Mathematische Annalen (2025) 391:5809–5856
https://doi.org/10.1007/s00208-024-03056-8

Mathematische Annalen

# Boundary points, minimal $L^2$ integrals and concavity property

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Received: 2 April 2024 / Revised: 17 November 2024 / Accepted: 19 November 2024 /
Published online: 23 December 2024


**Abstract**

For the purpose of proving the strong openness conjecture of multiplier ideal sheaves, Jonsson–Mustaţă posed an enhanced conjecture and proved the two-dimensional case, which says that: the Lebesgue measure of the set $\left\{c_o^F(\psi)\psi - \log|F| < \log r\right\}$ divided by $r^2$ has a uniform positive lower bound independent of $r$, for a plurisubharmonic function $\psi$ and a holomorphic function $F$ near the origin $o$. After proving the strong openness conjecture, Guan–Zhou proved Jonsson–Mustaţă's conjecture based on the truth of the strong openness conjecture. In this article, we use an $L^2$ method with the weight functions $\psi - \log|F|$ and first consider a module at a boundary point of the sublevel sets of a plurisubharmonic function. By studying the minimal $L^2$ integrals on the sublevel sets of a plurisubharmonic function with respect to the module at the boundary point, we establish a concavity property of the minimal $L^2$ integrals. As applications, we obtain a sharp effectiveness result related to Jonsson–Mustaţă's conjecture independent of the truth of the strong openness conjecture, which completes the approach from Jonsson–Mustaţă's conjecture to the strong openness conjecture. We also obtain a strong openness property of the module and a lower semi-continuity property with respect to the module.

**Mathematics Subject Classification** 32D15 · 32E10 · 32L10 · 32U05 · 32W05

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# 1 Introduction

## 1.1 Backgrounds and motivations

Let $\psi$ be a plurisubharmonic function of a complex manifold $M$ (see [9]). Recall that multiplier ideal sheaf $\mathcal{I}(\psi)$ is the sheaf of germs of holomorphic functions $f$ such that $|f|^2e^{-\psi}$ is locally integrable, which was widely discussed in the study of several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [10–14, 33, 39, 40, 42, 44, 46–48]).

The strong openness property of multiplier ideal sheaves, i.e.

$$\mathcal{I}(\psi) = \mathcal{I}_+(\psi) := \bigcup_{\epsilon>0} \mathcal{I}((1+\epsilon)\psi),$$

is an important feature of multiplier ideal sheaves, and "opened the door to new types of approximation techniques" (see [41]) (see e.g. [4–8, 17, 18, 28, 31, 37, 38, 50–52]). The strong openness property was conjectured by Demailly [11], and proved by Guan–Zhou [28] (the 2-dimensional case was proved by Jonsson–Mustaţă [35]). After that, Guan–Zhou [30] established an effectiveness result of the strong openness property by considering the minimal $L^2$ integral on the pseudoconvex domain $D$.

When $\mathcal{I}(\psi)=\mathcal{O}$, the strong openness property degenerates to the openness property, which was conjectured by Demailly-Kollár [12] and proved by Berndtsson [2] (the 2-dimensional case was proved by Favre-Jonsson in [16]).

Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$ containing the origin $o\in\mathbb{C}^n$, and let $\psi$ be a plurisubharmonic function on $D$. For a holomorphic function $h$ on a neighborhood of $o$, recall that

$$c_o^h(\psi):=\sup\left\{c\geq 0: |h|^2e^{-2c\psi} \text{ is } L^1 \text{ on a neighborhood of } o\right\}$$

is the jumping number (see [36]). When $h\equiv 1$, $c_o^h(\psi)$ degenerates to the complex singularity exponent $c_o(\psi)$ (see [12, 48]).

In [12] (see also [36]), Demailly and Kollár conjectured the following more precise form of the openness property:

**Conjecture D–K:** *If* $c_o(\psi)<+\infty$, $\frac{1}{r^2}\mu(\{c_o(\psi)\psi<\log r\})$ *has a uniform positive lower bound independent of* $r\in(0,1)$, *where* $\mu$ *is the Lebesgue measure on* $\mathbb{C}^n$.

The 2-dimensional case of Conjecture D–K was proved by Favre–Jonsson in [16], which deduces the 2-dimensional case of the openness property. Depending on the openness property, Guan–Zhou [30] proved Conjecture D–K.

In [21], Guan gave a sharp effectiveness result related to Conjecture D–K independent of the openness property, which completed the approach from Conjecture D–K to the openness property.

In [49], Xu completed the algebraic approach to Conjecture D–K.

In order to prove the strong openness property, Jonsson and Mustaţă (see [36], see also [35]) posed the following conjecture, and proved the 2-dimensional case [35]:

**Conjecture J–M:** *If* $c_o^F(\psi)<+\infty$ *for a holomorphic function* $F$ *on* $D$, *then* $\frac{1}{r^2}\mu(\{c_o^F(\psi)\psi-\log|F|<\log r\})$ *has a uniform positive lower bound independent of* $r\in(0,1)$, *where* $\mu$ *is the Lebesgue measure on* $\mathbb{C}^n$.

Depending on the strong openness property, Guan–Zhou [30] proved Conjecture J–M:

**Theorem 1.1** [30] *If* $c_o^F(\psi)<+\infty$ *and* $\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi,2\log|F|\}}<+\infty$, *then*

$$\begin{aligned}
&\liminf_{r\to 0+0}\frac{1}{r^2}\mu(\{c_o^F(\psi)\psi-\log|F|<\log r\})\\
&\quad\geq\sup_{\delta\in\mathbb{Z}_{>0}}\frac{C_{F^{1+\delta},2c_o^F(\psi)\psi+\delta\max\{2c_o^F(\psi)\psi,2\log|F|\}}(o)}{\left(1+\frac{1}{\delta}\right)\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi,2\log|F|\}}},
\end{aligned}$$

*where for a plurisubharmonic function* $\phi$ *and a holomorphic function* $F_0$ *on* $D$,

$$C_{F_0,\phi}(o):=\inf\left\{\int_D|\tilde{F}|^2:\tilde{F}\in\mathcal{O}(D)\ \&\ (\tilde{F}-F_0,o)\in\mathcal{I}(\phi)_o\right\}.$$

A natural question is:

**Question 1.2** *Can one obtain a sharp effectiveness result related to Conjecture J–M* ***independent*** *of the strong openness property, which completes the approach from Conjecture J–M to the strong openness property?*

We give an affirmative answer to Question 1.2 in the present paper.

#### 1.1.1 Formulations

Recall that considering the minimal $L^2$ integrals on the sublevel sets of a given plurisubharmonic weight $\psi$, Guan [21] established a concavity property of the minimal $L^2$ integrals on the sublevel sets of $\psi$ (see also [20, 22–25, 32]), and gave a sharp effectiveness result related to Conjecture D–K independent of the openness property.

The minimal $L^2$ integrals on the sublevel sets of the plurisubharmonic function $\psi$ considered in [21] (see also [20, 22–25, 32]) are with respect to the modules(ideals) at the inner points of the sublevel sets.

However, in Conjecture J–M, we need to deal with the weight function $\psi - \log|F|$, which is not plurisubharmonic on the entire domain $D$, instead of a purely plurisubharmonic function $\psi$. In addition, the considered singular point (which is the origin $o$ in Conjecture J–M) can be not an inner point but a boundary point of the sublevel sets of the weight function $\psi - \log|F|$.

To cross the obstacles mentioned above, we need an (optimal) $L^2$ estimate result with the weight function $\psi - \log|F|$, and we also need a module at the boundary point to replace the (multiplier) ideal at the inner point. Motivated by the $L^2$ methods in [21, 29, 30], we give an $L^2$ method with the weight functions $\psi - \log|F|$ in Sect. 2.1, where the technical part of the proof is given in the Appendix. In Sect. 2.2, we first discuss a module $I(a\Psi)$ at the boundary point and give some useful properties of the module.

After these preparations, we consider the minimal $L^2$ integrals with respect to a module at a boundary point of the sublevel sets related to a function in the shape of $\psi - \log|F|$, and we obtain a concavity property of the minimal $L^2$ integrals. Then using the concavity property, we obtain a sharp effectiveness result related to Conjecture J–M.

*It is worth noting that the effectiveness result (Corollary* 1.8*) related to Conjecture J–M in the present paper implies Theorem* 1.1 *(see Remark* 1.9*), and its proof does not depend on the truth of the strong openness property.* Consequently, the present paper completes the approach from Conjecture J–M to the strong openness property moreover.

Using the concavity property of the minimal $L^2$ integrals with respect to a module at a boundary point, we also obtain a strong openness property of the module and a lower semi-continuity property with respect to the module.

### 1.2 Concavity property

Let $F$ be a holomorphic function on a pseudoconvex domain $D \subset \mathbb{C}^n$ containing the origin $o \in \mathbb{C}^n$, and let $\psi$ be a plurisubharmonic function on $D$. Denote

$$\Psi := \min\{\psi - 2\log|F|, 0\}.$$

If $F(z) = 0$ for $z \in D$, we set $\Psi(z) = 0$. Note that $\Psi = \psi + 2\log|1/F|$ is a plurisubharmonic function on $\{\Psi < 0\}$. Note that, *if* $F(o) = 0$, *then* $o$ *can be a boundary point of the sublevel set* $\{\Psi < -t\}$ *for any* $t > 0$.

Set

$$\tilde{J}(\Psi)_o := \left\{f \in \mathcal{O}(\{\Psi < -t\} \cap V) : t \in \mathbb{R} \text{ and } V \text{ is a neighborhood of } o\right\}.$$

We define the following **module at the boundary point** $o$ of the sublevel sets $\{\Psi < -t\} = \{\psi - 2\log|F| < -t\}$, where $t \geq 0$.

**Definition 1.3** Denote by $J(\Psi)_o$ the set of the equivalence classes $f_o$ of $f \in \tilde{J}(\Psi)_o$ under the equivalence relation $f \sim g$ if $f = g$ on $\{\Psi < -t\} \cap V$, where $t \gg 1$ and $V$ is some neighborhood of $o$.

If $o \in \bigcap_{t>0}\{\Psi < -t\}$, then $J(\Psi)_o$ equals to $\mathcal{O}_{\mathbb{C}^n,o}$, and $f_o$ is the germ $(f, o)$ of a holomorphic function $f$.

For any $f_o \in J(\Psi)_o$, $\tilde{f}_o \in J(\Psi)_o$ and $(h, o) \in \mathcal{O}_{\mathbb{C}^n,o}$, define

$$f_o + \tilde{f}_o = (f + \tilde{f})_o, \text{ and } (h, o) \cdot f_o = (hf)_o.$$

Then it is clear that $J(\Psi)_o$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module.

**Definition 1.4** For any $h_o \in J(\Psi)_o$ and any $a \geq 0$, we call $h_o \in I(a\Psi)_o$ if and only if there exist $t \gg 1$ and a neighborhood $V$ of $o$ such that $\int_{\{\Psi<-t\}\cap V} |h|^2 e^{-a\Psi} < +\infty$.

It is clear that $I(a\Psi)_o$ is an $\mathcal{O}_{\mathbb{C}^n,o}-$submodule of $J(\Psi)_o$. Denote $I_o := I(0\Psi)_o$.

Let $f$ be a holomorphic function on $V_0 \cap \{\Psi < -t_0\}$, where $V_0$ is a neighborhood of $o$ and $t_0 > 0$. Let $J$ be an $\mathcal{O}_{\mathbb{C}^n,o}-$submodule of $I_o$ such that $I(\Psi)_o \subset J$.

**Definition 1.5** We define the **minimal $L^2$ integrals on the sublevel sets** $\{\Psi < -t\} = \{\psi - 2\log|F| < -t\}$ **with respect to** $J$ (which is a module at the boundary point $o$ of the sublevel sets) as follows:

$$G(t; \Psi, J, f) := \inf\left\{\int_{\{\Psi<-t\}} |\tilde{f}|^2 : \tilde{f} \in \mathcal{O}(\{\Psi < -t\}) \;\&\; (\tilde{f} - f)_o \in J\right\},$$

where $t \in [0, +\infty)$.

In the present article, we establish the following concavity of $G(-\log r; \Psi, J, f)$.

**Theorem 1.6** *If there exists some* $t_0 \geq 0$ *such that* $G(t_0; \Psi, J, f) < +\infty$, *then* $G(-\log r; \Psi, J, f)$ *is concave with respect to* $r \in (0, 1]$ *and* $\lim_{t\to+\infty} G(t; \Psi, J, f) = 0$.

When $F \equiv 1$ and $\psi(o) = -\infty$, Theorem 1.6 can be referred to [21] (see also [20, 22–25, 32]). We also give an example in Appendix (Sect. 7.2) for the case that $F$ is not a constant function.

### 1.3 Applications

In this section, we present some applications of Theorem 1.6.

#### 1.3.1 A sharp effectiveness result related to Jonsson–Mustaţă's conjecture

Let $f$ be a holomorphic function on $D$. Denote

$$\Psi_1 := \min\left\{2c_o^{fF}(\psi)\psi - 2\log|F|, 0\right\},$$

and

$$I_+(\Psi_1)_o := \bigcup_{a>1} I(a\Psi_1)_o.$$

Recall the definition of the minimal $L^2$ integrals $G$ in Sect. 1.2. Theorem 1.6 implies the following result independent of the strong openness property.

**Theorem 1.7** *If $c_o^{fF}(\psi) < +\infty$, then*

$$\frac{1}{r^2}\int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 \geq G(0; \Psi_1, I_+(\Psi_1)_o, f) > 0$$

*holds for any $r \in (0, 1]$.*

The constant $G(0; \Psi_1, I_+(\Psi_1)_o, f)$ is sharp. Let $D = \Delta \subset \mathbb{C}$ be the unit disc, and let $\psi = \log|z|$. Let $F \equiv 1$, and let $f \equiv 1$. It is clear that $c_o^{fF}(\psi) = 1$, $\int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 = \pi r^2$ and $G(0; \Psi_1, I_+(\Psi_1)_o, f) = \pi$, which implies the sharpness of Theorem 1.7. A boundary point example can be seen in Appendix (Sect. 7.2).

When $f \equiv 1$, Theorem 1.7 is the following sharp effectiveness result related to Conjecture J–M.

**Corollary 1.8** *If $c_o^F(\psi) < +\infty$, then*

$$\frac{1}{r^2}\mu\{c_o^F(\psi)\psi - \log|F| < \log r\} \geq G(0; \Psi_1, I_+(\Psi_1)_o, 1) > 0$$

*holds for any $r \in (0, 1]$, where $\Psi_1 = \min\{2c_o^F(\psi)\psi - 2\log|F|, 0\}$.*

The proof of Corollary 1.8 is independent of the strong openness property, and Corollary 1.8 recovers Theorem 1.1 due to the following remark.

**Remark 1.9** For any $\delta \in \mathbb{Z}_{>0}$, it holds that

$$G(0; \Psi_1, I_+(\Psi_1)_o, 1) \geq \frac{C_{F^{1+\delta}, 2c_o^F(\psi)\psi + \delta\max\{2c_o^F(\psi)\psi, 2\log|F|\}}(o)}{\left(1 + \frac{1}{\delta}\right)\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}}},$$

hence Corollary 1.8 implies that

$$\frac{1}{r^2}\mu(\{c_o^F(\psi)\psi - \log|F| < \log r\}) \geq \sup_{\delta\in\mathbb{Z}_{>0}} \frac{C_{F^{1+\delta},2c_o^F(\psi)\psi+\delta\max\{2c_o^F(\psi)\psi,2\log|F|\}}(o)}{\left(1+\frac{1}{\delta}\right)\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi,2\log|F|\}}}.$$

holds for any $r \in (0, 1]$. We prove the remark in Sect. 4.2.

When $F \equiv 1$, Theorem 1.7 is the following sharp effectiveness result related to Conjecture D–K.

**Corollary 1.10** [21] *If $c_o^f(\psi) < +\infty$, then*

$$\frac{1}{r^2}\int_{\{c_o^f(\psi)\psi<\log r\}} |f|^2 \geq G(0; \Psi_1, I_+(\Psi_1)_o, f) > 0$$

*holds for any $r \in (0, 1]$, where $\Psi_1 = \min\{2c_o^f(\psi)\psi, 0\}$.*

#### 1.3.2 The strong openness property of $I(a\Psi)_o$

In this section, we present a strong openness property of the module $I(a\Psi)_o$.

Let $F$ be a holomorphic function on a pseudoconvex domain $D \subset \mathbb{C}^n$ containing the origin $o \in \mathbb{C}^n$, and let $\psi$ be a plurisubharmonic function on $D$. Denote

$$\Psi := \min\{\psi - 2\log|F|, 0\}.$$

Let $f$ be a holomorphic function on $\{\Psi < -t_0\}$ such that $f_o \in I_o$. Set

$$a_o^f(\Psi) := \sup\left\{a \geq 0 : f_o \in I(2a\Psi)_o\right\}.$$

Especially, when $F \equiv 1$ and $\psi(o) = -\infty$, $a_o^f(\Psi)$ is the jumping number $c_o^f(\psi)$.

Using Theorem 1.6, we obtain the following estimate for $L^2$ integrals on the sublevel sets of $\Psi$:

**Theorem 1.11** *Assume that $a_o^f(\Psi) < +\infty$, then we have $a_o^f(\Psi) > 0$ and*

$$\frac{1}{r^2}\int_{\{a_o^f(\Psi)\Psi<\log r\}} |f|^2 \geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f) > 0$$

*holds for any $r \in (0, e^{-a_o^f(\Psi)t_0}]$.*

Theorem 1.11 implies the following strong openness property of $I(a\Psi)_o$.

