CONCAVITY PROPERTY OF MINIMAL L^2 INTEGRALS WITH LEBESGUE MEASURABLE GAIN V–FIBRATIONS OVER OPEN RIEMANN SURFACES

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ABSTRACT. In this article, we present characterizations of the concavity property of minimal L^2 integrals degenerating to linearity in the case of fibrations over open Riemann surfaces. As applications, we obtain characterizations of the holding of equality in optimal jets L^2 extension problem from fibers over analytic subsets to fibrations over open Riemann surfaces, which implies characterizations of the fibration versions of the equality parts of Suita conjecture and extended Suita conjecture.

1. INTRODUCTION

The strong openness property of multiplier ideal sheaves [35] i.e. $\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \underset{\epsilon>0}{\cup} \mathcal{I}((1+\epsilon)\varphi)$ (conjectured by Demailly [10]) has opened the door to new types of approximation techniques in several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [35, 41, 4, 5, 17, 6, 54, 38, 2, 55, 56, 18, 42, 7]), where φ is a plurisubharmonic function on a complex manifold M (see [8]), and multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is locally integrable (see e.g. [51, 44, 47, 11, 12, 10, 13, 43, 48, 49, 9, 39]).

When $\mathcal{I}(\varphi) = \mathcal{O}$, the strong openness property degenerates to the openness property (conjectured by Demailly-Kollár [12]). Berndtsson [1] proved the openness property by establishing an effectiveness result of the openness property. Stimulated by Berndtsson's effectiveness result, and continuing the solution of the strong openness property [35], Guan-Zhou [37] established an effectiveness result of the strong openness property by considering the minimal L^2 integral on the pseudoconvex domain D.

Considering the minimal L^2 integrals on the sublevel sets of the weight φ, Guan [21] obtained a sharp version of Guan-Zhou's effectiveness result, and established a concavity property of the minimal L^2 integrals on the sublevel sets of the weight φ (with constant gain). The concavity property deduces a proof of Saitoh's conjecture for conjugate Hardy H^2 kernels [22], and the sufficient and necessary condition of the existence of decreasing equisingular approximations with analytic singularities for the multiplier ideal sheaves with weights $\log(|z_1|^{a_1} + \cdots + |z_n|^{a_n})$ [23].

For smooth gain, Guan [20] (see also [24]) obtained the concavity property on Stein manifolds (weakly pseudoconvex Kähler case was obtained by Guan-Mi[25]). The concavity property deduces an optimal support function related to the strong

openness property (obtained by Guan-Yuan) [28], and an effectiveness result of the strong openness property in L^p (obtained by Guan-Yuan) [29]. For Lebesgue measurable gain, Guan-Yuan [27] obtained the concavity property on Stein manifolds (weakly pseudoconvex Kähler case was obtained by Guan-Mi-Yuan [26]), which deduces a twisted L^p version of strong openness property [30].

Note that the linearity is a degenerate concavity. A natural problem was posed in [31]:

Problem 1.1 ([31]). *How to characterize the concavity property degenerating to linearity?*

Recall that for 1-dim case, Guan-Yuan [27] gave an answer to Problem 1.1 for single point, i.e. for weights may not be subharmonic (the case of subharmonic weights was answered by Guan-Mi [24]), and Guan-Yuan [31] gave an answer to Problem 1.1 for finite points. For the case of products of open Riemann surfaces, Guan-Yuan [32] gave an answer to Problem 1.1 for product of finite points.

In this article, we give answers to Problem 1.1 for the case of fibrations over open Riemann surfaces.

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω, and let K_Ω be the canonical (holomorphic) line bundle on Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y.

Let Z_0^1 be a (closed) analytic subset of Ω, denote $Z_0 := \pi_1^{-1}(Z_0^1)$ be an analytic subset of M. Let ψ_1 be a negative subharmonic function on Ω such that $\psi_1(z) = -\infty$ for any $z \in Z_0^1$, and let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω. Let φ_2 be a plurisubharmonic function on Y. Denote that $\psi := \pi_1^*(\psi_1)$, $\varphi = \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$. Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$, $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $M \setminus Z_0$. Let f be a holomorphic $(n, 0)$ form on a neighborhood of Z_0. Denote

$$\inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, (z, y)) \in (\mathcal{O}(K_M))_{(z,y)} \otimes \mathcal{I}(\varphi + \psi)_{(z,y)} \right.$$

$$\text{for any } (z, y) \in Z_0$$

$$\left. \& \ \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right\}$$

by $G(t; c)$ for any $t \in [0, +\infty)$. Here $|f|^2 := \sqrt{-1}^{n^2} f \wedge \bar{f}$ for any $(n, 0)$ form f. We may denote $G(t; c)$ by $G(t)$ if there are no misunderstandings.

Recall that $G(h^{-1}(r))$ is concave with respect to r ([26]), where $h(t) = \int_t^{+\infty} c(l)e^{-l}dl$ for any $t \in [0, +\infty)$.

Recall some notation related to open Riemann surfaces (see [15], see also [36, 27, 26]). Let $P : \Delta \to \Omega$ be the universal covering from the unite disc Δ to Ω. The holomorphic function $\hat{f}$ (resp. holomorphic $(1, 0)$ form $\hat{F}$) on Δ is called as a multiplicative function (resp. multiplicative differential (Prym differential)), if there is a character χ, which is the representation of the fundamental group of Ω such that $g^*\hat{f} = \chi(g)\hat{f}$ (resp. $g^*(\hat{F}) = \chi(g)\hat{F}$), where $|\chi| = 1$ and g is an element of the fundamental group of Ω. Denote the set of such $\hat{f}$ by $\mathcal{O}^\chi(\Omega)$ (resp. $\Gamma^\chi(\Omega)$).

It is known that for any harmonic function u of Ω, there exists a χ_u and a multiplicative function $f_u \in \mathcal{O}^{\chi_u}(\Omega)$ such that $|f_u| = P^* e^u$. If $u_1 - u_2 = \log|\hat{f}|$, then $\chi_{u_1} = \chi_{u_2}$, where u_1 and u_2 are harmonic functions on Ω and $\hat{f}$ is a holomorphic function on Ω. Recall that for the Green function $G_\Omega(z, z_0)$, there exists a χ_{z_0} and a multiplicative function $f_{z_0} \in \mathcal{O}^{\chi_{z_0}}(\Omega)$ such that $|f_{z_0}(z)| = P^* e^{G_\Omega(z, z_0)}$ (see [50]).

1.1. Main result : linearity of the minimal L^2 integrals on fibrations over open Riemann surfaces.

In this section, we present characterizations of the concavity property of minimal L^2 integrals degenerating to linearity in the case of fibrations over open Riemann surfaces.

Let $Z_0^1 = \{z_0\} \subset \Omega$. Let w be a local coordinate on a neighborhood V_{z_0} of z_0 satisfying $w(z_0) = 0$. Let f be a holomorphic $(n, 0)$ form on $V_{z_0} \times Y$ which is a neighborhood of $Z_0 = \pi_1^{-1}(z_0)$, and $G(t)$ be the minimal L^2 integral on M with respect to f for any $t \geq 0$.

We present a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the fibers over single point sets as follows.

Theorem 1.2. *Assume that $G(0) \in (0, +\infty)$ and $(\psi_1 - 2pG_\Omega(\cdot, z_0))(z_0) > -\infty$, where $p = \frac{1}{2}v(dd^c(\psi_1), z_0) > 0$. Then $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1). $\psi_1 = 2pG_\Omega(\cdot, z_j)$;

(2). On $V_{z_0} \times Y$, $f = \pi_1^*(w^k dw) \wedge \pi_2^*(f_Y) + f_0$, where k is a nonnegative integer, f_Y is a holomorphic $(n - 1, 0)$ form on Y such that $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$, and $(f_0, (z_0, y)) \in (\mathcal{O}(K_M))_{(z_0, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_0, y)}$ for any $(z_0, y) \in Z_0$;

(3). $\varphi_1 + \psi_1 = 2\log|g| + 2G_\Omega(\cdot, z_0) + 2u$, where g is a holomorphic function on Ω such that $\mathrm{ord}_{z_0}(g) = k$ and u is a harmonic function on Ω;

(4). $\chi_{z_0} = \chi_{-u}$, where χ_{-u} and χ_{z_0} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_0)$ respectively.

When Y is a single point, Theorem 1.2 can be referred to [27].

Remark 1.3. *When the four statements in Theorem 1.2 hold,*

$$c_0 \pi_1^*(gP_*(f_u df_{z_0})) \wedge \pi_2^*(f_Y)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, (z_0, x)) \in (\mathcal{O}(K_M))_{(z_0, x)} \otimes \mathcal{I}(\varphi + \psi)_{(z_0, x)}$ for any $(z_0, x) \in Z_0$ and

$$G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s)e^{-s}ds\right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2} \int_Y |f_Y|^2 e^{-\varphi_2}$$

for any $t \geq 0$, where f_u is a holomorphic function on Δ such that $|f_u| = P^*(e^u)$, f_{z_0} is a holomorphic function on Δ such that $|f_{z_0}| = P^*(e^{G_\Omega(\cdot, z_0)})$,

$$c_0 := \lim_{z \to z_0} \frac{w^k dw}{gP_*(f_u df_{z_0})} \in \mathbb{C} \setminus \{0\},$$

and

$$d := \lim_{z \to z_0} \frac{g}{w^k}(z).$$

Let $Z_0^1 := \{z_j : j \in \mathbb{N} \& 1 \leq j \leq m\}$ be a finite subset of the open Riemann surface Ω. Let Y be an $n - 1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on

M. Let π_1, π_2 be the natural projections from M to Ω and Y and $Z_0 := \pi_1^{-1}(Z_0^1)$. Let ψ_1 be a subharmonic function on Ω such that $p_j = \frac{1}{2}v(dd^c\psi_1, z_j) > 0$, and let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω. Let φ_2 be a plurisubharmonic function on Y. Denote that $\psi := \pi_1^*(\psi_1)$, $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \subset\subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0^1$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any j, k, $j \neq k$. Denote that $V_0 := \bigcup_{1 \leq j \leq m} V_{z_j}$. Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y$ and $G(t)$ be the minimal L^2 integral on M with respect to f for any $t \geq 0$.

We present a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the fibers over sets of finite points as follows.

Theorem 1.4. *Assume that $G(0) \in (0, +\infty)$ and $(\psi_1 - 2p_j G_\Omega(\cdot, z_j))(z_j) > -\infty$, where $p_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$ for any $j \in \{1, 2, \ldots, m\}$. Then $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1). $\psi_1 = 2\sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j)$;

(2). for any $j \in \{1, 2, \ldots, m\}$, $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$ on $V_{z_j} \times Y$, where $a_j \in \mathbb{C} \backslash \{0\}$ is a constant, k_j is a nonnegative integer, f_Y is a holomorphic $(n-1, 0)$ form on Y such that $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$, and $(f_j, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$ for any $j \in \{1, 2, \ldots, m\}$ and $y \in Y$;

(3). $\varphi_1 + \psi_1 = 2\log|g| + 2\sum_{j=1}^{m} G_\Omega(\cdot, z_j) + 2u$, where g is a holomorphic function on Ω such that $\mathrm{ord}_{z_j}(g) = k_j$ and u is a harmonic function on Ω;

(4). $\prod_{j=1}^{m} \chi_{z_j} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_j)$ respectively;

(5). for any $j \in \{1, 2, \ldots, m\}$,

$$\lim_{z \to z_j} \frac{a_j w_j^{k_j} dw_j}{gP_*\left(f_u\left(\prod_{l=1}^{m} f_{z_l}\right)\left(\sum_{l=1}^{m} p_l \frac{df_{z_l}}{f_{z_l}}\right)\right)} = c_0, \tag{1.1}$$

where $c_0 \in \mathbb{C} \backslash \{0\}$ is a constant independent of j.

Remark 1.5. *When the five statements in Theorem 1.2 hold,*

$$c_0 \pi_1^*\left(gP_*\left(f_u\left(\prod_{l=1}^{m} f_{z_l}\right)\left(\sum_{l=1}^{m} p_l \frac{df_{z_l}}{f_{z_l}}\right)\right)\right) \wedge \pi_2^*(f_Y)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s)e^{-s}ds\right)\left(\sum_{j=1}^{m} \frac{2\pi|a_j|^2 e^{-2u(z_j)}}{p_j|d_j|^2}\right)\int_Y |f_Y|^2 e^{-\varphi_2}$$

for any $t \geq 0$, where f_u is a holomorphic function on Δ such that $|f_u| = P^*(e^u)$, f_{z_j} is a holomorphic function on Δ such that $|f_{z_j}| = P^*(e^{G_\Omega(\cdot, z_j)})$ for any $j \in \{1, 2, \ldots, m\}$, and $d_j := \lim_{z \to z_j} \frac{g}{w^{k_j}}(z_j)$.

Let $Z_0^1 := \{z_j : j \in \mathbb{N}_+\}$ be an infinite discrete subset of the open Riemann surface Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y and $Z_0 := \pi_1^{-1}(Z_0^1)$. Let ψ_1 be a subharmonic function on Ω such that $p_j = \frac{1}{2}v(dd^c\psi_1, z_j) > 0$, and let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω. Let φ_2 be a plurisubharmonic function on Y. Denote that $\psi := \pi_1^*(\psi_1)$, $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \subset\subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0^1$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any j, k, $j \neq k$. Denote that $V_0 := \bigcup_{j=1}^\infty V_{z_j}$. Let f be a holomorphic $(n,0)$ form on $V_0 \times Y$ and $G(t)$ be the minimal L^2 integral on M with respect to f for any $t \geq 0$.

We present a necessary condition such that $G(h^{-1}(r))$ is linear for the fibers over infinite analytic subsets.

Proposition 1.6. *Assume that $G(0) \in (0, +\infty)$ and $(\psi_1 - 2p_j G_\Omega(\cdot, z_j))(z_j) > -\infty$, where $p_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$ for any $j \in \mathbb{N}_+$. Assume that $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$, then the following statements hold:*

(1). $\psi_1 = 2\sum_{j=1}^\infty p_j G_\Omega(\cdot, z_j)$;

(2). for any $j \in \mathbb{N}_+$, $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$ on $V_{z_j} \times Y$, where $a_j \in \mathbb{C} \setminus \{0\}$ is a constant, k_j is a nonnegative integer, f_Y is a holomorphic $(n-1,0)$ form on Y such that $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$, and $(f_j, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$ for any $j \in \mathbb{N}_+$ and $y \in Y$;

(3). $\varphi_1 + \psi_1 = 2\log|g|$, where g is a holomorphic function on Ω such that $ord_{z_j}(g) = k_j + 1$ for any $j \in \mathbb{N}_+$;

(4). for any $j \in \mathbb{N}_+$,

$$\frac{p_j}{ord_{z_j} g} \lim_{z \to z_j} \frac{dg}{a_j w_j^{k_j} dw_j} = c_0, \tag{1.2}$$

where $c_0 \in \mathbb{C} \setminus \{0\}$ is a constant independent of j;

(5). $\sum_{j \in \mathbb{N}_+} p_j < +\infty$.

Let $Z_0^1 := \{z_j : j \in \mathbb{N} \& 1 \leq j < \gamma\}$ be a discrete subset of the open Riemann surface Ω, where $\gamma \in \mathbb{N}_+ \cup \{+\infty\}$. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold. Assume that $\tilde{M} \subset M$ is an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0 \subset \tilde{M}$. Let f be a holomorphic $(n,0)$ form on a neighborhood of Z_0 in $\tilde{M}$. Denote that

$$\tilde{G}(t) := \inf \left\{ \int_{\{\psi < -t\} \cap \tilde{M}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, (z, y)) \in (\mathcal{O}(K_{\tilde{M}}))_{(z,y)} \otimes \mathcal{I}(\varphi + \psi)_{(z,y)} \right.$$

$$\text{for any } (z, y) \in Z_0$$

$$\left. \& \ \tilde{f} \in H^0(\{\psi < -t\} \cap \tilde{M}, \mathcal{O}(K_{\tilde{M}})) \right\}$$

for any $t \geq 0$, where $K_{\tilde{M}}$ is the canonical line bundle on $\tilde{M}$, and ψ, φ are as above.

We present a necessary condition such that $\tilde{G}(h^{-1}(r))$ is linear for the fibrations over analytic subsets.

Proposition 1.7. *Assume that $\tilde{G}(0) \in (0, +\infty)$ and $(\psi_1 - 2p_j G_\Omega(\cdot, z_j))(z_j) > -\infty$, where $p_j = \frac{1}{2}v(dd^c(\psi_1), z_j) > 0$ for any $j \in \mathbb{N}$. If $\tilde{G}(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$, then $\tilde{M} = M$.*

1.2. Applications: optimal L^2 extension problem from fibers over analytic subsets to fibrations over open Riemann surfaces.

In this section, we give characterizations of the holding of equality in optimal jets L^2 extension problem from fibers over analytic subsets to fibrations over open Riemann surfaces.

Let $Z_0^1 := \{z_j : j \in \mathbb{N}\&1 \leq j \leq m\}$ be a finite subset of the open Riemann surface Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y, and $Z_0 := \pi_1^{-1}(Z_0^1)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \subset\subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0^1$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any j, k, $j \neq k$. Let $c_\beta(z)$ be the logarithmic capacity (see [46]) on Ω which is defined by

$$c_\beta(z_j) := \exp \lim_{z \to z_0}(G_\Omega(z, z_j) - \log|w_j(z)|).$$

Denote that $V_0 := \bigcup_{1 \leq j \leq m} V_{z_j}$. Assume that $\tilde{M} \subset M$ is an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0 \subset \tilde{M}$.

As an application of Theorem 1.4, we give a characterization of the holding of equality in optimal L^2 extension problem from analytic subsets to fibrations over open Riemann surfaces, where the analytic subsets are fibers over finite points on open Riemann surfaces.

Theorem 1.8. *Let k_j be a nonnegative integer for any $j \in \{1, 2, ..., m\}$. Let ψ_1 be a negative subharmonic function on Ω satisfying that $\frac{1}{2}v(dd^c\psi_1, z_j) = p_j > 0$ for any $j \in \{1, 2, ..., m\}$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω, $\frac{1}{2}v(dd^c(\varphi_1 + \psi_1), z_j) = k_j + 1$ and $\alpha_j := (\varphi_1 + \psi_1 - 2(k_j + 1)G_\Omega(\cdot, z_j))(z_j) > -\infty$ for any j. Let φ_2 be a plurisubharmonic function on Y. Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n-1, 0)$ form on Y such that $\int_Y |F_j|^2 e^{-\varphi_2} < +\infty$ for any j.*

Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying that $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_j)$ on $V_{z_j} \times Y$. Then there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \quad (1.3)$$

Moreover, equality $\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} = \inf\left\{\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \text{ form on } \tilde{M} \text{ such that } (\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)} \text{ for any } (z_j, y) \in Z_0\right\}$ holds if and only if the following statements hold:

(1). $\psi_1 = 2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j)$;

(2). $\varphi_1 + \psi_1 = 2\log|g| + 2\sum_{j=1}^{m}(k_j + 1)G_\Omega(\cdot, z_j) + 2u$, where g is a holomorphic function on Ω such that $g(z_j) \neq 0$ for any $j \in \{1, 2, \ldots, m\}$ and u is a harmonic function on Ω;

(3). $\prod_{j=1}^{m} \chi_{z_j}^{k_j+1} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_j)$ respectively;

(4). for any $j \in \{1, 2, \ldots, m\}$,

$$\lim_{z \to z_j} \frac{a_j w_j^{k_j} dw_j}{gP_*\left(f_u\left(\prod_{l=1}^{m} f_{z_l}^{k_l+1}\right)\left(\sum_{l=1}^{m} p_l \frac{df_{z_l}}{f_{z_l}}\right)\right)} = c_j \in \mathbb{C} \setminus \{0\}, \qquad (1.4)$$

and there exist $c_0 \in \mathbb{C} \setminus \{0\}$ and a holomorphic $(n-1, 0)$ form F_Y on Y which are independent of j such that $c_0 F_Y = c_j F_j$ for any $j \in \{1, 2, \ldots, m\}$;

(5). $\tilde{M} = M$.

Remark 1.9. *When the five statements in Theorem 1.8 hold,*

$$c_0 \pi_1^*\left(gP_*\left(f_u\left(\prod_{l=1}^{m} f_{z_l}^{k_l+1}\right)\left(\sum_{l=1}^{m} p_l \frac{df_{z_l}}{f_{z_l}}\right)\right)\right) \wedge \pi_2^*(F_Y)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{m} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

Let $Z_0^1 := \{z_j : j \in \mathbb{N}_+\}$ be an infinite discrete subset of the open Riemann surface Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y, and $Z_0 := \pi_1^{-1}(Z_0^1)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \subset\subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0^1$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any j, k, $j \neq k$. Denote that $V_0 := \bigcup_{j=1}^{\infty} V_{z_j}$.

We give an L^2 extension result from fibers over analytic subsets to fibrations over open Riemann surfaces, where the analytic subsets are infinite points on open Riemann surfaces.

Theorem 1.10. *Let k_j be a nonnegative integer for any $j \in \mathbb{N}_+$. Let ψ_1 be a negative subharmonic function on Ω satisfying that $\frac{1}{2}v(dd^c\psi_1, z_j) = k_j + 1 > 0$ for any $j \in \mathbb{N}_+$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω, $\frac{1}{2}v(dd^c(\varphi_1 + \psi_1), z_j) = k_j + 1$ and $\alpha_j := (\varphi_1 + \psi_1 - 2(k_j + 1)G_\Omega(\cdot, z_j))(z_j) > -\infty$ for any j. Let φ_2 be a plurisubharmonic function on Y. Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n-1, 0)$ form on Y such that $\int_Y |F_j|^2 e^{-\varphi_2} < +\infty$ for any j.*

Let f be a holomorphic $(n,0)$ form on $V_0 \times Y$ satisfying that $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_j)$ on $V_{z_j} \times Y$. If

$$\sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j+1) c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} < +\infty,$$

then there exists a holomorphic $(n,0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) < \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j+1) c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

$$(1.5)$$

Remark 1.11. *Assume that $\tilde{M} \subset M$ is an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0 \subset \tilde{M}$. Then according to Theorem 1.10 there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and*

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) < \int_M |F|^2 e^{-\varphi} c(-\psi)$$
$$\leq \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j+1) c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

1.3. **Suita conjecture and extended Suita conjecture.**

In this section, we present the characterizations of the fibration versions of the equality parts of Suita conjecture and extended Suita conjecture.

