* \left( \sum_{j \geq 1} 2p_j G_\Omega(z, z_j) \right)$  is a plurisubharmonic function on  $M$ . We assume that  $N \leq 0$  and  $N|_{Z_j}$  is not identically  $-\infty$  for any  $j$ .

Let  $\varphi_1$  be a Lebesgue measurable function on  $\Omega$  such that  $\psi + \pi_1^*(\varphi)$  is a plurisubharmonic function on  $M$ . With Similar discussion as above, by Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j \geq 1} 2\tilde{q}_j [Z_j] + \sum_{i \geq 1} \tilde{\lambda}_i [\tilde{A}_i] + \tilde{R},$$

where  $\tilde{q}_j \geq 0$  for any  $1 \leq j < \gamma$ .

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function  $g$  on  $\Omega$  such that  $\text{ord}_{z_j}(g) = q_j := [\tilde{q}_j]$  for any  $z_j \in Z_\Omega$  and  $g(z) \neq 0$  for any  $z \notin Z_\Omega$ , where  $[q]$  equals to the integer part of the nonnegative real number  $q$ . Then we know that there exists a plurisubharmonic function  $\tilde{\psi}_2 \in Psh(M)$  such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \log |g|) + \tilde{\psi}_2.$$

Let  $\varphi_2 \in Psh(Y)$ . Denote  $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$ .

For  $1 \leq j < \gamma$ , let  $(V_j, \tilde{z}_j)$  be a local coordinated open neighborhood of  $z_j$  in  $\Omega$  satisfying  $V_j \Subset \Omega$ ,  $\tilde{z}_j(z_j) = 0$  under the local coordinate and  $V_j \cap V_k = \emptyset$  for any  $j \neq k$ . Denote  $V_0 := \cup_{1 \leq j < \gamma} V_j$ . We assume that  $g = d_j \tilde{z}_j^{q_j} h_j(z)$  on  $V_j$ , where  $d_j$  is a constant,  $h_j(z)$  is a holomorphic function on  $V_j$  and  $h_j(z_j) = 1$ .

Let  $c(t)$  be a positive measurable function on  $(0, +\infty)$  satisfying that  $c(t)e^{-t}$  is decreasing and  $\int_0^{+\infty} c(s)e^{-s} ds < +\infty$ .

**Lemma 27** Assume that  $c(t)$  is increasing near  $+\infty$ . Let  $F$  be a holomorphic  $(n, 0)$  form on  $M$  such that  $F = \sum_{l=k_j}^{+\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$  on  $V_j \times Y$  ( $1 \leq j < \gamma$ ), where  $k_j$  is a nonnegative integer,  $F_{j,l}$  is a holomorphic  $(n - 1, 0)$  form on  $Y$ , for any  $l, j$ , and  $F_j := F_{j,k_j} \not\equiv 0$  on  $Y$ . Denote that

$$I_F := \{j : k_j + 1 - q_j \leq 0 \& 1 \leq j < \gamma\}.$$

Assume that

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty.$$

Then  $k_j + 1 - q_j = 0$  for any  $j \in I_F$ , and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (53)$$

**Proof** As  $\tilde{\psi}_2$  and  $N$  are upper semi-continuous functions on  $M$ , there exists continuous functions  $\tilde{\psi}_{2,\alpha}$  and  $N_\beta$  on  $M$  decreasingly convergent to  $\tilde{\psi}_2$  and  $N$  respectively.

There exists some  $t_1 \geq 0$  such that  $c(t)$  is increasing on  $[t_1, +\infty)$ . Recall that  $\psi = \pi_1^*(\sum_{j \geq 1} 2p_j \pi_1^*(G_\Omega(z, z_j))) + N$  and denote  $G = \sum_{j \geq 1} 2p_j \pi_1^*(G_\Omega(z, z_j))$ . For any  $t > t_1$ ,

$$\begin{aligned} & \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \\ &= \int_{\{G+N < -t\}} |F|^2 e^{-2\pi_1^*(\log|g|) - \tilde{\psi}_2 + \pi_1^*(G) + N - \pi_2^*(\varphi_2)} c(-(G + N)) \\ &\geq \int_{\{G+N_\beta < -t\}} |F|^2 e^{-2\pi_1^*(\log|g|) - \tilde{\psi}_{2,\alpha} + \pi_1^*(G) + N - \pi_2^*(\varphi_2)} c(-(G + N_\beta)). \end{aligned} \quad (54)$$

For any  $y \in Y$ , let  $(U_y, w)$  be a local coordinated open neighborhood of  $y$  in  $Y$  satisfying  $U_y \Subset Y$ . We assume that  $F = \tilde{z}_j^{k_j} \tilde{h}_j(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$  on  $V_j \times U_y$ .

Let  $m \in \mathbb{N}$ . For any  $0 < \epsilon < \frac{1}{2}$ , let  $s_0$  be large enough and shrink  $U_y$  if necessary such that,

(1) for any  $j \in \{1, 2, \dots, m\}$ ,

$$\tilde{V}_j := \{|\tilde{z}_j| < s_0, z \in V_j\} \Subset V_j,$$

(2) for any  $j \in \{1, 2, \dots, m\}$  and any  $w \in U_y$ , we have

$$\sup_{\tilde{z} \in \tilde{V}_j} |\tilde{\psi}_{2,\alpha}(\tilde{z}, w) - \tilde{\psi}_{2,\alpha}(z_0, w)| < \epsilon.$$

(3) Denote  $H_j(\tilde{z}_j, w) := G - 2p_j \log|\tilde{z}_j| + N_\beta(\tilde{z}_j, w) + \epsilon$  on  $\tilde{V}_j \times U_y$ . Then for any  $w \in U_y$ , we have

$$\sup_{\tilde{z} \in \tilde{V}_j} |H(\tilde{z}_j, w) - H(z_j, w)| < \epsilon.$$

(4) Recall that  $g = d_j \tilde{z}^{q_j} h_j(\tilde{z}_j)$  on  $V_j$ , where  $d_j$  is a constant,  $h_j(\tilde{z})$  is a holomorphic function on  $V_j$  and  $h(z_j) = 1$ . We assume that

$$\sup_{\tilde{z} \in \tilde{V}_j} |h_j(\tilde{z}_j) - 1| < \epsilon.$$

(5) Denote that  $2g_j(\tilde{z}_j) = G - 2p_j \log|\tilde{z}_j|$  on  $\tilde{V}_j$ . Note that  $g_j$  is a harmonic function on  $\tilde{V}_j$ .

It follows from Lemma 5 that there exists  $t' > t_1$  such that

$$\left( \{G + N_\beta < -t\} \cap (\tilde{V}_j \times U_y) \right) \Subset \tilde{V}_j \times U_y,$$

for any  $t > t'$ . We also have  $G + N_\beta \leq 2p_j \log |\tilde{z}_j| + H_j(z_j, w)$  on  $V_j \times U_y$ .

Following from (54), for any  $t > t'$ , direct calculation shows that

$$\begin{aligned} & \int_{\{\psi < -t\} \cap (V_0 \times U_y)} |F|^2 e^{-\varphi} c(-\psi) \\ & \geq \int_{\{G + N_\beta < -t\} \cap (V_0 \times U_y)} |F|^2 e^{-2\pi_1^*(\log |g|) - \tilde{\psi}_{2,\alpha} + \pi_1^*(G) + N - \pi_2^*(\varphi_2)} c(-(G + N_\beta)) \\ & \geq \sum_{j=1}^m \int_{\{2p_j \log |\tilde{z}_j| + H_j(z_j, w) < -t\} \cap (\tilde{V}_j \times U_y)} |\tilde{z}_j|^{2k_j + 2p_j - 2q_j} \frac{|\tilde{h}_j(\tilde{z}_j, w)|^2}{|d_j|^2 |\tilde{h}_j(\tilde{z}_j)|^2} \times \\ & \quad e^{-\tilde{\psi}_{2,\alpha}(z_j, w) - \epsilon + 2g_j(\tilde{z}_j) + N - \pi_2^*(\varphi_2)} c(-2p_j \log |\tilde{z}_j| - H_j(z_j, w)) \\ & \geq \sum_{j=1}^m \int_{w \in U_y} \left( \int_{\{2p_j \log |\tilde{z}_j| + H_j(z_j, w) < -t\}} |\tilde{h}_j(\tilde{z}_j, w)|^2 |\tilde{z}_j|^{2k_j + 2p_j - 2q_j} e^{2g_j(\tilde{z}_j) + N} \times \right. \\ & \quad \left. c(-2p_j \log |\tilde{z}_j| - H_j(z_j, w)) |d\tilde{z}_j|^2 \right) \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j, w) - \epsilon - \varphi_2(w)}}{|d|^2 |1 + \epsilon|^2} |dw|^2 \\ & = \sum_{j=1}^m \int_{w \in U_y} \left( 2 \int_0^{\frac{-t - H_j(z_j, w)}{2p_j}} \int_0^{2\pi} |\tilde{h}_j(re^{-\theta}, w)|^2 r^{2k_j + 2p_j - 2q_j + 1} e^{2g_j(re^{i\theta}) + N(re^{i\theta}, w)} \times \right. \\ & \quad \left. c(-2p_j \log r - H_j(z_j, w)) d\theta dr \right) \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j, w) - \epsilon - \varphi_2(w)}}{|d|^2 |1 + \epsilon|^2} |dw|^2 \\ & \geq \sum_{j=1}^m \int_{w \in U_y} 4\pi \left( \int_0^{\frac{-t - H_j(z_j, w)}{2p_j}} r^{2k_j + 2p_j - 2q_j + 1} c(-2p_j \log r - H_j(z_j, w)) \right) \times \\ & \quad |\tilde{h}_j(z_j, w)|^2 e^{2g_j(z_j) + N(z_j, w)} \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j, w) - \epsilon - \varphi_2(w)}}{|d|^2 |1 + \epsilon|^2} |dw|^2 \\ & = \sum_{j=1}^m \frac{2\pi}{p_j |d_j|^2 |1 + \epsilon|^2} \left( \int_t^{+\infty} c(s) e^{-\left(\frac{k_j+1-q_j}{p_j}+1\right)s} ds \right) \times \\ & \quad \int_{w \in U_y} |\tilde{h}_j(z_j, w)|^2 e^{-\left(\frac{k_j+1-q_j}{p_j}+1\right)H_j(z_j, w)} e^{2g_j(z_j) + N(z_j, w) - \tilde{\psi}_{2,\alpha}(z_j, w) - \epsilon - \varphi_2(w)} |dw|^2. \end{aligned} \tag{55}$$