**Corollary 1.12** *$I(a\Psi)_o = I_+(a\Psi)_o$ holds for any $a \geq 0$.*

When $F \equiv 1$ and $\psi(o) = -\infty$, Corollary 1.12 is the strong openness property of multiplier ideal sheaves ([28]).

#### 1.3.3 A lower semicontinuity property

Let $\{\psi_m\}_{m\in\mathbb{Z}_{>0}}$ be a sequence of plurisubharmonic functions on a pseudoconvex domain $\Delta^n$, and let $\{F_m\}_{m\in\mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on $\Delta^n$. Denote $\Psi_m := \min\{\psi_m - 2\log|F_m|, 0\}$. Assume that $\{\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ converges to a Lebesgue measurable function $\Psi$ on $\Delta^n$ in Lebesgue measure. Let $\{f_m\}_{m\in\mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on $\{\Psi_m < -t_m\}$, where $t_m > 0$.

In this section, we present a lower semicontinuity property of $\Psi_m$ with respect to $I(\Psi_m)_o$.

**Proposition 1.13** *Assume that there exist $t_0 \geq t_m$ for any $m \in \mathbb{Z}_{>0}$ and a sequence of Lebesgue functions $\{\tilde{f}_m\}_{\mathbb{Z}_{>0}}$ on $\Delta^n$ with uniform bound satisfying that $\tilde{f}_m = f_m$ on $\{\Psi_m < -t_0\}$ for any $m \in \mathbb{Z}_{>0}$ and $\{\tilde{f}_m\}_{\mathbb{Z}_{>0}}$ converges to a Lebesgue measurable function $\tilde{f}$ on $\Delta^n$ in Lebesgue measure. Assume that for any pseudoconvex domain $D \subset \Delta^n$ containing the origin o,*

$$\inf_m G(0; \Psi_m, I(\Psi_m)_o, f_m) > 0.$$

*Then we have*

$$|\tilde{f}|^2 e^{-\Psi} \notin L^1(U),$$

*where $U$ is any neighborhood of o.*

*If $\{\psi_m\}_{m\in\mathbb{Z}_{>0}}$ and $\{\log|F_m|\}_{m\in\mathbb{Z}_{>0}}$ converge to a plurisubharmonic function $\psi$ and $\log|F|$ in Lebesgue measure respectively, where $F$ is a holomorphic function, and there exists a holomorphic function $f$ on $\{\psi - 2\log|F| < -t\} \cap V$ such that $f = \tilde{f}$ on $\{\psi - 2\log|F| < -t\} \cap V$, where $t > 0$ and $V$ is a neighborhood of o, we have*

$$f_o \notin I(\Psi)_o.$$

When $F_m \equiv 1$ and $\psi_m(o) = -\infty$ for any $m \in \mathbb{Z}_{>0}$, Proposition 1.13 is Proposition 1.8 in [30].

## 2 Preparations

### 2.1 $L^2$ methods

Let $F$ be a holomorphic function on a pseudoconvex domain $D \subset \mathbb{C}^n$, and let $\psi$ be a plurisubharmonic function on $D$. Let $\delta$ be a positive integer. Denote

$$\varphi := (1 + \delta) \max\{\psi, 2\log|F|\},$$

and

$$\Psi := \min\{\psi - 2\log|F|, 0\}.$$

If $F(z) = 0$ for $z \in D$, we set $\Psi(z) = 0$. Then $\varphi + \Psi$ and $\varphi + (1+\delta)\Psi$ are both plurisubharmonic functions on $D$.

**Lemma 2.1** *Let $B \in (0, +\infty)$ and $t_0 \in (0, +\infty)$ be arbitrarily given. Let $f$ be a holomorphic function on $\{\Psi < -t_0\}$ such that*

$$\int_{\{\Psi<-t_0\}} |f|^2 < +\infty.$$

*Then there exists a holomorphic function $\tilde{F}$ on $D$ such that*

$$\begin{aligned}&\int_D |\tilde{F} - (1 - b_{t_0,B}(\Psi)) f F^{1+\delta}|^2 e^{-\varphi+v_{t_0,B}(\Psi)-\Psi} \\ &\quad\le \left(\frac{1}{\delta} + 1 - e^{-t_0-B}\right) \int_D \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |f|^2 e^{-\Psi},\end{aligned}$$

*where $b_{t_0,B}(t) = \int_{-\infty}^{t} \frac{1}{B} \mathbb{I}_{\{-t_0-B<s<-t_0\}} ds$ and $v_{t_0,B}(t) = \int_0^t b_{t_0,B}(s) ds$.*

We give the proof of Lemma 2.1 in Appendix. Lemma 2.1 implies the following lemma.

**Lemma 2.2** (see [21], see also [24, 25]) *Let $B \in (0, +\infty)$ and $t_0 > t_1 \ge 0$ be arbitrarily given. Let $f$ be a holomorphic function on $\{\Psi < -t_0\}$ such that*

$$\int_{\{\Psi<-t_0\}} |f|^2 < +\infty.$$

*Then there exists a holomorphic function $\tilde{F}$ on $\{\Psi < -t_1\}$ such that*

$$\begin{aligned}&\int_{\{\Psi<-t_1\}} |\tilde{F} - (1 - b_{t_0,B}(\Psi)) f|^2 e^{v_{t_0,B}(\Psi)-\Psi} \\ &\quad\le \left(e^{-t_1} - e^{-t_0-B}\right) \int_D \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |f|^2 e^{-\Psi},\end{aligned}$$

*where $b_{t_0,B}(t) = \int_{-\infty}^{t} \frac{1}{B} \mathbb{I}_{\{-t_0-B<s<-t_0\}} ds$ and $v_{t_0,B}(t) = \int_0^t b_{t_0,B}(s) ds$.*

***Proof*** Let $\delta$ be any positive integer. Set $\tilde{\varphi} := (1+\delta)\max\{\psi + t_1, 2\log|F|\}$ and $\tilde{\Psi} := \{\psi + t_1 - 2\log|F|, 0\}$. Note that $\{\Psi < -t_0\} = \{\tilde{\Psi} < -(t_0 - t_1)\}$ and $\tilde{\Psi} = \Psi + t_1$ on $\{\Psi < -t_1\}$. Lemma 2.1 shows that there exists a holomorphic function $\tilde{F}_\delta$ on $D$ such that

$$\begin{aligned}&\int_D |\tilde{F}_\delta - (1 - b_{t_0-t_1,B}(\tilde{\Psi})) f F^{1+\delta}|^2 e^{-\tilde{\varphi}+v_{t_0-t_1,B}(\tilde{\Psi})-\tilde{\Psi}} \\ &\quad\le \left(\frac{1}{\delta} + 1 - e^{-t_0-B+t_1}\right) \int_D \frac{1}{B} \mathbb{I}_{\{-t_0-B+t_1<\tilde{\Psi}<-t_0+t_1\}} |f|^2 e^{-\tilde{\Psi}},\end{aligned}$$

which implies that

$$
\begin{aligned}
&\int_{\{\Psi<-t_1\}} |\tilde{F}_\delta-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2e^{-\varphi+v_{t_0,B}(\Psi)-\Psi}\\
&\quad=\int_{\{\Psi<-t_1\}} |\tilde{F}_\delta-(1-b_{t_0-t_1,B}(\tilde{\Psi}))fF^{1+\delta}|^2e^{-\tilde{\varphi}+v_{t_0-t_1,B}(\tilde{\Psi})-\tilde{\Psi}}\\
&\quad\le\int_D |\tilde{F}_\delta-(1-b_{t_0-t_1,B}(\tilde{\Psi}))fF^{1+\delta}|^2e^{-\tilde{\varphi}+v_{t_0-t_1,B}(\tilde{\Psi})-\tilde{\Psi}}\\
&\quad\le\left(\frac{1}{\delta}e^{-t_1}+e^{-t_1}-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}.
\end{aligned}
\tag{2.1}
$$

Let $F_\delta=\frac{\tilde{F}_\delta}{F^{1+\delta}}$ be a holomorphic function on $\{\Psi<-t_1\}$. Since $|F|^{2(1+\delta)}e^{-\varphi}=1$ on $\{\Psi<-t_1\}$, inequality (2.1) becomes

$$
\begin{aligned}
&\int_{\{\Psi<-t_1\}} |F_\delta-(1-b_{t_0,B}(\Psi))f|^2e^{v_{t_0,B}(\Psi)-\Psi}\\
&\quad\le\left(\frac{1}{\delta}e^{-t_1}+e^{-t_1}-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}.
\end{aligned}
\tag{2.2}
$$

As $\int_{\{\Psi<-t_0\}}|f|^2<+\infty$ and $v_{t_0,B}(\Psi)-\Psi\ge 0$, it follows from inequality (2.2) that

$$
\begin{aligned}
&\sup_\delta\int_{\{\Psi<-t_1\}}|F_\delta|^2\\
&\le 2\int_{\{\Psi<-t_1\}}|(1-b_{t_0,B}(\Psi))f|^2+2\sup_\delta\int_{\{\Psi<-t_1\}}|F_\delta-(1-b_{t_0,B}(\Psi))f|^2\\
&\le 2\int_{\{\Psi<-t_0\}}|f|^2+2\sup_\delta\left(\frac{1}{\delta}e^{-t_1}+e^{-t_1}-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}\\
&<+\infty,
\end{aligned}
$$

yielding that there exists a subsequence of $\{F_\delta\}$ (also denoted by $\{F_\delta\}$) compactly convergent to a holomorphic function $\tilde{F}$ on $\{\Psi<-t_1\}$. It follows from Fatou's Lemma and inequality (2.2) that

$$
\begin{aligned}
&\int_{\{\Psi<-t_1\}} |F-(1-b_{t_0,B}(\Psi))f|^2e^{v_{t_0,B}(\Psi)-\Psi}\\
&\quad=\int_{\{\Psi<-t_1\}}\lim_{\delta\to+\infty}|F_\delta-(1-b_{t_0,B}(\Psi))f|^2e^{v_{t_0,B}(\Psi)-\Psi}\\
&\quad\le\liminf_{\delta\to+\infty}\int_{\{\Psi<-t_1\}}|F_\delta-(1-b_{t_0,B}(\Psi))f|^2e^{v_{t_0,B}(\Psi)-\Psi}\\
&\quad\le\liminf_{\delta\to+\infty}\left(\frac{1}{\delta}e^{-t_1}+e^{-t_1}-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}\\
&\quad=\left(e^{-t_1}-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}.
\end{aligned}
$$

Thus, Lemma 2.2 holds. □

### 2.2 $\mathcal{O}_{\mathbb{C}^n,o}$-module $I(a\Psi)_o$

Let $F$ be a holomorphic function on a pseudoconvex domain $D \subset \mathbb{C}^n$ containing the origin $o \in \mathbb{C}^n$, and let $\psi$ be a plurisubharmonic function on $D$. Denote $\Psi := \min\{\psi - 2\log|F|, 0\}$ and $\varphi := 2\max\{\psi, 2\log|F|\}$. In this section, we discuss the $\mathcal{O}_{\mathbb{C}^n,o}$-module $I(a\Psi)_o$, where $a \geq 0$.

Let $k$ be a positive integer. Denote $I_o := I(0\Psi)_o$.

**Lemma 2.3** *For any $f_o \in I_o$, there exists a pseudoconvex domain $D_0 \subset D$ containing $o$ and a holomorphic function $\tilde{F}$ on $D_0$ such that $(\tilde{F}, o) \in \mathcal{I}(k\varphi)_o$ and $\int_{\{\Psi<-t\}\cap D_0} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty$ for some $t > 0$.*

***Proof*** For any $f_o \in I_o$, there exists a pseudoconvex domain $D_0 \subset D$ containing $o$ and $t_0 > 0$ such that $f \in \mathcal{O}(\{k\Psi < -t_0\} \cap D_0)$ and $\int_{\{k\Psi<-t_0\}\cap D_0} |f|^2 < +\infty$. It follows from Lemma 2.1 that there exists a holomorphic function $\tilde{F}$ on $D_0$ such that

$$\begin{aligned}&\int_{D_0} |\tilde{F} - (1-b(k\Psi))fF^{2k}|^2 e^{-k\varphi+v(k\Psi)-k\Psi}\\&\leq (2-e^{-t_0-1})\int_{D_0} \mathbb{I}_{\{-t_0-1<k\Psi<-t_0\}}|f|^2 e^{-k\Psi},\end{aligned} \tag{2.3}$$

where $b(t) = \int_{-\infty}^{t} \mathbb{I}_{\{-t_0-1<s<-t_0\}}ds$ and $v(t) = \int_0^t b(s)ds$. Note that $v(t) \geq -t_0 - 1$ on $\mathbb{R}$ and $b(t) = 0$ on $(-\infty, -t_0-1)$. Inequality (2.3) implies that

$$\begin{aligned}&\int_{D_0\cap\{k\Psi<-t_0-1\}} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi-k\Psi}\\&\leq e^{t_0+1}\int_{D_0} |\tilde{F} - (1-b(k\Psi))fF^{2k}|^2 e^{-k\varphi+v(k\Psi)-k\Psi}\\&\leq (2e^{t_0+1}-1)\int_{D_0} \mathbb{I}_{\{-t_0-1<k\Psi<-t_0\}}|f|^2 e^{-k\Psi}\\&< +\infty.\end{aligned}$$

Consequently, since $v(k\Psi) - k\Psi \geq 0$, $b(t) = 1$ on $[-t_0, +\infty)$, and $|F|^{4k}e^{-k\varphi} = 1$ on $\{k\Psi < -t_0\}$, we deduce from Inequality (2.3) that

$$\begin{aligned}&\int_{D_0} |\tilde{F}|^2 e^{-k\varphi}\\&\leq 2\int_{D_0} |(1-b(k\Psi))fF^{2k}|^2 e^{-k\varphi} + 2\int_{D_0} |\tilde{F} - (1-b(k\Psi))fF^{2k}|^2 e^{-k\varphi}\\&\leq 2\int_{D_0\cap\{k\Psi<-t_0\}} |f|^2 + 2(2-e^{-t_0-1})\int_{D_0} \mathbb{I}_{\{-t_0-1<k\Psi<-t_0\}}|f|^2 e^{-k\Psi}\\&< +\infty.\end{aligned}$$

Thus, Lemma 2.3 holds. □

Let $f_o \in I_o$. Taking any $(\tilde{F}, o) \in \mathcal{I}(k\varphi)_o$ and $(\tilde{F}_1, o) \in \mathcal{I}(k\varphi)_o$, if there exist $t_1 > 0$ and a neighborhood $D_1$ of $o$ such that $\int_{\{\Psi<-t_1\}\cap D_1} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty$ and $\int_{\{\Psi<-t_1\}\cap D_1} |\tilde{F}_1 - fF^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty$, then we have

$$\int_{\{\Psi<-t_1\}\cap D_1} |\tilde{F}_1 - \tilde{F}|^2 e^{-k\varphi-k\Psi} < +\infty, \tag{2.4}$$

and there exists a neighborhood $D_2$ of $o$ such that

$$\int_{D_2} |\tilde{F} - \tilde{F}_1|^2 e^{-k\varphi} < +\infty. \tag{2.5}$$

Combining inequality (2.4) and inequality (2.5), we obtain that $(\tilde{F} - \tilde{F}_1, o) \in \mathcal{I}(k\varphi + k\Psi)_o$. Thus, according to Lemma 2.3, there exists a map

$$\tilde{P} : I_o \to \mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi + k\Psi)_o,$$

given by

$$\tilde{P}(f_o) = [(\tilde{F}, o)],$$

for any $f_o \in I_o$, where $(\tilde{F}, o) \in \mathcal{I}(k\varphi)_o$ such that $\int_{\{\Psi<-t\}\cap D_0} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty$ for some $t > 0$ and some neighborhood $D_0$ of $o$, and $[(\tilde{F}, o)]$ is the equivalence class of $(\tilde{F}, o)$ in $\mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi + k\Psi)_o$.

**Lemma 2.4** *$\tilde{P}$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module homomorphism, and* $\mathrm{Ker}(\tilde{P}) = I(k\Psi)_o$.

***Proof*** For any $f_o \in I_o$ and $\tilde{f}_o \in I_o$, take $[(\tilde{F}, o)] = \tilde{P}(f_o)$ and $[(\tilde{F}_1, o)] = \tilde{P}(\tilde{f}_o)$. Then we have $(\tilde{F} + \tilde{F}_1, o) \in \mathcal{I}(k\varphi + k\Psi)_o$, and

$$\begin{aligned}
&\int_{\{\Psi<-t_1\}\cap D_1} |\tilde{F}_1 + \tilde{F} - (f + \tilde{f})F^{2k}|^2 e^{-k\varphi-k\Psi} \\
&\leq 2\int_{\{\Psi<-t_1\}\cap D_1} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi-k\Psi} + 2\int_{\{\Psi<-t_1\}\cap D_1} |\tilde{F}_1 - \tilde{f}F^{2k}|^2 e^{-k\varphi-k\Psi} \\
&< +\infty,
\end{aligned}$$

for some $t_1 > 0$ and some neighborhood $D_1$ of $o$, which shows that

$$\tilde{P}(f_o + \tilde{f}_o) = \tilde{P}((f + \tilde{f})_o) = [(\tilde{F} + \tilde{F}_1, o)] = \tilde{P}(f_o) + \tilde{P}(\tilde{f}_o).$$

For any $(h, o) \in \mathcal{O}_{\mathbb{C}^n,o}$, we have $(h\tilde{F}, o) \in \mathcal{I}(k\varphi + k\Psi)_o$ and

$$\int_{\{\Psi<-t_2\}\cap D_2} |h\tilde{F} - hfF^{2k}|^2 e^{-k\varphi-k\Psi} \leq C\int_{\{\Psi<-t_2\}\cap D_2} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty,$$

for some $C > 0$, $t_2 > 0$ and a neighborhood $D_2$ of $o$, which shows that

$$\tilde{P}((h,o)\cdot f_o) = \tilde{P}((hf)_o) = [(h\tilde{F},o)] = (h,o)\cdot \tilde{P}(f_o).$$

Thus, $\tilde{P}$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module homomorphism from $I_o$ to $\mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi+k\Psi)_o$.