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω, and let K_Ω be the canonical (holomorphic) line bundle on Ω. Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Denote the space of L^2 integrable holomorphic section of K_Ω by $A^2(\Omega, K_\Omega, dV_\Omega^{-1}, dV_\Omega)$. Let $\{\phi_l\}_{l=1}^{+\infty}$ be a complete orthogonal system of $A^2(\Omega, K_\Omega, dV_\Omega^{-1}, dV_\Omega)$ satisfying $(\sqrt{-1}) \int_M \frac{\phi_i}{\sqrt{2}} \wedge \frac{\overline{\phi_j}}{\sqrt{2}} = \delta_i^j$. Put $\kappa_\Omega = \sum_{l=1}^{+\infty} \phi_l \otimes \overline{\phi_l} \in C^\omega(\Omega, K_\Omega \otimes \overline{K_\Omega})$. Denote that

$$B_\Omega(z) dw \otimes \overline{dw} := \kappa_\Omega|_{V_{z_0}}.$$

Let $c_\beta(z)$ be the logarithmic capacity (see [46]) which is locally defined by

$$c_\beta(z_0) := \exp \lim_{z \to z_0} (G_\Omega(z, z_0) - \log|w(z)|)$$

on Ω. In [50], Suita stated a conjecture as below.

Conjecture 1.12. $c_\beta(z_0)^2 \leq \pi B_\Omega(z_0)$ *holds for any $z_0 \in \Omega$, and equality holds if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.*

The inequality part of Suita conjecture for bounded planar domain was proved by Błocki [3], and original form of the inequality was proved by Guan-Zhou [33]. The equality part of Suita conjecture was proved by Guan-Zhou [36], which completed the proof of Suita conjecture.

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω. Let Y be an $(n-1)-$dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = \Omega \times Y$ be an

$n-$dimensional complex manifold. Let π_1, π_2 be the natural projections from M to Ω and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M.

Denote the space of L^2 integrable holomorphic section of K_M (resp. K_Y) by $A^2(M, K_M, dV_M^{-1}, dV_M)$ (resp. $A^2(Y, K_Y, dV_Y^{-1}, dV_Y)$). Let $\{\sigma_l\}_{l=1}^{+\infty}$ (resp. $\{\tau_l\}_{l=1}^{+\infty}$) be a complete orthogonal system of $A^2(M, K_M, dV_M^{-1}, dV_M)$ (resp. $A^2(Y, K_Y, dV_Y^{-1}, dV_Y)$) satisfying $(\sqrt{-1})^{n^2} \int_M \frac{\sigma_i}{\sqrt{2^n}} \wedge \frac{\overline{\sigma}_j}{\sqrt{2^n}} = \delta_i^j$ (resp. $(\sqrt{-1})^{(n-1)^2} \int_Y \frac{\tau_i}{\sqrt{2^{n-1}}} \wedge \frac{\overline{\tau}_j}{\sqrt{2^{n-1}}} = \delta_i^j$). Put $\kappa_M = \sum_{l=1}^{+\infty} \sigma_l \otimes \overline{\sigma}_l \in C^\omega(M, K_M \otimes \overline{K_M})$ and $\kappa_Y = \sum_{l=1}^{+\infty} \tau_l \otimes \overline{\tau}_l \in C^\omega(Y, K_Y \otimes \overline{K_Y})$.

Let $z_0 \in \Omega$, $y_0 \in Y$. Let w be a local coordinate on a neighborhood V_{z_0} of z_0 in Ω satisfying $w(z_0) = 0$. Let $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_{n-1})$ be a local coordinate on a neighborhood U_{y_0} of y_0 in Y satisfying that $\tilde{w}_j(y_0) = 0$ for any $j \in \{1, 2, \ldots, n-1\}$. Denote that

$$B_M((z, y))dw \wedge d\tilde{w}_1 \wedge \ldots d\tilde{w}_{n-1} \otimes \overline{dw \wedge d\tilde{w}_1 \wedge \ldots d\tilde{w}_{n-1}} := \kappa_M$$

on $V_{z_0} \times U_{y_0}$ and

$$B_Y(y)d\tilde{w}_1 \wedge \ldots d\tilde{w}_{n-1} \otimes \overline{d\tilde{w}_1 \wedge \ldots d\tilde{w}_{n-1}} := \kappa_Y$$

on U_{y_0}. Let $c_\beta(z_0)$ be the logarithmic capacity which is locally defined by

$$c_\beta(z_0) := \exp \lim_{z \to z_0} (G_\Omega(z, z_0) - \log|w(z)|).$$

Assume that $B_Y(y_0) > 0$. We give the following theorem as a characterization of the holding of equality in the fibration version of Suita conjecture.

Theorem 1.13. $c_\beta(z_0)^2 B_Y(y_0) \le \pi B_M((z_0, y_0))$ holds, and equality holds if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

Let $\tilde{M} \subset M$ be an $n-$dimensional complex manifold satisfying that $\{z_0\} \times Y \subset \tilde{M}$. Similar to M, the Bergman kernel $B_{\tilde{M}}$ can be defined. Theorem 1.13 implies the following result.

Remark 1.14. $c_\beta(z_0)^2 B_Y(y_0) \le \pi B_{\tilde{M}}((z_0, y_0))$ holds, and equality holds if and only if $\tilde{M} = M$ and Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

Let Ω be an open Riemann surface which admits a nontrivial Green function G_Ω, and let K_Ω be the canonical (holomorphic) line bundle on Ω. Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let $\rho := e^{-2u}$ on Ω, where u is a harmonic function on Ω. Denote that

$$B_{\Omega,\rho}dw \otimes \overline{dw} := \sum_{l=1}^{+\infty} \sigma_l \otimes \overline{\sigma}_l|_{V_{z_0}} \in C^\omega(V_{z_0}, K_\Omega \otimes \overline{K_\Omega}),$$

where $\{\sigma_l\}_{l=1}^{+\infty}$ are holomorphic $(1, 0)$ forms on Ω satisfying

$$\sqrt{-1} \int_\Omega \rho \frac{\sigma_i}{\sqrt{2}} \wedge \frac{\overline{\sigma}_j}{\sqrt{2}} = \delta_i^j$$

and

$$\{F \in H^0(\Omega, K_\Omega) : \int_\Omega \rho|F|^2 < +\infty \ \& \ \int_\Omega \rho\sigma_l \wedge \overline{F} = 0 \ \text{for any} \ l \in \mathbb{N}_+\} = \{0\}.$$

In [53], Yamada stated a conjecture as below (so-called extended Suita conjecture).

Conjecture 1.15. $c_\beta(z_0)^2 \leq \pi\rho(z_0)B_{\Omega,\rho}(z_0)$ *holds for any* $z_0 \in \Omega$, *and equality holds if and only* $\chi_{-u} = \chi_{z_0}$, *where* χ_{-u} *and* χ_{z_0} *are the characters associated to the functions* $-u$ *and* $G_\Omega(\cdot, z_0)$ *respectively.*

The inequality part of extended Suita conjecture was proved by Guan-Zhou [34]. The equality part of extended Suita conjecture was proved by Guan-Zhou [36].

Let $\rho := e^{-2\pi_1^*(u)}$ on M, where u is a harmonic function on Ω. Denote that

$$B_{M,\rho}d \wedge d\tilde{w}_1 \wedge \ldots d\tilde{w}_{n-1} \otimes \overline{dw \wedge d\tilde{w}_1 \wedge \ldots d\tilde{w}_{n-1}} := \sum_{l=1}^{+\infty} e_l \otimes \overline{e}_l$$

on $V_0 \times Y$, where $\{e_l\}_{l=1}^{+\infty}$ are holomorphic $(n,0)$ forms on M satisfying

$$(\sqrt{-1})^{n^2} \int_M \rho \frac{e_i}{\sqrt{2^n}} \wedge \frac{\overline{e}_j}{\sqrt{2^n}} = \delta_i^j$$

and

$$\left\{ F \in H^0(M, K_M) : \int_M \rho|F|^2 < +\infty \ \& \ \int_M \rho e_l \wedge \overline{F} = 0 \text{ for any } l \in \mathbb{N}_+ \right\} = \{0\}.$$

Assume that $B_Y(y_0) > 0$. We give the following theorem as a characterization of the holding of equality in the fibration version of extended Suita conjecture.

Theorem 1.16. $c_\beta(z_0)^2 B_Y(y_0) \leq \pi\rho(z_0)B_{M,\rho}(z_0)$ *holds, and equality holds if and only if* $\chi_{-u} = \chi_{z_0}$, *where* χ_{-u} *and* χ_{z_0} *are the characters associated to the functions* $-u$ *and* $G_\Omega(\cdot, z_0)$ *respectively.*

Let $\tilde{M} \subset M$ be an $n-$dimensional complex manifold satisfying that $\{z_0\} \times Y \subset \tilde{M}$. Similar to M, the Bergman kernel $B_{\tilde{M},\rho}$ can be defined. Theorem 1.16 implies the following result.

Remark 1.17. $c_\beta(z_0)^2 B_Y(y_0) \leq \pi B_{\tilde{M},\rho}((z_0, y_0))$ *holds, and equality holds if and only if* $\tilde{M} = M$ *and* $\chi_{-u} = \chi_{z_0}$.

2. PREPARATION

2.1. Concavity property of minimal L^2 integrals.

Recall the concavity property of minimal L^2 integrals on weakly pseudoconvex Kähler manifolds in [26]. Let M be an $n-$dimensional complex manifold. Let X and Z be closed subsets of M. A triple (M, X, Z) satisfies condition (A), if the following statements hold:

(1). X is a closed subset of M and X is locally negligible with respect to L^2 holomorphic functions, i.e. for any coordinated neighborhood $U \subset M$ and for any L^2 holomorphic function f on $U \setminus X$, there exists an L^2 holomorphic function $\tilde{f}$ on U such that $\tilde{f}|_{U \setminus X} = f$ with the same L^2 norm;

(2). Z is an analytic subset of M and $M \setminus (X \cup Z)$ is a weakly pseudoconvex Kähler manifold.

Let (M, X, Z) be a triple satisfying condition (A). Let K_M be the canonical line bundle on M. Let ψ be a plurisubharmonic function on M such that $\{\psi < -t\} \setminus (X \cup Z)$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$, and let φ

be a Lebesgue measurable function on M such that $\varphi + \psi$ is a plurisubharmonic function on M. Denote $T = -\sup_M \psi$.

Recall the concept of "gain" in [26]. A positive measurable function c on $(T, +\infty)$ is in the class $\mathcal{P}_{T,M}$ if the following two statements hold:

(1). $c(t)e^{-t}$ is decreasing with respect to t;

(2). there is a closed subset E of M such that $E \subset Z \cap \{\psi = -\infty\}$ and for any compact subset $K \subset M \setminus E$, $e^{-\varphi}c(-\psi)$ has a positive lower bound on K.

Let Z_0 be a subset of M such that $Z_0 \cap \mathrm{Supp}(\mathcal{O}/\mathcal{I}(\varphi+\psi)) \neq \emptyset$. Let $U \supset Z_0$ be an open subset of M, and let f be a holomorphic $(n,0)$ form on U. Let $\mathcal{F}_{z_0} \supset \mathcal{I}(\varphi+\psi)_{z_0}$ be an ideal of $\mathcal{O}_{z_0}$ for any $z_0 \in Z_0$.

Denote that

$$G(t;c) := \inf \left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi}c(-\psi) : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right.$$
$$\left. \& (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0}) \right\}, \tag{2.1}$$

and

$$\mathcal{H}^2(c,t) := \left\{ \tilde{f} : \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi}c(-\psi) < +\infty, \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_M)) \right.$$
$$\left. \& (\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0}) \right\}, \tag{2.2}$$

where $t \in [T, +\infty)$, and c is a nonnegative measurable function on $(T, +\infty)$. Here $|\tilde{f}|^2 := \sqrt{-1}^{n^2} \tilde{f} \wedge \bar{\tilde{f}}$ for any $(n,0)$ form $\tilde{f}$. And $(\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$ means that $(\tilde{f} - f, z_0) \in (\mathcal{O}(K_M) \otimes \mathcal{F})_{z_0}$ for any $z_0 \in Z_0$. If there is no holomorphic $(n,0)$ form $\tilde{f}$ on $\{\psi < -t\}$ satisfying $(\tilde{f} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$, we set $G(t;c) = +\infty$. We may denote $G(t;c)$ by $G(t)$ if there are no misunderstandings.

In [26], Guan-Mi-Yuan obtained the following concavity of $G(t;c)$.

Theorem 2.1. *Let $c \in \mathcal{P}_{T,M}$ such that $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. If there exists $t \in [T, +\infty)$ satisfying that $G(t) < +\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt)$, $\lim_{t \to T+0} G(t) = G(T)$ and $\lim_{t \to +\infty} G(t) = 0$, where $h(t) = \int_T^{+\infty} c(t_1)e^{-t_1}dt_1$.*

Guan-Mi-Yuan also obtained the following corollary of Theorem 2.1, which is a neccessary condition for the concavity degenerating to linearity.

Lemma 2.2 ([26]). *Let $c(t) \in \mathcal{P}_{T,M}$ such that $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. If $G(t) \in (0, +\infty)$ for some $t \geq T$ and $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_T^{+\infty} c(s)e^{-s}ds)$, where $h(t) = \int_t^{+\infty} c(l)e^{-l}dl$, then there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$, and $G(t;c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi}c(-\psi)$ for any $t \geq T$.*

Furthermore, we have

$$\int_{\{-t_2 \leq \psi < -t_1\}} |F|^2 e^{-\varphi}a(-\psi) = \frac{G(T_1;c)}{\int_{T_1}^{+\infty} c(t)e^{-t}dt} \int_{t_1}^{t_2} a(t)e^{-t}dt, \tag{2.3}$$

for any nonnegative measurable function a on $(T, +\infty)$, where $T \leq t_1 < t_2 \leq +\infty$.

Especially, if $\mathcal{H}^2(\tilde{c}, t_0) \subset \mathcal{H}^2(c, t_0)$ for some $t_0 \geq T$, where $\tilde{c}$ is a nonnegative measurable function on $(T, +\infty)$, we have

$$G(t_0; \tilde{c}) = \int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} \tilde{c}(-\psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(s) e^{-s} ds} \int_{t_0}^{+\infty} \tilde{c}(s) e^{-s} ds. \qquad (2.4)$$

Remark 2.3. *([26]) Let $c(t) \in \mathcal{P}_{T,M}$. If $\mathcal{H}^2(\tilde{c}, t_1) \subset \mathcal{H}^2(c, t_1)$, then $\mathcal{H}^2(\tilde{c}, t_2) \subset \mathcal{H}^2(c, t_2)$, where $t_1 > t_2 > T$. In the following, we give some sufficient conditions of $\mathcal{H}^2(\tilde{c}, t_0) \subset \mathcal{H}^2(c, t_0)$ for $t_0 > T$:*

(1). $\tilde{c} \in \mathcal{P}_{T,M}$ and $\lim_{t \to +\infty} \frac{\tilde{c}(t)}{c(t)} > 0$. Especially, $\tilde{c} \in \mathcal{P}_{T,M}$, c and $\tilde{c}$ are smooth on $(T, +\infty)$ and $\frac{d}{dt}(\log \tilde{c}(t)) \geq \frac{d}{dt}(\log c(t))$;

(2). $\tilde{c} \in \mathcal{P}_{T,M}$, $\mathcal{H}^2(c, t_0) \neq \emptyset$ and there exists $t > t_0$ such that $\{\psi < -t\} \subset\subset \{\psi < -t_0\}$, $\{z \in \overline{\{\psi < -t\}} : \mathcal{I}(\varphi + \psi)_z \neq \mathcal{O}_z\} \subset Z_0$ and $\mathcal{F}|_{\overline{\{\psi < -t\}}} = \mathcal{I}(\varphi + \psi)|_{\overline{\{\psi < -t\}}}$.

2.2. Open Riemann surface case.

Let $M = \Omega$ be an open Riemann surface, which admits a nontrivial Green function G_Ω. Let $P : \Delta \to \Omega$ be the universal covering from unit disc Δ to Ω. It is known that for any harmonic function u on Ω, there exists a χ_u (the character associate to u) and a multiplicative function $f_u \in \mathcal{O}^{\chi_u}(\Omega)$, such that $|f_u| = P^* e^u$. Let $z_0 \in \Omega$. Recall that for the Green function $G_\Omega(z, z_0)$, there exist a χ_{z_0} and a multiplicative function $f_{z_0} \in \mathcal{O}^{\chi_{z_0}}(\Omega)$, such that $|f_{z_0}(z)| = P^* e^{G_\Omega(z, z_0)}$.

Let $Z_0 := \{z_1, z_2, ..., z_m\} \subset \Omega$ be a subset of Ω satisfying that $z_j \neq z_k$ for any $j \neq k$. Let w_j be a local coordinate on a neighborhood $V_{z_j} \Subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $j \in \{1, 2, ..., m\}$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any $j \neq k$. Denote that $V_0 := \cup_{1 \leq j \leq n} V_{z_j}$.

Let f be a holomorphic $(1, 0)$ form on V_0, and let $f = f_1 dw_j$ on V_{z_j}, where f_1 is a holomorphic function on V_0. Let ψ be a negative subharmonic function on Ω, and let φ be a Lebesgue measurable function on Ω such that $\varphi + \psi$ is subharmonic on Ω.

The following Theorem gives a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity.

Theorem 2.4 ([31]). *Let $c \in \mathcal{P}_{0,\Omega}$. Assume that $G(0) \in (0, +\infty)$ and $(\psi - 2p_j G_\Omega(\cdot, z_j))(z_j) > -\infty$ for $j \in \{1, 2, .., m\}$, where $p_j = \frac{1}{2} v(dd^c(\psi), z_j) > 0$. Then $G(h^{-1}(r))$ is linear with respect to r if and only if the following statements hold:*

(1) $\psi = 2 \sum_{1 \leq j \leq m} p_j G_\Omega(\cdot, z_j)$;

(2) $\varphi + \psi = 2 \log |g| + 2 \sum_{1 \leq j \leq m} G_\Omega(\cdot, z_j) + 2u$ and $\mathcal{F}_{z_j} = \mathcal{I}(\varphi + \psi)_{z_j}$ for any $j \in \{1, 2, ..., m\}$, where g is a holomorphic function on Ω such that $ord_{z_j}(g) = ord_{z_j}(f_1)$ for any $j \in \{1, 2, ..., m\}$ and u is a harmonic function on Ω;

(3) $\prod_{1 \leq j \leq m} \chi_{z_j} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_j)$ respectively;

(4)

$$\lim_{z \to z_k} \frac{f}{gP_* \left(f_u (\prod_{1 \leq j \leq m} f_{z_j})(\sum_{1 \leq j \leq m} p_j \frac{df_{z_j}}{f_{z_j}}) \right)} = c_0$$

for any $k \in \{1, 2..., m\}$, where $c_0 \in \mathbb{C} \backslash \{0\}$ is a constant independent of k.

Remark 2.5 ([31]). *When the four statements in Theorem 2.4 hold,*

$$c_0 g P_* \left(f_u \big(\prod_{1 \le l \le m} f_{z_l} \big) \big(\sum_{1 \le l \le m} p_l \frac{df_{z_l}}{f_{z_l}} \big) \right)$$

is the unique holomorphic $(n,0)$ form F on M such that $(F - f, z_j) \in (\mathcal{O}(K_\Omega))_{z_j} \otimes \mathcal{F}_{z_j}$ for any $j \in \{1, 2, \ldots, m\}$ and

$$G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-2u(z_j)}}{p_j |d_j|^2}$$

for any $t \ge 0$, where f_u is a holomorphic function on Δ such that $|f_u| = P^*(e^u)$, f_{z_j} is a holomorphic function on Δ such that $|f_{z_j}| = P^*(e^{G_\Omega(\cdot, z_j)})$ for any $j \in \{1, 2, \ldots, m\}$, $a_j := \lim_{z \to z_j} \frac{f_1}{w^{k_j}}(z_j)$ and $d_j := \lim_{z \to z_j} \frac{g}{w^{k_j}}(z_j)$, here $k_j := \operatorname{ord}_{z_j}(g) = \operatorname{ord}_{z_j}(f_1)$.

Remark 2.6 ([31]). *For any $\{z_1, z_2, .., z_m\}$, there exists a harmonic function u on Ω such that $\prod_{1 \le j \le m} \chi_{z_j} = \chi_{-u}$. In fact, as Ω is an open Riemann surface, then there exists a holomorphic function $\tilde{f}$ on Ω satisfying that $u := \log |\tilde{f}| - \sum_{1 \le j \le m} G_\Omega(\cdot, z_j)$ is harmonic on Ω, which implies that $\prod_{1 \le j \le m} \chi_{z_j} = \chi_{-u}$.*

We recall a characterization of the holding of equality in optimal jets L^2 extension problem from finite points to open Riemann surfaces.

Theorem 2.7 ([31]). *Let k_j be a nonnegative integer for any $j \in \{1, 2, ..., m\}$. Let ψ be a negative subharmonic function on Ω satisfying that $\frac{1}{2} v(dd^c \psi, z_j) = p_j > 0$ for any $j \in \{1, 2, ..., m\}$. Let φ be a Lebesgue measurable function on Ω such that $\varphi + \psi$ is subharmonic on Ω, $\frac{1}{2} v(dd^c(\varphi + \psi), z_j) = k_j + 1$ and $\alpha_j := (\varphi + \psi - 2(k_j + 1) G_\Omega(\cdot, z_j))(z_j) > -\infty$ for any j. Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying $c(t) e^{-t}$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} c(s) e^{-s} ds < +\infty$. Let a_j be a constant for any j.*

Let f be a holomorphic $(1,0)$ form on V_0 satisfying that $f = a_j w_j^{k_j} dw_j$ on V_{z_j}. Then there exists a holomorphic $(1,0)$ form F on Ω such that $(F - f, z_j) \in (\mathcal{O}(K_\Omega) \otimes \mathcal{I}(2(k_j + 1) G_\Omega(\cdot, z_j)))_{z_j}$ and

$$\int_\Omega |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{1 \le j \le m} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j + 1)}}. \tag{2.5}$$

Moreover, equality $(\int_0^{+\infty} c(s) e^{-s} ds) \sum_{1 \le j \le m} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j + 1)}} = \inf \big\{ \int_\Omega |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F}$ is a holomorphic $(1,0)$ form on Ω such that $(\tilde{F} - f, z_j) \in (\mathcal{O}(K_\Omega) \otimes \mathcal{I}(2(k_j + 1) G_\Omega(\cdot, z_j)))_{z_j}$ for any $j \big\}$ holds if and only if the following statements hold:

(1) $\psi = 2 \sum_{1 \le j \le m} p_j G_\Omega(\cdot, z_j)$;

(2) $\varphi + \psi = 2 \log |g| + 2 \sum_{1 \le j \le m} (k_j + 1) G_\Omega(\cdot, z_j) + 2u$, where g is a holomorphic function on Ω such that $g(z_j) \ne 0$ for any $j \in \{1, 2, ..., m\}$ and u is a harmonic function on Ω;

(3) $\prod_{1 \le j \le m} \chi_{z_j}^{k_j + 1} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_j)$ respectively;

(4)
$$\lim_{z \to z_k} \frac{f}{g P_* \left(f_u (\prod_{1 \le j \le m} f_{z_j}^{k_j+1})(\sum_{1 \le j \le m} p_j \frac{df_{z_j}}{f_{z_j}}) \right)} = c_0$$

for any $k \in \{1, 2 ..., m\}$, where $c_0 \in \mathbb{C} \backslash \{0\}$ is a constant independent of k.

