Peking Mathematical Journal (2024) 7:441–470
https://doi.org/10.1007/s42543-023-00070-8

ORIGINAL ARTICLE

Modules at Boundary Points, Fiberwise Bergman Kernels, and Log-Subharmonicity

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Received: 10 October 2022 / Revised: 2 March 2023 / Accepted: 18 March 2023 /
Published online: 3 May 2023

Abstract
In this article, we consider Bergman kernels with respect to modules at boundary points, and obtain a log-subharmonicity property of the Bergman kernels, which implies a concavity property related to the Bergman kernels. As applications, we reprove the sharp effectiveness result related to a conjecture posed by Jonsson–Mustaţă and the effectiveness result of strong openness property of the modules at boundary points.

Keywords Bergman kernels · L^2 extension · Strong openness property

Mathematics Subject Classification 32D15 · 14F18 · 32A36 · 32U05 · 32W05

1 Introduction

The strong openness property of multiplier ideal sheaves (i.e., $\mathcal{I}(\psi) = \mathcal{I}_+(\psi) := \bigcup_{\epsilon>0} \mathcal{I}((1+\epsilon)\psi)$) is an important feature and has been widely used in the study of several complex variables, complex algebraic geometry, and complex differential geometry (see e.g. [5, 7–10, 17, 18, 24, 25, 29, 30, 39–41]), where ψ is a plurisubharmonic function on a complex manifold M (see [11]) and multiplier ideal sheaf $\mathcal{I}(\psi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2e^{-\psi}$ is locally integrable (see e.g. [12–16, 26, 31–33, 35–38]).

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The strong openness property was conjectured by Demailly [13] and proved by Guan–Zhou [24] (the 2-dimensional case was proved by Jonsson–Mustaţă [27]). Recall that in order to prove the strong openness property, Jonsson and Mustaţă (see [28], see also [27]) posed the following conjecture, and proved the 2-dimensional case [27]:

Conjecture J-M: If $c_o^F(\psi) < +\infty$, $\frac{1}{r^2}\mu(\{c_o^F(\psi)\psi - \log|F| < \log r\})$ has a uniform positive lower bound independent of $r \in (0, 1)$, where μ is the Lebesgue measure on $\mathbb{C}^n$, and $c_o^F(\psi) := \sup\{c \geq 0 : |F|^2 e^{-2c\psi}$ is locally L^1 near $o\}$.

Using the strong openness property, Guan–Zhou [23] proved Conjecture J-M.

Independent of the strong openness property, Bao–Guan–Yuan [3] considered minimal L^2 integrals with respect to a module at a boundary point of the sublevel sets, and established a concavity property of the minimal L^2 integrals, which implies a sharp effectiveness result related to Conjecture J-M, and completed the approach from Conjecture J-M to the strong openness property.

As a generalization of Berndtsson's log-plurisubharmonicity result of fiberwise Bergman kernels (see [4]), in [1] (see also [2]), we obtained the log-plurisubharmonicity of fiberwise Bergman kernels with respect to functionals over the space of holomorphic germs by using the optimal L^2 extension theorem (see [22]) and Guan–Zhou method (see [34]). As applications, we gave new approaches to the effectiveness results of strong openness property [1] and L^p strong openness property [2].

As a continuation work of [1, 2], in this article, we consider Bergman kernels with respect to the modules at boundary points and obtain a log-subharmonicity property of the Bergman kernels (as a generalization of Berndtsson's log-subharmonicity result of fiberwise Bergman kernels in [4]), which implies a new approach from Conjecture J-M to the strong openness property. We also give a reproof for the effectiveness result of the strong openness property related to the modules at boundary points.

1.1 Main Result

Let D be a pseudoconvex domain in $\mathbb{C}^n$, and the origin $o \in D$. Let $F \not\equiv 0$ be a holomorphic function on D, and ψ be a negative plurisubharmonic function on D. Let φ_0 be a plurisubharmonic function on D. Denote by

$$\Psi := \min\{\psi - 2\log|F|, 0\}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$.

We recall some notations from [3]. Denote

$$\tilde{J}(\Psi)_o := \{f \in \mathcal{O}(\{\Psi < -t\} \cap V) : \exists t \in \mathbb{R}, \exists V \text{ is a neighborhood of } o\},$$

and

$$J(\Psi)_o := \tilde{J}(\Psi)_o / \sim,$$

where the equivalence relation '$\sim$' is as follows:

$$f \sim g \iff f = g \text{ on } \{\Psi < -t\} \cap V, \quad \text{where } t \gg 1, V \text{ is a neighborhood of } o.$$

For any $f \in \tilde{J}(\Psi)_o$, denote the equivalence class of f in $J(\Psi)_o$ by f_o. And for any $f_o, g_o \in J(\Psi)_o$, and $(h, o) \in \mathcal{O}_o$, define

$$f_o + g_o := (f + g)_o, \quad (h, o) \cdot f_o := (hf)_o.$$

It is clear that $J(\Psi)_o$ is an $\mathcal{O}_o$-module. For any $a \geq 0$, denote by $I(a\Psi+\varphi_0)_o := \big\{f_o \in J(\Psi)_o : \exists t \gg 1, V \text{ is a neighborhood of } o, \text{ s.t. } \int_{\{\Psi<-t\}\cap V} |f|^2 e^{-a\Psi-\varphi_0} < +\infty\big\}$. Then it is clear that $I(a\Psi + \varphi_0)_o$ is an $\mathcal{O}_o$-submodule of $J(\Psi)_o$. Especially, we denote by $I(\varphi_0)_o := I(0\Psi+\varphi_0)_o$, $I(\Psi)_o := I(\Psi+0)_o$, and $I_o := I(0\Psi+0)_o$. Then $I(a\Psi + \varphi_0)_o$ is an $\mathcal{O}_o$-submodule of $I(\varphi_0)_o$ for any $a > 0$.

Note that for general Ψ, the origin o can be a boundary point of the sublevel set $\{\Psi < -t\}$ for some t. Then the modules mentioned above could be modules at the boundary point.

For any $t \in [0, +\infty)$ and $\lambda > 0$, denote by

$$\Psi_{\lambda,t} := \lambda \max\{\Psi + t, 0\},$$

and for any $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ and $\lambda > 0$, denote by

$$\|f\|_{\lambda,t} := \left(\int_{\{\Psi<0\}} |f|^2 e^{-\varphi_0-\Psi_{\lambda,t}}\right)^{1/2},$$

where $A^2(\{\Psi < 0\}, e^{-\varphi_0}) := \{f \in \mathcal{O}(\{\Psi < 0\}) : \int_{\{\Psi<0\}} |f|^2 e^{-\varphi_0} < +\infty\}$ (if $\varphi_0 \equiv 0$, we may denote $A^2(\{\Psi < 0\}) := A^2(\{\Psi < 0\}, e^0)$). It is clear that $e^{-\lambda t/2}\|f\|_{\lambda,0} \leq \|f\|_{\lambda,t} \leq \|f\|_{\lambda,0} < +\infty$ for any $t \geq 0$.

For any $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ (the dual space of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$), denote the Bergman kernel related to ξ by

$$K^{\varphi_0}_{\xi,\Psi,\lambda}(t) := \sup_{f \in A^2(\{\Psi<0\},e^{-\varphi_0})} \frac{|\xi \cdot f|^2}{\|f\|^2_{\lambda,t}}$$

for any $t \in [0, +\infty)$.

Denote $E := \{w \in \mathbb{C} : \operatorname{Re} w > 0\} \subset \mathbb{C}$. We obtain the following log-subharmonicity property of the Bergman kernel $K^{\varphi_0}_{\xi,\Psi,\lambda}$.

Theorem 1.1 *Assume that* $K^{\varphi_0}_{\xi,\Psi,\lambda}(0) \in (0, +\infty)$. *Then* $\log K^{\varphi_0}_{\xi,\Psi,\lambda}(\operatorname{Re} w)$ *is subharmonic with respect to* $w \in E$.

Let J be an $\mathcal{O}_o$-submodule of $I(\varphi_0)_o$. Denote by

$$A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J := \{f \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) : f_o \in J\}.$$

Assume that $A^2(\{\Psi<0\},e^{-\varphi_0})\cap J$ is a proper subspace of $A^2(\{\Psi<0\},e^{-\varphi_0})$ (and we will prove that it is a closed subspace).

Using Theorem 1.1, we obtain the following concavity and monotonicity property related to $K^{\varphi_0}_{\xi,\Psi,\lambda}$.

Theorem 1.2 *Assume that $J\supset I(\Psi+\varphi_0)_o$, and assume that $\xi\in A^2(\{\Psi<0\},e^{-\varphi_0})^*$ such that $\xi|_{A^2(\{\Psi<0\},e^{-\varphi_0})\cap J}\equiv 0$ and $K^{\varphi_0}_{\xi,\Psi,\lambda}(0)\in(0,+\infty)$. Then $-\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t)+t$ is concave and non-decreasing with respect to $t\in[0,+\infty)$.*

Remark 1.3 Let $\xi\in A^2(\{\Psi<0\},e^{-\varphi_0})^*$. According to Theorem 1.1, if there exist $k>0$ and $T>0$, such that $e^{-kt}K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$ is non-decreasing and not a constant function on $[0,T]$, then $e^{-kt}K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$ is strictly increasing on $[T,+\infty)$.

1.2 Applications

As applications of Theorems 1.1 and 1.2, we give new proofs of some results in [3, 19].

Let D be a pseudoconvex domain in $\mathbb{C}^n$, and the origin $o\in D$. Let $F\not\equiv 0$ be a holomorphic function on D, and ψ be a negative plurisubharmonic function on D. Denote by

$$\Psi:=\min\{\psi-2\log|F|,0\}.$$

If $F(w)=0$ for $w\in D$, set $\Psi(w)=0$.

Let f be a holomorphic function on D. Recall the definition of the minimal L^2 integral related to J [3, 19]

$$G(t;\Psi,J,f):=\inf\left\{\int_{\{\Psi<-t\}}|\tilde f|^2:\tilde f\in\mathcal{O}(\{\Psi<-t\})\ \&\ (\tilde f-f)_o\in J\right\}$$

for any $\mathcal{O}_o$-submodule J of I_o and $t\in[0,+\infty)$. Denote by

$$\Psi_1:=\min\{2c_o^{fF}(\psi)\psi-2\log|F|,0\},$$

and

$$I_+(\Psi_1)_o:=\bigcup_{a>1}I(a\Psi_1)_o,$$

where $c_o^{fF}(\psi):=\sup\{c\ge 0:|fF|^2e^{-2c\psi}$ is locally L^1 near $o\}$.