Now, we prove $\mathrm{Ker}(\tilde{P}) = I(k\Psi)_o$. For any $f_o \in I(k\Psi)_o$, $\tilde{P}(f_o) = 0$ holds if and only if there exist a neighborhood $D_3$ of $o$ and $t_3 > 0$ such that

$$\int_{\{\Psi<-t_3\}\cap D_3} |f_o F^{2k}|^2 e^{-k\varphi-k\Psi} = \int_{\{\Psi<-t_3\}\cap D_3} |f_o|^2 e^{-k\Psi} < +\infty,$$

i.e. $f_o \in I(k\Psi)_o$. □

Following from Lemma 2.4, there exists an $\mathcal{O}_{\mathbb{C}^n,o}$-module homomorphism

$$P : I_o/I(k\Psi)_o \to \mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi+k\Psi)_o,$$

given by

$$P([f_o]_1) = \tilde{P}(f_o),$$

for any $f_o \in I_o$, where $[f_o]_1$ is the equivalence class of $f_o$ in $I_o/I(k\Psi)_o$.

Note that $|F^{2k}|^2e^{-k\varphi} = 1$ on $\{\Psi < 0\}$. For any $(\tilde{F},o) \in \mathcal{I}(k\varphi)_o$, it is clear that $\left(\frac{\tilde{F}}{F^{2k}}\right)_o \in I_o$. Moreover, if $(\tilde{F},o) \in \mathcal{I}(k\varphi+k\Psi)_o$, we have $\left(\frac{\tilde{F}}{F^{2k}}\right)_o \in I(k\Psi)_o$. Hence, there exists an $\mathcal{O}_{\mathbb{C}^n,o}$-module homomorphism

$$Q : \mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi+k\Psi)_o \to I_o/I(k\Psi)_o,$$

given by

$$Q([(\tilde{F},o)]) = \left[\left(\frac{\tilde{F}}{F^{2k}}\right)_o\right]_1,$$

for any $[(\tilde{F},o)] \in \mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi+k\Psi)_o$.

**Lemma 2.5** *$P$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module isomorphism from $I_o/I(k\Psi)_o$ to $\mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi+k\Psi)_o$ and $P^{-1} = Q$.*

***Proof*** According to the definitions of $Q$ and $P$, we have that

$$P\circ Q([(\tilde{F},o)]) = [(\tilde{F},o)]$$

for any $[(\tilde{F},o)] \in \mathcal{I}(k\varphi)_o/\mathcal{I}(k\varphi+k\Psi)_o$, which implies that $P$ is surjective. Lemma 2.4 shows that $P$ is injective. Thus, we get that $P$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module isomorphism and $P^{-1} = Q$. □

Let $a \in [0, k)$. Denote $P_a := P|_{I(a\Psi)_o/I(k\Psi)_o}$, where $I(a\Psi)_o/I(k\Psi)_o$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-submodule of $I_o/I(k\Psi)_o$. It is clear that $k\varphi + a\Psi$ is a plurisubharmonic function on $D$. The following lemma gives an isomorphism between $I(a\Psi)_o/I(k\Psi)_o$ and $\mathcal{I}(k\varphi + a\Psi)_o/\mathcal{I}(k\varphi + k\Psi)_o$.

**Lemma 2.6** *$P_a$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module isomorphism from $I(a\Psi)_o/I(k\Psi)_o$ to $\mathcal{I}(k\varphi + a\Psi)_o/\mathcal{I}(k\varphi + k\Psi)_o$.*

***Proof*** It suffices to prove $\mathrm{Im}(P_a) = \mathcal{I}(k\varphi+a\Psi)_o/\mathcal{I}(k\varphi+k\Psi)_o$. For any $f_o \in I(a\Psi)_o$, taking $[(\tilde{F}, o)] = P_a([f_o]_1) = P([f_o]_1)$, we have that there exist $t > 0$ and a neighborhood $D_0$ of $o$ such that

$$\begin{aligned} &\int_{\{\Psi<-t\}\cap D_0} |\tilde{F}|^2 e^{-k\varphi-a\Psi} \\ &\quad\le 2\int_{\{\psi<-t\}\cap D_0} |f|^2 e^{-a\Psi} + 2\int_{\{\Psi<-t\}\cap D_0} |\tilde{F} - fF^{2k}|^2 e^{-k\varphi-a\Psi} \\ &\quad< +\infty. \end{aligned} \tag{2.6}$$

Combining $(\tilde{F}, o) \in \mathcal{I}(k\varphi)_o$ and inequality (2.6), we get $(\tilde{F}, o) \in \mathcal{I}(k\varphi + a\Psi)_o$, which shows $\mathrm{Im}(P_a) \subset \mathcal{I}(k\varphi+a\Psi)_o/\mathcal{I}(k\varphi+k\Psi)_o$. For any $(\tilde{F}_1, o) \in \mathcal{I}(k\varphi+a\Psi)_o$, it is clear that $\left(\frac{\tilde{F}_1}{F^{2k}}\right)_o \in I(a\Psi)_o$. Then we obtain

$$P^{-1}([(\tilde{F}_1, o)]) = Q([(\tilde{F}_1, o)]) \in I(a\Psi)_o/I(k\Psi)_o,$$

which implies that $\mathrm{Im}(P_a) = \mathcal{I}(k\varphi + a\Psi)_o/\mathcal{I}(k\varphi + k\Psi)_o$. □

According to the definition of $I(a\Psi)_o$, we have that $I(a\Psi)_o \subset I(a'\Psi)_o$ for any $0 \le a' < a < +\infty$. Denote

$$I_+(a\Psi)_o := \bigcup_{p>a} I(p\Psi)_o,$$

which is an $\mathcal{O}_{\mathbb{C}^n,o}$-submodule of $I_o$, where $a \ge 0$.

**Lemma 2.7** *There exists $a' > a$ such that $I(a'\Psi)_o = I_+(a\Psi)_o$ for any $a \ge 0$.*

***Proof*** By the definition of $I_+(a\Psi)_o$, $I(p\Psi)_o \subset I_+(a\Psi)_o$ for any $p > a$. Then it suffices to prove that there exists $a' > a$ such that $I_+(a\Psi)_o \subset I(a'\Psi)_o$.

Let $k > a$ be an integer. It is clear that $\mathcal{I}(k\varphi + a\Psi)_o \subset \mathcal{I}(k\varphi + a'\Psi)$ for any $0 \le a' < a \le k$. Denote

$$\mathfrak{I} := \bigcup_{a<p<k} \mathcal{I}(k\varphi + p\Psi)_o,$$

which is an ideal of $\mathcal{O}_{\mathbb{C}^n,o}$. It follows from Lemma 2.5 that $P|_{I_+(a\Psi)_o/I(k\Psi)_o}$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module isomorphism from $I_+(a\Psi)_o/I(k\Psi)_o$ to $\mathfrak{I}/\mathcal{I}(k\varphi + k\Psi)_o$. As $\mathcal{O}_{\mathbb{C}^n,o}$ is a Noetherian ring (see [34]), we get that $\mathfrak{I}$ is finitely generated. Hence, there exists a finite set $\{(f_1)_o, \ldots, (f_m)_o\} \subset I_+(a\Psi)_o$, which satisfies that for any $f_o \in I_+(a\Psi)_o$, there exists $(h_j, o) \in \mathcal{O}_{\mathbb{C}^n,o}$ for any $1 \le j \le m$, such that

$$f_o - \sum_{j=1}^{m} (h_j, o) \cdot (f_j)_o \in I(k\Psi)_o.$$

By the definition of $I_+(a\Psi)_o$, there exists $a' \in (a, k)$ such that $\{(f_1)_o, \ldots, (f_m)_o\} \subset I(a'\Psi)_o$. As $(h_j, o) \cdot (f_j)_o \in I(a'\Psi)_o$ for any $1 \le j \le m$ and $I(k\Psi)_o \subset I(a'\Psi)_o$, we obtain that $I_+(a\Psi)_o \subset I(a'\Psi)_o$.

Thus, Lemma 2.7 holds. □

We recall the closedness of submodules, which can be referred to [19] and will be used in the proof of Lemma 2.9.

**Lemma 2.8** (see [19]) *Let $N$ be a submodule of $\mathcal{O}^q_{\mathbb{C}^n,o}$, $1 \le q < +\infty$, let $f_j \in \mathcal{O}_{\mathbb{C}^n,o}(U)^q$ be a sequence of $q-$tuples holomorphic in an open neighborhood $U$ of the origin $o$. Assume that the $f_j$ converge uniformly in $U$ towards to a $q-$tuples $f \in \mathcal{O}_{\mathbb{C}^n,o}(U)^q$, assume furthermore that all germs $(f_j, o)$ belong to $N$. Then $(f, o) \in N$.*

The following lemma will be used in the proof of Theorem 1.6.

**Lemma 2.9** *Let $J$ be an $\mathcal{O}_{\mathbb{C}^n,o}$-submodule of $I_o$ such that $I(k\Psi)_o \subset J$ for some positive integer $k$, and let $f \in J(\Psi)_o$. Let $D_0 \subset D$ containing $o$ be a pseudoconvex domain, and let $f_j$ be a sequence of holomorphic functions on $\{\Psi < -s_j\} \cap D_0$ for any $j \in \mathbb{Z}_{>0}$, where $s_j \ge 0$. Assume that $s_0 := \lim_{j\to+\infty} s_j \in [0, +\infty)$,*

$$\limsup_{j\to+\infty} \int_{\{\Psi<-s_j\}\cap D_0} |f_j|^2 \le C < +\infty,$$

*and $(f_j - f)_o \in J$. Then there exists a subsequence of $\{f_j\}_{j\in\mathbb{Z}_{>0}}$ compactly convergent to a holomorphic function $f_0$ on $\{\Psi < -s_0\} \cap D_0$, which satisfies that*

$$\int_{\{\Psi<-s_0\}\cap D_0} |f_0|^2 \le C,$$

*and $(f_0 - f)_o \in J$.*

***Proof*** Since $\limsup_{j\to+\infty} \int_{\{\Psi<-s_j\}\cap D_0} |f_j|^2 < +\infty$ and $\lim_{j\to+\infty} s_j = s_0$, we can extract a subsequence of $\{f_j\}_{j\in\mathbb{Z}_{>0}}$ (also denoted by $\{f_j\}_{j\in\mathbb{Z}_{>0}}$) compactly convergent to a holomorphic function $f_0$ on $\{\Psi < -s_0\} \cap D_0$ and

$$\int_{\{\Psi<-s_0\}\cap D_0} |f_0|^2 \le C,$$

which implies $(f_0)_o \in I_o$. Thus, it suffices to prove $(f_0 - f)_o \in J$.

Denote $\varphi := 2\max\{\psi, 2\log|F|\}$, and $t_j := ks_j$ for $j \in \mathbb{Z}_{\geq 0}$. For any $j \in \mathbb{Z}_{>0}$, it follows from Lemma 2.1 that there exists a holomorphic function $F_j$ on $D_0$ satisfying that

$$\begin{aligned}&\int_{D_0} |F_j - (1-b_{t_j,1}(k\Psi))f_jF^{2k}|^2 e^{-k\varphi+v_{t_j,1}(k\Psi)-k\Psi}\\&\leq (2-e^{-t_j-1})\int_{D_0} \mathbb{I}_{\{-t_j-1<k\Psi<-t_j\}}|f_j|^2 e^{-k\Psi}\\&\leq (2e^{t_j+1}-1)\int_{\{k\Psi<-t_j\}\cap D_0} |f_j|^2,\end{aligned} \tag{2.7}$$

where $b_{t_j,1}(t) = \int_{-\infty}^{t} \mathbb{I}_{\{-t_j-1<s<-t_j\}}ds$ and $v_{t_j,1}(t) = \int_0^t b_{t_j,1}(s)ds$. Note that $b_{t_j,1}(t) = 0$ for $t < -t_j - 1$. Then we have

$$\int_{\{k\Psi<-t_j-1\}\cap D_0} |F_j - f_jF^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty. \tag{2.8}$$

In addition, note that $b_{t_j,1}(t) = 1$ for $t \geq t_j$ and $|F|^{2k}e^{-k\varphi} = 1$ on $\{k\Psi < -t_j\} \cap D_0$. Then as $v_{t_j,1}(k\Psi) - k\Psi \geq 0$, it follows from inequality (2.7) that

$$\begin{aligned}&\int_{D_0} |F_j|^2 e^{-k\varphi}\\&\leq 2\int_{D_0} |(1-b_{t_j,1}(k\Psi))f_jF^{2k}|^2 e^{-k\varphi} + 2\int_{D_0} |F_j - (1-b_{t_j,1}(k\Psi))f_jF^{2k}|^2 e^{-k\varphi}\\&\leq (2e^{t_j+1}+1)\int_{\{k\Psi<-t_j\}\cap D_0} |f_j|^2.\end{aligned} \tag{2.9}$$

Since $\lim_{j\to+\infty} t_j = t_0$ and $\limsup_{j\to+\infty} \int_{\{k\Psi<-t_j\}\cap D_0} |f_j|^2 < +\infty$, inequality (2.9) and Fatou's Lemma imply that there exists a subsequence of $\{F_j\}_{j\in\mathbb{Z}_{>0}}$ denoted by $\{F_{j_l}\}_{l\in\mathbb{Z}_{>0}}$, which compactly converges to a holomorphic function $F_0$ on $D_0$ with

$$\int_{D_0} |F_0|^2 e^{-k\varphi} \leq \liminf_{l\to+\infty} \int_{D_0} |F_{j_l}|^2 e^{-k\varphi} < +\infty. \tag{2.10}$$

As $f_j$ converge to $f_0$, by inequality (2.7) and Fatou's Lemma, we obtain

$$\begin{aligned}&\int_{D_0} |F_0 - (1-b_{t_0,1}(k\Psi))f_0F^{2k}|^2 e^{-k\varphi+v_{t_j,1}(k\Psi)-k\Psi}\\&= \int_{D_0} \liminf_{l\to+\infty} |F_{j_l} - (1-b_{t_{j_l},1}(k\Psi))f_{j_l}F^{2k}|^2 e^{-k\varphi+v_{t_{j_l},1}(k\Psi)-k\Psi}\\&\leq \liminf_{l\to+\infty} \int_{D_0} |F_{j_l} - (1-b_{t_{j_l},1}(k\Psi))f_{j_l}F^{2k}|^2 e^{-k\varphi+v_{t_{j_l},1}(k\Psi)-k\Psi}\\&< +\infty,\end{aligned}$$

yielding that

$$\int_{\{k\Psi<-t_0-1\}\cap D_0} |F_0 - f_0F^{2k}|^2 e^{-k\varphi-k\Psi} < +\infty. \tag{2.11}$$

Combining inequality (2.8), inequality (2.9), inequality (2.10), inequality (2.11) and the definition of $P : I_o/I(k\Psi)_o \to \mathcal{I}(k\varphi)/\mathcal{I}(k\varphi + k\Psi)$, we get

$$P([(f_j)_o]_1) = [(F_j, o)],$$

for any $j \in \mathbb{Z}_{\geq 0}$. $(f_j - f)_o \in J$ for any $j \in \mathbb{Z}_{>0}$ implies $(f_j - f_1)_o \in J$ for any $j \in \mathbb{Z}_{>0}$. Lemma 2.5 shows that there exists an ideal $\tilde{J}$ of $\mathcal{O}_{\mathbb{C}^n,o}$ such that

$$\mathcal{I}(k\varphi + k\Psi)_o \subset \tilde{J} \subset \mathcal{I}(k\varphi)_o,$$

and

$$\tilde{J}/\mathcal{I}(k\varphi + k\Psi)_o = \mathrm{Im}(P|_{J/I(k\Psi)_o}).$$

Then $(F_j - F_1, o) \in \tilde{J}$, for any $j \in \mathbb{Z}_{>0}$. As $F_j$ compactly converge to $F_0$ on $D_0$, using Lemma 2.8, we obtain $(F_0 - F_1, o) \in \tilde{J}$. Recall that $P$ is an $\mathcal{O}_{\mathbb{C}^n,o}$-module isomorphism, and $\tilde{J}/\mathcal{I}(k\varphi + k\Psi)_o = \mathrm{Im}(P|_{J/I(k\Psi)_o})$. We deduce $(f_0 - f_1)_o \in J$, which verifies $(f_0 - f)_o \in J$.

