

MODULES AT BOUNDARY POINTS, FIBERWISE BERGMAN KERNELS, AND LOG-SUBHARMONICITY

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ABSTRACT. In this article, we consider Bergman kernels with respect to modules at boundary points, and obtain a log-subharmonicity property of the Bergman kernels, which deduces a concavity property related to the Bergman kernels. As applications, we reprove the sharp effectiveness result related to a conjecture posed by Jonsson-Mustăță and the effectiveness result of strong openness property of the modules at boundary points.

1. INTRODUCTION

The strong openness property of multiplier ideal sheaves (i.e. $\mathcal{I}(\psi) = \mathcal{I}_+(\psi) := \bigcup_{\epsilon > 0} \mathcal{I}((1 + \epsilon)\psi)$) is an important feature and has been widely used in the study of several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [39, 47, 9, 10, 22, 11, 60, 42, 7, 61, 62, 23, 48, 12]), where ψ is a plurisubharmonic function on a complex manifold M (see [13]) and multiplier ideal sheaf $\mathcal{I}(\psi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\psi}$ is locally integrable (see e.g. [58, 50, 53, 16, 17, 15, 18, 49, 54, 55, 14, 43]).

The strong openness property was conjectured by Demailly [15] and proved by Guan-Zhou [39] (the 2-dimensional case was proved by Jonsson-Mustăță [45]). Recall that in order to prove the strong openness property, Jonsson and Mustăță (see [46], see also [45]) posed the following conjecture, and proved the 2-dimensional case [45]:

Conjecture J-M: If $c_o^F(\psi) < +\infty$, $\frac{1}{r^\pi} \mu(\{c_o^F(\psi)\psi - \log |F| < \log r\})$ has a uniform positive lower bound independent of $r \in (0, 1)$, where μ is the Lebesgue measure on \mathbb{C}^n , and $c_o^F(\psi) := \sup\{c \geq 0 : |F|^2 e^{-2c\psi} \text{ is locally } L^1 \text{ near } o\}$.

Using the strong openness property, Guan-Zhou [41] proved Conjecture J-M.

Independent of the strong openness property, Bao-Guan-Yuan [3] considered minimal L^2 integrals with respect to a module at a boundary point of the sublevel sets, and established a concavity property of the minimal L^2 integrals, which deduced a sharp effectiveness result related to Conjecture J-M, and completed the approach from Conjecture J-M to the strong openness property.

As a generalization of Berndtsson's log-plurisubharmonicity result of fiberwise Bergman kernels (see [4]), in [1] (see also [2]), we obtained the log-plurisubharmonicity of fiberwise Bergman kernels with respect to functionals over the space of holomorphic germs by using the optimal L^2 extension theorem (see [40]) and Guan-Zhou method (see [51]). As applications, we gave new approaches to the effectiveness results of strong openness property ([1]) and L^p strong openness property ([2]).

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As continuity work of [1] and [2], in this article, we consider Bergman kernels with respect to the modules at boundary points, and obtain a log-subharmonicity property of the Bergman kernels (as a generalization of Berndtsson's log-subharmonicity result of fiberwise Bergman kernels in [4]), which deduces a new approach from Conjecture J-M to the strong openness property. We also give a reproof for the effectiveness result of the strong openness property related to the modules at boundary points.

1.1. Main result.

Let D be a pseudoconvex domain in \mathbb{C}^n , and the origin $o \in D$. Let $F \not\equiv 0$ be a holomorphic function on D , and ψ be a negative plurisubharmonic function on D . Let φ_0 be a plurisubharmonic function on D . Denote that

$$\Psi := \min\{\psi - 2 \log |F|, 0\}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$.

We recall some notations in [3]. Denote that

$$\tilde{J}(\Psi)_o := \{f \in \mathcal{O}(\{\Psi < -t\} \cap V) : t \in \mathbb{R}, V \text{ is a neighborhood of } o\},$$

and

$$J(\Psi)_o := \tilde{J}(\Psi)_o / \sim,$$

where the equivalence relation ' \sim ' is as follows:

$$f \sim g \Leftrightarrow f = g \text{ on } \{\Psi < -t\} \cap V, \text{ where } t \gg 1, V \text{ is a neighborhood of } o.$$

For any $f \in \tilde{J}(\Psi)_o$, denote the equivalence class of f in $J(\Psi)_o$ by f_o . And for any $f_o, g_o \in J(\Psi)_o$, and $(h, o) \in \mathcal{O}_o$, define

$$f_o + g_o := (f + g)_o, \quad (h, o) \cdot f_o := (hf)_o.$$

It is clear that $J(\Psi)_o$ is an \mathcal{O}_o -module. For any $a \geq 0$, denote that $I(a\Psi + \varphi_0)_o := \{f_o \in J(\Psi)_o : \exists t \gg 1, V \text{ is a neighborhood of } o, \text{ s.t. } \int_{\{\Psi < -t\} \cap V} |f|^2 e^{-a\Psi - \varphi_0} < +\infty\}$. Then it is clear that $I(a\Psi + \varphi_0)_o$ is an \mathcal{O}_o -submodule of $J(\Psi)_o$. Especially, we denote that $I(\varphi_0)_o := I(0\Psi + \varphi_0)_o$, $I(\Psi)_o := I(\Psi + 0)_o$, and $I_o := I(0\Psi + 0)_o$. Then $I(a\Psi + \varphi_0)_o$ is an \mathcal{O}_o -submodule of $I(\varphi_0)_o$ for any $a > 0$.

For any $t \in [0, +\infty)$ and $\lambda > 0$, denote that

$$\Psi_{\lambda,t} := \lambda \max\{\Psi + t, 0\},$$

and for any $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ and $\lambda > 0$, denote that

$$\|f\|_{\lambda,t} := \left(\int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0 - \Psi_{\lambda,t}} \right)^{1/2},$$

where $A^2(\{\Psi < 0\}, e^{-\varphi_0}) := \{f \in \mathcal{O}(\{\Psi < 0\}) : \int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0} < +\infty\}$ (if $\varphi_0 \equiv 0$, we may denote $A^2(\{\Psi < 0\}) := A^2(\{\Psi < 0\}, e^0)$). It is clear that $e^{-\lambda t/2} \|f\|_{\lambda,0} \leq \|f\|_{\lambda,t} \leq \|f\|_{\lambda,0} < +\infty$ for any $t \geq 0$.

For any $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ (the dual space of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$), denote that the Bergman kernel related to ξ is

$$K_{\xi, \Psi, \lambda}^{\varphi_0}(t) := \sup_{f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})} \frac{|\xi \cdot f|^2}{\|f\|_{\lambda,t}^2}$$

for any $t \in [0, +\infty)$.

Denote $E := \{w \in \mathbb{C} : \operatorname{Re} w \geq 0\} \subset \mathbb{C}$. We obtain the following log-subharmonicity property of the Bergman kernel $K_{\xi, \Psi, \lambda}^{\varphi_0}$.

Theorem 1.1. *Assume that $K_{\xi, \Psi, \lambda}^{\varphi_0}(0) \in (0, +\infty)$. Then $\log K_{\xi, \Psi, \lambda}^{\varphi_0}(\operatorname{Re} w)$ is subharmonic with respect to $w \in E$.*

Let J be an \mathcal{O}_o -submodule of $I(\varphi_0)_o$. Denote that

$$A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J := \{f \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) : f_o \in J\}.$$

Assume that $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is a proper subspace of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ (and we will state that it is a closed subspace).

Using Theorem 1.1, we obtain the following concavity and monotonicity property related to $K_{\xi, \Psi, \lambda}^{\varphi_0}$.

Theorem 1.2. *Assume that $J \supset I(\Psi + \varphi_0)_o$, and assume that $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ such that $\xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J} \equiv 0$ and $K_{\xi, \Psi, \lambda}^{\varphi_0}(0) \in (0, +\infty)$. Then $-\log K_{\xi, \Psi, \lambda}^{\varphi_0}(t) + t$ is concave and increasing with respect to $t \in [0, +\infty)$.*

Remark 1.3. *Let $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$. According to Theorem 1.1, if there exist $k > 0$ and $T > 0$, such that $e^{-kt}K_{\xi, \Psi, \lambda}^{\varphi_0}(t)$ is increasing and not a constant function on $[0, T]$, then $e^{-kt}K_{\xi, \Psi, \lambda}^{\varphi_0}(t)$ is strictly increasing on $[T, +\infty)$.*

1.2. Applications.

As applications of Theorem 1.1 and Theorem 1.2, we give new proofs of some results in [3, 31].

Let D be a pseudoconvex domain in \mathbb{C}^n , and the origin $o \in D$. Let $F \not\equiv 0$ be a holomorphic function on D , and ψ be a negative plurisubharmonic function on D . Denote that

$$\Psi := \min\{\psi - 2 \log |F|, 0\}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$.

Let f be a holomorphic function on D . Recall the definition of the minimal L^2 integral related to J ([3, 31])

$$G(t; \Psi, J, f) := \inf \left\{ \int_{\{\Psi < -t\}} |\tilde{f}|^2 : \tilde{f} \in \mathcal{O}(\{\Psi < -t\}) \text{ } \& \text{ } (\tilde{f} - f)_o \in J \right\}$$

for any \mathcal{O}_o -submodule J of I_o and $t \in [0, +\infty)$. Denote that

$$\Psi_1 := \min\{2c_o^{f_F}(\psi)\psi - 2 \log |F|, 0\},$$

and

$$I_+(\Psi_1)_o := \bigcup_{a>1} I(a\Psi_1)_o,$$

where $c_o^{f_F}(\psi) := \sup\{c \geq 0 : |fF|^2 e^{-2c\psi} \text{ is locally } L^1 \text{ near } o\}$.

Theorem 1.2 deduces a reproof of the following lower bound of L^2 integrals.

