SCIENCE CHINA
Mathematics

• ARTICLES •

https://doi.org/10.1007/s11425-024-2390-2

Concavity property of minimal L^2 integrals with Lebesgue measurable gain VI: Fibrations over products of open Riemann surfaces

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Received April 16, 2024; accepted February 19, 2025

Abstract In this article, we present characterizations of the concavity property of minimal L^2 integrals degenerating to linearity in the case of fibrations over products of open Riemann surfaces. As applications, we obtain characterizations of the holding of equality in optimal jets L^2 extension problem from fibers over products of analytic subsets to fibrations over products of open Riemann surfaces, which implies characterizations of the equality parts of Suita conjecture and extended Suita conjecture for fibrations over products of open Riemann surfaces.

Keywords fibration, multiplier ideal sheaf, minimal L^2 integral, concavity, optimal L^2 extension theorem

MSC(2020) 32D15, 32A10, 32L10, 32U05

Citation: Bao S J, Guan Q A, Yuan Z. Concavity property of minimal L^2 integrals with Lebesgue measurable gain VI: Fibrations over products of open Riemann surfaces. Sci China Math, 2025, 68, https://doi.org/10.1007/s11425-024-2390-2

1 Introduction

The strong openness property of multiplier ideal sheaves [43] (2-dimensional case [50]), i.e., $\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \bigcup_{\epsilon>0}\mathcal{I}((1+\epsilon)\varphi)$ (conjectured by Demailly [14]) has opened the door to new types of approximation techniques, which was used in the study of several complex variables, complex algebraic geometry and complex differential geometry (see, e.g., [5, 8–11, 24, 25, 43, 46, 51, 52, 74–76]), where φ is a plurisubharmonic function of a complex manifold M (see [15]), and the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is defined as the sheaf of germs of holomorphic functions f such that $|f|^2e^{-\varphi}$ is locally integrable (see, e.g., [14, 16, 18, 20, 21, 48, 53–55, 65, 67, 68, 70]).

When $\mathcal{I}(\varphi) = \mathcal{O}$, the strong openness property degenerates to the openness property conjectured by Demailly and Kollár [20]. Berndtsson [4] (2-dimensional case by Favre and Jonsson [22]) proved the openness property by establishing an effectiveness result of the openness property. Stimulated by Berndtsson's effectiveness result, and continuing the proof of the strong openness property [43], Guan

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and Zhou established an effectiveness result of the strong openness property by considering the minimal L^2 integral on the pseudoconvex domain D [45].

Considering the minimal L^2 integrals on the sublevel sets of the weight φ, Guan obtained a sharp version of Guan and Zhou's effectiveness result [29], and established a concavity property of the minimal L^2 integrals on the sublevel sets of the weight φ (with constant gain). The concavity property was applied to study the upper bound of the Bergman kernel, i.e., a proof of Saitoh's conjecture for conjugate Hardy H^2 kernels [30], and equisingular approximations for the multiplier ideal sheaves, i.e., the necessary and sufficient condition of the existence of decreasing equisingular approximations with analytic singularities for the multiplier ideal sheaves with weights $\log(|z_1|^{a_1}+\cdots+|z_n|^{a_n})$ [31].

For smooth gain, Guan [28] (see also [33]) presented the concavity property on Stein manifolds (the weakly pseudoconvex Kähler case was obtained by Guan and Mi [32]). The concavity property [28] (see also [33]) was applied by Guan and Yuan to deduce an optimal support function related to the strong openness property [37] and an effectiveness result of the strong openness property in L^p [39].

For Lebesgue measurable gain, Guan and Yuan [38] obtained the concavity property on Stein manifolds (the weakly pseudoconvex Kähler case was obtained by Guan, Mi and Yuan [34]). The concavity property [38] was applied by Guan and Yuan to deduce a twisted L^p version of the strong openness property [35].

As the linearity is a degenerate case of concavity, a natural problem was posed in [36]:

Problem 1.1 (See [36])**.** *How can one characterize the concavity property degenerating to linearity?*

For 1-dimensional case, Guan and Yuan [38] gave an answer to Problem 1.1 for single point, i.e., for the case where the weights may not be subharmonic (the case of subharmonic weights was answered by Guan and Mi [33]), and Guan and Yuan [36] gave an answer to Problem 1.1 for finite points. For products of open Riemann surfaces, Guan and Yuan [40] gave answers to Problem 1.1 for products of analytic subsets. Recently, Bao, Guan and Yuan [1] gave an answer to Problem 1.1 for fibrations over open Riemann surfaces.

The linearity case of minimal L^2 integrals is closely related to the equality problem in optimal (jets) L^2 extension theorems and generalized Suita conjecture, some related results can be seen in [1,33,34,36,38,40,72].

In the present article, we give answers to Problem 1.1 for fibrations over products of open Riemann surfaces. Using these results, we obtain characterizations of the holding of equality in optimal jets L^2 extension problem from fibers over products of analytic subsets to fibrations over products of open Riemann surfaces, and characterizations of the equality parts of Suita conjecture and extended Suita conjecture for fibrations over products of open Riemann surfaces.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1\leqslant j\leqslant n_1$. Consider an n_2-dimensional weakly pseudoconvex Kähler manifold Y with canonical (holomorphic) line bundle K_Y. Let $n=n_1+n_2$, and define the n-dimensional complex manifold

$$M=\left(\prod_{1\leqslant j\leqslant n_1}\Omega_j\right)\times Y$$

equipped with natural projections π_1, $\pi_{1,j}$ and π_2 from M to $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M.

Let Z_j be a (closed) analytic subset of Ω_j for any $j\in\{1,\dots,n_1\}$, and denote $Z_0:=(\prod_{1\leqslant j\leqslant n_1}Z_j)\times Y\subset M$. Let φ_j ($j\in\{1,\dots,n_1\}$) be a subharmonic function on Ω_j such that $\varphi_j(z)>-\infty$ for any $z\in Z_j$, φ_Y be a plurisubharmonic function on Y. Set

$$\varphi:=\sum_{1\leqslant j\leqslant n_1}\pi_{1,j}^*(\varphi_j)+\pi_2^*(\varphi_Y).$$

Let $\psi<0$ be a plurisubharmonic function on M such that $\psi(z)=-\infty$ for any $z\in Z_0$, c be a positive function on $(0,+\infty)$ satisfying: $\int_0^{+\infty}c(t)\mathrm{e}^{-t}dt<+\infty$; $c(t)\mathrm{e}^{-t}$ is decreasing on $(0,+\infty)$; $c(-\psi)$ has a positive lower bound on any compact subset of $M\backslash Z_0$. Choosing arbitrary holomorphic $(n,0)$ form f on a neighborhood of Z_0, we denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):(\tilde{f}-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z \text{ for any } z\in Z_0 \right.$$
$$\left. \&\ \tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right\}$$

by $G(t;c)$ (without misunderstanding, we denote $G(t;c)$ by $G(t)$), where $t\in[0,+\infty)$ and $|f|^2:=\sqrt{-1}^{n^2}f\wedge\bar{f}$ for any $(n,0)$ form f.

Theorem 1.2 (See [34]). *$G(h^{-1}(r))$ is concave with respect to r, where $h(t)=\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds$ for any $t\geqslant 0$.*

In the following section, we present the characterizations of the concavity of $G(h^{-1}(r))$ degenerating to linearity.

1.1 Main results

We recall some notations (see [26], see also [34, 38, 44]). Let $P_j:\Delta\to\Omega_j$ be the universal covering from unit disc Δ to Ω_j. we call the holomorphic function f (resp. holomorphic $(1,0)$ form F) on Δ a multiplicative function (resp. multiplicative differential (Prym differential)), if there is a character χ, which is the representation of the fundamental group of Ω_j, such that $g^*(f)=\chi(g)f$ (resp. $g^*(F)=\chi(g)F$), where $|\chi|=1$ and g is an element of the fundamental group of Ω. Denote the set of such kinds of f (resp. F) by $\mathcal{O}^\chi(\Omega_j)$ (resp. $\Gamma^\chi(\Omega_j)$).

It is known that for any harmonic function u on Ω_j, there exists a $\chi_{j,u}$ (called character associated to u) and a multiplicative function $f_u\in\mathcal{O}^{\chi_{j,u}}(\Omega_j)$, such that $|f_u|=P_j^*(\mathrm{e}^u)$. If $u_1-u_2=\log|f|$, then $\chi_{j,u_1}=\chi_{j,u_2}$, where u_1 and u_2 are harmonic functions on Ω_j and f is a holomorphic function on Ω_j. Let $z_j\in\Omega_j$. Recall that for the Green function $G_{\Omega_j}(z,z_j)$, there exist a χ_{j,z_j} and a multiplicative function $f_{z_j}\in\mathcal{O}^{\chi_{j,z_j}}(\Omega_j)$, such that $|f_{z_j}(z)|=P_j^*\left(\mathrm{e}^{G_{\Omega_j}(z,z_j)}\right)$ (see [69]).

Firstly, we consider the case where Z_0 is the product of a single point in $\prod_{1\leqslant j\leqslant n_1}\Omega_j$ and Y.

Let $Z_0=\{z_0\}\times Y=\{(z_1,\ldots,z_{n_1})\}\times Y\subset M$. Let

$$\psi=\max_{1\leqslant j\leqslant n_1}\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\},$$

where p_j is positive real number for $1\leqslant j\leqslant n_1$. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j\in\Omega_j$ satisfying $w_j(z_j)=0$. Denote $V_0:=\prod_{1\leqslant j\leqslant n_1}V_{z_j}$, $w:=(w_1,\ldots,w_{n_1})$ is a local coordinate on V_0 of $z_0\in\prod_{1\leqslant j\leqslant n_1}\Omega_j$ and $dw:=dw_1\wedge\ldots\wedge dw_{n_1}$. Denote $E:=\{(\alpha_1,\ldots,\alpha_{n_1}):\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_j}=1\ \&\ \alpha_j\in\mathbb{Z}_{\geqslant 0}\}$. Let f be a holomorphic $(n,0)$ form on $V_0\times Y\subset M$.

We present a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the case $Z_0=\{z_0\}\times Y$.

Theorem 1.3. *Assume that $G(0)\in(0,+\infty)$. $G(h^{-1}(r))$ is linear with respect to $r\in(0,\int_0^{+\infty}c(t)\mathrm{e}^{-t}dt]$ if and only if the following statements hold:*

(1) $f=\sum_{\alpha\in E}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha)+g_0$ on $V_0\times Y$, where g_0 is a holomorphic $(n,0)$ form on $V_0\times Y$ satisfying $(g_0,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$ and f_α is a holomorphic $(n_2,0)$ form on Y such that $\sum_{\alpha\in E}\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi_Y}\in(0,+\infty)$;

(2) $\varphi_j=2\log|g_j|+2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j)\neq 0$ and u_j is a harmonic function on Ω_j for any $1\leqslant j\leqslant n_1$;

(3) $\chi_{j,z_j}^{\alpha_j+1}=\chi_{j,-u_j}$ for any $j\in\{1,2,\ldots,n_1\}$ and $\alpha\in E$ satisfying $f_\alpha\not\equiv 0$.

Remark 1.4. Lemma 2.26 shows that the above result also holds when we replace sheaf $\mathcal{I}(\varphi+\psi)$ (in the definition of $G(t)$ and statement (1) in Theorem 1.3) by $\mathcal{I}(\psi)$.

Let $c_j(z)$ be the logarithmic capacity (see [64]) on Ω_j, which is locally defined by

$$c_j(z_j):=\exp\lim_{z\to z_j}(G_{\Omega_j}(z,z_j)-\log|w_j(z)|).$$

When the concavity property of $G(h^{-1}(r))$ degenerates to linearity, there exists a unique holomorphic $(n,0)$ form F on M such that $(F-f,z) \in (\mathcal{O}(K_M))_z \otimes \mathcal{I}(\varphi+\psi)_z$ for any $z \in Z_0$ and $G(t) = \int_{\{\psi<-t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)$ for any t (see [34]). The following remark gives the expression of the unique "F".

Remark 1.5. When the three statements in Theorem 1.3 hold,

$$\sum_{\alpha\in E} c_\alpha \left(\wedge_{1\leqslant j\leqslant n_1} \pi_{1,j}^* \left(g_j(P_j)_* \left(f_{u_j} f_{z_j}^{\alpha_j} df_{z_j}\right)\right)\right) \wedge \pi_2^*(f_\alpha)$$

is the unique holomorphic $(n,0)$ form F on M such that $(F-f,z) \in (\mathcal{O}(K_M))_z \otimes \mathcal{I}(\varphi+\psi)_z$ for any $z \in Z_0$ and

$$\begin{aligned} G(t) &= \int_{\{\psi<-t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) \\ &= \left(\int_t^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \sum_{\alpha\in E} \frac{(2\pi)^{n_1} \mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1} \varphi_j(z_j)}}{\prod_{1\leqslant j\leqslant n_1} (\alpha_j+1) c_j(z_j)^{2\alpha_j+2}} \int_Y |f_\alpha|^2 \mathrm{e}^{-\varphi_Y} \end{aligned}$$

for any $t \geqslant 0$, where c_α is a constant such that $c_\alpha = \prod_{1\leqslant j\leqslant n_1} \left(\lim_{z\to z_j} \frac{w_j^{\alpha_j} dw_j}{g_j(P_j)_*\left(f_{u_j} f_{z_j}^{\alpha_j} df_{z_j}\right)}\right)$ for any $\alpha \in E$.

We prove Remark 1.5 in Section 3.1.

Secondly, we consider the case where Z_0 is the fibration of product of finite points in Ω_j.

Let $Z_j = \{z_{j,1}, \ldots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1,\ldots,n_1\}$, where m_j is a positive integer. Let

$$\psi = \max_{1\leqslant j\leqslant n_1} \left\{\pi_{1,j}^*\left(2\sum_{1\leqslant k\leqslant m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\},$$

where $p_{j,k}$ is a positive real number. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1,\ldots,n_1\}$ and $k \in \{1,\ldots,m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote $I_1 := \{(\beta_1,\ldots,\beta_{n_1}) : 1 \leqslant \beta_j \leqslant m_j$ for any $j \in \{1,\ldots,n_1\}\}$ and $V_\beta := \prod_{1\leqslant j\leqslant n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1,\ldots,\beta_{n_1}) \in I_1$. Then $w_\beta := (w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}}) \in \prod_{1\leqslant j\leqslant n_1} \Omega_j$ satisfying $w_\beta(z_\beta) = 0$, and denote $dw_\beta := dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}$.

Let $\beta^* = (1,\ldots,1) \in I_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geqslant 0}^{n_1}$. Denote $E' := \left\{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. Let f be a holomorphic $(n,0)$ form on $\bigcup_{\beta\in I_1} V_\beta \times Y$ satisfying $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{\beta^*}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*}}\right) + \sum_{\alpha\in E'} \pi_1^*(w_{\beta^*}^\alpha dw_{\beta^*}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2,0)$ forms on Y.

We present a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the case where Z_j is a set of finite points.

Theorem 1.6. *Assume that $G(0) \in (0,+\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)\mathrm{e}^{-s} ds]$ if and only if the following statements hold:*

(1) *$\varphi_j = 2\log|g_j| + 2u_j$ for any $j \in \{1,\ldots,n_1\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k}) \neq 0$ for any $k \in \{1,\ldots,m_j\}$;*

(2) *There exists a nonnegative integer $\gamma_{j,k}$ for any $j \in \{1,\ldots,n_1\}$ and $k \in \{1,\ldots,m_j\}$, which satisfies that $\prod_{1\leqslant k\leqslant m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ and $\sum_{1\leqslant j\leqslant n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ for any $\beta \in I_1$;*

(3) *$f = \pi_1^*\left(c_\beta \left(\prod_{1\leqslant j\leqslant n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}}\right) dw_\beta\right) \wedge \pi_2^*(f_0) + g_\beta$ on $V_\beta \times Y$ for any $\beta \in I_1$, where c_β is a constant, $f_0 \not\equiv 0$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y |f_0|^2 \mathrm{e}^{-\varphi_2} < +\infty$, and g_β is a holomorphic $(n,0)$ form on $V_\beta \times Y$ such that $(g_\beta, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))_z$ for any $z \in \{z_\beta\} \times Y$;*

(4) *$c_\beta \prod_{1\leqslant j\leqslant n_1} \left(\lim_{z\to z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\leqslant k\leqslant m_j} f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\leqslant k\leqslant m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)}\right) = c_0$ for any $\beta \in I_1$, where $c_0 \in \mathbb{C}\backslash\{0\}$ is a constant independent of β.*

Denote

$$c_{j,k} := \exp \lim_{z\to z_{j,k}} \left(\frac{\sum_{1\leqslant k_1\leqslant m_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log|w_{j,k}(z)| \right)$$

for any $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, m_j\}$.

Remark 1.7. When the four statements in Theorem 1.6 hold,

$$c_0 \left(\wedge_{1\leqslant j\leqslant n_1} \pi_{1,j}^* \left(g_j(P_j)_* \left(f_{u_j} \left(\prod_{k=1}^{m_j} f_{z_{j,k}}^{\gamma_{j,k}+1} \right) \left(\sum_{k=1}^{m_j} p_{j,k} \frac{df_{z_{j,k}}}{f_{z_{j,k}}} \right) \right) \right) \right) \wedge \pi_2^*(f_0)$$

is the unique holomorphic $(n,0)$ form F on M such that $(F - f, z) \in (\mathcal{O}(K_M))_z \otimes \mathcal{I}(\varphi + \psi)_z$ for any $z \in Z_0$ and

$$\begin{aligned} G(t) &= \int_{\{\psi<-t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) \\ &= \left(\int_t^{+\infty} c(s)\mathrm{e}^{-s} ds \right) \sum_{\beta\in I_1} \frac{|c_\beta|^2 (2\pi)^{n_1} \mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1\leqslant j\leqslant n_1} (\gamma_{j,\beta_j}+1) c_{j,\beta_j}^{2\gamma_{j,\beta_j}+2}} \int_Y |f_0|^2 \mathrm{e}^{-\varphi_Y} \end{aligned}$$

for any $t \geqslant 0$.

We prove Remark 1.7 in Section 4.

Finally, let us discuss the case where Z_0 is the fibration of product of discrete subset of Ω_j (maybe infinite points).

Let $Z_j = \{z_{j,k} : 1 \leqslant k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geqslant 2} \cup \{+\infty\}$. Let $p_{j,k}$ be a positive number for any $1 \leqslant j \leqslant n_1$ and $1 \leqslant k < \tilde{m}_j$ such that $\sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j. Let

$$\psi = \max_{1\leqslant j\leqslant n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}.$$

Assume that $\limsup_{t\to+\infty} c(t) < +\infty$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \ldots, n_1\}$ and $1 \leqslant k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Define $\tilde{I}_1 := \{(\beta_1, \ldots, \beta_{n_1}) : 1 \leqslant \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \ldots, n_1\}\}$, $V_\beta := \prod_{1\leqslant j\leqslant n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1\leqslant j\leqslant n_1} \Omega_j$.

Let $\beta^* = (1, \ldots, 1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \ldots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geqslant 0}^{n_1}$. Define $E' := \Big\{ \alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}} \Big\}$. Let f be a holomorphic $(n,0)$ form on $\bigcup \beta \in I_1 V_\beta \times Y$ satisfying $f = \pi_1^* \left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{\beta^*} \right) \wedge \pi_2^* \left(f_{\alpha_{\beta^*}} \right) + \sum_{\alpha\in E'} \pi_1^*(w_{\beta^*}^{\alpha} dw_{\beta^*}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2,0)$ forms on Y.

We present that $G(h^{-1}(r))$ is not linear when there exists $j_0 \in \{1, \ldots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$ as follows.

Theorem 1.8. *If $G(0) \in (0,+\infty)$ and there exists $j_0 \in \{1, \ldots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$, then $G(h^{-1}(r))$ is not linear with respect to $r \in (0, \int_0^{+\infty} c(s)\mathrm{e}^{-s} ds]$.*

Let $Z_j = \{z_{j,k} : 1 \leqslant k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geqslant 2} \cup \{+\infty\}$. Let $p_{j,k}$ be a positive number for any $1 \leqslant j \leqslant n_1$ and $1 \leqslant k < \tilde{m}_j$ such that $\sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j. Let

$$\psi = \max_{1\leqslant j\leqslant n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}.$$

Let $M_1 \subset M$ be an n-dimensional weakly pseudoconvex Kähler manifold satisfying that $Z_0 \subset M_1$. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0 \subset M_1$ of Z_0. Replace M in the definition of $G(t)$ by M_1, and the concavity property of $G(h^{-1}(r))$ also holds (see [34]).

Proposition 1.9. *If $G(0) \in (0,+\infty)$ and $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$, we have $\mu(M\backslash M_1) = 0$, where μ is the Lebesgue measure on M.*

1.2 Applications

In this section, we give some applications on equality problem in optimal jets L^2 extension theorems and generalized Suita conjecture.

1.2.1 *Equality problem in optimal jets L^2 extension theorems*

The celebrated Ohsawa-Takegoshi L^2 extension theorem [61] had many generalizations and applications in the study of several complex variables and complex geometry (see [?, ?, 2, 12, 13, 17, 19, 56–58, 60, 65, 66], etc.).

In [47] and [78], Guan, Zhou and Zhu introduced a method of undetermined functions to study the L^2 extension problem with optimal estimate. Blocki [7] developed the equation in this method and obtained an optimal L^2 extension theorem on bounded pseudoconvex domains. Using the method of undetermined functions, Guan and Zhou [44] (see also [41, 42]) established an optimal L^2 extension theorem on Stein manifolds. In [74, 75], Zhou and Zhu proved optimal L^2 extension theorems on weakly pseudoconvex Kähler manifolds.

In [62], Popovici proved a jets L^2 extension theorem on weakly pseudoconvex Kähler manifolds, which generalized the Ohsawa-Takegoshi L^2 extension theorem [61]. After that, some jets versions of L^2 extension theorems (with optimal estimates) in different settings were obtained (see, e.g., [17, 49, 63, 77]).

In this section, we focus on the equality problem in optimal jets L^2 extension theorems, which is explore the characterizations for the holding of the equality in optimal jets L^2 extension theorems. Some related results about this problem can be seen in [1, 33, 34, 36, 38, 40].

Let Ω_j, Y, M, Z_0, π_1, $\pi_{1,j}$ and π_2 be as in Theorem 1.2. Let $M_1 \subset M$ be an n-dimensional complex manifold satisfying that $Z_0 \subset M_1$. Assume that $\overline{M_1}^\circ = M_1$, where $\overline{M_1}^\circ$ is the inner set of the closure set in M.

Firstly, we consider the case where Z_0 is the fibration of a single point in $\prod_{1\leqslant j\leqslant n_1}\Omega_j$.

Let $Z_0 = \{z_0\}\times Y \subset M_1$, where $z_0 = (z_1,\ldots,z_{n_1}) \in \prod_{1\leqslant j\leqslant n_1}\Omega_j$. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote $V_0 := \prod_{1\leqslant j\leqslant n_1} V_{z_j}$, and $w := (w_1,\ldots,w_{n_1})$ is a local coordinate on V_0 of $z_0 \in \prod_{1\leqslant j\leqslant n_1}\Omega_j$. Let $\Psi \leqslant 0$ be a plurisubharmonic function on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, and let φ_j be a Lebesgue measurable function on Ω_j such that $\Psi + \sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)$ is plurisubharmonic on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leqslant j\leqslant n_1}\Omega_j$ to Ω_j. Let φ_Y be a plurisubharmonic function on Y. Denote

$$\psi := \max_{1\leqslant j\leqslant n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\} + \pi_1^*(\Psi)$$

and $\varphi := \sum_{1\leqslant j\leqslant n_1}\pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on M, where p_j is a positive real number for $1 \leqslant j \leqslant n_1$. Denote $E := \left\{(\alpha_1,\ldots,\alpha_{n_1}) : \sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_j} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geqslant 0}\right\}$ and $\tilde{E} := \left\{(\alpha_1,\ldots,\alpha_{n_1}) : \sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_j} \geqslant 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geqslant 0}\right\}$. Let

$$f = \sum_{\alpha\in\tilde{E}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha)$$

be a holomorphic $(n,0)$ form on a neighborhood $U_0 \subset (V_0\times Y)\cap M_1$ of Z_0, where f_α is a holomorphic $(n_2,0)$ form on Y. Let $c_j(z)$ be the logarithmic capacity on Ω_j.

We obtain a characterization of the holding of equality in optimal jets L^2 extension problem for the case $Z_0 = \{z_0\}\times Y$.

Theorem 1.10. *Let c be a positive function on $(0,+\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0,+\infty)$. Assume that*

$$\|f\|_{Z_0}^2 := \sum_{\alpha\in E}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_0)}\int_Y |f_\alpha|^2e^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}} \in (0,+\infty).$$

Then there exists a holomorphic $(n,0)$ form F on M_1 satisfying that $(F-f,z)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\max_{1\leqslant j\leqslant n_1}\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\}))_z$ for any $z\in Z_0$ and

$$\int_{M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|_{Z_0}^2.$$

Moreover, equality $\inf\big\{\int_{M_1}|\tilde F|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde F\in H^0(M_1,\mathcal{O}(K_{M_1}))\,\&\,(\tilde F-f,z)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\max_{1\leqslant j\leqslant n_1}\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\}))_z$ *for any* $z\in Z_0\big\}=\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\times\|f\|_{Z_0}^2$ *holds if and only if the following statements hold:*

(1) $M=M_1$ and $\Psi\equiv0$;

(2) $\varphi_j=2\log|g_j|+2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j)\neq0$ and u_j is a harmonic function on Ω_j for any $1\leqslant j\leqslant n_1$;

(3) $\chi_{j,z_j}^{\alpha_j+1}=\chi_{j,-u_j}$ for any $j\in\{1,2,\ldots,n_1\}$ and $\alpha\in E$ satisfying $f_\alpha\not\equiv0$.

Remark 1.11. If $(f_\alpha,y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ for any $y\in Y$ and $\alpha\in\tilde E\backslash E$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\}\right)$ by $\mathcal{I}(\varphi+\psi)$.

We prove Remark 1.11 in Section 6.2.

Remark 1.12. Let f be a holomorphic $(n,0)$ form on a neighborhood of Z_0. It follows from Lemma 2.23 that there exists a sequence of holomorphic $(n_2,0)$ form $\{f_\alpha\}_{\alpha\in\mathbb{Z}_{\geqslant0}^{n_1}}$ on Y such that $f=\sum_{\alpha\in\mathbb{Z}_{\geqslant0}^{n_1}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha)$ on a neighborhood of Z_0. In the setting of Theorem 1.10, we assume that $f_\alpha\equiv0$ for $\alpha\in\mathbb{Z}_{\geqslant0}^{n_1}$ satisfying $\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_j}<1$.

Remark 1.13. Let $\tilde\psi=\max_{1\leqslant j\leqslant n_1}\{2n_1\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\}$. It follows from Lemma 2.18 that $(H_1-H_2,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\tilde\psi))_z$ for any $z\in Z_0$ if and only if $(H_1-H_2)|_{Z_0}=0$, where H_1 and H_2 are holomorphic $(n,0)$ forms on a neighborhood of Z_0. Thus, Theorem 1.10 gives a characterization of the holding of equality in optimal L^2 extension theorem when $p_j=n_1$ for any $1\leqslant j\leqslant n_1$.

Secondly, we consider the case where Z_0 is the fibration of the product of finite points in Ω_j.

Let $Z_j=\{z_{j,1},\ldots,z_{j,m_j}\}\subset\Omega_j$ for any $j\in\{1,\ldots,n_1\}$, where m_j is a positive integer. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}}\Subset\Omega_j$ of $z_{j,k}\in\Omega_j$ satisfying $w_{j,k}(z_{j,k})=0$ for any $j\in\{1,\ldots,n_1\}$ and $k\in\{1,\ldots,m_j\}$, where $V_{z_{j,k}}\cap V_{z_{j,k'}}=\emptyset$ for any j and $k\neq k'$. Denote $I_1:=\{(\beta_1,\ldots,\beta_{n_1}):1\leqslant\beta_j\leqslant m_j$ for any $j\in\{1,\ldots,n_1\}\}$, $V_\beta:=\prod_{1\leqslant j\leqslant n_1}V_{z_{j,\beta_j}}$ and $w_\beta:=(w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta:=(z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}})\in\prod_{1\leqslant j\leqslant n_1}\Omega_j$ for any $\beta=(\beta_1,\ldots,\beta_{n_1})\in I_1$. Then $Z_0=\{(z_\beta,y):\beta\in I_1\,\&\,y\in Y\}\subset M_1$.

Let $\Psi\leqslant0$ be a plurisubharmonic function on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, and let φ_j be a Lebesgue measurable function on Ω_j such that $\Psi+\sum_{1\leqslant j\leqslant n_1}\tilde\pi_j^*(\varphi_j)$ is plurisubharmonic on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, where $\tilde\pi_j$ is the natural projection from $\prod_{1\leqslant j\leqslant n_1}\Omega_j$ to Ω_j. Let φ_Y be a plurisubharmonic function on Y. Define

$$\psi:=\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k\leqslant m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}+\pi_1^*(\Psi)$$

and $\varphi:=\sum_{1\leqslant j\leqslant n_1}\pi_{1,j}^*(\varphi_j)+\pi_2^*(\varphi_Y)$ on M, where $p_{j,k}$ is a positive real number for $1\leqslant j\leqslant n_1$ and $1\leqslant k\leqslant m_j$.