Theorem 1.2 gives a reproof of the following lower bound of L^2 integrals.

Corollary 1.4 [3] *If $f\in A^2(\{\Psi_1<0\})$, and $c_o^{fF}(\psi)<+\infty$, then for any $r\in(0,1]$,*

$$\frac{1}{r^2}\int_{\{c_o^{fF}(\psi)\psi-\log|F|<\log r\}}|f|^2\ge G(0;\Psi_1,I_+(\Psi_1)_o,f)>0.$$

Remark 1.5 The proof of the inequality $G(0; \Psi_1, I_+(\Psi_1)_o, f) > 0$ is given in [3].

When $f \equiv 1$, Corollary 1.4 gives a reproof of the sharp effectiveness result related to a conjecture posed by Jonsson–Mustaţă.

Corollary 1.6 [3] *If $f \in A^2(\{\Psi_1 < 0\})$ and $c_o^F(\psi) < +\infty$, then for any $r \in (0, 1]$,*

$$\frac{1}{r^2}\mu(\{c_o^F(\psi)\psi - \log|F| < \log r\}) \geq G(0; \Psi_1, I_+(\Psi_1)_o, 1) > 0,$$

where $\Psi_1 := \min\{2c_o^F(\psi)\psi - 2\log|F|, 0\}$, and $c_o^F(\psi) := \sup\{c \geq 0 : |F|^2e^{-2c\psi}$ is locally L^1 near $o\}$.

Let φ_0 be a plurisubharmonic function on D, and let f be a holomorphic function on $\{\Psi < 0\}$. Denote by $a_o^f(\Psi; \varphi_0) := \sup\{a \geq 0 : f_o \in I(2a\Psi + \varphi_0)_o\}$,

$$I_+(a\Psi_1 + \varphi_0)_o := \bigcup_{a' > a} I(a'\Psi_1 + \varphi_0)_o$$

for any $a \geq 0$, and

$$C(\Psi, \varphi_0, J, f) := \inf\left\{\int_{\{\Psi<0\}} |\tilde{f}|^2 e^{-\varphi_0} : (\tilde{f} - f)_o \in J \ \& \ \tilde{f} \in \mathcal{O}(\{\Psi < 0\})\right\}$$

for any $\mathcal{O}_o$-submodule J of $I(\varphi_0)_o$. The following effectiveness result of strong openness property of the module $I(a\Psi + \varphi_0)_o$ can be reproved by Theorem 1.2.

Corollary 1.7 [19] *Let C_1 and C_2 be two positive constants. If*

(1) $\int_{\{\Psi<0\}} |f|^2 e^{-\varphi_0 - \Psi} \leq C_1$*;*
(2) $C(\Psi, \varphi_0, I_+(2a_o^f(\Psi; \varphi_0)\Psi + \varphi_0)_o, f) \geq C_2$*,*

then for any $q > 1$ satisfying

$$\theta(q) > \frac{C_1}{C_2},$$

we have $f_o \in I(q\Psi + \varphi_0)_o$, where $\theta(q) = \frac{q}{q-1}$.

2 Preparations

2.1 L^2 Methods

We recall the optimal L^2 extension theorem.

Let Ω be a pseudoconvex domain in $\mathbb{C}^{n+1}$ with coordinate (z, t), where $z \in \mathbb{C}^n$, $t \in \mathbb{C}$. Let p be the natural projection $p(z, t) = t$ on Ω. Denote by $\omega := p(\Omega)$ and $\Omega_t := p^{-1}(t) \cap \Omega$ for any $t \in \omega$. Let φ be a plurisubharmonic function on Ω.

Lemma 2.1 (Optimal L^2 extension theorem ([6], see [20–22])) *Assume that the above $\omega = \Delta_{t_0,r}$(the disc in the complex plane centered at t_0 with radius r). Then for any f in $A^2(\Omega_{t_0}, e^{-\varphi})$, there exists a holomorphic function $\tilde{f}$ on Ω, such that $\tilde{f}|_{\Omega_{t_0}} = f$, and*

$$\frac{1}{\pi r^2}\int_{\Omega}|\tilde{f}|^2 e^{-\varphi} \le \int_{\Omega_{t_0}}|f|^2 e^{-\varphi}.$$

The following L^2 method will be used to prove Theorem 1.2.

Let D be a pseudoconvex domain in $\mathbb{C}^n$, and the origin $o \in D$. Let $F \not\equiv 0$ be a holomorphic function on D, and ψ be a negative plurisubharmonic function on D. Let φ_0 be a plurisubharmonic function on D. Denote by

$$\varphi := \varphi_0 + 2\max\{\psi, 2\log|F|\},$$

and

$$\Psi := \min\{\psi - 2\log|F|, 0\}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$.

Lemma 2.2 (see [3, 19, 22, 23]) *Let $t_0 \in (0, +\infty)$ be fixed. Let f be a holomorphic function on $\{\Psi < -t_0\}$ such that*

$$\int_{\{\Psi<-t_0\}\cap K}|f|^2 e^{-\varphi_0} < +\infty$$

for any compact subset $K \subset D$. Then there exists a holomorphic function $\tilde{F}$ on D such that

$$\int_D |\tilde{F} - (1 - b_{t_0}(\Psi)) f F^2|^2 e^{-\varphi + v_{t_0}(\Psi) - \Psi} \le C\int_D \mathbb{I}_{\{-t_0-1<\Psi<-t_0\}}|f|^2 e^{-\varphi_0-\Psi},$$

where $b_{t_0}(t) = \int_{-\infty}^{t}\mathbb{I}_{\{-t_0-1<s<-t_0\}}\mathrm{d}s$, $v_{t_0}(t) = \int_0^t b_{t_0}(s)\mathrm{d}s$ and C is a positive constant.

2.2 Some Lemmas About Submodules of $I(\varphi_0)_o$

Recall that D is a pseudoconvex domain in $\mathbb{C}^n$, and the origin $o \in D$. Let $F \not\equiv 0$ be a holomorphic function on D, and ψ be a negative plurisubharmonic function on D. Let φ_0 be a plurisubharmonic function on D. Denote by

$$\Psi := \min\{\psi - 2\log|F|, 0\}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$. We recall the following lemma.

Lemma 2.3 [19] *Let J_o be an $\mathcal{O}_{\mathbb{C}^n,o}$-submodule of $I(\varphi_0)_o$ such that $I(\Psi+\varphi_0)_o \subset J_o$. Assume that $f_o \in J(\Psi)_o$. Let U_0 be a pseudoconvex open neighborhood of o. Let $\{f_j\}_{j\geq 1}$ be a sequence of holomorphic functions on $U_0 \cap \{\Psi < -t_j\}$ for any $j \geq 1$, where $t_j \in (T, +\infty)$. Assume that $t_0 = \lim_{j\to+\infty} t_j \in [T, +\infty)$,*

$$\limsup_{j\to+\infty} \int_{U_0\cap\{\Psi<-t_j\}} |f_j|^2 e^{-\varphi_0} \leq C < +\infty,$$

and $(f_j - f)_o \in J_o$. Then there exists a subsequence of $\{f_j\}_{j\geq 1}$ compactly convergent to a holomorphic function f_0 on $\{\Psi < -t_0\} \cap U_0$ which satisfies

$$\int_{U_0\cap\{\Psi<-t_0\}} |f_0|^2 e^{-\varphi_0} \leq C,$$

and $(f_0 - f)_o \in J_o$.

It is well-known that $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ is a Hilbert space. Let J be an $\mathcal{O}_o$-submodule of $I(\varphi_0)_o$. We state that $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J := \{f \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) : f_o \in J\}$ is a closed subspace of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ if $J \supset I(\Psi + \varphi_0)_o$.

Lemma 2.4 *If $J \supset I(\Psi + \varphi_0)_o$, then $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is closed in $A^2(\{\Psi < 0\}, e^{-\varphi_0})$.*

Proof Let $\{f_j\}$ be a sequence of holomorphic functions in $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$, such that $\lim_{j\to+\infty} f_j = f_0$ in the topology of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$. Then $\{f_j\}$ compactly converges to f_0 on $\{\Psi < 0\}$, and $(f_j - 0)_o \in J$ for any j. According to Lemma 2.3, we can get that $(f_0 - 0)_o \in J$, which means that $(f_0)_o \in J$, i.e., $f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$. Then we get that $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is closed in $A^2(\{\Psi < 0\}, e^{-\varphi_0})$. □

The following remark and lemma can be found in [3] or [19].

Remark 2.5 [19] For any $a \geq 0$, there exists $a' > a$ such that $I(a'\Psi + \varphi_0)_o = I_+(a\Psi + \varphi_0)_o$.

Let f be a holomorphic function on D. Denote by

$$\Psi_1 := \min\{2c_o^{fF}(\psi)\psi - 2\log|F|, 0\},$$

where $c_o^{fF}(\psi) := \sup\{c \geq 0 : |fF|^2 e^{-2c\psi}$ is locally L^1 near $o\}$.

Lemma 2.6 ([3], see also [19]) *$f_o \notin I_+(\Psi_1)_o$.*

2.3 Some Lemmas About Functionals on $A^2(\{\Psi < 0\}, e^{-\varphi_0})$

The following two lemmas will be used in the proof of Theorem 1.1. For the convenience of readers, we recall the proofs.