Corollary 1.4 ([3]). *If $f \in A^2(\{\Psi_1 < 0\})$, and $c_o^{f_F}(\psi) < +\infty$, then for any $r \in (0, 1]$,*

$$\frac{1}{r^2} \int_{\{c_o^{f_F}(\psi)\psi - \log |F| < \log r\}} |f|^2 \geq G(0; \Psi_1, I_+(\Psi_1)_o, f) > 0.$$

Remark 1.5. *The proof of the inequality $G(0; \Psi_1, I_+(\Psi_1)_o, f) > 0$ can be referred to [3].*

When $f \equiv 1$, Corollary 1.4 deduces a reproof of the sharp effectiveness result related to a conjecture posed by Jonsson-Mustăță.

Corollary 1.6 ([3]). *If $f \in A^2(\{\Psi_1 < 0\})$ and $c_o^F(\psi) < +\infty$, then for any $r \in (0, 1]$,*

$$\frac{1}{r^2} \mu(\{c_o^F(\psi)\psi - \log |F| < \log r\}) \geq G(0; \Psi_1, I_+(\Psi_1)_o, 1) > 0,$$

where $\Psi_1 := \min\{2c_o^F(\psi)\psi - 2\log |F|, 0\}$, and $c_o^F(\psi) := \sup\{c \geq 0 : |F|^2 e^{-2c\psi} \text{ is locally } L^1 \text{ near } o\}$.

Let φ_0 be a plurisubharmonic function on D , and let f be a holomorphic function on $\{\Psi < 0\}$. Denote that $a_o^f(\Psi; \varphi_0) := \sup\{a \geq 0 : f_o \in I(2a\Psi + \varphi_0)_o\}$,

$$I_+(a\Psi_1 + \varphi_0)_o := \bigcup_{a' > a} I(a'\Psi_1 + \varphi_0)_o$$

for any $a \geq 0$, and

$$C(\Psi, \varphi_0, J, f) := \inf \left\{ \int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0} : (\tilde{f} - f)_o \in J \& \tilde{f} \in \mathcal{O}(\{\Psi < 0\}) \right\}$$

for any \mathcal{O}_o -submodule J of $I(\varphi_0)_o$. The following effectiveness result of strong openness property of the module $I(a\Psi + \varphi_0)_o$ can be reproved by Theorem 1.2.

Corollary 1.7 ([31]). *Let C_1 and C_2 be two positive constants. If*

- (1) $\int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0 - \Psi} \leq C_1$;
 - (2) $C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f) \geq C_2$,
- then for any $q > 1$ satisfying

$$\theta(q) > \frac{C_1}{C_2},$$

we have $f_o \in I(q\Psi + \varphi_0)_o$, where $\theta(q) = \frac{q}{q-1}$.

2. PREPARATIONS

2.1. L^2 methods.

We recall the optimal L^2 extension theorem.

Let Ω be a pseudoconvex domain in \mathbb{C}^{n+1} with coordinate (z, t) , where $z \in \mathbb{C}^n$, $t \in \mathbb{C}$. Let p be the natural projections $p(z, t) = t$ on Ω . Denote that $\omega := p(\Omega)$ and $\Omega_t := p^{-1}(t)$ for any $t \in \omega$. Let φ be a plurisubharmonic function on Ω .

Lemma 2.1 (Optimal L^2 extension theorem ([8], see [37, 40, 38])). *Let $\omega = \Delta_{t_0, r}$ be the disc in the complex plane centered at t_0 with radius r . Then for any f in $A^2(\Omega_{t_0}, e^{-\varphi})$, there exists a holomorphic function \tilde{f} on Ω , such that $\tilde{f}|_{\Omega_{t_0}} = f$, and*

$$\frac{1}{\pi r^2} \int_{\Omega} |\tilde{f}|^2 e^{-\varphi} \leq \int_{\Omega_{t_0}} |f|^2 e^{-\varphi}.$$

The following L^2 method will be used to prove Theorem 1.2.

Let D be a pseudoconvex domain in \mathbb{C}^n , and the origin $o \in D$. Let $F \not\equiv 0$ be a holomorphic function on D , and ψ be a negative plurisubharmonic function on D . Let φ_0 be a plurisubharmonic function on D . Denote that

$$\varphi := \varphi_0 + 2 \max\{\psi, 2 \log |F|\},$$

and

$$\Psi := \min\{\psi - 2 \log |F|, 0\}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$.

Lemma 2.2 (see [40, 41, 3, 31]). *Let $t_0 \in (0, +\infty)$ be arbitrary given. Let f be a holomorphic function on $\{\Psi < -t_0\}$ such that*

$$\int_{\{\Psi < -t_0\} \cap K} |f|^2 e^{-\varphi_0} < +\infty$$

for any compact subset $K \subset D$. Then there exists a holomorphic function \tilde{F} on D such that

$$\int_D |\tilde{F} - (1 - b_{t_0}(\Psi))f|^2 e^{-\varphi + v_{t_0}(\Psi) - \Psi} \leq C \int_D \mathbb{I}_{\{-t_0-1 < \Psi < -t_0\}} |f|^2 e^{-\varphi_0 - \Psi},$$

where $b_{t_0}(t) = \int_{-\infty}^t \mathbb{I}_{\{-t_0-1 < s < -t_0\}} ds$, $v_{t_0}(t) = \int_0^t b_{t_0}(s) ds$ and C is a positive constant.

2.2. Some lemmas about submodules of $I(\varphi_0)_o$.

Recall that D is a pseudoconvex domain in \mathbb{C}^n , and the origin $o \in D$. Let $F \not\equiv 0$ be a holomorphic function on D , and ψ be a negative plurisubharmonic function on D . Let φ_0 be a plurisubharmonic function on D . Denote that

$$\Psi := \min\{\psi - 2 \log |F|, 0\}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$. We recall the following lemma.

Lemma 2.3 ([31]). *Let J_o be an $\mathcal{O}_{\mathbb{C}^n, o}$ -submodule of $I(\varphi_0)_o$ such that $I(\Psi + \varphi_0)_o \subset J_o$. Assume that $f_o \in J(\Psi)_o$. Let U_0 be a Stein open neighborhood of o . Let $\{f_j\}_{j \geq 1}$ be a sequence of holomorphic functions on $U_0 \cap \{\Psi < -t_j\}$ for any $j \geq 1$, where $t_j \in (T, +\infty)$. Assume that $t_0 = \lim_{j \rightarrow +\infty} t_j \in [T, +\infty)$,*

$$\limsup_{j \rightarrow +\infty} \int_{U_0 \cap \{\Psi < -t_j\}} |f_j|^2 e^{-\varphi_0} \leq C < +\infty,$$

and $(f_j - f)_o \in J_o$. Then there exists a subsequence of $\{f_j\}_{j \geq 1}$ compactly convergent to a holomorphic function f_0 on $\{\Psi < -t_0\} \cap U_0$ which satisfies

$$\int_{U_0 \cap \{\Psi < -t_0\}} |f_0|^2 e^{-\varphi_0} \leq C,$$

and $(f_0 - f)_o \in J_o$.

It is well-known that $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ is a Hilbert space. Let J be an \mathcal{O}_o -submodule of $I(\varphi_0)_o$. We state that $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J := \{f \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) : f_o \in J\}$ is a closed subspace of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ if $J \supset I(\Psi + \varphi_0)_o$.

Lemma 2.4. *If $J \supset I(\Psi + \varphi_0)_o$, then $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is closed in $A^2(\{\Psi < 0\}, e^{-\varphi_0})$.*

Proof. Let $\{f_j\}$ be a sequence of holomorphic functions in $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$, such that $\lim_{j \rightarrow +\infty} f_j = f_0$ under the topology of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$. Then $\{f_j\}$ compactly converges to f_0 on $\{\Psi < 0\}$, and $(f_j - 0)_o \in J$ for any j . According to Lemma 2.3, we can get that $(f_0 - 0)_o \in J$, which means that $(f_0)_o \in J$, i.e. $f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$. Then we know that $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is closed in $A^2(\{\Psi < 0\}, e^{-\varphi_0})$. \square

The following two lemmas can be referred to [3] or [31].

Lemma 2.5 ([31]). *For any $a \geq 0$, there exists $a' > a$ such that $I(a'\Psi + \varphi_0)_o = I_+(a\Psi + \varphi_0)_o$.*

Let f be a holomorphic function on D . Denote that

$$\Psi_1 := \min\{2c_o^{fF}(\psi)\psi - 2\log|F|, 0\},$$

where $c_o^{fF}(\psi) := \sup\{c \geq 0 : |fF|^2 e^{-2c\psi} \text{ is locally } L^1 \text{ near } o\}$.

Lemma 2.6 ([3], see also [31]). *$f_o \notin I_+(\Psi_1)_o$.*

2.3. Some lemmas about functionals on $A^2(\{\Psi < 0\}, e^{-\varphi_0})$.

The following two lemmas will be used in the proof of Theorem 1.1. For the convenience of readers, we recall the proofs.