Remark 2.8 ([31])**.** *When the four statements in Theorem 2.7 hold,*

$$c_0 g P_* \left(f_u (\prod_{1 \le j \le m} f_{z_j}^{k_j+1})(\sum_{1 \le j \le m} p_j \frac{df_{z_j}}{f_{z_j}}) \right)$$

is the unique holomorphic $(1,0)$ form F on Ω such that $(F - f, z_j) \in (\mathcal{O}(K_\Omega) \otimes \mathcal{I}(2(k_j + 1)G_\Omega(\cdot, z_j)))_{z_j}$ and

$$\int_\Omega |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{1 \le j \le m} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta (z_j)^{2(k_j+1)}}.$$

Let $Z_0 := \{z_j : j \in \mathbb{Z}_{\ge 1}\} \subset \Omega$ be a discrete set of infinite points. Let w_j be a local coordinate on a neighborhood $V_{z_j} \Subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $j \in \mathbb{Z}_{\ge 1}$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any $j \neq k$. Denote that $V_0 := \cup_{j \in \mathbb{Z}_{\ge 1}} V_{z_j}$. Let f be a holomorphic $(1,0)$ form on V_0, and let $f = f_1 dw_j$ on V_{z_j}, where f_1 is a holomorphic function on V_0. Let ψ be a negative subharmonic function on Ω, and let φ be a Lebesgue measurable function on Ω such that $\varphi + \psi$ is subharmonic on Ω.

The following result gives a necessary condition for $G(h^{-1}(r))$ is linear.

Proposition 2.9 ([31])**.** *Let $c \in \mathcal{P}_{0,\Omega}$. Assume that $G(0) \in (0, +\infty)$ and $(\psi - 2p_j G_\Omega(\cdot, z_j))(z_j) > -\infty$ for $j \in \mathbb{Z}_{\ge 1}$, where $p_j = \frac{1}{2} v(dd^c(\psi), z_j) > 0$. Assume that $G(h^{-1}(r))$ is linear with respect to r. Then the following statements hold:*

(1) $\psi = 2 \sum_{j \in \mathbb{Z}_{\ge 1}} p_j G_\Omega(\cdot, z_j)$;

(2) $\varphi + \psi = 2 \log |g|$ and $\mathcal{F}_{z_j} = \mathcal{I}(\varphi + \psi)_{z_j}$ for any $j \in \mathbb{Z}_{\ge 1}$, where g is a holomorphic function on Ω such that $ord_{z_j}(g) = ord_{z_j}(f_1) + 1$ for any $j \in \mathbb{Z}_{\ge 1}$;

(3) $\frac{p_j}{ord_{z_j} g} \lim_{z \to z_j} \frac{dg}{f} = c_0$ for any $j \in \mathbb{Z}_{\ge 1}$, where $c_0 \in \mathbb{C} \backslash \{0\}$ is a constant independent of j;

(4) $\sum_{j \in \mathbb{Z}_{\ge 1}} p_j < +\infty$.

2.3. Basic properties of the Green functions.

In this subsection, we recall some basic properties of the Green functions. Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω, and let $z_0 \in \Omega$.

Lemma 2.10 (see [46], see also [52])**.** *Let w be a local coordinate on a neighborhood of z_0 satisfying $w(z_0) = 0$. $G_\Omega(z, z_0) = \sup_{v \in \Delta_\Omega^*(z_0)} v(z)$, where $\Delta_\Omega^*(z_0)$ is the set of negative subharmonic function on Ω such that $v - \log |w|$ has a locally finite upper bound near z_0. Moreover, $G_\Omega(\cdot, z_0)$ is harmonic on $\Omega \backslash \{z_0\}$ and $G_\Omega(\cdot, z_0) - \log |w|$ is harmonic near z_0.*

Lemma 2.11 (see [31])**.** *Let $K = \{z_j : j \in \mathbb{Z}_{\ge 1} \& j < \gamma\}$ be a discrete subset of Ω, where $\gamma \in \mathbb{Z}_{>1} \cup \{+\infty\}$. Let ψ be a negative subharmonic function on Ω such that $\frac{1}{2} v(dd^c \psi, z_j) \ge p_j$ for any j, where $p_j > 0$ is a constant. Then $2 \sum_{1 \le j < \gamma} p_j G_\Omega(\cdot, z_j)$ is a subharmonic function on Ω satisfying that $2 \sum_{1 \le j < \gamma} p_j G_\Omega(\cdot, z_j) \ge \psi$ and $2 \sum_{1 \le j < \gamma} p_j G_\Omega(\cdot, z_j)$ is harmonic on $\Omega \backslash K$.*

Lemma 2.12 (see [27]). *For any open neighborhood U of z_0, there exists $t > 0$ such that $\{G_\Omega(z, z_0) < -t\}$ is a relatively compact subset of U.*

Lemma 2.13 (see [31]). *Let Ω be an open Riemann surface which admits a non-trivial Green function G_Ω. Let $Z_0' = \{z_j : j \in \mathbb{Z}_+ \& j < \gamma\}$ be a discrete subset of Ω, where $\gamma \in \mathbb{Z}_+ \cup \{+\infty\}$. Let ψ be a negative plurisubharmonic function on Ω satisfying $\frac{1}{2}v(dd^c\psi, z_0) \geq p_j > 0$ for any j, where p_j is a constant. Assume that $\psi \not\equiv 2\sum_{1 \leq k < \gamma} p_j G_\Omega(\cdot, z_j)$. Let $l(t)$ be a positive Lebesgue measurable function on $(0, +\infty)$ satisfying l is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} l(t)dt < +\infty$. Then there exists a Lebesgue measurable subset V of M such that $l(-\psi(z)) < l(-2\sum_{1 \leq k < \gamma} p_j G_\Omega(z, z_j))$ for any $z \in V$ and $\mu(V) > 0$, where μ is the Lebesgue measure on Ω.*

Lemma 2.14 (see [31]). *There exists a sequence of open Riemann surfaces $\{\Omega_l\}_{l \in \mathbb{Z}^+}$ such that $z_0 \in \Omega_l \Subset \Omega_{l+1} \Subset \Omega$, $\cup_{l \in \mathbb{Z}^+}\Omega_l = \Omega$, Ω_l has a smooth boundary $\partial\Omega_l$ in Ω and $e^{G_{\Omega_l}(\cdot, z_0)}$ can be smoothly extended to a neighborhood of $\overline{\Omega_l}$ for any $l \in \mathbb{Z}^+$, where G_{Ω_l} is the Green function of Ω_l. Moreover, $\{G_{\Omega_l}(\cdot, z_0) - G_\Omega(\cdot, z_0)\}$ is decreasingly convergent to 0 on Ω.*

2.4. **Some other required results.**

Lemma 2.15 ([26]). *Let c be a positive function on $(0, +\infty)$, such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Let $B \in (0, +\infty)$ and $t_0 \geq 0$ be arbitrarily given. Let M be an $n-$dimensional weakly pseudoconvex Kähler manifold. Let $\psi < 0$ be a plurisubharmonic function on M. Let φ be a plurisubharmonic function on M. Let F be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\}$, such that*

$$\int_{K \cap \{\psi < -t_0\}} |F|^2 < +\infty \tag{2.6}$$

for any compact subset K of M, and

$$\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2 e^{-\varphi} \leq C < +\infty. \tag{2.7}$$

Then there exists a holomorphic $(n, 0)$ form $\tilde{F}$ on M, such that

$$\int_M |\tilde{F} - (1 - b_{t_0,B}(\psi))F|^2 e^{-\varphi + v_{t_0,B}(\psi)} c(-v_{t_0,B}(\psi)) \leq C \int_0^{t_0+B} c(t)e^{-t}dt. \tag{2.8}$$

where $b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B}\mathbb{I}_{\{-t_0-B<s<-t_0\}}ds$ and $v_{t_0,B}(t) = \int_{-t_0}^t b_{t_0,B}(s)ds - t_0$.

Follow the assumptions of Lemma 2.1 in the following two lemmas.

Lemma 2.16 ([26]). *The following three statements are equivalent:*
(1). $f \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$;
(2). $G(t) = 0$ for some $t \geq T$;
(3). $G(t) = 0$ for any $t \geq T$.

Lemma 2.17 ([26]). *Assume that $G(t) < +\infty$ for some $t \in [T, +\infty)$. Then there exists a unique holomorphic $(n, 0)$ form F_t on $\{\psi < -t\}$ satisfying*

$$\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) = G(t) \tag{2.9}$$

and $(F_t - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$.

Furthermore, for any holomorphic $(n, 0)$ form $\hat{F}$ on $\{\psi < -t\}$ satisfying

$$\int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi) < +\infty \tag{2.10}$$

and $(\hat{F} - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{F})|_{Z_0})$, we have the following equality,

$$\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) + \int_{\{\psi < -t\}} |\hat{F} - F_t|^2 e^{-\varphi} c(-\psi)$$
$$= \int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi). \tag{2.11}$$

We also need the following lemmas.

Lemma 2.18 (see [19]). *Let N be a submodule of $\mathcal{O}_{\mathbb{C}^n, o}^q$, $q \in \mathbb{Z}_+$. Let $\{f_j\} \subset \mathcal{O}_{\mathbb{C}^n}(U)^q$ be a sequence of $q-$tuples holomorphic functions in an open neighborhood U of the origin o. Assume that $\{f_j\}$ converges uniformly in U towards a $q-$tuples $f \in \mathcal{O}_{\mathbb{C}^n, o}^q$, and assume furthermore that all the germs (f_j, o) belong to N. Then $(f, o) \in N$.*

Lemma 2.19 (see [36]). *Let M be a complex manifold. Let S be an analytic subset of M. Let $\{g_j\}_{j=1,2,\ldots}$ be a sequence of nonnegative Lebesgue measurable functions on M, which satisfies that g_j are almost everywhere convergent to g on M when $j \to +\infty$, where g is a nonnegative Lebesgue measurable function on M. Assume that for any compact subset K of $M \backslash S$, there exist $s_K \in (0, +\infty)$ and $C_K \in (0, +\infty)$ such that*

$$\int_K g_j^{-s_K} dV_M \le C_K \tag{2.12}$$

for any j, where dV_M is a continuous volume form on M.

Let $\{F_j\}_{j=1,2,\ldots}$ be a sequence of holomorphic $(n, 0)$ form on M. Assume that $\liminf_{j \to +\infty} \int_M |F_j|^2 g_j \le C$, where C is a positive constant. Then there exists a subsequence $\{F_{j_l}\}_{l=1,2,\ldots}$, which satisfies that $\{F_{j_l}\}$ is uniformly convergent to a holomorphic $(n, 0)$ form F on M on any compact subset of M when $l \to +\infty$, such that

$$\int_M |F|^2 g \le C. \tag{2.13}$$

Lemma 2.20 ([16]). *Let X be a Stein manifold and $\varphi \in PSH(X)$. Then there exists a sequence $\{\varphi_n\}_{n=1,\ldots}$ of smooth strongly plurisubharmonic functions such that $\varphi_n \downarrow \varphi$.*

Lemma 2.21 (see [31]). *If $c(t)$ is a positive measurable function on $(T, +\infty)$ such that $c(t)e^{-t}$ is decreasing on $(T, +\infty)$ and $\int_{T_1}^{+\infty} c(s)e^{-s} ds < +\infty$ for some $T_1 > T$, then there exists a positive measurable function $\tilde{c}$ on $(T, +\infty)$ satisfying the following statements:*

(1). $\tilde{c} \ge c$ on $(T, +\infty)$;

(2). $\tilde{c}(t)e^{-t}$ is strictly decreasing on $(T, +\infty)$ and $\tilde{c}$ is increasing on $(a, +\infty)$, where $a > T$ is a real number;

(3). $\int_{T_1}^{+\infty} \tilde{c}(s)e^{-s} ds < +\infty$.

Moreover, if $\int_T^{+\infty} c(s)e^{-s} ds < +\infty$ and $c \in \mathcal{P}_T$, we can choose $\tilde{c}$ satisfying the above conditions, $\int_T^{+\infty} \tilde{c}(s)e^{-s} ds < +\infty$ and $\tilde{c} \in \mathcal{P}_T$.

Lemma 2.22. *Let $c(t)$ be a positive measurable function on $(0, +\infty)$, and let $a \in \mathbb{R}$. Assume that $\int_t^{+\infty} c(s)e^{-s}ds \in (0, +\infty)$ when t near $+\infty$. Then we have*

(1) $\lim_{t \to +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = 1$ *if and only if $a = 1$;*

(2) $\lim_{t \to +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = 0$ *if and only if $a > 1$;*

(3) $\lim_{t \to +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = +\infty$ *if and only if $a < 1$.*

Proof. If $a = 1$, it clear that $\lim_{t \to +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = 1$.

If $a > 1$, then $c(s)e^{-as} \le e^{(1-a)s_0}c(s)e^{-s}$ for $s \ge s_0 > 0$, which implies that

$$\limsup_{t \to +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} \le e^{(1-a)s_0}.$$

Let $s_0 \to +\infty$, we have

$$\lim_{t \to +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = 0.$$

If $a < 1$, then $c(s)e^{-as} \ge e^{(1-a)s_0}c(s)e^{-s}$ for $a > s_0 > 0$, which implies that

$$\liminf_{t \to +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} \ge e^{(1-a)s_0}.$$

Let $s_0 \to +\infty$, we have

$$\lim_{t \to +\infty} \frac{\int_t^{+\infty} c(s)e^{-as}ds}{\int_t^{+\infty} c(s)e^{-s}ds} = +\infty.$$

$\square$

3. SOME REQUIRED LEMMAS

Before the proofs of the main results, we need some lemmas, and we present them in the following.

3.1. Some lemmas in the local case.
We need the following lemmas in the local cases.

Let $X = \Delta^m, Y = \Delta^{n-m}$ be the unit polydiscs in $\mathbb{C}^m$ and $\mathbb{C}^{n-m}$. Let $M = X \times Y$. Let π_1, π_2 be the natural projections from M to X and Y. Let ψ_1 be a negative plurisubharmonic function on X such that $\psi_1(o) = -\infty$, where o is the origin of $\mathbb{C}^m$, and let φ_1 be a Lebesgue measurable function on X such that $\varphi_1 + \psi_1$ is plurisubharmonic on X. Let φ_2 be a plurisubharmonic function on Y. Denote that $\psi := \pi_1^*(\psi_1)$, $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$. Let $f(z, w)$ be a holomorphic function on M, where $z = (z_1, \ldots, z_m) \in X$, $w = (w_1, \ldots, w_{n-m}) \in Y$.

Lemma 3.1. *Assume that $f \in \mathcal{I}(\varphi + \psi)_{(o,w)}$ for any $(o, w) \in Y_o$, then $(f(\cdot, w), o) \in \mathcal{I}(\varphi_1 + \psi_1)_o$ on X for any $w \in Y$, and $(f(o, \cdot), w) \in \mathcal{I}(\varphi_2)_w$ on Y for any $w \in Y$.*

Proof. Since $f \in \mathcal{I}(\varphi + \psi)_{(o,w)}$, we can find some $r > 0$ such that

$$\int_{\Delta^n((o,w),r)} |f|^2 e^{-\varphi - \psi} < +\infty,$$

where $\Delta^n((o, w), r)$ is the polydisc centered on (o, w) with radius r. Then according to the Fubini's Theorem, we have

$$\int_{z\Delta^m(o,r)} |f(z,w)|^2 e^{-\varphi_1-\psi_1}$$

$$\leq \frac{1}{(\pi r^2)^{n-m}} \int_{w'\in\Delta^{n-m}(w,r)} \left(\int_{z\in\Delta^m(o,r)} |f(z,w')|^2 e^{-\varphi_1-\psi_1} \right)$$

$$\leq \frac{C}{(\pi r^2)^{n-m}} \int_{w'\in\Delta^{n-m}(w,r)} \left(\int_{z\in\Delta^m(o,r)} |f(z,w')|^2 e^{-\varphi_1-\psi_1} \right) e^{-\varphi_2}$$

$$= \frac{C}{(\pi r^2)^{n-m}} \int_{(z,w')\in\Delta^n((o,w),r)} |f(z,w')|^2 e^{-\varphi-\psi} < +\infty,$$

where C is a positive constant. And

$$\int_{w'\in\Delta^{n-m}(w,r)} |f(o,w')|^2 e^{-\varphi_2}$$

$$\leq \frac{1}{(\pi r^2)^m} \int_{w'\in\Delta^{n-m}(w,r)} \left(\int_{z\in\Delta^m(o,r)} |f(z,w')|^2 \right) e^{-\varphi_2}$$

$$\leq \frac{\sup_{\Delta^m(o,r)} e^{\varphi_1+\psi_1}}{(\pi r^2)^m} \int_{(z,w')\in\Delta^n((o,w),r)} |f(z,w')|^2 e^{-\varphi-\psi} < +\infty.$$

It means that $(f(\cdot, w), o) \in \mathcal{I}(\varphi_1 + \psi_1)_o$ on X, and $(f(o, \cdot), w) \in \mathcal{I}(\varphi_2)_w$ on Y. □

Lemma 3.2. *Assume that*

$$\int_M |f|^2 e^{-\varphi} c(-\psi) < +\infty.$$

Then for any $w \in \Delta^{n-m}$,

$$\int_{z\in\Delta^m} |f(z,w)|^2 e^{-\varphi_1} c(-\psi_1) < +\infty.$$

Proof. According to the Fubini's Theorem, we have

$$\int_{z\in\Delta^m} |f(z,w)|^2 e^{-\varphi_1} c(-\psi_1)$$

$$\leq \frac{1}{(\pi r^2)^{n-m}} \int_{w'\in\Delta^{n-m}(w,r)} \left(\int_{z\in\Delta^m} |f(z,w')|^2 e^{-\varphi_1} c(-\psi_1) \right)$$

$$\leq \frac{e^T}{(\pi r^2)^{n-m}} \int_{w'\in\Delta^{n-m}(w,r)} \left(\int_{z\in\Delta^m} |f(z,w')|^2 e^{-\varphi_1} c(-\psi_1) \right) e^{-\varphi_2}$$

$$\leq \frac{e^T}{(\pi r^2)^{n-m}} \int_{\Delta^n} |f|^2 e^{-\varphi} c(-\psi) < +\infty,$$

where $r > 0$ such that $\Delta^{n-m}(w, r) \subset\subset \Delta^{n-m}$, and $T := -\sup_{w'\in\Delta^{n-m}(w,r)} \varphi_2(w')$. □

Lemma 3.3. *Let $f_1(z)$ be a holomorphic function on X such that $(f_1, o) \in \mathcal{I}(\varphi_1 + \psi_1)_o$, and $f_2(w)$ be a holomorphic function on Y such that $(f_2, w) \in \mathcal{I}(\varphi_2)_w$ for any $w \in Y$. Let $\tilde{f}(z, w) = f_1(z)f_2(w)$ on M, then $(\tilde{f}, (o, w)) \in \mathcal{I}(\varphi + \psi)_{(o,w)}$ for any $(o, w) \in Y_o$.*

Proof. According to $(f_1, o) \in \mathcal{I}(\varphi_1 + \psi_1)_o$ and $(f_2, w) \in \mathcal{I}(\varphi_2)_w$, we can find some $r > 0$ such that

$$\int_{\Delta^m(o,r)} |f_1|^2 e^{-\varphi_1 - \psi_1} < +\infty,$$

and

$$\int_{\Delta^{n-m}(w,r)} |f_2|^2 e^{-\varphi_2} < +\infty.$$

Then using Fubini's Theorem, we get

$$\int_{\Delta^n((o,w),r)} |\tilde{f}|^2 e^{-\varphi - \psi} = \int_{\Delta^m(o,r)} |f_1|^2 e^{-\varphi_1 - \psi_1} \int_{\Delta^{n-m}(w,r)} |f_2|^2 e^{-\varphi_2} < +\infty,$$

which means that $(\tilde{f}, (o, w)) \in \mathcal{I}(\varphi + \psi)_{(o,w)}$. $\qquad\square$

Let $X = \Delta^m$ be the unit polydisc, where the coordinate is $w = (w_1, \ldots, w_m)$. Let Y be an $(n - m)$ dimensional complex manifold, and let $M = X \times Y$.

Lemma 3.4. *For any $(n, 0)$ holomorphic form F on M, there exists a unique sequence of $(n - m, 0)$ holomorphic forms $\{F_\alpha\}_{\alpha \in \mathbb{N}^m}$ on Y such that*

$$F = \sum_{\alpha \in \mathbb{N}^m} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(F_\alpha),$$

where the right side is uniformly convergent on any compact subset of U. Here π_1, π_2 are the natural projections from M to X and Y, $dw = dw_1 \wedge \cdots \wedge dw_m$, and $w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_m}$ for any $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$.