Define $E_\beta:=\left\{(\alpha_1,\ldots,\alpha_{n_1}):\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}=1\,\&\,\alpha_j\in\mathbb{Z}_{\geqslant0}\right\}$ and $\tilde E_\beta:=\left\{(\alpha_1,\ldots,\alpha_{n_1}):\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}\geqslant1\,\&\,\alpha_j\in\mathbb{Z}_{\geqslant0}\right\}$ for any $\beta\in I_1$. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset M_1$ of Z_0 such that

$$f=\sum_{\alpha\in\tilde E_\beta}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(f_{\alpha,\beta})$$

on $U_0\cap(V_\beta\times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha\in E_\beta$ and $\beta\in I_1$. Let $\beta^*=(1,\ldots,1)\in I_1$, and let $\alpha_{\beta^*}=(\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1})\in E_{\beta^*}$. Define $E':=\left\{\alpha\in\mathbb{Z}_{\geqslant0}^{n_1}:\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,1}}>1\right\}$.

Assume that $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{\beta^*}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right) + \sum_{\alpha\in E'} \pi_1^*(w_{\beta^*}^{\alpha} dw_{\beta^*}) \wedge \pi_2^*(f_{\alpha,\beta})$ on $U_0 \cap (V_{\beta^*} \times Y)$. Define

$$c_{j,k} := \exp \lim_{z\to z_{j,k}} \left(\frac{\sum_{1\leqslant k_1\leqslant m_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log|w_{j,k}(z)| \right)$$

for any $j \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,m_j\}$.

We obtain a characterization of the holding of equality in optimal jets L^2 extension problem for the case where Z_j is finite.

Theorem 1.14. *Let c be a positive function on $(0,+\infty)$ such that $\int_0^{+\infty} c(t)\mathrm{e}^{-t}dt < +\infty$ and $c(t)\mathrm{e}^{-t}$ is decreasing on $(0,+\infty)$. Assume that*

$$\|f\|_{Z_0}^2 := \sum_{\beta\in I_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} \mathrm{e}^{-\left(\Psi + \sum_{1\leqslant j\leqslant n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1} (\alpha_j+1) c_{j,\beta_j}^{2\alpha_j+2}} \in (0,+\infty).$$

Then there exists a holomorphic $(n,0)$ form F on M_1 satisfying that $(F-f,z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k\leqslant m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}))_z$ for any $z \in Z_0$ and

$$\int_{M_1} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) \leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \|f\|_{Z_0}^2.$$

Moreover, equality $\inf\Big\{ \int_{M_1} |\tilde{F}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{F} \in H^0(M_1, \mathcal{O}(K_{M_1}))\,\&\,(\tilde{F}-f,z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\max_{1\leqslant j\leqslant n_1}\{2\sum_{1\leqslant k\leqslant m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\}))_z$ *for any* $z \in Z_0\Big\} = \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right)\|f\|_{Z_0}^2$ *holds if and only if the following statements hold:*

(1) *$M = M_1$ and $\Psi \equiv 0$;*

(2) *$\varphi_j = 2\log|g_j| + 2u_j$ for any $j \in \{1,\ldots,n_1\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k}) \neq 0$ for any $k \in \{1,\ldots,m_j\}$;*

(3) *There exists a nonnegative integer $\gamma_{j,k}$ for any $j \in \{1,\ldots,n_1\}$ and $k \in \{1,\ldots,m_j\}$, which satisfies that $\prod_{1\leqslant k\leqslant m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ and $\sum_{1\leqslant j\leqslant n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ for any $\beta \in I_1$;*

(4) *$f_{\alpha,\beta} = c_\beta f_0$ holds for $\alpha = (\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})$ and $f_{\alpha,\beta} \equiv 0$ holds for any $\alpha \in E_\beta \backslash \{(\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})\}$, where $\beta \in I_1$, c_β is a constant and $f_0 \not\equiv 0$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y |f_0|^2 \mathrm{e}^{-\varphi_2} < +\infty$;*

(5) *$c_\beta \prod_{1\leqslant j\leqslant n_1} \left(\lim_{z\to z_{j,\beta_j}} \frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}} dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\leqslant k\leqslant m_j} f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\leqslant k\leqslant m_j} p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)} \right) = c_0$ for any $\beta \in I_1$, where $c_0 \in \mathbb{C}\backslash\{0\}$ is a constant independent of β.*

Remark 1.15. If $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ holds for any $y \in Y$, $\alpha \in \tilde{E}_\beta \backslash E_\beta$ and $\beta \in I_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\big(\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k\leqslant m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\big)$ by $\mathcal{I}(\varphi+\psi)$.

We prove Remark 1.15 in Section 7.2.

Finally, let us discuss the case where Z_0 is the fibration of product of discrete subset of Ω_j (maybe infinite points).

Let $Z_j = \{z_{j,k} : 1 \leqslant k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1,\ldots,n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geqslant 2} \cup \{+\infty\}$. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \leqslant j \leqslant n_1$ and $1 \leqslant k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote $\tilde{I}_1 := \{(\beta_1,\ldots,\beta_{n_1}) : 1 \leqslant \beta_j < \tilde{m}_j$ for any $j \in \{1,\ldots,n_1\}\}$, $V_\beta := \prod_{1\leqslant j\leqslant n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}}) \in \prod_{1\leqslant j\leqslant n_1} \Omega_j$ for any $\beta = (\beta_1,\ldots,\beta_{n_1}) \in \tilde{I}_1$. Then $Z_0 = \{(z_\beta, y) : \beta \in \tilde{I}_1\,\&\, y \in Y\} \subset M_1$.

Let $\Psi \leqslant 0$ be a plurisubharmonic function on $\prod_{1\leqslant j\leqslant n_1} \Omega_j$, and let φ_j be a Lebesgue measurable function on Ω_j such that $\Psi + \sum_{1\leqslant j\leqslant n_1} \tilde{\pi}_j^*(\varphi_j)$ is plurisubharmonic on $\prod_{1\leqslant j\leqslant n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leqslant j\leqslant n_1} \Omega_j$ to Ω_j. Let φ_Y be a plurisubharmonic function on Y. Let $p_{j,k}$ be a positive number for any $1 \leqslant j \leqslant n_1$ and $1 \leqslant k < \tilde{m}_j$, which satisfies that $\sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \leqslant j \leqslant n_1$. Denote

$$\psi := \max_{1\leqslant j\leqslant n_1} \left\{ 2 \sum_{1\leqslant k<\tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\} + \pi_1^*(\Psi)$$

and $\varphi := \sum_{1\leqslant j\leqslant n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on M.

Denote $E_\beta := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geqslant 0}\right\}$ and $\tilde{E}_\beta := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \geqslant 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geqslant 0}\right\}$ for any $\beta \in \tilde{I}_1$. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset M_1$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$. Let $\beta^* = (1, \ldots, 1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \ldots, \alpha_{\beta^*,n_1}) \in E_{\beta^*}$. Denote $E' := \left\{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,1}} > 1\right\}$. Assume that $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{\beta^*}\right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right) + \sum_{\alpha\in E'} \pi_1^*(w_{\beta^*}^\alpha dw_{\beta^*}) \wedge \pi_2^*(f_{\alpha,\beta})$ on $U_0 \cap (V_{\beta^*} \times Y)$. Denote

$$c_{j,k} := \exp \lim_{z\to z_{j,k}} \left(\frac{\sum_{1\leqslant k_1<\tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log|w_{j,k}(z)| \right)$$

for any $j \in \{1, \ldots, n\}$ and $1 \leqslant k < \tilde{m}_j$ (following from Lemma 2.12 and Lemma 2.13, we get that the above limit exists).

We obtain that the equality in optimal jets L^2 extension problem could not hold when there exists $j_0 \in \{1, \ldots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$.

Theorem 1.16. *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)\mathrm{e}^{-t}dt < +\infty$ and $c(t)\mathrm{e}^{-t}$ is decreasing on $(0, +\infty)$. Assume that*

$$\|f\|_{Z_0}^2 := \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} \mathrm{e}^{-\left(\Psi + \sum_{1\leqslant j\leqslant n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \in (0, +\infty)$$

and there exists $j_0 \in \{1, \ldots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$.

Then there exists a holomorphic $(n, 0)$ form F on M_1 satisfying that $(F - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\max_{1\leqslant j\leqslant n_1}\{2\sum_{1\leqslant k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\}))_z$ for any $z \in Z_0$ and

$$\int_{M_1} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) < \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds \right) \|f\|_{Z_0}^2.$$

Remark 1.17. If $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ holds for any $y \in Y$, $\alpha \in \tilde{E}_\beta \backslash E_\beta$ and $\beta \in \tilde{I}_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)$ by $\mathcal{I}(\varphi + \psi)$.

We prove Remark 1.17 in Section 8.2.

1.2.2 ***Suita conjecture and extended Suita conjecture***

Ohsawa [59] observed that the L^2 extension theorem with an optimal estimate can prove the inequality part of Suita conjecture [69]. Under the observation, we know that the equality part of Suita conjecture is related to the equality problem in optimal L^2 extension theorems. In this section, we present characterizations of the equality parts of Suita conjecture and extended Suita conjecture for fibrations over products of open Riemann surfaces.

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω. Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let κ_Ω be the Bergman kernel for holomorphic $(1, 0)$ form on Ω. We define that

$$B_\Omega(z)dw \otimes \overline{dw} := \kappa_\Omega|_{V_{z_0}}.$$

Let $c_\beta(z)$ be the logarithmic capacity on Ω (see [64], see also Section 1.1). In [69], Suita stated a conjecture as below.

Conjecture 1.18. $c_\beta(z_0)^2 \leqslant \pi B_\Omega(z_0)$ holds for any $z_0 \in \Omega$, and equality holds if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

The inequality part of Suita conjecture for bounded planar domains was proved by Błocki [7], and original form of the inequality was proved by Guan and Zhou [41]. The equality part of Suita conjecture was proved by Guan and Zhou [44], which completed the proof of Suita conjecture.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leqslant j \leqslant n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = \left(\prod_{1\leqslant j\leqslant n_1} \Omega_j\right) \times Y$ be an n-dimensional complex manifold, where $n = n_1 + n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M.

Denote the space of L^2 integrable holomorphic section of K_M (resp. K_Y) by $A^2(M, K_M, dV_M^{-1}, dV_M)$ (resp. $A^2(Y, K_Y, dV_Y^{-1}, dV_Y)$). Let $\{\sigma_l\}_{l=1}^{+\infty}$ (resp. $\{\tau_l\}_{l=1}^{+\infty}$) be a complete orthogonal system of $A^2(M, K_M, dV_M^{-1}, dV_M)$ (resp. $A^2(Y, K_Y, dV_Y^{-1}, dV_Y)$) satisfying $(\sqrt{-1})^{n^2}\int_M \frac{\sigma_i}{\sqrt{2^n}} \wedge \frac{\overline{\sigma}_j}{\sqrt{2^n}} = \delta_i^j$. Put $\kappa_M = \sum_{l=1}^{+\infty} \sigma_l \otimes \overline{\sigma}_l \in C^\omega(M, K_M \otimes \overline{K_M})$ and $\kappa_Y = \sum_{l=1}^{+\infty} \tau_l \otimes \overline{\tau}_l \in C^\omega(Y, K_Y \otimes \overline{K_Y})$.

Let $z_0 = (z_1, \ldots, z_{n_1}) \in \prod_{1\leqslant j\leqslant n_1} \Omega_j$, and let $y_0 \in Y$. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote $V_0 := \prod_{1\leqslant j\leqslant n_1} V_{z_j}$, and $w := (w_1, \ldots, w_{n_1})$ is a local coordinate on V_0 of z_0. Let $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_{n_2})$ be a local coordinate on a neighborhood U_0 of y_0 satisfying $\tilde{w}(y_0) = 0$. We define

$$B_M((z,y))dw \wedge d\tilde{w} \otimes \overline{dw \wedge d\tilde{w}} := \kappa_M$$

on $V_0 \times U_0$ and

$$B_Y(y)d\tilde{w} \otimes \overline{d\tilde{w}} := \kappa_Y$$

on U_0. Let $c_j(z_j)$ be the logarithmic capacity on Ω_j.

Assume that $B_Y(y_0) > 0$. Theorem 1.10 gives a characterization of the holding of equality in Suita conjecture for fibrations over products of open Riemann surfaces.

Theorem 1.19. $\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) \leqslant \pi^{n_1} B_M((z_0, y_0))$ *holds, and equality holds if and only if* Ω_j *is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero for any* $j \in \{1, \ldots, n_1\}$.

Let $M_1 \subset M$ be an n-dimensional complex manifold satisfying that $\{z_0\} \times Y \subset M_1$ and $\overline{M_1}^\circ = M_1$. Similar to M, we can define the Bergman kernel B_{M_1}. Theorem 1.19 implies the following result.

Remark 1.20. $\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) \leqslant \pi^{n_1} B_{M_1}((z_0, y_0))$ holds, and equality holds if and only if $M = M_1$ and Ω_j is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero for any $j \in \{1, \ldots, n_1\}$.

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω, and let K_Ω be the canonical (holomorphic) line bundle on Ω. Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let $\rho = \mathrm{e}^{-2u}$ on Ω, where u is a harmonic function on Ω. We define that

$$B_{\Omega,\rho} dw \otimes \overline{dw} := \sum_{l=1}^{+\infty} (\sigma_l \otimes \overline{\sigma}_l)|_{V_{z_0}} \in C^\omega(V_{z_0}, K_\Omega \otimes \overline{K_\Omega}),$$

where $\{\sigma_l\}_{l=1}^{+\infty}$ are holomorphic $(1,0)$ forms on Ω satisfying $\sqrt{-1}\int_\Omega \rho \frac{\sigma_i}{\sqrt{2}} \wedge \frac{\overline{\sigma}_j}{\sqrt{2}} = \delta_i^j$ and $\big\{F \in H^0(\Omega, K_\Omega) : \int_\Omega \rho|F|^2 < +\infty \,\&\, \int_\Omega \rho\sigma_l \wedge \overline{F} = 0$ for any $l \in \mathbb{Z}_{>0}\big\} = \{0\}$.

In [73], Yamada stated a conjecture as below (the so-called extended Suita conjecture).

Conjecture 1.21. $c_\beta(z_0)^2 \leqslant \pi\rho(z_0) B_{\Omega,\rho}(z_0)$ holds for any $z_0 \in \Omega$, and equality holds if and only $\chi_{-u} = \chi_{z_0}$, where χ_{-u} and χ_{z_0} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_0)$ respectively.

The extended Suita conjecture was proved by Guan and Zhou [42,44].

Let $\rho = \mathrm{e}^{-2\sum_{1\leqslant j\leqslant n_1}\pi_{1,j}^*(u_j)}$ on M, where u_j is a harmonic function on Ω_j for any $j \in \{1,\ldots,n_1\}$. We define that

$$B_{M,\rho}dw \wedge d\tilde{w} \otimes \overline{dw \wedge d\tilde{w}} := \sum_{l=1}^{+\infty} e_l \otimes \bar{e}_l$$

on $V_0 \times Y$, where $\{e_l\}_{l=1}^{+\infty}$ are holomorphic $(n,0)$ forms on M satisfying $(\sqrt{-1})^{n^2}\int_M \rho\frac{e_i}{\sqrt{2^n}} \wedge \frac{\bar{e}_j}{\sqrt{2^n}} = \delta_i^j$ and $\{F \in H^0(M,K_M) : \int_M \rho|F|^2 < +\infty \,\&\, \int_M \rho e_l \wedge \overline{F} = 0 \text{ for any } l \in \mathbb{Z}_{>0}\} = \{0\}$.

Assume that $B_Y(y_0) > 0$. Theorem 1.10 gives a characterization of the holding of equality in the extended Suita conjecture for fibrations over products of open Riemann surfaces.

Theorem 1.22. *$\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) \leqslant \pi^{n_1}\rho(z_0)B_{M,\rho}((z_0,y_0))$ holds, and equality holds if and only if $\chi_{j,-u_j} = \chi_{j,z_j}$ for any $j \in \{1,\ldots,n_1\}$, where $\chi_{j,-u_j}$ and χ_{j,z_j} are the characters associated to the functions $-u_j$ and $G_{\Omega_j}(\cdot,z_j)$ respectively.*

Let $M_1 \subset M$ be an n-dimensional complex manifold satisfying that $\{z_0\} \times Y \subset M_1$ and $\overline{M_1}^{\circ} = M_1$. Similar to M, we can define the Bergman kernel $B_{M_1,\rho}$. Theorem 1.22 implies the following result.

Remark 1.23. $\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) \leqslant \pi^{n_1}\rho(z_0)B_{M_1,\rho}((z_0,y_0))$ holds, and equality holds if and only if $\chi_{j,-u_j} = \chi_{j,z_j}$ for any $j \in \{1,\ldots,n_1\}$ and $M = M_1$.

2 Preparation

2.1 Concavity property of minimal L^2 integrals

In this section, we recall some results about the concavity property of minimal L^2 integrals (see [34,40]).

Let M be a complex manifold. Let X and Z be closed subsets of M. We say that a triple (M,X,Z) satisfies condition (A), if the following statements hold:

I. X is a closed subset of M and X is locally negligible with respect to L^2 holomorphic functions, i.e., for any local coordinated neighborhood $U \subset M$ and for any L^2 holomorphic function f on $U\backslash X$, there exists an L^2 holomorphic function $\tilde{f}$ on U such that $\tilde{f}|_{U\backslash X} = f$ with the same L^2 norm;

II. Z is an analytic subset of M and $M\backslash(X \cup Z)$ is a weakly pseudoconvex Kähler manifold.

Let M be an n-dimensional complex manifold, and let (M,X,Z) satisfy condition (A). Let K_M be the canonical line bundle on M. Let ψ be a plurisubharmonic function on M, and let φ be a Lebesgue measurable function on M such that $\psi+\varphi$ is a plurisubharmonic function on M. Denote $T = -\sup_M \psi$.

Definition 2.1. We call a positive measurable function c on $(T,+\infty)$ in class $P_{T,M}$ if the following two statements hold:

(1) $c(t)\mathrm{e}^{-t}$ is decreasing with respect to t;

(2) there is a closed subset E of M such that $E \subset Z \cap \{\psi(z) = -\infty\}$ and for any compact subset $K \subset M\backslash E$, $\mathrm{e}^{-\varphi}c(-\psi)$ has a positive lower bound on K.

Let Z_0 be a subset of $\{\psi = -\infty\}$ such that $Z_0 \cap Supp(\mathcal{O}/\mathcal{I}(\varphi+\psi)) \neq \emptyset$. Let $U \supset Z_0$ be an open subset of M, and let f be a holomorphic $(n,0)$ form on U. Let $\mathcal{F}_z \supset \mathcal{I}(\varphi+\psi)_z$ be an ideal of $\mathcal{O}_z$ for any $z \in Z_0$.

Denote

$$\inf\Bigg\{\int_{\{\psi<-t\}} |\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi) : (\tilde{f}-f) \in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0}) \\ \& \, \tilde{f} \in H^0(\{\psi<-t\},\mathcal{O}(K_M))\Bigg\}$$

by $G(t;c)$ ($G(t)$ for short), where $t \in [T,+\infty)$, c is a nonnegative function on $(T,+\infty)$, $|f|^2 := \sqrt{-1}^{n^2} f \wedge \bar{f}$ for any $(n,0)$ form f and $(\tilde{f}-f) \in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ means $(\tilde{f}-f,z) \in \mathcal{O}(K_M)_z \otimes \mathcal{F}_z$ for all $z \in Z_0$.

The following theorem shows the concavity for $G(t)$.

Theorem 2.2 ([34]). *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)\mathrm{e}^{-s}ds < +\infty$. If there exists $t \in [T,+\infty)$ satisfying that $G(t) < +\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(s)\mathrm{e}^{-s}ds)$, $\lim_{t\to T+0} G(t) = G(T)$ and $\lim_{t\to+\infty} G(t) = 0$, where $h(t) = \int_t^{+\infty} c(s)\mathrm{e}^{-s}ds$.*

Define

$$\mathcal{H}^2(c,t) := \left\{ \tilde{f} : \int_{\{\psi<-t\}} |\tilde{f}|^2 \mathrm{e}^{-\varphi} c(-\psi) < +\infty, (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0}) \right.$$
$$\left. \& \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\},$$

where $t \in [T,+\infty)$ and c is a nonnegative measurable function on $(T,+\infty)$.

Corollary 2.3 ([34]). *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)\mathrm{e}^{-s}ds < +\infty$. If $G(t) \in (0,+\infty)$ for some $t \geqslant T$ and $G(h^{-1}(r))$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)\mathrm{e}^{-s}ds)$, then there is a unique holomorphic $(n,0)$ form F on M satisfying $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $G(t;c) = \int_{\{\psi<-t\}} |F|^2\mathrm{e}^{-\varphi}c(-\psi)$ for any $t \geqslant T$. Furthermore,*

$$\int_{\{-t_1\leqslant\psi<-t_2\}} |F|^2\mathrm{e}^{-\varphi}a(-\psi) = \frac{G(T_1;c)}{\int_{T_1}^{+\infty} c(t)\mathrm{e}^{-t}dt} \int_{t_2}^{t_1} a(t)\mathrm{e}^{-t}dt$$

for any nonnegative measurable function a on $(T,+\infty)$, where $+\infty \geqslant t_1 > t_2 \geqslant T$ and $T_1 > T$.

Especially, if $\mathcal{H}^2(\tilde{c},t_0) \subset \mathcal{H}^2(c,t_0)$ for some $t_0 \geqslant T$, where $\tilde{c}$ is a nonnegative measurable function on $(T,+\infty)$, we have

$$G(t_0;\tilde{c}) = \int_{\{\psi<-t_0\}} |F|^2\mathrm{e}^{-\varphi}\tilde{c}(-\psi) = \frac{G(T_1;c)}{\int_{T_1}^{+\infty} c(s)\mathrm{e}^{-s}ds} \int_{t_0}^{+\infty} \tilde{c}(s)\mathrm{e}^{-s}ds.$$

The following lemma is a characterization of $G(t) = 0$, where $t \geqslant T$.

Lemma 2.4 ([34]). *The following two statements are equivalent:*

(1) $(f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$.

(2) $G(t) = 0$.

Lemma 2.5 ([34]). *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)\mathrm{e}^{-s}ds < +\infty$. Assume that $G(t) < +\infty$ for some $t \in [T,+\infty)$. Then there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi < -t\}$ satisfying $(F_t - f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}} |F_t|^2\mathrm{e}^{-\varphi}c(-\psi) = G(t)$. Furthermore, for any holomorphic $(n,0)$ form $\hat{F}$ on $\{\psi < -t\}$ satisfying $(\hat{F} - f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}} |\hat{F}|^2\mathrm{e}^{-\varphi}c(-\psi) < +\infty$, we have the following equality*

$$\int_{\{\psi<-t\}} |F_t|^2\mathrm{e}^{-\varphi}c(-\psi) + \int_{\{\psi<-t\}} |\hat{F} - F_t|^2\mathrm{e}^{-\varphi}c(-\psi)$$
$$= \int_{\{\psi<-t\}} |\hat{F}|^2\mathrm{e}^{-\varphi}c(-\psi).$$

The following result will be used in the proof of Theorem 1.10.

Lemma 2.6. *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)\mathrm{e}^{-s}ds < +\infty$. Assume $G(t) \in (0,+\infty)$ for some $t \geqslant T$ and $G(h^{-1}(r))$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)\mathrm{e}^{-s}ds)$. Let $\tilde{c}$ be a nonnegative function on $(T,+\infty)$, and let $t_0 \geqslant T$. If there is a holomorphic $(n,0)$ form $\tilde{F} \in \mathcal{H}^2(\tilde{c},t_0)$ such that*

$$G(t_0;\tilde{c}) = \int_{\{\psi<-t_0\}} |\tilde{F}|^2\mathrm{e}^{-\varphi}\tilde{c}(-\psi)$$

and $\tilde{F} \in \mathcal{H}^2(c,t_0)$, then we have

$$G(t_0;\tilde{c}) = \int_{\{\psi<-t_0\}} |F|^2\mathrm{e}^{-\varphi}\tilde{c}(-\psi) = \frac{G(T_1;c)}{\int_{T_1}^{+\infty} c(s)\mathrm{e}^{-s}ds} \int_{t_0}^{+\infty} \tilde{c}(s)\mathrm{e}^{-s}ds,$$

where $T_1 > T$.

The proof of Lemma 2.6 is the same as the proof of Remark 1.5 in [34] (replacing $F_{t_0,\tilde{c}}$ in [34] by $\tilde{F}$), so we omit the proof.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leqslant j \leqslant n$. Let $M = \prod_{1 \leqslant j \leqslant n} \Omega_j$ be an n-dimensional complex manifold, and let π_j be the natural projection from M to Ω_j. Let K_M be the canonical (holomorphic) line bundle on M. Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \ldots, n\}$, and let $Z_0 = \prod_{1 \leqslant j \leqslant n} Z_j$. For any $j \in \{1, \ldots, n\}$, let φ_j be a subharmonic function on Ω_j such that $\varphi_j(z) > -\infty$ for any $z \in Z_j$, and let $\varphi = \sum_{1 \leqslant j \leqslant n} \pi_j^*(\varphi_j)$. Let ψ be a plurisubharmonic function on M such that $\psi(z) = -\infty$ for any $z \in Z_0$ and ψ is continuous on $M \backslash Z_0$. Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)\mathrm{e}^{-t}dt < +\infty$ and $c(t)\mathrm{e}^{-t}$ is decreasing on $(0, +\infty)$. Let $\mathcal{F}_z = \mathcal{I}(\psi)_z$ for any $z \in Z_0$.

In the following, we recall some results about the concavity of $G(h^{-1}(r))$ degenerating to linearity.

Let $Z_0 = \{z_0\} = \{(z_1, \ldots, z_n)\} \subset M$. Let $\psi = \max_{1 \leqslant j \leqslant n} \{2p_j \pi_j^*(G_{\Omega_j}(\cdot, z_j))\}$, where p_j is positive real number. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote $V_0 := \prod_{1 \leqslant j \leqslant n} V_{z_j}$, and $w := (w_1, \ldots, w_n)$ is a local coordinate on V_0 of $z_0 \in M$. Let f be a holomorphic $(n, 0)$ form on V_0. Denote $E := \left\{(\alpha_1, \ldots, \alpha_n) : \sum_{1 \leqslant j \leqslant n} \frac{\alpha_j + 1}{p_j} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geqslant 0}\right\}$.

We recall a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the case Z_0 is a single point set as follows.

Theorem 2.7 ([40])**.** *Assume that $G(0) \in (0, +\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)\mathrm{e}^{-t}dt]$ if and only if the following statements hold:*

(1) *$f = \left(\sum_{\alpha \in E} d_\alpha w^\alpha + g_0\right) dw$ on V_0, where $d_\alpha \in \mathbb{C}$ such that $\sum_{\alpha \in E} |d_\alpha| \neq 0$ and g_0 is a holomorphic function on V_0 such that $(g_0, z_0) \in \mathcal{I}(\psi)_{z_0}$;*

(2) *$\varphi_j = 2\log|g_j| + 2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j) \neq 0$ and u_j is a harmonic function on Ω_j for any $1 \leqslant j \leqslant n$;*

(3) *$\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$ for any $j \in \{1, 2, \ldots, n\}$ and $\alpha \in E$ satisfying $d_\alpha \neq 0$, χ_{j,z_j} and $\chi_{j,-u_j}$ are the characters associated to functions $G_{\Omega_j}(\cdot, z_j)$ and $-u_j$ respectively.*

Remark 2.8 ([40])**.** When the three statements in Theorem 2.7 hold,

$$\sum_{\alpha \in E} \tilde{d}_\alpha \wedge_{1 \leqslant j \leqslant n} \pi_j^* \left(g_j(P_j)_* \left(f_{u_j} f_{z_j}^{\alpha_j} df_{z_j}\right)\right)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, z_0) \in (\mathcal{O}(K_M))_{z_0} \otimes \mathcal{I}(\psi)_{z_0}$ and

$$G(t) = \int_{\{\psi < -t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s)\mathrm{e}^{-s}ds\right) \sum_{\alpha \in E} \frac{|d_\alpha|^2 (2\pi)^n \mathrm{e}^{-\varphi(z_0)}}{\prod_{1 \leqslant j \leqslant n} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}}$$

for any $t \geqslant 0$, where $P_j : \Delta \to \Omega_j$ is the universal covering, f_{u_j} is a holomorphic function on Δ such that $|f_{u_j}| = P_j^*(\mathrm{e}^{u_j})$ for any $j \in \{1, \ldots, n\}$, f_{z_j} is a holomorphic function on Δ such that $|f_{z_j}| = P_j^*\left(\mathrm{e}^{G_{\Omega_j}(\cdot, z_j)}\right)$ for any $j \in \{1, \ldots, n\}$ and $\tilde{d}_\alpha$ is a constant such that $\tilde{d}_\alpha = \lim_{z \to z_0} \frac{d_\alpha w^\alpha dw}{\wedge_{1 \leqslant j \leqslant n} \pi_j^*\left(g_j(P_j)_*\left(f_{u_j} f_{z_j}^{\alpha_j} df_{z_j}\right)\right)}$ for any $\alpha \in E$.