Lemma 2.7. *Let D be a domain in \mathbb{C}^n , and let φ_0 be a plurisubharmonic function on D . Let $\{f_j\}$ be a sequence in $A^2(D, e^{-\varphi_0})$, such that $\int_D |f_j|^2 e^{-\varphi_0}$ is uniformly bounded for any $j \in \mathbb{N}_+$. Assume that f_j compactly converges to $f_0 \in A^2(D, e^{-\varphi_0})$. Then for any $\xi \in A^2(D, e^{-\varphi_0})^*$,*

$$\lim_{j \rightarrow +\infty} \xi \cdot f_j = \xi \cdot f_0.$$

Proof. For any $f \in A^2(D, e^{-\varphi_0})$, denote that $\|f\|^2 := \int_D |f|^2 e^{-\varphi_0}$. Let $\{f_{k_j}\}$ be any subsequence of $\{f_j\}$. Since $A^2(D, e^{-\varphi_0})$ is a Hilbert space, and $\|f_{k_j}\|^2$ is uniformly bounded, there exists a subsequence of $\{f_{k_j}\}$ (denoted by $\{f_{k_{l_j}}\}$) weakly convergent to some $\tilde{f} \in A^2(D, e^{-\varphi_0})$. Note that for any $z \in D$, the functional $e_z \in A^2(D, e^{-\varphi_0})^*$, where

$$\begin{aligned} e_z : A^2(D, e^{-\varphi_0}) &\longrightarrow \mathbb{C} \\ f &\longmapsto f(z). \end{aligned}$$

Then we have

$$f_0(z) = \lim_{j \rightarrow +\infty} e_z \cdot f_j = \lim_{j \rightarrow +\infty} e_z \cdot f_{k_{l_j}} = e_z \cdot \tilde{f} = \tilde{f}(z), \quad \forall z \in D,$$

thus $f_0 = \tilde{f}$. It means that $\{f_{k_j}\}$ has a subsequence weakly convergent to f_0 . Since $\{f_{k_j}\}$ is an arbitrary subsequence of $\{f_j\}$, we get that $\{f_j\}$ weakly converges to f_0 . In other words, for any $\xi \in A^2(D, e^{-\varphi_0})^*$,

$$\lim_{j \rightarrow +\infty} \xi \cdot f_j = \xi \cdot f_0.$$

□

Let $\Omega := D \times \omega \subset \mathbb{C}^{n+1}$, where D is a domain in \mathbb{C}^n , ω is a domain in \mathbb{C} . Denote the coordinate on Ω by (z, τ) , where $z \in D$, $\tau \in \omega$. Let φ_0 be a plurisubharmonic function on D . Let f be a holomorphic function on Ω , such that

$$\int_{\Omega} |f(z, \tau)|^2 e^{-\varphi_0(z)} < +\infty.$$

Denote $f_{\tau} := f|_{D \times \{\tau\}}$.

Lemma 2.8. *For any $\xi \in A^2(D, e^{-\varphi_0})^*$, $\xi \cdot f_{\tau}$ is holomorphic with respect to $\tau \in \omega$.*

Proof. We only need to prove that $h(\tau) := \xi \cdot f_\tau$ is holomorphic near any $\tau_0 \in \omega$. Since $\tau_0 \in \omega$, there exists $r > 0$ such that $\Delta(\tau_0, 2r) \subset \subset \omega$. Then for any $\tau \in \Delta(\tau_0, r)$, according to sub-mean value inequality of subharmonic functions, we have

$$\int_D |f_\tau(z)|^2 e^{-\varphi_0(z)} \leq \frac{1}{\pi r^2} \int_{D \times \Delta(t, r)} |f(z, \tau)|^2 e^{-\varphi_0(z)} \leq \frac{1}{\pi r^2} \int_{\Omega} |f|^2 e^{-\varphi_0} < +\infty,$$

which implies that $f_\tau \in A^2(D, e^{-\varphi_0})$ and there exists $M > 0$ such that $\int_D |f_\tau|^2 e^{-\varphi_0} \leq M$ for any $\tau \in \Delta(\tau_0, r)$.

Fix $z_0 \in D$. According to Lemma 7.1 in Appendix, we can find a sequence $\{\xi_k\} \subset \ell_0 \subset A^2(D, e^{-\varphi_0})^*$, such that

$$\lim_{k \rightarrow +\infty} \|\xi_k - \xi\|_{A^2(D, e^{-\varphi_0})^*} = 0,$$

where

$$\ell_0 := \{\eta = (\eta_\alpha)_{\alpha \in \mathbb{N}^n} : \exists k \in \mathbb{N}, \text{ such that } \eta_\alpha = 0, \forall |\alpha| \geq k\}.$$

Here for any $\eta = (\eta_\alpha)_{\alpha \in \mathbb{N}^n} \in \ell_0$ and $f \in A^2(D, e^{-\varphi_0})$, define that

$$\eta \cdot f := \sum_{\alpha \in \mathbb{N}^n} \eta_\alpha \frac{f^{(\alpha)}(z_0)}{\alpha!}.$$

Note that for any $\xi_k \in \ell_0$, $\xi_k \cdot f_\tau$ can be written as

$$\xi_k \cdot f_\tau = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq l_k} c_{\alpha, k} \frac{\partial^\alpha f(z, \tau)}{\partial z^\alpha}(z_0, \tau),$$

where l_k is a finite integer, and $c_{\alpha, k} \in \mathbb{C}$ are constants. It is clear that $h_k(\tau) := \xi_k \cdot f_\tau$ is holomorphic with respect to $\tau \in \omega$ for any $k \in \mathbb{N}_+$, since any $\frac{\partial^\alpha f(z, \tau)}{\partial z^\alpha}(z_0, \tau)$ is holomorphic with respect to τ for any $\alpha \in \mathbb{N}_+$. Note that for any $\tau \in \Delta(\tau_0, r)$, we have

$$\begin{aligned} & |h_k(\tau) - h(\tau)|^2 \\ &= |(\xi_k - \xi) \cdot f_\tau|^2 \\ &\leq \|\xi_k - \xi\|_{A^2(D, e^{-\varphi_0})^*}^2 \int_D |f_\tau|^2 e^{-\varphi_0} \\ &\leq M \|\xi_k - \xi\|_{A^2(D, e^{-\varphi_0})^*}^2, \end{aligned}$$

which means that h_k uniformly converges to h on $\Delta(\tau_0, r)$. According to Weierstrass theorem, we know that h is holomorphic on $\Delta(\tau_0, r)$, i.e. near τ_0 . Then we get that $\xi \cdot f_\tau$ is holomorphic with respect to $\tau \in \omega$. \square

2.4. Some properties of $K_{\xi, \Psi, \lambda}^{\varphi_0}(t)$.

In this section, we prove some properties of the Bergman kernel $K_{\xi, \Psi, \lambda}^{\varphi_0}(t)$.

Let $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\}$.

Lemma 2.9. *For any $t \in [0, +\infty)$, if $K_{\xi, \Psi, \lambda}^{\varphi_0}(t) \in (0, +\infty)$, then there exists $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, such that*

$$K_{\xi, \Psi, \lambda}^{\varphi_0}(t) = \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{\lambda, t}^2}.$$

Proof. By the definition of $K_{\xi, \Psi, \lambda}^{\varphi_0}(t)$, there exists a sequence $\{f_j\}$ of holomorphic functions in $A^2(\{\Psi < 0\}, e^{-\varphi_0})$, such that $\|f_j\|_{\lambda, t} = 1$, and $\lim_{j \rightarrow +\infty} |\xi \cdot f_j|^2 = K_{\xi, \Psi, \lambda}^{\varphi_0}(t)$. Then $\int_{\{\Psi < 0\}} |f_j|^2 e^{-\varphi_0}$ is uniformly bounded. Following from Montel's theorem, we can get a subsequence of $\{f_j\}$ compactly convergent to a holomorphic function \tilde{f} on $\{\Psi < 0\}$. According to Fatou's lemma, we have $\|\tilde{f}\|_{\lambda, t} \leq 1$, and according to Lemma 2.7, we have $|\xi \cdot \tilde{f}|^2 = K_{\xi, \Psi, \lambda}^{\varphi_0}(t)$, thus $K_{\xi, \Psi, \lambda}^{\varphi_0}(t) \leq \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{\lambda, t}^2}$. Note that $\|\tilde{f}\|_{\lambda, t} \leq 1$ implies $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, which means $K_{\xi, \Psi, \lambda}^{\varphi_0}(t) \geq \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{\lambda, t}^2}$. We get that $K_{\xi, \Psi, \lambda}^{\varphi_0}(t) = \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{\lambda, t}^2}$. \square

Let J be an \mathcal{O}_o -submodule of $I(\varphi_0)_o$ such that $J \supset I(\Psi + \varphi_0)_o$, and let $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, such that $f_o \notin J$. Recall the minimal L^2 integral ([3, 31])

$$C(\Psi, \varphi_0, J, f) := \inf \left\{ \int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0} : (\tilde{f} - f)_o \in J \text{ & } \tilde{f} \in \mathcal{O}(\{\Psi < 0\}) \right\}.$$

Then the following lemma holds.

Lemma 2.10. *Assume that $C(\Psi, \varphi_0, J, f) \in (0, +\infty)$, then*

$$C(\Psi, \varphi_0, J, f) = \sup_{\substack{\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J} \equiv 0}} \frac{|\xi \cdot f|^2}{K_{\xi, \Psi, \lambda}^{\varphi_0}(0)}. \quad (2.1)$$

Proof. Note that $\xi \cdot \tilde{f} = \xi \cdot f$ for any $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ with $(\tilde{f} - f)_o \in J$ and $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ satisfying $\xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J} \equiv 0$. Then we have

$$\begin{aligned} K_{\xi, \Psi, \lambda}^{\varphi_0}(0) &= \sup_{h \in A^2(\{\Psi < 0\}, e^{-\varphi_0})} \frac{|\xi \cdot h|^2}{\int_{\{\Psi < 0\}} |h|^2 e^{-\varphi_0}} \\ &\geq \sup_{\substack{\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \\ (\tilde{f} - f)_o \in J}} \frac{|\xi \cdot \tilde{f}|^2}{\int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0}} \\ &= \sup_{\substack{\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \\ (\tilde{f} - f)_o \in J}} \frac{|\xi \cdot f|^2}{\int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0}}. \end{aligned}$$

Thus we get that

$$\begin{aligned} &\sup_{\substack{\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J} \equiv 0}} \frac{|\xi \cdot f|^2}{K_{\xi, \Psi, \lambda}^{\varphi_0}(0)} \\ &\leq \inf_{\substack{\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \\ (\tilde{f} - f)_o \in J}} \int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0} \\ &= C(\Psi, \varphi_0, J, f). \end{aligned}$$