As  $\int_0^{+\infty} c(s) e^{-s} ds < +\infty$ , hence we have

$$\begin{aligned}
& \frac{\int_{\{\psi < -t\} \cap (V_0 \times U_y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\
& \geq \sum_{j=1}^m \frac{2\pi}{p_j |d_j|^2 |1 + \epsilon|^2} \left( \frac{\int_t^{+\infty} c(s) e^{-(\frac{k+1-q_2}{p}+1)s} ds}{\int_t^{+\infty} c(s) e^{-s} ds} \right) \times \\
& \quad \int_{w \in U_y} |\tilde{h}_j(z_j, w)|^2 e^{-(\frac{k_j+1-q_j}{p_j}+1)H_j(z_j, w)} e^{2g_j(z_j) + N(z_j, w) - \tilde{\psi}_{2,\alpha}(z_j, w) - \epsilon - \varphi_2(w)} |dw|^2. \tag{56}
\end{aligned}$$

Denote

$$I_m := \{1 \leq j \leq m : k_j + 1 - q_j \leq 0\}.$$

Note that

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty,$$

$N(z_j, w) \not\equiv -\infty$  and  $|\tilde{h}(z_j, w)|^2 \not\equiv 0$ . It follows from Lemma 15 and (56) that we know  $k_j + 1 - q_j = 0$  for any  $j \in I_m$  and

$$\begin{aligned}
& \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap (V_0 \times U_y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\
& \geq \sum_{j \in I_m} \frac{2\pi}{p_j |d_j|^2} \int_{w \in U_y} |\tilde{h}_j(z_j, w)|^2 e^{-H(z_j, w)} e^{2g_j(z_j) + N(z_j, w) - \tilde{\psi}_{2,\alpha}(z_j, w) - \varphi_2(w)} |dw|^2 \\
& = \sum_{j \in I_m} \frac{2\pi}{p_j |d_j|^2} \int_{w \in U_y} |\tilde{h}_j(z_j, w)|^2 e^{-2g_j(z_j, 0) + N_\beta(z_j, w)} e^{2g_j(z_j) + N(z_j, w) - \tilde{\psi}_{2,\alpha}(z_j, w) - \varphi_2(w)} |dw|^2 \\
& = \sum_{j \in I_m} \frac{2\pi}{p_j |d_j|^2} \int_{w \in U_y} |\tilde{h}_j(z_j, w)|^2 e^{N(z_j, w) - N_\beta(z_j, w) - \tilde{\psi}_{2,\alpha}(z_j, w) - \varphi_2(w)} |dw|^2.
\end{aligned}$$

By Monotone convergence theorem, letting  $\alpha \rightarrow +\infty$  and  $\beta \rightarrow +\infty$ , we have

$$\begin{aligned}
& \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap (V_0 \times U_y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\
& \geq \sum_{j \in I_m} \frac{2\pi}{p_j |d_j|^2} \int_{w \in U_y} |\tilde{h}_j(z_j, w)|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2(w)} |dw|^2. \tag{57}
\end{aligned}$$

As  $y$  and  $U_y$  are arbitrarily chosen, it follows from  $Y$  is a weakly pseudoconvex Kähler manifold that we have

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \sum_{j \in I_m} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2(w)}. \end{aligned} \quad (58)$$

Let  $m \rightarrow +\infty$  and we have

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2(w)}. \quad (59)$$

Especially, for any  $j \in I_F$ , we have

$$\int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j, w) - \varphi_2(w)} < +\infty.$$

Denote  $M = \prod_{1 \leq j \leq n_1} \Omega_j \times Y$ . Let  $p_{j,k}$  be a positive number for any  $1 \leq j \leq n_1$  and  $1 \leq k < \tilde{m}_j$ , which satisfies that  $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$  for any  $1 \leq j \leq n_1$ . Recall that

$$G := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}.$$

Let  $N \leq 0$  be a plurisubharmonic function on  $M$  satisfying that  $N|_{Z_0} \not\equiv -\infty$ . Denote  $\psi := G + N$ .

Let  $\varphi_X := \sum_{1 \leq j \leq n_1} \varphi_j(z)$ , where each  $\varphi_j$  is an upper semi-continuous function on  $\Omega_j$  satisfying  $\varphi_j(z_j) \neq -\infty$  for any  $z_j \in \Omega_j$ . We assume that  $\pi_1^*(\varphi_X) + N$  is a plurisubharmonic function on  $M$ . Let  $\varphi_Y$  be a plurisubharmonic function on  $Y$ . Denote  $\varphi := \pi_1^*(\varphi_X) + \pi_2^*(\varphi_Y)$ .

It follows from Lemmas 3 and 4 that there exists a local coordinate  $w_{j,k}$  on a neighborhood  $V_{z_{j,k}} \Subset \Omega_j$  of  $z_{j,k} \in \Omega_j$  satisfying  $w_{j,k}(z_{j,k}) = 0$  and

$$\log |w_{j,k}| = \frac{1}{p_{j,k}} \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})$$

for any  $j \in \{1, \dots, n_1\}$  and  $1 \leq k < \tilde{m}_j$ , where  $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$  for any  $j$  and  $k \neq k'$ .

Denote that  $\tilde{I}_1 := \{(\beta_1, \dots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \dots, n_1\}\}$ ,  $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$  for any  $\beta = (\beta_1, \dots, \beta_n) \in \tilde{I}_1$  and  $w_\beta := (w_{1,\beta_1}, \dots, w_{n,\beta_n})$  is a local coordinate on  $V_\beta$  of  $z_\beta := (z_{1,\beta_1}, \dots, z_{n,\beta_n}) \in M$ .

Let

$$G_1 = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on  $\prod_{1 \leq j \leq n_1} \Omega_j$ , where  $\tilde{\pi}_j$  is the natural projection from  $\prod_{1 \leq j \leq n_1} \Omega_j$  to  $\Omega_j$ . Note that  $G = \pi_1^*(G_1)$ .

Let  $F$  be a holomorphic  $(n, 0)$  form on  $\{\psi < -t_0\} \subset M$  for some  $t_0 > 0$  satisfying  $\int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$ . For any  $\beta \in \tilde{I}_1$ , it follows from Lemma 19 that there exists a sequence of holomorphic  $(n_2, 0)$  forms  $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$  on  $Y$  such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_{\beta}^{\alpha} dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on  $V_{\beta} \times Y$ .

Denote that  $E_{\beta} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \right\}$ ,  $E_{1,\beta} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} < 1 \right\}$  and  $E_{2,\beta} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} > 1 \right\}$ .

Assume that  $c(t)$  is increasing near  $+\infty$ .

**Lemma 28** If  $\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty$ , we have  $F_{\alpha,\beta} \equiv 0$  for any  $\alpha \in E_{1,\beta}$  and  $\beta \in \tilde{I}_1$ , and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_{\beta}} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_{\beta}, w')}.$$

**Proof** As  $\varphi_X$  is an upper semi-continuous functions on  $X := \prod_{1 \leq j \leq n_1} \Omega_j$  and  $N$  is an upper semi-continuous function on  $M$ , there exist continuous functions  $\varphi_{X,l}$  and  $N_{\gamma}$  on  $X$  and  $M$  decreasingly convergent to  $\varphi_X$  and  $N$  respectively.

When  $t$  is large enough,  $c(t)$  is increasing, then we have

$$\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \geq \int_{\{G + N_{\gamma} < -t\}} |F|^2 e^{-\pi_1^*(\varphi_{X,l}) - \pi_2^*(\varphi_Y)} c(-G - N_{\gamma}).$$

As  $Y$  is a weakly pseudoconvex Kähler manifold, there exist open weakly pseudoconvex Kähler manifolds  $Y_1 \Subset \dots \Subset Y_s \Subset Y_{s+1} \Subset \dots$  such that  $\cup_{s \in \mathbb{Z}_{\geq 1}} Y_s = Y$ .

Fix  $Y_s$ . For any  $\beta \in \tilde{I}_1$ , there exists  $t_{\beta} > t_0$  such that  $\{G + N_{\gamma} < -t_{\beta}\} \cap (V_{\beta} \times Y_s) \Subset (V_{\beta} \times Y_{s+1})$ .

On  $V_{\beta} \times Y_s$ , for any  $\epsilon > 0$ , there exists  $t_{\epsilon}$  large enough such that when  $t > t_{\epsilon}$ ,

(1) for any  $(w, w') \in \{G + N_{\gamma} < -t\}$ ,

$$|\varphi_{X,l}(w) - \varphi_{X,l}(z_{\beta})| < \epsilon.$$

(2) Denote  $H(w, w') := N_{\gamma}(w, w') + \epsilon$ . For any  $(w, w') \in \{G + N_{\gamma} < -t\}$ ,

$$|H(w, w') - H(z_{\beta}, w')| < \epsilon.$$

Then we have  $G + H(z_\beta, w') \geq G + N_s(w, w')$ , for any  $(w, w') \in \{G + N_\gamma < -t_\beta\} \cap (V_\beta \times Y_s)$ .