Proof. Firstly we consider the local case. Assume that $Y = \Delta^{n-m}$, and the coordinate is $w' = (w_1', \ldots, w_{n-m}')$. Then there exists a holomorphic function $\tilde{F}(w, w')$ on $M = \Delta^n$ such that

$$F = \tilde{F}(w, w') dw \wedge dw'.$$

Considering the Taylor's expansion of $\tilde{F}$, we can assume that

$$\begin{aligned}
\tilde{F}(w, w') &= \sum_{\alpha \in \mathbb{N}^m, \alpha' \in \mathbb{N}^{n-m}} d_{\alpha, \alpha'} w^\alpha w'^{\alpha'} \\
&= \sum_{\alpha \in \mathbb{N}^m} w^\alpha \left(\sum_{\alpha' \in \mathbb{N}^{n-m}} d_{\alpha, \alpha'} w'^{\alpha'} \right) \\
&= \sum_{\alpha \in \mathbb{N}^m} w^\alpha \tilde{F}_\alpha(w'),
\end{aligned}$$

where

$$\tilde{F}_\alpha(w') = \frac{1}{\alpha!} \cdot \left(\left. \frac{\partial^\alpha \tilde{F}(w, w')}{\partial w^\alpha} \right|_{w=0} \right) (w'),$$

and the summations are uniformly convergent on any compact subset of M. Let $F_\alpha = \tilde{F}_\alpha(w') dw'$ on $Y = \Delta^{n-m}$, then we have

$$F = \sum_{\alpha \in \mathbb{N}^m} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(F_\alpha).$$

Secondly, we need to prove that F_α is independent of the choices of the local coordinates of Y. Assume that $z' = (z_1' \ldots, z_{n-m}')$ is another coordinate on $Y =$

Δ^{n-m}, and $F = \tilde{F}_0(w, z')dw \wedge dz'$. Then we have $\tilde{F}(w, w'(z'))dw' = \tilde{F}_0(w, z')dz'$, which induces that

$$F_\alpha = \frac{1}{\alpha!} \cdot \left(\left. \frac{\partial^\alpha \tilde{F}(w, w')}{\partial w^\alpha} \right|_{w=0} \right) (w')dw'$$

$$= \frac{1}{\alpha!} \cdot \left(\left. \frac{\partial^\alpha \tilde{F}_0(w, z')}{\partial w^\alpha} \right|_{w=0} \right) (z')dz'.$$

It means that F_α is independent of the choices of the coordinates for any $\alpha \in \mathbb{N}^m$.

For general Y, we can find holomorphic $(n - m, 0)$ forms F_α on Y such that $F = \sum_{\alpha \in \mathbb{N}^m} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(F_\alpha)$ with gluing.

Finally, for the uniqueness, in the local case, we have

$$\tilde{F}(w, w')dw \wedge dw' = F = \sum_{\alpha \in \mathbb{N}^m} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(F_\alpha),$$

which implies that

$$F_\alpha(w') = \frac{1}{\alpha!} \cdot \left(\frac{\partial^\alpha \tilde{F}(w, w')}{\partial w^\alpha} |_{w=0} \right) (w')dw'.$$

Then we get the uniqueness in the local case, which can also implies the uniqueness for the general Y. $\qquad \square$

Let $\tilde{M}$ be an $n-$dimensional complex submanifold of $M = \Delta^m \times Y$ such that $\{o\} \times Y \subset \tilde{M}$, where o is the origin of Δ^m.

Lemma 3.5. *For any holomorphic $(n, 0)$ form F on $\tilde{M}$, there exist a unique sequence of holomorphic $(n - m, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{N}}$ on Y and a neighborhood $U \subset \tilde{M}$ of $\{o\} \times Y$ such that*

$$F = \sum_{\alpha \in \mathbb{N}^m} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(F_\alpha)$$

on U, where the right side is uniformly convergent on any compact subset of U.

Proof. For any open subset V of Y with $V \subset\subset Y$, there exists $r_V \in (0, 1)$ such that $\Delta_{r_V}^m \subset \tilde{M}$. It follows from Lemma 3.4 that there exists a unique sequence of holomorphic $(n - m, 0)$ forms $\{F_{\alpha,V}\}_{\alpha \in \mathbb{N}}$ on V such that

$$F = \sum_{\alpha \in \mathbb{N}^m} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(F_{\alpha,V})$$

on $\Delta_{r_V}^m \times V$, where the right side is uniformly convergent on any compact subset of $\Delta_{r_V}^m \times V$. According to the uniqueness of the decomposition in Lemma 3.4, we get that there exists a unique sequence of holomorphic $(n - m, 0)$ forms $\{F_\alpha\}_{\alpha \in \mathbb{N}}$ on Y and a neighborhood $U \subset \tilde{M}$ of $\{o\} \times Y$, such that

$$F = \sum_{\alpha \in \mathbb{N}^m} \pi_1^*(w^\alpha dw) \wedge \pi_2^*(F_\alpha)$$

on U, where the right side is uniformly convergent on any compact subset of U. $\quad \square$

Let X be an n_1-dimensional complex manifold, and let Y be an n_2-dimensional complex manifold. Let $M = X \times Y$ be an $n-$dimensional complex manifold, where $n = n_1 + n_2$. Let π_1 and π_2 be the natural projections from M to X and Y respectively.

Lemma 3.6. *Let $F \not\equiv 0$ be a holomorphic $(n,0)$ form on M. Let f_1 be a holomorphic $(n_1, 0)$ form on an open subset U of X, and let f_2 be a holomorphic $(n_2, 0)$ form on an open subset V of Y. If*

$$F = \pi_1^*(f_1) \wedge \pi_2^*(f_2)$$

on $U \times V$, there exist a holomorphic $(n_1, 0)$ form F_1 on X and a holomorphic $(n_2, 0)$ form F_2 on Y such that $F_1 = f_1$ on U, $F_2 = f_2$ on V, and

$$F = \pi_1^*(F_1) \wedge \pi_2^*(F_2)$$

on M.

Proof. As $F \not\equiv 0$ and $F = \pi_1^*(f_1) \wedge \pi_2^*(f_2)$, we have $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$. Choosing $x \in U$ such that $f_1(x) \neq 0$, there exists a local coordinate $w = (w_1, \ldots, w_{n_1})$ on a neighborhood $U_1 \subset U$ satisfying $w(x) = 0$ and $w(U_1) = \Delta^{n_1}$. Then $f_1 = \sum_{\alpha \in \mathbb{N}^{n_1}} b_\alpha w^\alpha dw_1 \wedge \ldots \wedge dw_{n_1}$, where b_α is a constant for any α and $b_0 \neq 0$, hence

$$F = \sum_{\alpha \in \mathbb{N}^{n_1}} b_\alpha \pi_1^*(w^\alpha dw_1 \wedge \ldots \wedge dw_{n_1}) \wedge \pi_2^*(f_2)$$

on $U_1 \times V$. It follows from Lemma 3.4 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{h_\alpha\}_{\alpha \in \mathbb{N}^{n_1}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{N}^{n_1}} \pi_1^*(w^\alpha dw_1 \wedge \ldots \wedge dw_{n_1}) \wedge \pi_2^*(h_\alpha)$$

on $U_1 \times Y$. It follows from the uniqueness of the decomposition in Lemma 3.4 that $b_\alpha f_2 = h_\alpha$ on V, thus there exists a holomorphic $(n_2, 0)$ form F_2 on Y such that $F_2 = f_2$ on V. With a similar discussion, we know that there exists a holomorphic $(n_1, 0)$ form on X such that $F_1 = f_1$ on X. As $F = \pi_1^*(f_1) \wedge \pi_2^*(f_2)$ on $U \times V$, we have $F = \pi_1^*(F_1) \wedge \pi_2^*(F_2)$ on M. □

Let $\Omega = \Delta$ be the unit disk in $\mathbb{C}$, where the coordinate is w. Let $Y = \Delta^{n-1}$ be the unit polydisc in $\mathbb{C}^n$, where the coordinate is $w' = (w_1', \ldots, w_{n-1}')$. Let $M = \Omega \times Y$. Let π_1, π_2 be the natural projections from M to Ω and Y.

Let $\psi_1 = 2p \log |w| + \psi_0$ on Ω, where $p > 0$ and ψ_0 is a negative subharmonic function on Ω with $\psi_0(0) > -\infty$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1 = 2 \log |g| + 2 \log |w| + 2u$, where g is a holomorphic function on Ω with $ord(g)_0 = k_0 \in \mathbb{N}$, u is a subharmonic function on Ω such that $v(dd^c u, z) \in [0, 1)$ for any $z \in \Omega$. Let φ_2 be a plurisubharmonic function on Y. Let $\psi := \pi_1^*(\psi_1)$ and $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$ on M. Let F be a holomorphic $(n, 0)$ form on M, where

$$F = \sum_{j=k}^{\infty} \pi_1^*(w^j dw) \wedge \pi_2^*(F_j)$$

according to Lemma 3.4. Here $k \in \mathbb{N}$ and F_j is a holomorphic $(n-1, 0)$ form on Y for any $j \geq k$.

Lemma 3.7. *Let c be a positive measurable function on $(0, +\infty)$ such that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, c is increasing near $+\infty$, and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Assume that $k > k_0$, and*

$$\int_M |F|^2 e^{-\varphi} c(-\psi) < +\infty.$$

Then

$$(F, (0, y)) \in (\mathcal{O}(K_M))_{(0,y)} \otimes \mathcal{I}(\varphi + \psi)_{(0,y)}$$

for any $y \in Y$.

Proof. It follows from Ω is a Stein manifold and Lemma 2.20 that there exist smooth subharmonic functions u_l and Ψ_l on Ω such that u_l are decreasingly convergent to u, and Ψ_l are decreasingly convergent to ψ_0.

Then there exists some $t_1 \geq 0$ such that c is increasing on $(t_1, +\infty)$, and for any $t > t_1$,

$$
\begin{aligned}
&\int_{\{\psi < -t\}} |F| e^{-\varphi} c(-\psi) \\
&= \int_{\{\pi_1^*(2p \log |w| + \psi_0) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g| - 2u + (2p-2) \log |w| + \psi_0) - \pi_2^*(\varphi_2)} c(\pi_1^*(-2p \log |w| - \psi_0)) \\
&\geq \int_{\{\pi_1^*(2p \log |w| + \Psi_l) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g| - 2u_l + (2p-2) \log |w| + \psi_0) - \pi_2^*(\varphi_2)} c(\pi_1^*(-2p \log |w| - \Psi_l)).
\end{aligned}
$$

$$(3.1)$$

For any $\varepsilon > 0$, there exists $\rho > 0$ such that

(1).
$$\sup_{|z| < \rho} 2|u_l(z) - u_l(0)| < \varepsilon;$$

(2).
$$\sup_{|z| < \rho} |\Psi_l(z) - \Psi_l(0)| < \varepsilon.$$

Then there exists $t' > 0$ such that for any $t > t'$, we have

$$
\begin{aligned}
&\int_{\{\psi < -t\}} |F| e^{-\varphi} c(-\psi) \\
&\geq \int_{\{\pi_1^*(2p \log |w| + \Psi_l) < -t\}} |F|^2 e^{\pi_1^*(-2 \log |g| - 2u_l + (2p-2) \log |w| + \psi_0) - \pi_2^*(\varphi_2)} c(\pi_1^*(-2p \log |w| - \Psi_l)) \\
&\geq \int_{\{2p \log |w| + \Psi_l(0) < -t-\varepsilon, w' \in Y\}} |w^k \tilde{h}(w, w')|^2 |w|^{2p-2} e^{-2 \log |g(w)| - 2u_l(0) - \varepsilon + \psi_0(w)} e^{-\varphi_2(w')} \\
&\quad \cdot c(-2p \log |w| - \Psi_l(0) - \varepsilon) |dw \wedge dw'|^2 \\
&\geq \int_{w' \in Y} \left(\int_0^{e^{-\frac{t + \Psi_l(0) + \varepsilon}{2p}}} \int_0^{2\pi} 2r^{2k+1} \left| \frac{\tilde{h}(re^{i\theta}, w')}{g(re^{i\theta})} \right|^2 r^{2p-2} c(-2p \log r - \Psi_l(0) - \varepsilon) e^{\psi_0(re^{i\theta})} dr d\theta \right) \\
&\quad \cdot e^{-2u_l(0) - \varepsilon} e^{-\varphi_2(w')} |dw'|^2 \\
&\geq \frac{2\pi}{p|d|^2} e^{-2u_l(0) - \varepsilon + \psi_0(0) - (\frac{k-k_0}{p} + 1)(\Psi_l(0) + \varepsilon)} \int_t^{+\infty} c(s) e^{-(\frac{k-k_0}{p} + 1)s} ds \\
&\quad \cdot \int_{w' \in Y} |\tilde{h}(0, w')|^2 e^{-\varphi_2(w')} |dw'|^2,
\end{aligned}
$$

$$(3.2)$$

where $\tilde{h}(w, w')$ is a holomorphic function on M such that $F = w^k \tilde{h}(w, w') dw \wedge dw'$ on M, and $d := \lim\limits_{w \to 0} \frac{g(w)}{w^{k_0}}$. Since $k > k_0$ and $\int_0^{+\infty} c(s) e^{-s} ds < +\infty$, we have

$$\int_t^{+\infty} c(s) e^{-(\frac{k-k_0}{p}+1)s} ds < +\infty.$$

Then it follows from $\int_{\{\psi < -t\}} |F| e^{-\varphi} c(-\psi) < +\infty$ and inequality (3.2) that

$$\int_Y |F_k|^2 e^{-\varphi_2} = \int_{w' \in Y} |\tilde{h}(0, w')|^2 e^{-\varphi_2(w')} |dw'|^2 < +\infty. \tag{3.3}$$

In addition, we get that $w^k \in \mathcal{I}(\varphi_1 + \psi_1)_0$ according to $k > k_0$. Then it follows from Lemma 3.1 that

$$(\pi_1^*(w^k dw) \wedge \pi_2^*(F_k), (0, y)) \in (\mathcal{O}(K_M))_{(0,y)} \otimes \mathcal{I}(\varphi + \psi)_{(0,y)}. \tag{3.4}$$

According to the Fubini's Theorem and inequality (3.3), there exists $r > 0$ such that

$$\begin{aligned}
&\int_{\Delta(0,r) \times Y} |\pi_1^*(w^k dw) \wedge \pi_2^*(F_k)|^2 e^{-\varphi} c(-\psi) \\
&= \int_{\Delta(0,r)} |w^k|^2 e^{-\varphi_1} c(-\psi_1) |dw|^2 \cdot \int_Y |F_k|^2 e^{-\varphi_2} \\
&\leq C \int_{\Delta(0,r)} |w^k|^2 e^{-\varphi_1 - \psi_1} |dw|^2 \cdot \int_Y |F_k|^2 e^{-\varphi_2} \\
&< +\infty,
\end{aligned} \tag{3.5}$$

where C is a positive constant independent of F. Then we have

$$\int_{\Delta(0,r) \times Y} |F - \pi_1^*(w^k dw) \wedge \pi_2^*(F_k)| e^{-\varphi} c(-\psi) < +\infty,$$

since $\int_{\Delta(0,r) \times Y} |F| e^{-\varphi} c(-\psi) < +\infty$. Note that

$$F - \pi_1^*(w^k dw) \wedge \pi_2^*(F_k) = \sum_{j=k+1}^{\infty} \pi_1^*(w^j dw) \wedge \pi_2^*(F_j)$$

and $k + 1 > k_0$. Using the same methods we can get that

$$(\pi_1^*(w^{k+1} dw) \wedge \pi_2^*(F_{k+1}), (0, y)) \in (\mathcal{O}(K_M))_{(0,y)} \otimes \mathcal{I}(\varphi + \psi)_{(0,y)}$$

for any $y \in Y$ and

$$\int_{\Delta(0,r') \times Y} \left| F - \pi_1^*(w^k dw) \wedge \pi_2^*(F_k) - \pi_1^*(w^{k+1} dw) \wedge \pi_2^*(F_{k+1}) \right|^2 e^{-\varphi} c(-\psi) < +\infty$$

for some $r' \in (0, r)$. Now with inductions, we can get that

$$(\pi_1^*(w^j dw) \wedge \pi_2^*(F_j), (0, y)) \in (\mathcal{O}(K_M))_{(0,y)} \otimes \mathcal{I}(\varphi + \psi)_{(0,y)}$$

for any $j \geq k$, $y \in Y$. Then it follows from Lemma 2.18 that

$$(F, (0, y)) = (\sum_{j=k}^{\infty} \pi_1^*(w^j dw) \wedge \pi_2^*(F_j), (0, y)) \in (\mathcal{O}(K_M))_{(0,y)} \otimes \mathcal{I}(\varphi + \psi)_{(0,y)}$$

for any $y \in Y$. $\qquad\qquad\square$

3.2. **Some other lemmas.** The following lemmas will be used in the proofs of the main results and their applications.

Let $Z_0' := \{z_j : j \in \mathbb{N}_+ \ \& \ j < \gamma\}$ be a discrete subset of the open Riemann surface Ω, where $\gamma \in \mathbb{N}_+ \cup \{+\infty\}$. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y. Let ψ_1 be a subharmonic function on Ω such that $p_j = \frac{1}{2}v(dd^c\psi_1, z_j) > 0$, and let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω. Let φ_2 be a plurisubharmonic function on Y. Denote that $\psi := \pi_1^*(\psi_1)$, $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$. Using the Weierstrass Theorem on open Riemann surfaces (see [15]) and the Siu's Decomposition Theorem, we have

$$\varphi_1 + \psi_1 = 2\log|g_0| + 2u_0,$$

where g_0 is a holomorphic function on Ω and u_0 is a subharmonic function on Ω such that $v(dd^c u_0, z) \in [0, 1)$ for any $z \in \Omega$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \subset\subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0'$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any $j,k, \ j \neq k$. Denote that $V_0 := \bigcup_{1 \leq j < \gamma} V_{z_j}$. Assume that $g_0 = d_j w_j^{k_j} h_j$ on V_{z_j}, where d_j is a constant, k_j is a nonnegative integer, and h_j is a holomorphic function on V_{z_j} such that $h_j(z_j) = 1$ for any j $(1 \leq j < \gamma)$.

Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$. We can get the following lemmas.

Lemma 3.8. *Let F be a holomorphic $(n,0)$ form on $V_0 \times Y$ such that $F = \pi_1^*(w_j^{\tilde{k}_j} f_j dw_j) \wedge \pi_2^*(\tilde{F}_j)$ on $V_{z_j} \times Y$, where $\tilde{k}_j$ is a nonnegative integer, f_j is a holomorphic function on V_{z_j} such that $f_j(z_j) = a_j \in \mathbb{C} \setminus \{0\}$, and $\tilde{F}_j$ is a holomorphic $(n-1,0)$ form on Y for any j $(1 \leq j < \gamma)$.*

Denote that $I_F := \{j : 1 \leq j < \gamma \& \tilde{k}_j + 1 - k_j \leq 0\}$. Assume that $\tilde{k}_j + 1 - k_j = 0$ and $u_0(z_j) > -\infty$ for $j \in I_F$. If

$$\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty \tag{3.6}$$

for any j with $1 \leq j < \gamma$, and

$$\sum_{j \in I_F} \frac{2\pi|a_j|^2 e^{-2u_0(z_j)}}{p_j|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty, \tag{3.7}$$

then there exists a holomorphic $(n,0)$ form $\tilde{F}$ on M, such that $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any j $(1 \leq j < \gamma)$ and $y \in Y$, and

$$\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j \in I_F} \frac{2\pi|a_j|^2 e^{-2u_0(z_j)}}{p_j|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}. \tag{3.8}$$

Proof. Lemma 2.11 tells us that $\psi_1 \leq 2\sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j)$. Since $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, we have

$$e^{-\varphi}c(-\psi) \leq e^{-\pi_1^*\left(\varphi_1 + \psi_1 - 2\sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j)\right) - \pi_2^*(\varphi_2)} c\left(-\pi_1^*(2\sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j))\right).$$

Thus we can assume that $\psi_1 = 2\sum_{1 \leq j < \gamma} p_j G_\Omega(\cdot, z_j)$.

The following remark shows that we only need to prove Lemma 3.8 when Z_0' is a finite set.

Remark 3.9. *It follows from Lemma 2.14 that there exists a sequence of Riemann surfaces $\{\Omega_l\}_{l\in\mathbb{N}_+}$ such that $\Omega_l \subset\subset \Omega_{l+1} \subset\subset \Omega$ for any l, $\bigcup_{l\in\mathbb{N}_+} \Omega_l = \Omega$, and $\{G_{\Omega_l}(\cdot, z_0) - G_\Omega(\cdot, z_0)\}$ is decreasingly convergent to 0 for any $z_0 \in \Omega$. As Z_0' is a discrete subset of Ω, $Z_l := \Omega_l \cap Z_0'$ is a set of finite points. Denote that*

$$\psi_{1,l} := 2 \sum_{z_j \in Z_l} p_j G_{\Omega_l}(\cdot, z_j),$$

and

$$\varphi_{1,l} := \varphi_1 + \psi_1 - \psi_{1,l}.$$

Then we have $\varphi_{1,l} + \psi_{1,l} = \varphi_1 + \psi_1$ on Ω_l. Denote that $\varphi_l' := \pi_1^*(\varphi_{1,l})$, and $\psi_l' := \pi_1^*(\psi_{1,l})$. Denote that $I_l := I_F \cap \{j : z_j \in Z_l\}$. After this remark we will prove that there exists a holomorphic $(n, 0)$ form F_l on M_l such that $(F_l - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi_l' + \psi_l'))_{(z_j,y)} = (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $z_j \in Z_l$, $y \in Y$ and

$$\int_{M_l} |F_l|^2 e^{-\varphi_l'} c(-\psi_l') \le \left(\int_0^{+\infty} c(t)e^{-t}dt \right) \sum_{j\in I_l} \frac{2\pi |a_j|^2 e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}$$

where $M_l := \Omega_l \times Y$.

Since $\psi \le \psi_l'$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, we have

$$
\begin{aligned}
&\int_{M_l} |F_l|^2 e^{-\varphi} c(-\psi) \\
\le &\int_{M_l} |F_l|^2 e^{-\varphi_l'} c(-\psi_l') \\
\le &\left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j\in I_l} 2\pi \frac{|a_j|^2 e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}.
\end{aligned}
\tag{3.9}
$$

Note that ψ is smooth on $M \setminus (Z_0' \times Y)$. For any compact subset K of $M \setminus (Z_0' \times Y)$, there exsits $s_K > 0$ such that $\int_K e^{-s_K \psi} dV_M < +\infty$, where dV_M is a continuous volume form on M. Then we have

$$\int_K \left(\frac{e^\varphi}{c(-\psi)} \right)^{s_K} dV_M = \int_K \left(\frac{e^{\varphi+\psi}}{c(-\psi)} \right)^{s_K} e^{-s_K \psi} dV_M \le C \int_K e^{-s_K \psi} dV_M < +\infty,$$

where C is a constant. It follows from Lemma 2.19 and the diagonal method that there exists a subsequence of $\{F_l\}$, denoted also by $\{F_l\}$, which is uniformly convergent to a holomorphic $(n, 0)$ form $\tilde{F}$ on M on any compact subset of M and

$$
\begin{aligned}
&\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\
\le &\lim_{l\to+\infty} \int_{M_l} |F_l|^2 e^{-\varphi} c(-\psi) \\
\le &\lim_{l\to+\infty} \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j\in I_l} 2\pi \frac{|a_j|^2 e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}. \\
= &\left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j\in I_F} 2\pi \frac{|a_j|^2 e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}.
\end{aligned}
\tag{3.10}
$$

Since $\{F_l\}$ is uniformly convergent to $\tilde{F}$ on any compact subset of M and $(F_l - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any l, $z_j \in Z'_l$, and $y \in Y$. Then it follows from Lemma 2.18 that $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $z_j \in Z'_0$, and $y \in Y$.