Since $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ is a Hilbert space, and $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is a closed proper subspace of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ (using Lemma 2.4), there exists a closed subspace H of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ such that $H = (A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J)^\perp \neq \{0\}$. Then for $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, we can make the decomposition $f = f_J + f_H$,

such that $f_J \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$, and $f_H \in H$. Note that the linear functional ξ_f defined as follows:

$$\xi_f \cdot g := \int_{\{\Psi < 0\}} g \overline{f_H} e^{-\varphi_0}, \quad \forall g \in A^2(\{\Psi < 0\}, e^{-\varphi_0}),$$

satisfies that $\xi_f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\}$ and $\xi_f|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J} \equiv 0$. Then we have

$$\sup_{\substack{\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J} \equiv 0}} \frac{|\xi \cdot f|^2}{K_{\xi, \Psi, \lambda}^{\varphi_0}(0)} \geq \frac{|\xi_f \cdot f|^2}{K_{\xi_f, \Psi, \lambda}^{\varphi_0}(0)}.$$

Besides, we can know that

$$K_{\xi_f, \Psi, \lambda}^{\varphi_0}(0) = \sup_{h \in A^2(\{\Psi < 0\}, e^{-\varphi_0})} \frac{|\int_{\{\Psi < 0\}} h \overline{f_H} e^{-\varphi_0}|^2}{\int_{\{\Psi < 0\}} |h|^2 e^{-\varphi_0}} \leq \int_{\{\Psi < 0\}} |f_H|^2 e^{-\varphi_0},$$

and

$$\xi_f \cdot f = \xi_f \cdot (f_J + f_H) = \xi_f \cdot f_H = \int_{\{\Psi < 0\}} |f_H|^2 e^{-\varphi_0}.$$

Then we have

$$\frac{|\xi_f \cdot f|^2}{K_{\xi_f, \Psi, \lambda}^{\varphi_0}(0)} \geq \int_{\{\Psi < 0\}} |f_H|^2 e^{-\varphi_0} \geq C(\Psi, \varphi_0, J, f),$$

which implies that

$$\sup_{\substack{\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J} \equiv 0}} \frac{|\xi \cdot f|^2}{K_{\xi, \Psi, \lambda}^{\varphi_0}(0)} \geq C(\Psi, \varphi_0, J, f).$$

Lemma 2.10 is proved. \square

3. PROOF OF THEOREM 1.1

We prove Theorem 1.1 by using Lemma 2.1 (optimal L^2 extension theorem).

Proof of Theorem 1.1. Denote that $\Omega := \{\Psi < 0\} \times E = \{\Psi < 0\} \times \{w \in \mathbb{C} : \operatorname{Re} w \geq 0\}$, and the coordinate on Ω is (z, w) , where $z \in \{\Psi < 0\} \subset \mathbb{C}^n$, $w \in E = \{w \in \mathbb{C} : \operatorname{Re} w \geq 0\}$. Note that $D \setminus \{F = 0\}$ is a pseudoconvex domain in \mathbb{C}^n , and $\{\Psi < 0\} = \{\psi + 2 \log |1/F| < 0\}$ on $D \setminus \{F = 0\}$. Then $\{\Psi < 0\}$ is a pseudoconvex domain in \mathbb{C}^n , and Ψ is a plurisubharmonic function on $\{\Psi < 0\}$. We get that Ω is a pseudoconvex domain in \mathbb{C}^{n+1} . For any $(z, w) \in \Omega$, let

$$\tilde{\Psi}(z, w) := \varphi_0(z) + \Psi_{\lambda, \operatorname{Re} w} = \varphi_0(z) + \lambda \max\{\Psi(z) + \operatorname{Re} w, 0\}.$$

Then $\tilde{\Psi}$ is a plurisubharmonic function on Ω .

Denote that

$$K(w) := K_{\xi, \Psi, \lambda}^{\varphi_0}(\operatorname{Re} w)$$

for any $w \in E$. We prove that $\log K(w)$ is a subharmonic function with respect to $w \in E$.

Firstly we prove that $\log K(w)$ is upper semicontinuous. Let $w_j \in E$ such that $\lim_{j \rightarrow +\infty} w_j = w_0 \in E$. We assume that $\{w_{k_j}\}$ is the subsequence of $\{w_j\}$ such that

$$\lim_{j \rightarrow +\infty} K(w_{k_j}) = \limsup_{j \rightarrow +\infty} K(w_j).$$

By Lemma 2.9, there exists a sequence of holomorphic functions $\{f_j\}$ on $\{\Psi < 0\}$ such that $f_j \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, $\|f_j\|_{\lambda, \operatorname{Re} w_j} = 1$, and $|\xi \cdot f_j|^2 = K(w_j)$, for any $j \in \mathbb{N}_+$. Since $\{w_j\}$ is bounded in \mathbb{C} , there exists some $s_0 > 0$, such that $\operatorname{Re} w_j < s_0$ for any j , which implies that

$$\int_{\{\Psi < 0\}} |f_j|^2 e^{-\varphi_0} \leq e^{\lambda s_0} \|f_j\|_{\lambda, \operatorname{Re} w_j}^2 = e^{\lambda s_0}, \quad \forall j \in \mathbb{N}_+.$$

Then following from Montel's theorem, we can get a subsequence of $\{f_{k_j}\}$ (denoted by $\{f_{k_j}\}$ itself) compactly convergent to a holomorphic function f_0 on $\{\Psi < 0\}$. According to Fatou's lemma, we have

$$\begin{aligned} \|f_0\|_{\lambda, \operatorname{Re} w_0} &= \int_{\{\Psi < 0\}} |f_0(z)|^2 e^{-\varphi_0(z) - \lambda \max\{\Psi(z) + \operatorname{Re} w_0, 0\}} \\ &= \int_{\{\Psi < 0\}} \lim_{j \rightarrow +\infty} |f_{k_j}(z)|^2 e^{-\varphi_0(z) - \lambda \max\{\Psi(z) + \operatorname{Re} w_{k_j}, 0\}} \\ &\leq \liminf_{j \rightarrow +\infty} \int_{\{\Psi < 0\}} |f_{k_j}(z)|^2 e^{-\varphi_0(z) - \lambda \max\{\Psi(z) + \operatorname{Re} w_{k_j}, 0\}} \\ &= \liminf_{j \rightarrow +\infty} \|f_{k_j}\|_{\lambda, \operatorname{Re} w_j} = 1. \end{aligned}$$

Then $\int_{\{\Psi < 0\}} |f_0|^2 e^{-\varphi_0} \leq e^{\lambda \operatorname{Re} w_0} \|f_0\|_{\lambda, \operatorname{Re} w_0}^2 \leq e^{\lambda s_0} < +\infty$, which implies that $f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$. Lemma 2.7 shows that $|\xi \cdot f_0|^2 = \lim_{j \rightarrow +\infty} |\xi \cdot f_{k_j}|^2 = \limsup_{j \rightarrow +\infty} K(w_j)$. Thus

$$K(w_0) \geq \frac{|\xi \cdot f_0|^2}{\|f_0\|_{\lambda, \operatorname{Re} w_0}^2} \geq \limsup_{j \rightarrow +\infty} K(w_j),$$

which means

$$\log K(w_0) \geq \limsup_{j \rightarrow +\infty} \log K(w_j).$$

Then we get that $\log K(w)$ is upper semicontinuous with respect to $w \in E$.

Secondly we prove that $\log K(w)$ satisfies the sub-mean value inequality.

Let $\Delta(w_0, r) \subset E$ be the disc centered at w_0 with radius r , and let $\Omega' := \{\Psi < 0\} \times \Delta(w_0, r) \subset \mathbb{C}^{n+1}$. Let $f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ such that

$$K(w_0) = \frac{|\xi \cdot f_0|^2}{\|f_0\|_{\lambda, \operatorname{Re} w_0}^2}$$

by Lemma 2.9.

Note that Ω' is a pseudoconvex domain in \mathbb{C}^{n+1} , and $\tilde{\Psi}(z, w) = \varphi_0(z) + \Psi_{\lambda, \operatorname{Re} w} = \varphi_0(z) + \lambda \max\{\Psi(z) + \operatorname{Re} w, 0\}$ is a plurisubharmonic function on Ω' . Using Lemma 2.1 (optimal L^2 extension theorem), we can get a holomorphic function \tilde{f} on Ω' such that $\tilde{f}(z, w_0) = f_0(z)$ for any $z \in \{\Psi < 0\}$, and

$$\frac{1}{\pi r^2} \int_{\Omega'} |\tilde{f}(z, w)|^2 e^{-\tilde{\Psi}(z, w)} \leq \int_{\{\Psi < 0\}} |f_0(z)|^2 e^{-\Psi(z, w_0)}. \quad (3.1)$$

Denote that $\tilde{f}_w(z) = \tilde{f}(z, w) = \tilde{f}|_{\{\Psi < 0\} \times \{w\}}$. Since the function $y = \log x$ is concave, according to Jensen's inequality and inequality (3.1), we have

$$\begin{aligned} \log \|f_0\|_{\lambda, \operatorname{Re} w_0}^2 &= \log \left(\int_{\{\Psi < 0\}} |f_0(z)|^2 e^{-\Psi(z, w_0)} \right) \\ &\geq \log \left(\frac{1}{\pi r^2} \int_{\Omega'} |\tilde{f}(z, w)|^2 e^{-\tilde{\Psi}(z, w)} \right) \\ &= \log \left(\frac{1}{\pi r^2} \int_{\Delta(w_0, r)} \int_{\{\Psi < 0\} \times \{w\}} |\tilde{f}_w(z)|^2 e^{-\tilde{\Psi}(z, w)} \right) \\ &\geq \frac{1}{\pi r^2} \int_{\Delta(w_0, r)} \log \left(\|\tilde{f}_w\|_{\lambda, \operatorname{Re} w}^2 \right) \\ &\geq \frac{1}{\pi r^2} \int_{\Delta(w_0, r)} \left(\log |\xi \cdot \tilde{f}_w|^2 - \log K(w) \right). \end{aligned} \quad (3.2)$$