Let

$$G_1 = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on  $\prod_{1 \leq j \leq n_1} \Omega_j$ , where  $\tilde{\pi}_j$  is the natural projection from  $\prod_{1 \leq j \leq n_1} \Omega_j$  to  $\Omega_j$ . Note that  $G = \pi_1^*(G_1)$ .

For any  $t \geq \max\{t_\beta, t_\epsilon\}$ , note that  $\{G_1 < -t\} = \prod_{1 \leq j \leq n_1} \{|w_{j,\beta_j}| < e^{-\frac{t}{2p_{j,\beta_j}}}\}$  and  $F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$  on  $\{G_1 < -t\} \times Y$ , then we have

$$\begin{aligned} & \int_{\{\psi < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi) \\ & \geq \int_{\{G + N_\gamma < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\pi_1^*(\varphi_{X,l}) - \pi_2^*(\varphi_Y)} c(-G - N_\gamma) \\ & \geq e^{-\epsilon} \int_{\{G + H(z_\beta, w') < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\pi_1^*(\varphi_{X,l}(z_\beta, w')) - \pi_2^*(\varphi_Y)} c(-G - H(z_\beta, w')) \\ & = e^{-\epsilon} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \int_{w' \in Y_s} \left( \int_{\{G_1 + H(z_\beta, w') < -t\}} |w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}|^2 c(-G - H(z_\beta, w')) \right) \times \\ & \quad e^{-\varphi_{X,l}(z_\beta)} |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned} \tag{60}$$

Denote that  $q_\alpha := \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} - 1$ . It follows from Lemma 12 and inequality (60) that

$$\begin{aligned} & \int_{\{\psi < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi) \\ & \geq e^{-\varphi_{X,l}(z_\beta) - \epsilon} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \left( \int_t^{+\infty} c(s) e^{-(q_\alpha + 1)s} ds \right) \frac{(q_\alpha + 1)(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_{Y_s} |F_{\alpha,\beta}|^2 e^{-\varphi_Y - (q_\alpha + 1)H(z_\beta, w')}. \end{aligned}$$

It follows from  $\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty$  and Lemma 15 that

$$F_{\alpha,\beta} \equiv 0$$

for any  $\alpha$  satisfying  $q_\alpha < 0$  and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap (V_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq e^{-\varphi_{X,l}(z_\beta) - \epsilon} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_{Y_s} |F_{\alpha,\beta}|^2 e^{-\varphi_Y - H(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)}.$$

Note that  $H(w, w') := N_\gamma(w, w') + \epsilon$ . Letting  $\epsilon \rightarrow 0$ ,  $\gamma \rightarrow +\infty$  and  $s \rightarrow +\infty$ , we have

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap (V_\beta \times Y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')} . \end{aligned} \quad (61)$$

Note that  $V_\beta \cap V_{\tilde{\beta}} = \emptyset$  for any  $\beta \neq \tilde{\beta}$  and  $\{\psi_1 < -t_\beta\} \cap V_\beta \Subset V_\beta$  for any  $\beta \in \tilde{I}_1$ . It follows from inequality (61) that

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')} .$$

Thus, Lemma 28 holds.

Let  $\tilde{M}$  be an open complex submanifold of  $M$  satisfying that  $Z_0 = \{z_\beta : \beta \in \tilde{I}_1\} \times Y \subset \tilde{M}$ , and let  $K_{\tilde{M}}$  be the canonical (holomorphic) line bundle on  $M_1$ . Let  $F_1$  be a holomorphic  $(n, 0)$  form on  $\{\psi < -t_0\} \cap \tilde{M}$  for  $t_0 > 0$  satisfying that  $\int_{\{\psi < -t_0\} \cap \tilde{M}} |F_1|^2 e^{-\varphi} c(-\psi) < +\infty$ . For any  $\beta \in \tilde{I}_1$ , it follows from Lemma 20 that there exist a sequence of holomorphic  $(n_2, 0)$  forms  $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$  on  $Y$  and an open subset  $U_\beta$  of  $\{\psi < -t_0\} \cap \tilde{M} \cap (V_\beta \times Y)$  such that

$$F_1 = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on  $U_\beta$  and

$$\int_K |F_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$$

for any  $\alpha \in \mathbb{Z}_{\geq 0}^{n_1}$  and compact subset  $K$  of  $Y$ . Using the similar method in Lemma 28, we have the following Remark.

**Remark 22** If  $\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap \tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty$ , we have  $F_{\alpha,\beta} \equiv 0$  for any  $\alpha \in E_{1,\beta}$  and  $\beta \in \tilde{I}_1$ , and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap \tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N} .$$

**Proof** As  $\varphi_X$  is an upper semi-continuous function on  $X := \prod_{1 \leq j \leq n_1} \Omega_j$  and  $N$  is an upper semi-continuous function on  $M$ , there exist continuous functions  $\varphi_{X,l}$  and  $N_\gamma$  on  $X$  and  $M$  decreasingly convergent to  $\varphi_X$  and  $N$  respectively.

When  $t$  is large enough,  $c(t)$  is increasing, then we have

$$\int_{\{\psi < -t\} \cap \tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \geq \int_{\{G + N_\gamma < -t\} \cap \tilde{M}} |F|^2 e^{-\pi_1^*(\varphi_{X,l}) - \pi_2^*(\varphi_Y)} c(-G - N_\gamma).$$

As  $Y$  is a weakly pseudoconvex Kähler manifold, there exist open weakly pseudoconvex Kähler manifolds  $Y_1 \Subset \dots \Subset Y_s \Subset Y_{s+1} \Subset \dots$  such that  $\cup_{s \in \mathbb{Z}_{\geq 1}} Y_s = Y$ .

For any  $\beta \in \tilde{I}_1$  and any  $Y_s$ , there exists open subset  $\hat{V}_\beta \Subset V_\beta$  and  $t_{\beta,s} > t_0$  such that  $\{G + N_\gamma < -t_{\beta,s}\} \cap (\hat{V}_\beta \times Y_s) \Subset (\hat{V}_\beta \times Y_{s+1}) \Subset U_\beta$ .

On  $\hat{V}_\beta \times Y_s$ , for any  $\epsilon > 0$ , there exists  $t_\epsilon$  large enough such that when  $t > t_\epsilon$ ,  
(1) for any  $(w, w') \in \{G + N_\gamma < -t\}$ ,

$$|\varphi_{X,l}(w) - \varphi_{X,l}(z_\beta)| < \epsilon.$$

(2) Denote  $H(w, w') := N_\gamma(w, w') + \epsilon$ . For any  $(w, w') \in \{G + N_\gamma < -t\}$ ,

$$|H(w, w') - H(z_\beta, w')| < \epsilon.$$

Then we have  $G + H(z_\beta, w') \geq G + N_s(w, w')$ , for any  $(w, w') \in \{G + N_\gamma < -t\} \cap (\hat{V}_\beta \times Y_s)$ .

Recall that  $G_1 = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$  on  $\prod_{1 \leq j \leq n_1} \Omega_j$ , where  $\tilde{\pi}_j$  is the natural projection from  $\prod_{1 \leq j \leq n_1} \Omega_j$  to  $\Omega_j$ . Note that  $G = \pi_1^*(G_1)$ .

For any  $t \geq \max \{t_{\beta,s}, t_\epsilon\}$ , note that  $\{G_1 < -t\} = \prod_{1 \leq j \leq n_1} \left\{ |w_{j,\beta_j}| < e^{-\frac{t}{2p_{j,\beta_j}}} \right\}$  and  $F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$  on  $U_\beta$ , then we have

$$\begin{aligned} & \int_{\{\psi < -t\} \cap (\hat{V}_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi) \\ & \geq \int_{\{G + N_\gamma < -t\} \cap (\hat{V}_\beta \times Y_s)} |F|^2 e^{-\pi_1^*(\varphi_{X,l}) - \pi_2^*(\varphi_Y)} c(-G - N_\gamma) \\ & \geq e^{-\epsilon} \int_{\{G + H(z_\beta, w') < -t\} \cap (\hat{V}_\beta \times Y_s)} |F|^2 e^{-\pi_1^*(\varphi_{X,l}(z_\beta, w')) - \pi_2^*(\varphi_Y)} c(-G - H(z_\beta, w')) \\ & = e^{-\epsilon} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \int_{w' \in Y_s} \left( \int_{\{G_1 + H(z_\beta, w') < -t\}} |w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}|^2 c(-G - H(z_\beta, w')) \right) \times \\ & \quad e^{-\varphi_{X,l}(z_\beta)} |F_{\alpha,\beta}|^2 e^{-\varphi_Y}. \end{aligned} \tag{62}$$

Denote that  $q_\alpha := \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} - 1$ . It follows from Lemma 12 and inequality (62) that

$$\begin{aligned} & \int_{\{\psi < -t\} \cap (\hat{V}_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi) \\ & \geq e^{-\varphi_{X,l}(z_\beta) - \epsilon} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \left( \int_t^{+\infty} c(s) e^{-(q_\alpha + 1)s} ds \right) \frac{(q_\alpha + 1)(2\pi)^{n_1}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_{Y_s} |F_{\alpha,\beta}|^2 e^{-\varphi_Y - (q_\alpha + 1)H(z_\beta, w')}. \end{aligned}$$

It follows from  $\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap \tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty$  and Lemma 15 that

$$F_{\alpha, \beta} \equiv 0$$

for any  $\alpha$  satisfying  $q_\alpha < 0$  and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap (\hat{V}_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq e^{-\varphi_{X,l}(z_\beta) - \epsilon} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_{Y_s} |F_{\alpha, \beta}|^2 e^{-\varphi_Y - H(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)}.$$

Note that  $H(w, w') := N_\gamma(w, w') + \epsilon$ . Letting  $\epsilon \rightarrow 0$ ,  $\gamma \rightarrow +\infty$  and  $s \rightarrow +\infty$ , we have

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap (\hat{V}_\beta \times Y_s)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \\ & \geq \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j, \beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha, \beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}. \end{aligned} \quad (63)$$

Note that  $V_\beta \cap V_{\tilde{\beta}} = \emptyset$  implies that  $\hat{V}_\beta \cap \hat{V}_{\tilde{\beta}} = \emptyset$  for any  $\beta \neq \tilde{\beta}$ . It follows from inequality (63) that

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\} \cap \tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j, \beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1)} \int_Y |F_{\alpha, \beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}.$$

Thus, Remark 22 holds.