Thus, Lemma 2.9 holds. □

Let $f$ be a holomorphic function on $D$. Denote that

$$\Psi_1 := \min\{2c_o^{fF}(\psi)\psi - 2\log|F|, 0\},$$

and

$$I_+(\Psi_1)_o := \bigcup_{a>1} I(a\Psi_1)_o.$$

**Lemma 2.10** $f_o \notin I_+(\Psi_1)_o$.

***Proof*** We prove Lemma 2.10 by contradiction: if not, there exists $p_0 > 1$ such that $f_o \in I(p_0\Psi_1)_o$, which implies that

$$\int_{\{\Psi_1<-t\}\cap V} |f|^2|F|^{2p_0}e^{-2p_0c_o^{fF}(\psi)\psi} = \int_{\{\Psi_1<-t\}\cap V} |f|^2 e^{-p_0\Psi} < +\infty \tag{2.12}$$

holds for some $t > 0$ and some neighborhood $V \Subset D$ of $o$. Note that

$$\sup_{V\cap\{\Psi_1\geq t\}} |f|^2|F|^{2p_0}e^{-2p_0c_o^{fF}(\psi)\psi} < +\infty.$$

Then inequality (2.12) indicates

$$\int_V |f|^2|F|^{2p_0}e^{-2p_0c_o^{fF}(\psi)\psi} < +\infty. \tag{2.13}$$

As $\log|F|$ is a plurisubharmonic function on $D$, there exist a neighborhood $U \subset V$ of $o$ and $p_0' \in (1, p_0)$ such that

$$\int_U |F|^{2(1-p_0')\frac{p_0}{p_0-p_0'}} = \int_U e^{-2(p_0'-1)\frac{p_0}{p_0-p_0'}\log|F|} < +\infty. \tag{2.14}$$

It follows from inequality (2.13), inequality (2.14) and Hölder's inequality that

$$\begin{aligned}
&\int_U |fF|^2e^{-2p_0'c_o^{fF}(\psi)\psi}\\
&= \int_U |f|^2|F|^{2p_0'}e^{-2p_0'c_o^{fF}(\psi)\psi}|F|^{2-2p_0'}\\
&\le \left(\int_U |f|^{2\frac{p_0}{p_0'}}|F|^{2p_0}e^{-2p_0c_o^{fF}(\psi)\psi}\right)^{\frac{p_0'}{p_0}} \times \left(\int_U |F|^{2(1-p_0')\frac{p_0}{p_0-p_0'}}\right)^{\frac{p_0-p_0'}{p_0}}\\
&< +\infty,
\end{aligned}$$

which contradicts the definition of $c_o^{fF}(\psi)$. Thus, we have $f_o \notin I_+(\Psi_1)_o$. □

### 2.3 Some properties of $G(t)$

Following the notations and assumptions in Sect. 1.2, we present some properties related to $G(t; \Psi, J, f)$ ($G(t)$ for short) in this section.

**Lemma 2.11** *$f_o \in J$ if and only if $G(t; \Psi, J, f) = 0$.*

***Proof*** If $f_o \in J$, it follows from the definition of $G(t; \Psi, J, f)$ that $G(t; \Psi, J, f) = 0$.

If $G(t; \Psi, J, f) = 0$, there exists a sequence of holomorphic functions $\{f_j\}_{j\in\mathbb{Z}_{>0}}$ on $\{\Psi < -t\}$ such that $\lim_{j\to+\infty}\int_{\{\Psi<-t\}}|f_j|^2 = 0$ and $(f_j - f)_o \in J$. Then Lemma 2.9 implies that there exists a holomorphic function $f_0$ on $\{\Psi < -t\}$ such that $\int_{\{\Psi<-t\}}|f_0|^2 = 0$ and $(f_0 - f)_o \in J$, which shows $f \in J$. □

The following lemma shows the existence and uniqueness of the minimal holomorphic function.

**Lemma 2.12** *Assume that $G(t) < +\infty$. Then there exists a unique holomorphic function $f_t$ on $\{\Psi < -t\}$ satisfying $(f_t - f)_o \in J$ and $\int_{\{\Psi<-t\}}|f_t|^2 = G(t)$. Furthermore, for any holomorphic function $\hat{f}$ on $\{\Psi < -t\}$ satisfying $(\hat{f} - f)_o \in J$ and $\int_{\{\Psi<-t\}}|\hat{f}|^2 < +\infty$, we have the following equality*

$$\int_{\{\Psi<-t\}}|f_t|^2 + \int_{\{\Psi<-t\}}|\hat{f} - f_t|^2 = \int_{\{\Psi<-t\}}|\hat{f}|^2. \tag{2.15}$$

***Proof*** Firstly, we prove the existence of $f_t$. As $G(t) < +\infty$, there exist holomorphic functions $\{F_j\}_{j\in\mathbb{Z}_{>0}}$ on $\{\Psi < -t\}$ such that $\lim_{j\to+\infty}\int_{\{\Psi<-t\}}|F_j|^2 = G(t)$ and $(F_j - f)_o \in J$. Lemma 2.9 indicates that there exists a holomorphic function $f_t$ on $\{\Psi < -t\}$ such that $(f_t - f)_o \in J$ and $\int_{\{\Psi<-t\}}|f_t|^2 \leq G(t)$. By the definition of $G(t)$, we have $\int_{\{\Psi<-t\}}|f_t|^2 = G(t)$.

Secondly, we prove the uniqueness of $f_t$ by contradiction: if not, there exist two different holomorphic functions $\tilde{f}_1$ and $\tilde{f}_2$ on $\{\Psi < -t\}$ satisfying $\int_{\{\Psi<-t\}}|\tilde{f}_1|^2 = \int_{\{\Psi<-t\}}|\tilde{f}_2|^2 = G(t)$, $(\tilde{f}_1 - f)_o \in J$, and $(\tilde{f}_2 - f)_o \in J$. Note that

$$\begin{aligned} &\int_{\{\Psi<-t\}}\left|\frac{\tilde{f}_1+\tilde{f}_2}{2}\right|^2 + \int_{\{\Psi<-t\}}\left|\frac{\tilde{f}_1-\tilde{f}_2}{2}\right|^2 \\ &= \frac{\int_{\{\Psi<-t\}}|\tilde{f}_1|^2 + \int_{\{\Psi<-t\}}|\tilde{f}_2|^2}{2} = G(t). \end{aligned} \tag{2.16}$$

Then we obtain that

$$\int_{\{\psi<-t\}}\left|\frac{f_1+f_2}{2}\right|^2 < G(t),$$

and $(\frac{\tilde{f}_1+\tilde{f}_2}{2} - f)_o \in J$, which contradicts the definition of $G(t)$.

Finally, we prove equality (2.15). For any holomorphic function $h$ on $\{\Psi < -t\}$ satisfying $h_o \in J$ and $\int_{\{\Psi<-t\}}|h|^2 < +\infty$, it is clear that for any complex number $\alpha$, $f_t + \alpha h$ satisfies $(f_t + \alpha h - f)_o \in J$. The definition of $G(t)$ shows that

$$\int_{\{\Psi<-t\}}|f_t|^2 \leq \int_{\{\Psi<-t\}}|f_t + \alpha h|^2 < +\infty,$$

yielding that

$$\mathrm{Re}\int_{\{\psi<-t\}} f_t\overline{h} = 0.$$

Then we get

$$\int_{\{\Psi<-t\}}|f_t + h|^2 = \int_{\{\Psi<-t\}}|f_t|^2 + \int_{\{\Psi<-t\}}|h|^2.$$

Choosing $h = \hat{f} - f_t$, we obtain equality (2.15). □

We present the monotonicity and the lower semicontinuity of $G(t)$.

**Lemma 2.13** *$G(t)$ is decreasing on $[0, +\infty)$, such that $\lim_{t\to t_0+0} G(t) = G(t_0)$ for $t_0 \in [0, +\infty)$. If there exists $t_1 \geq 0$ such that $G(t_1) < +\infty$, then $\lim_{t\to+\infty} G(t) = 0$. Especially, $G(t)$ is lower semicontinuous on $[0, +\infty)$ if $G(0) < +\infty$.*

***Proof*** By the definition of $G(t)$, it is clear that $G(t)$ is decreasing on $[0, +\infty)$. The dominated convergence theorem verifies that $\lim_{t\to+\infty} G(t) = 0$ if there exists $t_1 \geq 0$ such that $G(t_1) < +\infty$.

It suffices to prove $\lim_{t\to t_0+0} G(t) = G(t_0)$. We prove it by contradiction: if not, then $\lim_{t\to t_0+0} G(t) < G(t_0)$ for some $t_0 \in [0, +\infty)$. By Lemma 2.12, there exists a unique holomorphic function $f_t$ on $\{\Psi < -t\}$ satisfying that $(f_t - f)_o \in J$ and $\int_{\{\Psi<-t\}} |f_t|^2 = G(t)$. Note that $G_{F,\psi,c}(t)$ is decreasing, indicating that $\lim_{t\to t_0+0} \int_{\{\Psi<-t\}} |f_t|^2 < G(t_0)$. It follows from Lemma 2.9 that there exists a holomorphic function $\tilde{f}_{t_0}$ on $\{\Psi < -t_0\}$ such that $(\tilde{f}_{t_0} - f)_o \in J$ and $\int_{\{\Psi<-t\}} |\tilde{f}_{t_0}|^2 \leq \lim_{t\to t_0+0} \int_{\{\Psi<-t\}} |f_t|^2 < G(t_0)$, which contradicts the definition of $G(t_0)$. Thus, we have $\lim_{t\to t_0+0} G(t) = G(t_0)$. □

We consider the derivatives of $G(t)$ in the following lemma.

**Lemma 2.14** *Assume that $G(t_1) < +\infty$, where $t_1 \in [0, +\infty)$. Then for any $t_0 > t_1$, we have*

$$\frac{G(t_1) - G(t_0)}{e^{-t_1} - e^{-t_0}} \leq \liminf_{B\to 0+0} \frac{G(t_0) - G(t_0 + B)}{e^{-t_0} - e^{-t_0-B}}.$$

***Proof*** Lemma 2.13 tells that $G(t) < +\infty$ for any $t \geq t_1$. By Lemma 2.12, there exists a holomorphic function $f_{t_0}$ on $\{\Psi < -t_0\}$, such that $(f_{t_0} - f)_o \in J$ and $\int_{\{\Psi<-t_0\}} |f_{t_0}|^2 = G(t_0)$.

It suffices to consider that $\liminf_{B\to 0+0} \frac{G(t_0)-G(t_0+B)}{e^{-t_0}-e^{-t_0-B}} \in [0, +\infty)$ because of the decreasing property of $G(t)$. Then there exists $B_j \to 0 + 0$ $(j \to +\infty)$ such that

$$\begin{aligned} e^{t_0} \lim_{j\to+\infty} \frac{G(t_0) - G(t_0 + B_j)}{B_j} &= \lim_{j\to+\infty} \frac{G(t_0) - G(t_0 + B_j)}{e^{-t_0} - e^{-t_0-B_j}} \\ &= \liminf_{B\to 0+0} \frac{G(t_0) - G(t_0 + B)}{e^{-t_0} - e^{-t_0-B}}, \end{aligned} \tag{2.17}$$

and $\left\{\frac{G(t_0)-G(t_0+B_j)}{B_j}\right\}_{j\in\mathbb{Z}_{>0}}$ is bounded.

Since $t \leq v_{t_0,B_j}(t)$, Lemma 2.2 shows that for any $B_j$, there exists a holomorphic function $\tilde{F}_j$ on $\{\Psi < -t_1\}$, such that

$$\begin{aligned} &\int_{\{\Psi<-t_1\}} |\tilde{F}_j - (1 - b_{t_0,B_j}(\Psi)) f_{t_0}|^2 \\ &\leq \int_{\{\Psi<-t_1\}} |\tilde{F}_j - (1 - b_{t_0,B_j}(\Psi)) f_{t_0}|^2 e^{-\Psi+v_{t_0,B_j}(\Psi)} \\ &\leq (e^{-t_1} - e^{-t_0-B_j}) \int_{\{\Psi<-t_1\}} \frac{1}{B_j} (\mathbb{I}_{\{-t_0-B_j<\Psi<-t_0\}}) |f_{t_0}|^2 e^{-\Psi} \\ &\leq (e^{t_0-t_1+B_j} - 1) \left( \int_{\{\Psi<-t_0\}} \frac{1}{B_j} |f_{t_0}|^2 - \int_{\{\Psi<-t_0-B_j\}} \frac{1}{B_j} |f_{t_0}|^2 \right) \end{aligned}$$

$$\leq (e^{t_0-t_1+B_j}-1)\frac{G(t_0)-G(t_0+B_j)}{B_j}. \tag{2.18}$$

Note that $b_{t_0,B_j}(t)=0$ for $t\leq -t_0-B_j$ and $b_{t_0,B_j}(t)=1$ for $t\geq t_0$. According to inequality (2.18), we have $(\tilde{F}_j-f_{t_0})_o\in J$, and

$$\begin{aligned}
&\int_{\{\Psi<-t_1\}}|\tilde{F}_j|^2\\
&\leq 2\int_{\{\Psi<-t_1\}}|(1-b_{t_0,B_j}(\Psi))f_{t_0}|^2+2\int_{\{\Psi<-t_1\}}|\tilde{F}_j-(1-b_{t_0,B_j}(\Psi))f_{t_0}|^2\\
&\leq 2\int_{\{\Psi<-t_0\}}|f_{t_0}|^2+2\left(e^{t_0-t_1+B_j}-1\right)\frac{G(t_0)-G(t_0+B_j)}{B_j}.
\end{aligned} \tag{2.19}$$

Following from Lemma 2.9 and inequality (2.19), we get that there exists a subsequence of $\{\tilde{F}_j\}_{j\in\mathbb{Z}_{>0}}$ (also denoted by $\{\tilde{F}_j\}_{j\in\mathbb{Z}_{>0}}$), which compactly converges to a holomorphic function $\tilde{F}_0$ on $\{\Psi<-t_1\}$, such that $(\tilde{F}_0-f)_o\in J$ and

$$\int_{\{\Psi<-t_1\}}|\tilde{F}_0|^2\leq \liminf_{j\to+\infty}\int_{\{\Psi<-t_1\}}|\tilde{F}_j|^2<+\infty.$$

Note that

$$\lim_{j\to+\infty}b_{t_0,B_j}(t)=\lim_{j\to+\infty}\int_{-\infty}^{t}\frac{1}{B_j}\mathbb{I}_{\{-t_0-B_j<s<-t_0\}}ds=\begin{cases}0 \text{ if } x\in(-\infty,-t_0)\\ 1 \text{ if } x\in[-t_0,+\infty)\end{cases},$$

and

$$\lim_{j\to+\infty}v_{t_0,B_j}(t)=\lim_{j\to+\infty}\int_{-t_0}^{t}b_{t_0,B_j}ds-t_0=\begin{cases}-t_0 & \text{if } x\in(-\infty,-t_0)\\ t & \text{if } x\in[-t_0,+\infty)\end{cases}.$$

According to inequality (2.18) and Fatou's Lemma, we have

$$\begin{aligned}
&\int_{\{\Psi<-t_0\}}|\tilde{F}_0-f_{t_0}|+\int_{\{-t_0\leq\Psi<-t_1\}}|\tilde{F}_0|^2\\
&=\int_{\{\Psi<-t_1\}}\lim_{j\to+\infty}|\tilde{F}_j-(1-b_{t_0,B_j}(\Psi))f_{t_0}|^2\\
&\leq \liminf_{j\to+\infty}\int_{\{\Psi<-t_1\}}|\tilde{F}_j-(1-b_{t_0,B_j}(\Psi))f_{t_0}|^2\\
&\leq \left(e^{t_0-t_1}-1\right)\frac{G(t_0)-G(t_0+B_j)}{B_j}.
\end{aligned} \tag{2.20}$$

Then it follows from Lemma 2.12, equality (2.17) and inequality (2.20) that

$$
\begin{aligned}
&(e^{-t_1} - e^{-t_0}) \liminf_{B \to 0+0} \frac{G(t_0) - G(t_0 + B)}{e^{-t_0} - e^{-t_0 - B}} \\
&= (e^{t_0 - t_1} - 1) \frac{G(t_0) - G(t_0 + B_j)}{B_j} \\
&\geq \int_{\{\Psi < -t_0\}} |\tilde{F}_0 - f_{t_0}| + \int_{\{-t_0 \leq \Psi < -t_1\}} |\tilde{F}_0|^2 \\
&= \int_{\{\Psi < -t_1\}} |\tilde{F}_0|^2 - \int_{\{\Psi < -t_0\}} |f_{t_0}|^2 \\
&\geq G(t_1) - G(t_0).
\end{aligned}
$$

This proves Lemma 2.14. □

The following well-known property of concave functions will be used in the proof of Theorem 1.6.

**Lemma 2.15** (see [21]) *Let $a(r)$ be a lower semicontinuous function on $(0, R]$. Then $a(r)$ is concave if and only if*

$$
\frac{a(r_1) - a(r_2)}{r_1 - r_2} \geq \limsup_{r \to r_2 + 0} \frac{a(r) - a(r_2)}{r - r_2}, \tag{2.21}
$$

*holds for any $0 < r_2 < r_1 \leq R$.*

## 3 Proof of theorem 1.6

Firstly, we prove that $G(t_1) < +\infty$ for any $t_1 \in [0, t_0)$. It follows from Lemma 2.12 that there exists a holomorphic function $f_{t_0}$ on $\{\Psi < -t_0\}$ satisfying $(f_{t_0} - f)_o \in J$ and $\int_{\{\Psi < -t_0\}} |f_{t_0}|^2 = G(t_0) < +\infty$. Using Lemma 2.2, we get a holomorphic function $\tilde{F}$ on $\{\Psi < -t_1\}$, such that

$$
\begin{aligned}
&\int_{\{\Psi < -t_1\}} |\tilde{F} - (1 - b_{t_0,B}(\Psi)) f_{t_0}|^2 \\
&\leq \int_{\{\Psi < -t_1\}} |\tilde{F} - (1 - b_{t_0,B}(\Psi)) f_{t_0}|^2 e^{-\Psi + v_{t_0,B}(\Psi)} \\
&\leq (e^{-t_1} - e^{-t_0 - B}) \int_{\{\Psi < -t_1\}} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f_{t_0}|^2 e^{-\Psi},
\end{aligned} \tag{3.1}
$$

which implies that $(\tilde{F} - f_{t_0})_o \in J$ and

$$\begin{aligned}
&\int_{\{\Psi<-t_1\}} |\tilde{F}|^2 \\
&\le 2\int_{\{\Psi<-t_1\}} |(1-b_{t_0,B}(\Psi))f_{t_0}|^2 + 2\int_{\{\Psi<-t_1\}} |\tilde{F} - (1-b_{t_0,B}(\Psi))f_{t_0}|^2 \\
&\le 2\int_{\{\Psi<-t_1\}} |f_{t_0}|^2 + \frac{2(e^{t_0+B-t_1}-1)}{B}\int_{\{\Psi<-t_0\}} |f_{t_0}|^2 \\
&< +\infty.
\end{aligned}$$

Then we obtain $G(t_1) \le \int_{\{\Psi<-t_1\}} |\tilde{F}|^2 < +\infty$.