Lemma 2.7 *Let D be a domain in $\mathbb{C}^n$, and let φ_0 be a plurisubharmonic function on D. Let $\{f_j\}$ be a sequence in $A^2(D, e^{-\varphi_0})$, such that $\int_D |f_j|^2 e^{-\varphi_0}$ is uniformly bounded for any $j \in \mathbb{N}_+$. Assume that f_j compactly converges to $f_0 \in A^2(D, e^{-\varphi_0})$. Then for any $\xi \in A^2(D, e^{-\varphi_0})^*$,*

$$\lim_{j \to +\infty} \xi \cdot f_j = \xi \cdot f_0.$$

Proof For any $f \in A^2(D, e^{-\varphi_0})$, denote by $\|f\|^2 := \int_D |f|^2 e^{-\varphi_0}$. Let $\{f_{k_j}\}$ be any subsequence of $\{f_j\}$. Since $A^2(D, e^{-\varphi_0})$ is a Hilbert space, and $\|f_{k_j}\|^2$ is uniformly bounded, there exists a subsequence of $\{f_{k_j}\}$ (denoted by $\{f_{k_{l_j}}\}$) weakly convergent to some $\tilde{f} \in A^2(D, e^{-\varphi_0})$. Note that for any $z \in D$, the functional $e_z \in A^2(D, e^{-\varphi_0})^*$, where

$$\begin{aligned} e_z \ : \ A^2(D, e^{-\varphi_0}) &\longrightarrow \mathbb{C}, \\ f &\longmapsto f(z). \end{aligned}$$

Then we have

$$f_0(z) = \lim_{j \to +\infty} e_z \cdot f_j = \lim_{j \to +\infty} e_z \cdot f_{k_{l_j}} = e_z \cdot \tilde{f} = \tilde{f}(z), \quad \forall z \in D,$$

thus $f_0 = \tilde{f}$. It means that $\{f_{k_j}\}$ has a subsequence weakly convergent to f_0. Since $\{f_{k_j}\}$ is an arbitrary subsequence of $\{f_j\}$, we get that $\{f_j\}$ weakly converges to f_0. In other words, for any $\xi \in A^2(D, e^{-\varphi_0})^*$,

$$\lim_{j \to +\infty} \xi \cdot f_j = \xi \cdot f_0. \qquad \square$$

Let $\Omega := D \times \omega \subset \mathbb{C}^{n+1}$, where D is a domain in $\mathbb{C}^n$, ω is a domain in $\mathbb{C}$. Denote the coordinate on Ω by (z, τ), where $z \in D$, $\tau \in \omega$. Let φ_0 be a plurisubharmonic function on D. Let f be a holomorphic function on Ω, such that

$$\int_\Omega |f(z,\tau)|^2 e^{-\varphi_0(z)} < +\infty.$$

Denote $f_\tau := f|_{D \times \{\tau\}}$.

Lemma 2.8 *For any $\xi \in A^2(D, e^{-\varphi_0})^*$, $\xi \cdot f_\tau$ is holomorphic with respect to $\tau \in \omega$.*

Proof We only need to prove that $h(\tau) := \xi \cdot f_\tau$ is holomorphic near any $\tau_0 \in \omega$. Since $\tau_0 \in \omega$, there exists $r > 0$ such that $\Delta(\tau_0, 2r) \subset\subset \omega$. Then for any $\tau \in \Delta(\tau_0, r)$, according to sub-mean value inequality of subharmonic functions, we have

$$\begin{aligned} \int_D |f_\tau(z)|^2 e^{-\varphi_0(z)} &\le \frac{1}{\pi r^2} \int_{D \times \Delta(\tau_0, r)} |f(z,\tau)|^2 e^{-\varphi_0(z)} \\ &\le \frac{1}{\pi r^2} \int_\Omega |f|^2 e^{-\varphi_0} < +\infty, \end{aligned}$$

which implies that $f_\tau \in A^2(D, e^{-\varphi_0})$ and there exists $M > 0$ such that $\int_D |f_\tau|^2 e^{-\varphi_0} \leq M$ for any $\tau \in \Delta(\tau_0, r)$.

Fix $z_0 \in D$. According to Lemma A.1 in the Appendix, we can find a sequence $\{\xi_k\} \subset \ell_0 \subset A^2(D, e^{-\varphi_0})^*$, such that

$$\lim_{k \to +\infty} \|\xi_k - \xi\|_{A^2(D, e^{-\varphi_0})^*} = 0,$$

where

$$\ell_0 := \{\eta = (\eta_\alpha)_{\alpha \in \mathbb{N}^n} : \exists k \in \mathbb{N}, \text{ such that } \eta_\alpha = 0,\ \forall \alpha, |\alpha| \geq k\}.$$

Here for any $\eta = (\eta_\alpha)_{\alpha \in \mathbb{N}^n} \in \ell_0$ and $f \in A^2(D, e^{-\varphi_0})$, define that

$$\eta \cdot f := \sum_{\alpha \in \mathbb{N}^n} \eta_\alpha \frac{f^{(\alpha)}(z_0)}{\alpha!}.$$

Note that for any $\xi_k \in \ell_0$, $\xi_k \cdot f_\tau$ can be written as

$$\xi_k \cdot f_\tau = \sum_{\alpha \in \mathbb{N}^n,\ |\alpha| \leq l_k} c_{\alpha,k} \frac{\partial^\alpha f(z, \tau)}{\partial z^\alpha}(z_0, \tau),$$

where l_k is a finite integer, and $c_{\alpha,k} \in \mathbb{C}$ are constants. It is clear that $h_k(\tau) := \xi_k \cdot f_\tau$ is holomorphic with respect to $\tau \in \omega$ for any $k \in \mathbb{N}_+$, since any $\frac{\partial^\alpha f(z,\tau)}{\partial z^\alpha}(z_0, \tau)$ is holomorphic with respect to τ for any $\alpha \in \mathbb{N}^n$. Note that for any $\tau \in \Delta(\tau_0, r)$, we have

$$\begin{aligned}
&\|h_k(\tau) - h(\tau)\|^2 \\
&\quad = \|(\xi_k - \xi) \cdot f_\tau\|^2 \\
&\quad \leq \|\xi_k - \xi\|^2_{A^2(D, e^{-\varphi_0})^*} \int_D |f_\tau|^2 e^{-\varphi_0} \\
&\quad \leq M \|\xi_k - \xi\|^2_{A^2(D, e^{-\varphi_0})^*},
\end{aligned}$$

which means that h_k uniformly converges to h on $\Delta(\tau_0, r)$. According to the Weierstrass theorem, h is holomorphic on $\Delta(\tau_0, r)$, i.e., near τ_0. Then we get that $\xi \cdot f_\tau$ is holomorphic with respect to $\tau \in \omega$. □

2.4 Some Properties of $K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$

In this section, we prove some properties of the Bergman kernel $K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$.

Let $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\}$.

Lemma 2.9 *For any $t \in [0, +\infty)$, if $K^{\varphi_0}_{\xi,\Psi,\lambda}(t) \in (0, +\infty)$, then there exists $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, such that*

$$K^{\varphi_0}_{\xi,\Psi,\lambda}(t) = \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|^2_{\lambda,t}}.$$

Proof By the definition of $K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$, there exists a sequence $\{f_j\}$ of holomorphic functions in $A^2(\{\Psi < 0\}, e^{-\varphi_0})$, such that $\|f_j\|_{\lambda,t} = 1$, and $\lim_{j\to+\infty} |\xi \cdot f_j|^2 = K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$. Then $\int_{\{\Psi<0\}} |f_j|^2 e^{-\varphi_0}$ is uniformly bounded. Following from Montel's theorem, we can get a subsequence of $\{f_j\}$ compactly convergent to a holomorphic function $\tilde{f}$ on $\{\Psi < 0\}$. According to Fatou's lemma, we have $\|\tilde{f}\|_{\lambda,t} \leq 1$, and according to Lemma 2.7, we have $|\xi \cdot \tilde{f}|^2 = K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$, thus $K^{\varphi_0}_{\xi,\Psi,\lambda}(t) \leq \frac{|\xi\cdot\tilde{f}|^2}{\|\tilde{f}\|^2_{\lambda,t}}$. Note that $\|\tilde{f}\|_{\lambda,t} \leq 1$ implies $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, which means $K^{\varphi_0}_{\xi,\Psi,\lambda}(t) \geq \frac{|\xi\cdot\tilde{f}|^2}{\|\tilde{f}\|^2_{\lambda,t}}$. We get that $K^{\varphi_0}_{\xi,\Psi,\lambda}(t) = \frac{|\xi\cdot\tilde{f}|^2}{\|\tilde{f}\|^2_{\lambda,t}}$. □

Let J be an $\mathcal{O}_o$-submodule of $I(\varphi_0)_o$ such that $J \supset I(\Psi + \varphi_0)_o$, and let $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, such that $f_o \notin J$. Recall the minimal L^2 integral ([3, 19])

$$C(\Psi, \varphi_0, J, f) := \inf\left\{\int_{\{\Psi<0\}} |\tilde{f}|^2 e^{-\varphi_0} : (\tilde{f} - f)_o \in J \;\&\; \tilde{f} \in \mathcal{O}(\{\Psi < 0\})\right\}.$$

Then the following lemma holds.

Lemma 2.10 *Assume that $C(\Psi, \varphi_0, J, f) \in (0, +\infty)$, then*

$$C(\Psi, \varphi_0, J, f) = \sup_{\substack{\xi \in A^2(\{\Psi<0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi<0\}, e^{-\varphi_0}) \cap J} \equiv 0}} \frac{|\xi \cdot f|^2}{K^{\varphi_0}_{\xi,\Psi,\lambda}(0)}. \tag{2.1}$$

Proof Note that $\xi \cdot \tilde{f} = \xi \cdot f$ for any $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ with $(\tilde{f} - f)_o \in J$ and $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ satisfying $\xi|_{A^2(\{\Psi<0\},e^{-\varphi_0})\cap J} \equiv 0$. Then we have

$$\begin{aligned}
K^{\varphi_0}_{\xi,\Psi,\lambda}(0) &= \sup_{h\in A^2(\{\Psi<0\},e^{-\varphi_0})} \frac{|\xi \cdot h|^2}{\int_{\{\Psi<0\}} |h|^2 e^{-\varphi_0}} \\
&\geq \sup_{\substack{\tilde{f} \in A^2(\{\Psi<0\}, e^{-\varphi_0}) \\ (\tilde{f}-f)_o \in J}} \frac{|\xi \cdot \tilde{f}|^2}{\int_{\{\Psi<0\}} |\tilde{f}|^2 e^{-\varphi_0}} \\
&= \sup_{\substack{\tilde{f} \in A^2(\{\Psi<0\}, e^{-\varphi_0}) \\ (\tilde{f}-f)_o \in J}} \frac{|\xi \cdot f|^2}{\int_{\{\Psi<0\}} |\tilde{f}|^2 e^{-\varphi_0}}.
\end{aligned}$$