## 4 Proofs of Theorem 1, Propositions 1 and 2

In this section, we prove Theorem 1, Propositions 1 and 2.

### 4.1 Proof of Theorem 1

**Proof** We firstly give the proof of the sufficiency in Theorem 1.

As  $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1)$  is a plurisubharmonic function on  $M$ , by definition, we know that  $2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1$  is a subharmonic function on  $\Omega$ . By the construction of  $g \in O_\Omega$ , we have  $2u(z) := 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1 - 2 \log |g|$  is a subharmonic function on  $\Omega$  which satisfies  $v(dd^c u, z) \in [0, 1)$  for any  $z \in Z_\Omega$ . Then it follows from Lemma 23 that we know  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} =$

$\mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . It follows from Theorem 11 that we know the sufficiency part of Theorem 1 holds.

Now we prove the necessity part of Theorem 1.

Assume that  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ . Then according to Lemma 2, there exists a unique holomorphic  $(n, 0)$  form  $F$  on  $M$  satisfying  $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))|_{Z_0})$ , and  $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$  for any  $t \geq 0$ . Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that  $c$  is increasing near  $+\infty$ .

Following from Lemma 19, for any  $j \in \{1, 2, \dots, m\}$ , we assume that

$$F = \sum_{l=k_j}^{\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$$

on  $V_j \times Y$ , where  $k_j \in \mathbb{N}$ ,  $F_{j,l}$  is a holomorphic  $(n-1, 0)$  form on  $Y$  for any  $l \geq k_j$ , and  $\tilde{F}_j := F_{j,k_j} \not\equiv 0$ .

We firstly recall the construction of  $\psi$  and  $\varphi$ .

Recall that  $\psi = \pi_1^*\left(\sum_{j=1}^m 2p_j G_\Omega(z, z_j)\right) + N$  is a plurisubharmonic function on  $M$ . We assume that  $N \leq 0$  is a plurisubharmonic function on  $M$  and  $N|_{Z_j}$  is not identically  $-\infty$  for any  $j$ .  $\varphi_1$  is a Lebesgue measurable function on  $\Omega$  such that  $\psi + \pi_1^*(\varphi)$  is a plurisubharmonic function on  $M$ . We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surfaces, we have

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \log |g|) + \tilde{\psi}_2,$$

where  $g$  is a holomorphic function on  $\Omega$  and  $\tilde{\psi}_2 \in Psh(M)$ . Denote  $ord_{z_j}(g) = p_j$  and  $d_j := \lim_{z \rightarrow z_j} (g/\tilde{z}_j^{q_j})(z)$ .

Now we prove that  $\psi = \pi_1^*\left(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j)\right)$ . As  $G(h^{-1}(r))$  is linear and  $c$  is increasing near  $+\infty$ , using Lemma 27, we can get that  $k_j - q_j + 1 = 0$  for any  $j \in I_F$  and

$$\frac{\int_M |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}, \quad (64)$$

where  $I_F := \{j : k_j - q_j + 1 \leq 0\}$ . Especially,  $\sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty$  and  $\tilde{\psi}_2$  is not identically  $-\infty$  on  $Z_j$  for any  $j \in I_F$ .

Let  $V_j \times U_y$  be an open neighborhood of  $(z_j, y)$ . Assume that  $F = \tilde{z}_j^{k_j} h(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$  on  $V_j \times U_y$ , where  $h(z_j, w)dw = \tilde{F}_j$ . Note that  $h(z_j, w)$  is not identically zero on  $\{z_j\} \times U_y$ , there must exist  $\hat{w} \in U_y$  such that  $h(z_j, \hat{w}) \neq 0$ . Then we know  $|h(z_j, w)|^2$  has a positive lower bound on  $\tilde{V}_j \times U_{\hat{w}}$ , where  $\tilde{V}_j \times U_{\hat{w}}$  is a small open neighborhood of  $(z_j, \hat{w})$ . Then, according to Lemma 17,  $\psi \leq \pi_1^*\left(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j)\right)$  and  $c(t)$  is increasing near  $+\infty$ , we know that for any

$w \in U_{\tilde{w}}$ , we have  $\int_{\tilde{V}_j} |\tilde{z}_j^{k_j} d\tilde{z}_j|^2 e^{-\varphi_1} c(-\sum_{j=1}^m 2p_j G_\Omega(\cdot, z_j)) < +\infty$ . It follows from Lemma 18 that we have

$$\int_{\tilde{V}_j \times U_{\tilde{w}}} |\pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j)|^2 e^{-\varphi} c\left(-\pi_1^*\left(\sum_{j=1}^m 2p_j G_\Omega(\cdot, z_j)\right)\right) < +\infty.$$

Then by Lemma 22, we have

$$(F - \pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$$

for any  $j \in I_F$ ,  $y \in Y$ . And according to Lemma 22 we also have

$$(F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$$

for any  $j \in \{1, 2, \dots, m\} \setminus I_F$ ,  $y \in Y$ .

Denote that  $\tilde{\psi} := \pi_1^*(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j))$ ,  $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$ , and  $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$ . Then according to Lemma 25 ( $F = \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j)$  on  $V_j \times Y$  for  $j \in I_F$  and  $F \equiv 0$  on  $V_j \times Y$  for  $j \notin I_F$  in Lemma 25), there exists a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $M$  such that  $(\tilde{F} - \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $j \in I_F$ ,  $y \in Y$ ,  $(\tilde{F}, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $j \in \{1, 2, \dots, m\} \setminus I_F$ ,  $y \in Y$ , and

$$\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\varphi}_2(z_j, w)}. \quad (65)$$

Then  $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Combining inequality (64) with inequality (65), we have that

$$\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \geq \int_M |F|^2 e^{-\varphi} c(-\psi) \geq \int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}). \quad (66)$$

As  $c(t)e^{-t}$  is decreasing with respect to  $t$  and  $\psi \leq \tilde{\psi}$ , we have  $\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \geq \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi)$ . Then all “ $\geq$ ” in (66) should be “ $=$ ”. It follows from Lemma 7 that  $N = 0$  and  $\psi = \pi_1^*(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j))$ .

As  $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1)$  is a plurisubharmonic function on  $M$ , by definition, we know that  $2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1$  is a subharmonic function on  $\Omega$ . By the construction of  $g \in \mathcal{O}_\Omega$ , we have  $2u(z) := 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1 - 2 \log |g|$  is a subharmonic function on  $\Omega$  which satisfies  $v(dd^c u, z) \in [0, 1]$  for any  $z \in Z_0$ . Then it follows from Lemma 23 that we know  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ .

Then it follows from Theorem 11 that we know Theorem 1 holds.

## 4.2 Proof of Proposition 1

**Proof (Proof of Proposition 1)** Assume that  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ . Then according to Lemma 2, there exists a unique holomorphic  $(n, 0)$  form  $F$  on  $M$  satisfying  $(F - f) \in H^0(Z_0, (O(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)))|_{Z_0})$ , and  $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$  for any  $t \geq 0$ . Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that  $c$  is increasing near  $+\infty$ .

Following from Lemma 19, for any  $j \geq 1$ , we assume that

$$F = \sum_{l=k_j}^{\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$$

on  $V_j \times Y$ , where  $k_j \in \mathbb{N}$ ,  $F_{j,l}$  is a holomorphic  $(n-1, 0)$  form on  $Y$  for any  $l \geq k_j$ , and  $\tilde{F}_j := F_{j,k_j} \not\equiv 0$ .

We firstly recall the construction of  $\psi$  and  $\varphi$ .

Recall that  $\psi = \pi_1^*\left(\sum_{j \geq 1} 2p_j G_\Omega(z, z_j)\right) + N$  is a plurisubharmonic function on  $M$ . We assume that  $N \leq 0$  is a plurisubharmonic function on  $M$  and  $N|_{Z_j}$  is not identically  $-\infty$  for any  $j$ .  $\varphi_1$  is a Lebesgue measurable function on  $\Omega$  such that  $\psi + \pi_1^*(\varphi)$  is a plurisubharmonic function on  $M$ . We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surface, we have

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2,$$

where  $g$  is a holomorphic function on  $\Omega$  and  $\tilde{\psi}_2 \in Psh(M)$ . Denote  $ord_{z_j}(g) = p_j$  and  $d_j := \lim_{z \rightarrow z_j} (g/\tilde{z}_j^{q_j})(z)$ .

Now we prove that  $\psi_1 = 2 \sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j)$ . As  $G(h^{-1}(r))$  is linear and  $c$  is increasing near  $+\infty$ , using Lemma 27, we can get that  $k_j - q_j + 1 = 0$  for any  $j \in I_F$  and

$$\frac{\int_M |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}, \quad (67)$$

where  $I_F := \{j : k_j - q_j + 1 \leq 0\}$ . Especially,  $\sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty$  and  $\tilde{\psi}_2$  is not identically  $-\infty$  on  $Z_j$  for any  $j \in I_F$ .