Now, as $G(h^{-1}(r))$ is lower semicontinuous (Lemma 2.13), Lemma 2.14 and Lemma 2.15 imply the concavity of $G(-\log r)$. Lemma 2.13 shows $\lim_{t\to+\infty} G(t) = 0$. Hence the proof of Theorem 1.6 is completed.

## 4 Proofs of theorem 1.7 and remark 1.9

In this section, we prove Theorem 1.7 and Remark 1.9.

### 4.1 Proof of theorem 1.7

As $D$ is a pseudoconvex domain, there exist a sequence of pseudoconvex domains $D_1 \Subset \ldots D_j \Subset D_{j+1} \Subset \ldots$ such that $\bigcup_{j=1}^{+\infty} D_j = D$. Denote

$$\inf\left\{\int_{\{p\Psi_1<-t\}\cap D_j} |\tilde{f}|^2 : \tilde{f} \in \mathcal{O}(\{p\Psi_1 < -t\}\cap D_j) \ \&\ (\tilde{f} - f)_o \in I(p\Psi_1)_o\right\}$$

by $G_{j,p}(t)$, where $t \in [0, +\infty)$, $j \in \mathbb{Z}_{>0}$ and $p \in (1, 2)$. Note that

$$p\Psi_1 = \min\left\{(2pc_o^{fF}(\psi)\psi + (4-2p)\log|F|) - 2\log|F^2|, 0\right\},$$

and

$$G_{j,p}(0) \le \int_{D_j} |f|^2 < +\infty.$$

Theorem 1.6 shows that $G_{j,p}(-\log r)$ is concave with respect to $r \in (0, 1]$, and $\lim_{t\to+\infty} G_{j,p}(t) = 0$, yielding that

$$\frac{1}{r_1^2}\int_{\{p\Psi_1<2\log r_1\}\cap D_j} |f|^2 \ge \frac{1}{r_1^2} G_{j,p}(2\log r_1) \ge G_{j,p}(0), \tag{4.1}$$

where $0 < r_1 \le 1$.

Denote

$$C_j := \inf\left\{\int_{\{\Psi_1<0\}\cap D_j}|\tilde{f}|^2 : \tilde{f}\in\mathcal{O}(\{\Psi_1<0\}\cap D_j)\ \&\ (\tilde{f}-f)_o\in I_+(\Psi_1)_o\right\}.$$

Since $\int_{D_j}|f|^2<+\infty$ and $G_{j,p}(0)\geq C_j$ for any $j\in\mathbb{Z}_{>0}$, according to the dominated convergence theorem and inequality (4.1), we obtain that

$$\begin{aligned}&\frac{1}{r_1^2}\int_{\{\Psi_1\leq 2\log r_1\}\cap D_j}|f|^2\\&=\lim_{p\to 1+0}\frac{1}{r_1^2}\int_{\{p\Psi_1<2\log r_1\}\cap D_j}|f|^2\\&\geq\limsup_{p\to 1+0}G_{j,p}(0)\\&\geq C_j\end{aligned}\tag{4.2}$$

holds for any $j\in\mathbb{Z}_{>0}$ and $r_1\in(0,1]$. Without loss of generality, assume that there exists $r_0\in(0,1]$ such that $\frac{1}{r_0^2}\int_{\{\Psi_1\leq 2\log r_0\}}|f|^2<+\infty$. Following from inequality (4.2), we get that $\sup_j C_j<+\infty$.

Lemma 2.7 tells us that there exists $p_0\in(1,2)$ such that $I_+(\Psi_1)_o=I(p_0\Psi_1)_o$. Then it follows from Lemma 2.12 that there exists a holomorphic function $\tilde{f}_j$ on $\{\Psi_1<0\}\cap D_j$ such that $C_j=\int_{\{\Psi_1<0\}\cap D_j}|\tilde{f}_j|^2$ and $(\tilde{f}_j-f)_o\in I(p_0\Psi_1)_o$ for any $j\in\mathbb{Z}_{>0}$. Note that $\sup_j\int_{\{\Psi_1<0\}\cap D_j}|\tilde{f}_j|^2<+\infty$. Then we can apply Lemma 2.9 and the diagonal method to extract a subsequence of $\{\tilde{f}_j\}_{j\in\mathbb{Z}_{>0}}$ (also denoted by $\{\tilde{f}_j\}_{j\in\mathbb{Z}_{>0}}$) compactly convergent to a holomorphic function $\tilde{f}_0$ on $\{\Psi_1<0\}$, which satisfies $(\tilde{f}_0-f)_o\in I(p_0\Psi)_o$. As $C_j$ is increasing with respect to $j$, using Fatou's Lemma and the definition of $G(0;\Psi_1,I_+(\Psi_1)_o,f)$, we obtain that

$$G(0;\Psi_1,I_+(\Psi_1)_o,f)\leq\int_{\{\Psi<0\}}|\tilde{f}_0|^2\leq\liminf_{j\to+\infty}\int_{\{\Psi_1<0\}\cap D_j}|\tilde{f}_j|^2=\lim_{j\to+\infty}C_j.\tag{4.3}$$

Combining inequality (4.2) and inequality (4.3), we get

$$\begin{aligned}&\frac{1}{r_1^2}\int_{\{\Psi_1\leq 2\log r_1\}}|f|^2\\&\geq\lim_{j\to+\infty}\frac{1}{r_1^2}\int_{\{\Psi_1\leq 2\log r_1\}\cap D_j}|f|^2\\&\geq\lim_{j\to+\infty}C_j\\&\geq G(0;\Psi_1,I_+(\Psi_1)_o,f).\end{aligned}\tag{4.4}$$

Note that

$$\left\{2c_o^{fF}(\psi)\psi - 2\log|F| < 2\log r\right\} = \bigcup_{0<r_1<r}\{\Psi_1 \le 2\log r_1\}$$

where $r \in (0, 1]$. Then inequality (4.4) implies

$$\begin{aligned}\int_{\{c_o^{fF}(\psi)\psi-\log|F|<\log r\}}|f|^2 &\ge \sup_{r_1\in(0,r)} r_1^2 G(0;\Psi_1, I_+(\Psi_1)_o, f)\\ &= r^2 G(0;\Psi_1, I_+(\Psi_1)_o, f).\end{aligned}$$

Lemma 2.10 shows $f_o \notin I_+(\Psi_1)_o$. In addition, $I_+(\Psi_1)_o = I(p_0\Psi_1)_o$ and Lemma 2.11 verify $G(0;\Psi_1, I_+(\Psi_1)_o, f) > 0$. Thus, Theorem 1.7 holds.

### 4.2 Proof of remark 1.9

Note that $G(0;\Psi_1, I_+(\Psi_1)_o, 1) \ge G(0;\Psi_1, I(\Psi_1)_o, 1)$. Then it suffices to prove

$$G(0;\Psi_1, I(\Psi_1)_o, 1) \ge \frac{C_{F^{1+\delta},2c_o^F(\psi)\psi+\delta\max\{2c_o^F(\psi)\psi,2\log|F|\}}(o)}{\left(1+\frac{1}{\delta}\right)\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi,2\log|F|\}}},$$

for any $\delta \in \mathbb{Z}_{>0}$. Without loss of generality, assume that $G(0;\Psi_1, I(\Psi_1)_o, 1) < +\infty$. Denote $G(t;\Psi_1, I(\Psi_1)_o, 1)$ by $G(t)$ for short.

Lemma 2.12 indicates that there exists a holomorphic function $f_0$ on $\{\Psi_1 < 0\}$ such that $(f_0 - 1)_o \in I(\Psi_1)_o$ and $G(0) = \int_{\{\Psi_1<0\}}|f_0|^2$. Denote

$$\varphi := (1+\delta)\max\{2c_o^F(\psi)\psi, 2\log|F|\}.$$

For any $B > 0$, it follows from Lemma 2.1 that there exists a holomorphic function $\tilde{F}_B$ on $D$ such that

$$\begin{aligned}&\int_D |\tilde{F}_B - (1-b_B(\Psi_1))f_0F^{1+\delta}|^2 e^{-\varphi+v_B(\Psi_1)-\Psi_1}\\ &\le \left(\frac{1}{\delta}+1-e^{-B}\right)\int_D \frac{1}{B}\mathbb{I}_{\{-B<\Psi_1<0\}}|f_0|^2 e^{-\Psi_1}\\ &\le \left(\left(\frac{1}{\delta}+1\right)e^B - 1\right)\int_D \frac{1}{B}\mathbb{I}_{\{-B<\Psi_1<0\}}|f_0|^2\\ &\le \left(\left(\frac{1}{\delta}+1\right)e^B - 1\right)\frac{G(0)-G(B)}{B},\end{aligned} \tag{4.5}$$

where $b_B(t)=\int_{-\infty}^{t}\frac{1}{B}\mathbb{I}_{\{-B<s<0\}}ds$ and $v_B(t)=\int_0^t b_B(s)ds$. Theorem 1.6 shows that $G(-\log r)$ is concave with respect to $r$ and $\lim_{t\to+\infty}G(t)=0$, which implies that

$$\begin{aligned}
&\lim_{B\to0+0}\left(\left(\frac{1}{\delta}+1\right)e^B-1\right)\frac{G(0)-G(B)}{B}\\
&=\lim_{B\to0+0}\left(\left(\frac{1}{\delta}+1\right)e^B-1\right)\frac{G(-\log 1)-G(-\log e^{-B})}{1-e^{-B}}\cdot\frac{1-e^{-B}}{B}\\
&\le\frac{1}{\delta}G(0).
\end{aligned}\tag{4.6}$$

Combining inequality (4.5), inequality (4.6) and the fact $v_B(\Psi_1)-\Psi_1\ge0$, we have

$$\begin{aligned}
&\lim_{B\to0+0}\int_D|\tilde{F}_B|^2e^{-\varphi}\\
&\le\lim_{B\to0+0}2\left(\int_D|\tilde{F}_B-(1-b_B(\Psi_1))f_0F^{1+\delta}|^2e^{-\varphi}+\int_D|(1-b_B(\Psi_1))f_0F^{1+\delta}|^2e^{-\varphi}\right)\\
&\le2\int_{\{\Psi_1<0\}}|f_0|^2+2\lim_{B\to0+0}\left(\left(\frac{1}{\delta}+1\right)e^B-1\right)\frac{G(0)-G(B)}{B}\\
&\le2\left(1+\frac{1}{\delta}\right)G(0)\\
&<+\infty.
\end{aligned}\tag{4.7}$$

This implies that there exists a subsequence of $\{\tilde{F}_B\}_{B>0}$, denoted by $\{\tilde{F}_{B_j}\}_{j\in\mathbb{Z}_{>0}}$, satisfying that $\lim_{j\to+\infty}B_j=0$ and $\{\tilde{F}_{B_j}\}_{j\in\mathbb{Z}_{>0}}$ converges to a holomorphic function $\tilde{F}$ on $D$. It follows from Fatou's Lemma and inequality (4.7) that

$$\int_D|\tilde{F}|^2e^{-\varphi}\le\liminf_{j\to+\infty}\int_D|\tilde{F}_{B_j}|^2e^{-\varphi}\le\left(1+\frac{1}{\delta}\right)G(0)<+\infty.\tag{4.8}$$

Using Fatou's Lemma, inequality (4.5) and inequality (4.6), we obtain

$$\begin{aligned}
&\int_{\{\Psi_1<0\}}|\tilde{F}-f_0F^{1+\delta}|^2e^{-\varphi-\Psi_1}+\int_{\{\Psi_1\ge0\}}|\tilde{F}|^2e^{-\varphi}\\
&\le\liminf_{j\to+\infty}\int_D|\tilde{F}_{B_j}-(1-b_{B_j}(\Psi_1))f_0F^{1+\delta}|^2e^{-\varphi+v_{B_j}(\Psi_1)-\Psi_1}\\
&\le\liminf_{j\to+\infty}\left(\left(\frac{1}{\delta}+1\right)e^{B_j}-1\right)\frac{G(0)-G(B_j)}{B_j}\\
&\le\frac{1}{\delta}G(0).
\end{aligned}\tag{4.9}$$

Note that

$$\int_{\{\Psi_1<0\}}\left|\frac{\tilde{F}}{F^{1+\delta}}\right|^2=\int_{\{\Psi_1<0\}}|\tilde{F}|^2e^{-\varphi}<+\infty,$$

and

$$\int_{\{\Psi_1<0\}}\left|\frac{\tilde{F}}{F^{1+\delta}}-f_0\right|^2e^{-\Psi_1}=\int_{\{\Psi_1<0\}}|\tilde{F}-f_0F^{1+\delta}|^2e^{-\varphi-\Psi_1}<+\infty.$$

Then Lemma 2.12 indicates that

$$\begin{aligned}\int_{\{\Psi_1<0\}}|\tilde{F}-f_0F^{1+\delta}|^2e^{-\varphi}&=\int_{\{\Psi_1<0\}}\left|\frac{\tilde{F}}{F^{1+\delta}}-f_0\right|^2\\&=\int_{\{\Psi_1<0\}}\left|\frac{\tilde{F}}{F^{1+\delta}}\right|^2-\int_{\{\Psi_1<0\}}|f_0|^2.\end{aligned}\tag{4.10}$$

Combining inequality (4.9) and equality (4.10), we get

$$\begin{aligned}&\int_D|\tilde{F}|^2e^{-\varphi}\\&=\int_{\{\Psi_1<0\}}|\tilde{F}|^2e^{-\varphi}-\int_{\{\Psi_1<0\}}|f_0|^2+\int_{\{\Psi_1\geq0\}}|\tilde{F}|^2e^{-\varphi}+\int_{\{\Psi_1<0\}}|f_0|^2\\&=\int_{\{\Psi_1<0\}}|\tilde{F}-f_0F^{1+\delta}|^2e^{-\varphi}+\int_{\{\Psi_1\geq0\}}|\tilde{F}|^2e^{-\varphi}+G(0)\\&\leq\left(1+\frac{1}{\delta}\right)G(0).\end{aligned}\tag{4.11}$$

As $(f-1)_o\in I(\Psi_1)_o$ and

$$\int_{\{\Psi_1<0\}}\left|\frac{\tilde{F}}{F^{1+\delta}}-f_0\right|^2e^{-\Psi_1}=\int_{\{\Psi_1<0\}}|\tilde{F}-f_0F^{1+\delta}|^2e^{-\varphi-\Psi_1}<+\infty,$$

we have $(\frac{\tilde{F}}{F^{1+\delta}}-1)_o\in I(\Psi_1)_o$, yielding that there exist a neighborhood $U\Subset D$ of $o$ and $t>0$ such that

$$\int_{\{\Psi_1<-t\}\cap U}|\tilde{F}-F^{1+\delta}|^2e^{-\varphi-\Psi_1}=\int_{\{\Psi_1<-t\}\cap U}\left|\frac{\tilde{F}}{F^{1+\delta}}-1\right|^2e^{-\Psi_1}<+\infty.\tag{4.12}$$

Sicne

$$\int_{\{\Psi_1>-t\}\cap U}|\tilde{F}|^2e^{-\varphi-\Psi_1}\leq e^t\int_D|\tilde{F}|^2e^{-\varphi}<+\infty,$$

and

$$\int_{\{\Psi_1>-t\}\cap U} |F^{1+\delta}|^2 e^{-\varphi-\Psi_1} \le \int_{U\cap\{\Psi_1\ge -t\}} e^{-\Psi_1} < +\infty,$$

inequality (4.12) verifies that

$$\begin{aligned}(\tilde{F} - F^{1+\delta}, o) &\in \mathcal{I}(\varphi + \Psi_1)_o\\ &= \mathcal{I}\Big(2c_o^F(\psi)\psi + \delta \max\left\{2c_o^F(\psi)\psi, 2\log|F|\right\}\Big)_o.\end{aligned}$$

Recall the definition of $C_{F^{1+\delta},2c_o^F(\psi)\psi+\delta\max\{2c_o^F(\psi)\psi,2\log|F|\}}(o)$. According to inequality (4.11), we obtain that

$$\frac{C_{F^{1+\delta},2c_o^F(\psi)\psi+\delta\max\{2c_o^F(\psi)\psi,2\log|F|\}}(o)}{\sup_D e^{(1+\delta)\max\{2c_o^F(\psi)\psi,2\log|F|\}}} \le \int_D |\tilde{F}|^2 e^{-\varphi} \le \left(1 + \frac{1}{\delta}\right) G(0).$$

Thus, Remark 1.9 holds.

## 5 Proofs of theorem 1.11 and corollary 1.12

In this section, we prove Theorem 1.11 and Corollary 1.12.