Let $Z_j = \{z_{j,1}, \ldots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \ldots, n\}$, where m_j is a positive integer. Let $\psi = \max_{1 \leqslant j \leqslant n} \left\{\pi_j^*\left(2 \sum_{1 \leqslant k \leqslant m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote $I_1 := \{(\beta_1, \ldots, \beta_n) : 1 \leqslant \beta_j \leqslant m_j$ for any $j \in \{1, \ldots, n\}\}$, $V_\beta := \prod_{1 \leqslant j \leqslant n} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \ldots, \beta_n) \in I_1$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n,\beta_n})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n,\beta_n}) \in M$. Let f be a holomorphic $(n, 0)$ form on $\bigcup \beta \in I_1 V_\beta$ such that $f = w_{\beta^*}^{\alpha_{\beta^*}} dw_{\beta^*}$ on V_{β^*}, where $\beta^* = (1, \ldots, 1) \in I_1$.

We recall a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the case Z_j is a set of finite points as follows.

Theorem 2.9 ([40])**.** *Assume that $G(0) \in (0, +\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)\mathrm{e}^{-s}ds]$ if and only if the following statements hold:*

(1) $\varphi_j = 2\log|g_j| + 2u_j$ *for any* $j \in \{1,\dots,n\}$, *where* u_j *is a harmonic function on* Ω_j *and* g_j *is a holomorphic function on* Ω_j *satisfying* $g_j(z_{j,k}) \neq 0$ *for any* $k \in \{1,\dots,m_j\}$;

(2) *There exists a nonnegative integer* $\gamma_{j,k}$ *for any* $j \in \{1,\dots,n\}$ *and* $k \in \{1,\dots,m_j\}$, *which satisfies that* $\prod_{1\leqslant k\leqslant m_j}\chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ *and* $\sum_{1\leqslant j\leqslant n}\frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ *for any* $\beta \in I_1$, *where* $\chi_{j,z_{j,k}}$ *and* $\chi_{j,-u_j}$ *are the characters associated to* $G_{\Omega_j}(\cdot, z_{j,k})$ *and* $-u_j$ *respectively;*

(3) $f = \left(c_\beta\prod_{1\leqslant j\leqslant n}w_{j,\beta_j}^{\gamma_{j,\beta_j}} + g_\beta\right)dw_\beta$ *on* V_β *for any* $\beta \in I_1$, *where* c_β *is a constant and* g_β *is a holomorphic function on* V_β *such that* $(g_\beta, z_\beta) \in \mathcal{I}(\psi)_{z_\beta}$;

(4) $\lim_{z\to z_\beta}\frac{c_\beta\prod_{1\leqslant j\leqslant n}w_{j,\beta_j}^{\gamma_{j,\beta_j}}dw_\beta}{\wedge_{1\leqslant j\leqslant n}\pi_j^*\left(g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\leqslant k\leqslant m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\leqslant k\leqslant m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)\right)} = c_0$ *for any* $\beta \in I_1$, *where* $c_0 \in \mathbb{C}\backslash\{0\}$ *is a constant independent of* β.

Define

$$c_{j,k} := \exp\lim_{z\to z_{j,k}}\left(\frac{\sum_{1\leqslant k_1\leqslant m_j}p_{j,k_1}G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log|w_{j,k}(z)|\right)$$

for any $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,m_j\}$.

Remark 2.10 ([40]). When the four statements in Theorem 2.9 hold,

$$c_0\wedge_{1\leqslant j\leqslant n}\pi_j^*\left(g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\leqslant k\leqslant m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\leqslant k\leqslant m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)\right)$$

is the unique holomorphic $(n,0)$ form F on M such that $(F - f, z_\beta) \in (\mathcal{O}(K_M))_{z_\beta}\otimes\mathcal{I}(\psi)_{z_\beta}$ for any $\beta \in I_1$ and

$$G(t) = \int_{\{\psi<-t\}}|F|^2\mathrm{e}^{-\varphi}c(-\psi) = \left(\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\sum_{\beta\in I_1}\frac{|c_\beta|^2(2\pi)^n\mathrm{e}^{-\varphi(z_\beta)}}{\prod_{1\leqslant j\leqslant n}(\gamma_{j,\beta_j}+1)c_{j,\beta_j}^{2\gamma_{j,\beta_j}+2}}$$

for any $t \geqslant 0$.

Let $Z_j = \{z_{j,k} : 1 \leqslant k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1,\dots,n\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geqslant 2}\cup\{+\infty\}$. Let $p_{j,k}$ be a positive number for any $1 \leqslant j \leqslant n$ and $1 \leqslant k < \tilde{m}_j$ such that $\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j. Let $\psi = \max_{1\leqslant j\leqslant n}\left\{\pi_j^*\left(2\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$. Assume that $\limsup_{t\to+\infty}c(t) < +\infty$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1,\dots,n\}$ and $1 \leqslant k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote $\tilde{I}_1 := \{(\beta_1,\dots,\beta_n) : 1 \leqslant \beta_j < \tilde{m}_j$ for any $j \in \{1,\dots,n\}\}$, $V_\beta := \prod_{1\leqslant j\leqslant n}V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1,\dots,\beta_n) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1},\dots,w_{n,\beta_n})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1},\dots,z_{n,\beta_n}) \in M$. Let f be a holomorphic $(n,0)$ form on $\bigcup\beta \in \tilde{I}_1 V_\beta$ such that $f = w_{\beta^*}^{\alpha_{\beta^*}}dw_{\beta^*}$ on V_{β^*}, where $\beta^* = (1,\dots,1) \in \tilde{I}_1$.

We recall that $G(h^{-1}(r))$ is not linear when there exists $j_0 \in \{1,\dots,n\}$ such that $\tilde{m}_{j_0} = +\infty$ as follows.

Theorem 2.11 ([40]). *If* $G(0) \in (0,+\infty)$ *and there exists* $j_0 \in \{1,\dots,n\}$ *such that* $\tilde{m}_{j_0} = +\infty$, *then* $G(h^{-1}(r))$ *is not linear with respect to* $r \in (0, \int_0^{+\infty}c(s)\mathrm{e}^{-s}ds]$.

2.2 Some basic properties of the Green functions

In this Section, we recall some basic properties of the Green functions. Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω, and let $z_0 \in \Omega$.

Lemma 2.12 (see [64], see also [71]). *Let* w *be a local coordinate on a neighborhood of* z_0 *satisfying* $w(z_0) = 0$. $G_\Omega(z, z_0) = \sup_{v\in\Delta_\Omega^*(z_0)}v(z)$, *where* $\Delta_\Omega^*(z_0)$ *is the set of negative subharmonic function on* Ω *such that* $v - \log|w|$ *has a locally finite upper bound near* z_0. *Moreover,* $G_\Omega(\cdot, z_0)$ *is harmonic on* $\Omega\backslash\{z_0\}$ *and* $G_\Omega(\cdot, z_0) - \log|w|$ *is harmonic near* z_0.

Lemma 2.13 (see [36]). *Let* $K = \{z_j : j \in \mathbb{Z}_{\geqslant 1}\,\&\, j < \gamma\}$ *be a discrete subset of* Ω, *where* $\gamma \in \mathbb{Z}_{>1}\cup\{+\infty\}$. *Let* ψ *be a negative subharmonic function on* Ω *such that* $\frac{1}{2}v(dd^c\psi, z_j) \geqslant p_j$ *for any* j, *where* $p_j > 0$ *is a constant. Then* $2\sum_{1\leqslant j<\gamma}p_jG_\Omega(\cdot, z_j)$ *is a subharmonic function on* Ω *satisfying that* $2\sum_{1\leqslant j<\gamma}p_jG_\Omega(\cdot, z_j) \geqslant \psi$ *and* $2\sum_{1\leqslant j<\gamma}p_jG_\Omega(\cdot, z_j)$ *is harmonic on* $\Omega\backslash K$.

Lemma 2.14 (see [38]). *For any open neighborhood U of z_0, there exists $t>0$ such that $\{G_\Omega(z,z_0)<-t\}$ is a relatively compact subset of U.*

Lemma 2.15 (see [36]). *There exists a sequence of open Riemann surfaces $\{\Omega_l\}_{l\in\mathbb{Z}^+}$ such that $z_0\in\Omega_l\Subset\Omega_{l+1}\Subset\Omega$, $\bigcup l\in\mathbb{Z}^+\Omega_l=\Omega$, Ω_l has a smooth boundary $\partial\Omega_l$ in Ω and $\mathrm{e}^{G_{\Omega_l}(\cdot,z_0)}$ can be smoothly extended to a neighborhood of $\overline{\Omega_l}$ for any $l\in\mathbb{Z}^+$, where G_{Ω_l} is the Green function of Ω_l. Moreover, $\{G_{\Omega_l}(\cdot,z_0)-G_\Omega(\cdot,z_0)\}$ converges decreasingly to 0 on Ω with respect to l.*

Let Ω_j be an open Riemann surface for any $1\leqslant j\leqslant n$, which admits a nontrivial Green function G_Ω. Let $\{z_{j,k}:1\leqslant k<\tilde{m}_j\}$ be a discrete subset of Ω_j for any $1\leqslant j\leqslant n$, where $\tilde{m}_j\in\mathbb{Z}_{\geqslant 2}\cup\{+\infty\}$. The following lemma will be used in the proof of the applications.

Lemma 2.16 (see [40]). *Let $\psi=\max_{1\leqslant j\leqslant n}\{\pi_j^*(2\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k}))\}$ be a plurisubharmonic function on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, where $\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\not\equiv-\infty$ for any $j\in\{1,\ldots,n\}$. Let $\Psi\leqslant 0$ be a plurisubharmonic function on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, and denote $\tilde{\psi}:=\psi+\Psi$. Let $l(t)$ be a positive Lebesgue measurable function on $(0,+\infty)$ satisfying that $l(t)$ is decreasing on $(0,+\infty)$ and $\int_0^{+\infty}l(t)dt<+\infty$. If $\Psi\not\equiv 0$ on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, there exists a Lebesgue measurable subset V of $\prod_{1\leqslant j\leqslant n_1}\Omega_j$ such that $l(-\tilde{\psi}(z))<l(-\psi(z))$ for any $z\in V$ and $\mu(V)>0$, where μ is the Lebesgue measure on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$.*

2.3 Some results related to $\max_{1\leqslant j\leqslant n}\{2p_j\log|w_j|\}$

In this section, we recall some basic property related to $\max_{1\leqslant j\leqslant n}\{2p_j\log|w_j|\}$. In the following lemma, we recall a closedness of the submodules of $\mathcal{O}^q_{\mathbb{C}^n,o}$.

Lemma 2.17 (see [27]). *Let N be a submodule of $\mathcal{O}^q_{\mathbb{C}^n,o}$, $1\leqslant q<+\infty$, let $f_j\in\mathcal{O}_{\mathbb{C}^n}(U)^q$ be a sequence of $q-$tuples holomorphic in an open neighborhood U of the origin o. Assume that the f_j converge uniformly in U towards a $q-$tuples $f\in\mathcal{O}_{\mathbb{C}^n}(U)^q$, assume furthermore that all germs (f_j,o) belong to N. Then $(f,o)\in N$.*

Let $f=\sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $D=\{w\in\mathbb{C}^n:|w_j|<r_0$ for any $j\in\{1,\ldots,n\}\}$, where $r_0>0$. Let

$$\psi=\max_{1\leqslant j\leqslant n_1}\{2p_j\log|w_j|\}$$

be a plurisubharmonic function on $\mathbb{C}^n$, where $n_1\leqslant n$ and $p_j>0$ is a constant for any $j\in\{1,\ldots,n_1\}$. We recall a characterization of $\mathcal{I}(\psi)_o$, where o is the origin in $\mathbb{C}^n$.

Lemma 2.18 (see [31]). *$(f,o)\in\mathcal{I}(\psi)_o$ if and only if $\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_j}>1$ for any $\alpha\in\mathbb{Z}^n_{\geqslant 0}$ satisfying $b_\alpha\neq 0$.*

For any $y\in D'=\{y\in\mathbb{C}^{n-n_1}:|y_k|<r_0$ for $1\leqslant k\leqslant n-n_1\}$, denote $f_y=f(\cdot,y)$ is a holomorphic function on $D''=\{x\in\mathbb{C}^{n_1}:|x_j|<r_0$ for any $j\in\{1,\ldots,n_1\}\}$. It follows from Lemma 2.18 that the following lemma holds.

Lemma 2.19. *$(f,(o_1,y))\in\mathcal{I}(\psi)_{(o_1,y)}$ for any $y\in D'$ if and only if $(f_y,o_1)\in\mathcal{I}(\psi)_{o_1}$ for any $y\in D'$, where o_1 is the origin in $\mathbb{C}^{n_1}$.*

In the following, let $\psi=\max_{1\leqslant j\leqslant n}\{2p_j\log|w_j|\}$ be a plurisubharmonic function on $\mathbb{C}^n$, where $p_j>0$. Let $f=\sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $\{\psi<-t_0\}$, where $t_0>0$.

Lemma 2.20. *Let $c(t)$ be a nonnegative measurable function on $(t_0,+\infty)$. Denote $q_\alpha:=\sum_{1\leqslant j\leqslant n}\frac{\alpha_j+1}{p_j}-1$ for any $\alpha\in\mathbb{Z}^n_{\geqslant 0}$. Then*

$$\int_{\{\psi<-t\}}|f|^2c(-\psi)d\lambda_n=\sum_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}\left(\int_t^{+\infty}c(s)\mathrm{e}^{-(q_\alpha+1)s}ds\right)\frac{(q_\alpha+1)|b_\alpha|^2\pi^n}{\prod_{1\leqslant j\leqslant n}(\alpha_j+1)}$$

holds for any $t\geqslant t_0$.

Proof. By direct calculations, we obtain that

$$
\begin{aligned}
&\int_{\{\psi<-t\}}|w^{\alpha}|^2c(-\psi)d\lambda_n\\
=&(2\pi)^n\int_{\left\{\max_{1\leqslant j\leqslant n}\{s_j^{p_j}\}<\mathrm{e}^{-\frac{t}{2}}\ \&\ s_j>0\right\}}\prod_{1\leqslant j\leqslant n}s_j^{2\alpha_j+1}\cdot c\big(-\log\max_{1\leqslant j\leqslant n}\{s_j^{2p_j}\}\big)ds_1ds_2\dots ds_n\\
=&(2\pi)^n\frac{1}{\prod_{1\leqslant j\leqslant n}p_j}\\
&\times\int_{\left\{\max_{1\leqslant j\leqslant n}\{r_j\}<\mathrm{e}^{-\frac{t}{2}}\ \&\ r_j>0\right\}}\prod_{1\leqslant j\leqslant n}r_j^{\frac{2\alpha_j+2}{p_j}-1}\cdot c\big(-\log\max_{1\leqslant j\leqslant n}\{r_j^2\}\big)dr_1dr_2\dots dr_n.
\end{aligned}
\tag{2.1}
$$

By Fubini's theorem, we have

$$
\begin{aligned}
&\int_{\left\{\max_{1\leqslant j\leqslant n}\{r_j\}<\mathrm{e}^{-\frac{t}{2}}\ \&\ r_j>0\right\}}\prod_{1\leqslant j\leqslant n}r_j^{\frac{2\alpha_j+2}{p_j}-1}\cdot c\big(-\log\max_{1\leqslant j\leqslant n}\{r_j^2\}\big)dr_1dr_2\dots dr_n\\
=&\sum_{j'=1}^{n}\int_0^{\mathrm{e}^{-\frac{t}{2}}}\left(\int_{\{0\leqslant r_j<r_{j'},j\neq j'\}}\prod_{j\neq j'}r_j^{\frac{2\alpha_j+2}{p_j}-1}\cdot\wedge_{j\neq j'}dr_j\right)r_{j'}^{\frac{2\alpha_{j'}+2}{p_{j'}}-1}c(-2\log r_{j'})dr_{j'}\\
=&\sum_{j'=1}^{n}\left(\prod_{j\neq j'}\frac{p_j}{2\alpha_j+2}\right)\int_0^{\mathrm{e}^{-\frac{t}{2}}}r_{j'}^{\sum_{1\leqslant k\leqslant n}\frac{2\alpha_k+2}{p_k}-1}c(-2\log r_{j'})dr_{j'}\\
=&(q_\alpha+1)\left(\int_t^{+\infty}c(s)\mathrm{e}^{-(q_\alpha+1)s}ds\right)\prod_{1\leqslant j\leqslant n}\frac{p_j}{2\alpha_j+2}.
\end{aligned}
\tag{2.2}
$$

Following from $\int_{\{\psi<-t\}}|f|^2c(-\psi)d\lambda_n=\sum_{\alpha\in\mathbb{Z}^n_{\geqslant0}}|b_\alpha|^2\int_{\{\psi<-t\}}|w^\alpha|^2c(-\psi)d\lambda_n$, equality (2.1) and equality (2.2), we obtain that

$$
\int_{\{\psi<-t\}}|f|^2c(-\psi)d\lambda_n=\sum_{\alpha\in\mathbb{Z}^n_{\geqslant0}}\left(\int_t^{+\infty}c(s)\mathrm{e}^{-(q_\alpha+1)s}ds\right)\frac{(q_\alpha+1)|b_\alpha|^2\pi^n}{\prod_{1\leqslant j\leqslant n}(\alpha_j+1)}.
$$

□

The following lemma will be used in the proof of Proposition 2.37.

Lemma 2.21 (see [40])**.** *Denote* $q_\alpha:=\sum_{1\leqslant j\leqslant n}\frac{\alpha_j+1}{p_j}-1$ *for any* $\alpha\in\mathbb{Z}^n_{\geqslant0}$ *and* $E_1:=\{\alpha\in\mathbb{Z}^n_{\geqslant0}:q_\alpha=0\}$*. Then*

$$
\begin{aligned}
\int_{\{-t-1<\psi<-t\}}|f|^2\mathrm{e}^{-\psi}d\lambda_n=&\sum_{\alpha\in E_1}\frac{|b_\alpha|^2\pi^n}{\prod_{1\leqslant j\leqslant n}(\alpha_j+1)}\\
&+\sum_{\alpha\notin E_1}\frac{|b_\alpha|^2\pi^n(q_\alpha+1)(\mathrm{e}^{-q_\alpha t}-\mathrm{e}^{-q_\alpha(t+1)})}{q_\alpha\prod_{1\leqslant j\leqslant n}(\alpha_j+1)}
\end{aligned}
$$

for any $t>t_0$*.*

2.4 Some results about fibrations

In this section, we discuss the fibrations.

Let $\Delta^{n_1}=\{w\in\mathbb{C}^{n_1}:|w_j|<1$ for any $j\in\{1,\dots,n_1\}\}$ be product of the unit disks. Let Y be an n_2-dimensional complex manifold, and let $M=\Delta^{n_1}\times Y$. Denote $n=n_1+n_2$. Let π_1 and π_2 be the natural projections from M to Δ^{n_1} and Y respectively. Let ρ_1 be a nonnegative Lebesgue measurable function on Δ^{n_1} satisfying that $\rho_1(w)=\rho_1(|w_1|,\dots,|w_{n_1}|)$ for any $w\in\Delta^{n_1}$ and the Lebesgue measure of $\{w\in\Delta^{n_1}:\rho_1(w)>0\}$ is positive. Let ρ_2 be a nonnegative Lebesgue measurable function on Y, and denote $\rho=\pi_1^*(\rho_1)\times\pi_2^*(\rho_2)$ on M.

Lemma 2.22. *For any holomorphic $(n,0)$ form F on M, there exists a unique sequence of holomorphic $(n_2,0)$ forms $\{F_\alpha\}_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}$ on Y such that*

$$F=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha), \tag{2.3}$$

where the right term of the above equality is uniformly convergent on any compact subset of M. Moreover, if $\int_M|F|^2\rho<+\infty$, we have

$$\int_Y|F_\alpha|^2\rho_2<+\infty \tag{2.4}$$

for any $\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}$.

Proof. Firstly, we consider the local case. Assume that $Y=\Delta^{n_2}$, and the coordinate is $\tilde{w}=(\tilde{w}_1,\ldots,\tilde{w}_{n_2})$. Then there exists a holomorphic function $\tilde{F}(w,\tilde{w})$ on Δ^n such that $F=\tilde{F}(w,\tilde{w})dw\wedge d\tilde{w}$. Let

$$F_\alpha=\frac{1}{\alpha!}\left(\left(\frac{\partial}{\partial w}\right)^\alpha\tilde{F}\right)\bigg|_{w=0}d\tilde{w}$$

be a holomorphic $(n_2,0)$ form on Y. Considering Taylor's expansion of $\tilde{F}$, we can assume that

$$\tilde{F}(w,\tilde{w})=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0},\tilde{\alpha}\in\mathbb{Z}^{n_2}_{\geqslant 0}}d_{\alpha,\tilde{\alpha}}w^\alpha\tilde{w}^{\tilde{\alpha}}=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}\frac{1}{\alpha!}\left(\left(\frac{\partial}{\partial w}\right)^\alpha\tilde{F}\right)\bigg|_{w=0}\cdot w^\alpha,$$

where the summations are uniformly convergent on any compact subset of M, then we have

$$F=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha).$$

Secondly, we need to prove that the gluing is independent of the choices of the local coordinates of Y. Assume that $y=(y_1,\ldots,y_{n_2})$ is another coordinate on $Y=\Delta^{n_2}$, and $F=\tilde{F}_0(w,y)dw\wedge dy$, thus we have $\tilde{F}(w,\tilde{w}(y))\frac{\partial(\tilde{w}_1,\ldots,\tilde{w}_{n_2})}{\partial(y_1,\ldots,y_{n_2})}=\tilde{F}_0(w,y)$. By direct calculations, we have

$$\begin{aligned}F_\alpha&=\frac{1}{\alpha!}\left(\left(\frac{\partial}{\partial w}\right)^\alpha\tilde{F}\right)\bigg|_{w=0}d\tilde{w}\\&=\frac{1}{\alpha!}\left(\left(\frac{\partial}{\partial w}\right)^\alpha\tilde{F}\right)\bigg|_{w=0}\frac{\partial(\tilde{w}_1,\ldots,\tilde{w}_{n_2})}{\partial(y_1,\ldots,y_{n_2})}dy\\&=\frac{1}{\alpha!}\left(\left(\frac{\partial}{\partial w}\right)^\alpha\tilde{F}_0\right)\bigg|_{w=0}dy,\end{aligned}$$

which means that F_α is independent of the choices of the coordinates for any $\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}$. For general Y, we can find holomorphic $(n_2,0)$ forms F_α on Y such that $F=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha)$.

Then, for the uniqueness, it suffices to prove the local case $Y=\Delta^{n_2}$. There exists a holomorphic function $\tilde{F}(w,\tilde{w})$ on Δ^n such that $F=\tilde{F}(w,\tilde{w})dw\wedge d\tilde{w}$. If

$$F=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha)$$

for a holomorphic $(n_2,0)$ form F_α on Y, we have

$$F_\alpha=\frac{1}{\alpha!}\left(\left(\frac{\partial}{\partial w}\right)^\alpha\tilde{F}\right)\bigg|_{w=0}d\tilde{w}.$$

Thus, the uniqueness holds.

Finally, we prove inequality (2.4). Let $f = \sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}} b_\alpha w^\alpha$ be a holomorphic function on Δ^{n_1}. As $\rho_1(w) = \rho_1(|w_1|,\dots,|w_{n_1}|)$ for any $w \in \Delta^{n_1}$, we have

$$\begin{aligned}&\int_{\Delta^{n_1}} |f|^2\rho_1 d\lambda_{n_1}\\ &= \sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}} (2\pi)^{n_1}|b_\alpha|^2 \int_{\{0\leqslant r_1\leqslant 1\}\times\ldots\{0\leqslant r_{n_1}\leqslant 1\}} \left(\prod_{1\leqslant j\leqslant n_1} r_j^{2\alpha_j}\right)\rho_1(r_1,\dots,r_{n_1})dr_1\ldots dr_{n_1}\\ &= \sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}} |b_\alpha|^2\int_{\Delta^{n_1}} |w^\alpha|^2\rho_1 d\lambda_{n_1}.\end{aligned} \tag{2.5}$$

It follows from equality (2.3), equality (2.5) and Fubini's theorem that

$$\begin{aligned}\int_M |F|^2\rho &= \int_{\Delta^{n_1}\times Y}\left|\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}} \pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha)\right|^2 \pi_1^*(\rho_1)\pi_2^*(\rho_2)\\ &= \sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}\left(\int_{\Delta^{n_1}}|w^\alpha dw|^2\rho_1\right)\left(\int_Y |F_\alpha|^2\rho_2\right).\end{aligned} \tag{2.6}$$

As $\int_M |F|^2\rho < +\infty$ and the Lebesgue measure of $\{w \in \Delta^{n_1} : \rho_1(w) > 0\}$ is a positive number, equality (2.6) implies that $\int_Y |F_\alpha|^2\rho_2 < +\infty$ for any $\alpha \in \mathbb{Z}^{n_1}_{\geqslant 0}$. □

Let $M_1 \subset M$ be an n-dimensional complex manifold satisfying that $\{o\}\times Y \subset M_1$, where o is the origin in Δ^{n_1}.

Lemma 2.23. *For any holomorphic $(n,0)$ form F on M_1, there exist a unique sequence of holomorphic $(n_2,0)$ forms $\{F_\alpha\}_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}$ on Y and a neighborhood $M_2 \subset M_1$ of $\{o\}\times Y$, such that*

$$F = \sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}} \pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha)$$

on M_2, where the right term of the above equality is uniformly convergent on any compact subset of M_2. Moreover, if $\int_{M_1}|F|^2\rho < +\infty$, we have

$$\int_K |F_\alpha|^2\rho_2 < +\infty$$

for any compact subset K of Y and $\alpha \in \mathbb{Z}^{n_1}_{\geqslant 0}$.

Proof. For any open subset V of Y satisfying $V \Subset Y$, there exists $s_V \in (0,1)$ such that $\Delta^{n_1}_{s_V}\times V \subset M_1$, where $\Delta_{s_V} = \{w \in \mathbb{C} : |w| < s_V\}$. It follows from Lemma 2.22 that there exists a sequence of holomorphic $(n_2,0)$ forms $\{F_{V,\alpha}\}_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}$ on V such that

$$F = \sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}} \pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_{V,\alpha})$$

on $\Delta^{n_1}_{s_V}\times V$, where the right term of the above equality is uniformly convergent on any compact subset of $\Delta^{n_1}_{s_V}\times V$. If $\int_{M_1}|F|^2\rho < +\infty$, Lemma 2.22 shows that

$$\int_V |F_{V,\alpha}|^2\rho_2 < +\infty.$$

Following from the uniqueness of decomposition in Lemma 2.22, we get that there exists a unique sequence of holomorphic $(n_2,0)$ forms $\{F_\alpha\}_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}$ on Y and a neighborhood $M_2 \subset M_1$ of $\{o\}\times Y$, such that

$$F = \sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}} \pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha)$$

on M_2, where the right term of the above equality is uniformly convergent on any compact subset of M_2. Moreover, if $\int_{M_1}|F|^2\rho<+\infty$, we have

$$\int_K |F_\alpha|^2\rho_2<+\infty$$

for any compact subset K of Y and $\alpha\in\mathbb{Z}_{\geqslant 0}^{n_1}$. $\square$

Let $M=X\times Y$ be n-dimensional complex manifold, and let K_M be the canonical (holomorphic) line bundle on M, where X is an n_1-dimensional weakly pseudoconvex Kähler manifold, Y is an n_2-dimensional weakly pseudoconvex Kähler manifold, and $n=n_1+n_2$. Let K_X and K_Y be the canonical (holomorphic) line bundles on X and Y respectively. Let π_X and π_Y be the natural projections from M to X and Y respectively. It is clear that $(M,\emptyset,\emptyset)$ satisfies condition (A).

Let ψ_1 be a plurisubharmonic function on X, and let φ_1 be a Lebesgue measurable function on X such that $\varphi_1+\psi_1$ is plurisubharmonic. Let φ_2 be a plurisubharmonic function on Y. Denote $\varphi:=\pi_X^*(\varphi_1)+\pi_Y^*(\varphi_2)$ and $\psi:=\pi_X^*(\psi_1)$ on M. Let $T=-\sup_M\psi$, and let $c\in\mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty}c(s)\mathrm{e}^{-s}ds<+\infty$.