According to the above discussions, we assume that $\gamma = m + 1$ and $I_F = \{1, 2, \ldots, m_1\}$, where m is a positive integer and $m_1 < m$ is a nonnegative integer ($I_F = \emptyset$ if and only is $m_1 = 0$). And we can replace Ω_l, ψ'_l, φ'_l by Ω, ψ, φ_1.

Since Ω is a Stein manifold, there exist smooth subharmonic functions u_l on Ω, which are decreasingly convergent to u_0 with respect to l.

Using Lemma 2.12, we know that there exists $t_0 > 0$ such that $\{\psi_1 < -t\} \subset\subset V_0$ for any $t \geq t_0$, which implies that $\int_{\{\psi < -t\}} |F|^2 < +\infty$ according to the Fubini's Theorem and $\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty$ for any j, $1 \leq j < \gamma$. Using Lemma 2.15, there exists a holomorphic $(n, 0)$ form $F_{l,t}$ on M, such that

$$
\int_M |F_{l,t} - (1 - b_{t,1}(\psi))F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l + v_{t,1}(\psi_1)) - \pi_2^*(\varphi_2)} c(-v_{t,1}(\psi))
$$
$$
\leq \left(\int_0^{t+1} c(s)e^{-s} ds \right) \int_M \mathbb{I}_{\{-t-1 < \psi < -t\}} |F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l) - \pi_2^*(\varphi_2)} \tag{3.11}
$$

for any $t \geq t_0$, where according to the Fubini's Theorem we can know that

$$
\int_M \mathbb{I}_{\{-t-1 < \psi < -t\}} |F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l) - \pi_2^*(\varphi_2)} < +\infty.
$$

Note that $b_{t,1}(s) = 0$ for large enough s, then $(F_{l,t} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g_0|) + \pi_2^*(\varphi_2)))_{(z_j,y)} = \mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)_{(z_j,y)}$ for any $j \in \{1, 2, \ldots, m\}$.

For any $y \in Y$, we can choose some neighborhood $U_y \subset\subset Y$ such that $\tilde{F}_j = \tilde{h}_j(w')dw'$ on U_y, where $w' = (w'_1, \ldots, w'_{n-1})$ is a local coordinate on U_y such that $w'_1(y) = \cdots = w'_{n-1}(y) = 0$, and $\tilde{h}_j$ is a holomorphic function on U_y.

For any $\varepsilon > 0$, there exists $t_1 > t_0$ such that

(1). for any $j \in \{1, 2, \ldots, m\}$,

$$
\sup_{z \in \{\psi_1 < -t_1\} \cap V_{z_j}} |g_1(z) - g_1(z_j)| < \varepsilon,
$$

where g_1 is a smooth function on V_0 satisfying that

$$
g_1|_{V_{z_j}} = \psi_1 - 2p_j \log|w_j|;
$$

(2). for any $j \in \{1, 2, \ldots, m\}$,

$$
\sup_{z \in \{\psi_1 < -t_1\} \cap V_{z_j}} 2|u_l(z) - u_l(z_j)| < \varepsilon;
$$

(3). for any $j \in \{1, 2, \ldots, m\}$,

$$
\sup_{z \in \{\psi_1 < -t_1\} \cap V_{z_j}} |f_j(z) - a_j| < \varepsilon.
$$

Note that $\tilde{k}_j + 1 - k_j = 0$ for $1 \leq j \leq m_1$ and $\tilde{k}_j + 1 - k_j > 0$ for $m_1 < j \leq m$. With direct calculating, we have

$$\limsup_{t \to +\infty} \int_{\Omega \times U_y} \mathbb{I}_{\{-t-1 \leq \psi_1 < -t\}} |F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l) - \pi_2^*(\varphi_2)}$$

$$\leq \limsup_{t \to +\infty} \sum_{1 \leq j \leq m} \int_{\{-t-1-\varepsilon < 2p_j \log|w_j| + g_1(z_j) < -t+\varepsilon, w' \in U_y\}} \left| \frac{\tilde{h}_j(w')}{d_j} \right|^2 (|a_j| + \varepsilon)^2$$

$$\cdot |w_j|^{2(\tilde{k}_j - k_j)} e^{-2u_l(z_j) + \varepsilon} e^{-\varphi_2(w')} |dw'|^2$$

$$= \limsup_{t \to +\infty} \sum_{1 \leq j \leq m} \int_{w' \in U_y} \int_{\{-t-1-\varepsilon < 2p_j \log|w_j| + g_1(z_j) < -t+\varepsilon\}} \left| \frac{\tilde{h}_j(w')}{d_j} \right|^2 (|a_j| + \varepsilon)^2$$

$$\cdot |w_j|^{2(\tilde{k}_j - k_j)} e^{-2u_l(z_j) + \varepsilon} e^{-\varphi_2(w')} |dw'|^2$$

$$\leq \limsup_{t \to +\infty} \sum_{1 \leq j \leq m} \int_{w' \in U_y} \left| \frac{\tilde{h}_j(w')}{d_j} \right|^2 (|a_j| + \varepsilon)^2 e^{-2u_l(z_j) + \varepsilon} e^{-\varphi_2(w')} |dw'|^2$$

$$\cdot 4\pi \int_{e^{-\frac{t+1+\varepsilon+g_1(z_j)}{2p_j}}}^{e^{-\frac{t-\varepsilon+g_1(z_j)}{2p_j}}} r^{2(\tilde{k}_j - k_j) + 1} dr$$

$$= \sum_{1 \leq j \leq m} \int_{w' \in U_y} \left| \frac{\tilde{h}_j(w')}{d_j} \right|^2 (|a_j| + \varepsilon)^2 e^{-2u_l(z_j) + \varepsilon} e^{-\varphi_2(w')} |dw'|^2$$

$$\cdot 4\pi \limsup_{t \to +\infty} \int_{e^{-\frac{t+1+\varepsilon+g_1(z_j)}{2p_j}}}^{e^{-\frac{t-\varepsilon+g_1(z_j)}{2p_j}}} r^{2(\tilde{k}_j - k_j) + 1} dr$$

$$= \sum_{1 \leq j \leq m_1} 2\pi \int_{w' \in U_y} \left| \frac{\tilde{h}_j(w')}{d_j} \right|^2 (|a_j| + \varepsilon)^2 e^{-2u_l(z_j) + \varepsilon} e^{-\varphi_2(w')} |dw'|^2 \frac{1 + 2\varepsilon}{p_j}.$$

Letting $\varepsilon \to 0+$, we get that

$$\limsup_{t \to +\infty} \int_{\Omega \times U_y} \mathbb{I}_{\{-t-1 \leq \psi_1 < -t\}} |F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l) - \pi_2^*(\varphi_2)}$$
$$\leq \sum_{1 \leq j \leq m_1} 2\pi \frac{|a_j|^2 e^{-2u_l(z_j)}}{p_j |d_j|^2} \int_{w' \in U_y} |\tilde{h}_j(w')|^2 e^{-\varphi_2(w')} |dw'|^2. \tag{3.12}$$

According to the arbitrariness of y and U_y, it follows from the above calculations in local case that

$$\limsup_{t \to +\infty} \int_M \mathbb{I}_{\{-t-1 \leq \psi_1 < -t\}} |F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l) - \pi_2^*(\varphi_2)}$$
$$\leq \sum_{1 \leq j \leq m_1} 2\pi \frac{|a_j|^2 e^{-2u_l(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty. \tag{3.13}$$

Since $v_{t,1}(\psi) \geq \psi$ and $c(t)e^{-t}$ is decreasing, combining inequality (3.11) with inequality (3.13), we have

$$
\limsup_{t \to +\infty} \int_M |F_{l,t} - (1 - b_{t,1}(\psi))F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l + \psi_1) - \pi_2^*(\varphi_2)} c(-\psi)
$$
$$
\leq \limsup_{t \to +\infty} \int_M |F_{l,t} - (1 - b_{t,1}(\psi))F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l + v_{t,1}(\psi_1)) - \pi_2^*(\varphi_2)} c(-v_{t,1}(\psi))
$$
$$
\leq \limsup_{t \to +\infty} \left(\int_0^{t+1} c(s)e^{-s} ds \right) \int_M \mathbb{I}_{\{-t-1 \leq \psi_1 < -t\}} |F|^2 e^{-\pi_1^*(2\log|g_0| - 2u_l) - \pi_2^*(\varphi_2)}
$$
$$
\leq \left(\int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{1 \leq j \leq m_1} 2\pi \frac{|a_j|^2 e^{-2u_l(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty.
$$
$$(3.14)$$

As $\tilde{k}_j - k_j = -1$ for $1 < j \leq m_1$, $\tilde{k}_j - k_j \geq 0$ for $m_1 < j \leq m$, $\{\psi_1 < -t_0\} \subset\subset V_0$ and $\psi_1 = 2\sum_{1 \leq j < m} p_j G_\Omega(\cdot, z_j)$, we have

$$
\int_{\{\psi < -t'\}} |F|^2 e^{\pi_1^*(-2\log|g_0| + \psi_1) - \pi_2^*(\varphi_2)} c(-\psi)
$$
$$
\leq \sum_{j=1}^m \int_{\{\psi_1 < -t'\} \cap V_{z_j}} |w_j^{\tilde{k}_j} f_j dw_j|^2 e^{-2\log|g_0| + \psi_1} c(-\psi_1) \cdot \int_Y |F_j|^2 e^{-\varphi_2}
$$
$$
\leq C \sum_{j=1}^m \int_{\{2p_j \log|w_j| < -C'\}} |w_j|^{-2} e^{2p_j \log|w_j|} c(-2p_j \log|w_j|) \cdot \int_Y |F_j|^2 e^{-\varphi_2} < +\infty,
$$
$$(3.15)$$

for some $t' > 0$, where C, C' are positive constants. Inequality (3.15) implies that

$$
\limsup_{t \to +\infty} \int_M |(1 - b_{t,1}(\psi))F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l + \psi_1) - \pi_2^*(\varphi_2)} c(-\psi) < +\infty.
$$

Thus we have

$$
\limsup_{t \to +\infty} \int_M |F_{l,t}|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l + \psi_1) - \pi_2^*(\varphi_2)} c(-\psi) < +\infty.
$$

According to Lemma 2.19, we obtain that there exists a subsequence of $\{F_{l,t}\}_{t \to +\infty}$ (also denoted by $\{F_{l,t}\}_{t \to +\infty}$) compactly convergent to a holomorphic $(n, 0)$ form on M denoted by F_l. Then it follows from inequality (3.14) and Fatou's Lemma that

$$
\int_M |F_l|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l + \psi_1) - \pi_2^*(\varphi_2)} c(-\psi)
$$
$$
= \int_M \liminf_{t \to +\infty} |F_l - (1 - b_{t,1}(\psi))F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l + \psi_1) - \pi_2^*(\varphi_2)} c(-\psi)
$$
$$
\leq \liminf_{t \to +\infty} \int_M |F_l - (1 - b_{t,1}(\psi))F|^2 e^{\pi_1^*(-2\log|g_0| - 2u_l + \psi_1) - \pi_2^*(\varphi_2)} c(-\psi)
$$
$$
\leq \left(\int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{1 \leq j \leq m_1} 2\pi \frac{|a_j|^2 e^{-2u_l(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty.
$$
$$(3.16)$$

Since u_l decreasingly converges to u_0, which implies that $\lim_{l\to+\infty} u_l(z_j) = u_0(z_j) > -\infty$ for $1 \le j \le m_1$, then we have

$$\limsup_{l\to+\infty} \int_M |F_l|^2 e^{\pi_1^*(-2\log|g_0|-2u_l+\psi_1)-\pi_2^*(\varphi_2)} c(-\psi)$$
$$\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{1\le j\le m_1} 2\pi \frac{|a_j|^2 e^{-2u_0(z_j)}}{p_j|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty. \tag{3.17}$$

According to Lemma 2.19, we obtain that there exists a subsequence of $\{F_l\}$ (also denoted by $\{F_l\}$) compactly convergent to a holomorphic $(n,0)$ form on M denoted by $\tilde{F}$, which satisfies that $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $j \in \{1, 2, \ldots, m\}$, $y \in Y$, and

$$\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi)$$
$$= \int_M |\tilde{F}|^2 e^{\pi_1^*(-2\log|g_0|-2u_0+\psi_1)-\pi_2^*(\varphi_2)} c(-\psi) \tag{3.18}$$
$$\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{1\le j\le m_1} 2\pi \frac{|a_j|^2 e^{-2u_0(z_j)}}{p_j|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}.$$

Finally combining with Remark 3.9, the proof of Lemma 3.8 is completed. $\square$

Lemma 3.10. *Assume that $(\psi_1 - 2p_j G_\Omega(\cdot, z_j))(z_j) > -\infty$ for any j $(1 \le j < \gamma)$ and c is increasing near $+\infty$. Let F be a holomorphic $(n,0)$ form on M such that $F = \sum_{l=\tilde{k}_j} \pi_1^*(w_j^l dw_j) \wedge F_{j,l}$ on $V_{z_j} \times Y$ for any j $(1 \le j < \gamma)$ according to Lemma 3.4, where $\tilde{k}_j$ is a nonnegative integer, $F_{j,l}$ is a holomorphic $(n-1,0)$ form on $V_{z_j} \times Y$ for any j, l, and $\tilde{F}_j := F_{j,\tilde{k}_j} \not\equiv 0$ on Y. Denote that*

$$I_F := \{j : 1 \le j < \gamma \ \& \ \tilde{k}_j + 1 - k_j \le 0\}.$$

Assume that

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} < +\infty.$$

Then $\tilde{k}_j + 1 - k_j = 0$ for any $j \in I_F$, and

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \ge \sum_{j\in I_F} \frac{2\pi e^{-2u_0(z_j)}}{p_j|d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}. \tag{3.19}$$

Proof. According to the Siu's Decomposition Theorem and Lemma 2.10, we can assume that

$$\psi_1 = 2 \sum_{1\le j<\gamma} p_j G_\Omega(\cdot, z_j) + \psi_0, \tag{3.20}$$

where ψ_0 is a negative subharmonic function on Ω such that $\psi_0(z_j) > -\infty$ for any j, $1 \le j < \gamma$.

It follows from Ω is a Stein manifold and Lemma 2.20 that there exist smooth subharmonic functions u_l and Ψ_l on Ω such that u_l are decreasingly convergent to u, and Ψ_l are decreasingly convergent to ψ_0.

Denote that

$$G := 2 \sum_{1\le j<\gamma} p_j G_\Omega(\cdot, z_j).$$

Then there exists some $t_1 \geq 0$ such that c is increasing on $(t_1, +\infty)$ and for any $t > t_1$,

$$\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)$$

$$= \int_{\{\pi_1^*(G+\psi_0)<-t\}} |F|^2 e^{\pi_1^*(-2\log|g_0|-2u_0+G+\psi_0)-\pi_2^*(\varphi_2)} c(\pi_1^*(-G-\psi_0)) \qquad (3.21)$$

$$\geq \int_{\{\pi_1^*(G+\Psi_l)<-t\}} |F|^2 e^{\pi_1^*(-2\log|g_0|-2u_l+G+\psi_0)-\pi_2^*(\varphi_2)} c(\pi_1^*(-G-\Psi_l)).$$

For any $\varepsilon > 0$, $m \in \mathbb{N}$, there exists $s_0 > 0$ such that
(1). for any $j \in \{1, 2, \ldots, m\}$,

$$U_j := \{|w_j(z)| < s_0 : z \in V_{z_j}\} \subset\subset V_{z_j};$$

(2). for any $j \in \{1, 2, \ldots, m\}$, there exists a holomorphic function $\tilde{g}_j$ on U_j such that $|\tilde{g}_j|^2 = e^{\frac{G}{p_j}}$;
(3). for any $j \in \{1, 2, \ldots, m\}$,

$$\sup_{z \in U_j} 2|u_l(z) - u_l(z_j)| < \varepsilon;$$

(4). for any $j \in \{1, 2, \ldots, m\}$,

$$\sup_{z \in U_j} |H_j(z) - H_j(z_j)| < \varepsilon,$$

where $H_j := G - 2p_j \log|w_j| + \Psi_l + \varepsilon$ is smooth function on U_j.

Note that $G + \Psi_l \leq 2p_j \log|w_j| + H_j(z_j)$ on U_j, then it follows from Lemma 2.12 that there exists $t' > t_1$ such that

$$(\{G + \Psi_l < -t'\} \cap U_j) \subset\subset U_j.$$

Then for $t > t'$, following from inequality (3.21) we get

$$\int_{\{\psi<-t\}} |F| e^{-\varphi} c(-\psi)$$

$$\geq \int_{\{\pi_1^*(G+\Psi_l)<-t\}} |F|^2 e^{\pi_1^*(-2\log|g_0|-2u_l+G+\psi_0)-\pi_2^*(\varphi_2)} c(\pi_1^*(-G-\Psi_l))$$

$$\geq \sum_{1 \leq j \leq m} \int_{(\{G+\Psi_l<-t\}\cap U_j)\times Y} |F|^2 e^{\pi_1^*(-2\log|g_0|-2u_l+G+\psi_0)-\pi_2^*(\varphi_2)} c(\pi_1^*(-G-\Psi_l))$$

$$\geq \sum_{1 \leq j \leq m} \int_{(\{2p_j \log|w_j|+H_j(z_j)<-t\}\cap U_j)\times Y} |F|^2 e^{\pi_1^*(-2\log|g_0|-2u_l+G+\psi_0)-\pi_2^*(\varphi_2)} c(\pi_1^*(-G-\Psi_l))$$

$$\geq \sum_{1 \leq j \leq m} \int_{(\{2p_j \log|w_j|+H_j(z_j)<-t\}\cap U_j)\times Y} |F|^2 |\pi_1^*(\tilde{g}_j)|^{2p_j} e^{\pi_1^*(-2\log|g_0|-2u_l(z_j)-\varepsilon+\psi_0)-\pi_2^*(\varphi_2)}$$

$$\cdot c(\pi_1^*(-2p_j \log|w_j| - H_j(z_j))).$$

$$(3.22)$$

For any $y \in Y$, we can choose some neighborhood $U_y \subset\subset Y$ such that $F = w_j^{\tilde{k}_j} \tilde{h}_j(w_j, w') dw_j \wedge dw'$ on $V_j \times U_y$, where $w' = (w_1', \ldots, w_{n-1}')$ is a local coordinate on U_y such that $w_1'(y) = \cdots = w_{n-1}'(y) = 0$, and $\tilde{h}_j$ is a holomorphic function on $V_j \times U_y$. Note that $ord_{z_j} \tilde{g}_j = 1$ for any j $(1 \leq j \leq m)$, then we can assume that

$\tilde{g}_j = \hat{d}_j w_j \hat{h}_j$ on U_j, where $\hat{d}_j \neq 0$ are constants, $\hat{h}_j$ are holomorphic functions on U_j such that $\hat{h}_j(z_j) = 1$ for $j \in \{1, 2, \ldots, m\}$.

Then it follows from inequality (3.22) that

$$
\int_{\{\psi_1 < -t\} \times U_y} |F|^2 e^{-\varphi} c(-\psi)
$$

$$
\geq \sum_{1 \leq j \leq m} \int_{\{w_j \in \{2p_j \log |w_j| + H_j(z_j) < -t\} \cap U_j, w' \in U_y\}} |\tilde{h}_j(w_j, w')|^2 |\tilde{g}_j(w_j)|^{2p_j} |w_j|^{2\tilde{k}_j}
$$

$$
\cdot e^{-2\log|g_0(w_j)| - 2u_l(z_j) - \varepsilon + \psi_0(w_j)) - \varphi_2(w')} c(-2p_j \log|w_j| - H_j(z_j)) |dw_j \wedge dw'|^2
$$

$$
\geq \sum_{1 \leq j \leq m} \frac{2|\hat{d}_j|^{2p_j} e^{-2u_l(z_j) - \varepsilon}}{|d_j|^2} \int_{w' \in U_y} \left(\int_0^{e^{-\frac{t + H_j(z_j)}{2p_j}}} \int_0^{2\pi} |r|^{2(\tilde{k}_j + p_j - k_j) + 1} \right.
$$

$$
\cdot \left| \frac{\tilde{h}_j(re^{i\theta}, w')}{h_j(re^{i\theta})} \right|^2 |\hat{h}_j(re^{i\theta})|^{2p_j} e^{\psi_0(re^{i\theta})} c(-2p_j \log r - H_j(z_j)) d\theta dr \right) e^{-\varphi_2(w')} |dw'|^2
$$

$$
\geq \sum_{1 \leq j \leq m} \frac{2\pi |\hat{d}_j|^{2p_j}}{p_j |d_j|^2} e^{-2u_l(z_j) - \varepsilon + \psi_0(z_j) - \frac{\tilde{k}_j - k_j + 1 + p_j}{p_j} H_j(z_j)}
$$

$$
\cdot \int_{w' \in U_y} \left(|\tilde{h}_j(z_j, w')|^2 \int_t^{+\infty} c(s) e^{-\left(\frac{\tilde{k}_j + 1 - k_j}{p_j} + 1 \right) s} ds \right) e^{-\varphi_2(w')} |dw'|^2.
$$

(3.23)

where $k_j = \mathrm{ord}_{z_j}(g)$, and $d_j = \lim_{z \to z_j} (g/w_j^{k_j})(z)$.