It follows from Lemma 2.8 that $\xi \cdot \tilde{f}_w$ is holomorphic with respect to w , which implies that $\log |\xi \cdot \tilde{f}_w|^2$ is subharmonic with respect to w . Combining with $\tilde{f}_{w_0} = f_0$, we have

$$\log |\xi \cdot f_0|^2 \leq \frac{1}{\pi r^2} \int_{\Delta(w_0, r)} \log |\xi \cdot \tilde{f}_w|^2.$$

Combining with inequality (3.2), we get

$$\log \|f_0\|_{\lambda, \operatorname{Re} w_0}^2 \geq \log |\xi \cdot f_0|^2 - \frac{1}{\pi r^2} \int_{\Delta(w_0, r)} \log K(w),$$

which means

$$\log K(w_0) \leq \frac{1}{\pi r^2} \int_{\Delta(w_0, r)} \log K(w).$$

Since $\log K(w)$ is upper semicontinuous and satisfies the sub-mean value inequality, we know that $\log K(w)$ is a subharmonic function on the interior of E . In addition, since $\log K(w)$ is upper semicontinuous near $\{0\} + \sqrt{-1}\mathbb{R}$, and $\log K(w)$ is only dependent on the real part of w , we know that $\log K(w)$ is a subharmonic function on E . \square

4. PROOFS OF THEOREM 1.2 AND REMARK 1.3

In this section, we give the proofs of Theorem 1.2 and Remark 1.3. We need the following lemma.

Lemma 4.1 (see [13]). *Let $D = I + \sqrt{-1}\mathbb{R} := \{z = x + \sqrt{-1}y \in \mathbb{C} : x \in I, y \in \mathbb{R}\}$ be a subset of \mathbb{C} , where I is an interval in \mathbb{R} . Let $\phi(z)$ be a subharmonic function on D which is only dependent on $x = \operatorname{Re} z$. Then $\phi(x) := \phi(x + \sqrt{-1}\mathbb{R})$ is a convex function with respect to $x \in I$.*

Proof of Theorem 1.2. It follows from Theorem 1.1 that $\log K_{\xi, \Psi, \lambda}^{\varphi_0}(\operatorname{Re} w)$ is subharmonic with respect to $w \in [0, +\infty) + \sqrt{-1}\mathbb{R}$. Note that $\log K_{\xi, \Psi, \lambda}^{\varphi_0}(\operatorname{Re} w)$ is only dependent on $\operatorname{Re} w$, then following from Lemma 4.1, we get that $\log K_{\xi, \Psi, \lambda}^{\varphi_0}(t) = \log K_{\xi, \Psi, \lambda}^{\varphi_0}(t + \sqrt{-1}\mathbb{R})$ is convex with respect to $t \in [0, +\infty)$, which implies that $-\log K_{\xi, \Psi, \lambda}^{\varphi_0}(t) + t$ is concave with respect to $t \in [0, +\infty)$. Then for any $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ with $\xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J} \equiv 0$, to prove that $\log -K_{\xi, \Psi, \lambda}^{\varphi_0}(t) + t$

is increasing, we only need to prove that $\log -K_{\xi, \Psi, \lambda}^{\varphi_0}(t) + t$ has a lower bound on $[0, +\infty)$.

Using Lemma 2.9, we obtain that there exists $f_t \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ for any $t \in [0, +\infty)$, such that $\xi \cdot f_t = 1$ and

$$K_{\xi, \Psi, \lambda}^{\varphi_0}(t) = \frac{1}{\|f_t\|_{\lambda, t}^2}. \quad (4.1)$$

In addition, according to Lemma 2.2, there exists a holomorphic function \tilde{F} on D such that

$$\int_D |\tilde{F} - (1 - b_t(\Psi))f_t F^2|^2 e^{-\varphi + v_t(\Psi) - \Psi} \leq C \int_D \mathbb{I}_{\{-t-1 < \Psi < -t\}} |f_t|^2 e^{-\varphi_0 - \Psi}, \quad (4.2)$$

where

$$\varphi = \varphi_0 + 2 \max\{\psi, 2 \log |F|\},$$

and C is a positive constant. Then it follows from inequality (4.2) that

$$\begin{aligned} & \int_{\{\Psi < 0\}} |\tilde{F} - (1 - b_t(\Psi))f_t F^2|^2 e^{-\varphi + v_t(\Psi) - \Psi} \\ & \leq \int_D |\tilde{F} - (1 - b_t(\Psi))f_t F^2|^2 e^{-\varphi + v_t(\Psi) - \Psi} \\ & \leq C \int_D \mathbb{I}_{\{-t-1 < \Psi < -t\}} |f_t|^2 e^{-\varphi_0 - \Psi} \\ & \leq C e^{t+1} \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0}. \end{aligned} \quad (4.3)$$

Denote that $\tilde{F}_t := \tilde{F}/F^2$ on $\{\Psi < 0\}$, then \tilde{F}_t is a holomorphic function on $\{\Psi < 0\}$. Note that $|F|^4 e^{-\varphi} = e^{-\varphi_0}$ on $\{\Psi < 0\}$. Then inequality (4.3) implies that

$$\int_{\{\Psi < 0\}} |\tilde{F}_t - (1 - b_t(\Psi))f_t|^2 e^{-\varphi_0 + v_t(\Psi) - \Psi} \leq C e^{t+1} \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0} < +\infty. \quad (4.4)$$

According to inequality (4.4), we can get that $(\tilde{F}_t - f_t)_o \in I(\Psi + \varphi_0) \subset J$, which means that $\xi \cdot \tilde{F}_t = \xi \cdot f_t = 1$. Besides, since $v_t(\Psi) \geq \Psi$, we have

$$\begin{aligned} & \left(\int_{\{\Psi < 0\}} |\tilde{F}_t - (1 - b_t(\Psi))f_t|^2 e^{-\varphi_0 + v_t(\Psi) - \Psi} \right)^{1/2} \\ & \geq \left(\int_{\{\Psi < 0\}} |\tilde{F}_t - (1 - b_t(\Psi))f_t|^2 e^{-\varphi_0} \right)^{1/2} \\ & \geq \left(\int_{\{\Psi < 0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \right)^{1/2} - \left(\int_{\{\Psi < 0\}} |(1 - b_t(\Psi))f_t|^2 e^{-\varphi_0} \right)^{1/2} \\ & \geq \left(\int_{\{\Psi < 0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \right)^{1/2} - \left(\int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0} \right)^{1/2}. \end{aligned}$$

Combining with inequality (4.4), we have

$$\begin{aligned} & \int_{\{\Psi < 0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \\ & \leq 2 \int_{\{\Psi < 0\}} |\tilde{F}_t - (1 - b_t(\Psi)) f_t|^2 e^{-\varphi_0 + v_t(\Psi) - \Psi} + 2 \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0} \\ & \leq 2(Ce^{t+1} + 1) \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0}. \end{aligned}$$

Note that

$$\begin{aligned} \|f_t\|_{\lambda,t}^2 &= \int_{\{\Psi < 0\}} |f_t|^2 e^{-\varphi_0 - \Psi_{\lambda,t}} \\ &= \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0} + \int_{\{0 > \Psi \geq -t\}} |f_t|^2 e^{-\varphi_0 - \lambda(\Psi+t)} \\ &\geq \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0}. \end{aligned}$$

Then we have

$$\int_{\{\Psi < 0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \leq 2(Ce^{t+1} + 1) \|f_t\|_{\lambda,t}^2 = C_1 \frac{e^t}{K_{\xi,\Psi,\lambda}^{\varphi_0}(t)},$$

where $C_1 := 2(eC + 1)$ is a positive constant. In addition, $\xi \cdot \tilde{F}_t = 1$ implies that

$$\int_{\{\Psi < 0\}} |\tilde{F}_t|^2 e^{-\varphi_0} = \|\tilde{F}_t\|_{\lambda,0}^2 \geq (K_{\xi,\Psi,\lambda}^{\varphi_0}(0))^{-1}.$$

Then we get that

$$-\log K_{\xi,\Psi,\lambda}^{\varphi_0}(t) + t \geq C_2, \quad \forall t \in [0, +\infty),$$

where $C_2 := \log(C_1^{-1} K_{\xi,\Psi,\lambda}^{\varphi_0}(0))$ is a finite constant. Since $-\log K_{\xi,\Psi,\lambda}^{\varphi_0}(t) + t$ is concave, we get that $-\log K_{\xi,\Psi,\lambda}^{\varphi_0}(t) + t$ is increasing with respect to $t \in [0, +\infty)$. \square

In the following we give the proof of Remark 1.3.

Proof of Remark 1.3. Denote that $K(t) := K_{\xi,\Psi,\lambda}^{\varphi_0}(t)$ for any $t \in [0, +\infty)$. According to Theorem 1.1 and Lemma 4.1, we can know that $\log K(t) - kt$ is convex on $[0, +\infty)$. Combining with that $e^{-kt} K(t)$ is increasing and not a constant function on $[0, T]$, which implies that $\log K(t) - kt = \log(e^{-kt} K(t))$ is increasing and not a constant function on $[0, T]$, we have that $\log K(t) - kt$ is strictly increasing on $[T, +\infty)$. Then $e^{-kt} K_{\xi,\Psi,\lambda}^{\varphi_0}(t) = \exp(\log K(t) - kt)$ is strictly increasing on $[T, +\infty)$. \square

5. PROOF OF COROLLARY 1.4

In this section, we give the proof of Corollary 1.4.