Let $Z_0\subset X$ be a subset of $\{\psi_1=-\infty\}$ such that $Z_0\cap Supp\,(\mathcal{O}_X/\mathcal{I}(\varphi_1+\psi_1))\neq\emptyset$, and let $\tilde{Z}_0=Z_0\times Y\subset M$. Let $U\supset Z_0$ be an open subset of X, and let f_1 be a holomorphic $(n_1,0)$ form on U. Let f_2 be a holomorphic $(n_2,0)$ form on Y, and let $f=\pi_X^*(f_1)\wedge\pi_Y^*(f_2)$ on $U\times Y$. Let $\mathcal{F}_x\supset\mathcal{I}(\varphi_1+\psi_1)_x$ be an ideal of $\mathcal{O}_{X,x}$ for any $x\in Z_0$. Let $\tilde{\mathcal{F}}_z\supset\mathcal{I}(\varphi+\psi)_z$ be an ideal of $\mathcal{O}_{M,z}$ for any $z\in\tilde{Z}_0$. For any $x\in Z_0$ and any holomorphic function g, assume that $(g,(x,y))\in\tilde{\mathcal{F}}_{(x,y)}$ for any $y\in Y$ if and only if $(g(\cdot,y),x)\in\mathcal{F}_x$ for any $y\in Y$.

Denote

$$\inf\Bigg\{\int_{\{\psi_1<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi_1}c(-\psi_1):(\tilde{f}-f_1)\in H^0(Z_0,(\mathcal{O}(K_X)\otimes\mathcal{F})|_{Z_0})\\ \&\ \tilde{f}\in H^0(\{\psi_1<-t\},\mathcal{O}(K_X))\Bigg\}$$

by $G_X(t)$, where $t\in[T,+\infty)$, $|f|^2:=\sqrt{-1}^{n_1^2}f\wedge\bar{f}$ for any $(n_1,0)$ form f and $(\tilde{f}-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ means $(\tilde{f}-f,x)\in\mathcal{O}(K_X)_x\otimes\mathcal{F}_x$ for all $x\in Z_0$. Denote

$$\inf\Bigg\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):(\tilde{f}-f)\in H^0(\tilde{Z}_0,(\mathcal{O}(K_M)\otimes\tilde{\mathcal{F}})|_{\tilde{Z}_0})\\ \&\ \tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\Bigg\}$$

by $G_M(t)$, where $t\in[T,+\infty)$.

Theorem 2.2 shows that $G_X(h^{-1}(r))$ and $G_M(h^{-1}(r))$ are concave with respect to r, where $h(t)=\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds$. The following Proposition gives a property of the minimal L^2 integrals on fibration, which implies that $G_M(h^{-1}(r))$ is linear with respect to r if and only if $G_X(h^{-1}(r))$ is linear with respect to r.

Proposition 2.24. *$G_M(t)=G_X(t)\int_Y|f_2|^2\mathrm{e}^{-\varphi_2}$ holds for any $t\geqslant T$. Moreover, if $G_X(t)<+\infty$, there exists a holomorphic $(n_1,0)$ form F_1 on $\{\psi_1<-t\}$ such that $(F_{1,t}-f_1)\in H^0(Z_0,(\mathcal{O}(K_X)\otimes\mathcal{F})|_{Z_0})$, $G_X(t)=\int_{\{\psi_1<-t\}}|F_{1,t}|^2\mathrm{e}^{-\varphi_1}c(-\psi_1)$ and $G_M(t)=\int_{\{\psi<-t\}}|\pi_X^*(F_{1,t})\wedge\pi_2^*(f_2)|^2\mathrm{e}^{-\varphi}c(-\psi)$.*

Proof. Let $\tilde{f}_1$ be a holomorphic $(n_1,0)$ form on $\{\psi_1<-t\}$ satisfying $(\tilde{f}_1-f_1)\in H^0(Z_0,(\mathcal{O}(K_X)\otimes\mathcal{F})|_{Z_0})$, where $t\geqslant T$. As $f=\pi_X^*(f_1)\wedge\pi_Y^*(f_2)$ and $\tilde{Z}_0=Z_0\times Y$, it follows from the relationship between $\tilde{\mathcal{F}}$ and $\mathcal{F}$ that $(\pi_X^*(\tilde{f}_1)\wedge\pi_Y^*(f_2)-f)\in H^0(\tilde{Z}_0,(\mathcal{O}(K_M)\otimes\tilde{\mathcal{F}})|_{\tilde{Z}_0})$. By the definitions of $G_X(t)$ and $G_M(t)$, we obtain that

$$G_M(t)\leqslant G_X(t)\int_Y|f_2|^2\mathrm{e}^{-\varphi_2} \tag{2.7}$$

for any $t\geqslant T$.

Let $t \geqslant T$. If $G_M(t) = +\infty$, inequality (2.7) implies that $G_X(t)\int_Y |f_2|^2 \mathrm{e}^{-\varphi_2} = G_M(t) = +\infty$. Thus, assume that $G_M(t) < +\infty$. Lemma 2.5 shows that there exists a holomorphic $(n,0)$ form F_t on $\{\psi < -t\}$ such that $(F_t - f) \in H^0(\tilde{Z}_0, (\mathcal{O}(K_M) \otimes \tilde{\mathcal{F}})|_{\tilde{Z}_0})$ and $G_M(t) = \int_{\{\psi<-t\}} |F_t|^2 \mathrm{e}^{-\varphi} c(-\psi)$. For any $y_0 \in Y$, let $w = (w_1, \ldots, w_{n_2})$ be a coordinate on a neighborhood U of y_0 satisfying $w(y_0) = 0$ and $w(U) = \Delta^{n_2}$. Lemma 2.22 implies that $F_t|_{X\times U} = \sum_{\alpha\in\mathbb{Z}^{n_2}_{\geqslant 0}} \pi_X^*(f_\alpha) \wedge \pi_Y^*(w^\alpha dw)$, where f_α is a holomorphic $(n_1,0)$ form on $\{\psi_1 < -t\}$ for any $\alpha \in \mathbb{Z}^{n_2}_{\geqslant 0}$. There exists a holomorphic function $\tilde{f}_2(w)$ on U such that $f_2 = \tilde{f}_2(w)dw$ on U. As $(g,(x,y)) \in \tilde{\mathcal{F}}_{(x,y)}$ for any $y \in Y$ if and only if $(g(\cdot,y),x) \in \mathcal{F}_x$ for any $y \in Y$, where $x \in Z_0$ and g is a holomorphic function, it follows from $(F_t - f) \in H^0(\tilde{Z}_0, (\mathcal{O}(K_M)\otimes\tilde{\mathcal{F}})|_{\tilde{Z}_0})$ and $f = \pi_X^*(f_1)\wedge\pi_Y^*(f_2)$ that $(\sum_{\alpha\in\mathbb{Z}^{n_2}_{\geqslant 0}} w^\alpha f_\alpha - \tilde{f}_2(w)f_1) \in H^0(Z_0, (\mathcal{O}(K_X)\otimes\mathcal{F})|_{Z_0})$ for any $w \in \Delta^{n_2}$. Let U_1 be an open subset of U, and let $V = w(U_1) \subset \Delta^{n_2}$. Following Fubini's theorem and the definition of $G_X(t)$, we have

$$\begin{aligned}\int_{\{\psi_1<-t\}\times U_1} |F_t|^2 \mathrm{e}^{-\varphi} c(-\psi) &= \int_V \left(\int_{\{\psi_1<-t\}} |\sum_{\alpha\in\mathbb{Z}^{n_2}_{\geqslant 0}} w^\alpha f_\alpha|^2 \mathrm{e}^{-\varphi_1} c(-\psi_1)\right) \mathrm{e}^{-\varphi_2} |dw|^2 \\ &\geqslant G_X(t) \int_V |\tilde{f}_2(w)dw|^2 \mathrm{e}^{-\varphi_2} \\ &= G_X(t) \int_{U_1} |f_2|^2 \mathrm{e}^{-\varphi_2},\end{aligned}$$

which implies $G_M(t) = \int_{\{\psi<-t\}} |F_t|^2 \mathrm{e}^{-\varphi} c(-\psi) \geqslant G_X(t)\int_Y |f_2|^2 \mathrm{e}^{-\varphi_2}$. Thus, we have $G_M(t) = G_X(t)\int_Y |f_2|^2 \mathrm{e}^{-\varphi_2}$ for any $t \geqslant T$. If $G_X(t) < +\infty$, it follows from Lemma 2.5 that there exists a holomorphic $(n_1,0)$ for $F_{1,t}$ on $\{\psi_1 < -t\}$ satisfying that $(F_{1,t} - f_1) \in H^0(Z_0, (\mathcal{O}(K_X)\otimes\mathcal{F})|_{Z_0})$ and $G_X(t) = \int_{\{\psi<-t\}} |F_{1,t}|^2 \mathrm{e}^{-\varphi_1} c(-\psi_1)$, hence $G_M(t) = G_X(t)\int_Y |f_2|^2 \mathrm{e}^{-\varphi_2} = \int_{\{\psi<-t\}} |\pi_X^*(F_{1,t}) \wedge \pi_2^*(f_2)|^2 \mathrm{e}^{-\varphi} c(-\psi)$. □

We recall a result about multiplier ideal sheaves.

Lemma 2.25. *Let Φ_1 and Φ_2 be plurisubharmonic functions on Δ^n satisfying $\Phi_2(o) > -\infty$, where $n \in \mathbb{Z}_{>0}$ and o is the origin in Δ^n. Then $\mathcal{I}(\Phi_1)_o = \mathcal{I}(\Phi_1+\Phi_2)_o$.*

Proof. For convenience of the reader, we give the proof. It is clear that $\mathcal{I}(\Phi_1+\Phi_2)_o \subset \mathcal{I}(\Phi_1)_o$. Let f be a holomorphic function on a neighborhood of o satisfying $(f,o) \in \mathcal{I}(\Phi_1)_o$. Following from the strong openness property of multiplier ideal sheaves ([43]) and $\Phi_2(o) > -\infty$, there exist $s > 1$ and $r > 0$ such that

$$\int_{|w|<r} |f|^{2s} \mathrm{e}^{-s\Phi_1} d\lambda_n < +\infty \tag{2.8}$$

and

$$\int_{|w|<r} \mathrm{e}^{-\frac{s}{s-1}\Phi_2} d\lambda_n < +\infty, \tag{2.9}$$

where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$. Combining inequality (2.8), inequality (2.9) and the Hölder inequality, we have

$$\int_{|w|<r} |f|^2 \mathrm{e}^{-\Phi_1-\Phi_2} d\lambda_n \leqslant \left(\int_{|w|<r} |f|^{2s} \mathrm{e}^{-s\Phi_1} d\lambda_n\right)^{\frac{1}{s}} \left(\int_{|w|<r} \mathrm{e}^{-\frac{s}{s-1}\Phi_2} d\lambda_n\right)^{1-\frac{1}{s}} < +\infty,$$

which implies that $(f,o) \in \mathcal{I}(\Phi_1+\Phi_2)_o$. Thus, we have $\mathcal{I}(\Phi_1)_o = \mathcal{I}(\Phi_1+\Phi_2)_o$. □

In the following, we consider fibrations over products of open Riemann surfaces. Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leqslant j \leqslant n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold. Let $M = \left(\prod_{1\leqslant j\leqslant n_1} \Omega_j\right) \times Y$ be an n-dimensional complex manifold, where $n = n_1 + n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\leqslant j\leqslant n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M.

Let $Z_j = \{z_{j,k} : 1 \leqslant k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1,\ldots,n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geqslant 2} \cup \{+\infty\}$. Denote $Z_0 := \left(\prod_{1\leqslant j\leqslant n_1} Z_j\right) \times Y \subset M$. Let $p_{j,k}$ be a positive number such that $\sum_{1\leqslant k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\not\equiv-\infty$ for any j, and let

$$\psi=\max_{1\leqslant j\leqslant n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$$

on M. For any $j\in\{1,\ldots,n_1\}$, let φ_j be a subharmonic function on Ω_j such that $\varphi_j(z)>-\infty$ for any $z\in Z_j$. Let φ_Y be a plurisubharmonic function on Y, and denote $\varphi:=\sum_{1\leqslant j\leqslant n_1}\pi_{1,j}^*(\varphi_j)+\pi_2^*(\varphi_Y)$. Let c be a positive function on $(0,+\infty)$ such that $\int_0^{+\infty}c(t)\mathrm{e}^{-t}dt<+\infty$ and $c(t)\mathrm{e}^{-t}$ is decreasing on $(0,+\infty)$. Let f be a holomorphic $(n,0)$ form on $\{\psi<-t_0\}$ satisfying $\int_{\{\psi<-t_0\}}|f|^2\mathrm{e}^{-\varphi}c(-\psi)<+\infty$, where $t_0>0$ is constant.

Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):(\tilde{f}-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z\text{ for any }z\in Z_0\right.$$
$$\left.\&\ \tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right\}$$

by $G(t)$, where $t\in[0,+\infty)$, and denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):(\tilde{f}-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z\text{ for any }z\in Z_0\right.$$
$$\left.\&\ \tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right\}$$

by $\tilde{G}(t)$, where $t\in[0,+\infty)$.

Lemma 2.26. *Let $t\geqslant 0$. If $\tilde{G}(t)<+\infty$, there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ satisfying that $(F_t-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$ and $G(t)=\tilde{G}(t)=\int_{\{\psi<-t\}}|F|^2\mathrm{e}^{-\varphi}c(-\psi)$.*

Proof. As $\mathcal{I}(\varphi+\psi)\subset\mathcal{I}(\psi)$, we have $\tilde{G}(t)\leqslant G(t)$. It follows from Lemma 2.5 that there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ satisfying that $\tilde{G}(t)=\int_{\{\psi<-t\}}|F_t|^2\mathrm{e}^{-\varphi}c(-\psi)$ and $(F_t-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z$ for any $z\in Z_0$.

Let $z_0=(z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}})\in\prod_{1\leqslant j\leqslant n_1}\Omega_j$, where $1\leqslant\beta_j<\tilde{m}_j$ for any $1\leqslant j\leqslant n_1$. It follows from Lemma 2.12 and Lemma 2.13 that there exists a local coordinate w_j on a neighborhood $V_{z_{j,\beta_j}}\Subset\Omega_j$ of $z_{j,\beta_j}\in\Omega_j$ satisfying $w_j(z_{j,\beta_j})=0$ and

$$\log|w_j|=\frac{1}{p_{j,\beta_j}}\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})$$

on $V_{z_{j,\beta_j}}$ for any $j\in\{1,\ldots,n_1\}$. Denote $V_0:=\prod_{1\leqslant j\leqslant n_1}V_{z_{j,\beta_j}}$ and $w:=(w_1,\ldots,w_{n_1})$ on V_0. Thus, there exists $t_1>\max\{t,t_0\}$ such that

$$\left\{z\in\Omega_j:2\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(z,z_{j,k})<-t_1\right\}\cap V_{z_{j,\beta_j}}\Subset V_{z_{j,\beta_j}}$$

for any $1\leqslant j\leqslant n_1$. Let $\psi_1=\max_{1\leqslant j\leqslant n_1}\left\{\tilde{\pi}_j^*\left(2\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$ on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leqslant j\leqslant n_1}\Omega_j$ to Ω_j. Note that

$$\{\psi<-t_1\}=\{\psi_1<-t_1\}\times Y$$

and

$$\{\psi_1<-t_1\}\cap V_0=\prod_{1\leqslant j\leqslant n_1}\left\{|w_j|<\mathrm{e}^{-\frac{t_1}{2p_{j,\beta_j}}}\right\}.$$

As φ_j is a subharmonic function on Ω_j, $\int_{\{\psi<-t_1\}}|f|^2\mathrm{e}^{-\varphi}c(-\psi)\leqslant\int_{\{\psi<-t_0\}}|f|^2\mathrm{e}^{-\varphi}c(-\psi)<+\infty$ implies that $\int_{\{\psi<-t_1\}}|f|^2\mathrm{e}^{-\pi_2^*(\varphi_Y)}c(-\psi)<+\infty$ and $\int_{\{\psi<-t_1\}}|F_t|^2\mathrm{e}^{-\varphi}c(-\psi)<+\infty$ implies that

$\int_{\{\psi<-t_1\}}|F_t|^2\mathrm{e}^{-\pi_2^*(\varphi_Y)}c(-\psi)<+\infty$. It follows from Lemma 2.22 that there exist a sequence of holomorphic $(n_2,0)$ forms $\{f_\alpha\}_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}$ on Y and a sequence of holomorphic $(n_2,0)$ forms $\{F_\alpha\}_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}$ on Y such that

$$f=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha) \tag{2.10}$$

on $(\{\psi_1<-t_1\}\cap V_0)\times Y$,

$$F_t=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha) \tag{2.11}$$

on $(\{\psi_1<-t_1\}\cap V_0)\times Y$,

$$\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi_Y}<+\infty \tag{2.12}$$

for any $\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}$ and

$$\int_Y|F_\alpha|^2\mathrm{e}^{-\varphi_Y}<+\infty \tag{2.13}$$

for any $\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}$, where the right terms of the equalities (2.10) and (2.11) are uniformly convergent on any compact subset of $(\{\psi_1<-t_1\}\cap V_0)\times Y$. As $(F_t-f,(z_0,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_{(z_0,y)}$ for any $y\in Y$, it follows from Lemma 2.18 that

$$f_\alpha=F_\alpha$$

for any $\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}$ satisfying $\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}\leqslant 1$. Define

$$R:=\bigg\{\alpha\in\mathbb{Z}^{n_1}_{\geqslant 0}:\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}>1\bigg\}.$$

Lemma 2.18 shows that $(w^\alpha,z_0)\in\mathcal{I}(\psi_1)_{z_0}$ for any $\alpha\in R$. It follows from inequality (2.12) and inequality (2.13) that $(\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha),(z_0,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi+\pi_2^*(\varphi_Y)))_{(z_0,y)}$ and $(\pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha),(z_0,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi+\pi_2^*(\varphi_Y)))_{(z_0,y)}$ for any $y\in Y$ and any $\alpha\in R$. As $\varphi_j(z_{j,\beta_j})>-\infty$, using Lemma 2.25, we obtain that

$$(\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha),(z_0,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_{(z_0,y)}$$

and

$$(\pi_1^*(w^\alpha dw)\wedge\pi_2^*(F_\alpha),(z_0,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_{(z_0,y)}$$

for any $y\in Y$ and any $\alpha\in R$. It follows from equality (2.10), equality (2.11) and Lemma 2.17 that

$$\begin{aligned}(f-F_t,(z_0,y))&=\Big(\sum_{\alpha\in R}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha-F_\alpha),(z_0,y)\Big)\\&\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_{(z_0,y)}\end{aligned}$$

holds for any $y\in Y$. Hence we have $(F_t-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$, which implies that $G(t)\leqslant\int_{\{\psi<-t\}}|F_t|^2\mathrm{e}^{-\varphi}c(-\psi)=\tilde{G}(t)$. Thus, we obtain that $G(t)=\tilde{G}(t)=\int_{\{\psi<-t\}}|F_t|^2\mathrm{e}^{-\varphi}c(-\psi)$. □

The following two lemmas will be used in the proof of Lemma 2.29.

Lemma 2.27 (see [40]). *Let $c(t)$ be a positive measurable function on $(0,+\infty)$, and let $a\in\mathbb{R}$. Assume that $\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds\in(0,+\infty)$ when t near $+\infty$. Then we have*

(1) $\lim_{t\to+\infty}\frac{\int_t^{+\infty}c(s)\mathrm{e}^{-as}ds}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}=1$ *if and only if $a=1$;*

(2) $\lim_{t\to+\infty}\frac{\int_t^{+\infty}c(s)\mathrm{e}^{-as}ds}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}=0$ *if and only if $a>1$;*

(3) $\lim_{t\to+\infty}\frac{\int_t^{+\infty}c(s)\mathrm{e}^{-as}ds}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}=+\infty$ *if and only if $a<1$.*

The following Lemma belongs to Fornaess and Narasimhan on approximation property of plurisubharmonic functions of Stein manifolds.

Lemma 2.28 ([23]). *Let X be a Stein manifold and $\varphi \in PSH(X)$. Then there exists a sequence $\{\varphi_n\}_{n=1,\cdots}$ of smooth strongly plurisubharmonic functions such that $\varphi_n \downarrow \varphi$.*

It follows from Lemma 2.12 and Lemma 2.13 that there eixsts a local coordinate $w_{j,k}$ on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ and

$$\log |w_{j,k}| = \frac{1}{p_{j,k}} \sum_{1 \leqslant k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})$$

for any $j \in \{1, \ldots, n_1\}$ and $1 \leqslant k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote $\tilde{I}_1 := \{(\beta_1, \ldots, \beta_{n_1}) : 1 \leqslant \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \ldots, n_1\}\}$, $V_\beta := \prod_{1 \leqslant j \leqslant n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \ldots, \beta_n) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n,\beta_n})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n,\beta_n}) \in M$. Let

$$\psi_1 = \max_{1 \leqslant j \leqslant n_1} \left\{ \tilde{\pi}_j^* \left(2 \sum_{1 \leqslant k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on $\prod_{1 \leqslant j \leqslant n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leqslant j \leqslant n_1} \Omega_j$ to Ω_j. Note that $\psi = \pi_1^*(\psi_1)$.

Let F be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\} \subset M$ for some $t_0 > 0$ satisfying $\int_{\{\psi < -t_0\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) < +\infty$. Without loss of generality, we can assume $\bigcup \beta \in \tilde{I}_1 V_\beta \times Y \subset \{\psi < -t_0\}$. For any $\beta \in \tilde{I}_1$, it follows from Lemma 2.22 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(F_{\alpha,\beta})$$

on $V_\beta \times Y$ and

$$\int_Y |F_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y} < +\infty$$

for any $\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1}$. Denote $E_\beta := \left\{ \alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1 \leqslant j \leqslant n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1 \right\}$, $E_{1,\beta} := \left\{ \alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1 \leqslant j \leqslant n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} < 1 \right\}$ and $E_{2,\beta} := \left\{ \alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1 \leqslant j \leqslant n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} > 1 \right\}$.

Lemma 2.29. *If $\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) \mathrm{e}^{-s} ds} < +\infty$, we have $F_{\alpha,\beta} \equiv 0$ for any $\alpha \in E_{1,\beta}$ and $\beta \in \tilde{I}_1$, and*

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) \mathrm{e}^{-s} ds} \geqslant \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} \mathrm{e}^{-\sum_{1 \leqslant j \leqslant n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leqslant j \leqslant n_1} (\alpha_j + 1)} \int_Y |F_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y}.$$

Proof. Fix any $\beta \in \tilde{I}_1$. As $\Phi := \sum_{1 \leqslant j \leqslant n_1} \tilde{\pi}_j^*(\varphi_j)$ is upper semi-continuous, for any $\epsilon > 0$, there exists $t_\beta > t_0$ such that $\{\psi_1 < -t_\beta\} \cap V_\beta \Subset V_\beta$ and

$$\sup_{z \in \{\psi_1 < -t_\beta\} \cap V_\beta} \Phi(z) < \Phi(z_\beta) + \epsilon.$$

For any $t \geqslant t_\beta$, note that $\{\psi_1 < -t\} = \prod_{1 \leqslant j \leqslant n_1} \left\{ |w_{j,\beta_j}| < \mathrm{e}^{-\frac{t}{2p_{j,\beta_j}}} \right\}$ and $F = \sum_{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(F_{\alpha,\beta})$ on $\{\psi_1 < -t\} \times Y$, then we have

$$\begin{aligned}
&\int_{\{\psi < -t\} \cap (V_\beta \times Y)} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) \\
\geqslant &\mathrm{e}^{-\Phi(z_\beta) - \epsilon} \int_{(\{\psi_1 < -t\} \cap V_\beta) \times Y} |F|^2 \mathrm{e}^{-\pi_2^*(\varphi_Y)} c(-\pi_1^*(\psi_1)) \\
= &\mathrm{e}^{-\Phi(z_\beta) - \epsilon} \sum_{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1}} \int_{\{\psi_1 < -t\}} |w_\beta^\alpha dw_\beta|^2 c(-\psi) \times \int_Y |F_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y}.
\end{aligned}$$

Denote $q_\alpha := \sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}-1$. It follows from Lemma 2.20 that

$$\begin{aligned}&\int_{\{\psi<-t\}\cap(V_\beta\times Y)}|F|^2\mathrm{e}^{-\varphi}c(-\psi)\\ \geqslant&\mathrm{e}^{-\Phi(z_\beta)-\epsilon}\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant0}}\left(\int_t^{+\infty}c(s)\mathrm{e}^{-(q_\alpha+1)s}ds\right)\frac{(q_\alpha+1)(2\pi)^{n_1}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}.\end{aligned}$$

Taking $t\to+\infty$ and $\epsilon\to0+0$, by $\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2\mathrm{e}^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}<+\infty$ and Lemma 2.27, we have

$$F_{\alpha,\beta}\equiv0$$

for any α satisfying $q_\alpha<0$ and

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(V_\beta\times Y)}|F|^2\mathrm{e}^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}\geqslant\mathrm{e}^{-\Phi(z_\beta)}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|F_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)}.$$

Note that $V_\beta\cap V_{\tilde\beta}=\emptyset$ for any $\beta\neq\tilde\beta$ and $\{\psi_1<-t_\beta\}\cap V_\beta\Subset V_\beta$ for any $\beta\in\tilde I_1$. Then we have

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2\mathrm{e}^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}\geqslant\sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}.$$

Thus, Lemma 2.29 holds. □

Let M_1 be an open complex submanifold of M satisfying that $Z_0=\{z_\beta:\beta\in\tilde I_1\}\times Y\subset M_1$, and let K_{M_1} be the canonical (holomorphic) line bundle on M_1. Let F_1 be a holomorphic $(n,0)$ form on $\{\psi<-t_0\}\cap M_1$ for $t_0>0$ satisfying that $\int_{\{\psi<-t_0\}\cap M_1}|F_1|^2\mathrm{e}^{-\varphi}c(-\psi)<+\infty$. For any $\beta\in\tilde I_1$, it follows from Lemma 2.23 that there exist a sequence of holomorphic $(n_2,0)$ forms $\{F_{\alpha,\beta}\}_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant0}}$ on Y and an open subset U_β of $\{\psi<-t_0\}\cap M_1\cap(V_\beta\times Y)$ such that

$$F_1=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geqslant0}}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(F_{\alpha,\beta})$$

on U_β and

$$\int_K|F_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}<+\infty$$

for any $\alpha\in\mathbb{Z}^{n_1}_{\geqslant0}$ and compact subset K of Y.

Lemma 2.30. *If $\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}<+\infty$, we have $F_{\alpha,\beta}\equiv0$ for any $\alpha\in E_{1,\beta}$ and $\beta\in\tilde I_1$ and*

$$\begin{aligned}&\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}\\ \geqslant&\sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}.\end{aligned}$$

Proof. Note that $\psi_1=\max_{1\leqslant j\leqslant n_1}\left\{\tilde\pi_j^*\left(2\sum_{1\leqslant k<\tilde m_j}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$ on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$. For any $\beta\in\tilde I_1$ and any open subset V of Y satisfying $V\Subset Y$, it follows from Lemma 2.12 and Lemma 2.13 that there exists $t_{\beta,V}>t_0$ such that $\{\psi_1<-t_{\beta,V}\}\times V\Subset U_\beta$. $\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}<+\infty$ implies that

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi_1<-t\}\times V}|F|^2\mathrm{e}^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}<+\infty.\tag{2.14}$$

It follows from equality (2.14) and Lemma 2.29 that $F_{\alpha,\beta} \equiv 0$ on V for any $\alpha \in E_{1,\beta}$ and

$$\begin{aligned}\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}\cap U_\beta} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)\mathrm{e}^{-s}ds} &\geqslant \liminf_{t\to+\infty} \frac{\int_{\{\psi_1<-t\}\times V} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)\mathrm{e}^{-s}ds} \\ &\geqslant \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} \mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)} \int_V |F_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y}.\end{aligned}$$

Following from the arbitrariness of V, we have

$$F_{\alpha,\beta} \equiv 0$$

on Y for any $\alpha \in E_{1,\beta}$ and

$$\begin{aligned}&\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}\cap U_\beta} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)\mathrm{e}^{-s}ds} \\ &\geqslant \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} \mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)} \int_Y |F_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y}.\end{aligned} \tag{2.15}$$

$V_\beta \cap V_{\beta'} = \emptyset$ for any $\beta \neq \beta'$ implies that $U_\beta \cap U_{\beta'} = \emptyset$ for any $\beta \neq \beta'$. It follows from inequality (2.15) that

$$\begin{aligned}&\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}\cap M_1} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)\mathrm{e}^{-s}ds} \\ &\geqslant \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} \mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)} \int_Y |F_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y}.\end{aligned}$$

Thus, Lemma 2.30 holds. □

In the following, we consider the case where Z_j is a single point set. Let $M' = \prod_{1\leqslant j\leqslant n_1} \Omega_j$ be an n_1-dimensional complex manifold, and let $K_{M'}$ be the canonical (holomorphic) line bundle on M'. Let $z_j \in \Omega_j$ and $z_0 = (z_1, \ldots, z_{n_1}) \in M'$. Let φ_j be subharmonic functions on Ω_j such that $\varphi_j(z_j) > -\infty$. Denote

$$\psi_1 := \max_{1\leqslant j\leqslant n_1} \left\{2p_j \tilde{\pi}_j^*(G_{\Omega_j}(\cdot, z_j))\right\}$$

and $\tilde{\varphi} := \sum_{1\leqslant j\leqslant n_1} \tilde{\pi}_j^*(\varphi_j)$ on M', where p_j is a positive real number for $1 \leqslant j \leqslant n_1$ and $\tilde{\pi}_j$ is the natural projection from M' to Ω_j.

Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote $V_0 := \prod_{1\leqslant j\leqslant n_1} V_{z_j}$, and $w := (w_1, \ldots, w_{n_1})$ is a local coordinate on V_0 of $z_0 \in M'$. Take $E = \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_j} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geqslant 0}\right\}$.

Let $c_j(z)$ be the logarithmic capacity on Ω_j (see [64], see also Section 1.1).

Lemma 2.31 (see [34]). *Let $c(t)$ be a positive function on $(0, +\infty)$ satisfying that $c(t)\mathrm{e}^{-t}$ is decreasing and $\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds < +\infty$. For any $\alpha \in E$, there exists a holomorphic $(n_1, 0)$ form F on M', which satisfies that $(F - w^\alpha dw, z_0) \in (\mathcal{O}(K_{\Omega_j}) \otimes \mathcal{I}(\psi_1))_{z_0}$ and*

$$\int_{M'} |F|^2 \mathrm{e}^{-\tilde{\varphi}} c(-\psi_1) \leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right) \frac{(2\pi)^{n_1} \mathrm{e}^{-\tilde{\varphi}(z_0)}}{\Pi_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}.$$

As φ_j is subharmonic on Ω_j, it follows from Lemma 2.31 and Lemma 2.5 that there exists a holomorphic $(1, 0)$ form f_{j,α_j} on Ω_j such that $(f_{j,\alpha_j} - w_j^{\alpha_j} dw_j, z_j) \in (\mathcal{O}(K_{\Omega_j}) \otimes \mathcal{I}(2(\alpha_j+1)G_{\Omega_j}(\cdot, z_j)))_{z_j}$ and $\int_{\Omega_j} |f_{j,\alpha_j}|^2 \mathrm{e}^{-\varphi_j} = \inf\big\{\int_{\Omega_j} |\tilde{f}|^2 \mathrm{e}^{-\varphi_j} : \tilde{f} \in H^0(\Omega_j, \mathcal{O}(K_{\Omega_j})) \,\&\, (\tilde{f} - w_j^{\alpha_j} dw_j, z_j) \in (\mathcal{O}(K_{\Omega_j}) \otimes \mathcal{I}(2(\alpha_j + 1)G_{\Omega_j}(\cdot, z_j)))_{z_j}\big\} < +\infty$ for any $\alpha \in E$ and $j \in \{1, \ldots, n_1\}$.

Lemma 2.32 (see [40]). *$F=\sum_{\alpha\in E}d_\alpha\prod_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(f_{j,\alpha_j})$ is a holomorphic $(n_1,0)$ form on M' satisfying that $(F-\sum_{\alpha\in E}d_\alpha w^\alpha dw,z_0)\in\mathcal{O}(K_{M'})\otimes\mathcal{I}(\psi_1))_{z_0}$,*

$$\int_{M'}|F|^2\mathrm{e}^{-\tilde{\varphi}}=\sum_{\alpha\in E}|d_\alpha|^2\int_{M'}\bigg|\prod_{1\leqslant j\leqslant n_1}\pi_j^*(f_{j,\alpha_j})\bigg|^2\mathrm{e}^{-\tilde{\varphi}}$$

and $\int_{M'}|F|^2\mathrm{e}^{-\tilde{\varphi}}=\inf\big\{\int_{M'}|\tilde{F}|^2\mathrm{e}^{-\tilde{\varphi}}:\tilde{F}$ is a holomorphic $(n_1,0)$ form on M' such that $(\tilde{F}-\sum_{\alpha\in E}d_\alpha w^\alpha dw,z_0)\in\mathcal{O}(K_{M'})\otimes\mathcal{I}(\psi_1))_{z_0}\big\}$, where d_α is a constant for any $\alpha\in E$.

Let φ_Y be a plurisubharmonic function on Y. Let f_α be a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi_Y}<+\infty$ for any $\alpha\in E$. Let $f=\sum_{\alpha\in E}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha)$ be a holomorphic $(n,0)$ form on $V_0\times Y\subset M=M'\times Y$. Denote $\varphi:=\pi_1^*(\tilde{\varphi})+\pi_2^*(\varphi_Y)$ and $\psi:=\pi_1^*(\psi_1)$ on M.

Lemma 2.33. *$F=\sum_{\alpha\in E}\pi_{1,1}^*(f_{1,\alpha_1})\wedge\ldots\wedge\pi_{1,n_1}^*(f_{n_1,\alpha_{n_1}})\wedge\pi_2^*(f_\alpha)$ is a holomorphic $(n,0)$ form on M, and satisfies that $(F-f,(z_0,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_{(z_0,y)}$ for any $y\in Y$,*

$$\int_M|F|^2\mathrm{e}^{-\varphi}=\sum_{\alpha\in E}\left(\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi_Y}\right)\prod_{1\leqslant j\leqslant n_1}\int_{\Omega_j}|f_{j,\alpha_j}|^2\mathrm{e}^{-\varphi_j}$$

and $\int_M|F|^2\mathrm{e}^{-\varphi}=\inf\big\{\int_M|\tilde{F}|^2\mathrm{e}^{-\varphi}:\tilde{F}$ is a holomorphic $(n,0)$ form on M such that $(\tilde{F}-f,(z_0,y))\in\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_{(z_0,y)}$ for any $y\in Y\big\}$.

Proof. It follows from Lemma 2.19 that $(h,(z_0,y))\in\mathcal{I}(\psi)_{(z_0,y)}$ for any $y\in Y$ if and only if $(h(\cdot,y),z_0)\in\mathcal{I}(\psi_1)_{z_0}$. For any $\alpha\in E$, using Proposition 2.24 and Lemma 2.32, we obtain that $F_\alpha=\pi_{1,1}^*(f_{1,\alpha_1})\wedge\ldots\wedge\pi_{1,n_1}^*(f_{n_1,\alpha_{n_1}})\wedge\pi_2^*(f_\alpha)$ satisfies that $\int_M|F_\alpha|^2\mathrm{e}^{-\varphi}=\big(\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi_Y}\big)\prod_{1\leqslant j\leqslant n_1}\int_{\Omega_j}|f_{j,\alpha_j}|^2\mathrm{e}^{-\varphi_j}=\inf\big\{\int_M|\tilde{F}|^2\mathrm{e}^{-\varphi}:\tilde{F}$ is a holomorphic $(n,0)$ form on M such that $(\tilde{F}-\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha),(z_0,y))\in\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_{(z_0,y)}$ for any $y\in Y\big\}$, i.e.,

$$\int_M F_\alpha\wedge\overline{\tilde{F}}\mathrm{e}^{-\varphi}=0 \tag{2.16}$$

for any holomorphic $(n,0)$ form $\tilde{F}$ satisfying $\int_M|\tilde{F}|^2\mathrm{e}^{-\varphi}<+\infty$ and $(\tilde{F},(z_0,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_{(z_0,y)}$ for any $y\in Y$. It follows from Fubini's theorem and Lemma 2.32 that

$$\int_M F_\alpha\wedge\overline{F_{\tilde{\alpha}}}\mathrm{e}^{-\varphi}=0 \tag{2.17}$$

for any $\alpha\neq\tilde{\alpha}$. Note that $F=\sum_{\alpha\in E}F_\alpha$ and $(F-f,(z_0,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_{(z_0,y)}$ for any $y\in Y$. It follows from equality (2.16) and equality (2.17) that

$$\int_M|F|^2\mathrm{e}^{-\varphi}=\sum_{\alpha\in E}\left(\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi_Y}\right)\prod_{1\leqslant j\leqslant n_1}\int_{\Omega_j}|f_{j,\alpha_j}|^2\mathrm{e}^{-\varphi_j}$$

and $\int_M|F|^2\mathrm{e}^{-\varphi}=\inf\big\{\int_M|\tilde{F}|^2\mathrm{e}^{-\varphi}:\tilde{F}$ is a holomorphic $(n,0)$ form on M such that $(\tilde{F}-f,(z_0,y))\in\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_{(z_0,y)}$ for any $y\in Y\big\}$. □

Let X be an n_1-dimensional complex manifold, and let Y be an n_2-dimensional complex manifold. Let $M=X\times Y$ be an n-dimensional complex manifold, where $n=n_1+n_2$. Let π_1 and π_2 be the natural projections from M to X and Y respectively. We recall the following lemma.

Lemma 2.34 (see [1]). *Let $F\not\equiv 0$ be a holomorphic $(n,0)$ form on M. Let f_1 be a holomorphic $(n_1,0)$ form on an open subset U of X, and let f_2 be a holomorphic $(n_2,0)$ form on an open subset V of Y. If*

$$F=\pi_1^*(f_1)\wedge\pi_2^*(f_2)$$

on $U\times V$, there exist a holomorphic $(n_1,0)$ form F_1 on X and a holomorphic $(n_2,0)$ form F_2 on Y such that $F_1=f_1$ on U, $F_2=f_2$ on V, and

$$F=\pi_1^*(F_1)\wedge\pi_2^*(F_2)$$

on M.

2.5 Optimal jets L^2 extension

In this section, we give an optimal jets L^2 extension result, i.e., Proposition 2.37. We recall two lemmas, which will be used in the proof of Proposition 2.37.

Lemma 2.35 ([34]). *Let c be a positive function on $(0,+\infty)$, such that $\int_0^{+\infty} c(t)\mathrm{e}^{-t}dt < +\infty$ and $c(t)\mathrm{e}^{-t}$ is decreasing on $(0,+\infty)$. Let $B \in (0,+\infty)$ and $t_0 \geqslant 0$ be arbitrarily given. Let M be an n-dimensional weakly pseudoconvex Kähler manifold. Let $\psi < 0$ be a plurisubharmonic function on M. Let φ be a plurisubharmonic function on M. Let F be a holomorphic $(n,0)$ form on $\{\psi < -t_0\}$, such that*

$$\int_{K\cap\{\psi<-t_0\}} |F|^2 < +\infty$$

for any compact subset K of M, and

$$\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2\mathrm{e}^{-\varphi} \leqslant C < +\infty.$$

Then there exists a holomorphic $(n,0)$ form $\tilde{F}$ on M, such that

$$\int_M |\tilde{F} - (1-b_{t_0,B}(\psi))F|^2\mathrm{e}^{-\varphi+v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi)) \leqslant C\int_0^{t_0+B} c(t)\mathrm{e}^{-t}dt,$$

where $b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B}\mathbb{I}_{\{-t_0-B<s<-t_0\}}ds$ and $v_{t_0,B}(t) = \int_{-t_0}^t b_{t_0,B}(s)ds - t_0$.

It is clear that $\mathbb{I}_{(-t_0,+\infty)} \leqslant b_{t_0,B}(t) \leqslant \mathbb{I}_{(-t_0-B,+\infty)}$ and $\max\{t,-t_0-B\} \leqslant v_{t_0,B}(t) \leqslant \max\{t,-t_0\}$.

Lemma 2.36 (see [38]). *Let M be a complex manifold. Let S be an analytic subset of M. Let $\{g_j\}_{j\in\mathbb{Z}_{\geqslant 1}}$ be a sequence of nonnegative Lebesgue measurable functions on M, which satisfies that g_j converges almost everywhere to g on M when $j \to +\infty$, where g is a nonnegative Lebesgue measurable function on M. Assume that for any compact subset K of $M\backslash S$, there exist $s_K \in (0,+\infty)$ and $C_K \in (0,+\infty)$ such that*

$$\int_K {g_j}^{-s_K} dV_M \leqslant C_K$$

for any j, where dV_M is a continuous volume form on M.

Let $\{F_j\}_{j\in\mathbb{Z}_{\geqslant 1}}$ be a sequence of holomorphic $(n,0)$ form on M. Assume that $\liminf_{j\to+\infty}\int_M |F_j|^2 g_j \leqslant C$, where C is a positive constant. Then there exists a subsequence $\{F_{j_l}\}_{l\in\mathbb{Z}_{\geqslant 1}}$, which satisfies that $\{F_{j_l}\}$ converges uniformly to a holomorphic $(n,0)$ form F on M on any compact subset of M when $l \to +\infty$, such that

$$\int_M |F|^2 g \leqslant C.$$

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leqslant j \leqslant n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = (\prod_{1\leqslant j\leqslant n_1}\Omega_j) \times Y$ be an n-dimensional complex manifold, where $n = n_1 + n_2$, and let K_M be the canonical (holomorphic) line bundle on M. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\leqslant j\leqslant n_1}\Omega_j$, Ω_j and Y respectively. Let $Z_j = \{z_{j,k} : 1 \leqslant k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1,\ldots,n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geqslant 2}\cup\{+\infty\}$. Denote $Z_0 := (\prod_{1\leqslant j\leqslant n_1} Z_j) \times Y$.

Let φ_X be a plurisubharmonic function on $\prod_{1\leqslant j\leqslant n_1}\Omega_j$ satisfying that $\varphi_X(z) > -\infty$ for any $z \in \prod_{1\leqslant j\leqslant n_1} Z_j$, and let φ_Y be a plurisubharmonic function on Y. Let $p_{j,k}$ be a positive number for any $1 \leqslant j \leqslant n_1$ and $1 \leqslant k < \tilde{m}_j$, which satisfies that $\sum_{1\leqslant k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \leqslant j \leqslant n_1$. Denote

$$\psi := \max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}$$

and $\varphi := \pi_1^*(\varphi_X) + \pi_2^*(\varphi_Y)$ on M.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \leqslant j \leqslant n_1$ and $1 \leqslant k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$.

Denote $\tilde{I}_1 := \{(\beta_1,\ldots,\beta_{n_1}) : 1 \leqslant \beta_j < \tilde{m}_j$ for any $j \in \{1,\ldots,n_1\}\}$, $V_\beta := \prod_{1\leqslant j\leqslant n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}}) \in \prod_{1\leqslant j\leqslant n_1}\Omega_j$ for any $\beta = (\beta_1,\ldots,\beta_{n_1}) \in \tilde{I}_1$. Denote $E_\beta := \left\{(\alpha_1,\ldots,\alpha_{n_1}) : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geqslant 0}\right\}$ and $\tilde{E}_\beta := \left\{(\alpha_1,\ldots,\alpha_{n_1}) : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \geqslant 1 \,\&\, \alpha_j \in \mathbb{Z}_{\geqslant 0}\right\}$ for any $\beta \in \tilde{I}_1$. Let f be a holomorphic $(n,0)$ form on a neighborhood U_0 of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha \in \tilde{E}_\beta$ and $\beta \in \tilde{I}_1$. Denote

$$c_{j,k} := \exp \lim_{z\to z_{j,k}} \left(\frac{\sum_{1\leqslant k_1<\tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log|w_{j,k}(z)| \right)$$

for any $j \in \{1,\ldots,n\}$ and $1 \leqslant k < \tilde{m}_j$ (following from Lemma 2.12 and Lemma 2.13, we get that the above limit exists).

Proposition 2.37. *Let c be a positive function on $(0,+\infty)$ such that $\int_0^{+\infty} c(t)\mathrm{e}^{-t}dt < +\infty$ and $c(t)\mathrm{e}^{-t}$ is decreasing on $(0,+\infty)$. Assume that*

$$\|f\|_{Z_0}^2 := \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\mathrm{e}^{-\varphi_X(z_\beta)}\int_Y |f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} < +\infty.$$

Then there exists a holomorphic $(n,0)$ form F on M satisfying that $(F-f,z) \in (\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z$ for any $z \in Z_0$ and

$$\int_M |F|^2\mathrm{e}^{-\varphi}c(-\psi) \leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right)\|f\|_{Z_0}^2.$$

Proof. The following Remark shows that it suffices to prove Proposition 2.37 for the case $\tilde{m}_j < +\infty$ for any $j \in \{1,\ldots,n_1\}$.

Remark 2.38. Assume that Proposition 2.37 holds for the case $\tilde{m}_j < +\infty$ for any $j \in \{1,\ldots,n_1\}$. For any $j \in \{1,\ldots,n_1\}$, it follows from Lemma 2.15 that there exists a sequence of Riemann surfaces $\{\Omega_{j,l}\}_{l\in\mathbb{Z}_{\geqslant 1}}$, which satisfies that $\Omega_{j,l} \Subset \Omega_{j,l+1} \Subset \Omega_j$ for any l, $\bigcup l \in \mathbb{Z}_{\geqslant 1}\Omega_{j,l} = \Omega_j$ and $\{G_{\Omega_{j,l}}(\cdot,z) - G_{\Omega_j}(\cdot,z)\}_{l\in\mathbb{Z}_{\geqslant 1}}$ converges decreasingly to 0 with respect to l for any $z \in \Omega_j$. As Z_j is a discrete subset of Ω_j, $Z_{j,l} := \Omega_{j,l} \cap Z_j$ is a set of finite points. Define $M_l := (\prod_{1\leqslant j\leqslant n_1}\Omega_{j,l}) \times Y$ and $\psi_l := \max_{1\leqslant j\leqslant n_1}\left\{\pi_{1,j}^*\big(\sum_{z_{j,k}\in Z_{j,l}} 2p_{j,k}G_{\Omega_{j,l}}(\cdot, z_{j,k})\big)\right\}$ on M_l. Define

$$c_{j,k,l} = \exp \lim_{z\to z_{j,k}} \left(\frac{\sum_{z_{j,k_1}\in Z_{j,l}} p_{j,k_1} G_{\Omega_{j,l}}(z, z_{j,k_1})}{p_{j,k}} - \log|w_{j,k}(z)| \right)$$

for any $1 \leqslant j \leqslant n_1$, $l \in \mathbb{Z}_{\geqslant 1}$ and $1 \leqslant k < \tilde{m}_j$ satisfying $z_{j,k} \in Z_{j,l}$. Hence $c_{j,k,l}$ converges decreasingly to $c_{j,k}$ with respect to l, ψ_l converges decreasingly to ψ with respect to l and $\bigcup l \in \mathbb{Z}_{\geqslant 1}M_l = M$.

Then there exists a holomorphic $(n,0)$ form F_l on M_l such that $(F_l - f, (z_\beta, y)) \in (\mathcal{O}(K_{M_l}) \otimes \mathcal{I}(\psi_l))_{(z_\beta,y)} = (\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_{(z_\beta,y)}$ for any $\beta \in \{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1\leqslant j\leqslant n_1}\Omega_{j,l}\}$ and $y \in Y$, and F_l satisfies

$$\begin{aligned}
&\int_{M_l} |F_l|^2\mathrm{e}^{-\varphi}c(-\psi_l) \\
&\leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right) \sum_{\beta\in\{\tilde{\beta}\in\tilde{I}_1 : z_{\tilde{\beta}}\in\prod_{1\leqslant j\leqslant n_1}\Omega_{j,l}\}} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\mathrm{e}^{-\varphi_X(z_\beta)}\int_Y |f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j,l}^{2\alpha_j+2}} \\
&\leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right)\|f\|_{Z_0}^2 < +\infty.
\end{aligned}$$

Since $\psi \leqslant \psi_l$ and $c(t)\mathrm{e}^{-t}$ is decreasing on $(0,+\infty)$, we have

$$\int_{M_l} |F_l|^2 \mathrm{e}^{-\varphi-\psi_l+\psi} c(-\psi) \leqslant \int_{M_l} |F_l|^2 \mathrm{e}^{-\varphi} c(-\psi_l) \leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \|f\|^2_{Z_0}. \tag{2.18}$$

Note that ψ is continuous on $M\backslash Z_0$, ψ_l is continuous on $M_l\backslash Z_0$ and Z_0 is a closed complex submanifold of M. For any compact subset K of $M\backslash Z_0$, there exist $l_K > 0$ such that $K \Subset M_{l_K}$ and $C_K > 0$ such that $\frac{\mathrm{e}^{\varphi+\psi_l-\psi}}{c(-\psi)} \leqslant C_K$ for any $l \geqslant l_K$. It follows from Lemma 2.36 and the diagonal method that there exists a subsequence of $\{F_l\}$, denoted still by $\{F_l\}$, which converges uniformly to a holomorphic $(n,0)$ form F on M on any compact subset of M. It follows from the Fatou's Lemma and inequality (2.18) that

$$\begin{aligned}\int_M |F|^2 \mathrm{e}^{-\varphi} c(-\psi) &= \int_M \lim_{l\to+\infty} |F_l|^2 \mathrm{e}^{-\varphi-\psi_l+\psi} c(-\psi)\\ &\leqslant \liminf_{l\to+\infty} \int_{M_l} |F_l|^2 \mathrm{e}^{-\varphi-\psi_l+\psi} c(-\psi)\\ &\leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \|f\|^2_{Z_0}.\end{aligned}$$

Since $\{F_l\}$ converges uniformly to F on any compact subset of M and $(F_l - f, (z_\beta, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_\beta,y)}$ for any $\beta \in \left\{\tilde{\beta} \in \tilde{I}_1 : z_{\tilde{\beta}} \in \prod_{1\leqslant j\leqslant n_1} \Omega_{j,l}\right\}$ and $y \in Y$, it follows from Lemma 2.17 that $(F - f, (z_\beta, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_{(z_\beta,y)}$ for any $\beta \in \tilde{I}_1$ and $y \in Y$.

In the following, we assume that $\tilde{m}_j < +\infty$ for any $1 \leqslant j \leqslant n_1$. Denote $m_j = \tilde{m}_j - 1$. As $\prod_{1\leqslant j\leqslant n_1} \Omega_j$ is a Stein manifold, it follows from Lemma 2.28 that there exist smooth plurisubharmonic functions Φ_l on $\prod_{1\leqslant j\leqslant n_1} \Omega_j$, which converge decreasingly to φ_X with respect to l. Denote

$$\varphi_l := \pi_1^*(\Phi_l) + \pi_2^*(\varphi_Y).$$

As Y is a weakly pseudoconvex Kähler manifold, it is well-known that there exist open weakly pseudoconvex Kähler manifolds $D_1 \Subset \ldots \Subset D_{l'} \Subset D_{l'+1} \Subset \ldots$ such that $\bigcup l' \in \mathbb{Z}_{\geqslant 1} D_{l'} = Y$. Denote $M_{l'} := \left(\prod_{1\leqslant j\leqslant n_1} \Omega_j\right) \times D_{l'}$.

It follows from Lemma 2.12 and Lemma 2.13 that there exists a local coordinate $\tilde{w}_{j,k}$ on a neighborhood $\tilde{V}_{z_{j,k}} \Subset V_{z_{j,k}}$ of $z_{j,k}$ satisfying $\tilde{w}_{j,k}(z_{j,k}) = 0$ and

$$|\tilde{w}_{j,k}| = \exp\left(\frac{\sum_{1\leqslant k_1\leqslant m_j} p_{j,k_1} G_{\Omega_j}(\cdot, z_{j,k_1})}{p_{j,k}}\right)$$

on $\tilde{V}_{z_{j,k}}$. Denote $\tilde{V}_\beta := \prod_{1\leqslant j\leqslant n_1} \tilde{V}_{j,\beta_j}$ for any $\beta \in \tilde{I}_1$. Let $\tilde{f}$ be a holomorphic $(n,0)$ form on $\bigcup \beta \in \tilde{I}_1 \tilde{V}_\beta \times Y$ satisfying

$$\tilde{f} = \sum_{\alpha\in E_\beta} c_{\alpha,\beta} \pi_1^*(\tilde{w}_\beta^\alpha d\tilde{w}_\beta) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $\tilde{V}_\beta \times Y$, where $\tilde{w}_\beta = (\tilde{w}_{1,\beta_1}, \tilde{w}_{2,\beta_2}, \ldots, \tilde{w}_{n_1,\beta_{n_1}})$ and $c_{\alpha,\beta} = \prod_{1\leqslant j\leqslant n_1} \left(\lim_{z\to z_{j,\beta_j}} \frac{w_{j,\beta_j}(z)}{\tilde{w}_{j,\beta_j}(z)}\right)^{\alpha_j+1}$. It follows from Lemma 2.18 that

$$(f - \tilde{f}, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$$

for any $z \in Z_0$. Denote

$$\psi_1 := \max_{1\leqslant j\leqslant n_1} \left\{2 \sum_{1\leqslant k\leqslant m_j} p_{j,k} \tilde{\pi}_j^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}$$

on $\prod_{1\leqslant j\leqslant n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leqslant j\leqslant n_1} \Omega_j$ to Ω_j. Note that $\psi = \pi_1^*(\psi_1)$. It follows from Lemma 2.14 and Lemma 2.13 that there exists $t_0 > 0$ such that $\{\psi_1 < -t_0\} \Subset \bigcup \beta \in \tilde{I}_1 \tilde{V}_\beta$, which implies that $\int_{\{\psi_1<-t_0\}\times D_{l'}} |\tilde{f}|^2 < +\infty$.

Using Lemma 2.35, there exists a holomorphic $(n,0)$ form $F_{l,l',t}$ on $M_{l'}$ such that

$$\begin{aligned}&\int_{M_{l'}}|F_{l,l',t}-(1-b_{t,1}(\psi))\tilde{f}|^2\mathrm{e}^{-\varphi_l-\psi+v_{t,1}(\psi)}c(-v_{t,1}(\psi))\\&\leqslant\left(\int_0^{t+1}c(s)\mathrm{e}^{-s}ds\right)\int_{M_{l'}}\mathbb{I}_{\{-t-1<\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi_l-\psi},\end{aligned}\tag{2.19}$$

where $t\geqslant t_0$. Note that $b_{t,1}(s)=0$ for large enough s, then $(F_{l,l',t}-\tilde{f},z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z$ for any $z\in Z_0\cap M_{l'}$.