Thus we get that

$$\begin{aligned}
&\sup_{\substack{\xi \in A^2(\{\Psi<0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi<0\}, e^{-\varphi_0}) \cap J} \equiv 0}} \frac{|\xi \cdot f|^2}{K^{\varphi_0}_{\xi,\Psi,\lambda}(0)} \\
&\leq \inf_{\substack{\tilde{f} \in A^2(\{\Psi<0\}, e^{-\varphi_0}) \\ (\tilde{f}-f)_o \in J}} \int_{\{\Psi<0\}} |\tilde{f}|^2 e^{-\varphi_0} \\
&= C(\Psi, \varphi_0, J, f).
\end{aligned}$$

Since $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ is a Hilbert space, and $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is a closed proper subspace of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ (using Lemma 2.4), there exists a closed subspace H of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ such that $H = (A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J)^{\perp} \neq \{0\}$. Then for $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, we can make the decomposition $f = f_J + f_H$, such that $f_J \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$, and $f_H \in H$. Note that the linear functional ξ_f defined as follows:

$$\xi_f \cdot g := \int_{\{\Psi<0\}} g\overline{f_H} e^{-\varphi_0}, \quad \forall g \in A^2(\{\Psi < 0\}, e^{-\varphi_0}),$$

satisfies that $\xi_f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \backslash \{0\}$ and $\xi_f|_{A^2(\{\Psi<0\}, e^{-\varphi_0}) \cap J} \equiv 0$. Then we have

$$\sup_{\substack{\xi \in A^2(\{\Psi<0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi<0\}, e^{-\varphi_0}) \cap J} \equiv 0}} \frac{|\xi \cdot f|^2}{K^{\varphi_0}_{\xi,\Psi,\lambda}(0)} \geq \frac{|\xi_f \cdot f|^2}{K^{\varphi_0}_{\xi_f,\Psi,\lambda}(0)}.$$

Besides, we can see that

$$K^{\varphi_0}_{\xi_f,\Psi,\lambda}(0) = \sup_{h \in A^2(\{\Psi<0\}, e^{-\varphi_0})} \frac{|\int_{\{\Psi<0\}} h\overline{f_H} e^{-\varphi_0}|^2}{\int_{\{\Psi<0\}} |h|^2 e^{-\varphi_0}} \leq \int_{\{\Psi<0\}} |f_H|^2 e^{-\varphi_0},$$

and

$$\xi_f \cdot f = \xi_f \cdot (f_J + f_H) = \xi_f \cdot f_H = \int_{\{\Psi<0\}} |f_H|^2 e^{-\varphi_0}.$$

Then we have

$$\frac{|\xi_f \cdot f|^2}{K^{\varphi_0}_{\xi_f,\Psi,\lambda}(0)} \geq \int_{\{\Psi<0\}} |f_H|^2 e^{-\varphi_0} \geq C(\Psi, \varphi_0, J, f),$$

which implies that

$$\sup_{\substack{\xi \in A^2(\{\Psi<0\}, e^{-\varphi_0})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi<0\}, e^{-\varphi_0}) \cap J} \equiv 0}} \frac{|\xi \cdot f|^2}{K^{\varphi_0}_{\xi,\Psi,\lambda}(0)} \geq C(\Psi, \varphi_0, J, f).$$

Lemma 2.10 is proved. □

We follow the assumption of Theorem 1.1 in the following lemma.

Lemma 2.11 *$K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$ is upper-semicontinuous with respect to $t \in [0, +\infty)$, i.e., for any sequence $t_j \in [0, +\infty)$ such that $\lim_{j\to+\infty} t_j = t_0 \in [0, +\infty)$, we have*

$$\limsup_{j\to+\infty} K^{\varphi_0}_{\xi,\Psi,\lambda}(t_j) \leq K^{\varphi_0}_{\xi,\Psi,\lambda}(t_0).$$

Proof Denote by

$$K(t) := K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$$

for any $t \in [0, +\infty)$. Let $t_j \in [0, +\infty)$ such that $\lim_{j\to+\infty} t_j = t_0 \in [0, +\infty)$. We assume that $\{t_{k_j}\}$ is the subsequence of $\{t_j\}$ such that

$$\lim_{j\to+\infty} K(t_{k_j}) = \limsup_{j\to+\infty} K(t_j).$$

By Lemma 2.9, there exists a sequence of holomorphic functions $\{f_j\}$ on $\{\Psi < 0\}$ such that $f_j \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, $\|f_j\|_{\lambda,t_j} = 1$, and $|\xi \cdot f_j|^2 = K(t_j)$, for any $j \in \mathbb{N}_+$. Since $\{t_j\}$ is bounded in $\mathbb{R}$, there exists some $s_0 > 0$, such that $t_j < s_0$ for any j, which implies that

$$\int_{\{\Psi<0\}} |f_j|^2 e^{-\varphi_0} \leq e^{\lambda s_0} \|f_j\|^2_{\lambda,t_j} = e^{\lambda s_0}, \quad \forall j \in \mathbb{N}_+.$$

According to Montel's theorem, we can get a subsequence of $\{f_{k_j}\}$ (denoted by $\{f_{k_j}\}$ itself) compactly convergent to a holomorphic function f_0 on $\{\Psi < 0\}$. According to Fatou's lemma, we have

$$\begin{aligned}
\|f_0\|_{\lambda,t_0} &= \int_{\{\Psi<0\}} |f_0(z)|^2 e^{-\varphi_0(z)-\lambda\max\{\Psi(z)+t_0,0\}} \\
&= \int_{\{\Psi<0\}} \lim_{j\to+\infty} |f_{k_j}(z)|^2 e^{-\varphi_0(z)-\lambda\max\{\Psi(z)+t_{k_j},0\}} \\
&\leq \liminf_{j\to+\infty} \int_{\{\Psi<0\}} |f_{k_j}(z)|^2 e^{-\varphi_0(z)-\lambda\max\{\Psi(z)+t_{k_j},0\}} \\
&= \liminf_{j\to+\infty} \|f_{k_j}\|_{\lambda,t_j} = 1.
\end{aligned}$$

Then $\int_{\{\Psi<0\}}|f_0|^2e^{-\varphi_0} \leq e^{\lambda t_0}\|f_0\|^2_{\lambda,t_0} \leq e^{\lambda s_0} < +\infty$, which implies that $f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$. Lemma 2.7 shows that $|\xi \cdot f_0|^2 = \lim_{j\to+\infty}|\xi \cdot f_{k_j}|^2 = \limsup_{j\to+\infty} K(t_j)$, thus

$$K(t_0) \geq \frac{|\xi \cdot f_0|^2}{\|f_0\|^2_{\lambda,t_0}} \geq \limsup_{j\to+\infty} K(t_j),$$

Now we get that $K(t)$ is upper semicontinuous with respect to $t \in [0, +\infty)$. □

3 Proof of Theorem 1.1

We prove Theorem 1.1 by using Lemma 2.1 (optimal L^2 extension theorem).

Proof of Theorem 1.1 Denote by $\Omega := \{\Psi < 0\} \times E = \{\Psi < 0\} \times \{w \in \mathbb{C} : \operatorname{Re} w \geq 0\}$, and the coordinate on Ω is (z, w), where $z \in \{\Psi < 0\} \subset \mathbb{C}^n$, $w \in E = \{w \in \mathbb{C} : \operatorname{Re} w \geq 0\}$. Note that $D \setminus \{F = 0\}$ is a pseudoconvex domain in $\mathbb{C}^n$, and $\{\Psi < 0\} = \{\psi + 2\log|1/F| < 0\}$ on $D \setminus \{F = 0\}$. Then $\{\Psi < 0\}$ is a pseudoconvex domain in $\mathbb{C}^n$, and Ψ is a plurisubharmonic function on $\{\Psi < 0\}$. We get that Ω is a pseudoconvex domain in $\mathbb{C}^{n+1}$. For any $(z, w) \in \Omega$, let

$$\tilde{\Psi}(z, w) := \varphi_0(z) + \Psi_{\lambda,\operatorname{Re} w} = \varphi_0(z) + \lambda \max\{\Psi(z) + \operatorname{Re} w, 0\}.$$

Then $\tilde{\Psi}$ is a plurisubharmonic function on Ω.

Denote by

$$K(w) := K^{\varphi_0}_{\xi,\Psi,\lambda}(\operatorname{Re} w)$$

for any $w \in E$. We prove that $\log K(w)$ is a subharmonic function with respect to $w \in E$.

First, we prove that $\log K(w)$ is upper semicontinuous. Note that $K(w)$ only depends on $\operatorname{Re} w \in (0, +\infty)$. According to Lemma 2.11, we have that $K(w)$ is upper-semicontinuous with respect to $w \in E$, which yields that $\log K(w)$ is upper semicontinuous with respect to $w \in E$.

Next, we prove that $\log K(w)$ satisfies the sub-mean value inequality.

Let $\Delta(w_0, r) \subset E$ be the disc centered at w_0 with radius r, and let $\Omega' := \{\Psi < 0\} \times \Delta(w_0, r) \subset \mathbb{C}^{n+1}$. Let $f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ such that

$$K(w_0) = \frac{|\xi \cdot f_0|^2}{\|f_0\|^2_{\lambda,\operatorname{Re} w_0}}$$

by Lemma 2.9.