Proof of Corollary 1.4. For any $p \in (1, 2)$, $\lambda > 0$, let $\xi \in A^2(\{\Psi_1 < 0\})^* \setminus \{0\}$, such that $\xi|_{A^2(\{\Psi_1 < 0\}) \cap J_p} \equiv 0$, where $J_p := I(p\Psi_1)_o$. Denote that

$$K_{\xi,p,\lambda}(t) := \sup_{\tilde{f} \in A^2(\{\Psi_1 < 0\})} \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{p,\lambda,t}^2},$$

where

$$\|\tilde{f}\|_{p,\lambda,t} := \left(\int_{\{\Psi_1 < 0\}} |\tilde{f}|^2 e^{-\lambda \max\{p\Psi_1 + t, 0\}} \right)^{1/2},$$

and $t \in [0, +\infty)$. Note that

$$p\Psi_1 = \min\{(2pc_o^{fF}(\psi)\psi + (4 - 2p)\log|F|) - 2\log|F^2|, 0\}$$

and Lemma 2.6 shows $f_o \notin J_p$, which implies that $A^2(\{\Psi_1 < 0\}) \cap J_p$ is a proper subspace of $A^2(\{\Psi_1 < 0\})$, and $K_{\xi,p,\lambda}(0) \in (0, +\infty)$. Theorem 1.2 tells us that $-\log K_{\xi,p,\lambda}(t) + t$ is increasing with respect to $t \in [0, +\infty)$, which implies that

$$-\log K_{\xi,p,\lambda}(t) + t \geq -\log K_{\xi,p,\lambda}(0), \quad \forall t \in [0, +\infty). \quad (5.1)$$

Since $f \in A^2(\{\Psi_1 < 0\})$, following from inequality (5.1), we get that

$$\|f\|_{p,\lambda,t}^2 \geq \frac{|\xi \cdot f|^2}{K_{\xi,p,\lambda}(t)} \geq e^{-t} \frac{|\xi \cdot f|^2}{K_{\xi,p,\lambda}(0)}, \quad \forall t \in [0, +\infty).$$

In addition, since $f_o \notin J_p$, according to Lemma 2.10, we have

$$\begin{aligned} \|f\|_{p,\lambda,t}^2 &\geq \sup_{\substack{\xi \in A^2(\{\Psi_1 < 0\})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi_1 < 0\}) \cap J_p} \equiv 0}} e^{-t} \frac{|\xi \cdot f|^2}{K_{\xi,p,\lambda}(0)} \\ &= e^{-t} C(p\Psi_1, 0, J_p, f), \quad \forall t \in [0, +\infty). \end{aligned} \quad (5.2)$$

Note that for any $t \in [0, +\infty)$,

$$\|f\|_{p,\lambda,t}^2 = \int_{\{p\Psi_1 < -t\}} |f|^2 + \int_{\{0 > p\Psi_1 \geq -t\}} |f|^2 e^{-\lambda(p\Psi_1 + t)}. \quad (5.3)$$

Since for any $\lambda > 0$,

$$\int_{\{0 > p\Psi_1 \geq -t\}} |f|^2 e^{-\lambda(p\Psi_1 + t)} \leq \int_{\{0 > p\Psi_1 \geq -t\}} |f|^2 < +\infty,$$

and $\lim_{\lambda \rightarrow +\infty} e^{-\lambda(p\Psi_1 + t)} = 0$ on $\{0 > p\Psi_1 \geq -t\}$, according to Lebesgue's dominated convergence theorem, we have

$$\lim_{\lambda \rightarrow +\infty} \int_{\{0 > p\Psi_1 \geq -t\}} |f|^2 e^{-\lambda(p\Psi_1 + t)} = 0.$$

Then equality (5.3) implies

$$\lim_{\lambda \rightarrow +\infty} \|f\|_{p,\lambda,t}^2 = \int_{\{p\Psi_1 < -t\}} |f|^2, \quad \forall t \in [0, +\infty). \quad (5.4)$$

Letting $\lambda \rightarrow +\infty$ in inequality (5.2), we get that for any $t \in [0, +\infty)$,

$$\int_{\{p\Psi_1 < -t\}} |f|^2 \geq e^{-t} C(p\Psi_1, 0, J_p, f). \quad (5.5)$$

Note that $\{p\Psi_1 < 0\} = \{\Psi_1 < 0\}$ and $J_p \subset I_+(\Psi_1)_o$ for any $p \in (1, 2)$. Then we have

$$C(p\Psi_1, 0, J_p, f) \geq C(\Psi_1, 0, I_+(\Psi_1)_o, f), \quad \forall p \in (1, 2).$$

Since $\int_{\{\Psi_1 < 0\}} |f|^2 < +\infty$, it follows from Lebesgue's dominated convergence theorem and inequality (5.5) that

$$\begin{aligned} & \int_{\{\Psi_1 < -t\}} |f|^2 \\ &= \lim_{p \rightarrow 1+0} \int_{\{p\Psi_1 < -t\}} |f|^2 \\ &\geq \limsup_{p \rightarrow 1+0} e^{-t} C(p\Psi_1, 0, J_p, f) \\ &\geq e^{-t} C(\Psi_1, 0, I_+(\Psi_1)_o, f), \quad \forall t \in [0, +\infty). \end{aligned} \tag{5.6}$$

Let $r = e^{-t/2}$, and we get that

$$\frac{1}{r^2} \int_{\{\Psi_1 < 2\log r\}} |f|^2 \geq C(\Psi_1, 0, I_+(\Psi_1)_o, f), \quad \forall r \in (0, 1]. \tag{5.7}$$

Note that $C(\Psi_1, 0, I_+(\Psi_1)_o, f) = G(0; \Psi_1, I_+(\Psi_1)_o, f) > 0$, thus Corollary 1.4 holds. \square

6. PROOF OF COROLLORY 1.7

In this section, we give the proof of Corollary 1.7.

Proof of Corollary 1.7. Let $\Psi_q := q\Psi$ for any $q > 2a_o^f(\Psi; \varphi_0) \geq 1$. Note that

$$q\Psi = \min\{2q\psi + (2\lceil q \rceil - 2q) \log |F| - 2\log |F^{\lceil q \rceil}|, 0\},$$

where $\lceil q \rceil = \min\{m \in \mathbb{Z} : m \geq q\}$. By the definition of $a_o^f(\Psi; \varphi_0)$, we have $f_o \notin I(2q\Psi + \varphi_0)_o$ for any $q > 2a_o^f(\Psi; \varphi_0)$. For any fixed $q > 2a_o^f(\Psi; \varphi_0)$, $\lambda > 0$, let $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\}$, such that $\xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J_q} \equiv 0$, where $J_q := I(q\Psi + \varphi_0)_o$. Denote that

$$K_{\xi, q, \lambda}(t) := \sup_{\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})} \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{q, \lambda, t}^2},$$

where

$$\|\tilde{f}\|_{q, \lambda, t} := \left(\int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0 - \lambda \max\{q\Psi + t, 0\}} \right)^{1/2},$$

and $t \in [0, +\infty)$. Theorem 1.2 tells us that $-\log K_{\xi, q, \lambda}(t) + t$ is increasing with respect to $t \in [0, +\infty)$, which implies that

$$-\log K_{\xi, q, \lambda}(t) + t \geq -\log K_{\xi, q, \lambda}(0), \quad \forall t \in [0, +\infty). \tag{6.1}$$

Since $\int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0} \leq \int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0 - \Psi} < +\infty$, following from inequality (6.1), we get that

$$\|f\|_{q, \lambda, t}^2 \geq \frac{|\xi \cdot f|^2}{K_{\xi, q, \lambda}(t)} \geq e^{-t} \frac{|\xi \cdot f|^2}{K_{\xi, q, \lambda}(0)}, \quad \forall t \in [0, +\infty).$$

According to Lemma 2.10, we have

$$\begin{aligned} \|f\|_{q, \lambda, t}^2 &\geq \sup_{\substack{\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J_q} \equiv 0}} e^{-t} \frac{|\xi \cdot f|^2}{K_{\xi, q, \lambda}(0)} \\ &= e^{-t} C(q\Psi, \varphi_0, J_q, f), \quad \forall t \in [0, +\infty). \end{aligned} \tag{6.2}$$

Note that for any $t \in [0, +\infty)$,

$$\|f\|_{q,\lambda,t}^2 = \int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0} + \int_{\{0 > q\Psi \geq -t\}} |f|^2 e^{-\varphi_0 - \lambda(q\Psi + t)}. \quad (6.3)$$

Since for any $\lambda > 0$,

$$\int_{\{0 > q\Psi \geq -t\}} |f|^2 e^{-\varphi_0 - \lambda(q\Psi + t)} \leq \int_{\{0 > q\Psi \geq -t\}} |f|^2 e^{-\varphi_0} < +\infty,$$

and $\lim_{\lambda \rightarrow +\infty} e^{-\lambda(q\Psi + t)} = 0$ on $\{0 > q\Psi \geq -t\}$, according to Lebesgue's dominated convergence theorem, we have

$$\lim_{\lambda \rightarrow +\infty} \int_{\{0 > q\Psi \geq -t\}} |f|^2 e^{-\varphi_0 - \lambda(q\Psi + t)} = 0.$$

Then equality (6.3) implies

$$\lim_{\lambda \rightarrow +\infty} \|f\|_{q,\lambda,t}^2 = \int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0}, \quad \forall t \in [0, +\infty). \quad (6.4)$$

Thus letting $\lambda \rightarrow +\infty$ in inequality (6.2), we get that for any $t \in [0, +\infty)$,

$$\int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0} \geq e^{-t} C(\Psi, \varphi_0, J_q, f) = e^{-t} C(\Psi, \varphi_0, J_q, f). \quad (6.5)$$

Note that $J_q \subset I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o$ for any $q > 2a_o^f(\Psi; \varphi_0)$. Then we have

$$C(\Psi, \varphi_0, J_q, f) \geq C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f), \quad \forall q > 2a_o^f(\Psi; \varphi_0).$$

Then it follows from inequality (6.5) that

$$\int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0} \geq e^{-t} C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f) \quad (6.6)$$

for any $q > 2a_o^f(\Psi; \varphi_0)$ and $t \in [0, +\infty)$.