Denote that

$$
I_m := \{j \in \{1, 2, \ldots, m\} : \tilde{k}_j + 1 - k_j \leq 0\}.
$$

As

$$
\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} < +\infty,
$$

$\int_0^{+\infty} c(s) e^{-s} ds < +\infty$, and $\tilde{h}_j(z_j, w') \not\equiv 0$ on U_y, we have

$$
\tilde{k}_j + 1 - k_j = 0, \quad \forall j \in I_m
$$

(3.24)

according to Lemma 2.22. And we have

$$
\liminf_{t \to +\infty} \frac{\int_{\{\psi_1 < -t\} \times U_y} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds}
$$

$$
\geq \sum_{j \in I_m} \frac{2\pi |\hat{d}_j|^{2p_j}}{p_j |d_j|^2} e^{-2u_l(z_j) - \varepsilon + \psi_0(z_j) - \frac{\tilde{k}_j - k_j + p_j + 1}{p_j} H_j(z_j)}
$$

$$
\cdot \int_{w' \in U_y} \left(|\tilde{h}_j(z_j, w')|^2 \cdot \lim_{t \to +\infty} \frac{\int_t^{+\infty} c(s) e^{-\left(\frac{\tilde{k}_j + 1 - k_j}{p_j} + 1 \right) s} ds}{\int_t^{+\infty} c(s) e^{-s} ds} \right) e^{-\varphi_2(w')} |dw'|^2
$$

$$
= \sum_{j \in I_m} \frac{2\pi |\hat{d}_j|^{2p_j}}{p_j |d_j|^2} e^{-2u_l(z_j) - \varepsilon + \psi_0(z_j) - H_j(z_j)} \int_{w' \in U_y} |\tilde{h}_j(z_j, w')|^2 e^{-\varphi_2(w')} |dw'|^2.
$$

(3.25)

Note that

$$H_j(z_j) = \Psi_l(z_j) + \varepsilon + \lim_{z \to z_j}(G - 2p_j \log|w_j|)$$

$$= \Psi_l(z_j) + \varepsilon + \log\left(\lim_{z \to z_j}\frac{|\tilde{g}|^{2p_j}}{|w_j|^{2p_j}}\right)$$

$$= \Psi_l(z_j) + \varepsilon + 2p_j \log|\hat{d}_j|.$$

Letting $\varepsilon \to 0+$, from inequality (3.25) we get

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi_1 < -t\} \times U_y}|F|^2 e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds} \geq \sum_{j \in I_m}\frac{2\pi}{p_j|d_j|^2}e^{-2u_l(z_j)+\psi_0(z_j)-\Psi_l(z_j)} \tag{3.26}$$
$$\cdot \int_{w' \in U_y}|\tilde{h}_j(z_j,w')|^2 e^{-\varphi_2(w')}|dw'|^2.$$

Letting $l \to +\infty$, from inequality (3.26) we get

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi_1 < -t\} \times U_y}|F|^2 e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds} \geq \sum_{j \in I_m}\frac{2\pi}{p_j|d_j|^2}e^{-2u_0(z_j)} \tag{3.27}$$
$$\cdot \int_{w' \in U_y}|\tilde{h}_j(z_j,w')|^2 e^{-\varphi_2(w')}|dw'|^2.$$

Letting $m \to +\infty$, from inequality (3.27) we get

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi_1 < -t\} \times U_y}|F|^2 e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds} \geq \sum_{j \in I_F}\frac{2\pi}{p_j|d_j|^2}e^{-2u_0(z_j)} \tag{3.28}$$
$$\cdot \int_{w' \in U_y}|\tilde{h}_j(z_j,w')|^2 e^{-\varphi_2(w')}|dw'|^2.$$

And we also have $\tilde{k}_j + 1 - k_j = 0$ for any $j \in I_F$ from equality (3.24). According to the arbitrariness of y and U_y, from the above calculations in the local cases we get

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\}}|F|^2 e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds} \geq \sum_{j \in I_F}\frac{2\pi e^{-2u_0(z_j)}}{p_j|d_j|^2}\int_Y |\tilde{F}_j|^2 e^{-\varphi_2}.$$

Especially, we have $\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty$ and $u_0(z_j) > -\infty$ for any $j \in I_F$. $\qquad\square$

4. Proof of Theorem 1.2 and Remark 1.3

In this section, we give the proof of Theorem 1.2 and Remark 1.3.

4.1. **Proof of the sufficiency in Theorem 1.2 and Remark 1.3.** we give the proof of the sufficiency in Theorem 1.2 and the proof of Remark 1.3.

Proof of the sufficiency in Theorem 1.2 and Remark 1.3. It follows from the statement (2) in Theorem 1.2 that we can assume that $f = \pi_1^*(w^k dw) \wedge \pi_2^*(f_Y)$ on $V_{z_0} \times Y$. Denote that

$$F := c_0 \pi_1^*(gP_*(f_u df_{z_0})) \wedge \pi_2^*(f_Y),$$

where f_u is a holomorphic function on Δ such that $|f_u| = P^*(e^u)$, f_{z_0} is a holomorphic function on Δ such that $|f_{z_0}| = P^*(e^{G_\Omega(\cdot, z_0)})$, and

$$c_0 := \lim_{z \to z_0} \frac{w^k dw}{g P_*(f_u df_{z_0})} \in \mathbb{C} \setminus \{0\}.$$

Then Lemma 3.3 implies that $(F - f, (z_0, y)) \in (\mathcal{O}(K_M))_{(z_0, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_0, y)}$ for any $(z_0, y) \in Z_0$.

For any $y \in Y$, let $w' = (w'_1, \dots, w'_{n-1})$ be a local coordinate on a neighborhood U_y of y satisfying $w'_j(y) = 0$ for any $j \in \{1, \dots, n-1\}$. Assume that $f_Y = h(w')dw'$ on U_y, where h is a holomorphic function on U_y, $dw' = dw'_1 \wedge \cdots \wedge dw'_{n-1}$. Then we have

$$f = w^k h(w') dw \wedge dw' \tag{4.1}$$

on $V_{z_0} \times U_y$, and

$$F = c_0 \pi_1^*(g P_*(f_u df_{z_0})) \wedge h(w') dw' \tag{4.2}$$

on $\Omega \times U_y$.

For any $w' \in U_y$, $t \geq 0$, denote that

$$G_{w'}(t) := \inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\varphi_1} c(-\psi_1) : (\tilde{f} - h(w')w^k dw, z_0) \in (\mathcal{O}(K_\Omega))_{z_0} \otimes \mathcal{I}(\varphi_1 + \psi_1)_{z_0} \right.$$

$$\left. \& \ \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_\Omega)) \right\}.$$

Denote that

$$F_{w'} := c_0 h(w') g P_*(f_u df_{z_0})$$

on Ω. Then it follows from Theorem 2.4 and Remark 2.5 that $G_{w'}(h^{-1}(r))$ is linear and

$$\begin{aligned} G_{w'}(t) &= \int_{\{\psi_1 < -t\}} |F_{w'}|^2 e^{-\varphi_1} c(-\psi_1) \\ &= \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2} |h(w')|^2 \end{aligned} \tag{4.3}$$

for any $t \geq 0$, $w' \in U_y$, where

$$d := \lim_{z \to z_0} \frac{g}{w^k}(z).$$

Then following from the Fubini's Theorem, we have

$$\begin{aligned} \int_{\{\psi_1 < -t\} \times U_y} |F|^2 e^{-\varphi} c(-\psi) &= \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2} \int_{U_y} |h(w')|^2 e^{-\varphi_2(w')} |dw'|^2 \\ &= \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2} \int_{U_y} |f_Y|^2 e^{-\varphi_2}. \end{aligned}$$

According to the arbitrariness of y and U_y, for general Y and any $t \geq 0$ we have

$$\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2} \int_Y |f_Y|^2 e^{-\varphi_2}. \tag{4.4}$$

Assume that $t \geq 0$. According to Lemma 2.17, let $\tilde{F}$ be the holomorphic $(n, 0)$ form on $\{\psi < -t\}$ such that $(\tilde{F} - f, (z_0, y)) \in (\mathcal{O}(K_M))_{(z_0, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_0, y)}$ for

any $(z_0, y) \in Z_0$ and

$$\int_{\{\psi < -t\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) = G(t; c).$$

Following from Lemma 3.4, we assume that

$$\tilde{F} = \sum_{j=\tilde{k}}^{\infty} \pi_1^*(w^j dw) \wedge \pi_2^*(\tilde{F}_j)$$

on $V_{z_0} \times Y$, where $\tilde{k} \in \mathbb{N}$, $\tilde{F}_j$ is a holomorphic $(n-1, 0)$ form on Y for any $j \geq \tilde{k}$, and $\tilde{F}_{\tilde{k}} \not\equiv 0$. Now we prove that $\tilde{k} = k$ and $\tilde{F}_{\tilde{k}} = f_Y$.

For any $y \in Y$, let $w' = (w_1', \dots, w_{n-1}')$ be a local coordinate on a neighborhood U_y of y satisfying $w_j'(y) = 0$ for any $j \in \{1, \dots, n-1\}$. Assume that $f_Y = h(w')dw'$ on U_y, $\tilde{F} = w^{\tilde{k}} \tilde{h}(w, w')dw \wedge dw'$ on $V_{z_0} \times U_y$, where h is a holomorphic function on U_y, $\tilde{h}$ is a holomorphic function on $V_{z_0} \times U_y$. Then we have

$$f = w^k h(w')dw \wedge dw', \ \tilde{F} = w^{\tilde{k}} \tilde{h}(w, w')dw \wedge dw'$$

on $V_{z_0} \times U_y$, and $\tilde{F}_{\tilde{k}} = \tilde{h}(z_j, w')dw'$ on U_y. According to Lemma 3.1, we have $w^{\tilde{k}} \tilde{h}(w, w') - w^k h(w') \in \mathcal{I}(\varphi_1 + \psi_1)_{z_0}$, which implies that $\tilde{k} = k$ and $\tilde{h}(z_0, w') = h(w')$ if $h(w') \neq 0$. Since $\int_Y |f_Y| e^{-\varphi_2} \in (0, +\infty)$ and h is holomorphic, we have $\tilde{k} = k$ and $\tilde{h}(z_0, w') = h(w')$ for any $w' \in U_y$. According to the arbitrariness of y and U_y, we get that $\tilde{k} = k$ and $\tilde{F}_{\tilde{k}} = f_Y$.

Besides, following from Lemma 3.4, we can also assume that

$$\tilde{F} = \sum_{\alpha' \in \mathbb{N}^{n-1}} \pi_1^*(\tilde{F}_{\alpha'}) \wedge \pi_2^*(w'^{\alpha'} dw')$$

on $\{\psi_1 < -t\} \times U_y$, where $\tilde{F}_{\alpha'}$ is a holomorphic $(1, 0)$ form on $\{\psi_1 < -t\}$ for any $\alpha' \in \mathbb{N}^{n-1}$. According to the discussions above, we can assume that

$$\tilde{F}_{\alpha'} = w^k \tilde{f}_{\alpha'} dw$$

on $V_{z_0} \cap \{\psi_1 < -t\}$, where $\tilde{f}_{\alpha'}$ is a holomorphic function on $V_{z_0} \cap \{\psi_1 < -t\}$. And we also have

$$\sum_{\alpha' \in \mathbb{N}^{n-1}} \tilde{f}_{\alpha'}(z_0) w'^{\alpha'} = \tilde{h}(z_0, w') = h(w'),$$

which implies that

$$\sum_{\alpha' \in \mathbb{N}^{n-1}} \tilde{f}_{\alpha'}(z_0) w'^{\alpha'} dw' = f_Y$$

on U_y. Then by the definition of $G_{w'}(t)$, it follows from equality (4.3) that

$$\int_{\{\psi_1 < -t\}} \left| \sum_{\alpha' \in \mathbb{N}^{n-1}} \tilde{F}_{\alpha'} w'^{\alpha'} \right|^2 e^{-\varphi_1} c(-\psi_1) \geq \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2} |h(w')|^2.$$

Then we have

$$\int_{\{\psi_1<-t\}\times U_y}|\tilde{F}|^2e^{-\varphi}c(-\psi)$$

$$=\int_{\{\psi_1<-t\}\times U_y}\left|\sum_{\alpha'\in\mathbb{N}^{n-1}}\pi_1^*(\tilde{F}_{\alpha'})\wedge\pi_2^*(w'^{\alpha'}dw')\right|^2e^{-\varphi}c(-\psi)$$

$$=\int_{w'\in U_y}\left(\int_{\{\psi_1<-t\}}\left|\sum_{\alpha'\in\mathbb{N}^{n-1}}\tilde{F}_{\alpha'}w'^{\alpha'}\right|^2e^{-\varphi_1}c(-\psi_1)\right)e^{-\varphi_2(w')}|dw'|^2$$

$$\geq\int_{w'\in U_y}\left(\int_t^{+\infty}c(s)e^{-s}ds\right)\frac{2\pi e^{-2u(z_0)}}{p|d|^2}|h(w')|^2e^{-\varphi_2(w')}|dw'|^2$$

$$=\int_{\{\psi_1<-t\}\times U_y}|F|^2e^{-\varphi}c(-\psi).$$

According to the arbitrariness of y and U_y, we get that

$$\int_{\{\psi<-t\}}|\tilde{F}|^2e^{-\varphi}c(-\psi)\geq\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi),$$

which implies that $F\equiv\tilde{F}$ on $\{\psi<-t\}$ by the choice of $\tilde{F}$.

Now we get that

$$G(t;c)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)=\left(\int_t^{+\infty}c(s)e^{-s}ds\right)\frac{2\pi e^{-2u(z_0)}}{p|d|^2}\int_Y|f_Y|^2e^{-\varphi_2}$$

$$(4.5)$$

for any $t\geq 0$. Then $G(h^{-1}(r))$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$, and $F=c_0\pi_1^*(gP_*(f_udf_{z_0}))\wedge\pi_2^*(f_Y)$ is the unique holomorphic $(n,0)$ form F on M such that $(F-f,(z_0,y))\in(\mathcal{O}(K_M))_{(z_0,y)}\otimes\mathcal{I}(\varphi+\psi)_{(z_0,y)}$ for any $(z_0,y)\in Z_0$ and

$$G(t)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)=\left(\int_t^{+\infty}c(s)e^{-s}ds\right)\frac{2\pi e^{-2u(z_0)}}{p|d|^2}\int_Y|f_Y|^2e^{-\varphi_2}$$

for any $t\geq 0$. Then the proof of the sufficiency in Theorem 1.2 and the proof of Remark 1.3 are done. $\qquad\square$

4.2. **Proof of the necessity in Theorem 1.2.** we give the proof of the necessity in Theorem 1.2.

Proof of the necessity in Theorem 1.2. Assume that $G(h^{-1}(r))$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$. Then according to Lemma 2.2, there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f)\in H^0(Z_0,(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))|_{Z_0})$, and $G(t;c)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$ for any $t\geq 0$. Then according to Lemma 2.2, Remark 2.3 and Lemma 2.21, we can assume that c is increasing near $+\infty$.

Following from Lemma 3.4, we assume that

$$F=\sum_{j=k}^{\infty}\pi_1^*(w^jdw)\wedge\pi_2^*(F_j)$$

on $V_{z_0}\times Y$, where $k\in\mathbb{N}$, F_j is a holomorphic $(n-1,0)$ form on Y for any $j\geq k$, and $F_k\not\equiv 0$.

Using the Weierstrass Theorem on open Riemann surfaces (see [15]) and the Siu's Decomposition Theorem, we have

$$\varphi_1 + \psi_1 = 2\log|g| + 2\log G_\Omega(\cdot, z_0) + 2u,$$

where g is a holomorphic function on Ω and u is a subharmonic function on Ω such that $v(dd^c u, z) \in [0, 1)$ for any $z \in \Omega$. Denote that $k_0 := ord_{z_0}(g)$.

We prove that $k = k_0$. Firstly, as $G(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 3.10, we can get that $k \geq k_0$ and $\int_Y |F_k|^2 e^{-\varphi_2} < +\infty$. Secondly, if $k > k_0$, then it follows from Lemma 3.7 that $(F, (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_0, y)}$ for any $(z_0, y) \in Z_0$, which contradicts to $G(0) \neq 0$ according to Lemma 2.16. Thus we get that $k = k_0$.

Since $k = k_0$, according to Lemma 3.2, Lemma 3.3 and Lemma 3.7, we have

$$(F - \pi_1^*(w^k dw) \wedge \pi_2^*(F_k), (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_0, y)} \qquad (4.6)$$

for any $(z_0, y) \in Z_0$. Then it follows from $(F - f, (z_0, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_0, y)}$ that $f = \pi_1^*(w^k dw) \wedge \pi_2^*(F_k) + f_0$, where $(f_0, (z_0, y)) \in (\mathcal{O}(K_M))_{(z_0, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_0, y)}$ for any $(z_0, y) \in Z_0$.

For $t \geq 0$, denote that

$$G_\Omega(t) := \inf\left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\varphi_1} c(-\psi_1) : (\tilde{f} - w^k dw, z_0) \in (\mathcal{O}(K_\Omega))_{z_0} \otimes \mathcal{I}(\varphi_1 + \psi_1)_{z_0} \right.$$

$$\left. \& \; \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_\Omega)) \right\}.$$

Then it follows from Lemma 2.17 that there exists a unique holomorphic $(1, 0)$ form $\tilde{f}_t$ on $\{\psi_1 < -t\}$ such that $(\tilde{f}_t - w^k dw, z_0) \in (\mathcal{O}(K_\Omega))_{z_0} \otimes \mathcal{I}(\varphi_1 + \psi_1)_{z_0}$ and

$$G_\Omega(t) = \int_{\{\psi_1 < -t\}} |\tilde{f}_t|^2 e^{-\varphi_1} c(-\psi_1)$$

for any $t \geq 0$. Following from Lemma 3.8 (when Y is a single point), we have

$$\frac{\int_{\{\psi_1 < -t\}} |\tilde{f}_t|^2 e^{-\varphi_1} c(-\psi_1)}{\int_t^{+\infty} c(s) e^{-s} ds} = \frac{G_\Omega(t)}{\int_t^{+\infty} c(s) e^{-s} ds} \leq \frac{2\pi e^{-2u(z_0)}}{p|d|^2}.$$

Denote that $\tilde{F}_t := \pi_1^*(\tilde{f}_t) \wedge \pi_2^*(F_k)$. Then according to the Fubini's theorem we can get that

$$\frac{G(t)}{\int_t^{+\infty} c(s) e^{-s} ds} \leq \frac{\int_{\{\psi < -t\}} |\tilde{F}_t|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \leq \frac{2\pi e^{-2u(z_0)}}{p|d|^2} \int_Y |F_k|^2 e^{-\varphi}, \qquad (4.7)$$

which means that

$$\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2} \int_Y |F_k|^2 e^{-\varphi}. \qquad (4.8)$$

Besides, according to $ord_{z_0}(g) = k$ and Lemma 3.10, we have

$$\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) \geq \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2} \int_Y |F_k|^2 e^{-\varphi} \qquad (4.9)$$

for any $t \geq 0$. Combining inequality (4.7), inequality (4.8) and inequality (4.9), we have

$$
\begin{aligned}
\frac{G(t)}{\int_t^{+\infty} c(s)e^{-s}ds}
&= \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \\
&= \frac{\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \\
&= \frac{2\pi e^{-2u(z_0)}}{p|d|^2} \int_Y |F_k|^2 e^{-\varphi}.
\end{aligned}
\tag{4.10}
$$

According to the uniqueness of F, we have $F \equiv F_t$ on $\{\psi < -t\}$. In addition, we have

$$
\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} c(-\psi) = \int_{\{\psi_1<-t\}} |\tilde{f}_t|^2 e^{-\varphi_1} c(-\psi_1) \cdot \int_Y |F_k|^2 e^{-\varphi_2}
$$

according to the Fubini's Theorem. It means that

$$
G_\Omega(t) = \int_{\{\psi_1<-t\}} |\tilde{f}_t|^2 e^{-\varphi_1} c(-\psi_1) = \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \frac{2\pi e^{-2u(z_0)}}{p|d|^2}
$$

for any $t \geq 0$, i.e. $G_\Omega(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. It follows from Theorem 2.4 that the statements (1), (3) and (4) in Theorem 1.2 hold. Denote that $f_Y := F_k$, then $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$.

Now the proof of the necessity of Theorem 1.2 is done.

$\square$

5. Proofs of Theorem 1.4 and Remark 1.5

In this section, we give the proofs of Theorem 1.4 and Remark 1.5.

5.1. **Proofs of the sufficiency in Theorem 1.4 and Remark 1.5.** we give the proof of the sufficiency in Theorem 1.4 and the proof of Remark 1.5.

Proofs of the sufficiency in Theorem 1.4 and Remark 1.5. We may assume that $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y)$ on $V_{z_j} \times Y$ for any $j \in \{1, 2, \ldots, m\}$ according to the statement (2) in Theorem 1.4. Denote that

$$
F := c_0 \pi_1^* \left(g P_* \left(f_u \left(\prod_{l=1}^m f_{z_l} \right) \left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right) \right) \wedge \pi_2^*(f_Y),
$$

where f_u is a holomorphic function on Δ such that $|f_u| = P^*(e^u)$, f_{z_j} is a holomorphic function on Δ such that $|f_{z_j}| = P^*(e^{G_\Omega(\cdot, z_j)})$ for any $j \in \{1, 2, \ldots, m\}$, and c_0 is the constant in statement (5) of Theorem 1.4.

Then Theorem 2.4, Remark 2.5 and Lemma 3.3 implie that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j,y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$.

For any $y \in Y$, let $w' = (w_1', \ldots, w_{n-1}')$ be a local coordinate on a neighborhood U_y of y satisfying $w_l'(y) = 0$ for any $l \in \{1, \ldots, n-1\}$. Assume that $f_Y = h(w')dw'$

on U_y, where h is a holomorphic function on U_y, $dw' = dw'_1 \wedge \cdots \wedge dw'_{n-1}$. Then we have

$$f = a_j w_j^{k_j} h(w') dw_j \wedge dw' \tag{5.1}$$

on $V_{z_j} \times U_y$, and

$$F = c_0 \pi_1^* \left(g P_* \left(f_u \left(\prod_{l=1}^m f_{z_l} \right) \left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right) \right) \wedge h(w') dw' \tag{5.2}$$

on $\Omega \times U_y$.

For any $w' \in U_y$, $t \geq 0$, denote that

$$G_{w'}(t) := \inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\varphi_1} c(-\psi_1) : (\tilde{f} - a_j h(w') w_j^{k_j} dw, z_j) \in (\mathcal{O}(K_\Omega))_{z_j} \otimes \mathcal{I}(\varphi_1 + \psi_1)_{z_j} \right.$$

$$\left. \text{for any } j \in \{1, 2, \ldots, m\} \& \ \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_\Omega)) \right\}.$$

Denote that

$$F_{w'} := c_0 h(w') g P_* \left(f_u \left(\prod_{l=1}^m f_{z_l} \right) \left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)$$

on Ω. Then it follows from Theorem 2.4 and Remark 2.5 that $G_{w'}(h^{-1}(r))$ is linear and

$$G_{w'}(t) = \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \left(\sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-2u(z_j)}}{p_j |d_j|^2} \right) |h(w')|^2 \tag{5.3}$$

for any $t \geq 0$, $w' \in U_y$, where

$$d_j := \lim_{z \to z_j} \frac{g}{w_j^{k_j}}(z).$$

Then following from the Fubini's Theorem, we have

$$\int_{\{\psi_1 < -t\} \times U_y} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \left(\sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-2u(z_j)}}{p_j |d_j|^2} \right) \int_{U_y} |f_Y|^2 e^{-\varphi_2}.$$

According to the arbitrariness of y and U_y, for general Y and any $t \geq 0$ we have

$$\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \left(\sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-2u(z_j)}}{p_j |d_j|^2} \right) \int_Y |f_Y|^2 e^{-\varphi_2}. \tag{5.4}$$

Assume that $t \geq 0$. According to Lemma 2.17, let $\tilde{F}$ be the holomorphic $(n, 0)$ form on $\{\psi < -t\}$ such that $(\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\{\psi < -t\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) = G(t; c).$$

Following from Lemma 3.4, we assume that

$$\tilde{F} = \sum_{l=\tilde{k}_j}^\infty \pi_1^*(w_j^l dw_j) \wedge \pi_2^*(F_{j,l})$$

on $V_{z_j} \times Y$, where $\tilde{k}_j \in \mathbb{N}$, $\tilde{F}_{j,l}$ is a holomorphic $(n-1, 0)$ form on Y for any $l \geq \tilde{k}_j$, and $\tilde{F}_{j,\tilde{k}_j} \not\equiv 0$. Now we prove that $\tilde{k}_j = k_j$ and $\tilde{F}_{j,\tilde{k}_j} = a_j f_Y$.