For any $\epsilon>0$, there exists $t_1>t_0$, such that $\sup_{z\in\{\psi_1<-t_1\}\cap\tilde{V}_\beta}|\Phi_l(z)-\Phi_l(z_\beta)|<\epsilon$ for any $\beta\in\tilde{I}_1$. Note that $\varphi_l=\pi_1^*(\Phi_l)+\pi_2^*(\varphi_Y)$ and $|c_{\alpha,\beta}|=\frac{1}{\prod_{1\leqslant j\leqslant n_1}c_{j,\beta_j}^{\alpha_j+1}}$ for any $\beta\in\tilde{I}_1$ and $\alpha\in E_\beta$. As $\{\psi_1<-t_1\}\Subset\bigcup\beta\in I_1\tilde{V}_\beta$, it follows from Lemma 2.21, Fubini's theorem and

$$\int_Y|f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}<+\infty$$

that

$$\begin{aligned}\int_{M_{l'}}\mathbb{I}_{\{-t-1<\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi_l-\psi}&=\int_{\{-t-1<\psi_1<-t\}\times D_{l'}}|\tilde{f}|^2\mathrm{e}^{-\pi_1^*(\Phi_l+\psi)-\pi_2^*(\varphi_Y)}\\&\leqslant\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\Phi_l(z_\beta)+\epsilon}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\int_{D_{l'}}|f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}\end{aligned}\tag{2.20}$$

for $t>t_1$. Letting $t\to+\infty$ and $\epsilon\to0$, inequality (2.20) implies that

$$\limsup_{t\to+\infty}\int_{M_{l'}}\mathbb{I}_{\{-t-1<\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi_l-\psi}\leqslant\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\Phi_l(z_\beta)}\int_{D_{l'}}|f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\tag{2.21}$$

As $v_{t,1}(\psi)\geqslant\psi$ and $c(t)\mathrm{e}^{-t}$ is decreasing, combining inequality (2.19) and (2.21), then we have

$$\begin{aligned}&\limsup_{t\to+\infty}\int_{M_{l'}}|F_{l,l',t}-(1-b_{t,1}(\psi))\tilde{f}|^2\mathrm{e}^{-\varphi_l}c(-\psi)\\&\leqslant\limsup_{t\to+\infty}\int_{M_{l'}}|F_{l,l',t}-(1-b_{t,1}(\psi))\tilde{f}|^2\mathrm{e}^{-\varphi_l-\psi+v_{t,1}(\psi)}c(-v_{t,1}(\psi))\\&\leqslant\limsup_{t\to+\infty}\left(\int_0^{t+1}c(s)\mathrm{e}^{-s}ds\right)\int_{M_{l'}}\mathbb{I}_{\{-t-1<\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi_l-\psi}\\&\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\Phi_l(z_\beta)}\int_{D_{l'}}|f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\&<+\infty.\end{aligned}\tag{2.22}$$

Note that ψ is continuous on $M\backslash Z_0$. For any open set $K\Subset M_{l'}\backslash Z_0$, as $b_{t,1}(s)=1$ for any $s\geqslant-t$ and $c(s)\mathrm{e}^{-s}$ is decreasing with respect to s, we get that there exists a constant $C_K>0$ such that

$$\int_K|(1-b_{t,1}(\psi))\tilde{f}|^2\mathrm{e}^{-\varphi_l}c(-\psi)\leqslant C_K\int_{\{\psi<-t_1\}\cap K}|\tilde{f}|^2<+\infty$$

for any $t>t_1$, which implies that

$$\limsup_{t\to+\infty}\int_K|F_{l,l',t}|^2\mathrm{e}^{-\varphi_l}c(-\psi)<+\infty.$$

Using Lemma 2.36 and the diagonal method, we obtain that there exists a subsequence of $\{F_{l,l',t}\}_{t\to+\infty}$ denoted by $\{F_{l,l',t_m}\}_{m\to+\infty}$ uniformly convergent on any compact subset of $M_{l'}\backslash Z_0$. As Z_0 is a closed complex submanifold of M, we obtain that $\{F_{l,l',t_m}\}_{m\to+\infty}$ converges uniformly to a holomorphic $(n,0)$ form $F_{l,l'}$ on $M_{l'}$ on any compact subset of $M_{l'}$. Then it follows from inequality (2.22) and the Fatou's Lemma that

$$\begin{aligned}&\int_{M_{l'}}|F_{l,l'}|^2\mathrm{e}^{-\varphi_l}c(-\psi)\\&=\int_{M_{l'}}\liminf_{m\to+\infty}|F_{l,l',t_m}-(1-b_{t_m,1}(\psi))\tilde{f}|^2\mathrm{e}^{-\varphi_l}c(-\psi)\\&\leqslant\liminf_{m\to+\infty}\int_{M_{l'}}|F_{l,l',t_m}-(1-b_{t_m,1}(\psi))\tilde{f}|^2\mathrm{e}^{-\varphi_l}c(-\psi)\\&\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\Phi_l(z_\beta)}\int_{D_{l'}}|f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\&<+\infty.\end{aligned}$$

Note that $\lim_{l\to+\infty}\Phi_l(z_\beta)=\varphi_X(z_\beta)>-\infty$ for any $\beta\in I_1$, then we have

$$\begin{aligned}&\limsup_{l\to+\infty}\int_{M_{l'}}|F_{l,l'}|^2\mathrm{e}^{-\varphi_l}c(-\psi)\\&\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\varphi_X(z_\beta)}\int_{D_{l'}}|f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\&<+\infty.\end{aligned}$$

Note that ψ is continuous on $M\backslash Z_0$ and Z_0 is a closed complex submanifold of M. Using Lemma 2.36, we obtain that there exists a subsequence of $\{F_{l,l'}\}_{l\to+\infty}$ (also denoted by $\{F_{l,l'}\}_{l\to+\infty}$), which converges uniformly to a holomorphic $(n,0)$ form $F_{l'}$ on $M_{l'}$ on any compact subset of $M_{l'}$, and

$$\int_{M_{l'}}|F_{l'}|^2\mathrm{e}^{-\varphi}c(-\psi)\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\varphi_X(z_\beta)}\int_{D_{l'}}|f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

As $\bigcup l'\in\mathbb{Z}_{\geqslant1}D_{l'}=Y$, we have

$$\begin{aligned}&\limsup_{l'\to+\infty}\int_{M_{l'}}|F_{l'}|^2\mathrm{e}^{-\varphi}c(-\psi)\\&\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\lim_{l'\to+\infty}\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\mathrm{e}^{-\varphi_X(z_\beta)}\int_{D_{l'}}|f_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\&=\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}<+\infty.\end{aligned}\tag{2.23}$$

Note that ψ is continuous on $M\backslash Z_0$, Z_0 is a closed complex submanifold of M and $\bigcup l'\in\mathbb{Z}_{\geqslant1}M_{l'}=M$. Using Lemma 2.36 and the diagonal method, we get that there exists a subsequence of $\{F_{l'}\}$ (also denoted by $\{F_{l'}\}$), which converges uniformly to a holomorphic $(n,0)$ form F on M on any compact subset of M. Then it follows from inequality (2.23) and the Fatou's Lemma that

$$\begin{aligned}\int_M|F|^2\mathrm{e}^{-\varphi}c(-\psi)&=\int_M\liminf_{l'\to+\infty}\mathbb{I}_{M_{l'}}|F_{l'}|^2\mathrm{e}^{-\varphi}c(-\psi)\\&\leqslant\liminf_{l'\to+\infty}\int_{M_{l'}}|F_{l'}|^2\mathrm{e}^{-\varphi}c(-\psi)\\&\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}.\end{aligned}$$

Following from Lemma 2.17, we have $(F-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z$ for any $z\in Z_0$.

Thus, Proposition 2.37 holds. □

3 proofs of Theorem 1.3 and remark 1.5

In this section, we prove Theorem 1.3 and Remark 1.5.

3.1 Proofs of the sufficiency part of Theorem 1.3 and Remark 1.5

In this section, we prove the sufficiency part of Theorem 1.3 and Remark 1.5.

Denote $M' := \prod_{1\leqslant j\leqslant n_1}\Omega_j$, and let $\tilde{\pi}_j$ be the natural projection from M' to Ω_j. Denote $\psi_1 := \max_{1\leqslant j\leqslant n_1}\left\{\tilde{\pi}_j^*(2p_jG_{\Omega_j}(\cdot,z_j))\right\}$ and $\tilde{\varphi} := \sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)$ on M'. It follows from statements (2) and (3) in Theorem 1.3 that

$$\tilde{f}_\alpha = \wedge_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*\left(g_j(P_j)_*\left(f_{u_j}f_{z_j}^{\alpha_j}df_{z_j}\right)\right)$$

is a (single-value) holomorphic $(n_1,0)$ form on M' for any $\alpha\in E$ satisfying $f_\alpha\not\equiv 0$, where $P_j:\Delta\to\Omega_j$ is the universal covering, f_{u_j} is a holomorphic $(1,0)$ form on Δ satisfying $|f_{u_j}| = (P_j)^*(\mathrm{e}^{u_j})$ and f_{z_j} is a holomorphic $(1,0)$ form on Δ satisfying $|f_{z_j}| = (P_j)^*\left(\mathrm{e}^{G_{\Omega_j}(\cdot,z_j)}\right)$. Denote $\tilde{E} := \{\alpha\in E : f_\alpha\not\equiv 0\}$. Let

$$F = \sum_{\alpha\in\tilde{E}} c_\alpha\pi_1^*(\tilde{f}_\alpha)\wedge\pi_2^*(f_\alpha)$$

be a holomorphic $(n,0)$ form on M, where $c_\alpha = \lim_{z\to z_0}\frac{w^\alpha dw}{\tilde{f}_\alpha}$. As $\int_Y |f_\alpha|^2\mathrm{e}^{-\varphi_Y} < +\infty$ and $\tilde{\varphi}(z_0) > -\infty$, it follows Lemma 2.18 and Lemma 2.25 that

$$(F-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$$

for any $z\in Z_0$.

It follows from Remark 2.8 that $\sum_{\alpha\in\tilde{E}}c_\alpha d_\alpha\tilde{f}_\alpha$ is the unique holomorphic $(n_1,0)$ form on M' such that $\left(\sum_{\alpha\in\tilde{E}}c_\alpha d_\alpha\tilde{f}_\alpha - \sum_{\alpha\in\tilde{E}}d_\alpha w^\alpha dw, z_0\right) \in (\mathcal{O}(K_{M'})\otimes\mathcal{I}(\psi_1))_{z_0}$, $\int_{\{\psi_1<-t\}}|\sum_{\alpha\in\tilde{E}}c_\alpha d_\alpha\tilde{f}_\alpha|^2\mathrm{e}^{-\tilde{\varphi}}c(-\psi_1) = \inf\big\{\int_{\{\psi_1<-t\}}|\tilde{F}|^2\mathrm{e}^{-\tilde{\varphi}}c(-\psi_1) : \tilde{F}$ is a holomorphic $(n_1,0)$ form on $\{\psi_1<-t\}$ satisfying that $(\tilde{F}-\sum_{\alpha\in\tilde{E}}d_\alpha w^\alpha dw, z_0)\in(\mathcal{O}(K_{M'}))_{z_0}\otimes\mathcal{I}(\psi_1)_{z_0}\big\}$ and

$$\begin{aligned}&\int_{\{\psi_1<-t\}}\left|\sum_{\alpha\in\tilde{E}}c_\alpha d_\alpha\tilde{f}_\alpha\right|^2\mathrm{e}^{-\tilde{\varphi}}c(-\psi_1)\\ &=\left(\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\sum_{\alpha\in\tilde{E}}\frac{|d_\alpha|^2(2\pi)^{n_1}\mathrm{e}^{-\tilde{\varphi}(z_0)}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}\end{aligned}\tag{3.1}$$

for any $t\geqslant 0$, where $c_j(z_j) = \exp\lim_{z\to z_j}(G_{\Omega_j}(z,z_j) - \log|w_j(z)|)$. Following from equality (3.1) and Fubini's theorem, we obtain that

$$\begin{aligned}&\int_{\{\psi<-t\}}|F|^2\mathrm{e}^{-\varphi}c(-\psi)\\ &=\left(\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\varphi(z_j)}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi_Y}\\ &<+\infty\end{aligned}\tag{3.2}$$

for any $t\geqslant 0$. Thus, $G(t)\leqslant\int_{\{\psi<-t\}}|F|^2\mathrm{e}^{-\varphi}c(-\psi)<+\infty$ for any $t\geqslant 0$.

It follows from Lemma 2.5 that there exists a holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ satisfying that $(F_t-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$ and $G(t)=\int_{\{\psi<-t\}}|F_t|^2\mathrm{e}^{-\varphi}c(-\psi)$. For any $y_0\in Y$, let $u=(u_1,\ldots,u_{n_2})$ be a coordinate on a neighborhood U of y satisfying $u(y_0)=0$ and $u(U)=\Delta^{n_2}$. Lemma 2.22 implies that $F_t|_U = \sum_{\gamma\in\mathbb{Z}_{\geqslant 0}^{n_2}}\pi_1^*(f_{t,\gamma})\wedge\pi_2^*(u^\gamma du)$, where $f_{t,\gamma}$ is a holomorphic $(n_1,0)$ form on $\{\psi_1<-t\}$ for any $\gamma\in\mathbb{Z}_{\geqslant 0}^{n_2}$. There exists a holomorphic function $f_{u,\alpha}$ on U such that $f_\alpha = f_{u,\alpha}du$ on U for any $\alpha\in\tilde{E}$. Note that $f=\sum_{\alpha\in\tilde{E}}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha)+g_0$ on $V_0\times Y$, where g_0 is a holomorphic $(n,0)$

form on $V_0 \times Y$ satisfying $(g_0, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$. It follows from Lemma 2.19 and $(F_t - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ that $\left(\sum_{\gamma \in \mathbb{Z}^{n_2}_{\geqslant 0}} u^\gamma f_{t,\gamma} - \sum_{\alpha \in \tilde{E}} f_{u,\alpha}(u) w^\alpha dw\right) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}$ for any $u \in \Delta^{n_2}$. Let U_1 be an open subset of U, and let $V = u(U_1) \subset \Delta^{n_2}$. Note that $\left(\sum_{\alpha \in \tilde{E}} c_\alpha f_{u,\alpha}(u) \tilde{f}_\alpha - \sum_{\alpha \in \tilde{E}} f_{u,\alpha}(u) w^\alpha dw\right) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}$ for any $u \in \Delta^{n_2}$. Following Fubini's theorem and the minimal property of $\int_{\{\psi_1 < -t\}} |\sum_{\alpha \in \tilde{E}} c_\alpha f_{u,\alpha} \tilde{f}_\alpha|^2 \mathrm{e}^{-\tilde{\varphi}} c(-\psi_1)$, we have

$$
\begin{aligned}
&\int_{\{\psi_1 < -t\} \times U_1} |F_t|^2 \mathrm{e}^{-\varphi} c(-\psi) \\
&= \int_V \left(\int_{\{\psi_1 < -t\}} \Bigg| \sum_{\gamma \in \mathbb{Z}^{n_2}_{\geqslant 0}} u^\gamma f_{t,\gamma} \Bigg|^2 \mathrm{e}^{-\tilde{\varphi}} c(-\psi_1) \right) \mathrm{e}^{-\varphi_Y} |du|^2 \\
&\geqslant \int_V \left(\int_{\{\psi_1 < -t\}} \Bigg| \sum_{\alpha \in \tilde{E}} c_\alpha f_{u,\alpha}(u) \tilde{f}_\alpha \Bigg|^2 \mathrm{e}^{-\tilde{\varphi}} c(-\psi_1) \right) \mathrm{e}^{-\varphi_Y} |du|^2 \\
&= \int_{\{\psi_1 < -t\} \times U_1} \Bigg| \sum_{\alpha \in \tilde{E}} c_\alpha \pi_1^*(\tilde{f}_\alpha) \wedge \pi_2^*(f_\alpha) \Bigg|^2 \mathrm{e}^{-\varphi} c(-\psi),
\end{aligned}
$$

which implies $G(t) = \int_{\{\psi < -t\}} |F_t|^2 \mathrm{e}^{-\varphi} c(-\psi) \geqslant \int_{\{\psi < -t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)$. It follows from $G(t) \leqslant \int_{\{\psi < -t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)$ and inequality (3.2) that

$$
\begin{aligned}
G(t) &= \int_{\{\psi < -t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) \\
&= \left(\int_t^{+\infty} c(s) \mathrm{e}^{-s} ds \right) \sum_{\alpha \in E} \frac{(2\pi)^{n_1} \mathrm{e}^{-\sum_{1 \leqslant j \leqslant n_1} \varphi(z_j)}}{\prod_{1 \leqslant j \leqslant n_1} (\alpha_j + 1) c_j(z_j)^{2\alpha_j + 2}} \int_Y |f_\alpha|^2 \mathrm{e}^{-\varphi_Y},
\end{aligned}
$$

hence $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s) \mathrm{e}^{-s} ds]$. The uniqueness of F follows from Corollary 2.3.

Thus, the sufficiency part of Theorem 1.3 and Remark 1.5 hold.

3.2 Proof of the necessity part of Theorem 1.3

In this section, we prove the necessity part of Theorem 1.3 in three steps.

Step 1. $f = \sum_{\alpha \in E} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(f_\alpha) + g_0$.

Corollary 2.3 show that there is a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$ and $G(t) = \int_{\{\psi < -t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)$ for any $t \geqslant 0$. It follows from Lemma 2.12 that there exists a local coordinate $\tilde{w}_j$ on a neighborhood $\tilde{V}_{z_j} \Subset V_{z_j}$ of $z_j \in \Omega_j$ satisfying $\tilde{w}_j(z_j) = 0$ and

$$\log |\tilde{w}_j| = G_{\Omega_j}(\cdot, z_j)$$

on $\tilde{V}_{z_j}$ for any $j \in \{1, \ldots, n_1\}$. Denote $\tilde{V}_0 := \prod_{1 \leqslant j \leqslant n_1} \tilde{V}_{z_j}$ and $\tilde{w} := (\tilde{w}_1, \ldots, \tilde{w}_{n_1})$ on $\tilde{V}_0$. Using Lemma 2.14, we get that there exists $t_0 > 0$ such that

$$\{2p_j G_{\Omega_j}(\cdot, z_j) < -t_0\} \Subset \tilde{V}_{z_j}$$

for any $1 \leqslant j \leqslant n_1$. As φ_j is a subharmonic function on Ω_j, $\int_{\{\psi < -t_0\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) < +\infty$ implies that $\int_{\{\psi < -t_0\}} |F|^2 \mathrm{e}^{-\pi_2^*(\varphi_Y)} c(-\psi) < +\infty$. Note that $\{\psi < -t_0\} = \left(\prod_{1 \leqslant j \leqslant n_1} \left\{ |\tilde{w}_j| < \mathrm{e}^{-\frac{t_0}{2p_j}} \right\} \right) \times Y$. It follows from Lemma 2.22 that there exists a unique sequence of holomorphic $(n_2, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{Z}^{n_1}_{\geqslant 0}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{Z}^{n_1}_{\geqslant 0}} \pi_1^*(\tilde{w}^\alpha d\tilde{w}) \wedge \pi_2^*(F_\alpha)$$

on $\{\psi < -t_0\}$ and

$$\int_Y |F_\alpha|^2 e^{-\varphi_Y} < +\infty,$$

where the right term of the above equality is uniformly convergent on any compact subset of M. As $\frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds}$ is a positive number independent of t, Lemma 2.29 implies that $F_\alpha \equiv 0$ for any $\alpha \in \mathbb{Z}_{\geqslant 0}$ satisfying $\sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_j} < 1$. Denote $E_2 := \left\{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_j} > 1\right\}$. Note that $\varphi(z_j) > -\infty$ for any $1 \leqslant j \leqslant n_1$. It follows from Lemma 2.18 and Lemma 2.25 that $(\pi_1^*(\tilde{w}^\alpha d\tilde{w}) \wedge \pi_2^*(F_\alpha), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))_z$ for any $z \in Z_0$ and $\alpha \in E_2$, thus

$$(\sum_{\alpha\in E_2} \pi_1^*(\tilde{w}^\alpha d\tilde{w}) \wedge \pi_2^*(F_\alpha), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))_z$$

for any $z \in Z_0$ (by using Lemma 2.17). As $(F - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))_z$ for any $z \in Z_0$, we have

$$(f - \sum_{\alpha\in E} \pi_1^*(\tilde{w}^\alpha d\tilde{w}) \wedge \pi_2^*(F_\alpha), z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))_z$$

for any $z \in Z_0$. Denote

$$\psi_1 := \max_{1\leqslant j\leqslant n_1} \left\{\tilde{\pi}_j^*(2p_j G_{\Omega_j}(\cdot, z_j))\right\}$$

on $\prod_{1\leqslant j\leqslant n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leqslant j\leqslant n_1} \Omega_j$ to Ω_j. Taking $c_\alpha = \prod_{1\leqslant j\leqslant n_1} \left(\lim_{z\to z_j} \frac{\tilde{w}_j}{w_j}\right)^{\alpha_j+1}$, it follows from Lemma 2.18 and Lemma 2.25 that $(\tilde{w}^\alpha d\tilde{w} - c_\alpha w^\alpha dw, z_0) \in \mathcal{O}(K_{\prod_{1\leqslant j\leqslant n_1} \Omega_j})_{z_0} \otimes \mathcal{I}\left(\sum_{1\leqslant j\leqslant n_1} \tilde{\pi}_j(\varphi_j) + \psi_1\right)_{z_0}$ for any $\alpha \in E$, which implies that $\big(\sum_{\alpha\in E} \pi_1^*(\tilde{w}^\alpha d\tilde{w}) \wedge \pi_2^*(F_\alpha) - \sum_{\alpha\in E} \pi_1^*(c_\alpha w^\alpha dw) \wedge \pi_2^*(F_\alpha), z\big) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))_z$ for any $z \in Z_0$. Taking $f_\alpha = c_\alpha F_\alpha$, there exists a holomorphic $(n, 0)$ form g_0 on $V_0 \times Y$ such that

$$f = \sum_{\alpha\in E} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(f_\alpha) + g_0$$

and $(g_0, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))_z$ for any $z \in Z_0$. As $G(0) > 0$, we know that there exists $\alpha \in E$ such that $f_\alpha \not\equiv 0$.

Step 2. $G(-\log r; \tilde{c} \equiv 1)$ is linear with respect to r.

It follows from Corollary 2.3 that $G(t; \tilde{c} \equiv 1) \leqslant \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} = \frac{G(0;c)}{\int_0^{+\infty} c(s)e^{-s}ds} e^{-t} < +\infty$. Denote

$$\inf\left\{\int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} : (\tilde{f} - f, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right.$$
$$\left. \& \ \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M))\right\}$$

by $\tilde{G}(t)$, where $t \geqslant 0$. It follows from Lemma 2.26 that $G(t; \tilde{c} \equiv 1) = \tilde{G}(t)$ for any $t \geqslant 0$. Denote $M' := \prod_{1\leqslant j\leqslant n_1} \Omega_j$, and let $K_{M'}$ be the canonical (holomorphic) line bundle on M'. Using Lemma 2.26, Lemma 2.32 and Lemma 2.33, we obtain that there exists a unique holomorphic $(n, 0)$ form $F_t = \sum_{\alpha\in E} \pi_1^*(h_{t,\alpha}) \wedge \pi_2^*(f_\alpha)$ on $\{\psi < -t\}$ satisfying

$$G(t; \tilde{c} \equiv 1) = \tilde{G}(t) = \int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} = \sum_{\alpha\in E} \int_{\{\psi<-t\}} |\pi_1^*(h_{t,\alpha}) \wedge \pi_2^*(f_\alpha)|^2 e^{-\varphi}, \tag{3.3}$$

where $h_{t,\alpha}$ is a holomorphic $(n_1, 0)$ form on $\{\psi_1 < -t\}$ satisfying

$$(h_{t,\alpha} - w^\alpha dw, z_0) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}$$

and $\int_{\{\psi_1<-t\}} |h_{t,\alpha}|^2 e^{-\sum_{1\leqslant j\leqslant n_1} \tilde{\pi}_j^*(\varphi_j)} = \inf\big\{\int_{\{\psi_1<-t\}} |\tilde{F}|^2 e^{-\sum_{1\leqslant j\leqslant n_1} \tilde{\pi}_j^*(\varphi_j)} : \tilde{F}$ is a holomorphic $(n_1, 0)$ form on $\{\psi_1 < -t\}$ satisfying $(\tilde{F} - w^\alpha dw, z_0) \in (\mathcal{O}(K_{M'}) \otimes \mathcal{I}(\psi_1))_{z_0}\big\} < +\infty$. It follows from Lemma 2.31

that there exists a holomorphic $(n_1,0)$ form $\tilde{h}_\alpha$ on M' such that $\int_{M'}|\tilde{h}_\alpha|^2\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)}c(-\psi_1)<+\infty$ and $(\tilde{h}_\alpha-w^\alpha dw,z_0)\in(\mathcal{O}(K_{M'})\otimes\mathcal{I}(\psi_1))_{z_0}$. As $\varphi_j(z_j)>-\infty$ for any $1\leqslant j\leqslant n_1$, it follows from Lemma 2.25 that there exists $t_1>t$ such that

$$\int_{\{\psi_1<-t_1\}}|h_{t,\alpha}-\tilde{h}_\alpha|^2\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)-\psi_1}<+\infty$$

for any $\alpha\in E$. As $c(s)\mathrm{e}^{-s}$ is a positive decreasing function on $(0,+\infty)$, for any $t>0$, we obtain that

$$\begin{aligned}
&\int_{\{\psi_1<-t\}}|h_{t,\alpha}|^2\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)}c(-\psi_1)\\
\leqslant&C\int_{\{\psi_1<-t_1\}}|h_{t,\alpha}-\tilde{h}_\alpha|^2\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)-\psi_1}\\
&+2\int_{\{\psi_1<-t_1\}}|\tilde{h}_\alpha|^2\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)}c(-\psi_1)\\
&+\sup_{s\in(t,t_1]}c(s)\times\int_{\{-t_1\leqslant\psi_1<-t\}}|h_{t,\alpha}|^2\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)}\\
<&+\infty
\end{aligned}$$

for any $\alpha\in E$, which implies that

$$\begin{aligned}
&\int_{\{\psi<-t\}}|F_t|^2\mathrm{e}^{-\varphi}c(-\psi)\\
\leqslant&C\sum_{\alpha\in E}\int_{\{\psi_1<-t\}}|h_{t,\alpha}|^2\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)}c(-\psi_1)\times\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi_Y}\\
<&+\infty.
\end{aligned}\tag{3.4}$$

It follows from Lemma 2.6 and inequality (3.4) that

$$G(t;\tilde{c}\equiv 1)=\int_{\{\psi<-t\}}|F|^2\mathrm{e}^{-\varphi}=\frac{G(0;c)}{\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds}\mathrm{e}^{-t}$$

for any $t>0$. Theorem 2.2 shows that $\lim_{t\to0+}G(t;\tilde{c}\equiv1)=G(0;\tilde{c}\equiv1)$, hence we get $G(-\log r;\tilde{c}\equiv1)$ is linear with respect to $r\in(0,1]$.

Step 3. Proofs of statements (2) and (3) in Theorem 1.3.

Denote

$$\inf\left\{\int_{\{\psi_1<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}:(\tilde{f}-w^\alpha dw,z_0)\in(\mathcal{O}(K_{M'})\otimes\mathcal{I}(\psi_1))_{z_0}\right.$$
$$\left.\&\,\tilde{f}\in H^0(\{\psi_1<-t\},\mathcal{O}(K_{M'}))\right\}$$

by $G_\alpha(t)$, where $t\geqslant0$. Lemma 2.4 and Lemma 2.18 show that $G_\alpha(t)\neq0$ for any $\alpha\in E$. It follows from equality (3.3) that

$$G(t,\tilde{c}\equiv1)=\sum_{\alpha\in E}G_\alpha(t)\int_Y|f_\alpha|^2\mathrm{e}^{-\varphi}.\tag{3.5}$$

Theorem 2.2 tells us that $G_\alpha(-\log r)$ is concave with respect to r. It follows from the linearity of $G(-\log r;\tilde{c})$ and equality (3.5) that $G_\alpha(-\log r)$ is linear with respect to r for any $\alpha\in E$ satisfying $f_\alpha\not\equiv0$. It follows from Theorem 2.7 and the linearity of $G_\alpha(-\log r)$ that statements (2) and (3) in Theorem 1.3 hold.

Thus, the necessity part of Theorem 1.3 holds.

4 Proofs of Theorem 1.6 and Reamrk 1.7

In this section, we prove Theorem 1.6 and Remark 1.7.

Denote $M' := \prod_{1\leqslant j\leqslant n_1}\Omega_j$, and let $K_{M'}$ be the conanical (holomorphic) line bundle on M'. Denote

$$\psi_1 := \max_{1\leqslant j\leqslant n_1}\left\{\tilde{\pi}_j^*\left(2\sum_{1\leqslant k\leqslant m_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$$

on M', where $\tilde{\pi}_j$ is the natural projection from M' to Ω_j. For any $\beta\in I_1$ and any holomorphic function h, it follows from Lemma 2.19 that $(h,(z_\beta,y))\in\mathcal{I}(\psi)_{(z_\beta,y)}$ for any $y\in Y$ if and only if $(h(\cdot,y),z_\beta)\in\mathcal{I}(\psi_1)_{z_\beta}$ for any $y\in Y$. The sufficiency part of Theorem 1.6 follows from Proposition 2.24, Theorem 2.9 and Lemma 2.26. In the following, we prove the necessity part of Theorem 1.6 and Remark 1.7.

Following from the linearity of $G(h^{-1}(r))$ and Corollary 2.3, there exists a holomorphic $(n,0)$ form F on M, such that $(F-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$ and

$$G(t)=\int_{\{\psi<-t\}}|F|^2\mathrm{e}^{-\varphi}c(-\psi). \tag{4.1}$$

It follows from Lemma 2.13 and Lemma 2.14 that there exists $t_0>0$ such that $\{\psi_1<-t_0\}\Subset\bigcup\beta\in I_1V_\beta$ and $\left\{z\in\Omega_j : 2\sum_{1\leqslant k\leqslant m_j}p_{j,k}G_{\Omega_j}(z,z_{j,k})<-t_0\right\}\cap V_{z_{j,k}}$ is simply connected for any $j\in\{1,\ldots,n_1\}$ and $k\in\{1,\ldots,m_j\}$. For any $\beta\in I_1$, denote

$$\inf\left\{\int_{\{\psi<-t\}\cap(V_\beta\times Y)}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{f}\in H^0\left(\{\psi<-t\}\cap(V_\beta\times Y),\mathcal{O}(K_M)\right)\right.$$
$$\left.\&\,(\tilde{f}-f,(z_\beta,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_{(z_\beta,y)},\forall y\in Y\right\}$$

by $G_\beta(t)$, where $t\in[t_0,+\infty)$. Note that $\{\psi<-t\}=\bigcup\beta\in I_1(\{\psi<-t\}\cap(V_\beta\times Y))$ for any $t\geqslant t_0$. Following from the definition of $G(t)$ and $G_\beta(t)$, we have $G(t)=\sum_{\beta\in I_1}G_\beta(t)$ for $t\geqslant t_0$. Thus, we have

$$G_\beta(t)=\int_{\{\psi<-t\}\cap(V_\beta\times Y)}|F|^2\mathrm{e}^{-\varphi}c(-\psi)$$

for any $t\geqslant t_0$. Theorem 2.2 tells us that $G_\beta(h^{-1}(r))$ is concave with respect to $r\in(0,\int_{t_0}^{+\infty}c(s)\mathrm{e}^{-s}ds]$. As $G(h^{-1}(r))$ is linear with respect to r, we have $G_\beta(h^{-1}(r))$ is linear with respect to $r\in(0,\int_{t_0}^{+\infty}c(s)\mathrm{e}^{-s}ds]$.