According to Fubini's theorem, we have

$$\begin{aligned} & \int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0 - \Psi} \\ &= \int_{\{\Psi < 0\}} \left(|f|^2 e^{-\varphi_0} \int_0^{e^{-\Psi}} ds \right) \\ &= \int_0^{+\infty} \left(\int_{\{\Psi < 0\} \cap \{s < e^{-\Psi}\}} |f|^2 e^{-\varphi_0} \right) ds \\ &= \int_{-\infty}^{+\infty} \left(\int_{\{q\Psi < -qt\} \cap \{\Psi < 0\}} |f|^2 e^{-\varphi_0} \right) e^t dt. \end{aligned}$$

Inequality (6.6) implies that for any $q > 2a_o^f(\Psi; \varphi_0)$,

$$\begin{aligned} & \int_0^{+\infty} \left(\int_{\{q\Psi < -qt\} \cap \{\Psi < 0\}} |f|^2 e^{-\varphi_0} \right) e^t dt \\ &\geq \int_0^{+\infty} e^{-qt} C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f) \cdot e^t dt \\ &= \frac{1}{q-1} C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f), \end{aligned}$$

and

$$\begin{aligned} & \int_{-\infty}^0 \left(\int_{\{q\Psi < -qt\} \cap \{\Psi < 0\}} |f|^2 e^{-\varphi_0} \right) e^t dt \\ & \geq \int_{-\infty}^0 C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f) \cdot e^t dt \\ & = C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f). \end{aligned}$$

Then we have

$$\int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0 - \Psi} \geq \frac{q}{q-1} C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f). \quad (6.7)$$

for any $q > 2a_o^f(\Psi; \varphi_0)$. Let $q \rightarrow 2a_o^f(\Psi; \varphi_0) + 0$, then inequality (6.7) also holds for $q \geq 2a_o^f(\Psi; \varphi_0)$. Thus if $q > 1$ satisfying

$$\int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0 - \Psi} < \frac{q}{q-1} C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f), \quad (6.8)$$

we have $q < 2a_o^f(\Psi; \varphi_0)$, which means that $f_o \in I(q\Psi + \varphi_0)_o$. Proof of Corollary 1.7 is done. \square

7. APPENDIX

Let D be a domain in \mathbb{C}^n , and φ be a plurisubharmonic function on D . Denote that

$$\ell_0 := \{\eta = (\eta_\alpha)_{\alpha \in \mathbb{N}^n} : \exists k \in \mathbb{N}, \text{ such that } \eta_\alpha = 0, \forall |\alpha| \geq k\},$$

where for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \dots + \alpha_n$. Let $z_0 \in D$, and $\eta = (\eta_\alpha) \in \ell_0$. For any $f \in \mathcal{O}(D)$, denote that

$$\eta \cdot f := \sum_{\alpha \in \mathbb{N}^n} \eta_\alpha \frac{f^{(\alpha)}(z_0)}{\alpha!}. \quad (7.1)$$

It can be shown that for any $\eta \in \ell_0$, there is a finite constant $C_\eta > 0$, such that

$$|\eta \cdot f|^2 \leq C_\eta \int_D |f|^2 e^{-\varphi},$$

for any $f \in A^2(D, e^{-\varphi})$ (see [1, 2]). Then any $\eta \in \ell_0$ can be seen as an element in $A^2(D, e^{-\varphi})^*$ by equality (7.1). Note that $A^2(D, e^{-\varphi})$ is a Hilbert space. By Riesz representation theorem, there exists $g_\eta \in A^2(D, e^{-\varphi})$, such that

$$\eta \cdot f = \int_D f \overline{g_\eta} e^{-\varphi}, \quad \forall f \in A^2(D, e^{-\varphi}),$$

which induces a map from ℓ_0 to $A^2(D, e^{-\varphi})$. We denote the map by T_{φ, z_0} :

$$\begin{aligned} T_{\varphi, z_0} : \ell_0 & \longrightarrow A^2(D, e^{-\varphi}) \\ \eta & \longmapsto g_\eta. \end{aligned}$$

We state the following lemma.

Lemma 7.1. *The image of T_{φ, z_0} is dense in $A^2(D, e^{-\varphi})$, i.e., $\overline{T_{\varphi, z_0}(\ell_0)} = A^2(D, e^{-\varphi})$, under the topology of $A^2(D, e^{-\varphi})$.*

Remark 7.2. *It follows from Lemma 7.1 that ℓ_0 is dense in $A^2(D, e^{-\varphi})^*$, under the strong topology of $A^2(D, e^{-\varphi})^*$.*

Proof of Lemma 7.1. We introduce some notations before the proof.

For $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, denote that $\alpha < \beta$, if $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ but there exists k with $1 \leq k \leq n$, such that $\alpha_1 = \beta_1, \dots, \alpha_{k-1} = \beta_{k-1}, \alpha_k < \beta_k$.

We may assume that $z_0 = o \in D$ is the origin in \mathbb{C}^n , and denote T_{φ, z_0} by T . We will choose a countable sequence $\{\eta[\alpha]\}$ of elements in ℓ_0 , such that

$$\overline{\text{span}\{T(\eta[\alpha])\}} = A^2(D, e^{-\varphi}),$$

which can imply Lemma 7.1. For any $\alpha \in \mathbb{N}^n$, we set $\eta[\alpha] \in \ell_0$, with $\eta[\alpha]_\gamma = \gamma! \cdot b_\gamma^\alpha \in \mathbb{C}$ (which will be determined in the following discussions) for any $\gamma < \alpha$, $\eta[\alpha]_\alpha = \alpha!$, and $\eta[\alpha]_\gamma = 0$ for any $\gamma > \alpha$. Denote that $g_\alpha := g_{\eta_\alpha} = T(\eta[\alpha]) \in A^2(D, e^{-\varphi})$. We will choose b_γ^α such that

$$\begin{aligned} \int_D g_\alpha \overline{g_\beta} e^{-\varphi} &= 0, \quad \forall \alpha \neq \beta; \\ g_\beta^{(\gamma)}(o) &= 0, \quad \forall \gamma < \beta; \\ g_\alpha^{(\alpha)}(o) &= \int_D |g_\alpha|^2 e^{-\varphi}, \quad \forall \alpha. \end{aligned} \tag{7.2}$$

And for any $\alpha \in \mathbb{N}^n$, denote that $\alpha \in S_1$ if $g_\alpha \equiv 0$. Otherwise we denote that $\alpha \in S_2$.

Firstly, for $\alpha = (0, \dots, 0)$, we set

$$\eta((0, \dots, 0)) = (1, 0, \dots, 0, \dots) \in \ell_0.$$

Denote that

$$T((1, 0, \dots, 0, \dots)) = g_{(0, \dots, 0)} \in A^2(D, e^{-\varphi}),$$

then for any $f \in A^2(D, e^{-\varphi_0})$,

$$f(o) = \int_D f \overline{g_{(0, \dots, 0)}} e^{-\varphi}. \tag{7.3}$$

Let $f = g_{(0, \dots, 0)}$ in equality (7.3), we get

$$g_{(0, \dots, 0)}(o) = \int_D |g_{(0, \dots, 0)}|^2 e^{-\varphi}.$$

For some $\alpha \in \mathbb{N}^n$, we assume that for any $\beta < \alpha$, $\eta(\beta) \in \ell_0$ (i.e. the complex number sequence $\{b_\gamma^\beta\}$) has been chosen to satisfy

$$\begin{aligned} \int_D g_\beta \overline{g_{\beta_2}} e^{-\varphi} &= 0, \quad \forall \beta_1 \neq \beta_2, \beta_1, \beta_2 < \alpha; \\ g_\beta^{(\gamma)}(o) &= 0, \quad \forall \gamma < \beta; \\ g_\beta^{(\beta)}(o) &= \int_D |g_\beta|^2 e^{-\varphi}, \quad \forall \beta < \alpha. \end{aligned} \tag{7.4}$$

By the choice of $\eta[\alpha]$, for any $f \in A^2(D, e^{-\varphi})$,

$$\sum_{\gamma < \alpha} b_\gamma^\alpha f^{(\gamma)}(o) + f^{(\alpha)}(o) = \int_D f \overline{g_\alpha} e^{-\varphi}. \tag{7.5}$$

Since we want

$$\int_D g_\beta \overline{g_\alpha} e^{-\varphi} = 0, \quad \forall \beta < \alpha,$$

then there must be

$$\sum_{\gamma < \alpha} b_\gamma^\alpha g_\beta^{(\gamma)}(o) + g_\beta^{(\alpha)}(o) = 0, \quad \forall \beta < \alpha,$$

which is equivalent to

$$b_\beta^\alpha g_\beta^{(\beta)}(o) + \sum_{\beta < \gamma < \alpha} b_\gamma^\alpha g_\beta^{(\gamma)}(o) = -g_\beta^{(\alpha)}(o), \quad \forall \beta < \alpha, \quad (7.6)$$

by the choice of $\{g_\beta\}$. Equality (7.6) can be seen as a linear equations system for $(b_\gamma^\alpha)_{\gamma < \alpha}$. Note that for $\beta \in S_1$, $g_\beta^{(\beta)} = 0$. We set $b_\beta^\alpha = 0$ for any $\beta < \alpha$ with $\beta \in S_1$. And we also note that

$$\prod_{\beta < \alpha, \beta \in S_2} g_\beta^{(\beta)}(o) = \prod_{\beta < \alpha, \beta \in S_2} \int_D |g_\beta|^2 e^{-\varphi} > 0.$$