For any $y \in Y$, let $w' = (w'_1, \ldots, w'_{n-1})$ be a local coordinate on a neighborhood U_y of y satisfying $w'_j(y) = 0$ for any $j \in \{1, \ldots, n-1\}$. Assume that $f_Y = h(w')dw'$ on U_y, $F = w_j^{\tilde{k}_j} \tilde{h}_j(w_j, w')dw'$ on $V_{z_j} \times U_y$, where h is a holomorphic function on U_y, $\tilde{h}_j$ is a holomorphic function on $V_{z_j} \times U_y$. Then we have

$$f = a_j w_j^{k_j} h(w')dw_j \wedge dw', \ \tilde{F} = w_j^{\tilde{k}_j} \tilde{h}_j(w_j, w')dw_j \wedge dw'$$

on $V_{z_j} \times U_y$, and $\tilde{F}_{j,\tilde{k}_j} = \tilde{h}_j(z_j, w')dw'$ on U_y. According to Lemma 3.1, we have $w_j^{\tilde{k}_j} \tilde{h}_j(w_j, w') - a_j w_j^{k_j} h(w') \in \mathcal{I}(\varphi_1 + \psi_1)_{z_j}$, which implies that $\tilde{k}_j = k_j$ and $\tilde{h}_j(z_j, w') = a_j h(w')$ if $h(w') \neq 0$. Since $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0, +\infty)$ and h is holomorphic, we have $\tilde{k}_j = k_j$ and $\tilde{h}_j(z_j, w') = a_j h(w')$ for any $w' \in U_y$. According to the arbitrariness of y and U_y, we get that $\tilde{k}_j = k_j$ and $\tilde{F}_{j,\tilde{k}_j} = a_j f_Y$.

Besides, following from Lemma 3.4, we can also assume that

$$\tilde{F} = \sum_{\alpha' \in \mathbb{N}^{n-1}} \pi_1^*(\tilde{F}_{\alpha'}) \wedge \pi_2^*(w'^{\alpha'} dw')$$

on $\{\psi_1 < -t\} \times U_y$, where $\tilde{F}_{\alpha'}$ is a holomorphic $(1, 0)$ form on $\{\psi_1 < -t\}$ for any $\alpha' \in \mathbb{N}^{n-1}$. According to the discussions above, we can assume that

$$\tilde{F}_{\alpha'} = w_j^{k_j} \tilde{f}_{j,\alpha'} dw_j$$

on $V_{z_j} \cap \{\psi_1 < -t\}$, where $\tilde{f}_{j,\alpha'}$ is a holomorphic function on $V_{z_j} \cap \{\psi_1 < -t\}$. And we also have

$$\sum_{\alpha' \in \mathbb{N}^{n-1}} \tilde{f}_{j,\alpha'}(z_j)w'^{\alpha'} = \tilde{h}_j(z_j, w') = a_j h(w'),$$

which implies that

$$\sum_{\alpha' \in \mathbb{N}^{n-1}} \tilde{f}_{j,\alpha'}(z_j)w'^{\alpha'} dw' = a_j f_Y$$

on U_y. Then by the definition of $G_{w'}(t)$, it follows from equality (5.3) that

$$\int_{\{\psi_1 < -t\}} \left| \sum_{\alpha' \in \mathbb{N}^{n-1}} \tilde{F}_{\alpha'} w'^{\alpha'} \right|^2 e^{-\varphi_1} c(-\psi_1) \geq \left(\int_t^{+\infty} c(s)e^{-s}ds \right)$$
$$\cdot \left(\sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-2u(z_j)}}{p_j |d_j|^2} \right) |h(w')|^2.$$

Then we have

$$
\int_{\{\psi_1<-t\}\times U_y} |\tilde{F}|^2 e^{-\varphi} c(-\psi)
$$

$$
= \int_{\{\psi_1<-t\}\times U_y} \left| \sum_{\alpha'\in\mathbb{N}^{n-1}} \pi_1^*(\tilde{F}_{\alpha'}) \wedge \pi_2^*(w'^{\alpha'} dw') \right|^2 e^{-\varphi} c(-\psi)
$$

$$
= \int_{w'\in U_y} \left(\int_{\{\psi_1<-t\}} \left| \sum_{\alpha'\in\mathbb{N}^{n-1}} \tilde{F}_{\alpha'} w'^{\alpha'} \right|^2 e^{-\varphi_1} c(-\psi_1) \right) e^{-\varphi_2(w')} |dw'|^2
$$

$$
\geq \int_{w'\in U_y} \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \cdot \left(\sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-2u(z_j)}}{p_j|d_j|^2} \right) |h(w')|^2 e^{-\varphi_2(w')} |dw'|^2
$$

$$
= \int_{\{\psi_1<-t\}\times U_y} |F|^2 e^{-\varphi} c(-\psi).
$$

According to the arbitrariness of y and U_y, we get that

$$
\int_{\{\psi<-t\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) \geq \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi),
$$

which implies that $F \equiv \tilde{F}$ on $\{\psi < -t\}$ by the choice of $\tilde{F}$.

Now we get that

$$
G(t) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \left(\sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-2u(z_j)}}{p_j|d_j|^2} \right) \int_Y |f_Y|^2 e^{-\varphi_2}
$$

$$
\tag{5.5}
$$

for any $t \geq 0$. Then $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$, and

$$
F = c_0 \pi_1^* \left(g P_* \left(f_u \left(\prod_{l=1}^m f_{z_l} \right) \left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right) \right) \wedge \pi_2^*(f_Y)
$$

is the unique holomorphic $(n,0)$ form F on M such that $(F-f, (z_j,y)) \in (\mathcal{O}(K_M))_{(z_j,y)} \otimes \mathcal{I}(\varphi+\psi)_{(z_j,y)}$ for any $(z_j,y) \in Z_0$ and

$$
G(t) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) = \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \left(\sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-2u(z_j)}}{p_j|d_j|^2} \right) \int_Y |f_Y|^2 e^{-\varphi_2}
$$

for any $t \geq 0$. Then the proof of the sufficiency in Theorem 1.4 and the proof of Remark 1.5 are done.

$\square$

5.2. **Proof of the necessity in Theorem 1.4.** we give the proof of the necessity in Theorem 1.4.

Proof of the necessity in Theorem 1.4. Assume that $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2.2, there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi+\psi))|_{Z_0})$, and $G(t;c) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq 0$. Then according to Lemma 2.2, Remark 2.3 and Lemma 2.21, we can assume that c is increasing near $+\infty$.

Following from Lemma 3.4, for any $j \in \{1, 2, \ldots, m\}$, we assume that

$$F = \sum_{l=\tilde{k}_j}^{\infty} \pi_1^*(w_j^l dw_j) \wedge \pi_2^*(F_{j,l})$$

on $V_{z_j} \times Y$, where $\tilde{k}_j \in \mathbb{N}$, $F_{j,l}$ is a holomorphic $(n-1, 0)$ form on Y for any $l \geq \tilde{k}_j$, and $\tilde{F}_j := F_{j,\tilde{k}_j} \not\equiv 0$.

Using the Weierstrass Theorem on open Riemann surfaces (see [15]) and the Siu's Decomposition Theorem, we have

$$\varphi_1 + \psi_1 = 2 \log |g_0| + 2u,$$

where g_0 is a holomorphic function on Ω and u is a subharmonic function on Ω such that $v(dd^c u, z) \in [0, 1)$ for any $z \in \Omega$. Denote that $k_j := ord_{z_j}(g_0)$, and $d_j := \lim_{z \to z_j}(g_0/w_j^{k_j})(z)$.

We prove that $\psi_1 = 2 \sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j)$. As $G(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 3.10, we can get that $\tilde{k}_j - k_j + 1 = 0$ for any $j \in I_F$ and

$$\frac{\int_M |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s} ds} = \liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s} ds} \geq \sum_{j \in I_F} \frac{2\pi e^{-2u(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2},$$
(5.6)

where $I_F := \{j : \tilde{k}_j - k_j + 1 \leq 0\}$. Especially, $\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty$ and $u_0(z_j) > -\infty$ for any $j \in I_F$. Then according to Lemma 3.2, and Lemma 3.7, we have

$$(F - \pi_1^*(w_j^{\tilde{k}_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$$

for any $j \in I_F$, $y \in Y$. And according to Lemma 3.7 we also have

$$(F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$$

for any $j \in \{1, 2, \ldots, m\} \setminus I_F$, $y \in Y$. Denote that $\tilde{\psi}_1 := 2 \sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j)$, $\tilde{\varphi}_1 := \varphi_1 + \psi_1 - \tilde{\psi}_1$, $\tilde{\psi} := \pi_1^*(\tilde{\psi}_1)$, and $\tilde{\varphi} := \pi_1^*(\tilde{\varphi}_1) + \pi_2^*(\varphi_2)$. Then according to Lemma 3.8, there exists a holomorphic $(n, 0)$ form $\tilde{F}$ on M such that $(\tilde{F} - \pi_1^*(w_j^{\tilde{k}_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\tilde{\varphi} + \tilde{\psi}))_{(z_j, y)}$ for any $j \in I_F$, $y \in Y$, $(\tilde{F}, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\tilde{\varphi} + \tilde{\psi}))_{(z_j, y)}$ for any $j \in \{1, 2, \ldots, m\} \setminus I_F$, $y \in Y$, and

$$\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \leq \left(\int_0^{+\infty} c(s)e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi e^{-2u(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2} \quad (5.7)$$

(Here in Lemma 3.8 letting $\tilde{F}_j \equiv 0$ for $j \notin I_F$). Then $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$. Combining inequality (5.6) with inequality (5.7), we have that

$$\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \geq \int_M |F|^2 e^{-\varphi} c(-\psi) \geq \int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}).$$

According to Lemma 2.13, we get that $\psi_1 = 2 \sum_{j=1}^{m} p_j G_\Omega(\cdot, z_j)$.

It follows from Lemma 2.12 that there exists $s_0 > 0$ such that $\{\psi_1 < -s_0\} \subset\subset V_0 = \bigcup_{j=1}^{m} V_{z_j}$, and $U_j := V_{z_j} \cap \{\psi_1 < -s_0\}$ is conformally equivalent to the unit

disc. Denote that

$$G_j(t) := \inf \left\{ \int_{(\{\psi_1 < -t\} \cap U_j) \times Y} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)} \right.$$

$$\left. \& \ \tilde{f} \in H^0((\{\psi_1 < -t\} \cap U_j) \times Y, \mathcal{O}(K_M)) \right\},$$

where $t > s_0$ and $j \in \{1, 2, \ldots, m\}$. Theorem 2.1 shows that $G_j(h^{-1}(r))$ is concave respect to $r \in (0, \int_{s_0}^{+\infty} c(s) e^{-s} ds]$ for any $j \in \{1, 2, \ldots, m\}$, where $h(t) = \int_t^{+\infty} c(s) e^{-s} ds$. Since

$$G(h^{-1}(r)) = \sum_{j=1}^m G_j(h^{-1}(r)) \tag{5.8}$$

and $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_{s_0}^{+\infty} c(s) e^{-s} ds]$, then $G_j(h^{-1}(r))$ is linear with respect to $(0, \int_{s_0}^{+\infty} c(s) e^{-s} ds]$ and

$$G_j(h^{-1}(r)) = \int_{(\{\psi_1 < -t\} \cap U_j) \times Y} |F|^2 e^{-\varphi} c(-\psi) \tag{5.9}$$

for any $j \in \{1, 2, \ldots, m\}$. As $G(0) \in (0, +\infty)$, according to equality (5.8) and Lemma 2.16 there exists $j_0 \in \{1, 2, \ldots, m\}$ such that $G_{j_0}(s_0) \neq 0$. Then it follows from Theorem 1.2, Remark 1.3 and equality (5.9) that there exist a holomorphic $(1, 0)$ form f_0 on U_{j_0} and a holomorphic $(n - 1, 0)$ form F_Y on Y such that $\int_Y |F_Y|^2 e^{-\varphi_2} \in (0, +\infty)$ and

$$F = \pi_1^*(f_0) \wedge \pi_2^*(F_Y)$$

on $U_{j_0} \times Y$. By Lemma 3.6, there exists a holomorphic $(1, 0)$ form F_0 on Ω such that $F_0 = f_0$ on U_{j_0} and

$$F = \pi_1^*(F_0) \wedge \pi_2^*(F_Y) \tag{5.10}$$

on M.

For any $t \geq 0$, denote that

$$G_\Omega(t) := \inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\varphi_1} c(-\psi_1) : (\tilde{f} - F_0, z_j) \in (\mathcal{O}(K_\Omega))_{z_j} \otimes \mathcal{I}(\varphi_1 + \psi_1)_{z_j} \right.$$

$$\left. \text{for any } j \in \{1, 2, \ldots, m\} \& \ \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_\Omega)) \right\}.$$

Then it follows from Lemma 2.17 that there exists a unique holomorphic $(1, 0)$ form $\tilde{f}_t$ on $\{\psi_1 < -t\}$ such that $(\tilde{f}_t - F_0, z_j) \in (\mathcal{O}(K_\Omega))_{z_j} \otimes \mathcal{I}(\varphi_1 + \psi_1)_{z_j}$ for any $j \in \{1, 2, \ldots, m\}$ and

$$G_\Omega(t) = \int_{\{\psi_1 < -t\}} |\tilde{f}_t|^2 e^{-\varphi_1} c(-\psi_1).$$

According to the definition of $G_\Omega(t)$, we have

$$\int_{\{\psi_1 < -t\}} |\tilde{f}_t|^2 e^{-\varphi} c(-\psi) = G_\Omega(t) \leq \int_{\{\psi_1 < -t\}} |F_0|^2 e^{-\varphi} c(-\psi). \tag{5.11}$$

Denote that $\tilde{F}_t := \pi_1^*(\tilde{f}_t) \wedge \pi_2^*(F_Y)$. Lemma 3.3 shows that $(\tilde{F}_t - F, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$ for any $(z_j, y) \in Z_0$, which implies that

$$\int_{\{\psi < -t\}} |\tilde{F}_t|^2 e^{-\varphi} c(-\psi) \geq \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi).$$

However, it follows from equality (5.10), inequality (5.11) and the Fubini's Theorem that

$$
\begin{aligned}
&\int_{\{\psi<-t\}} |\tilde{F}_t|^2 e^{-\varphi} c(-\psi) \\
&= \int_{\{\psi_1<-t\}} |\tilde{f}_t|^2 e^{-\varphi_1} c(-\psi_1) \cdot \int_Y |F_Y|^2 e^{-\varphi_2} \\
&\leq \int_{\{\psi_1<-t\}} |F_0|^2 e^{-\varphi_1} c(-\psi_1) \cdot \int_Y |F_Y|^2 e^{-\varphi_2} \\
&= \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi).
\end{aligned}
$$

Thus we have

$$
\int_{\{\psi<-t\}} |\tilde{F}_t|^2 e^{-\varphi} c(-\psi) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi),
$$

then $F \equiv \tilde{F}_t$ on $\{\psi < -t\}$, which implies that $F_0 \equiv \tilde{f}_t$ on $\{\psi_1 < -t\}$. Then we have

$$
\begin{aligned}
G(t) &= \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) \\
&= \int_{\{\psi_1<-t\}} |F_0|^2 e^{-\varphi_1} c(-\psi_1) \cdot \int_Y |F_Y|^2 e^{-\varphi_2} \\
&= G_\Omega(t) \cdot \int_Y |F_Y|^2 e^{-\varphi_2}
\end{aligned}
$$

for any $t \geq 0$, which implies that $G_\Omega(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. According to Theorem 2.4 and Remark 2.5, we have

(1). $\varphi_1 + \psi_1 = 2\log|g| + 2\sum_{j=1}^m G_\Omega(\cdot, z_j) + 2u$, where g is a holomorphic function on Ω such that $ord_{z_j}(g) = ord_{z_j}(f_1)$ for any $j \in \{1, 2, ..., m\}$ and u is a harmonic function on Ω, here $F_0 = f_1 dw_j$ on V_{z_j};

(2). $\prod_{j=1}^m \chi_{z_j} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot, z_j)$ respectively;

(3).

$$
F_0 = c_0 g P_* \left(f_u (\prod_{j=1}^m f_{z_j})(\sum_{j=1}^m p_j \frac{df_{z_j}}{f_{z_j}}) \right)
$$

where $c_0 \in \mathbb{C}\backslash\{0\}$ is a constant, f_u is a holomorphic function on Δ such that $|f_u| = P^*(e^u)$, and f_{z_j} is a holomorphic function on Δ such that $|f_{z_j}| = P^*(e^{G_\Omega(\cdot, z_j)})$ for any $j \in \{1, 2, \ldots, m\}$;

(4).

$$
G_\Omega(t) = \int_{\{\psi<-t\}} |F_0|^2 e^{-\varphi_1} c(-\psi_1) = \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-2u(z_j)}}{p_j|d_j|^2}
$$

for any $t \geq 0$, where $a_j := \lim_{z \to z_j} \frac{F_0}{w^{k_j} dw_j}(z)$ and $d_j := \lim_{z \to z_j} \frac{g}{w^{k_j}}(z)$, here $k_j := ord_{z_j}(g)$.

Then we have

$$c_0 = \lim_{z \to z_j} \frac{a_j w_j^{k_j} dw_j}{g P_* \left(f_u \left(\prod_{l=1}^m f_{z_l} \right) \left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)},$$

and $a_j \neq 0$. Thus the proofs of statements (1), (3), (4) and (5) in Theorem 1.4 are done. Besides, we have

$$(F_0 - a_j w_j^{k_j} dw_j) \in (\mathcal{O}(K_\Omega) \otimes \mathcal{I}(\varphi_1 + \psi_1))_{z_j}$$

for any $j \in \{1, 2, \ldots, m\}$, which implies that

$$(F - \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_Y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$$

for any $(z_j, y) \in Z_0$ according to $\int_Y |F_Y|^2 e^{-\varphi_2} < +\infty$ and Lemma 3.3. Thus we have $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$ on $V_{z_j} \times Y$ for any $j \in \{1, 2, \ldots, m\}$, where $(f_j, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$ for any $j \in \{1, 2, \ldots, m\}$ and $y \in Y$. Then the proof of statement (2) in Theorem 1.4 is done, and the proof of necessity in Theorem 1.4 is completed. □

6. Proof of Proposition 1.6

In this section, we give the proof of Proposition 1.6.

Proof of Proposition 1.6. Assume that $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2.2, there exists a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f) \in H^0(Z_0, (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))|_{Z_0})$, and $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq 0$. Then according to Lemma 2.2, Remark 2.3 and Lemma 2.21, we can assume that c is increasing near $+\infty$.

Following from Lemma 3.4, for any $j \in \{1, 2, \ldots, m\}$, we assume that

$$F = \sum_{l=\tilde{k}_j}^\infty \pi_1^*(w_j^l dw_j) \wedge \pi_2^*(F_{j,l})$$

on $V_{z_j} \times Y$, where $\tilde{k}_j \in \mathbb{N}$, $F_{j,l}$ is a holomorphic $(n-1, 0)$ form on Y for any $l \geq \tilde{k}_j$, and $\tilde{F}_j := F_{j, \tilde{k}_j} \not\equiv 0$.

Using the Weierstrass Theorem on open Riemann surfaces (see [15]) and the Siu's Decomposition Theorem, we have

$$\varphi_1 + \psi_1 = 2 \log |g_0| + 2u,$$

where g_0 is a holomorphic function on Ω and u is a subharmonic function on Ω such that $v(dd^c u, z) \in [0, 1)$ for any $z \in \Omega$. Denote that $k_j := ord_{z_j}(g_0)$, and $d_j := \lim_{z \to z_j} (g_0/w_j^{k_j})(z)$.

We prove that $\psi_1 = 2 \sum_{j=1}^\infty p_j G_\Omega(\cdot, z_j)$. As $G(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 3.10, we can get that $\tilde{k}_j - k_j + 1 = 0$ for any $j \in I_F$ and

$$\frac{\int_M |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} = \liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{j \in I_F} \frac{2\pi e^{-2u(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2},$$

(6.1)

where $I_F := \{j \in \mathbb{N}_+ : \tilde{k}_j - k_j + 1 \le 0\}$. Especially, $\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty$ and $u_0(z_j) > -\infty$ for any $j \in I_F$. Then according to Lemma 3.2, and Lemma 3.7, we have

$$(F - \pi_1^*(w_j^{\tilde{k}_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$$

for any $j \in I_F$, $y \in Y$. And according to Lemma 3.7 we also have

$$(F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$$

for any $j \in \mathbb{N}_+ \setminus I_F$, $y \in Y$. Denote that $\tilde{\psi}_1 := 2\sum_{j=1}^\infty p_j G_\Omega(\cdot, z_j)$, $\tilde{\varphi}_1 := \varphi_1 + \psi_1 - \tilde{\psi}_1$, $\tilde{\psi} := \pi_1^*(\tilde{\psi}_1)$, and $\tilde{\varphi} := \pi_1^*(\tilde{\varphi}_1) + \pi_2^*(\varphi_2)$. Then according to Lemma 3.8, there exists a holomorphic $(n, 0)$ form $\tilde{F}$ on M such that $(\tilde{F} - \pi_1^*(w_j^{\tilde{k}_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\tilde{\varphi} + \tilde{\psi}))_{(z_j, y)}$ for any $j \in I_F$, $y \in Y$, $(\tilde{F}, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\tilde{\varphi} + \tilde{\psi}))_{(z_j, y)}$ for any $j \in \mathbb{N}_+ \setminus I_F$, $y \in Y$, and

$$\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \le \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2} \qquad (6.2)$$

(Here in Lemma 3.8 letting $\tilde{F}_j \equiv 0$ for $j \notin I_F$). Then $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\tilde{\varphi} + \tilde{\psi}))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$. Combining inequality (6.1) with inequality (6.2), we have that

$$\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \ge \int_M |F|^2 e^{-\varphi} c(-\psi) \ge \int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}).$$

According to Lemma 2.13, we get that $\psi_1 = 2\sum_{j=1}^\infty p_j G_\Omega(\cdot, z_j)$.