Let  $V_j \times U_y$  be an open neighborhood of  $(z_j, y)$ . Assume that  $F = \tilde{z}_j^{k_j} h(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$  on  $V_j \times U_y$ , where  $h(z_j, w)dw = \tilde{F}_j$ . Note that  $h(z_j, w)$  is not identically zero on  $\{z_j\} \times U_y$ , there must exist  $\hat{w} \in U_y$  such that  $h(z_j, \hat{w}) \neq 0$ . Then we know  $|h(z_j, w)|^2$  has a positive lower bound on  $\tilde{V}_j \times U_{\hat{w}}$ , where  $\tilde{V}_j \times U_{\hat{w}}$  is a small open neighborhood of  $(z_j, \hat{w})$ . Then, according to Lemma 17,  $\psi \leq \pi_1^*(\sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j))$  and  $c(t)$  is increasing near  $+\infty$ , we know that for any  $w \in U_{\hat{w}}$ , we have

$\int_{\tilde{V}_j} |\tilde{z}_j^{k_j} d\tilde{z}_j|^2 e^{-\varphi_1} c(-\sum_{j=1}^m 2p_j G_\Omega(\cdot, z_j)) < +\infty$ . It follows from Lemma 18 that we have

$$\int_{\tilde{V}_j \times U_{\tilde{w}}} |\pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j)|^2 e^{-\varphi} c(-\pi_1^*(\sum_{j=1}^m 2p_j G_\Omega(\cdot, z_j))) < +\infty.$$

Then by Lemma 22, we have

$$(F - \pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (O(K_M) \otimes I(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$$

for any  $j \in I_F$ ,  $y \in Y$ . And according to Lemma 22 we also have

$$(F, (z_j, y)) \in (O(K_M) \otimes I(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$$

for any  $j \in \{1, 2, \dots, m\} \setminus I_F$ ,  $y \in Y$ .

Denote that  $\tilde{\psi} := \pi_1^*(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j))$ ,  $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$ , and  $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$ . Then according to Lemma 25 ( $F = \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j)$  on  $V_j \times Y$  for  $j \in I_F$  and  $F \equiv 0$  on  $V_j \times Y$  for  $j \notin I_F$  in Lemma 25), there exists a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $M$  such that  $(\tilde{F} - \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (O(K_M) \otimes I(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $j \in I_F$ ,  $y \in Y$ ,  $(\tilde{F}, (z_j, y)) \in (O(K_M) \otimes I(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $j \in \{1, 2, \dots, m\} \setminus I_F$ ,  $y \in Y$ , and

$$\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\varphi}_2(z_j, w)}. \quad (68)$$

Then  $(\tilde{F} - F, (z_j, y)) \in (O(K_M) \otimes I(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Combining inequality (67) with inequality (68), we have that

$$\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \geq \int_M |F|^2 e^{-\varphi} c(-\psi) \geq \int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}). \quad (69)$$

As  $c(t)e^{-t}$  is decreasing with respect to  $t$  and  $\psi \leq \tilde{\psi}$ , we have  $\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \geq \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi)$ . Then all “ $\geq$ ” in (69) should be “ $=$ ”. It follows from Lemma 7 that  $N = 0$  and  $\psi = \pi_1^*(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j))$ .

As  $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1)$  is a plurisubharmonic function on  $M$ , by definition, we know that  $2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1$  is a subharmonic function on  $\Omega$ . By the construction of  $g \in O_\Omega$ , we have  $2u(z) := 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1 - 2 \log |g|$  is a subharmonic function on  $\Omega$  which satisfies  $v(dd^c u, z) \in [0, 1)$

for any  $z \in Z_0$ . Then it follows from Lemma 23 that we know  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ .

Then it follows from Proposition 4 that we know Proposition 1 holds.

### 4.3 Proof of Proposition 2

**Proof** Assume that  $\tilde{G}(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ . Then according to Lemma 2, there exists a unique holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying  $(F - f) \in H^0(Z_0, (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))|_{Z_0})$ , and  $\tilde{G}(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$  for any  $t \geq 0$ . Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that  $c$  is increasing near  $+\infty$ .

We firstly recall the construction of  $\psi$  and  $\varphi$ .

Recall that  $\psi = \pi_1^*\left(\sum_{j \geq 1} 2p_j G_{\Omega}(z, z_j)\right) + N$  is a plurisubharmonic function on  $\tilde{M}$ . We assume that  $N \leq 0$  is a plurisubharmonic function on  $\tilde{M}$  and  $N|_{Z_j}$  is not identically  $-\infty$  for any  $j$ .  $\varphi_1$  is a Lebesgue measurable function on  $\Omega$  such that  $\psi + \pi_1^*(\varphi)$  is a plurisubharmonic function on  $\tilde{M}$ . We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surface, we have

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \log |g|) + \tilde{\psi}_2,$$

where  $g$  is a holomorphic function on  $\Omega$  and  $\tilde{\psi}_2 \in Psh(\tilde{M})$ . Denote  $ord_{z_j}(g) = p_j$  and  $d_j := \lim_{z \rightarrow z_j} (g/\tilde{z}_j^{q_j})(z)$ .

Following from Lemma 20, for any  $j \in \{1, 2, \dots, m\}$ , we assume that

$$F = \sum_{l=k_j}^{\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$$

on  $U_j \Subset (V_j \times Y) \cap \tilde{M}$  is a neighborhood of  $Z_j := \{z_j\} \times Y$  in  $\tilde{M}$ , and  $k_j \in \mathbb{N}$ ,  $F_{j,l}$  is a holomorphic  $(n-1, 0)$  form on  $Y$  for any  $l \geq k_j$ , and  $\tilde{F}_j := F_{j,k_j} \not\equiv 0$ .

Let  $W$  be an open subset of  $Y$  such that  $W \Subset Y$ . Then for any  $j$ ,  $1 \leq j < \gamma$ , there exists  $r_{j,W} > 0$  such that  $V_{j,W} \times W \subset \tilde{M}$ , where  $V_{j,W} := \{z \in \Omega : |\tilde{z}_j(z)| < r_{j,W}\}$ .

As  $\tilde{G}(h^{-1}(r))$  is linear and  $c$  is increasing near  $+\infty$ , using Lemma 27, we can get that  $k_j - q_j + 1 = 0$  for any  $j \in I_F$  and

$$\frac{\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_W |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)},$$

where  $I_F := \{j : 1 \leq j < \gamma \text{ \& } k_j + 1 - k_j \leq 0\}$ .

By the arbitrariness of  $W$ , we have  $\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty$  for any  $j \in I_F$ , and

$$\frac{\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s) e^{-s} ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (70)$$

Especially, we know  $\sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} < +\infty$  and  $\tilde{\psi}_2$  is not identically  $-\infty$  on  $Z_j$  for any  $j \in I_F$ .

Let  $V_j \times U_y$  be an open neighborhood of  $(z_j, y)$  in  $\tilde{M}$ . Assume that  $F = \tilde{z}_j^{k_j} h(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$  on  $V_j \times U_y$ , where  $h(z_j, w) dw = \tilde{F}_j$ . Note that  $h(z_j, w)$  is not identically zero on  $\{z_j\} \times U_y$ , there must exist  $\hat{w} \in U_y$  such that  $h(z_j, \hat{w}) \neq 0$ . Then we know  $|h(z_j, w)|^2$  has a positive lower bound on  $\tilde{V}_j \times U_{\hat{w}}$ , where  $\tilde{V}_j \times U_{\hat{w}} \Subset V_j \times U_y$  is a small open neighborhood of  $(z_j, \hat{w})$ . Then, according to Lemma 17,  $\psi \leq \sum_{j=1}^Y 2p_j G_\Omega(\cdot, z_j)$  and  $c(t)$  is increasing near  $+\infty$ , we know that for any  $w \in U_{\hat{w}}$ , we have  $\int_{\tilde{V}_j} |\tilde{z}_j^{k_j} d\tilde{z}_j|^2 e^{-\varphi_1} c(-\sum_{j=1}^Y 2p_j G_\Omega(\cdot, z_j)) < +\infty$ . It follows from Lemma 18 that we have

$$\int_{\tilde{V}_j \times U_{\hat{w}}} |\pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j)|^2 e^{-\varphi} c(-\pi_1^*(\sum_{j=1}^Y 2p_j G_\Omega(\cdot, z_j))) < +\infty.$$

Then by Lemma 22, we have

$$(F - \pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$$

for any  $j \in I_F$ ,  $y \in Y$ . And according to Lemma 22 we also have

$$(F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$$

for any  $j \in \{1, 2, \dots, m\} \setminus I_F$ ,  $y \in Y$ .

Now we prove that  $N \equiv 0$ . Denote that  $\tilde{\psi} := \pi_1^*(2 \sum_{j=1}^Y p_j G_\Omega(\cdot, z_j))$ ,  $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$ , and  $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$ . Then according to Lemma 25 ( $F = \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j)$  on  $U_j$  for  $j \in I_F$  and  $F \equiv 0$  on  $U_j$  for  $j \notin I_F$  in Lemma 25), there exists a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $\tilde{M}$  such that  $(\tilde{F} - \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $j \in I_F$ ,  $y \in Y$ ,  $(\tilde{F}, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $j \geq 1 \& j \notin I_F$  and any  $y \in Y$ , and

$$\int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (71)$$

Note that we have  $(\tilde{F} - F, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ .

It follows from (70) and (73) that, we have

$$\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) \geq \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \geq \int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}). \quad (72)$$

As  $c(t)e^{-t}$  is decreasing with respect to  $t$  and  $\psi \leq \tilde{\psi}$ , we have  $\int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \geq \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi)$ . Then all “ $\geq$ ” in (72) should be “ $=$ ”. It follows from Lemma 7 that  $N = 0$  and  $\psi = \pi_1^*(2 \sum_{j=1}^{\gamma} p_j G_{\Omega}(\cdot, z_j))$ .