Note that Ω' is a pseudoconvex domain in $\mathbb{C}^{n+1}$, and $\tilde{\Psi}(z, w) = \varphi_0(z) + \Psi_{\lambda,\operatorname{Re} w} = \varphi_0(z) + \lambda \max\{\Psi(z) + \operatorname{Re} w, 0\}$ is a plurisubharmonic function on Ω'. Using Lemma

2.1 (optimal L^2 extension theorem), we can get a holomorphic function $\tilde{f}$ on Ω' such that $\tilde{f}(z, w_0) = f_0(z)$ for any $z \in \{\Psi < 0\}$, and

$$\frac{1}{\pi r^2}\int_{\Omega'} |\tilde{f}(z, w)|^2 e^{-\tilde{\Psi}(z,w)} \leq \int_{\{\Psi<0\}} |f_0(z)|^2 e^{-\Psi(z,w_0)}. \tag{3.1}$$

Denote by $\tilde{f}_w(z) = \tilde{f}(z, w) = \tilde{f}|_{\{\Psi<0\}\times\{w\}}$. Since the function $y = \log x$ is concave and increasing, according to Jensen's inequality and inequality (3.1), we have

$$\begin{aligned}
\log \|f_0\|^2_{\lambda,\mathrm{Re}\, w_0} &= \log\left(\int_{\{\Psi<0\}} |f_0(z)|^2 e^{-\Psi(z,w_0)}\right)\\
&\geq \log\left(\frac{1}{\pi r^2}\int_{\Omega'} |\tilde{f}(z, w)|^2 e^{-\tilde{\Psi}(z,w)}\right)\\
&= \log\left(\frac{1}{\pi r^2}\int_{\Delta(w_0,r)}\int_{\{\Psi<0\}\times\{w\}} |\tilde{f}_w(z)|^2 e^{-\tilde{\Psi}(z,w)}\right)\\
&\geq \frac{1}{\pi r^2}\int_{\Delta(w_0,r)} \log\left(\|\tilde{f}_w\|^2_{\lambda,\mathrm{Re}\, w}\right)\\
&\geq \frac{1}{\pi r^2}\int_{\Delta(w_0,r)} \left(\log|\xi\cdot\tilde{f}_w|^2 - \log K(w)\right).
\end{aligned} \tag{3.2}$$

It follows from Lemma 2.8 that $\xi \cdot \tilde{f}_w$ is holomorphic with respect to w, which implies that $\log|\xi \cdot \tilde{f}_w|^2$ is subharmonic with respect to w. Combining with $\tilde{f}_{w_0} = f_0$, we have

$$\log|\xi\cdot f_0|^2 \leq \frac{1}{\pi r^2}\int_{\Delta(w_0,r)} \log|\xi\cdot\tilde{f}_w|^2.$$

Combining with inequality (3.2), we get

$$\log\|f_0\|^2_{\lambda,\mathrm{Re}\, w_0} \geq \log|\xi\cdot f_0|^2 - \frac{1}{\pi r^2}\int_{\Delta(w_0,r)} \log K(w),$$

which means

$$\log K(w_0) \leq \frac{1}{\pi r^2}\int_{\Delta(w_0,r)} \log K(w).$$

Since $\log K(w)$ is upper semicontinuous and satisfies the sub-mean value inequality, $\log K(w)$ is a subharmonic function on E. □

4 Proofs of Theorem 1.2 and Remark 1.3

In this section, we give the proofs of Theorem 1.2 and Remark 1.3. We need the following lemma.

Lemma 4.1 (see [11]) *Let $D = I + \sqrt{-1}\mathbb{R} := \{z = x + \sqrt{-1}y \in \mathbb{C} : x \in I, y \in \mathbb{R}\}$ be a subset of $\mathbb{C}$, where I is an interval in $\mathbb{R}$. Let $\phi(z)$ be a subharmonic function on D which is only dependent on $x = \operatorname{Re} z$. Then $\phi(x) := \phi(x + \sqrt{-1}\mathbb{R})$ is a convex function with respect to $x \in I$.*

Proof of Theorem 1.2 It follows from Theorem 1.1 that $\log K^{\varphi_0}_{\xi,\Psi,\lambda}(\operatorname{Re} w)$ is subharmonic with respect to $w \in (0, +\infty) + \sqrt{-1}\mathbb{R}$. Note that $\log K^{\varphi_0}_{\xi,\Psi,\lambda}(\operatorname{Re} w)$ is only dependent on $\operatorname{Re} w$, then following from Lemma 4.1, we get that $\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$ is convex with respect to $t \in (0, +\infty)$. Combining with Lemma 2.11, we have that $\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$ is convex on $[0, +\infty)$, which implies that $-\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t) + t$ is concave with respect to $t \in [0, +\infty)$. Then for any $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ with $\xi|_{A^2(\{\Psi<0\},e^{-\varphi_0})\cap J} \equiv 0$, to prove that $-\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t) + t$ is non-decreasing, we only need to prove that $-\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t) + t$ has a lower bound on $[0, +\infty)$.

Using Lemma 2.9, we obtain that there exists $f_t \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ for any $t \in [0, +\infty)$, such that $\xi \cdot f_t = 1$ and

$$K^{\varphi_0}_{\xi,\Psi,\lambda}(t) = \frac{1}{\|f_t\|^2_{\lambda,t}}. \tag{4.1}$$

In addition, according to Lemma 2.2, there exists a holomorphic function $\tilde{F}$ on D such that

$$\int_D |\tilde{F} - (1 - b_t(\Psi)) f_t F^2|^2 e^{-\varphi+v_t(\Psi)-\Psi} \le C \int_D \mathbb{I}_{\{-t-1<\Psi<-t\}} |f_t|^2 e^{-\varphi_0-\Psi}, \tag{4.2}$$

where

$$\varphi = \varphi_0 + 2\max\{\psi, 2\log|F|\},$$

and C is a positive constant. Then it follows from inequality (4.2) that

$$\begin{aligned}
&\int_{\{\Psi<0\}} |\tilde{F} - (1 - b_t(\Psi)) f_t F^2|^2 e^{-\varphi+v_t(\Psi)-\Psi} \\
&\le \int_D |\tilde{F} - (1 - b_t(\Psi)) f_t F^2|^2 e^{-\varphi+v_t(\Psi)-\Psi} \\
&\le C \int_D \mathbb{I}_{\{-t-1<\Psi<-t\}} |f_t|^2 e^{-\varphi_0-\Psi} \\
&\le Ce^{t+1} \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0}.
\end{aligned} \tag{4.3}$$

Denote by $\tilde{F}_t := \tilde{F}/F^2$ on $\{\Psi < 0\}$, then $\tilde{F}_t$ is a holomorphic function on $\{\Psi < 0\}$. Note that $|F|^4 e^{-\varphi} = e^{-\varphi_0}$ on $\{\Psi < 0\}$. Then inequality (4.3) implies that

$$\int_{\{\Psi<0\}} |\tilde{F}_t - (1-b_t(\Psi)) f_t|^2 e^{-\varphi_0 + v_t(\Psi) - \Psi} \le Ce^{t+1} \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0} < +\infty. \tag{4.4}$$

According to inequality (4.4), we can get that $(\tilde{F}_t - f_t)_o \in I(\Psi + \varphi_0) \subset J$, which means that $\xi \cdot \tilde{F}_t = \xi \cdot f_t = 1$. Besides, since $v_t(\Psi) \ge \Psi$, we have

$$\begin{aligned}
&\left(\int_{\{\Psi<0\}} |\tilde{F}_t - (1-b_t(\Psi)) f_t|^2 e^{-\varphi_0 + v_t(\Psi) - \Psi}\right)^{1/2} \\
&\ge \left(\int_{\{\Psi<0\}} |\tilde{F}_t - (1-b_t(\Psi)) f_t|^2 e^{-\varphi_0}\right)^{1/2} \\
&\ge \left(\int_{\{\Psi<0\}} |\tilde{F}_t|^2 e^{-\varphi_0}\right)^{1/2} - \left(\int_{\{\Psi<0\}} |(1-b_t(\Psi)) f_t|^2 e^{-\varphi_0}\right)^{1/2} \\
&\ge \left(\int_{\{\Psi<0\}} |\tilde{F}_t|^2 e^{-\varphi_0}\right)^{1/2} - \left(\int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0}\right)^{1/2}.
\end{aligned}$$

Combining with inequality (4.4), we have

$$\begin{aligned}
&\int_{\{\Psi<0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \\
&\le 2\int_{\{\Psi<0\}} |\tilde{F}_t - (1-b_t(\Psi)) f_t|^2 e^{-\varphi_0 + v_t(\Psi) - \Psi} + 2\int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0} \\
&\le 2(Ce^{t+1} + 1) \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0}.
\end{aligned}$$

Note that

$$\begin{aligned}
\|f_t\|^2_{\lambda,t} &= \int_{\{\Psi<0\}} |f_t|^2 e^{-\varphi_0 - \Psi_{\lambda,t}} \\
&= \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0} + \int_{\{0>;\Psi\ge -t\}} |f_t|^2 e^{-\varphi_0 - \lambda(\Psi+t)} \\
&\ge \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0}.
\end{aligned}$$

Then we have

$$\int_{\{\Psi<0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \le 2(Ce^{t+1}+1)\|f_t\|^2_{\lambda,t} = C_1 \frac{e^t}{K^{\varphi_0}_{\xi,\Psi,\lambda}(t)},$$

where $C_1 := 2(eC + 1)$ is a positive constant. In addition, $\xi \cdot \tilde{F}_t = 1$ implies that

$$\int_{\{\Psi<0\}} |\tilde{F}_t|^2 e^{-\varphi_0} = \|\tilde{F}_t\|^2_{\lambda,0} \geq (K^{\varphi_0}_{\xi,\Psi,\lambda}(0))^{-1}.$$

Then we get that

$$-\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t) + t \geq C_2, \quad \forall t \in [0, +\infty),$$

where $C_2 := \log(C_1^{-1} K^{\varphi_0}_{\xi,\Psi,\lambda}(0))$ is a finite constant. Since $-\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t) + t$ is concave, we get that $-\log K^{\varphi_0}_{\xi,\Psi,\lambda}(t)+t$ is non-decreasing with respect to $t \in [0, +\infty)$. □

In the following, we give the proof of Remark 1.3.

Proof of Remark 1.3 Denote by $K(t) := K^{\varphi_0}_{\xi,\Psi,\lambda}(t)$ for any $t \in [0, +\infty)$. According to Theorem 1.1 and Lemma 4.1, we can see that $\log K(t) - kt$ is convex on $[0, +\infty)$. Combining with that $e^{-kt}K(t)$ is non-decreasing and not a constant function on $[0, T]$, which implies that $\log K(t) - kt = \log(e^{-kt}K(t))$ is non-decreasing and not a constant function on $[0, T]$, we have that $\log K(t) - kt$ is strictly increasing on $[T, +\infty)$. Then $e^{-kt}K^{\varphi_0}_{\xi,\Psi,\lambda}(t) = \exp(\log K(t) - kt)$ is strictly increasing on $[T, +\infty)$. □

5 Proof of Corollary 1.4

In this section, we give the proof of Corollary 1.4.