### 5.1 Proof of theorem 1.11

Lemma 2.7 shows that there exists some $p_0 > 2a_o^f(\Psi)$ such that $I(p_0\Psi)_o = I_+(2a_o^f(\Psi)\Psi)_o$. According to the definition of $a_o^f(\Psi)$ and Lemma 2.11, we obtain that

$$G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f) > 0. \tag{5.1}$$

Without loss of generality, assume that there exists $t > t_0$ such that $\int_{\{\Psi<-t\}} |f|^2 < +\infty$. Denote

$$t_1 := \inf\left\{t \ge t_0 : \int_{\{\Psi<-t\}} |f|^2 < +\infty\right\},$$

and

$$G_p(t) := \inf\left\{ \int_{\{p\Psi<-t\}} |\tilde{f}|^2 : \tilde{f} \in \mathcal{O}(\{p\Psi < -t\})\ \&\ (\tilde{f} - f)_o \in I(p\Psi)_o\right\},$$

where $t \in [0, +\infty)$ and $p > 2a_o^f(\Psi)$. By the definition of $G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f)$, we have that $G_p(0) \geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f)$ for any $p > 2a_o^f(\Psi)$. Note that

$$p\Psi = \min\left\{2\psi + (2\lceil p\rceil - 2p)\log|F| - 2\log\left|F^{\lceil p\rceil}\right|, 0\right\},$$

where $\lceil p\rceil = \min\{n \in \mathbb{Z} : n \geq p\}$, and

$$G_p(pt) \leq \int_{\{\Psi<-t\}} |f|^2 < +\infty,$$

for any $t > t_1$. Theorem 1.6 indicates that $G_{j,p}(-\log r)$ is concave with respect to $r \in (0, 1]$ and $\lim_{t\to+\infty} G_{j,p}(t) = 0$, which verifies that

$$\begin{aligned}\frac{1}{r_1^2}\int_{\{p\Psi<2\log r_1\}} |f|^2 \geq \frac{1}{r_1^2}G_p(-2\log r_1) \geq G_p(0)&\\ \geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f),&\end{aligned} \tag{5.2}$$

where $0 < r_1 \leq e^{-\frac{pt_0}{2}}$.

We prove $a_o^f(\Psi) > 0$ by contradiction: if $a_o^f(\Psi) = 0$, as $\int_{\{\Psi<-t_1-1\}} |f|^2 < +\infty$, the dominated convergence theorem and inequality (5.2) show that

$$\begin{aligned}\frac{1}{r_1^2}\int_{\{\Psi=-\infty\}} |f|^2 = \lim_{p\to 0+0}\frac{1}{r_1^2}\int_{\{p\Psi<2\log r_1\}} |f|^2&\\ \geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f).&\end{aligned} \tag{5.3}$$

Note that $\mu(\{\Psi = -\infty\}) = \mu(\{\psi = -\infty\}) = 0$, where $\mu$ is the Lebesgue measure on $\mathbb{C}^n$. Then inequality (5.3) implies that $G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f) = 0$, which contradicts inequality (5.1). Thus, we get $a_o^f(\Psi) > 0$.

For any $r_2 \in (0, e^{-a_o^f(\Psi)t_1})$ and $p \in (2a_0^f(\Psi), -\frac{2\log r_2}{t_1})$, since $\frac{2\log r_2}{p} < -t_1$, we have $\int_{\{p\Psi<2\log r_2\}} |f|^2 < +\infty$. Then it follows from the dominated convergence theorem and inequality (5.2) that

$$\begin{aligned}\frac{1}{r_2^2}\int_{\{2a_0^f(\Psi)\Psi\leq 2\log r_2\}} |f|^2 = \lim_{p\to 2a_0^f(\Psi)+0}\frac{1}{r_2^2}\int_{\{p\Psi<2\log r_2\}} |f|^2&\\ \geq G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f).&\end{aligned} \tag{5.4}$$

Set $r \in (0, e^{-a_o^f(\Psi)t_0}]$. If $r > e^{-a_o^f(\Psi)t_1}$, we have

$$\int_{\{a_0^f(\Psi)\Psi<\log r\}} |f|^2 = +\infty > G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f).$$

If $r \in (0, e^{-a_o^f(\Psi)t_1}]$, it follows from

$$\{a_o^f(\Psi)\Psi < \log r\} = \bigcup_{0<r_2<r} \{a_o^f(\Psi)\Psi < \log r_2\}$$

and inequality (5.4) that

$$\begin{aligned}\int_{\{a_0^f(\Psi)\Psi<\log r\}} |f|^2 &= \sup_{r_2\in(0,r)} \int_{\{2a_0^f(\Psi)\Psi\leq 2\log r_2\}} |f|^2 \\ &\geq \sup_{r_2\in(0,r)} r_2^2 G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f) \\ &= r^2 G(0; \Psi, I_+(2a_o^f(\Psi)\Psi)_o, f).\end{aligned}$$

Thus, Theorem 1.11 holds.

### 5.2 Proof of corollary 1.12

It is clear that $I_+(a\Psi)_o \subset I(a\Psi)_o$. Hence, it suffices to prove $I(a\Psi) \subset I_+(a\Psi)_o$.

If there exists a holomorphic function $f$ on $\{\Psi < -t_0\} \cap D_0$ such that $f_o \in I(a\Psi)_o$ and $f_o \notin I_+(a\Psi)_o$, where $t_0 > 0$ and $D_0 \subset D$ is a neighborhood of $o$, then $a_o^f(\Psi)_o = a/2 < +\infty$. Theorem 1.11 shows that $a > 0$. For any neighborhood $U \subset D_0$ of $o$, it follows from Theorem 1.11 that there exists $C_U > 0$ such that

$$\frac{1}{r^2} \int_{\{a\Psi<2\log r\}\cap U} |f|^2 \geq C_U, \tag{5.5}$$

for any $r \in (0, e^{-\frac{at_0}{2}}]$. Given any $t > at_0$, Fubini's Theorem and inequality (5.5) imply that

$$\begin{aligned}\int_{\{a\Psi<-t\}\cap U} |f|^2 e^{-a\Psi} &= \int_{\{a\Psi<-t\}\cap U} \left( |f|^2 \int_0^{e^{-a\Psi}} dl \right) \\ &= \int_0^{+\infty} \left( \int_{\{l<e^{-a\Psi}\}\cap\{a\Psi<-t\}\cap U} |f|^2 \right) dl \\ &\geq \int_{e^t}^{+\infty} \left( \int_{\{a\Psi<-\log l\}\cap U} |f|^2 \right) dl \\ &\geq C_U \int_{e^t}^{+\infty} \frac{1}{l} dl \\ &= +\infty,\end{aligned}$$

which contradicts $f_o \in I(a\Psi)_o$. Thus, there does not exist $f_o \in I(a\Psi)_o$ such that $f_o \notin I_+(a\Psi)_o$, i.e. $I(a\Psi) = I_+(a\Psi)_o$ for any $a \geq 0$.

### 5.2.1 Another proof of Corollary 1.12

We give another proof of Corollary 1.12 by using Lemma 2.6 and the strong openness property of multiplier ideal sheaves.

According to Lemma 2.6, we only need to prove that

$$\mathcal{I}(k\varphi + a\Psi)_o = \bigcup_{p\in(a,k)} \mathcal{I}(k\varphi + p\Psi)_o,$$

where $\varphi = 2\max\{\psi, 2\log|F|\}$, and $k > a$ is an integer. It is easy to see

$$\bigcup_{p\in(a,k)} \mathcal{I}(k\varphi + p\Psi)_o \subset \mathcal{I}(k\varphi + a\Psi)_o.$$

In the following, we prove that

$$\mathcal{I}(k\varphi + a\Psi)_o \subset \bigcup_{p\in(a,k)} \mathcal{I}(k\varphi + p\Psi)_o.$$

Note that $k\varphi + p\Psi$ is a plurisubharmonic function on $D$ for any $p \in [0, k]$.

When $a > 0$, for any $(f, o) \in \mathcal{I}(k\varphi + a\Psi)_o$, it follows from the strong openness property of multiplier ideal sheaves ([28]) that there exists $r > 1$ such that $(f, o) \in \mathcal{I}(rk\varphi + ar\Psi)_o$. As

$$\mathcal{I}(rk\varphi + ar\Psi)_o \subset \bigcup_{p\in(a,k)} \mathcal{I}(k\varphi + p\Psi)_o,$$

we have

$$(f, o) \in \bigcup_{p\in(a,k)} \mathcal{I}(k\varphi + p\Psi)_o.$$

When $a = 0$, for any $(f, o) \in \mathcal{I}(k\varphi + a\Psi)_o$, as $\psi$ is plurisubharmonic, it follows from the strong openness property of multiplier ideal sheaves ([28]) and Hölder's inequality that there exist $r_1 > 1$ and a neighborhood $U \Subset D$ of $o$ satisfying that

$$\int_U |f|^2 e^{-k\varphi - r_1\psi} \le \left(\int_U |f|^{2q_1} e^{-q_1 k\varphi}\right)^{\frac{1}{q_1}} \cdot \left(\int_U e^{-q_2 r_1\psi}\right)^{\frac{1}{q_2}} < +\infty, \tag{5.6}$$

where $q_1 > 1$ and $q_2 > 1$ satisfy $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Since $\varphi/2 + \Psi = \psi$, inequality (5.6) implies

$$(f, o) \in \bigcup_{p\in(a,k)} \mathcal{I}(k\varphi + p\Psi)_o.$$

Thus, Corollary 1.12 holds.

## 6 Proof of proposition 1.13

Denote

$$C := \inf_m G(0; \Psi_m, I(\Psi_m)_o, f_m) > 0.$$

It follows from Theorem 1.6 that $G(-\log r; \Psi_m, I(\Psi_m)_o, f_m)$ is concave with respect to $r \in (0, 1]$. Then for some $t_0 > 0$, it holds that

$$\int_{\{\Psi_m < -t\}} |f_m|^2 \geq e^{-t} G(0; \Psi_m, I(\Psi_m)_o, f_m) \geq e^{-t} C, \ \forall t \geq t_0. \tag{6.1}$$

For any $\epsilon > 0$, as $\{\Psi_m\}_{m \in \mathbb{Z}_{>0}}$ converges to $\Psi$, there exists $m_0 > 0$ such that

$$\mu\{|\Psi_m - \Psi| \geq 1\} < \epsilon, \ \forall m \geq m_0, \tag{6.2}$$

where $\mu$ is the Lebesgue measure on $\mathbb{C}^n$. Denote

$$M := \sup_m \sup_D |\tilde{f}_m|^2 < +\infty.$$

Note that $\tilde{f}_m = f_m$ on $\{\Psi_m < -t_0\}$. Then inequality (6.1) and inequality (6.2) indicate that

$$\begin{aligned} &\int_{\{\Psi < -t+1\}} |\tilde{f}_m|^2 \\ &\geq \int_{\{\Psi_m < -t\}} |\tilde{f}_m|^2 - \int_{\{|\Psi_m - \Psi| \geq 1\}} |\tilde{f}_m|^2 \\ &\geq e^{-t} C - M\epsilon, \end{aligned}$$

for any $t \geq t_0$ and $m \geq m_0$, yielding that

$$\liminf_{m \to +\infty} \int_{\{\Psi < -t+1\}} |\tilde{f}_m|^2 \geq e^{-t} C, \ \forall t \geq t_0. \tag{6.3}$$

As $\{\tilde{f}_m\}_{m \in \mathbb{Z}_{>0}}$ converges to $\tilde{f}$ in Lebesgue measure, and $\sup_m \sup_D |\tilde{f}_m| < +\infty$, the dominated convergence theorem and inequality (6.3) verify that

$$\int_{\{\Psi < -t+1\}} |\tilde{f}|^2 \geq e^{-t} C, \ \forall t \geq t_0.$$

Using Fubini's Theorem, we get that

$$\begin{aligned}\int_D |\tilde{f}|^2 e^{-\Psi} \geq \int_{\{\Psi<-t_0\}} |\tilde{f}|^2 e^{-\Psi} &= \int_{\{\Psi<-t_0\}} \left( |\tilde{f}|^2 \int_0^{e^{-\Psi}} dl \right) \\ &= \int_0^{+\infty} \left( \int_{\{l<e^{-\Psi}\}\cap\{\Psi<-t_0\}} |\tilde{f}|^2 \right) dl \\ &\geq \int_{e^{t_0}}^{+\infty} \left( \int_{\{\Psi<-\log l\}} |\tilde{f}|^2 \right) dl \\ &\geq e^{-1} C \int_{e^{t_0}}^{+\infty} \frac{1}{l} dl \\ &= +\infty,\end{aligned}$$

which holds for any pseudoconvex domain $D \subset \Delta^n$ containing $o$. Consequently, we conclude that

$$|\tilde{f}|^2 e^{-\Psi} \notin L^1(U),$$

where $U$ is any neighborhood of $o$.

## 7 Appendix

### 7.1 Proof of lemma 2.1

In the first part of the appendix, we give the proof of Lemma 2.1.

Firstly, we do some preparations. We recall some lemmas on $L^2$ estimates for $\bar{\partial}$-equations. In the following, $\bar{\partial}^*$ denotes the Hilbert adjoint operator of $\bar{\partial}$.

**Lemma 7.1** (see [46], see also [3]) *Let $\Omega \Subset \mathbb{C}^n$ be a domain with $C^\infty$ boundary $b\Omega$, $\Phi \in C^\infty(\bar{\Omega})$. Let $\rho$ be a $C^\infty$ defining function for $\Omega$ such that $|d\rho| = 1$ on $b\Omega$. Let $\eta$ be a smooth function on $\bar{\Omega}$. For any $(0,1)$-form $\alpha = \sum_{j=1}^n \alpha_{\bar{j}} d\bar{z}^j \in Dom_\Omega(\bar{\partial}^*) \cap C^\infty_{(0,1)}(\bar{\Omega})$,*

$$\begin{aligned}\int_\Omega \eta|\bar{\partial}^*_\Phi \alpha|^2 e^{-\Phi} + \int_\Omega \eta|\bar{\partial}\alpha|^2 e^{-\Phi} &= \sum_{i,j=1}^n \int_\Omega \eta|\bar{\partial}_i \alpha_{\bar{j}}|^2 e^{-\Phi} \\ &+ \sum_{i,j=1}^n \int_{b\Omega} \eta(\partial_i \bar{\partial}_j \rho)\alpha_{\bar{i}} \bar{\alpha}_{\bar{j}} e^{-\Phi} + \sum_{i,j=1}^n \int_\Omega \eta(\partial_i \bar{\partial}_j \Phi)\alpha_{\bar{i}} \bar{\alpha}_{\bar{j}} e^{-\Phi} \\ &+ \sum_{i,j=1}^n \int_\Omega -(\partial_i \bar{\partial}_j \eta)\alpha_{\bar{i}} \bar{\alpha}_{\bar{j}} e^{-\Phi} + 2\mathrm{Re}(\bar{\partial}^*_\Phi \alpha, \alpha \llcorner (\bar{\partial}\eta)^\sharp)_{\Omega,\Phi},\end{aligned} \tag{7.1}$$

*where $\alpha \llcorner (\bar{\partial}\eta)^\sharp = \sum_j \alpha_{\bar{j}} \partial_j \eta$.*

The symbols and notations can be referred to [29]. See also [46] or [45].

**Lemma 7.2** (see [3], see also [29]) *Let $\Omega \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain with $C^\infty$ boundary $b\Omega$ and $\Phi \in C(\bar{\Omega})$. Let $\lambda$ be a $\bar\partial$ closed smooth form of bidegree $(n, 1)$ on $\bar{\Omega}$. Assume the inequality*

$$|(\lambda, \alpha)_{\Omega,\Phi}|^2 \le C \int_\Omega |\bar\partial^*_\Phi \alpha|^2 \frac{e^{-\Phi}}{\mu} < +\infty,$$

*where $\frac{1}{\mu}$ is an integrable positive function on $\Omega$ and $C$ is a constant, holds for all $(n, 1)$-form $\alpha \in Dom_\Omega(\bar\partial^*) \cap Ker(\bar\partial) \cap C^\infty_{(n,1)}(\bar{\Omega})$. Then there is a solution $u$ to the equation $\bar\partial u = \lambda$ such that*

$$\int_\Omega |u|^2 \mu e^{-\Phi} \le C.$$

The following lemma will be used in the proof of Lemma 7.4.

**Lemma 7.3** (see [9]) *For arbitrary $\eta = (\eta_1, \ldots, \eta_p) \in (0, +\infty)^p$, the function*

$$M_\eta(t_1, \ldots, t_p) = \int_{\mathbb{R}^p} \max\{t_1 + h_1, \ldots, t_p + h_p\} \prod_{1\le j\le p} \theta\left(\frac{h_j}{\eta_j}\right) dh_1 \ldots dh_p$$

*possesses the following properties:*

(1) *$M_\eta(t_1, \ldots, t_p)$ is non decreasing in all variables, smooth and convex on $\mathbb{R}^p$;*
(2) *$\max\{t_1, \ldots, t_p\} \le M_\eta(t_1, \ldots, t_p) \le \max\{t_1 + \eta_1, \ldots, t_p + \eta_p\}$;*
(3) *$M_\eta(t_1, \ldots, t_p) = M_{(\eta_1,\ldots,\hat{\eta}_j,\ldots,\eta_p)}(t_1, \ldots, \hat{t}_j, \ldots, t_p)$ if $t_j + \eta_j \le \max_{k\neq j}\{t_k - \eta_k\}$;*
(4) *$M_\eta(t_1 + a, \ldots, t_p + a) = M_\eta(t_1, \ldots, t_p) + a$ for any $a \in \mathbb{R}$;*
(5) *if $u_1, \ldots, u_p$ are plurisubharmonic functions, then $u = M_\eta(u_1, \ldots, u_p)$ is plurisubharmonic.*

Following the notations and assumptions in Lemma 2.1, we present the following approximation property of $\varphi$ and $\Psi$.