Using Lemma 25 ( $\tilde{M} \sim M$  and  $F = \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j)$  for  $j \in I_F$  and  $F \equiv 0$  for  $j \notin I_F$  in Lemma 25), there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $M$  such that  $(\hat{F} - \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $j \in I_F$ ,  $y \in Y$ ,  $(\hat{F}, (z_j, y)) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $j \geq 1 \& j \notin I_F$  and any  $y \in Y$ , and

$$\int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (73)$$

Then  $(\hat{F} - F, (z_j, y)) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ .

Now According to the choice of  $F$ , we have that

$$\begin{aligned} \tilde{G}(0) &= \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ &\leq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ &\leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \end{aligned} \quad (74)$$

Combining inequality (70) with inequality (74), we get that

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) = \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi).$$

As  $\tilde{F} \not\equiv 0$ , the above equality implies that  $\tilde{M} = M$ .

## 5 Proofs of Theorems 2, 3, 4 and Proposition 3

In this section, we prove Theorems 2, 3, 4 and Proposition 3.

### 5.1 Proof of Theorem 2

**Proof** We firstly give the proof of the sufficiency of Theorem 2. It follows from Theorem 14 that  $G(h^{-1}(r); \mathcal{I}(\varphi + \psi))$  is linear with respect to  $r$ .

When  $N \equiv 0$ , then  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$ , it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ .

Hence  $G(h^{-1}(r); \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))$  is linear with respect to  $r$ . The sufficiency of Theorem 2 is proved.

We prove the necessity part of Theorem 2.

Assume that  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ . Then according to Lemma 2, there exists a unique holomorphic  $(n, 0)$  form  $F$  on  $M$  satisfying  $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))|_{Z_0}))$ , and  $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$  for any  $t \geq 0$ . Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that  $c$  is increasing near  $+\infty$ .

It follows from Lemma 19 that there exists a sequence of holomorphic  $(n_2, 0)$  forms  $\{F_\alpha\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$  on  $Y$  such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}) \wedge \pi_2^*(F_\alpha)$$

on  $V_0 \times Y$ .

Denote that  $E_0 := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} = 1 \right\}$ ,  $E_1 := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} < 1 \right\}$  and  $E_2 := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} > 1 \right\}$ .

Now we prove  $N \equiv 0$ . It follows from Lemma 28 that we have  $F_\alpha \equiv 0$  for any  $\alpha \in E_1$ , and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\alpha \in E_0} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j^{2\alpha_j + 2}(z_j)} \int_Y |F_{\alpha, \beta}|^2 e^{-\varphi_Y - N(z_0, w')}.$$
(75)

Note that  $\psi = \hat{G} + N \leq \hat{G}$ ,  $c(t)$  is increasing near  $+\infty$ ,  $\varphi_X$  is upper semi-continuous on  $\prod_{j=1}^{n_1} \Omega_j$ . When  $t$  is large enough, we have

$$\begin{aligned} \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) &\geq \int_{\{\hat{G} < -t\}} |F|^2 e^{-\pi_1^*(-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)) - \pi_2^*(\varphi_Y)} c(-\hat{G}) \\ &\geq C_0 \int_{\{\hat{G} < -t\}} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\hat{G}), \end{aligned}$$

where  $C_0 > 0$  is a constant, then it follows from  $\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$  and Lemma 19 that for any  $\alpha \in E_2 := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_j} > 1 \right\}$ , we know  $F_\alpha \in \mathcal{I}(\varphi_Y)$  for any  $y \in Y$ .

Recall that

$$\hat{G} = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}.$$

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Lemma 26 that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $M$  satisfying that  $(\hat{F} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} &\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ &\leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E_0} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j)} \int_Y |f_\alpha|^2 e^{-\varphi_Y - N(z_0, w')} }{ \prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j^{2\alpha_j + 2}(z_j)}. \end{aligned} \quad (76)$$

It follows from (75) and (76) that we have

$$\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \leq \int_M |F|^2 e^{-\varphi} c(-\psi) \leq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \quad (77)$$

As  $c(t)e^{-t}$  is decreasing with respect to  $t$  and  $\psi \leq \hat{G}$ , we have  $\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \geq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)$ . Then all “ $\geq$ ” in (77) should be “ $=$ ”. It follows from Lemma 7 that we know  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ . It follows from Theorem 14 that the necessity part of Theorem 2 holds.

## 5.2 Proof of Theorem 3

**Proof** We firstly give the proof of the sufficiency of Theorem 3. It follows from Theorem 15 that  $G(h^{-1}(r); \mathcal{I}(\varphi + \psi))$  is linear with respect to  $r$ .

When  $N \equiv 0$ , then  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ .

Hence  $G(h^{-1}(r); \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))$  is linear with respect to  $r$ . The sufficiency of Theorem 3 is proved.

We prove the necessity part of Theorem 3.

Assume that  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ . Then according to Lemma 2, there exists a unique holomorphic  $(n, 0)$  form  $F$  on  $M$  satisfying  $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))|_{Z_0}))$ , and  $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$  for any  $t \geq 0$ . Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that  $c$  is increasing near  $+\infty$ .

It follows from Lemma 19 that there exists a sequence of holomorphic  $(n_2, 0)$  forms  $\{F_{\alpha, \beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$  on  $Y$  such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_{\beta}^{\alpha} dw_{1, \beta_1} \wedge \dots \wedge dw_{n_1, \beta_{n_1}}) \wedge \pi_2^*(F_{\alpha, \beta})$$

on  $V_{\beta} \times Y$ , for any  $\beta \in \tilde{I}_1$ . Denote that  $E_{\beta} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j, \beta_j}} = 1 \right\}$ ,  $E_{\beta, 1} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j, \beta_j}} < 1 \right\}$  and  $E_{\beta, 2} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j, \beta_j}} > 1 \right\}$ .

Now we prove  $N \equiv 0$ . It follows from Lemma 28 that we have we have  $F_{\alpha} \equiv 0$  for any  $\alpha \in E_{\beta, 1}$ , and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_{\beta}} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_j, \beta_j)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j + 2}} \int_Y |F_{\alpha, \beta}|^2 e^{-\varphi_Y - N(z_{\beta}, w')}. \quad (78)$$

Note that  $\psi = \hat{G} + N \leq \hat{G}$ ,  $c(t)$  is increasing near  $+\infty$ ,  $\varphi_X$  is upper semi-continuous on  $\prod_{j=1}^{n_1} \Omega_j$ . When  $t$  is large enough, we have

$$\begin{aligned} & \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \geq \int_{\{\hat{G} < -t\}} |F|^2 e^{-\pi_1^*\left(\sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j)\right) - \pi_2^*(\varphi_Y)} c(-\hat{G}) \\ & \geq C_0 \int_{\{\hat{G} < -t\}} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\hat{G}), \end{aligned}$$

then it follows from  $\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$  and Lemma 19 that for any  $\alpha \in E_{\beta, 2} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j, \beta_j}} > 1 \right\}$  and any  $\beta \in I_1$ , we know  $F_{\alpha, \beta} \in \mathcal{I}(\varphi_Y)$  for any  $y \in Y$ .

Recall that

$$\hat{G} := \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Lemma 26 that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $M$  satisfying that  $(\hat{F} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')} }{ \prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2} }. \end{aligned} \quad (79)$$

It follows from (78) and (79) that we have

$$\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \leq \int_M |F|^2 e^{-\varphi} c(-\psi) \leq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi). \quad (80)$$

As  $c(t)e^{-t}$  is decreasing with respect to  $t$  and  $\psi \leq \hat{G}$ , we have  $\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \geq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)$ . Then all “ $\geq$ ” in (80) should be “ $=$ ”. It follows from Lemma 7 that we know  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$ . It follows from Theorem 15 that the necessity part of Theorem 3 holds.

### 5.3 Proof of Theorem 4

**Proof** We prove Theorem 4 by contradiction.

Assume that  $G(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s) e^{-s} ds]$ . Then according to Lemma 2, there exists a unique holomorphic  $(n, 0)$  form  $F$  on  $M$  satisfying  $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))|_{Z_0})$ , and  $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$  for any  $t \geq 0$ . Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that  $c$  is increasing near  $+\infty$ .

It follows from Lemma 19 that there exists a sequence of holomorphic  $(n_2, 0)$  forms  $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$  on  $Y$  such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on  $V_\beta \times Y$  for any  $\beta \in \tilde{I}_1$ . Denote that  $E_\beta := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \right\}$ ,  $E_{\beta,1} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} < 1 \right\}$  and  $E_{\beta,2} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} > 1 \right\}$ .

Now we prove  $N \equiv 0$ . It follows from Lemma 28 that we have we have  $F_\alpha \equiv 0$  for any  $\alpha \in E_{\beta,1}$ , and

$$\liminf_{t \rightarrow +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}. \quad (81)$$

Note that  $\psi = \hat{G} + N \leq \hat{G}$ ,  $c(t)$  is increasing near  $+\infty$ ,  $\sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j)$  is upper semi-continuous on  $\prod_{j=1}^{n_1} \Omega_j$ . When  $t$  is large enough, we have

$$\begin{aligned} \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) &\geq \int_{\{G < -t\}} |F|^2 e^{-\sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) - \pi_2^*(\varphi_Y)} c(-\hat{G}) \\ &\geq C_0 \int_{\{\hat{G} < -t\}} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\hat{G}), \end{aligned}$$

where  $C_0$  is a constant. Then it follows from  $\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$  and Lemma 19 that for any  $\alpha \in E_{\beta,2} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} > 1 \right\}$  and any  $\beta \in I_1$ , we know  $F_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)$  for any  $y \in Y$ .