Proof of Corollary 1.4 For any $p \in (1, 2)$, $\lambda > 0$, let $\xi \in A^2(\{\Psi_1 < 0\})^* \backslash \{0\}$, such that $\xi|_{A^2(\{\Psi_1<0\})\cap J_p} \equiv 0$, where $J_p := I(p\Psi_1)_o$. Denote by

$$K_{\xi,p,\lambda}(t) := \sup_{\tilde{f}\in A^2(\{\Psi_1<0\})} \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|^2_{p,\lambda,t}},$$

where

$$\|\tilde{f}\|_{p,\lambda,t} := \left(\int_{\{\Psi_1<0\}} |\tilde{f}|^2 e^{-\lambda \max\{p\Psi_1+t,0\}}\right)^{1/2},$$

and $t \in [0, +\infty)$. Note that

$$p\Psi_1 = \min\{(2pc_o^{fF}(\psi)\psi + (4-2p)\log|F|) - 2\log|F^2|, 0\}$$

and Lemma 2.6 shows that $f_o \notin J_p$, which implies that $A^2(\{\Psi_1 < 0\}) \cap J_p$ is a proper subspace of $A^2(\{\Psi_1 < 0\})$, and $K_{\xi,p,\lambda}(0) \in (0, +\infty)$. Theorem 1.2 tells us that $-\log K_{\xi,p,\lambda}(t)+t$ is non-decreasing with respect to $t \in [0,+\infty)$, which implies that

$$-\log K_{\xi,p,\lambda}(t)+t \geq -\log K_{\xi,p,\lambda}(0), \quad \forall t \in [0,+\infty). \tag{5.1}$$

Since $f \in A^2(\{\Psi_1 < 0\})$, following from inequality (5.1), we get that

$$\|f\|^2_{p,\lambda,t} \geq \frac{|\xi \cdot f|^2}{K_{\xi,p,\lambda}(t)} \geq e^{-t}\frac{|\xi \cdot f|^2}{K_{\xi,p,\lambda}(0)}, \quad \forall t \in [0,+\infty).$$

In addition, since $f_o \notin J_p$, according to Lemma 2.10, we have

$$\begin{aligned} \|f\|^2_{p,\lambda,t} &\geq \sup_{\substack{\xi \in A^2(\{\Psi_1<0\})^* \setminus \{0\} \\ \xi|_{A^2(\{\Psi_1<0\})\cap J_p} \equiv 0}} e^{-t}\frac{|\xi \cdot f|^2}{K_{\xi,p,\lambda}(0)} \\ &= e^{-t}C(p\Psi_1, 0, J_p, f), \quad \forall t \in [0,+\infty). \end{aligned} \tag{5.2}$$

Note that for any $t \in [0,+\infty)$,

$$\|f\|^2_{p,\lambda,t} = \int_{\{p\Psi_1<-t\}} |f|^2 + \int_{\{0>p\Psi_1\geq -t\}} |f|^2 e^{-\lambda(p\Psi_1+t)}. \tag{5.3}$$

Since for any $\lambda > 0$,

$$\int_{\{0>p\Psi_1\geq -t\}} |f|^2 e^{-\lambda(p\Psi_1+t)} \leq \int_{\{0>p\Psi_1\geq -t\}} |f|^2 < +\infty,$$

and $\lim_{\lambda\to+\infty} e^{-\lambda(p\Psi_1+t)} = 0$ on $\{0 > p\Psi_1 \geq -t\}$, according to Lebesgue's dominated convergence theorem, we have

$$\lim_{\lambda\to+\infty} \int_{\{0>p\Psi_1\geq -t\}} |f|^2 e^{-\lambda(p\Psi_1+t)} = 0.$$

Then equality (5.3) implies

$$\lim_{\lambda\to+\infty} \|f\|^2_{p,\lambda,t} = \int_{\{p\Psi_1<-t\}} |f|^2, \quad \forall t \in [0,+\infty). \tag{5.4}$$

Letting $\lambda \to +\infty$ in inequality (5.2), we get that for any $t \in [0,+\infty)$,

$$\int_{\{p\Psi_1<-t\}} |f|^2 \geq e^{-t}C(p\Psi_1, 0, J_p, f). \tag{5.5}$$

Note that $\{p\Psi_1 < 0\} = \{\Psi_1 < 0\}$ and $J_p \subset I_+(\Psi_1)_o$ for any $p \in (1,2)$. Then we have

$$C(p\Psi_1, 0, J_p, f) \geq C(\Psi_1, 0, I_+(\Psi_1)_o, f), \quad \forall p \in (1,2).$$

Since $\int_{\{\Psi_1<0\}}|f|^2<+\infty$, it follows from Lebesgue's dominated convergence theorem and inequality (5.5) that

$$\begin{aligned}&\int_{\{\Psi_1<-t\}}|f|^2\\&=\lim_{p\to 1+0}\int_{\{p\Psi_1<-t\}}|f|^2\\&\geq \limsup_{p\to 1+0} e^{-t}C(p\Psi_1,0,J_p,f)\\&\geq e^{-t}C(\Psi_1,0,I_+(\Psi_1)_o,f),\quad \forall t\in[0,+\infty).\end{aligned}\tag{5.6}$$

Let $r=e^{-t/2}$, and we get that

$$\frac{1}{r^2}\int_{\{\Psi_1<2\log r\}}|f|^2\geq C(\Psi_1,0,I_+(\Psi_1)_o,f),\quad \forall r\in(0,1].\tag{5.7}$$

Note that $C(\Psi_1,0,I_+(\Psi_1)_o,f)=G(0;\Psi_1,I_+(\Psi_1)_o,f)>0$, thus Corollary 1.4 holds. □

6 Proof of Corollary 1.7

In this section, we give the proof of Corollary 1.7.

Proof of Corollary 1.7 Let $\Psi_q:=q\Psi$ for any $q>2a_o^f(\Psi;\varphi_0)\geq 1$. Note that

$$q\Psi=\min\{q\psi+(2\lceil q\rceil-2q)\log|F|-2\log|F^{\lceil q\rceil}|,0\},$$

where $\lceil q\rceil=\min\{m\in\mathbb{Z}:m\geq q\}$. By the definition of $a_o^f(\Psi;\varphi_0)$, we have $f_o\notin I(2q\Psi+\varphi_0)_o$ for any $q>2a_o^f(\Psi;\varphi_0)$. For any fixed $q>2a_o^f(\Psi;\varphi_0)$, $\lambda>0$, let $\xi\in A^2(\{\Psi<0\},e^{-\varphi_0})^*\backslash\{0\}$, such that $\xi|_{A^2(\{\Psi<0\},e^{-\varphi_0})\cap J_q}\equiv 0$, where $J_q:=I(q\Psi+\varphi_0)_o$. Denote by

$$K_{\xi,q,\lambda}(t):=\sup_{\tilde f\in A^2(\{\Psi<0\},e^{-\varphi_0})}\frac{|\xi\cdot\tilde f|^2}{\|\tilde f\|^2_{q,\lambda,t}},$$

where

$$\|\tilde f\|_{q,\lambda,t}:=\left(\int_{\{\Psi<0\}}|\tilde f|^2e^{-\varphi_0-\lambda\max\{q\Psi+t,0\}}\right)^{1/2},$$

and $t\in[0,+\infty)$. Theorem 1.2 tells us that $-\log K_{\xi,q,\lambda}(t)+t$ is non-decreasing with respect to $t\in[0,+\infty)$, which implies that

$$-\log K_{\xi,q,\lambda}(t)+t\geq -\log K_{\xi,q,\lambda}(0),\quad \forall t\in[0,+\infty).\tag{6.1}$$

Since $\int_{\{\Psi<0\}}|f|^2e^{-\varphi_0} \le \int_{\{\Psi<0\}}|f|^2e^{-\varphi_0-\Psi} < +\infty$, following from inequality (6.1), we get that

$$\|f\|^2_{q,\lambda,t} \ge \frac{|\xi\cdot f|^2}{K_{\xi,q,\lambda}(t)} \ge e^{-t}\frac{|\xi\cdot f|^2}{K_{\xi,q,\lambda}(0)}, \quad \forall t\in[0,+\infty).$$

According to Lemma 2.10, we have

$$\begin{aligned}\|f\|^2_{q,\lambda,t} &\ge \sup_{\substack{\xi\in A^2(\{\Psi<0\},e^{-\varphi_0})^*\setminus\{0\}\\ \xi|_{A^2(\{\Psi<0\},e^{-\varphi_0})\cap J_q}\equiv 0}} e^{-t}\frac{|\xi\cdot f|^2}{K_{\xi,q,\lambda}(0)}\\ &= e^{-t}C(q\Psi,\varphi_0,J_q,f), \quad \forall t\in[0,+\infty).\end{aligned} \tag{6.2}$$

Note that for any $t\in[0,+\infty)$,

$$\|f\|^2_{q,\lambda,t} = \int_{\{q\Psi<-t\}}|f|^2e^{-\varphi_0} + \int_{\{0>q\Psi\ge -t\}}|f|^2e^{-\varphi_0-\lambda(q\Psi+t)}. \tag{6.3}$$

Since for any $\lambda>0$,

$$\int_{\{0>q\Psi\ge -t\}}|f|^2e^{-\varphi_0-\lambda(q\Psi+t)} \le \int_{\{0>q\Psi\ge -t\}}|f|^2e^{-\varphi_0} < +\infty,$$

and $\lim_{\lambda\to+\infty}e^{-\lambda(q\Psi+t)}=0$ on $\{0>q\Psi\ge -t\}$, according to Lebesgue's dominated convergence theorem, we have

$$\lim_{\lambda\to+\infty}\int_{\{0>q\Psi\ge -t\}}|f|^2e^{-\varphi_0-\lambda(q\Psi+t)}=0.$$

Then equality (6.3) implies

$$\lim_{\lambda\to+\infty}\|f\|^2_{q,\lambda,t} = \int_{\{q\Psi<-t\}}|f|^2e^{-\varphi_0}, \quad \forall t\in[0,+\infty). \tag{6.4}$$

Thus letting $\lambda\to+\infty$ in inequality (6.2), we get that for any $t\in[0,+\infty)$,

$$\int_{\{q\Psi<-t\}}|f|^2e^{-\varphi_0} \ge e^{-t}C(q\Psi,\varphi_0,J_q,f) = e^{-t}C(\Psi,\varphi_0,J_q,f). \tag{6.5}$$

Note that $J_q\subset I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o$ for any $q>2a_o^f(\Psi;\varphi_0)$. Then we have

$$C(\Psi,\varphi_0,J_q,f)\ge C(\Psi,\varphi_0,I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o,f), \quad \forall q>2a_o^f(\Psi;\varphi_0).$$

Then it follows from inequality (6.5) that

$$\int_{\{q\Psi<-t\}}|f|^2e^{-\varphi_0}\geq e^{-t}C(\Psi,\varphi_0,I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o,f) \tag{6.6}$$

for any $q>2a_o^f(\Psi;\varphi_0)$ and $t\in[0,+\infty)$.