**Lemma 7.4** *For any $t_0 > 0$, there exist smooth plurisubharmonic functions $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ and smooth functions $\{\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ on $D\backslash\{F = 0\}$, such that*

(1) *$\{\varphi_m + \Psi_m\}_{m\in\mathbb{Z}_{>0}}$ and $\{\varphi_m + (1 + \delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ are smooth plurisubharmonic functions;*
(2) *the sequence $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ is convergent to $\varphi$, and there exists a smooth plurisubharmonic function $\varphi_0$ on $D\backslash\{F = 0\}$ such that $\varphi_0 \ge \varphi_m \ge \varphi$ on $D\backslash\{F = 0\}$ for any $m \in \mathbb{Z}_{>0}$;*
(3) *the sequence $\{\varphi_m + (1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ is decreasingly convergent to $\varphi + (1+\delta)\Psi$;*
(4) *$\{\Psi_m < -t_0\} \subset \{\Psi < -t_0\}$ and $\Psi_m \le 0$ on $D\backslash\{F = 0\}$ for any $m \in \mathbb{Z}_{>0}$.*

***Proof*** As $D$ is a pseudoconvex domain, there exists a sequence of plurisubharmonic functions $\{\psi_m\}_{m\in\mathbb{Z}_{>0}}$, which is decreasingly convergent to $\psi$. Let $\eta_m = (\frac{t_0}{3m}, \frac{t_0}{3m})$. Set

$$\varphi_m = (1+\delta)M_{\eta_m}(\psi_m, 2\log|F|),$$

and

$$\Psi_m = \psi_m - \frac{\varphi_m}{1+\delta},$$

which are smooth functions on $D\backslash\{F=0\}$.

Now, we prove that $(\varphi_m, \Psi_m)$ satisfies the four conditions. Lemma 7.3 shows that $\varphi_m$ is plurisubharmonic. Note that

$$\varphi_m + \Psi_m = \psi_m + \frac{\delta}{1+\delta}\varphi_m$$

and

$$\varphi_m + (1+\delta)\Psi_m = (1+\delta)\psi_m$$

are smooth plurisubharmonic functions, and $\{\varphi_m + (1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ decreasingly converges to $\varphi + (1+\delta)\Psi = (1+\delta)\psi$. Set

$$\varphi_0 = (1+\delta)\left(\max\{\psi_1, 2\log|F|\} + \frac{t_0}{3}\right).$$

It follows from Lemma 7.3 that

$$\varphi \le (1+\delta)\max\{\psi_m, 2\log|F|\} \le \varphi_m \le (1+\delta)\left(\max\{\psi_m, 2\log|F|\} + \frac{t_0}{3m}\right),$$

which implies that $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ is convergent to $\varphi$ and $\varphi \le \varphi_m \le \varphi_0$ on $D\backslash\{F=0\}$. It can be verified by Lemma 7.3 that:

(1) if $\psi_m \le 2\log|F| - \frac{2t_0}{3m}$ holds, then $\varphi_m = 2(1+\delta)\log|F|$ and $\Psi_m = \psi_m - 2\log|F| \le 0$;
(2) if $\psi_m \ge 2\log|F| + \frac{2t_0}{3m}$ holds, then $\varphi_m = (1+\delta)\psi_m$ and $\Psi_m = 0$;
(3) if $2\log|F| - \frac{2t_0}{3m} < \psi_m < 2\log|F| + \frac{2t_0}{3m}$ holds, then $-\frac{t_0}{m} \le \Psi_m \le 0$, and

$$(1+\delta)\max\{\psi_m, 2\log|F|\} \le \varphi_m \le (1+\delta)\left(\psi_m + \frac{t_0}{m}\right).$$

Thus, we have

$$\{\Psi_m < -t_0\} \subset \{\psi - 2\log|F| < -t_0\} \subset \{\Psi < -t_0\},$$

and $\Psi_m \le 0$ on $D\backslash\{F=0\}$. □

Now we begin to prove Lemma 2.1.

Note that $D\backslash\{F=0\}$ is a pseudoconvex domain. The following remark shows that it suffices to consider the case of Lemma 2.1 that $F(z)\neq 0$ for any $z\in D$.

**Remark 7.5** Assume that there exists a holomorphic function $F_1$ on $D\backslash\{F=0\}$ such that

$$\begin{aligned}&\int_{D\backslash\{F=0\}}|F_1-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2e^{-\varphi+v_{t_0,B}(\Psi)-\Psi}\\&\quad\leq\left(\frac{1}{\delta}+1-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}.\end{aligned}\tag{7.2}$$

Note that $v_{t_0,B}(\Psi)-\Psi\geq 0$ on $D$ and $|F|^{2(1+\delta)}e^{-\varphi}=1$ on $\{\Psi<-t_0\}$. Then it follows from $\int_{\{\Psi<-t_0\}}|f|^2<+\infty$ and inequality (7.2) that

$$\begin{aligned}&\int_{D\backslash\{F=0\}}|F_1|^2e^{-\varphi}\\&\leq 2\int_{D\backslash\{F=0\}}|(1-b_{t_0,B}(\Psi))f|^2+2\int_{D\backslash\{F=0\}}|F_1-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2e^{-\varphi}\\&\leq 2\int_{\{\Psi<-t_0\}}|f|^2+\left(\frac{1}{\delta}+1-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}\\&<+\infty,\end{aligned}$$

which implies that there exists a holomorphic function $\tilde{F}$ on $D$ such that $\tilde{F}=F_1$ on $D\backslash\{F=0\}$. Thus, we have

$$\begin{aligned}&\int_D|\tilde{F}-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2e^{-\varphi+v_{t_0,B}(\Psi)-\Psi}\\&\quad\leq\left(\frac{1}{\delta}+1-e^{-t_0-B}\right)\int_D\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}.\end{aligned}$$

Let $D_1\Subset\ldots\Subset D_j\Subset D_{j+1}\Subset\ldots$ be a sequence of strongly pseudoconvex domains such that $\bigcup_{j=1}^{+\infty}D_j=D$. As discussed above, there exist smooth plurisubharmonic functions $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ and smooth functions $\{\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ on $D$, such that:

(1) $\{\varphi_m+\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ and $\{\varphi_m+(1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ are smooth plurisubharmonic functions;
(2) the sequence $\{\varphi_m\}_{m\in\mathbb{Z}_{>0}}$ is convergent to $\varphi$, and there exists a smooth plurisubharmonic function $\varphi_0$ on $D$ such that $\varphi_0\geq\varphi_m\geq\varphi$ on $D$ for any $m\in\mathbb{Z}_{>0}$;
(3) the sequence $\{\varphi_m+(1+\delta)\Psi_m\}_{m\in\mathbb{Z}_{>0}}$ is decreasingly convergent to $\varphi+(1+\delta)\Psi$;
(4) $\{\Psi_m<-t_0\}\subset\{\Psi<-t_0\}$ and $\Psi_m\leq 0$ on $D$ for any $m\in\mathbb{Z}_{>0}$.

In the following, we prove Lemma 2.1 in 6 steps.

*Step 1: Some Notations*

Let $\epsilon \in (0, \frac{1}{8}B)$. Let $\{v_\epsilon\}_{\epsilon\in(0,\frac{1}{8}B)}$ be a family of smooth increasing convex functions on $\mathbb{R}$, which are continuous functions on $\mathbb{R}\cup\{-\infty\}$, such that:

(1) $v_\epsilon(t) = t$ for $t \geq -t_0-\epsilon$, $v_\epsilon(t) = constant$ for $t < -t_0-B+\epsilon$ and are pointwise convergent to $v_{t_0,B}$, when $\epsilon \to 0$;
(2) $v''_\epsilon(t)$ are pointwise convergent to $\frac{1}{B}\mathbb{I}_{(-t_0-B,-t_0)}$, when $\epsilon \to 0$, and $0 \leq v''_\epsilon(t) \leq \frac{2}{B}\mathbb{I}_{(-t_0-B+\epsilon,-t_0-\epsilon)}$ for any $t \in \mathbb{R}$;
(3) $v'_\epsilon(t)$ are pointwise convergent to $b_{t_0,B}(t)$ which is a continuous function on $\mathbb{R}\cup\{-\infty\}$, when $\epsilon \to 0$, and $0 \leq v'_\epsilon(t) \leq 1$ for any $t \in \mathbb{R}$.

One can construct the family $\{v_\epsilon\}_{\epsilon\in(0,\frac{1}{8}B)}$ by the setting

$$v_\epsilon(t) := \int_0^t \left(\int_{-\infty}^{t_1}\left(\frac{1}{B-4\epsilon}\mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon}\right)(s)ds\right)dt_1, \tag{7.3}$$

where $\rho_{\frac{1}{4}\epsilon}$ is the kernel of convolution satisfying $\mathrm{supp}(\rho_{\frac{1}{4}\epsilon}) \subset (-\frac{1}{4}\epsilon, \frac{1}{4}\epsilon)$. Then it follows that

$$v''_\epsilon(t) = \frac{1}{B-4\epsilon}\mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon}(t),$$

and

$$v'_\epsilon(t) = \int_{-\infty}^{t}\left(\frac{1}{B-4\epsilon}\mathbb{I}_{(-t_0-B+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon}\right)(s)ds.$$

It is clear that $\lim_{\epsilon\to 0} v_\epsilon(t) = v_{t_0,B}(t)$, and $\lim_{\epsilon\to 0} v'_\epsilon(t) = b_{t_0,B}(t)$.

Let $\eta = s(-v_\epsilon(\Psi_m))$ and $\phi = u(-v_\epsilon(\Psi_m))$, where $s \in C^\infty((0,+\infty))$ satisfies $s \geq \frac{1}{\delta}$ and $s' > 0$, and $u \in C^\infty([0,+\infty))$, such that $u''s - s'' > 0$ and $s' - u's = 1$ on $(0,+\infty)$. Let $\Phi = \phi + \varphi_m + \Psi_m$.

*Step 2: Solving the $\bar\partial$-equation*

Now let $\alpha = \sum_{j=1}^n \alpha_{\bar j}d\bar z^j \in \mathrm{Dom}_{D_j}(\bar\partial^*)\cap\mathrm{Ker}(\bar\partial)\cap C^\infty_{(0,1)}(\overline{D_j})$. It follows from Cauchy–Schwarz's inequality that

$$2\mathrm{Re}(\bar\partial^*_\Phi\alpha, \alpha\llcorner(\bar\partial\eta)^\sharp)_{D_j,\Phi} \geq -\int_{D_j} g^{-1}|\bar\partial^*_\Phi\alpha|^2e^{-\Phi} + \sum_{j,k=1}^n\int_{D_j} -g(\partial_j\eta)(\bar\partial_k\eta)\alpha_{\bar j}\bar\alpha_{\bar k}e^{-\Phi}, \tag{7.4}$$

where $g$ is a positive continuous function on $\overline{D_j}$. Using Lemma 7.1 and inequality (7.4), since $D_j$ is a strongly pseudoconvex domain, we get

$$
\begin{aligned}
&\int_{D_j}(\eta+g^{-1})|\bar{\partial}^*_\Phi\alpha|^2e^{-\Phi}\\
&\quad\ge\sum_{j,k=1}^n\int_{D_j}(-\partial_j\bar{\partial}_k\eta+\eta\partial_j\bar{\partial}_k\Phi-g(\partial_j\eta)(\bar{\partial}_k\eta))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}e^{-\Phi}\\
&\quad=\sum_{j,k=1}^n\int_{D_j}(-\partial_j\bar{\partial}_k\eta+\eta\partial_j\bar{\partial}_k(\phi+\varphi_m+\Psi_m)-g(\partial_j\eta)(\bar{\partial}_k\eta))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}e^{-\Phi}.
\end{aligned}
\tag{7.5}
$$

We need some calculations to determine $g$.

Since

$$
\partial_j\bar{\partial}_k\eta=-s'(-v_\epsilon(\Psi_m))\partial_j\bar{\partial}_k(v_\epsilon(\Psi_m))+s''(-v_\epsilon(\Psi_m))\partial_jv_\epsilon(\Psi_m)\bar{\partial}_kv_\epsilon(\Psi_m),
$$

and

$$
\partial_j\bar{\partial}_k\phi=-u'(-v_\epsilon(\Psi_m))\partial_j\bar{\partial}_k(v_\epsilon(\Psi_m))+u''(-v_\epsilon(\psi_m))\partial_jv_\epsilon(\Psi_m)\bar{\partial}_kv_\epsilon(\Psi_m),
$$

for any $j,k$ $(1\le j,k\le n)$, we have

$$
\begin{aligned}
&\sum_{j,k=1}^n(-\partial_j\bar{\partial}_k\eta+\eta\partial_j\bar{\partial}_k\phi-g(\partial_j\eta)(\bar{\partial}_k\eta))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}\\
&\quad=(s'-su')\sum_{j,k=1}^n\partial_j\bar{\partial}_kv_\epsilon(\Psi_m)\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}\\
&\qquad+((u''s-s'')-gs'^2)\sum_{j,k=1}^n\partial_j(-v_\epsilon(\Psi_m))\bar{\partial}_k(-v_\epsilon(\Psi_m))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}\\
&\quad=(s'-su')\sum_{j,k=1}^n(v'_\epsilon(\Psi_m)\partial_j\bar{\partial}_k\Psi_m+v''_\epsilon(\Psi_m)\partial_j(\Psi_m)\bar{\partial}_k(\Psi_m))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}\\
&\qquad+((u''s-s'')-gs'^2)\sum_{j,k=1}^n\partial_j(-v_\epsilon(\Psi_m))\bar{\partial}_k(-v_\epsilon(\Psi_m))\alpha_{\bar{j}}\bar{\alpha}_{\bar{k}}.
\end{aligned}
\tag{7.6}
$$

We omit the composite item $-v_\epsilon(\Psi_m)$ after $s'-su'$ and $(u''s-s'')-gs'^2$ in the above equalities. As $\varphi_m+\Psi_m$ and $\varphi_m+(1+\delta)\Psi_m$ are plurisubharmonic functions and $v'_\epsilon\in[0,1]$, we have

$$
(1-v'_\epsilon(\Psi_m))\sqrt{-1}\partial\bar{\partial}(\varphi_m+\Psi_m)+v'_\epsilon(\Psi_m)\sqrt{-1}\partial\bar{\partial}(\varphi_m+(1+\delta)\Psi_m)\ge0
$$

on $\overline{D_j}$, which implies that

$$\begin{aligned}&\eta\sqrt{-1}\partial\bar\partial(\varphi_m+\Psi_m)+v'_\epsilon(\Psi_m)\sqrt{-1}\partial\bar\partial\Psi_m\\&\geq\frac{1}{\delta}\sqrt{-1}\partial\bar\partial(\varphi_m+\Psi_m)+v'_\epsilon(\Psi_m)\sqrt{-1}\partial\bar\partial\Psi_m\geq 0\end{aligned}\tag{7.7}$$

on $\overline{D_j}$.