Note that $f=\pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}}dw_{\beta^*}\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*}}\right)+\sum_{\alpha\in E'}\pi_1^*(w^\alpha dw_{\beta^*})\wedge\pi_2^*(f_\alpha)$ on $V_{\beta^*}\times Y$, where $E'=\left\{\alpha\in\mathbb{Z}_{\geqslant0}^{n_1}:\sum_{j=1}^{n_1}\frac{\alpha_j+1}{p_{j,1}}>\sum_{j=1}^{n_1}\frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. As

$$\frac{1}{2p_{j,1}}\left(2\sum_{1\leqslant k\leqslant m_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})+t_0\right)$$

is the Green function on $\left\{z\in\Omega_j:2\sum_{1\leqslant k\leqslant m_j}p_{j,k}G_{\Omega_j}(z,z_{j,k})<-t_0\right\}\cap V_{z_{j,1}}$, it follows from Theorem 1.3 that $\left(f-\sum_{\alpha\in E_{\beta^*}}\pi_1^*(w_{\beta^*}^\alpha dw_{\beta^*})\wedge\pi_2^*(\tilde{f}_\alpha),(z_{\beta^*},y)\right)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_{(z_{\beta^*},y)}$ for any $y\in Y$, where $E_{\beta^*}=\left\{\alpha\in\mathbb{Z}_{\geqslant0}^{n_1}:\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,\beta_j^*}}=1\right\}$ and $\tilde{f}_\alpha$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y|\tilde{f}_\alpha|^2\mathrm{e}^{-\varphi_Y}<+\infty$ for any $\alpha\in E_{\beta^*}$. Following from Lemma 2.18 and Lemma 2.19, we have $\alpha_{\beta^*}\in E_{\beta^*}$, $f_{\alpha_{\beta^*}}=\tilde{f}_{\alpha_{\beta^*}}$ and $\tilde{f}_\alpha\equiv0$ for any $\alpha\neq\alpha_{\beta^*}$. Using Theorem 1.3 and Remark 1.5, we obtain that there exists a holomorphic $(n_1,0)$ form h_0 on $\{\psi_1<-t_0\}\cap V_{\beta^*}$ such that

$$F=\pi_1^*(h_0)\wedge\pi_2^*(f_{\alpha_{\beta^*}})$$

on $(\{\psi_1<-t_0\}\cap V_{\beta^*})\times Y$. It follows from Lemma 2.34 that there exists a holomorphic $(n_1,0)$ form h_1 on M' such that

$$F=\pi_1^*(h_1)\wedge\pi_2^*(f_{\alpha_{\beta^*}}) \tag{4.2}$$

on M and $h_0=h_1$ on $\{\psi_1<-t_0\}\cap V_{\beta^*}$.

Denote $\tilde{\varphi}=\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)$ on M'. Denote

$$\inf\left\{\int_{\{\psi_1<-t\}}|\tilde{f}|^2\mathrm{e}^{-\tilde{\varphi}}c(-\psi_1):(\tilde{f}-h_1,z_\beta)\in(\mathcal{O}(K_{M'})\otimes\mathcal{I}(\psi_1))_{z_\beta},\forall\beta\in I_1\right.$$
$$\left.\&\ \tilde{f}\in H^0(\{\psi_1<-t\},\mathcal{O}(K_{M'}))\right\}$$

by $G'(t)$, where $t\in[0,+\infty)$. Note that $f_{\alpha_{\beta^*}}=\tilde{f}_{\alpha_{\beta^*}}$ satisfies $\int_Y|f_{\alpha_{\beta^*}}|^2\mathrm{e}^{-\varphi_Y}<+\infty$. For any $\beta\in I_1$ and any holomorphic function h, note that $(h,(z_\beta,y))\in\mathcal{I}(\psi)_{(z_\beta,y)}$ for any $y\in Y$ if and only if $(h(\cdot,y),z_\beta)\in\mathcal{I}(\psi_1)_{z_\beta}$ for any $y\in Y$. Following from Lemma 2.26, equality (4.2) and Proposition 2.24, we get that $G'(0)<+\infty$, $G'(h^{-1}(r))$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds]$ and

$$G'(t)=\int_{\{\psi_1<-t\}}|h_1|^2\mathrm{e}^{-\tilde{\varphi}}c(-\psi_1)\tag{4.3}$$

for any $t\geqslant 0$. Theorem 2.9 tells us that the following statements hold:

(1) $\varphi_j=2\log|g_j|+2u_j$ for any $j\in\{1,\dots,n\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k})\neq 0$ for any $k\in\{1,\dots,m_j\}$;

(2) There exists a nonnegative integer $\gamma_{j,k}$ for any $j\in\{1,\dots,n_1\}$ and $k\in\{1,\dots,m_j\}$, which satisfies that $\prod_{1\leqslant k\leqslant m_j}\chi_{j,z_{j,k}}^{\gamma_{j,k}+1}=\chi_{j,-u_j}$ and $\sum_{1\leqslant j\leqslant n_1}\frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}}=1$ for any $\beta\in I_1$;

(3) $h_1=\left(c_\beta\prod_{1\leqslant j\leqslant n_1}w_{j,\beta_j}^{\gamma_{j,\beta_j}}+\tilde{g}_\beta\right)dw_\beta$ on V_β for any $\beta\in I_1$, where c_β is a constant and g_β is a holomorphic function on V_β such that $(g_\beta,z_\beta)\in\mathcal{I}(\psi_1)_{z_\beta}$;

(4) $\lim_{z\to z_\beta}\frac{c_\beta\prod_{1\leqslant j\leqslant n_1}w_{j,\beta_j}^{\gamma_{j,\beta_j}}dw_\beta}{\wedge_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*\left(g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\leqslant k\leqslant m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\leqslant k\leqslant m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)\right)}=c_0$ for any $\beta\in I_1$, where $c_0\in\mathbb{C}\backslash\{0\}$ is a constant independent of β, f_{u_j} is a holomorphic function Δ such that $|f_{u_j}|=P_j^*(\mathrm{e}^{u_j})$ and $f_{z_{j,k}}$ is a holomorphic function on Δ such that $|f_{z_{j,k}}|=P_j^*\left(\mathrm{e}^{G_{\Omega_j}(\cdot,z_{j,k})}\right)$ for any $j\in\{1,\dots,n\}$ and $k\in\{1,\dots,m_j\}$.

As $\int_Y|f_{\alpha_{\beta^*}}|^2\mathrm{e}^{-\varphi_Y}<+\infty$ and $\tilde{\varphi}(z_\beta)>-\infty$ for any $\beta\in I_1$, it follows from Lemma 2.25 that $\pi_1^*(\tilde{g}_\beta dw_\beta)\wedge\pi_2^*(f_{\alpha_{\beta^*}}),z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$. As $(F-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$ and $F=\pi_1^*(h_1)\wedge\pi_2^*(f_{\alpha_{\beta^*}})$, we have

$$f=\pi_1^*\left(c_\beta\left(\prod_{1\leqslant j\leqslant n_1}w_{j,\beta_j}^{\gamma_{j,\beta_j}}\right)dw_\beta\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*}}\right)+g_\beta$$

on $V_\beta\times Y$ for any $\beta\in I_1$, where g_β is a holomorphic $(n,0)$ form on $V_\beta\times Y$ such that $(g_\beta,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in\{z_\beta\}\times Y$. Take $f_0=f_{\alpha_{\beta^*}}$. Thus, Theorem 1.6 holds.

Note that $G'(h^{-1}(r))$ is linear with respect to r. Following from Theorem 2.9, Remark 2.10 and equality (4.3), we have

$$h_1=c_0\wedge_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*\left(g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\leqslant k\leqslant m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\leqslant k\leqslant m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)\right)$$

and

$$\begin{aligned}G'(t)&=\int_{\{\psi_1<-t\}}|h_1|^2\mathrm{e}^{-\tilde{\varphi}}c(-\psi_1)\\&=\left(\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\sum_{\beta\in I_1}\frac{|c_\beta|^2(2\pi)^{n_1}\mathrm{e}^{-\tilde{\varphi}(z_\beta)}}{\prod_{1\leqslant j\leqslant n_1}(\gamma_{j,\beta_j}+1)c_{j,\beta_j}^{2\gamma_{j,\beta_j}+2}}.\end{aligned}$$

Thus, we have

$$F=c_0\left(\wedge_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*\left(g_j(P_j)_*\left(f_{u_j}\left(\prod_{k=1}^{m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{k=1}^{m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)\right)\right)\wedge\pi_2^*(f_0)$$

and

$$
\begin{aligned}
G(t) &= \int_{\{\psi<-t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) \\
&= \left(\int_t^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \sum_{\beta\in I_1} \frac{|c_\beta|^2 (2\pi)^{n_1} \mathrm{e}^{-\tilde{\varphi}(z_\beta)}}{\prod_{1\leqslant j\leqslant n_1} (\gamma_{j,\beta_j}+1) c_{j,\beta_j}^{2\gamma_{j,\beta_j}+2}} \int_Y |f_0|^2 \mathrm{e}^{-\varphi_Y}.
\end{aligned}
$$

The uniqueness of F follows from Corollary 2.3. Thus, Remark 1.7 holds.

5 Proofs of Theorem 1.8 and Proposition 1.9

In this section, we prove Theorem 1.8 and Proposition 1.9.

5.1 Proof of Theorem 1.8

In this section, we prove Theorem 1.8 by contradiction. Assume that $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)\mathrm{e}^{-s} ds]$.

Denote $M' := \prod_{1\leqslant j\leqslant n_1} \Omega_j$, and let $K_{M'}$ be the canonical (holomorphic) line bundle on M'. Denote $\psi_1 := \max_{1\leqslant j\leqslant n_1} \left\{\tilde{\pi}_j^*\left(2\sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$ on M', where $\tilde{\pi}_j$ is the natural projection from M' to Ω_j. Following from the linearity of $G(h^{-1}(r))$ and Corollary 2.3, there exists a holomorphic $(n,0)$ form F on M, such that $(F-f, z) \in (\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z \in Z_0$ and

$$
G(t) = \int_{\{\psi<-t\}} |F|^2 \mathrm{e}^{-\varphi} c(-\psi).
$$

For any $\beta \in \tilde{I}_1$, it follows from Lemma 2.13 and Lemma 2.14 that there exists $t_\beta > 0$ such that $\{\psi_1 < -t_\beta\} \cap V_\beta \Subset V_\beta$ and $\left\{z \in \Omega_j : 2\sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(z, z_{j,k}) < -t_0\right\} \cap V_{z_{j,k}}$ is simply connected for any $1 \leqslant j \leqslant n_1$ and $1 \leqslant k < \tilde{m}_j$. For any $\beta \in \tilde{I}_1$, denote

$$
\begin{aligned}
\inf\bigg\{ \int_{\{\psi<-t\}\cap(V_\beta\times Y)} |\tilde{f}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi<-t\}\cap(V_\beta\times Y), \mathcal{O}(K_M)) \\
\& \, (\tilde{f}-f, (z_\beta, y)) \in (\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_{(z_\beta,y)}, \forall y \in Y \bigg\}
\end{aligned}
$$

by $G_\beta(t)$, where $t \in [t_\beta, +\infty)$, and denote

$$
\begin{aligned}
\inf\bigg\{ \int_{\{\psi<-t\}\backslash(V_\beta\times Y)} |\tilde{f}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi<-t\}\backslash(V_\beta\times Y), \mathcal{O}(K_M)) \\
\& \, (\tilde{f}-f, z) \in (\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z, \forall z \in \left(\tilde{I}_1\backslash\{\beta\}\right)\times Y \bigg\}
\end{aligned}
$$

by $\tilde{G}_\beta(t)$, where $t \in [t_\beta, +\infty)$. By the definition of $G(t)$, $G_\beta(t)$ and $\tilde{G}_\beta(t)$, we have $G(t) = G_\beta(t) + \tilde{G}_\beta(t)$ for $t \geqslant t_\beta$. Thus, we have

$$
G_\beta(t) = \int_{\{\psi<-t\}\cap(V_\beta\times Y)} |F|^2 \mathrm{e}^{-\varphi} c(-\psi)
$$

for any $t \geqslant t_\beta$. Theorem 2.2 tells us that $G_\beta(h^{-1}(r))$ and $\tilde{G}_\beta(h^{-1}(r))$ are concave with respect to $r \in (0, \int_{t_\beta}^{+\infty} c(s)\mathrm{e}^{-s} ds]$. As $G(h^{-1}(r))$ is linear with respect to r, we have $G_\beta(h^{-1}(r))$ is linear with respect to $r \in (0, \int_{t_\beta}^{+\infty} c(s)\mathrm{e}^{-s} ds]$.

Following from Lemma 2.12 and 2.13, we know $\frac{1}{2p_{j,1}}\left(2\sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) + t_{\beta^*}\right)$ is the Green function on $\left\{z \in \Omega_j : 2\sum_{1\leqslant k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(z, z_{j,k}) < -t_{\beta^*}\right\} \cap V_{z_{j,1}}$. Note that $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{\beta^*}\right) \wedge$

$\pi_2^*\left(f_{\alpha_{\beta^*}}\right)+\sum_{\alpha\in E'}\pi_1^*(w_{\beta^*}^{\alpha}dw_{\beta^*})\wedge\pi_2^*(f_\alpha)$ on $V_{\beta^*}\times Y$, where $E'=\left\{\alpha\in\mathbb{Z}_{\geqslant 0}^{n_1}:\sum_{j=1}^{n_1}\frac{\alpha_j+1}{p_{j,1}}>\sum_{j=1}^{n_1}\frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. It follows from Theorem 1.3 that $\left(f-\sum_{\alpha\in E_{\beta^*}}\pi_1^*(w_{\beta^*}^{\alpha}dw_{\beta^*})\wedge\pi_2^*(\tilde{f}_\alpha),(z_{\beta^*},y)\right)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_{(z_{\beta^*},y)}$ for any $y\in Y$, where $E_{\beta^*}=\left\{\alpha\in\mathbb{Z}_{\geqslant 0}^{n_1}:\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,\beta_j^*}}=1\right\}$ and $\tilde{f}_\alpha$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y|\tilde{f}_\alpha|^2\mathrm{e}^{-\varphi_Y}<+\infty$ for any $\alpha\in E_{\beta^*}$. Following from Lemma 2.18 and Lemma 2.19, we have $\alpha_{\beta^*}\in E_{\beta^*}$, $f_{\alpha_{\beta^*}}=\tilde{f}_{\alpha_{\beta^*}}$ and $\tilde{f}_\alpha\equiv 0$ for any $\alpha\neq\alpha_{\beta^*}$. Using Theorem 1.3 and Remark 1.5, we obtain that there exists a holomorphic $(n_1,0)$ form h_0 on $\{\psi_1<-t_{\beta^*}\}\cap V_{\beta^*}$ such that

$$F=\pi_1^*(h_0)\wedge\pi_2^*(f_{\alpha_{\beta^*}})$$

on $(\{\psi_1<-t_{\beta^*}\}\cap V_{\beta^*})\times Y$. It follows from Lemma 2.34 that there exists a holomorphic $(n_1,0)$ form h_1 on M' such that

$$F=\pi_1^*(h_1)\wedge\pi_2^*(f_{\alpha_{\beta^*}})\tag{5.1}$$

on M and $h_0=h_1$ on $\{\psi_1<-t_{\beta^*}\}\cap V_{\beta^*}$.

Denote $\tilde{\varphi}=\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)$ on M'. Denote

$$\inf\left\{\int_{\{\psi_1<-t\}}|\tilde{f}|^2\mathrm{e}^{-\tilde{\varphi}}c(-\psi_1):(\tilde{f}-h_1,z_\beta)\in(\mathcal{O}(K_{M'})\otimes\mathcal{I}(\psi_1))_{z_\beta},\forall\beta\in\tilde{I}_1\right.$$
$$\left.\&\ \tilde{f}\in H^0(\{\psi_1<-t\},\mathcal{O}(K_{M'}))\right\}$$

by $G'(t)$, where $t\in[0,+\infty)$. Note that $f_{\alpha_{\beta^*}}=\tilde{f}_{\alpha_{\beta^*}}$ satisfies $\int_Y|f_{\alpha_{\beta^*}}|^2\mathrm{e}^{-\varphi_Y}<+\infty$. For any $\beta\in\tilde{I}_1$ and any holomorphic function h, it follows from Lemma 2.19 that $(h,(z_\beta,y))\in\mathcal{I}(\psi)_{(z_\beta,y)}$ for any $y\in Y$ if and only if $(h(\cdot,y),z_\beta)\in\mathcal{I}(\psi_1)_{z_\beta}$ for any $y\in Y$. Following from Lemma 2.26, equality (5.1) and Proposition 2.24, we get that $G'(0)<+\infty$ and $G'(h^{-1}(r))$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds]$, which contradicts to Theorem 2.11.

Thus, we obtain that $G(h^{-1}(r))$ is not linear.

5.2 Proof of Proposition 1.9

It follows from Corollary 2.3 that there exists a holomorphic $(n,0)$ form F on M_1, which satisfies that $(F-f,z)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$ and

$$G(t)=\int_{\{\psi<-t\}\cap M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)\tag{5.2}$$

for any $t\geqslant 0$.

It follows from Lemma 2.12 and Lemma 2.13 that there exists a local coordinate $w_{j,k}$ on a neighborhood $V_{z_{j,k}}\Subset\Omega_j$ of $z_{j,k}\in\Omega_j$ satisfying $w_{j,k}(z_{j,k})=0$ and

$$\log|w_{j,k}|=\frac{1}{p_{j,k}}\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})$$

for any $j\in\{1,\ldots,n_1\}$ and $1\leqslant k<\tilde{m}_j$, where $V_{z_{j,k}}\cap V_{z_{j,k'}}=\emptyset$ for any j and $k\neq k'$. Denote $\tilde{I}_1:=\{(\beta_1,\ldots,\beta_{n_1}):1\leqslant\beta_j<\tilde{m}_j\text{ for any }j\in\{1,\ldots,n_1\}\}$, $V_\beta:=\prod_{1\leqslant j\leqslant n_1}V_{z_{j,\beta_j}}$ for any $\beta=(\beta_1,\ldots,\beta_{n_1})\in\tilde{I}_1$ and $w_\beta:=(w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta:=(z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}})\in M$. It follows from Lemma 2.23 that

$$F=\sum_{\alpha\in\mathbb{Z}_{\geqslant 0}^{n_1}}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(F_{\alpha,\beta})$$

on a neighborhood $U_\beta\subset(V_\beta\times Y)\cap M_1$ of $\{z_\beta\}\times Y$ for any $\beta\in\tilde{I}_1$, where $F_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y. Following from Lemma 2.30 and equality (5.2), we obtain that

$$F_{\alpha,\beta}\equiv 0$$

for any $\alpha \in \left\{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} < 1\right\}$ and $\beta \in \tilde{I}_1$, and we have

$$\frac{G(0)}{\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds} \geqslant \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)} \int_Y |F_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}, \tag{5.3}$$

where $E_\beta = \left\{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1\right\}$ for any $\beta \in \tilde{I}_1$. Proposition 2.37 shows that there exists a holomorphic $(n, 0)$ form F_1 on M such that $(F_1 - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$ and

$$\begin{aligned}&\int_M |F_1|^2\mathrm{e}^{-\varphi}c(-\psi)\\ &\leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\mathrm{e}^{-\sum_{1\leqslant j\leqslant n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leqslant j\leqslant n_1}(\alpha_j+1)} \int_Y |F_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}.\end{aligned} \tag{5.4}$$

Denote $\tilde{E}_\beta := \left\{\alpha \in \mathbb{Z}_{\geqslant 0}^{n_1} : \sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \geqslant 1\right\}$ for any $\beta \in \tilde{I}_1$. As $(F_1 - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$. It follows from Lemma 2.22 and Lemma 2.18 that

$$F_1 = \sum_{\alpha\in E_\beta} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(F_{\alpha,\beta}) + \sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(\tilde{F}_{\alpha,\beta})$$

on a neighborhood of $\{z_\beta\} \times Y$ for any $\beta \in \tilde{I}_1$, where $\tilde{F}_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y. It follows from Lemma 2.23 that $(F_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ and $(\tilde{F}_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ for any $y \in Y$. Using Lemma 2.18 and Lemma 2.25, we obtain that $(F_1 - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$. Combining inequality (5.3) and (5.4), we have

$$\frac{G(0)}{\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds} = \int_M |F_1|^2\mathrm{e}^{-\varphi}c(-\psi) = \int_{M_1} |F_1|^2\mathrm{e}^{-\varphi}c(-\psi),$$

which implies that $\mu(M\backslash M_1) = 0$.

6 Proofs of Theorem 1.10 and Remark 1.11

In this section, we prove Theorem 1.10 and Remark 1.11.

6.1 Proof of Theorem 1.10

As $c(t)\mathrm{e}^{-t}$ is decreasing and $\Psi \leqslant 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M, which satisfies that $(F - f, z) \in \left(\mathcal{O}(K_M) \otimes \mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}\right)\right)_z$ for any $z \in Z_0$ and

$$\int_{M_1} |F|^2\mathrm{e}^{-\varphi}c(-\psi) \leqslant \int_M |F|^2\mathrm{e}^{-\varphi-\pi_1^*(\Psi)}c(-\psi + \pi_1^*(\Psi)) \leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right) \|f\|_{Z_0}^2. \tag{6.1}$$

If $\Psi \equiv 0$, as $(F - f, z) \in \left(\mathcal{O}(K_M) \otimes \mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}\right)\right)_z$ for any $z \in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F = \sum_{\alpha\in E} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(f_\alpha) + \sum_{\alpha\in\tilde{E}\backslash E} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(\tilde{f}_\alpha)$ on $V_0 \times Y$, where $\tilde{f}_\alpha$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_\alpha|^2\mathrm{e}^{-\varphi_Y} < +\infty$ for any $\alpha \in \tilde{E}\backslash E$. Note that $\left(\Psi + \sum_{1\leqslant j\leqslant n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_0) > -\infty$. It follows from Lemma 2.18, Lemma 2.25 and Lemma 2.17 that

$$\Big(\sum_{\alpha\in\tilde{E}\backslash E} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(\tilde{f}_\alpha), z\Big) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$$

for any $z \in Z_0$.

In the following, we prove the characterization of the holding of the equality in Theorem 1.10.

Firstly, we prove the necessity. Using inequality (6.1), we have

$$\int_{M_1} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) = \int_M |F|^2 \mathrm{e}^{-\varphi-\pi_1^*(\Psi)} c(-\psi+\pi_1^*(\Psi)).$$

Note that $c(t)\mathrm{e}^{-t}$ is decreasing. As $F \not\equiv 0$ and $\overline{M_1}^{\circ} = M_1$, we get that $M_1 = M$. As $\Psi \leqslant 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1 \leqslant j \leqslant n_1} \left\{ \pi_{1,j}^*(2p_j G_{\Omega_j}(\cdot, z_j)) \right\}.$$

Denote

$$\inf\left\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi<-t\}, \mathcal{O}(K_M)) \right.$$
$$\left. \& \, (\tilde{f}-F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))_z \text{ for any } z \in Z_0 \right\}$$

by $G(t)$, where $t \geqslant 0$. Denote

$$\inf\left\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi<-t\}, \mathcal{O}(K_M)) \right.$$
$$\left. \& \, (\tilde{f}-F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z \text{ for any } z \in Z_0 \right\}$$

by $\tilde{G}(t)$, where $t \geqslant 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geqslant 0$. Let $t \geqslant 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here $\sim$ means the former replaced by the latter) that there exists a holomorphic $(n, 0)$ form F_t on $\{\psi < -t\}$ satisfying that $(F_t - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$ and

$$\int_{\{\psi<-t\}} |F_t|^2 \mathrm{e}^{-\varphi} c(-\psi) \leqslant \left(\int_t^{+\infty} c(s)\mathrm{e}^{-s} ds \right) \|f\|_{Z_0}^2. \tag{6.2}$$

Following from inequality (6.2), we have $\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s)\mathrm{e}^{-s} ds} \leqslant \|f\|_{Z_0}^2$ for any $t \geqslant 0$. Note that

$$\tilde{G}(0) = \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds \right) \|f\|_{Z_0}^2.$$

Combining Theorem 2.2, we obtain that $\tilde{G}(h^{-1}(r))$ is linear with respect to r, which implies that $G(h^{-1}(r))$ is linear with respect to r, where $h(t) = \int_t^{+\infty} c(s)\mathrm{e}^{-s} ds$. It follows from Theorem 1.3 that statements (2) and (3) in Theorem 1.10 hold.

Now, we prove the sufficiency. Following from Remark 1.5 and $G(0) = \tilde{G}(0)$, we obtain that $\tilde{G}(0) = \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds \right) \|f\|_{Z_0}^2$.

Thus, Theorem 1.10 holds.

6.2 Proof of Remark 1.11

Note that $\left(\Psi + \sum_{1 \leqslant j \leqslant n_1} \tilde{\pi}_j^*(\varphi_j) \right)(z_0) > -\infty$. As $(f_\alpha, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ for any $y \in Y$ and $\alpha \in \tilde{E} \backslash E$, following from Lemma 2.25, Lemma 2.18 and Lemma 2.17, we get that $(\sum_{\alpha \in \tilde{E} \backslash E} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(f_\alpha), z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in Z_0$.

As $c(t)\mathrm{e}^{-t}$ is decreasing and $\Psi \leqslant 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M, which satisfies that $(F - f, z) \in \left(\mathcal{O}(K_M) \otimes \mathcal{I}\left(\max_{1 \leqslant j \leqslant n_1} \left\{ 2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j)) \right\} \right) \right)_z$ for any $z \in Z_0$ and

$$\int_{M_1} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) \leqslant \int_M |F|^2 \mathrm{e}^{-\varphi-\pi_1^*(\Psi)} c(-\psi+\pi_1^*(\Psi)) \leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds \right) \|f\|_{Z_0}^2. \tag{6.3}$$

If $\Psi\equiv 0$, as $(F-f,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)\right)_z$ for any $z\in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that $F=\sum_{\alpha\in E}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(f_\alpha)+\sum_{\alpha\in\tilde{E}\backslash E}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(\tilde{f}_\alpha)$ on $V_0\times Y$, where $\tilde{f}_\alpha$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y|\tilde{f}_\alpha|^2\mathrm{e}^{-\varphi_Y}<+\infty$ for any $\alpha\in\tilde{E}\backslash E$. Note that

$$(\Psi+\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j))(z_0)>-\infty.$$

It follows from Lemma 2.18, Lemma 2.25 and Lemma 2.17 that

$$(\sum_{\alpha\in\tilde{E}\backslash E}\pi_1^*(w^\alpha dw)\wedge\pi_2^*(\tilde{f}_\alpha),z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$$

for any $z\in Z_0$. Thus, we have $(F-f,z)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$.

In the following, we prove the characterization of the holding of the equality (replacing the ideal sheaf $\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)$ by $\mathcal{I}(\varphi+\psi)$) in Theorem 1.10.

Firstly, we prove the necessity. Using inequality (6.3), we have

$$\int_{M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)=\int_M|F|^2\mathrm{e}^{-\varphi-\pi_1^*(\Psi)}c(-\psi+\pi_1^*(\Psi)).$$

Note that $c(t)\mathrm{e}^{-t}$ is decreasing. As $F\not\equiv 0$, we get that

$$M_1=M=\left(\prod_{1\leqslant j\leqslant n_1}\Omega_j\right)\times Y.$$

As $\Psi\leqslant 0$, it follows from Lemma 2.16 that $\Psi\equiv 0$, i.e.,

$$\psi=\max_{1\leqslant j\leqslant n_1}\left\{\pi_{1,j}^*(2p_jG_{\Omega_j}(\cdot,z_j))\right\}.$$

Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right.$$
$$\left.\&\,(\tilde{f}-F,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z\text{ for any }z\in Z_0\right\}$$

by $G(t)$, where $t\geqslant 0$. Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right.$$
$$\left.\&\,(\tilde{f}-F,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z\text{ for any }z\in Z_0\right\}$$

by $\tilde{G}(t)$, where $t\geqslant 0$. It follows from Lemma 2.26 that $G(t)=\tilde{G}(t)$ for any $t\geqslant 0$. Let $t\geqslant 0$. It follows from Proposition 2.37 ($M\sim\{\psi<-t\}$, $\psi\sim\psi+t$ and $c(\cdot)\sim c(\cdot+t)$, here $\sim$ means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}\leqslant\|f\|_{Z_0}^2.$$

Note that $G(0)=\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|_{Z_0}^2$. Combining Theorem 2.2, we obtain that $G(h^{-1}(r))$ is linear with respect to r, where $h(t)=\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds$. It follows from Theorem 1.3 that statements (2) and (3) in Theorem 1.10 hold.

Now, we prove the sufficiency. Following from Remark 1.5, we obtain that $G(0)=\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|_{Z_0}^2$.

Thus, Remark 1.11 holds.

7 Proofs of Theorem 1.14 and Remark 1.15

In this section, we prove Theorem 1.14 and Remark 1.15.

7.1 Proof of Theorem 1.14

As $c(t)\mathrm{e}^{-t}$ is decreasing and $\Psi \leqslant 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n,0)$ form F on M, which satisfies that $(F-f,z) \in \big(\mathcal{O}(K_M)\otimes\mathcal{I}\big(\max_{1\leqslant j\leqslant n_1}\{2\sum_{1\leqslant k\leqslant m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\}\big)\big)_z$ for any $z\in Z_0$ and

$$\int_{M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)\leqslant\int_M|F|^2\mathrm{e}^{-\varphi-\pi_1^*(\Psi)}c(-\psi+\pi_1^*(\Psi))\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}. \tag{7.1}$$

If $\Psi\equiv0$, as $(F-f,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k\leqslant m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ for any $z\in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F=\sum_{\alpha\in E_\beta}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(f_{\alpha,\beta})+\sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(\tilde{f}_{\alpha,\beta})$ on $V_\beta\times Y$, where $\tilde{f}_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y|\tilde{f}_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}<+\infty$ for any $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in I_1$. Note that $\left(\Psi+\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)>-\infty$. It follows from Lemma 2.18, Lemma 2.25 and Lemma 2.17 that $\left(\sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(\tilde{f}_{\alpha,\beta}),z\right)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in\{z_\beta\}\times Y$, where $\beta\in I_1$.