It means that there exists $(b_\gamma^\alpha)_{\gamma < \alpha}$ satisfying equality (7.6), where $b_\beta^\alpha = 0$ for any $\beta < \alpha$ with $\beta \in S_1$. Now suppose that $(b_\gamma^\alpha)_{\gamma < \alpha}$ is the solution as we described above. Then g_α satisfies

$$\int_D g_\beta \overline{g_\alpha} e^{-\varphi} = 0, \quad \forall \beta < \alpha.$$

Note that in the process of induction, for any $\beta < \alpha$, we have

$$\sum_{\gamma < \beta} b_\gamma^\beta f^{(\gamma)}(o) + f^{(\beta)}(o) = \int_D f \overline{g_\beta} e^{-\varphi}.$$

Let $f = g_\alpha$, then we get

$$\sum_{\gamma < \beta} b_\gamma^\beta g_\alpha^{(\gamma)}(o) + g_\alpha^{(\beta)}(o) = \int_D g_\alpha \overline{g_\beta} e^{-\varphi}, \quad \forall \beta < \alpha. \quad (7.7)$$

In equality (7.7), by induction, we can know that for any $\beta < \alpha$,

$$g_\alpha^{(\beta)}(o) = \int_D g_\alpha \overline{g_\beta} e^{-\varphi} = \overline{\int_D g_\beta \overline{g_\alpha} e^{-\varphi}} = 0. \quad (7.8)$$

In addition, in equality (7.5), Letting $f = g_\alpha$, we have

$$\sum_{\gamma < \alpha} b_\gamma^\alpha g_\alpha^{(\gamma)}(o) + g_\alpha^{(\alpha)}(o) = \int_D |g_\alpha|^2 e^{-\varphi}.$$

Then it follows from equality (7.8) that

$$g_\alpha^{(\alpha)}(o) = \int_D |g_\alpha|^2 e^{-\varphi}.$$

Now, by induction, we can choose out $\eta[\alpha] \in \ell_0$ for any $\alpha \in \mathbb{N}^n$ satisfying what we described before equality (7.2), and $\{g_\alpha\}_{\alpha \in \mathbb{N}^n} \subset A^2(D, e^{-\varphi})$ satisfies equality (7.2). In the following we prove that

$$\overline{\text{span}\{g_\alpha : \alpha \in S_2\}} = A^2(D, e^{-\varphi}). \quad (7.9)$$

For any $f \in A^2(D, e^{-\varphi})$, let $\{a_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a sequence of complex numbers (which will be determined in the following). Denote

$$f_\alpha(z) = \sum_{\beta \leq \alpha} a_\beta g_\beta(z). \quad (7.10)$$

We choose a_β for $\beta \leq \alpha$, such that

$$f_\alpha^{(\beta)}(o) = f^{(\beta)}(o). \quad (7.11)$$

Firstly, for $\beta = (0, \dots, 0)$, if $(0, \dots, 0) \in S_2$, we can see that

$$a_{(0, \dots, 0)} = \frac{f(o)}{g_{(0, \dots, 0)}(o)}$$

satisfy equality (7.11), and if $(0, \dots, 0) \in S_1$, we have $f(o) = 0$ according to equality (7.3), which implies $a_{(0, \dots, 0)} = 0$ satisfies inequality (7.11).

Secondly, assume that for some $\gamma \leq \alpha$, all $\beta < \gamma$ have been choosen to satisfy equality (7.11). According to equality (7.10) and equality (7.2), we have

$$f_\alpha^{(\gamma)}(o) = \sum_{\beta < \gamma} a_\beta g_\beta^{(\gamma)}(o) + a_\gamma g_\gamma^{(\gamma)}(o). \quad (7.12)$$

Then

$$f_\alpha^{(\gamma)}(o) = f^{(\gamma)}(o) \Leftrightarrow f^{(\gamma)}(o) = \sum_{\beta < \gamma} a_\beta g_\beta^{(\gamma)}(o) + a_\gamma g_\gamma^{(\gamma)}(o). \quad (7.13)$$

Note that for $\gamma \in S_2$, $g_\gamma^{(\gamma)}(o) = \int_D |g_\gamma| e^{-\varphi} > 0$, then we can choose

$$a_\gamma = (g_\gamma^{(\gamma)}(o))^{-1} \left(f^{(\gamma)}(o) - \sum_{\beta < \gamma} a_\beta g_\beta^{(\gamma)}(o) \right)$$

to satisfy equality (7.11). If $\gamma \in S_1$, following from equality (7.5), we have

$$\begin{aligned} f^{(\gamma)}(o) &= - \sum_{\beta < \gamma} b_\beta^\gamma f^{(\beta)}(o) = - \sum_{\beta < \gamma} b_\beta^\gamma f_\alpha^{(\beta)}(o) \\ &= - \sum_{\beta < \gamma} b_\beta^\gamma \left(\sum_{\beta' \leq \beta} a_{\beta'} g_{\beta'}^{(\beta)}(o) \right) \\ &= - \sum_{\beta' < \gamma} a_{\beta'} \left(\sum_{\beta' \leq \beta < \gamma} b_\beta^\gamma g_{\beta'}^{(\beta)}(o) \right) \\ &= \sum_{\beta' < \gamma} a_{\beta'} g_{\beta'}^{(\gamma)}(o). \end{aligned}$$

Then we can choose $a_\gamma = 0$ according to equation (7.13).

Finally, by induction, we can know that the sequence $\{a_\beta\}$ can be choosen to satisfy equality (7.11). In addition, we have $a_\beta = 0$ for $\beta \in S_1$.

Now we have $(f - f_\alpha)^{(\beta)}(o) = 0$ for any $\beta \leq \alpha$. Then it follows from equality (7.5) that

$$\int_D (f - f_\alpha) \overline{g_\alpha} e^{-\varphi} = 0,$$

which means that

$$\int_D f \overline{g_\alpha} e^{-\varphi} = \int_D f_\alpha \overline{g_\alpha} e^{-\varphi} = a_\alpha.$$

Then combining with equality (7.2), we get that

$$\sum_\alpha |a_\alpha|^2 \int_D |g_\alpha|^2 e^{-\varphi} \leq \int_D |f|^2 e^{-\varphi}.$$

Denote

$$h(z) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha g_\alpha(z),$$

then $h \in A^2(\Omega, e^{-\varphi})$. In addition, we have

$$h(z) = \lim_{|\alpha| \rightarrow +\infty} f_\alpha \Rightarrow h^{(\beta)}(o) = \lim_{|\alpha| \rightarrow +\infty} f_\alpha^{(\beta)}(o) = f^{(\beta)}(o)$$

for any $\beta \in \mathbb{N}^n$, inducing that $f \equiv h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g_\alpha(z)$. Note that $a_\alpha = 0$ for $\alpha \in S_1$.

By the arbitrariness of $f \in A^2(D, e^{-\varphi})$, we get

$$\overline{\text{span}\{g_\alpha : \alpha \in S_2\}} = A^2(D, e^{-\varphi}), \quad (7.14)$$

which implies

$$\overline{\text{span}\{T(\eta[\alpha])\}} = A^2(D, e^{-\varphi}).$$

Then we know that Lemma 7.1 holds. \square

Note that for any $z \in D$, the functional $e_z \in A^2(D, e^{-\varphi})^*$, where

$$\begin{aligned} e_z : A^2(D, e^{-\varphi}) &\longrightarrow \mathbb{C} \\ f &\longmapsto f(z). \end{aligned}$$

Then it follows from Riesz representation theorem that there exists $\phi_z \in A^2(D, e^{-\varphi})$ such that

$$e_z \cdot f = \int_D f \overline{\phi_z} e^{-\varphi}, \quad \forall f \in A^2(D, e^{-\varphi}).$$

According to the following Lemma, we can also prove Lemma 2.8.

Lemma 7.3. *Under the topology of $A^2(D, e^{-\varphi})$, we have*

$$\overline{\text{span}\{\phi_z : z \in D\}} = A^2(D, e^{-\varphi}).$$

Proof. Denote that

$$H := \overline{\text{span}\{\phi_z : z \in D\}},$$

then H is a closed subspace of $A^2(D, e^{-\varphi})$. If $H \subsetneq A^2(D, e^{-\varphi})$, there exists $h \in A^2(D, e^{-\varphi})$ such that $h \neq 0$, and

$$\int_D h \overline{\phi_z} e^{-\varphi} = 0, \quad \forall z \in D.$$

However, we have $h(z) = e_z \cdot h = \int_D h \overline{\phi_z} e^{-\varphi} = 0$ for any $z \in D$, inducing that $h \equiv 0$, which is a contradiction. It means that

$$\overline{\text{span}\{\phi_z : z \in D\}} = A^2(D, e^{-\varphi}).$$

\square

Remark 7.4. *It follows from Lemma 7.3 that $\text{span}\{e_z : z \in D\}$ is dense in $A^2(D, e^{-\varphi})^*$, under the strong topology. And in Lemma 2.8, for any $\eta \in \text{span}\{e_z : z \in D\}$ such that*

$$\eta = \sum_{k=1}^N c_k e_{z_k},$$

where N is a finite positive integer, $c_k \in \mathbb{C}$, and $z_k \in D$ for any k , we have that

$$\eta \cdot f_\tau = \sum_{k=1}^N c_k e_{z_k} \cdot f_\tau = \sum_{k=1}^N c_k f(\tau, z_k)$$

is holomorphic with respect to τ . Then with a similar discussion in the proof of Lemma 2.8, we can know that Lemma 2.8 can also be induced by Lemma 7.3.

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