Let $g=\frac{u''s-s''}{s'^2}(-v_\epsilon(\Psi_m))$. It follows that $\eta+g^{-1}=\left(s+\frac{s'^2}{u''s-s''}\right)(-v_\epsilon(\Psi_m))$. As $s'-su'=1$, using inequality (7.5), inequality (7.6) and inequality (7.7), we obtain that

$$\int_{D_j}(\eta+g^{-1})|\bar\partial^*_\Phi\alpha|^2e^{-\Phi}\geq\int_{D_j}v''_\epsilon(\Psi_m)|\alpha\llcorner(\bar\partial\Psi_m)^\sharp|^2e^{-\Phi}.\tag{7.8}$$

As $fF^{1+\delta}$ is holomorphic on $\{\Psi<-t_0\}$ and $\overline{\{\Psi_m<-t_0-\epsilon\}}\subset\{\Psi_m<-t_0\}\subset\{\Psi<-t_0\}$,

$$\lambda:=\bar\partial\Big(\big(1-v'_\epsilon(\Psi_m)\big)fF^{1+\delta}\Big)$$

is well-defined and smooth on $D$. By the definition of contraction, Cauchy-Schwarz's inequality and inequality (7.8), it follows that

$$\begin{aligned}|(\lambda,\alpha)_{D_j,\Phi}|^2=&|(v''_\epsilon(\Psi_m)fF^{1+\delta}\bar\partial\Psi_m,\alpha)_{D_j,\Phi}|^2\\=&|(v''_\epsilon(\Psi_m)fF^{1+\delta},\alpha\llcorner(\bar\partial\Psi_m)^\sharp)_{D_j,\Phi}|^2\\\leq&\left(\int_{D_j}v''_\epsilon(\Psi_m)|fF^{1+\delta}|^2e^{-\Phi}\right)\left(\int_{D_j}v''_\epsilon(\Psi_m)|\alpha\llcorner(\bar\partial\Psi_m)^\sharp|^2e^{-\Phi}\right)\\\leq&\left(\int_{D_j}v''_\epsilon(\Psi_m)|fF^{1+\delta}|^2e^{-\Phi}\right)\left(\int_{D_j}(\eta+g^{-1})|\bar\partial^*_\Phi\alpha|^2e^{-\Phi}\right).\end{aligned}\tag{7.9}$$

Set $\mu:=(\eta+g^{-1})^{-1}$. By Lemma 7.2, there exists a locally $L^1$ function $u_{m,\epsilon,j}$ on $D_j$ such that $\bar\partial u_{m,\epsilon,j}=\lambda$, and

$$\int_{D_j}|u_{m,\epsilon,j}|^2(\eta+g^{-1})^{-1}e^{-\Phi}\leq\int_{D_j}v''_\epsilon(\Psi_m)|fF^{1+\delta}|^2e^{-\Phi}.\tag{7.10}$$

Assume that we can choose $\eta$ and $\phi$ such that $e^{v_\epsilon(\Psi_m)}e^{\phi}=(\eta+g^{-1})^{-1}$. Then inequality (7.10) becomes

$$\int_{D_j}|u_{m,\epsilon,j}|^2e^{v_\epsilon(\Psi_m)-\varphi_m-\Psi_m}\leq\int_{D_j}v''_\epsilon(\Psi_m)|F|^2e^{-\phi-\varphi_m-\Psi_m}.\tag{7.11}$$

Let

$$F_{m,\epsilon,j} := -u_{m,\epsilon,j} + \big(1 - v'_\epsilon(\Psi_m)\big) f F^{1+\delta}.$$

Then $F_{m,\epsilon,j}$ is holomorphic on $D_j$. Now inequality (7.11) becomes

$$\begin{aligned} &\int_{D_j} |F_{m,\epsilon,j} - (1 - v'_\epsilon(\Psi_m)) f F^{1+\delta}|^2 e^{v_\epsilon(\Psi_m) - \varphi_m - \Psi_m} \\ &\le \int_{D_j} (v''_\epsilon(\Psi_m)) |f F^{1+\delta}|^2 e^{-\phi - \varphi_m - \Psi_m}. \end{aligned} \tag{7.12}$$

*Step 3: $m \to +\infty$*

Note that

$$\sup_m \sup_{D_j} e^{-\phi} = \sup_m \sup_{D_j} e^{-u(-v_\epsilon(\Psi_m))} < +\infty.$$

As $\{\Psi_m < -t_0 - \epsilon\} \subset \{\Psi < -t_0\}$ and $|F^{1+\delta}|^2 e^{-\varphi} = 1$ on $\{\Psi < -t_0\}$, we have

$$\begin{aligned} (v''_\epsilon(\Psi_m)) |f F^{1+\delta}|^2 e^{-\phi - \varphi_m - \Psi_m} &\le e^{t_0 + B} (v''_\epsilon(\Psi_m)) |f F^{1+\delta}|^2 e^{-\phi - \varphi} \\ &\le \left( \sup_m \sup_{D_j} e^{-\phi} \right) e^{t_0 + B} \mathbb{I}_{\{\Psi < -t_0\}} |f|^2 \end{aligned}$$

on $D_j$. Then it follows from $\int_{\{\Psi < -t_0\}} |f|^2 < +\infty$ and the dominated convergence theorem that

$$\begin{aligned} &\lim_{m \to +\infty} \int_{D_j} (v''_\epsilon(\Psi_m)) |f F^{1+\delta}|^2 e^{-\phi - \varphi_m - \Psi_m} \\ &= \int_{D_j} (v''_\epsilon(\Psi)) |f F^{1+\delta}|^2 e^{-u(-v_\epsilon(\Psi)) - \varphi - \Psi} < +\infty. \end{aligned} \tag{7.13}$$

Since $\inf_m \inf_{D_j} e^{v_\epsilon(\Psi_m) - \varphi_m - \Psi_m} \ge \inf_{D_j} e^{-\varphi_0} > 0$, by inequality (7.12), we have

$$\sup_m \int_{D_j} |F_{m,\epsilon,j} - (1 - v'_\epsilon(\Psi_m)) f F^{1+\delta}|^2 < +\infty.$$

According to

$$\left| (1 - v'_\epsilon(\Psi_m)) f F^{1+\delta} \right| \le \left( \sup_{D_j} |F|^{1+\delta} \right) \mathbb{I}_{\{\Psi < -t_0\}} |f|,$$

and $\int_{\{\Psi < -t_0\}} |f|^2 < +\infty$, we get $\sup_m \int_{D_j} |F_{m,\epsilon,j}|^2 < +\infty$, which implies that there exists a subsequence of $\{F_{m,\epsilon,j}\}_{m \in \mathbb{Z}_{>0}}$ (also denoted by $F_{m,\epsilon,j}$) compactly convergent to a holomorphic function $F_{\epsilon,j}$ on $D_j$. Now Fatou's Lemma, inequality (7.12) and inequality (7.13) indicate that

$$\begin{aligned}
&\int_{D_j}|F_{\epsilon,j}-(1-v'_\epsilon(\Psi))fF^{1+\delta}|^2e^{v_\epsilon(\Psi)-\varphi-\Psi}\\
&\le \liminf_{m\to+\infty}\int_{D_j}\left|F_{m,\epsilon,j}-(1-v'_\epsilon(\Psi_m))fF^{1+\delta}\right|^2e^{v_\epsilon(\Psi_m)-\varphi_m-\Psi_m}\\
&\le \lim_{m\to+\infty}\int_{D_j}v''_\epsilon(\Psi_m)|fF^{1+\delta}|^2e^{-\phi-\varphi_m-\Psi_m}\\
&=\int_{D_j}v''_\epsilon(\Psi)|fF^{1+\delta}|^2e^{-u(-v_\epsilon(\Psi))-\varphi-\Psi}\\
&=\int_{D_j}v''_\epsilon(\Psi)|f|^2e^{-u(-v_\epsilon(\Psi))-\Psi}.
\end{aligned}\tag{7.14}$$

*Step 4:* $\epsilon\to 0+0$

Note that $\sup_\epsilon\sup_{D_j}e^{-u(-v_\epsilon(\Psi))}<+\infty$, and

$$(v''_\epsilon(\Psi))|f|^2e^{-u(-v_\epsilon(\Psi))-\Psi}\le\left(\sup_\epsilon\sup_{D_j}e^{-u(-v_\epsilon(\Psi))}\right)e^{t_0+B}\mathbb{I}_{\{\Psi<-t_0\}}|f|^2$$

on $D_j$. Then it follows from $\int_{\{\Psi<-t_0\}}|f|^2<+\infty$ and the dominated convergence theorem that

$$\begin{aligned}
&\lim_{\epsilon\to0}\int_{D_j}(v''_\epsilon(\Psi))|f|^2e^{-u(-v_\epsilon(\Psi))-\Psi}\\
&=\int_{D_j}\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-u(-v_{t_0,B}(\Psi))-\Psi}\\
&\le\left(\sup_{D_j}e^{-u(-v_{t_0,B}(\Psi))}\right)\int_{D_j}\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}\\
&<+\infty.
\end{aligned}\tag{7.15}$$

Since $\inf_\epsilon\inf_{D_j}e^{v_\epsilon(\Psi)-\varphi-\Psi}>0$, according to inequality (7.14) and (7.15), we have

$$\int_{D_j}\left|F_{\epsilon,j}-(1-v'_\epsilon(\Psi))fF^{1+\delta}\right|^2<+\infty.$$

Combining with

$$\sup_\epsilon\int_{D_j}|(1-v'_\epsilon(\Psi))fF^{1+\delta}|^2\le\left(\sup_{D_j}|F|^{2(1+\delta)}\right)\int_{D_j}\mathbb{I}_{\{\Psi<-t_0\}}|f|^2<+\infty,$$

one can obtain $\sup_\epsilon \int_{D_j} |F_{\epsilon,j}|^2 < +\infty$, which implies that there exists a subsequence of $\{F_{\epsilon,j}\}_{\epsilon\to 0+0}$ (denoted by $\{F_{\epsilon_l,j}\}_{l\in\mathbb{Z}_{>0}}$) compactly convergent to a holomorphic function $F_j$ on $D_j$. Now Fatou's Lemma, inequality (7.14) and inequality (7.15) verify that

$$\begin{aligned}
&\int_{D_j} |F_j - (1-b_{t_0,B}(\Psi)) f F^{1+\delta}|^2 e^{v_{t_0,B}(\Psi)-\varphi-\Psi} \\
&\le \liminf_{l\to+\infty} \int_{D_j} |F_{\epsilon_l,j} - (1-v'_{\epsilon_l}(\Psi)) f F^{1+\delta}|^2 e^{v_{\epsilon_l}(\Psi)-\varphi-\Psi} \\
&\le \liminf_{l\to+\infty} \int_{D_j} (v''_{\epsilon_l}(\Psi))|f|^2 e^{-u(-v_{\epsilon_l}(\Psi))-\Psi} \\
&\le \left(\sup_{D_j} e^{-u(-v_{t_0,B}(\Psi))}\right) \int_{D_j} \frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi}.
\end{aligned} \tag{7.16}$$

*Step 5: $j \to +\infty$*

Note that

$$\begin{aligned}
&\lim_{j\to+\infty} \left(\sup_{D_j} e^{-u(-v_{t_0,B}(\Psi))}\right) \int_{D_j} \frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi} \\
&\le \left(\sup_{D} e^{-u(-v_{t_0,B}(\Psi))}\right) \int_{D} \frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2 e^{-\Psi} < +\infty,
\end{aligned} \tag{7.17}$$

and $v_{t_0,B}(\Psi) - \Psi \ge 0$. Then it follows from inequality (7.16) that

$$\limsup_{j\to+\infty} \int_{D_j} |F_j - (1-b_{t_0,B}(\Psi)) f F^{1+\delta}|^2 e^{-\varphi} < +\infty.$$

Combining with

$$\int_{D_j} |(1-b_{t_0,B}(\Psi)) f F^{1+\delta}|^2 e^{-\varphi} \le \int_{\{\Psi<-t_0\}} |f|^2 < +\infty,$$

we obtain

$$\limsup_{j\to+\infty} \int_{D_j} |F_j|^2 e^{-\varphi} < +\infty.$$

Then we can extract a compactly convergent subsequence of $\{F_j\}$ (also denoted by $\{F_j\}$), which is convergent to a holomorphic function $\tilde{F}$ on $D$. It follows from Fatou's

Lemma, inequality (7.16) and inequality (7.17) that

$$\begin{aligned}
&\int_D |\tilde{F}-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{v_{t_0,B}(\Psi)-\varphi-\Psi} \\
&\leq \liminf_{j\to+\infty}\int_{D_j}|F_j-(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2 e^{v_{t_0,B}(\Psi)-\varphi-\Psi} \\
&\leq \lim_{j\to+\infty}\left(\sup_{D_j} e^{-u(-v_{t_0,B}(\Psi))}\right)\int_{D_j}\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi} \\
&\leq \left(\sup_{D} e^{-u(-v_{t_0,B}(\Psi))}\right)\int_{D}\frac{1}{B}\mathbb{I}_{\{-t_0-B<\Psi<-t_0\}}|f|^2e^{-\Psi}.
\end{aligned} \tag{7.18}$$

*Step 6: ODE system*

Finally, it suffices to find $\eta$ and $\phi$ such that $\eta+g^{-1}=e^{-v_\epsilon(\Psi_m)}e^{-\phi}$ on $D_j$ and $s'-u's=1$. As $\eta=s(-v_\epsilon(\Psi_m))$ and $\phi=u(-v_\epsilon(\Psi_m))$, we have

$$(\eta+g^{-1})e^{v_\epsilon(\Psi_m)}e^{\phi}=\left(\left(s+\frac{s'^2}{u''s-s''}\right)e^{-t}e^{u}\right)\circ\big(-v_\epsilon(\Psi_m)\big).$$

Summarizing the above discussions about $s$ and $u$, we are naturally led to a system of ODEs (see [26, 27, 29, 30]):

$$\begin{aligned}
&(1)\ \left(s+\frac{s'^2}{u''s-s''}\right)e^{u-t}=1, \\
&(2)\ s'-su'=1,
\end{aligned} \tag{7.19}$$

where $t\in(0,+\infty)$. Solving the ODE system (7.19), we get

$$u(t)=-\log\left(\frac{1}{\delta}+1-e^{-t}\right),$$

and

$$s(t)=\frac{\left(1+\frac{1}{\delta}\right)t+\frac{1}{\delta}\left(1+\frac{1}{\delta}\right)}{\frac{1}{\delta}+1-e^{-t}}-1,$$

(see [30]). It follows that $s\in C^\infty((0,+\infty))$ satisfying $s\geq\frac{1}{\delta}$ and $s'>0$, and $u\in C^\infty([0,+\infty))$ satisfying $u''s-s''>0$.

As $u(t)=-\log\left(\frac{1}{\delta}+1-e^{-t}\right)$ is decreasing with respect to $t$, and $0\geq v(t)\geq \max\{t,-t_0-B_0\}\geq -t_0-B_0$ for any $t\leq 0$, we can see that

$$\sup_D e^{-u(-v_{t_0,B}(\Psi))}\leq\sup_{t\in[0,t_0+B]}e^{-u(t)}=\frac{1}{\delta}+1-e^{-t_0-B}.$$

Therefore, we are done.

Eventually, we complete the proof of Lemma 2.1.

### 7.2 An example on the Hartogs triangle

In the second part of the appendix, for the sake of the completeness, we drag an explicit example for the objects described in the present paper. In this example, the function $F$ is not trivial (i.e. not a constant function). A similar example can also be seen in [1].

Let $D = \Delta^2$ be the unit polydisc in $\mathbb{C}^2$ with the coordinate $(z, w)$, $\psi = 2\log|z|$ a plurisubharmonic function on $\Delta^2$, and $F = w^k$ a holomorphic function on $\Delta^2$ for a positive integer $k$. Then we have

$$\Psi(z, w) := \min\{\psi - 2\log|F|, 0\} = \min\left\{2\log\left|\frac{z}{w^k}\right|, 0\right\},$$

where $\Psi(o, w) := 0$ for any $w \in \Delta$. It follows that

$$D_t := \{\Psi < -t\} = \left\{(z, w) \in \Delta \times \Delta : |z| < e^{-t/2}|w|^k\right\}, \ \forall t \in [0, +\infty).$$

Especially,

$$D_0 = \left\{(z, w) \in \mathbb{C}^2 : |z| < |w|^k < 1\right\},$$

which is also called the (generalized) *Hartogs triangle* (see [15, 43]). One can see that $o$ is a boundary point of $D_t$ for any $t \geq 0$.

Fix $t \geq 0$. Note that $D_t$ is a Reinhardt domain (which means that $D_t$ is invariant under the $\mathbb{T}^n-$action). Then any holomorphic function on $D_t$ can be written as its Laurent expansion. Let

$$f(z, w) = \sum_{(j,l)\in\mathbb{Z}^2} c_{j,l} z^j w^l \in \mathcal{O}(D_t).$$

One can verify that

$$\int_{D_t} |f|^2 = \sum_{(j,l)\in\mathbb{Z}^2} |c_{j,l}|^2 \int_{D_t} |z|^{2j}|w|^{2l},$$

where

$$\int_{D_t} |z|^{2j}|w|^{2l} = \begin{cases} \dfrac{\pi^2 e^{-(j+1)t}}{(j+1)(jk+k+l+1)}, & \text{if } j \geq 0, \ jk+k+l \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Similarly,

$$\int_{D_t} |f|^2 e^{-\Psi} = \sum_{(j,l)\in\mathbb{Z}^2} |c_{j,l}|^2 \int_{D_t} |z|^{2j-2}|w|^{2l+2k},$$

where

$$\int_{D_t} |z|^{2j-2}|w|^{2l+2k} = \begin{cases} \dfrac{\pi^2 e^{-jt}}{j(jk+k+l+1)}, & \text{if } j \geq 1,\ jk+k+l \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then it can be summarized that

$$I(\Psi)_o = \Big\{h(z,w) = \sum_{(j,l)\in\mathbb{Z}^2} a_{j,l} z^j w^l \text{ near } o : a_{j,l} = 0,\ \forall (j,l) \in \mathcal{S}\Big\},$$

where

$$\mathcal{S} = \big\{(j,l) \in \mathbb{Z}^2 : j \leq 0 \text{ or } l \leq -(jk+k+1)\big\}.$$

Set $J = I(\Psi)_o$, and $f = w^{-k}$. It is not hard to calculate that

$$G(t; \Psi, J, f) = \int_{D_t} |f|^2 = \pi^2 e^{-t},$$

which shows that $G(-\log r; \Psi, J, f)$ is concave with respect to $r \in (0,1]$ in this case.

Under the same assumptions above, one can also verify that $c_o^{fF}(\psi) = 1/2$, and

$$\frac{1}{r^2}\int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 = \frac{1}{r^2}\int_{D_{-2\log r}} |w|^{-2k} = \pi^2.$$

On the other hand, since

$$\Psi_1 := \min\{2c_o^{fF}(\psi)\psi - 2\log|F|, 0\} = \Psi,$$

with some similar computations for $a\Psi$ $(a > 1)$, one can see that

$$I_+(\Psi)_o = \bigcup_{a>1} I(a\Psi)_o = I(\Psi)_o.$$

Now according to the value of $G(t; \Psi, J, f)$, it can be obtained that

$$\frac{1}{r^2}\int_{\{c_o^{fF}(\psi)\psi - \log|F| < \log r\}} |f|^2 = \pi^2 = G(0; \Psi_1, I_+(\Psi)_o, f),$$

for any $r \in (0, 1]$. This also shows that the effectiveness result in Theorem 1.7 is sharp.

It is easy to check that $a_o^f(\Psi) = 1/2$, thus one can verify that the lower bound in Theorem 1.11 is also sharp. The strong openness property for $\Psi$ has been checked.

**Acknowledgements** The authors would like to express their sincere gratitude to the reviewers for their insightful comments and valuable suggestions. The second named author was supported by National Key R&D Program of China 2021YFA1003100, NSFC-12425101, NSFC-11825101, NSFC-11522101 and NSFC-11431013. The third named author was supported by China Postdoctoral Science Foundation BX20230402 and 2023M743719.

**Data Availibility** The authors declare that the manuscript has no associated data.

## Declarations

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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