In the following, we prove the characterization of the holding of the equality in Theorem 1.14.

Firstly, we prove the necessity. Using inequality (7.1), we have

$$\int_{M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)=\int_M|F|^2\mathrm{e}^{-\varphi-\pi_1^*(\Psi)}c(-\psi+\pi_1^*(\Psi)).$$

Note that $c(t)\mathrm{e}^{-t}$ is decreasing. As $F\not\equiv0$ and $\overline{M_1}^\circ=M_1$, we get that

$$M_1=M=\left(\prod_{1\leqslant j\leqslant n_1}\Omega_j\right)\times Y.$$

As $\Psi\leqslant0$, it follows from Lemma 2.16 that $\Psi\equiv0$, i.e.,

$$\psi=\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k\leqslant m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}.$$

Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right.$$
$$\left.\&\,(\tilde{f}-F,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z\text{ for any }z\in Z_0\right\}$$

by $G(t)$, where $t\geqslant0$. Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right.$$
$$\left.\&\,(\tilde{f}-F,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z\text{ for any }z\in Z_0\right\}$$

by $\tilde{G}(t)$, where $t\geqslant0$. It follows from Lemma 2.26 that $G(t)=\tilde{G}(t)$ for any $t\geqslant0$. Let $t\geqslant0$. It follows from Proposition 2.37 ($M\sim\{\psi<-t\}$, $\psi\sim\psi+t$ and $c(\cdot)\sim c(\cdot+t)$, here $\sim$ means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}\leqslant\|f\|^2_{Z_0}.$$

Note that $\tilde{G}(0)=\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}$. Combining Theorem 2.2, we obtain that $\tilde{G}(h^{-1}(r))$ is linear with respect to r, which implies that $G(h^{-1}(r))$ is linear with respect to r, where $h(t)=\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds$. As $f_{\alpha,\beta^*}\equiv 0$ for any $\alpha\neq\alpha_{\beta^*}$ satisfying $\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,1}}=1$, where $\beta^*=(1,\ldots,1)\in I_1$, it follows from Theorem 1.6 that statements (2), (3), (4) and (5) in Theorem 1.14 hold.

Now, we prove the sufficiency. Following from Remark 1.7 and $G(0)=\tilde{G}(0)$, we obtain that $\tilde{G}(0)=\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}$.

Thus, Theorem 1.14 holds.

7.2 Proof of Remark 1.15

Note that $\left(\Psi+\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)>-\infty$ for any $\beta\in I_1$. As $(f_{\alpha,\beta},y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ for any $y\in Y$, $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in I_1$, following from Lemma 2.25, Lemma 2.18 and Lemma 2.17, we get that $\left(\sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(f_{\alpha,\beta}),z\right)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in\{z_\beta\}\times Y$, where $\beta\in I_1$.

As $c(t)\mathrm{e}^{-t}$ is decreasing and $\Psi\leqslant 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n,0)$ form F on M, which satisfies that $(F-f,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k\leqslant m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ for any $z\in Z_0$ and

$$\int_{M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)\leqslant\int_M|F|^2\mathrm{e}^{-\varphi-\pi_1^*(\Psi)}c(-\psi+\pi_1^*(\Psi))\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}. \tag{7.2}$$

If $\Psi\equiv 0$, as $(F-f,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi)\right)_z$ for any $z\in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F=\sum_{\alpha\in E_\beta}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(f_{\alpha,\beta})+\sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(\tilde{f}_{\alpha,\beta})$ on $V_\beta\times Y$, where $\beta\in I_1$ and $\tilde{f}_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y|\tilde{f}_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}<+\infty$ for any $\alpha\in\tilde{E}_\beta\backslash E_\beta$. Note that $(\Psi+\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}_j^*(\varphi_j))(z_\beta)>-\infty$ for any $\beta\in I_1$. Following from Lemma 2.18, Lemma 2.25 and Lemma 2.17, we obtain that $\left(\sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta}\pi_1^*(w_\beta^\alpha dw_\beta)\wedge\pi_2^*(\tilde{f}_{\alpha,\beta}),z\right)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in\{z_\beta\}\times Y$. Thus, we have $(F-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$.

In the following, we prove the characterization of the holding of the equality (replacing the ideal sheaf $\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k\leqslant m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)$ by $\mathcal{I}(\varphi+\psi)$) in Theorem 1.14.

Firstly, we prove the necessity. Using inequality (7.2), we have

$$\int_{M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)=\int_M|F|^2\mathrm{e}^{-\varphi-\pi_1^*(\Psi)}c(-\psi+\pi_1^*(\Psi)).$$

Note that $c(t)\mathrm{e}^{-t}$ is decreasing. As $F\not\equiv 0$, we get that

$$M_1=M=\left(\prod_{1\leqslant j\leqslant n_1}\Omega_j\right)\times Y.$$

As $\Psi\leqslant 0$, it follows from Lemma 2.16 that $\Psi\equiv 0$, i.e.,

$$\psi=\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k\leqslant m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}.$$

Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right.$$
$$\left.\&\,(\tilde{f}-F,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z\text{ for any }z\in Z_0\right\}$$

by $G(t)$, where $t\geqslant 0$. Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right.$$
$$\left.\&\,(\tilde{f}-F,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z\text{ for any }z\in Z_0\right\}$$

by $\tilde{G}(t)$, where $t \geqslant 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geqslant 0$. Let $t \geqslant 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here $\sim$ means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s)\mathrm{e}^{-s} ds} \leqslant \|f\|_{Z_0}^2.$$

Note that $G(0) = \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \|f\|_{Z_0}^2$. Combining Theorem 2.2, we obtain that $G(h^{-1}(r))$ is linear with respect to r, where $h(t) = \int_t^{+\infty} c(s)\mathrm{e}^{-s} ds$. It follows from Theorem 1.3 that statements (2), (3), (4) and (5) in Theorem 1.14 hold.

Now, we prove the sufficiency. Following from Remark 1.7, we obtain that $G(0) = \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \|f\|_{Z_0}^2$. Thus, Remark 1.15 holds.

8 Proofs of Theorem 1.16 and Remark 1.17

In this section, we prove Theorem 1.16 and Remark 1.17.

8.1 Proof of Theorem 1.16

As $c(t)\mathrm{e}^{-t}$ is decreasing and $\Psi \leqslant 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n, 0)$ form F on M, which satisfies that $(F - f, z) \in \left(\mathcal{O}(K_M) \otimes \mathcal{I}\left(\max_{1 \leqslant j \leqslant n_1} \left\{2 \sum_{1 \leqslant k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\right)_z$ for any $z \in Z_0$ and

$$\int_{M_1} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) \leqslant \int_M |F|^2 \mathrm{e}^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)) \leqslant \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \|f\|_{Z_0}^2. \tag{8.1}$$

If $\Psi \equiv 0$, as $(F - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\psi))_z$ for any $z \in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F = \sum_{\alpha \in E_\beta} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(f_{\alpha,\beta}) + \sum_{\alpha \in \tilde{E}_\beta \backslash E_\beta} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(\tilde{f}_{\alpha,\beta})$ on $V_\beta \times Y$, where $\tilde{f}_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |\tilde{f}_{\alpha,\beta}|^2 \mathrm{e}^{-\varphi_Y} < +\infty$ for any $\alpha \in \tilde{E}_\beta \backslash E_\beta$ and $\beta \in \tilde{I}_1$. Note that $\left(\Psi + \sum_{1 \leqslant j \leqslant n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta) > -\infty$. For any $\beta \in \tilde{I}_1$, it follows from Lemma 2.18, Lemma 2.25 and Lemma 2.17 that $\left(\sum_{\alpha \in \tilde{E}_\beta \backslash E_\beta} \pi_1^*(w_\beta^\alpha dw_\beta) \wedge \pi_2^*(\tilde{f}_{\alpha,\beta}), z\right) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z$ for any $z \in \{z_\beta\} \times Y$.

Denote $\tilde{\psi} := \max_{1 \leqslant j \leqslant n_1} \left\{2 \sum_{1 \leqslant k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}$. Now, we assume $\left(\int_0^{+\infty} c(s)\mathrm{e}^{-s} ds\right) \|f\|_{Z_0}^2 = \inf \Big\{ \int_{M_1} |\tilde{F}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{F}$ is a holomorphic $(n, 0)$ form on M_1 such that $(\tilde{F} - f, z) \in (\mathcal{O}(K_{M_1}) \otimes \mathcal{I}(\tilde{\psi}))_z$ for any $z \in Z_0 \Big\}$ to get a contradiction.

Using inequality (8.1), we have

$$\int_{M_1} |F|^2 \mathrm{e}^{-\varphi} c(-\psi) = \int_M |F|^2 \mathrm{e}^{-\varphi - \pi_1^*(\Psi)} c(-\psi + \pi_1^*(\Psi)).$$

Note that $c(t)\mathrm{e}^{-t}$ is decreasing. As $F \not\equiv 0$, we get that

$$M_1 = M = \left(\prod_{1 \leqslant j \leqslant n_1} \Omega_j\right) \times Y.$$

As $\Psi \leqslant 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1 \leqslant j \leqslant n_1} \left\{2 \sum_{1 \leqslant k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}.$$

Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right.$$
$$\left. \& \, (\tilde{f} - F, z) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_z \text{ for any } z \in Z_0 \right\}$$

by $G(t)$, where $t \geqslant 0$. Denote

$$\inf\left\{\int_{\{\psi<-t\}}|\tilde{f}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{f}\in H^0(\{\psi<-t\},\mathcal{O}(K_M))\right.$$
$$\left.\&\,(\tilde{f}-F,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z \text{ for any } z\in Z_0\right\}$$

by $\tilde{G}(t)$, where $t\geqslant 0$. It follows from Lemma 2.26 that $G(t)=\tilde{G}(t)$ for any $t\geqslant 0$. Let $t\geqslant 0$. It follows from Proposition 2.37 ($M\sim\{\psi<-t\}$, $\psi\sim\psi+t$ and $c(\cdot)\sim c(\cdot+t)$, here $\sim$ means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds}\leqslant\|f\|^2_{Z_0}.$$

Note that $\tilde{G}(0)=\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}$. Combining Theorem 2.2, we obtain that $\tilde{G}(h^{-1}(r))$ is linear with respect to r, which implies that $G(h^{-1}(r))$ is linear with respect to r, where $h(t)=\int_t^{+\infty}c(s)\mathrm{e}^{-s}ds$. As $f_{\alpha,\beta^*}\equiv 0$ for any $\alpha\neq\alpha_{\beta^*}$ satisfying $\sum_{1\leqslant j\leqslant n_1}\frac{\alpha_j+1}{p_{j,1}}=1$, where $\beta^*=(1,\ldots,1)\in\tilde{I}_1$, the linearity of $G(h^{-1}(r))$ contradicts to Theorem 1.8. Thus, we obtain that there exists a holomorphic $(n,0)$ form $\tilde{F}$ on M_1, which satisfies that $\int_M|\tilde{F}|^2\mathrm{e}^{-\varphi}c(-\psi)<\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}$ and $(\tilde{F}-f,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}\pi^*_{1,j}(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ for any $z\in Z_0$.

8.2 Proof of Remark 1.17

Note that $\left(\Psi+\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}^*_j(\varphi_j)\right)(z_\beta)>-\infty$ for any $\beta\in\tilde{I}_1$. As $(f_{\alpha,\beta},y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ for any $y\in Y$, $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$, following from Lemma 2.25, Lemma 2.18 and Lemma 2.17, we get that $\left(\sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta}\pi^*_1(w^\alpha_\beta dw_\beta)\wedge\pi^*_2(f_{\alpha,\beta}),z\right)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in\{z_\beta\}\times Y$, where $\beta\in\tilde{I}_1$.

As $c(t)\mathrm{e}^{-t}$ is decreasing and $\Psi\leqslant 0$, it follows from Proposition 2.37 that there exists a holomorphic $(n,0)$ form F on M, which satisfies that $(F-f,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}\left(\max_{1\leqslant j\leqslant n_1}\left\{2\sum_{1\leqslant k<\tilde{m}_j}p_{j,k}\pi^*_{1,j}(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ for any $z\in Z_0$ and

$$\int_{M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)\leqslant\int_M|F|^2\mathrm{e}^{-\varphi-\pi^*_1(\Psi)}c(-\psi+\pi^*_1(\Psi))\leqslant\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}. \tag{8.2}$$

If $\Psi\equiv 0$, as $(F-f,z)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\psi))_z$ for any $z\in Z_0$, it follows from Lemma 2.18 and Lemma 2.22 that we have $F=\sum_{\alpha\in E_\beta}\pi^*_1(w^\alpha_\beta dw_\beta)\wedge\pi^*_2(f_{\alpha,\beta})+\sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta}\pi^*_1(w^\alpha_\beta dw_\beta)\wedge\pi^*_2(\tilde{f}_{\alpha,\beta})$ on $V_\beta\times Y$, where $\tilde{f}_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y|\tilde{f}_{\alpha,\beta}|^2\mathrm{e}^{-\varphi_Y}<+\infty$ for any $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$. Note that $\left(\Psi+\sum_{1\leqslant j\leqslant n_1}\tilde{\pi}^*_j(\varphi_j)\right)(z_\beta)>-\infty$. Following from Lemma 2.18, Lemma 2.25 and Lemma 2.17, we obtain that $\left(\sum_{\alpha\in\tilde{E}_\beta\backslash E_\beta}\pi^*_1(w^\alpha_\beta dw_\beta)\wedge\pi^*_2(\tilde{f}_{\alpha,\beta}),z\right)\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in\{z_\beta\}\times Y$, where $\beta\in\tilde{I}_1$. Hence, we have $(F-f,z)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\varphi+\psi))_z$ for any $z\in Z_0$.

In the following, we assume that $\inf\left\{\int_{M_1}|\tilde{F}|^2\mathrm{e}^{-\varphi}c(-\psi):\tilde{F}\text{ is a holomorphic }(n,0)\text{ form on }M_1\text{ such that }(\tilde{F}-f,z)\in(\mathcal{O}(K_{M_1})\otimes\mathcal{I}(\varphi+\psi))_z\text{ for any }z\in Z_0\right\}=\left(\int_0^{+\infty}c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}$ to get a contradiction.

Using inequality (8.2), we have

$$\int_{M_1}|F|^2\mathrm{e}^{-\varphi}c(-\psi)=\int_M|F|^2\mathrm{e}^{-\varphi-\pi^*_1(\Psi)}c(-\psi+\pi^*_1(\Psi)).$$

Note that $c(t)\mathrm{e}^{-t}$ is decreasing. As $F\not\equiv 0$, we get that

$$M_1=M=\left(\prod_{1\leqslant j\leqslant n_1}\Omega_j\right)\times Y.$$

As $\Psi \leqslant 0$, it follows from Lemma 2.16 that $\Psi \equiv 0$, i.e.,

$$\psi = \max_{1\leqslant j\leqslant n_1} \left\{ 2 \sum_{1\leqslant k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}.$$

Denote

$$\inf\left\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi<-t\}, \mathcal{O}(K_M)) \right.$$
$$\left. \& \, (\tilde{f}-F, z) \in (\mathcal{O}(K_M)\otimes \mathcal{I}(\varphi+\psi))_z \text{ for any } z\in Z_0 \right\}$$

by $G(t)$, where $t \geqslant 0$. Denote

$$\inf\left\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 \mathrm{e}^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi<-t\}, \mathcal{O}(K_M)) \right.$$
$$\left. \& \, (\tilde{f}-F, z) \in (\mathcal{O}(K_M)\otimes \mathcal{I}(\psi))_z \text{ for any } z\in Z_0 \right\}$$

by $\tilde{G}(t)$, where $t \geqslant 0$. It follows from Lemma 2.26 that $G(t) = \tilde{G}(t)$ for any $t \geqslant 0$. Let $t \geqslant 0$. It follows from Proposition 2.37 ($M \sim \{\psi < -t\}$, $\psi \sim \psi + t$ and $c(\cdot) \sim c(\cdot + t)$, here $\sim$ means the former replaced by the latter) that

$$\frac{\tilde{G}(t)}{\int_t^{+\infty} c(s)\mathrm{e}^{-s}ds} \leqslant \|f\|^2_{Z_0}.$$

Note that $G(0) = \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}$. Combining Theorem 2.2, we obtain that $G(h^{-1}(r))$ is linear with respect to r, which implies that $G(h^{-1}(r))$ is linear with respect to r, where $h(t) = \int_t^{+\infty} c(s)\mathrm{e}^{-s}ds$. As $f_{\alpha,\beta^*} \equiv 0$ for any $\alpha \neq \alpha_{\beta^*}$ satisfying $\sum_{1\leqslant j\leqslant n_1} \frac{\alpha_j+1}{p_{j,1}} = 1$, where $\beta^* = (1,\ldots,1) \in \tilde{I}_1$, the linearity of $G(h^{-1}(r))$ contradicts to Theorem 1.8. Thus, we obtain that there exists a holomorphic $(n,0)$ form $\tilde{F}$ on Ω such that $(\tilde{F} - f, z) \in (\varphi+\psi)_z$ for any $z \in Z_0$ and $\int_M |\tilde{F}|^2 \mathrm{e}^{-\varphi}c(-\psi) < \left(\int_0^{+\infty} c(s)\mathrm{e}^{-s}ds\right)\|f\|^2_{Z_0}$.

9 Proofs of Theorem 1.19, Remark 1.20, Theorem 1.22 and Remark 1.23

In this section, we prove Theorem 1.19, Remark 1.20, Theorem 1.22 and Remark 1.23.

9.1 Proofs of Theorem 1.19 and Remark 1.20

Let $f_1 = dw \wedge d\tilde{w}$ on $V_0 \times U_0$, and let $f_2 = d\tilde{w}$ on U_0. Let $\psi = \max_{1\leqslant j\leqslant n_1}\left\{\pi_{1,j}^*(2n_1 G_{\Omega_j}(\cdot, z_j))\right\}$. Following from Lemma 2.18, we get that $(H_1 - H_2, (z_0, y)) \in \mathcal{I}(\psi)_{(z_0,y)}$ for any $y \in Y$ if and only if $(H_1 - H_2)|_{\{z_0\}\times Y} = 0$, where H_1 and H_2 are holomorphic $(n,0)$ form on a neighborhood of $\{z_0\}\times Y$. Let f be a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y |f|^2 < +\infty$. It follows from Proposition 2.37 that there exists a holomorphic $(n,0)$ form F on M such that $F|_{\{z_0\}\times Y} = \pi_1^*(dw)\wedge \pi_2^*(f)$ and

$$\int_M |F|^2 \leqslant \frac{(2\pi)^{n_1}}{\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2} \int_Y |f|^2.$$

Note that

$$B_Y(y_0) = \frac{2^{n_2}}{\inf\left\{\int_Y |f|^2 : f \in H^0(Y, \mathcal{O}(K_Y)) \,\&\, f(y_0) = f_2(y_0)\right\}}$$

and

$$B_M((z_0,y_0)) = \frac{2^n}{\inf\left\{\int_M |F|^2 : F \in H^0(M, \mathcal{O}(K_M)) \,\&\, F((z_0,y_0)) = f_1((z_0,y_0))\right\}}.$$

Thus, we have $\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) \leqslant \pi^{n_1} B_M((z_0,y_0))$.

In the following, we prove the characterization of the holding of the equality $\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} B_M((z_0,y_0))$.

There exists a holomorphic $(n_2,0)$ form f_0 on Y such that $f_0(y_0)=f_2(y_0)$ and

$$B_Y(y_0)=\frac{2^{n_2}}{\int_Y |f_0|^2}>0.$$

It follows from Proposition 2.37 that there exists a holomorphic $(n,0)$ form F_0 on M such that $F_0 = \pi_1^*(dw)\wedge\pi_2^*(f_0)$ and

$$\int_M |F_0|^2 \leqslant \frac{(2\pi)^{n_1}}{\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2}\int_Y |f_0|^2. \tag{9.1}$$

Firstly, we prove the necessity. Note that $B_M((z_0,y_0)) \geqslant \frac{2^n}{\int_M |\tilde{F}|^2}$ for any holomorphic $(n,0)$ form $\tilde{F}$ on M satisfying that $\tilde{F} = \pi_1^*(dw)\wedge\pi_2^*(f_0)$ on $\{z_0\}\times Y$. Combining $\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} B_M((z_0,y_0))$, $B_Y(y_0)=\frac{2^{n_2}}{\int_Y |f_0|^2}$ and inequality (9.1), we obtain that

$$\begin{aligned}\frac{(2\pi)^{n_1}}{\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2}\int_Y |f_0|^2 = \inf\bigg\{\int_M |\tilde{F}|^2 : &\tilde{F}\in H^0(M,\mathcal{O}(K_M))\\ &\& \tilde{F}|_{\{z_0\}\times Y} = \pi_1^*(dw)\wedge\pi_2^*(f_0)\bigg\}.\end{aligned}$$

It follows from Theorem 1.10 that $\chi_{j,z_j}=1$ for any $1\leqslant j\leqslant n_1$. $\chi_{j,z_j}=1$ implies that there exists a holomorphic function f_j on Ω_j such that $|f_j| = \mathrm{e}^{G_{\Omega_j}(\cdot,z_j)}$, thus Ω_j is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero (see [69], see also [73] and [42]).

Now, we prove the sufficiency. As Ω_j is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero, we have $\chi_{j,z_j}=1$. We prove $\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} B_M((z_0,y_0))$ by contradiction: if not, there exists a holomorphic $(n,0)$ form $\tilde{F}_0$ on M such that $\tilde{F}_0((z_0,y_0)) = f_1((z_0,y_0))$ and

$$\int_M |\tilde{F}_0|^2 < \frac{(2\pi)^{n_1}}{\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2}\int_Y |f_0|^2. \tag{9.2}$$

There exists a holomorphic $(n_2,0)$ form $\tilde{f}_0$ on Y such that $\tilde{F}_0 = \pi_1^*(dw)\wedge\pi_2^*(\tilde{f}_0)$ on $\{z_0\}\times Y$. Hence $\tilde{f}_0(y_0) = f_2(y_0) = f_0(y_0)$, which implies that $\int_Y |\tilde{f}_0|^2 \geqslant \int_Y |f_0|^2$. Combining inequality (9.2), we have $\inf\{\int_M |\tilde{F}|^2 : F\in H^0(M,\mathcal{O}(K_M))\,\&\,\tilde{F}|_{\{z_0\}\times Y} = \pi_1^*(dw)\wedge\pi_2^*(\tilde{f}_0)\} < \frac{(2\pi)^{n_1}}{\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2}\int_Y |\tilde{f}_0|^2$, which contradicts to Theorem 1.10, hence $\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2 B_Y(y_0) = \pi^{n_1} B_M((z_0,y_0))$.

Thus, Theorem 1.19 holds.

Note that $B_{M_1}((z_0,y_0)) \geqslant B_M((z_0,y_0)) > 0$ and $B_{M_1}((z_0,y_0)) = B_M((z_0,y_0))$ if and only if $M=M_1$, thus Theorem 1.19 shows Remark 1.20 holds.

9.2 Proofs of Theorem 1.22 and Remark 1.23

Let $f_1 = dw\wedge d\tilde{w}$ on $V_0\times U_0$, and let $f_2 = d\tilde{w}$ on U_0. Let $\psi = \max_{1\leqslant j\leqslant n_1}\{\pi_{1,j}^*(2n_1 G_{\Omega_j}(\cdot,z_j))\}$. Following from Lemma 2.18, we get that $(H_1-H_2,(z_0,y))\in\mathcal{I}(\psi)_{(z_0,y)}$ for any $y\in Y$ if and only if $(H_1-H_2)|_{\{z_0\}\times Y}=0$, where H_1 and H_2 are holomorphic $(n,0)$ form on a neighborhood of $\{z_0\}\times Y$. Let f be a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y |f|^2 < +\infty$. It follows from Proposition 2.37 that there exists a holomorphic $(n,0)$ form F on M such that $F|_{\{z_0\}\times Y} = \pi_1^*(dw)\wedge\pi_2^*(f)$ and

$$\int_M |F|^2\rho \leqslant \frac{(2\pi)^{n_1}\rho(z_0)}{\prod_{1\leqslant j\leqslant n_1} c_j(z_j)^2}\int_Y |f|^2.$$

Note that

$$B_Y(y_0) = \frac{2^{n_2}}{\inf\{\int_Y |f|^2 : f\in H^0(Y,\mathcal{O}(K_Y))\,\&\, f(y_0)=f_2(y_0)\}}$$

and

$$B_{M,\rho}((z_0,y_0))=\frac{2^n}{\inf\left\{\int_M|F|^2\rho : F\in H^0(M,\mathcal{O}(K_M))\,\&\,F((z_0,y_0))=f_1((z_0,y_0))\right\}}.$$

Thus, we have $\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2B_Y(y_0)\leqslant\pi^{n_1}\rho(z_0)B_{M,\rho}((z_0,y_0))$.

In the following, we prove the characterization of the holding of the equality $\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2B_Y(y_0)=\pi^{n_1}\rho(z_0)B_{M,\rho}((z_0,y_0))$.

There exists a holomorphic $(n_2,0)$ form f_0 on Y such that $f_0(y_0)=f_2(y_0)$ and

$$B_Y(y_0)=\frac{2^{n_2}}{\int_Y|f_0|^2}>0.$$

It follows from Proposition 2.37 that there exists a holomorphic $(n,0)$ form F_0 on M such that $F_0=\pi_1^*(dw)\wedge\pi_2^*(f_0)$ and

$$\int_M|F_0|^2\rho\leqslant\frac{(2\pi)^{n_1}\rho(z_0)}{\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2}\int_Y|f_0|^2. \tag{9.3}$$

Firstly, we prove the necessity. Note that $B_{M,\rho}((z_0,y_0))\geqslant\frac{2^n}{\int_M|\tilde{F}|^2\rho}$ for any holomorphic $(n,0)$ form $\tilde{F}$ on M satisfying that $\tilde{F}=\pi_1^*(dw)\wedge\pi_2^*(f_0)$ on $\{z_0\}\times Y$. Combining $\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2B_Y(y_0)=\pi^{n_1}\rho(z_0)B_{M,\rho}((z_0,y_0))$, $B_Y(y_0)=\frac{2^{n_2}}{\int_Y|f_0|^2}$ and inequality (9.3), we obtain that $\frac{(2\pi)^{n_1}\rho(z_0)}{\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2}\int_Y|f_0|^2=\inf\left\{\int_M|\tilde{F}|^2\rho:\tilde{F}\in H^0(M,\mathcal{O}(K_M))\,\&\,\tilde{F}|_{\{z_0\}\times Y}=\pi_1^*(dw)\wedge\pi_2^*(f_0)\right\}$. It follows from Theorem 1.10 that $\chi_{j,z_j}=\chi_{j,-u_j}$ for any $1\leqslant j\leqslant n_1$.

Now, we prove $\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2B_Y(y_0)=\pi^{n_1}\rho(z_0)B_{M,\rho}((z_0,y_0))$ by contradiction: if not, there exists a holomorphic $(n,0)$ form $\tilde{F}_0$ on M such that $\tilde{F}_0((z_0,y_0))=f_1((z_0,y_0))$ and

$$\int_M|\tilde{F}_0|^2\rho<\frac{(2\pi)^{n_1}\rho(z_0)}{\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2}\int_Y|f_0|^2. \tag{9.4}$$

There exists a holomorphic $(n_2,0)$ form $\tilde{f}_0$ on Y such that $\tilde{F}_0=\pi_1^*(dw)\wedge\pi_2^*(\tilde{f}_0)$ on $\{z_0\}\times Y$. Hence $\tilde{f}_0(y_0)=f_2(y_0)=f_0(y_0)$, which implies that $\int_Y|\tilde{f}_0|^2\geqslant\int_Y|f_0|^2$. Combining inequality (9.4), we have $\inf\left\{\int_M|\tilde{F}|^2\rho:F\in H^0(M,\mathcal{O}(K_M))\,\&\,\tilde{F}|_{\{z_0\}\times Y}=\pi_1^*(dw)\wedge\pi_2^*(\tilde{f}_0)\right\}<\frac{(2\pi)^{n_1}\rho(z_0)}{\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2}\int_Y|\tilde{f}_0|^2$, which contradicts to Theorem 1.10, hence $\prod_{1\leqslant j\leqslant n_1}c_j(z_j)^2B_Y(y_0)=\pi^{n_1}\rho(z_0)B_{M,\rho}((z_0,y_0))$.

Thus, Theorem 1.22 holds.

Note that $B_{M_1,\rho}((z_0,y_0))\geqslant B_{M,\rho}((z_0,y_0))>0$ and $B_{M_1,\rho}((z_0,y_0))=B_{M,\rho}((z_0,y_0))$ if and only if $M=M_1$, thus Theorem 1.22 shows Remark 1.23 holds.

Acknowledgements The second author was supported by National Key R&D Program of China (Grant No. 2021YFA1003100) and National Natural Science Foundation of China (Grant Nos. 12425101, 11825101, 11522101 and 11431013). The third author was supported by China Postdoctoral Science Foundation (Grant Nos. BX20230402 and 2023M743719).

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