According to Fubini's theorem, we have

$$\begin{aligned}&\int_{\{\Psi<0\}}|f|^2e^{-\varphi_0-\Psi}\\&=\int_{\{\Psi<0\}}\left(|f|^2e^{-\varphi_0}\int_0^{e^{-\Psi}}\mathrm{d}s\right)\\&=\int_0^{+\infty}\left(\int_{\{\Psi<0\}\cap\{s<e^{-\Psi}\}}|f|^2e^{-\varphi_0}\right)\mathrm{d}s\\&=\int_{-\infty}^{+\infty}\left(\int_{\{q\Psi<-qt\}\cap\{\Psi<0\}}|f|^2e^{-\varphi_0}\right)e^t\,\mathrm{d}t.\end{aligned}$$

Inequality (6.6) implies that for any $q>2a_o^f(\Psi;\varphi_0)$,

$$\begin{aligned}&\int_0^{+\infty}\left(\int_{\{q\Psi<-qt\}\cap\{\Psi<0\}}|f|^2e^{-\varphi_0}\right)e^t\,\mathrm{d}t\\&\geq\int_0^{+\infty}e^{-qt}C(\Psi,\varphi_0,I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o,f)\cdot e^t\,\mathrm{d}t\\&=\frac{1}{q-1}C(\Psi,\varphi_0,I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o,f),\end{aligned}$$

and

$$\begin{aligned}&\int_{-\infty}^{0}\left(\int_{\{q\Psi<-qt\}\cap\{\Psi<0\}}|f|^2e^{-\varphi_0}\right)e^t\,\mathrm{d}t\\&\geq\int_{-\infty}^{0}C(\Psi,\varphi_0,I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o,f)\cdot e^t\,\mathrm{d}t\\&=C(\Psi,\varphi_0,I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o,f).\end{aligned}$$

Then we have

$$\int_{\{\Psi<0\}}|f|^2e^{-\varphi_0-\Psi}\geq\frac{q}{q-1}C(\Psi,\varphi_0,I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o,f) \tag{6.7}$$

for any $q > 2a_o^f(\Psi;\varphi_0)$. Let $q \to 2a_o^f(\Psi;\varphi_0)+0$, then inequality (6.7) also holds for $q \geq 2a_o^f(\Psi;\varphi_0)$. Thus if $q > 1$ satisfying

$$\int_{\{\Psi<0\}} |f|^2 e^{-\varphi_0-\Psi} < \frac{q}{q-1} C(\Psi, \varphi_0, I_+(2a_o^f(\Psi;\varphi_0)\Psi+\varphi_0)_o, f), \tag{6.8}$$

we have $q < 2a_o^f(\Psi;\varphi_0)$, which means that $f_o \in I(q\Psi+\varphi_0)_o$. Proof of Corollary 1.7 is done. □

Acknowledgements We would like to thank Zhitong Mi and Zheng Yuan for checking this paper. We thank the referees for their time and comments. The second named author was supported by National Key R&D Program of China 2021YFA1003100, NSFC-11825101, NSFC-11522101 and NSFC-11431013.

Appendix A

Let D be a bounded domain in $\mathbb{C}^n$, and φ be a plurisubharmonic function on D. Denote by

$$\ell_0 := \{\eta = (\eta_\alpha)_{\alpha\in\mathbb{N}^n} : \eta_\alpha \in \mathbb{C},\ \text{and } \exists k \in \mathbb{N},\ \text{such that } \eta_\alpha = 0,\ \forall\alpha, |\alpha| \geq k\},$$

where for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Let $z_0 \in D$, and $\eta = (\eta_\alpha) \in \ell_0$. For any $f \in \mathcal{O}(D)$, denote by

$$\eta \cdot f := \sum_{\alpha\in\mathbb{N}^n} \eta_\alpha \frac{f^{(\alpha)}(z_0)}{\alpha!}. \tag{A.1}$$

It can be shown that for any $\eta \in \ell_0$, there is a finite constant $C_\eta > 0$, such that

$$|\eta \cdot f|^2 \leq C_\eta \int_D |f|^2 e^{-\varphi},$$

for any $f \in A^2(D, e^{-\varphi})$ (see [1, 2]). Then any $\eta \in \ell_0$ can be seen as an element in $A^2(D, e^{-\varphi})^*$ by equality (A.1). Note that $A^2(D, e^{-\varphi})$ is a Hilbert space. By Riesz representation theorem, there exists $g_\eta \in A^2(D, e^{-\varphi})$, such that

$$\eta \cdot f = \int_D f\overline{g_\eta} e^{-\varphi}, \quad \forall f \in A^2(D, e^{-\varphi}),$$

which induces a map from ℓ_0 to $A^2(D, e^{-\varphi})$. We denote this map by T_{φ,z_0}:

$$\begin{aligned} T_{\varphi,z_0} : \ell_0 &\longrightarrow A^2(D, e^{-\varphi}), \\ \eta &\longmapsto g_\eta. \end{aligned}$$

We state the following lemma.

Lemma A.1 *The image of T_{φ,z_0} is dense in $A^2(D, e^{-\varphi})$, i.e., $\overline{T_{\varphi,z_0}(\ell_0)} = A^2(D, e^{-\varphi})$, in the topology of $A^2(D, e^{-\varphi})$.*

Remark A.2 It follows from Lemma A.1 that ℓ_0 is dense in $A^2(D, e^{-\varphi})^*$, in the strong topology of $A^2(D, e^{-\varphi})^*$.

Proof of Lemma A.1 We introduce some notations before the proof.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, denote by $\alpha < \beta$, if $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ but there exists k with $1 \le k \le n$, such that $\alpha_1 = \beta_1, \ldots, \alpha_{k-1} = \beta_{k-1}$, $\alpha_k < \beta_k$. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, denote $\alpha! := \alpha_1! \cdots \alpha_n!$.

We may assume that $z_0 = o \in D$ is the origin in $\mathbb{C}^n$, and denote T_{φ,z_0} by T. We will choose a countable sequence $\{\eta[\alpha]\}$ of elements in ℓ_0, such that

$$\overline{\operatorname{span}\{T(\eta[\alpha])\}} = A^2(D, e^{-\varphi}),$$

which yields Lemma A.1. For any $\alpha \in \mathbb{N}^n$, we set $\eta[\alpha] \in \ell_0$, with $\eta[\alpha]_\gamma = \gamma! \cdot b^\alpha_\gamma \in \mathbb{C}$ (which will be determined in the following discussions) for any $\gamma < \alpha$, $\eta[\alpha]_\alpha = \alpha!$, and $\eta[\alpha]_\gamma = 0$ for any $\gamma > \alpha$. Denote by $g_\alpha := g_{\eta[\alpha]} = T(\eta[\alpha]) \in A^2(D, e^{-\varphi})$. We will choose b^α_γ such that

$$\begin{aligned} &\int_D g_\alpha \overline{g_\beta} e^{-\varphi} = 0, \quad \forall \alpha \ne \beta; \\ &g^{(\gamma)}_\beta(o) = 0, \quad \forall \gamma < \beta; \\ &g^{(\alpha)}_\alpha(o) = \int_D |g_\alpha|^2 e^{-\varphi}, \quad \forall \alpha. \end{aligned} \tag{A.2}$$

And for any $\alpha \in \mathbb{N}^n$, denote by $\alpha \in S_1$ if $g_\alpha \equiv 0$. Otherwise we denote by $\alpha \in S_2$.

First, for $\alpha = (0, \ldots, 0)$, we set

$$\eta((0, \ldots, 0)) = (1, 0, \ldots, 0, \ldots) \in \ell_0.$$

Denote by

$$T((1, 0, \ldots, 0, \ldots)) = g_{(0,\ldots,0)} \in A^2(D, e^{-\varphi}),$$

then for any $f \in A^2(D, e^{-\varphi_0})$,

$$f(o) = \int_D f \overline{g_{(0,\ldots,0)}} e^{-\varphi}. \tag{A.3}$$

Letting $f = g_{(0,\ldots,0)}$ in equality (A.3), we get

$$g_{(0,\ldots,0)}(o) = \int_D |g_{(0,\ldots,0)}|^2 e^{-\varphi}.$$

For some $\alpha \in \mathbb{N}^n$, we assume that for any $\beta < \alpha$, $\eta(\beta) \in \ell_0$ (i.e., the complex number sequence $\{b_\gamma^\beta\}$) has been chosen to satisfy

$$\begin{aligned}
&\int_D g_{\beta_1}\overline{g_{\beta_2}}e^{-\varphi} = 0, \quad \forall \beta_1 \neq \beta_2,\ \beta_1, \beta_2 < \alpha; \\
&g_\beta^{(\gamma)}(o) = 0, \quad \forall \gamma < \beta; \\
&g_\beta^{(\beta)}(o) = \int_D |g_\beta|^2 e^{-\varphi}, \quad \forall \beta < \alpha.
\end{aligned} \tag{A.4}$$

By the choice of $\eta[\alpha]$, for any $f \in A^2(D, e^{-\varphi})$,

$$\sum_{\gamma<\alpha} b_\gamma^\alpha f^{(\gamma)}(o) + f^{(\alpha)}(o) = \int_D f\overline{g_\alpha}e^{-\varphi}. \tag{A.5}$$