Recall that

$$\hat{G} := \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}.$$

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Lemma 26 that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $M$  satisfying that  $(\hat{F} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} &\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ &\leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned} \quad (82)$$

It follows from (81) and (82) that we have

$$\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \leq \int_M |F|^2 e^{-\varphi} c(-\psi) \leq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi). \quad (83)$$

As  $c(t)e^{-t}$  is decreasing with respect to  $t$  and  $\psi \leq \hat{G}$ , we have  $\int_M |\hat{F}|^2 e^{-\bar{\varphi}} c(-\hat{G}) \geq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)$ . Then all “ $\geq$ ” in (83) should be “ $=$ ”. It follows from Lemma 7 that we know  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$ . It follows from Theorem 16 that we get a contradiction.

## 5.4 Proof of Proposition 3

**Proof** As  $\tilde{G}(h^{-1}(r))$  is linear with respect to  $r \in (0, \int_0^{+\infty} c(s)e^{-s} ds]$ , then according to Lemma 2, there exists a unique holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying  $(F - f) \in H^0(Z_0, (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))|_{Z_0})$ , and  $G(t) = \int_{\{\psi < -t\} \cap \tilde{M}} |F|^2 e^{-\varphi} c(-\psi)$  for any  $t \geq 0$ . Then according to Lemma 2, Remark 14 and Lemma 16, we can assume that  $c$  is increasing near  $+\infty$ .

It follows from Lemma 20 that there exists a sequence of holomorphic  $(n_2, 0)$  forms  $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$  on  $Y$  such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_{\beta}^{\alpha} dw_{1,\beta_1} \wedge \dots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on a open neighborhood  $U_{\beta} \subset (V_{\beta} \times Y) \cap \tilde{M}$  of  $\{z_{\beta}\} \times Y$  for any  $\beta \in \tilde{I}_1$ . Denote that  $E_{\beta} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \right\}$ ,  $E_{\beta,1} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} < 1 \right\}$  and  $E_{\beta,2} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} > 1 \right\}$ .

It follows from Remark 22 that we have we have  $F_{\alpha,\beta} \equiv 0$  for any  $\alpha \in E_{\beta,1}$ , and

$$\frac{G(0)}{\int_t^{+\infty} c(s)e^{-s} ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_{\beta}} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_{\beta}, w')} \quad (84)$$

Note that  $\psi = \hat{G} + N \leq \hat{G}$ ,  $c(t)$  is increasing near  $+\infty$ ,  $\sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j)$  is upper semi-continuous on  $\prod_{j=1}^{n_1} \Omega_j$ . When  $t$  is large enough, we have

$$\begin{aligned} \int_{\{\psi < -t\} \cap \tilde{M}} |F|^2 e^{-\varphi} c(-\psi) &\geq \int_{\{\hat{G} < -t\} \cap \tilde{M}} |F|^2 e^{-\sum_{1 \leq j \leq n_1} \pi_{1,j}^*(\varphi_j) - \pi_2^*(\varphi_Y)} c(-\hat{G}) \\ &\geq C_0 \int_{\{\hat{G} < -t\} \cap \tilde{M}} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\hat{G}), \end{aligned}$$

then it follows from  $\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$  and Lemma 20 that for any  $\alpha \in E_{\beta,2} := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} > 1 \right\}$  and any  $\beta \in \mathcal{I}_1$ , we know  $F_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)$  for any  $y \in Y$ .

Recall that  $\hat{G} := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$  and  $\tilde{\varphi} = \varphi + N$ .

It follows from Lemma 26 that there exists a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $\tilde{M}$  satisfying that  $(\tilde{F} - F, z) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_{\beta}} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_{\beta}, w')} }{ \prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2} }. \end{aligned} \quad (85)$$

Combining with (84) and (85), we have

$$\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) \geq \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \geq \int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}). \quad (86)$$

As  $c(t)e^{-t}$  is decreasing with respect to  $t$  and  $\psi \leq \hat{G}$ , we have  $\int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \geq \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi)$ . Then all “ $\geq$ ” in (86) should be “ $=$ ”. It follows from Lemma 7 that  $N = 0$  and  $\psi = \max_{1 \leq j \leq n_1} \left\{ \pi_{1,j}^* \left( 2 \sum_{1 \leq k \leq m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$ .

It follows from Lemma 26 ( $\tilde{M} \sim M$ ) that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $M$  satisfying that  $(\hat{F} - F, z) \in (O(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_{\beta}} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_{\beta}, w')} }{ \prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2} }. \end{aligned} \quad (87)$$

We also have

$$\frac{\tilde{G}(0)}{\int_t^{+\infty} c(s) e^{-s} ds} = \frac{\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \leq \frac{\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds}. \quad (88)$$

Combining (84), (87) and (88), we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) = \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi).$$

As  $\hat{F} \neq 0$ , the above equality shows that  $M = \tilde{M}$ .

## 6 Proofs of Theorems 5 and 6

In this section, we prove Theorems 5 and 6.

### 6.1 Proof of Theorem 5

**Proof** It follows from Lemma 25 that we know there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  such that  $(F - f, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (89)$$

In the following, we prove the characterization of the holding of the equality  $\left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} = \inf \{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F}$  is a holomorphic  $(n, 0)$  form on  $\tilde{M}$  such that  $(\tilde{F} - f, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0 \}$ .

We firstly give the proof of the sufficiency in Theorem 5.

As  $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \sum_{j=1}^m p_j G_{\Omega}(\cdot, z_j) + \varphi_1)$  is a plurisubharmonic function on  $M$ , by definition, we know that  $2 \sum_{j=1}^m p_j G_{\Omega}(\cdot, z_j) + \varphi_1$  is a subharmonic function on  $\Omega$ . By the construction of  $g \in O_{\Omega}$ , we have  $2u(z) := 2 \sum_{j=1}^m p_j G_{\Omega}(\cdot, z_j) + \varphi_1 - 2 \log |g|$  is a subharmonic function on  $\Omega$  which satisfies  $v(dd^c u, z) \in [0, 1)$  for any  $z \in Z_{\Omega}$ . Then it follows from Lemma 23 that we know  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . It follows from Theorem 12 that we know the sufficiency part of Theorem 5 holds.

Next we prove the necessity part of Theorem 5.

Denote that  $\tilde{\psi} := \pi_1^*(2 \sum_{j=1}^m p_j G_{\Omega}(\cdot, z_j))$ ,  $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$ , and  $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$ . It follows from Lemma 25 that there exists a holomorphic  $(n, 0)$  form  $\tilde{F}_1 \neq 0$  on  $\tilde{M}$  such that  $(\tilde{F}_1 - f, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \end{aligned} \quad (90)$$

As  $\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j, w)} = \inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } \tilde{M} \text{ such that } (\tilde{F} - f, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)} \text{ for any } (z_j, y) \in Z_0 \right\}$  holds, we know

$$\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\varphi} c(-\psi) \geq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j, w)}.$$

Note that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ ,  $\psi \leq \tilde{\psi}$ . Hence we must have

$$\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\tilde{\psi}} c(-\tilde{\psi}).$$

It follows from Lemma 7 that we have  $N \equiv 0$  and  $\psi = \pi_1^*(2 \sum_{j=1}^m p_j G_{\Omega}(\cdot, z_j))$ .

It follows from Lemma 25 ( $\tilde{M} \sim M$ ) that there exists a holomorphic  $(n, 0)$  form  $\hat{F} \neq 0$  on  $M$  such that  $(\hat{F} - f, (z_j, y)) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\begin{aligned} & \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j, w)}. \end{aligned} \tag{91}$$

As  $\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j, w)} = \inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } \tilde{M} \text{ such that } (\tilde{F} - f, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)} \text{ for any } (z_j, y) \in Z_0 \right\}$  holds, we know

$$\begin{aligned} & \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j, w)} \\ & \leq \int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \leq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^m \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j, w)}. \end{aligned}$$

As  $\hat{F} \neq 0$ , we have  $\tilde{M} = M$ .

Now  $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \sum_{j=1}^m p_j G_{\Omega}(\cdot, z_j) + \varphi_1)$  is a plurisubharmonic function on  $M$ . By definition, we know that  $2 \sum_{j=1}^m p_j G_{\Omega}(\cdot, z_j) + \varphi_1$  is a subharmonic

function on  $\Omega$ . By the construction of  $g \in \mathcal{O}_\Omega$ , we have  $2u(z) := 2 \sum_{j=1}^m p_j G_\Omega(\cdot, z_j) + \varphi_1 - 2 \log |g|$  is a subharmonic function on  $\Omega$  and satisfies  $v(dd^c u, z) \in [0, 1)$  for any  $z \in Z_\Omega$ . Then it follows from Lemma 23 that we know  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ .

Then it follows from the necessity part of Theorem 12 that we know Theorem 5 holds.

## 6.2 Proof of Theorem 6

**Proof** It follows from Lemma 25 that we know there exists a holomorphic  $(n, 0)$  form  $F$  on  $M$  such that  $(F - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}.$$

Now it suffices to show that the equality  $\left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} = \inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } \tilde{M} \text{ such that } (\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)} \text{ for any } (z_j, y) \in Z_0 \right\}$  can not hold. We assume that the equality holds to get a contradiction.

Denote that  $\tilde{\psi} := \pi_1^*(2 \sum_{j=1}^m (k_j + 1)G_\Omega(\cdot, z_j))$ ,  $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$ , and  $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$ . It follows from Lemma 25 that there exists a holomorphic  $(n, 0)$  form  $\tilde{F}_1 \neq 0$  on  $\tilde{M}$  such that  $(\tilde{F}_1 - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \end{aligned} \tag{92}$$

As  $\left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} = \inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } \tilde{M} \text{ such that } (\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)} \text{ for any } (z_j, y) \in Z_0 \right\}$ , we know

$$\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\varphi} c(-\psi) \geq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^{+\infty} \frac{2\pi |a_j|^2}{(k_j + 1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}.$$

Note that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$  and  $\psi \leq \tilde{\psi}$ . Hence we must have

$$\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}).$$

Hence, by Lemma 7, we have  $N \equiv 0$  and  $\psi = \pi_1^*(2 \sum_{j=1}^{+\infty} (k_j + 1) G_\Omega(\cdot, z_j))$ .