Since we want

$$\int_D g_\beta \overline{g_\alpha} e^{-\varphi} = 0, \quad \forall \beta < \alpha,$$

then there must be

$$\sum_{\gamma<\alpha} b_\gamma^\alpha g_\beta^{(\gamma)}(o) + g_\beta^{(\alpha)}(o) = 0, \quad \forall \beta < \alpha,$$

which is equivalent to

$$b_\beta^\alpha g_\beta^{(\beta)}(o) + \sum_{\beta<\gamma<\alpha} b_\gamma^\alpha g_\beta^{(\gamma)}(o) = -g_\beta^{(\alpha)}(o), \quad \forall \beta < \alpha, \tag{A.6}$$

by the choice of $\{g_\beta\}$. Equality (A.6) can be seen as a linear equations system for $(b_\gamma^\alpha)_{\gamma<\alpha}$. Note that for $\beta \in S_1$, $g_\beta^{(\beta)} = 0$. We set $b_\beta^\alpha = 0$ for any $\beta < \alpha$ with $\beta \in S_1$. And we also note that

$$\prod_{\beta<\alpha,\ \beta\in S_2} g_\beta^{(\beta)}(o) = \prod_{\beta<\alpha,\ \beta\in S_2} \int_D |g_\beta|^2 e^{-\varphi} > 0.$$

It means that there exists $(b_\gamma^\alpha)_{\gamma<\alpha}$ satisfying equality (A.6), where $b_\beta^\alpha = 0$ for any $\beta < \alpha$ with $\beta \in S_1$. Now suppose that $(b_\gamma^\alpha)_{\gamma<\alpha}$ is the solution as we described above. Then g_α satisfies

$$\int_D g_\beta \overline{g_\alpha} e^{-\varphi} = 0, \quad \forall \beta < \alpha.$$

Note that in the process of induction, for any $\beta < \alpha$, we have

$$\sum_{\gamma<\beta} b_\gamma^\beta f^{(\gamma)}(o) + f^{(\beta)}(o) = \int_D f\overline{g_\beta}e^{-\varphi}.$$

Let $f = g_\alpha$, then we get

$$\sum_{\gamma<\beta} b_\gamma^\beta g_\alpha^{(\gamma)}(o) + g_\alpha^{(\beta)}(o) = \int_D g_\alpha\overline{g_\beta}e^{-\varphi}, \quad \forall\beta < \alpha. \tag{A.7}$$

In equality (A.7), by induction, for any $\beta < \alpha$,

$$g_\alpha^{(\beta)}(o) = \int_D g_\alpha\overline{g_\beta}e^{-\varphi} = \overline{\int_D g_\beta\overline{g_\alpha}e^{-\varphi}} = 0. \tag{A.8}$$

In addition, in equality (A.5), letting $f = g_\alpha$, we have

$$\sum_{\gamma<\alpha} b_\gamma^\alpha g_\alpha^{(\gamma)}(o) + g_\alpha^{(\alpha)}(o) = \int_D |g_\alpha|^2 e^{-\varphi}.$$

Then it follows from equality (A.8) that

$$g_\alpha^{(\alpha)}(o) = \int_D |g_\alpha|^2 e^{-\varphi}.$$

Now, by induction, we can choose out $\eta[\alpha] \in \ell_0$ for any $\alpha \in \mathbb{N}^n$ satisfying what we described before equality (A.2), and $\{g_\alpha\}_{\alpha\in\mathbb{N}^n} \subset A^2(D, e^{-\varphi})$ satisfies equality (A.2). In the following, we prove that

$$\overline{\operatorname{span}\{g_\alpha : \alpha \in S_2\}} = A^2(D, e^{-\varphi}). \tag{A.9}$$

For any $f \in A^2(D, e^{-\varphi})$, let $\{a_\alpha\}_{\alpha\in\mathbb{N}^n}$ be a sequence of complex numbers (which will be determined in the following). Denote

$$f_\alpha(z) = \sum_{\beta\leq\alpha} a_\beta g_\beta(z). \tag{A.10}$$

We choose a_β for $\beta \leq \alpha$, such that

$$f_\alpha^{(\beta)}(o) = f^{(\beta)}(o). \tag{A.11}$$

First, for $\beta = (0, \ldots, 0)$, if $(0, \ldots, 0) \in S_2$, we can see that

$$a_{(0,\ldots,0)} = \frac{f(o)}{g_{(0,\ldots,0)}(o)}$$

satisfies equality (A.11), and if $(0,\ldots,0)\in S_1$, we have $f(o)=0$ according to equality (A.3), which implies $a_{(0,\ldots,0)}=0$ satisfies inequality (A.11).

Next, assume that for some $\gamma\le\alpha$, all $\beta<\gamma$ have been chosen to satisfy equality (A.11). According to equality (A.10) and equality (A.2), we have

$$f_\alpha^{(\gamma)}(o)=\sum_{\beta<\gamma}a_\beta g_\beta^{(\gamma)}(o)+a_\gamma g_\gamma^{(\gamma)}(o). \tag{A.12}$$

Then,

$$f_\alpha^{(\gamma)}(o)=f^{(\gamma)}(o)\Leftrightarrow f^{(\gamma)}(o)=\sum_{\beta<\gamma}a_\beta g_\beta^{(\gamma)}(o)+a_\gamma g_\gamma^{(\gamma)}(o). \tag{A.13}$$

Note that for $\gamma\in S_2$, $g_\gamma^{(\gamma)}(o)=\int_D|g_\gamma|^2e^{-\varphi}>0$, then we can choose

$$a_\gamma=(g_\gamma^{(\gamma)}(o))^{-1}\left(f^{(\gamma)}(o)-\sum_{\beta<\gamma}a_\beta g_\beta^{(\gamma)}(o)\right)$$

to satisfy equality (A.11). If $\gamma\in S_1$, following from equality (A.5), we have

$$\begin{aligned}
f^{(\gamma)}(o)&=-\sum_{\beta<\gamma}b_\beta^\gamma f^{(\beta)}(o)=-\sum_{\beta<\gamma}b_\beta^\gamma f_\alpha^{(\beta)}(o)\\
&=-\sum_{\beta<\gamma}b_\beta^\gamma\left(\sum_{\beta'\le\beta}a_{\beta'}g_{\beta'}^{(\beta)}(o)\right)\\
&=-\sum_{\beta'<\gamma}a_{\beta'}\left(\sum_{\beta'\le\beta<\gamma}b_\beta^\gamma g_{\beta'}^{(\beta)}(o)\right)\\
&=\sum_{\beta'<\gamma}a_{\beta'}g_{\beta'}^{(\gamma)}(o).
\end{aligned}$$

Then we can choose $a_\gamma=0$ according to equation (A.13).

Finally, by induction, the sequence $\{a_\beta\}$ can be chosen to satisfy equality (A.11). In addition, we have $a_\beta=0$ for $\beta\in S_1$.

Now we have $(f-f_\alpha)^{(\beta)}(o)=0$ for any $\beta\le\alpha$. Then it follows from equality (A.5) that

$$\int_D(f-f_\alpha)\overline{g_\alpha}e^{-\varphi}=0,$$

which means that

$$\int_D f\overline{g_\alpha}e^{-\varphi}=\int_D f_\alpha\overline{g_\alpha}e^{-\varphi}=a_\alpha.$$

Then combining with equality (A.2), we get that

$$\sum_{\alpha} |a_\alpha|^2 \int_D |g_\alpha|^2 e^{-\varphi} \leq \int_D |f|^2 e^{-\varphi}.$$

Denote

$$h(z) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha g_\alpha(z),$$

then $h \in A^2(\Omega, e^{-\varphi})$. In addition, we have

$$h(z) = \lim_{|\alpha| \to +\infty} f_\alpha \Rightarrow h^{(\beta)}(o) = \lim_{|\alpha| \to +\infty} f_\alpha^{(\beta)}(o) = f^{(\beta)}(o)$$

for any $\beta \in \mathbb{N}^n$, inducing that $f \equiv h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g_\alpha(z)$. Note that $a_\alpha = 0$ for $\alpha \in S_1$.

By the arbitrariness of $f \in A^2(D, e^{-\varphi})$, we get

$$\overline{\mathrm{span}\{g_\alpha : \alpha \in S_2\}} = A^2(D, e^{-\varphi}),$$

which implies

$$\overline{\mathrm{span}\{T(\eta[\alpha])\}} = A^2(D, e^{-\varphi}).$$

Then Lemma A.1 holds. □

For any $z \in D$, denote by the functional $e_z \in A^2(D, e^{-\varphi})^*$, where

$$\begin{aligned} e_z \ : \ A^2(D, e^{-\varphi}) &\longrightarrow \mathbb{C}, \\ f &\longmapsto f(z). \end{aligned}$$

Then it follows from Riesz representation theorem that there exists $\phi_z \in A^2(D, e^{-\varphi})$ such that

$$e_z \cdot f = \int_D f \overline{\phi_z} e^{-\varphi}, \quad \forall f \in A^2(D, e^{-\varphi}).$$

According to the following lemma, we can also prove Lemma 2.8.

Lemma A.3 *In the topology of $A^2(D, e^{-\varphi})$, we have*

$$\overline{\mathrm{span}\{\phi_z : z \in D\}} = A^2(D, e^{-\varphi}).$$

Proof Denote by

$$H := \overline{\mathrm{span}\{\phi_z : z \in D\}},$$

then H is a closed subspace of $A^2(D, e^{-\varphi})$. If $H \subsetneqq A^2(D, e^{-\varphi})$, there exists $h \in A^2(D, e^{-\varphi})$ such that $h \neq 0$, and

$$\int_D h\overline{\phi_z}e^{-\varphi} = 0, \quad \forall z \in D.$$

However, we have $h(z) = e_z \cdot h = \int_D h\overline{\phi_z}e^{-\varphi} = 0$ for any $z \in D$, inducing that $h \equiv 0$, which is a contradiction. It follows that

$$\overline{\mathrm{span}\{\phi_z : z \in D\}} = A^2(D, e^{-\varphi}).$$ □

Remark A.4 It follows from Lemma A.3 that $\mathrm{span}\{e_z : z \in D\}$ is dense in $A^2(D, e^{-\varphi})^*$, in the strong topology. And in Lemma 2.8, for any $\eta \in \mathrm{span}\{e_z : z \in D\}$ such that

$$\eta = \sum_{k=1}^{N} c_k e_{z_k},$$

where N is a finite positive integer, $c_k \in \mathbb{C}$, and $z_k \in D$ for any k, we have that

$$\eta \cdot f_\tau = \sum_{k=1}^{N} c_k e_{z_k} \cdot f_\tau = \sum_{k=1}^{N} c_k f(\tau, z_k)$$

is holomorphic with respect to τ. Then, with a similar discussion as in the proof of Lemma 2.8, we can see that Lemma 2.8 follows from Lemma A.3.

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