It follows from Lemma 25 ( $\tilde{M} \sim M$ ) that there exists a holomorphic  $(n, 0)$  form  $\hat{F} \neq 0$  on  $M$  such that  $(\hat{F} - f, (z_j, y)) \in (O(K_M) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$  and

$$\begin{aligned} & \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2}{(k_j + 1) |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \end{aligned} \quad (93)$$

As  $\left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} = \inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \right.$   
 $\tilde{F}$  is a holomorphic  $(n, 0)$  form on  $\tilde{M}$  such that  $(\tilde{F} - f, (z_j, y)) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2)))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0 \left. \right\}$  holds, we know

$$\begin{aligned} & \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2}{(k_j + 1) |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)} \\ & \leq \int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \leq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2}{(k_j + 1) |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \end{aligned}$$

As  $\hat{F} \neq 0$ , we have  $\tilde{M} = M$ .

Now  $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2 \sum_{j=1}^{+\infty} (k_j + 1) G_\Omega(\cdot, z_j) + \varphi_1)$  is a plurisubharmonic function on  $M$ , by definition, we know that  $2 \sum_{j=1}^m (k_j + 1) G_\Omega(\cdot, z_j) + \varphi_1$  is a subharmonic function on  $\Omega$ . By the construction of  $g \in O_\Omega$ , we have  $2u(z) := 2 \sum_{j=1}^{+\infty} (k_j + 1) G_\Omega(\cdot, z_j) + \varphi_1 - 2 \log |g|$  is a subharmonic function on  $\Omega$  and satisfies  $v(dd^c u, z) \in [0, 1]$  for any  $z \in Z_\Omega$ . Then it follows from Lemma 23 that we know  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\pi_1^*(2 \log |g|) + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ .

Then it follows from Theorem 13 that we know Theorem 6 holds.

## 7 Proofs of Theorems 7 and 8 and 9

In this section, we prove Theorems 7 and 8 and 9.

### 7.1 Proofs of Theorem 7 and Remark 10

**Proof (Proof of Theorem 7)** It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\})_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}. \end{aligned}$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 17 that we only need to show the necessity.

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $\tilde{M}$  satisfying that  $(\hat{F} - F, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G}))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}. \end{aligned} \quad (94)$$

When the equality  $\inf \{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \& (\tilde{F} - f, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}))_z \text{ for any } z \in Z_0 \} = \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \times \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}$  holds, we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \geq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}.$$

Note that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ ,  $\psi \leq \hat{G}$ . Hence we must have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}).$$

It follows from Lemma 7 that we have  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ . Then It follows from the necessity part of Theorem 17 that Theorem 7 holds.

**Proof (Proof of Remark 10)** As  $(f_\alpha, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  for any  $y \in Y$  and  $\alpha \in \tilde{E} \setminus E$ , it follows from Lemma 26 that there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, (z, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_{(z, y)}$  for any  $(z, y) \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j^{2\alpha_j + 2}(z_j)}. \end{aligned}$$

In the following, we prove the characterization of the holding of the equality.

When  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ . It follows from Remark 15 that we have the sufficiency of Remark 10.

Next we show the necessity of Remark 10.

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $\tilde{M}$  satisfying that  $(\hat{F} - F, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j^{2\alpha_j + 2}(z_j)}. \end{aligned} \quad (95)$$

Now we know the equality  $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \& (\tilde{F} - f, (z, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_{(z, y)} \text{ for any } (z, y) \in Z_0 \right\} = \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \times \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j^{2\alpha_j + 2}(z_j)}$  holds, then we have

$$\begin{aligned} & \int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \geq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \int_Y |f_\alpha|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_0, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_j^{2\alpha_j + 2}(z_j)}. \end{aligned} \quad (96)$$

It follows from (95) and (96) that we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}).$$

Note that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ ,  $\psi \leq \hat{G}$ . It follows from Lemma 7 that we have  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ . Then It follows from Remark 15 that Remark 10 holds.

## 7.2 Proofs of Theorem 8 and Remark 11

**Proof (Proof of Theorem 8)** It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\})_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned}$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 18 that we only need to show the necessity.

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $\tilde{M}$  satisfying that  $(\hat{F} - F, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G}))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned} \quad (97)$$

When the equality  $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \& (\tilde{F} - f, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I} \left( \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} \right))_z \text{ for any } z \in Z_0 \right\}$   
 $= \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}$  holds,  
we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \geq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}.$$

Note that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ ,  $\psi \leq \hat{G}$ . Hence we must have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}).$$

It follows from Lemma 7 that we have  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$ . Then It follows from the necessity part of Theorem 18 that Theorem 8 holds.

**Proof (Proof of Remark 11)** As  $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  holds for any  $y \in Y$ ,  $\alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , it follows from Lemma 26 that there exists a holomorphic  $(n, 0)$  form  $F$  on  $M$  satisfying that  $(F - f, (z, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_{(z, y)}$  for any  $(z, y) \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned}$$

In the following, we prove the characterization of the holding of the equality.

When  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \{2 p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\}$ . It follows from Remark 16 that we have the sufficiency of Remark 11.

Next we show the necessity of Remark 11.

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $\tilde{F}$  on  $\tilde{M}$  satisfying that  $(\tilde{F} - F, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_{(z, y)}$  for any  $(z, y) \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j + 2}}. \end{aligned} \quad (98)$$

Now we know the equality  $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \& (\tilde{F} - f, (z, y) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_{(z, y)} \text{ for any } (z, y) \in Z_0 \right\} = \left( \int_0^{+\infty} c(s) e^{-s} ds \right)$   
 $\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j + 2}}$  holds, then we have

$$\begin{aligned} & \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\ & \geq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j + 2}}. \end{aligned} \quad (99)$$

Note that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ ,  $\psi \leq \hat{G}$ . It follows from (98) and (99) that we have

$$\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi - N} c(-\hat{G}).$$

It follows from Lemma 7 that we have  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$ , it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$ . Then It follows from Remark 16 that Remark 11 holds.

### 7.3 Proofs of Theorem 9 and Remark 12

**Proof (Proof of Theorem 9)** It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, z) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned}$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 18 that we only need to show the necessity.

We show the equality  $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, O(K_{\tilde{M}})) \& (\tilde{F} - f, z) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\max_{1 \leq j \leq n_1} \{2 \sum_{1 \leq k \leq m_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}))_z \text{ for any } z \in Z_0 \right\} = \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}$

can not hold. We assume that the equality holds to get a contradiction.

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $\tilde{M}$  satisfying that  $(\hat{F} - F, z) \in (O(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G}))_z$  for any  $z \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned} \quad (100)$$

As the equality holds, we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \geq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}.$$

Note that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ ,  $\psi \leq \hat{G}$ . Hence we must have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}).$$

It follows from Lemma 7 that we have  $\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^* (G_{\Omega_j}(\cdot, z_{j,k})) \right\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(G + \pi_2^*(\varphi_Y))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^* (G_{\Omega_j}(\cdot, z_{j,k})) \right\}$ . Then It follows from Theorem 19 that we get the contradiction and Theorem 9 is proved.

**Proof (Proof of Remark 12)** As  $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$  holds for any  $y \in Y$ ,  $\alpha \in \tilde{E}_\beta \setminus E_\beta$  and  $\beta \in \tilde{I}_1$ , it follows Lemma 26 that there exists a holomorphic  $(n, 0)$  form  $F$  on  $\tilde{M}$  satisfying that  $(F - f, (z, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_{(z, y)}$  for any  $(z, y) \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned}$$

In the following, we assume that the equality  $\inf \left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}})) \& (\tilde{F} - f, (z, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(G + \pi_2^*(\varphi_Y)))_{(z, y)} \right\}$  for any  $(z, y) \in Z_0$

$$Z_0 = \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}$$

holds to get a contradiction.

Denote  $\tilde{\varphi} := \varphi + \psi - \hat{G}$ . It follows from Remark 20 that there exists a holomorphic  $(n, 0)$  form  $\hat{F}$  on  $\tilde{M}$  satisfying that  $(\hat{F} - F, z) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)))_{(z, y)}$  for any  $(z, y) \in Z_0$  and

$$\begin{aligned} & \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \\ & \leq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}}. \end{aligned} \tag{101}$$

Since the equality holds, we have

$$\begin{aligned} & \int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \\ & \geq \left( \int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha, \beta}|^2 e^{-\varphi_Y - (N + \pi_{1,j}^*(\sum_{1 \leq j \leq n_1} \varphi_j))(z_\beta, w)}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j, \beta_j}^{2\alpha_j + 2}}. \end{aligned} \quad (102)$$

Note that  $c(t)e^{-t}$  is decreasing on  $(0, +\infty)$ ,  $\psi \leq \hat{G}$ . It follows from (101) and (102) that we have

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |F|^2 e^{-\varphi - N} c(-\hat{G}).$$

It follows from Lemma 7 that we have  $N \equiv 0$  and  $\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$ .

As  $N \equiv 0$ , we know  $\varphi_j$  is a subharmonic function on  $\Omega_j$  satisfying  $\varphi_j(z_j) > -\infty$  for any  $1 \leq j \leq n_1$  and it follows from Lemma 24 that  $\mathcal{I}(\varphi + \psi)_{(z_j, y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_2))_{(z_j, y)}$  for any  $(z_j, y) \in Z_0$ . Note that  $\psi = \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$ . Then It follows from Remark 17 that Remark 12 holds.

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