For $p \in Z_0^1$ (without loss of generality, we can assume $p = z_1$), there exists $s_1 > 0$ such that $\{\psi_1 < -s_1\} \cap \partial U_1 = \emptyset$, where $U_1 := \{|w_1(z)| < r_1 : z \in V_{z_1}\} \subset\subset V_{z_1}$ is a neighborhood of z_1 in Ω. For $l \in \{1, 2\}$, denote that

$$G_l(t) := \inf \left\{ \int_{(\{\psi_1 < -t\} \cap D_l) \times Y} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : (\tilde{f} - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)} \right.$$

$$\left. \text{for any } z_j \in D_l \ \& \ \tilde{f} \in H^0((\{\psi_1 < -t\} \cap D_l) \times Y, \mathcal{O}(K_M)) \right\},$$

where $t \ge s_1$, $D_1 := \{\psi_1 < -s_1\} \cap U_1$, and $D_2 := \{\psi_1 < -s_1\} \setminus \overline{U_1}$. Theorem 2.1 shows that $G_1(h^{-1}(r))$ and $G_2(h^{-1}(r))$ are concave with respect to $r \in (0, \int_{s_1}^{+\infty} c(s) e^{-s} ds]$. As $G(t) = G_1(t) + G_2(t)$ for $t \ge s_1$ and $G(h^{-1}(r))$ is linear with respect to r, we have that $G_1(h^{-1}(r))$ and $G_2(h^{-1}(r))$ are linear with respect to $r \in (0, \int_{s_1}^{+\infty} c(s) e^{-s} ds]$, and

$$G_l(h^{-1}(r)) = \int_{(\{\psi_1 < -t\} \cap D_l) \times Y} |F|^2 e^{-\varphi} c(-\psi) \qquad (6.3)$$

for $l \in \{1, 2\}$. If $G_1(t) = 0$ for some $t \ge s_1$, we have $F \equiv 0$, which contradicts to $G(0) \ne 0$. Thus $G_1(t) \ne 0$ for any $t \ge s_1$. Note that $\{j \in \mathbb{N}_+ : z_j \in D_1\} = \{z_1\}$, then it follows from Theorem 1.2 that there exist a holomorphic $(1, 0)$ form f_0 on D_1 and a holomorphic $(n-1, 0)$ form F_Y on Y such that $\int_Y |F_Y|^2 e^{-\varphi_2} \in (0, +\infty)$ and

$$F = \pi_1^*(f_0) \wedge \pi_2^*(F_Y)$$

on $D_1 \times Y$. By Lemma 3.6, there exists a holomorphic $(1,0)$ form F_0 on Ω such that $F_0 = f_0$ on D_1 and

$$F = \pi_1^*(F_0) \wedge \pi_2^*(F_Y) \tag{6.4}$$

on M.

For any $t \geq 0$, denote that

$$G_\Omega(t) := \inf \left\{ \int_{\{\psi_1 < -t\}} |\tilde{f}|^2 e^{-\varphi_1} c(-\psi_1) : (\tilde{f} - F_0, z_j) \in (\mathcal{O}(K_\Omega))_{z_j} \otimes \mathcal{I}(\varphi_1 + \psi_1)_{z_j} \right.$$

$$\left. \text{for any } j \in \mathbb{N}_+ \& \ \tilde{f} \in H^0(\{\psi_1 < -t\}, \mathcal{O}(K_\Omega)) \right\}.$$

Then it follows from Lemma 2.17 that there exists a unique holomorphic $(1,0)$ form $\tilde{f}_t$ on $\{\psi_1 < -t\}$ such that $(\tilde{f}_t - F_0, z_j) \in (\mathcal{O}(K_\Omega))_{z_j} \otimes \mathcal{I}(\varphi_1 + \psi_1)_{z_j}$ for any $j \in \mathbb{N}_+$ and

$$G_\Omega(t) = \int_{\{\psi_1 < -t\}} |\tilde{f}_t|^2 e^{-\varphi_1} c(-\psi_1).$$

According to the definition of $G_\Omega(t)$, we have

$$\int_{\{\psi_1 < -t\}} |\tilde{f}_t|^2 e^{-\varphi} c(-\psi) = G_\Omega(t) \leq \int_{\{\psi_1 < -t\}} |F_0|^2 e^{-\varphi} c(-\psi). \tag{6.5}$$

Denote that $\tilde{F}_t := \pi_1^*(\tilde{f}_t) \wedge \pi_2^*(F_Y)$. Lemma 3.3 shows that $(\tilde{F}_t - F, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j,y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$, which implies that

$$\int_{\{\psi < -t\}} |\tilde{F}_t|^2 e^{-\varphi} c(-\psi) \geq \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi).$$

However, it follows from equality (6.4), inequality (6.5) and the Fubini's Theorem that

$$\int_{\{\psi < -t\}} |\tilde{F}_t|^2 e^{-\varphi} c(-\psi)$$

$$= \int_{\{\psi_1 < -t\}} |\tilde{f}_t|^2 e^{-\varphi_1} c(-\psi_1) \cdot \int_Y |F_Y|^2 e^{-\varphi_2}$$

$$\leq \int_{\{\psi_1 < -t\}} |F_0|^2 e^{-\varphi_1} c(-\psi_1) \cdot \int_Y |F_Y|^2 e^{-\varphi_2}$$

$$= \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi).$$

Thus we have

$$\int_{\{\psi < -t\}} |\tilde{F}_t|^2 e^{-\varphi} c(-\psi) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi),$$

then $F \equiv \tilde{F}_t$ on $\{\psi < -t\}$, which implies that $F_0 \equiv \tilde{f}_t$ on $\{\psi_1 < -t\}$. Then we have

$$G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$$

$$= \int_{\{\psi_1 < -t\}} |F_0|^2 e^{-\varphi_1} c(-\psi_1) \cdot \int_Y |F_Y|^2 e^{-\varphi_2}$$

$$= G_\Omega(t) \cdot \int_Y |F_Y|^2 e^{-\varphi_2}$$

for any $t \geq 0$, which implies that $G_\Omega(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. According to Proposition 2.9, we have

(1). $\varphi_1 + \psi_1 = 2\log|g|$, where g is a holomorphic function on Ω such that $ord_{z_j}(g) = ord_{z_j}(f_1) + 1$ for any $j \in \mathbb{N}_+$, here $F_0 = f_1 dw_j$ on V_{z_j};

(2). for any $j \in \mathbb{N}_+$,

$$\frac{p_j}{ord_{z_j} g} \lim_{z \to z_j} \frac{dg}{F_0} = c_0, \tag{6.6}$$

where $c_0 \in \mathbb{C} \setminus \{0\}$ is a constant independent of j;

(3). $\sum_{j \in \mathbb{N}_+} p_j < +\infty$.

Denote that $a_j := \lim_{z \to z_j} \frac{F_0}{w_j^{k_j} dw_j}(z)$, where $k_j := ord_{z_j}(f_1)$. Then we have that

$$(F_0 - a_j w_j^{k_j} dw_j) \in (\mathcal{O}(K_\Omega) \otimes \mathcal{I}(\varphi_1 + \psi_1))_{z_j}$$

and

$$\frac{p_j}{ord_{z_j} g} \lim_{z \to z_j} \frac{dg}{a_j w_j^{k_j} dw_j} = c_0, \tag{6.7}$$

for any $j \in \mathbb{N}_+$. Then we obtain that

$$(F - \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_Y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$$

for any $(z_j, y) \in Z_0$ according to $\int_Y |F_Y|^2 e^{-\varphi_2} < +\infty$ and Lemma 3.3. Thus we have $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(f_Y) + f_j$ on $V_{z_j} \times Y$ for any $j \in \mathbb{N}_+$, where $(f_j, (z_j, y)) \in (\mathcal{O}(K_M))_{(z_j, y)} \otimes \mathcal{I}(\varphi + \psi)_{(z_j, y)}$ for any $j \in \mathbb{N}_+$ and $y \in Y$. Then the proof of Proposition 1.6 is done.

$\square$

7. PROOF OF PROPOSITION 1.7

In this section, we prove Proposition 1.7.

Proof of Proposition 1.7. Assume that $\tilde{G}(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2.2, there exists a unique holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $(F - f) \in H^0(Z_0, (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))|_{Z_0})$, and $\tilde{G}(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq 0$. Then according to Lemma 2.2, Remark 2.3 and Lemma 2.21, we can assume that c is increasing near $+\infty$.

Using the Weierstrass Theorem on open Riemann surfaces (see [15]) and the Siu's Decomposition Theorem, we have

$$\varphi_1 + \psi_1 = 2\log|g_0| + 2u_0,$$

where g_0 is a holomorphic function on Ω and u_0 is a subharmonic function on Ω such that $v(dd^c u_0, z) \in [0, 1)$ for any $z \in \Omega$.

Assume that $F = \sum_{l=\tilde{k}_j}^{\infty} \pi_1^*(w_j^l dw_j) \wedge \pi_2^*(F_{j,l})$ on U according to Lemma 3.5, where $U \subset \tilde{M}$ is a neighborhood of Z_0, $\tilde{k}_j$ is a nonnegative integer, $F_{j,l}$ is a holomorphic $(n-1, 0)$ on Y for any j, l, and $\tilde{F}_j := F_{j, \tilde{k}_j} \not\equiv 0$. Assume that $g_0 = d_j w_j^{k_j} h_j$ on V_{z_j}, where d_j is a constant, k_j is a nonnegative integer, and h_j is a holomorphic function on V_{z_j} such that $h_j(z_j) = 1$ for any j $(1 \leq j < \gamma)$.

Let W be an open subset of Y such that $W \subset\subset Y$. Then for any j, $1 \leq j < \gamma$, there exists $r_{j,W} > 0$ such that $U_{j,W} \times W \subset \tilde{M}$, where $U_{j,W} := \{z \in \Omega : |w_j(z)| < r_{j,W}\}$.

According to Lemma 3.10, we have that $\tilde{k}_j + 1 - k_j = 0$ for any $j \in I_F$ and

$$
\begin{aligned}
&\frac{\tilde{G}(0)}{\int_0^{+\infty} c(s)e^{-s}ds} \\
&= \lim_{t \to +\infty} \frac{\tilde{G}(t)}{\int_t^{+\infty} c(s)e^{-s}ds} \\
&\geq \sum_{j \in I_F} \liminf_{t \to +\infty} \frac{\int_{(\{\psi_1 < -t\} \cap U_{j,W}) \times W} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \\
&\geq \sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_W |\tilde{F}_j|^2 e^{-\varphi_2},
\end{aligned}
\tag{7.1}
$$

where $I_F := \{j : 1 \leq j < \gamma \ \& \ \tilde{k}_j + 1 - k_j \leq 0\}$. Besides, we can know that $u_0(z_j) > -\infty$. According to the arbitrariness of W, we have $\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty$ for any $j \in I_F$, and

$$
\frac{\tilde{G}(0)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}.
\tag{7.2}
$$

Then according to Lemma 3.2, Lemma 3.3, Lemma 3.7 and $\tilde{k}_j + 1 - k_j = 0$, $\int_Y |F_j|^2 e^{-\varphi_2} < +\infty$ for $j \in I_F$, we know that $(F - \pi_1^*(w_j^{\tilde{k}_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $j \in I_F$ and $y \in Y$. According to Lemma 3.7, we also have $(F, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $j \notin I_F$ and $y \in Y$.

By Lemma 3.8, there exists a holomorphic $(n, 0)$ form $\tilde{F}$ on M, such that $(\tilde{F} - \pi_1^*(w_j^{\tilde{k}_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $j \in I_F$, $y \in Y$, $(\tilde{F}, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $j \notin I_F$, $y \in Y$ and

$$
\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}.
$$

Then $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any j $(1 \leq j < \gamma)$ and $y \in Y$.

Now According to the choice of F, we have that

$$
\begin{aligned}
\tilde{G}(0) &= \int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\
&\leq \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \\
&\leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j \in I_F} \frac{2\pi e^{-2u_0(z_j)}}{p_j |d_j|^2} \int_Y |\tilde{F}_j|^2 e^{-\varphi_2}.
\end{aligned}
\tag{7.3}
$$

Combining inequality (7.2) with inequality (7.3), we get that

$$
\int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) = \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi).
$$

As $\tilde{F} \not\equiv 0$, the equality above implies that $\tilde{M} = M$. $\qquad\square$

8. Proofs of Theorem 1.8 and Remark 1.9

In this section, we give the proofs of Theorem 1.8 and Remark 1.9.

Proof of Theorem 1.8. Using the Weierstrass Theorem on open Riemann surfaces (see [15]) and the Siu's Decomposition Theorem, we have

$$\varphi_1 + \psi_1 = 2\log|g_0| + 2u_0,$$

where g_0 is a holomorphic function on Ω and u_0 is a subharmonic function on Ω such that $v(dd^c u_0, z) \in [0, 1)$ for any $z \in \Omega$. Note that $ord_{z_j} g_0 = k_j + 1$ and

$$e^{2u_0(z_j)} \lim_{z \to z_j} \left| \frac{g_0}{w_j^{k_j+1}}(z) \right|^2 = e^{\alpha_j} c_\beta(z_j)^{2(k_j+1)}.$$

Using Lemma 3.8, we obtain that there exists a holomorphic $(n, 0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

Thus $(F - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

In the following, we prove the characterization of the holding of the equality $\left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} = \inf\left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a}\right.$ holomorphic $(n, 0)$ form on $\tilde{M}$ such that $(\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $\left. (z_j, y) \in Z_0 \right\}$.

According to the above discussions (replacing ψ_1 by $\psi_1 + t$, replacing $c(\cdot)$ by $c(\cdot + t)$ and replacing Ω by $\{\psi_1 < -t\}$), for any $t \geq 0$, there exists a holomorphic $(n, 0)$ form F_t on $\{\psi < -t\}$ such that $(F_t - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) \leq \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \tag{8.1}$$

Firstly, we prove the necessity. According to the above discussions, there exists a holomorphic $(n, 0)$ form $\tilde{F}_1 \neq 0$ on M such that $(\tilde{F}_1 - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |\tilde{F}_1|^2 e^{-\pi_1^*(\varphi_1 + \psi_1 - 2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j)) - \pi_2^*(\varphi_2)} c\left(\pi_1^*(-2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j)) \right)$$
$$\leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \tag{8.2}$$

As $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, $\psi_1 \leq 2\sum_{j=1}^m p_j G_\Omega(\cdot, z_j)$ and $\left(\int_0^{+\infty} c(s)e^{-s}ds \right)$ $\cdot \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} = \inf\left\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \right.$

form on $\tilde{M}$ such that $(\tilde{F}-f,(z_j,y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\varphi+\psi))_{(z_j,y)}$ for any $(z_j,y) \in Z_0\}$, we have

$$\int_{\tilde{M}} |\tilde{F}_1|^2 e^{-\varphi} c(-\psi)$$

$$= \int_M |\tilde{F}_1|^2 e^{-\pi_1^*(\varphi_1+\psi_1-2\sum_{j=1}^m p_j G_\Omega(\cdot,z_j))-\pi_2^*(\varphi_2)} c\left(\pi_1^*(-2\sum_{j=1}^m p_j G_\Omega(\cdot,z_j))\right).$$

As $\frac{1}{2}v(dd^c\psi_1,z_j) = p_j > 0$, $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$, we have $\tilde{M} = M$, and $\psi_1 = 2\sum_{j=1}^m p_j G_\Omega(\cdot,z_j)$ according to Lemma 2.13.

As $e^{-\varphi_1}c(-\psi_1) = e^{-\varphi_1-\psi_1}e^{\psi_1}c(-\psi_1)$, $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, and φ_2 is a plurisubharmonic function on Y, $e^{-\varphi}c(-\psi)$ has locally positive lower bound on $M \setminus Z_0$. Thus $G(h^{-1}(r))$ is concave with respect to r. According to the definition of $G(t)$ and inequality (8.1), we have

$$\frac{G(t)}{\int_t^{+\infty} c(s)e^{-s}ds} \le \sum_{j=1}^m \frac{2\pi|a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} = \frac{G(0)}{\int_0^{+\infty} c(s)e^{-s}ds} \tag{8.3}$$

for any $t \ge 0$. Then $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. From Theorem 1.4, we get that $f = \pi_1^*(a_j' w_j^{k_j'} dw_j) \wedge \pi_2^*(F_Y) + f_j$ on $V_{z_j} \times Y$, $\varphi_1 + \psi_1 = 2\log|g_1| + 2\sum_{j=1}^m G_\Omega(\cdot,z_j) + 2u_1$, $\prod_{j=1}^m \chi_{z_j} = \chi_{-u_1}$, and

$$\lim_{z \to z_j} \frac{a_j' w_j^{k_j'} dw_j}{g_1 P_*\left(f_{u_1}\left(\prod_{l=1}^m f_{z_l}\right)\left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}}\right)\right)} = c_0$$

for any $j \in \{1,2,\ldots,m\}$, where $a_j' \in \mathbb{C} \setminus \{0\}$ is a constant, k_j' is a nonnegative integer, F_Y is a holomorphic $(n-1,0)$ form on Y such that $\int_Y |F_Y|^2 e^{-\varphi_2} \in (0,+\infty)$, $(f_j,(z_j,y)) \in (\mathcal{O}(K_M))_{(z_j,y)} \otimes \mathcal{I}(\varphi+\psi)_{(z_j,y)}$ for any $(z_j,y) \in Z_0$, g_1 is a holomorphic function on Ω such that $ord_{z_j}(g_1) = k_j'$, u_1 is a harmonic function on Ω and $c_0 \in \mathbb{C} \setminus \{0\}$ is a constant independent of j. Since $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_j)$ on $V_{z_j} \times Y$ for any j, we have $k_j = k_j'$, and $c_j F_j = c_0 F_Y$ for any j, where

$$c_j := \lim_{z \to z_j} \frac{a_j w_j^{k_j} dw_j}{g_1 P_*\left(f_{u_1}\left(\prod_{l=1}^m f_{z_l}\right)\left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}}\right)\right)} \in \mathbb{C} \setminus \{0\}. \tag{8.4}$$

Thus $a_j \ne 0$ and $c_j \ne 0$ for any $j \in \{1,2,\ldots,m\}$. As Ω is an open Riemann surface, then there exists a holomorphic function g_2 on Ω such that $g_2(z) \ne 0$ for any $z \in \Omega \setminus \{z_1, z_2, \ldots, z_m\}$ and $ord_{z_j}(g_2) = k_j$ for any j. Denote that $g := g_1/g_2$ and $u := u_1 + \log|g_1| - \sum_{j=1}^m k_j G_\Omega(\cdot,z_j)$. Then g is a holomorphic function on Ω such that $g(z_j) \ne 0$ for any j, u is a harmonic function on Ω,

$$\varphi_1 + \psi_1 = 2\log|g| + 2\sum_{j=1}^m (k_j+1)G_\Omega(\cdot,z_j) + 2u$$

and

$$\prod_{j=1}^{m} \chi_{z_j}^{k_j+1} = \chi_{-u}.$$

Note that there exists a holomorphic function

$$f_u := f_{u_1} \frac{P^*(g_2)}{\prod_{j=1}^{m} f_{z_j}^{k_j}}$$

on Δ such that $|f_u| = P^* e^u$. Then it follows from equality (8.4) that

$$c_j = \lim_{z \to z_j} \frac{a_j w_j^{k_j} dw_j}{g P_* \left(f_u \left(\prod_{l=1}^{m} f_{z_l}^{k_l+1} \right) \left(\sum_{l=1}^{m} p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)}.$$

Thus the five statements in Theorem 1.8 hold.

Secondly, we prove the sufficiency. Assume that the five statements in Theorem 1.8 hold. As Ω is an open Riemann surface, then there exists a holomorphic function g_2 on Ω such that $g_2(z) \neq 0$ for any $z \in \Omega \setminus \{z_1, z_2, \ldots, z_m\}$ and $ord_{z_j}(g_2) = k_j$ for any j. Denote that $g_1 := g g_2$ is a holomorphic function on Ω such that $ord_{z_j}(g_1) = k_j$ for any j, and $u_1 := u + \sum_{j=1}^{m} k_j G_\Omega(\cdot, z_j) - \log|g_2|$ is a harmonic function Ω. It follows from $\varphi_1 + \psi_1 = 2 \log|g| + 2 \sum_{j=1}^{m} (k_j+1) G_\Omega(\cdot, z_j) + 2u$ and $\chi_{-u} = \prod_{j=1}^{m} \chi_{z_j}^{k_j+1}$ that

$$\varphi_1 + \psi_1 = 2 \log|g_1| + 2 \sum_{j=1}^{m} G_\Omega(\cdot, z_j) + 2u_1$$

and

$$\chi_{-u_1} = \prod_{j=1}^{m} \chi_{z_j}.$$

Note that there exists a holomorphic function

$$f_{u_1} := f_u \frac{\prod_{j=1}^{m} f_{z_j}^{k_j}}{P^*(g_2)}$$

on Δ such that $|f_{u_1}| = P^* e^{u_1}$. As

$$\lim_{z \to z_j} \frac{a_j w_j^{k_j} dw_j}{g P_* \left(f_u \left(\prod_{l=1}^{m} f_{z_l}^{k_l+1} \right) \left(\sum_{l=1}^{m} p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)} = c_j \in \mathbb{C} \setminus \{0\},$$

Then we have $a_j \neq 0$ and

$$\lim_{z \to z_j} \frac{a_j w_j^{k_j} dw_j}{g_1 P_* \left(f_{u_1} \left(\prod_{l=1}^{m} f_{z_l} \right) \left(\sum_{l=1}^{m} p_l \frac{df_{z_l}}{f_{z_l}} \right) \right)} = c_j. \tag{8.5}$$

Besides, we have $c_j F_j = c_0 F_Y$ and $a_j F_j = a_j' F_Y$ for some $a_j' \in \mathbb{C} \setminus \{0\}$. Then Theorem 1.4 shows that $G(h^{-1}(r))$ is linear with respect to r. It follows from Lemma 2.2 that there exists a holomorphic $(n, 0)$ form $\tilde{F}$ such that $(\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\{\psi < -t\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) = G(t)$$

for any $t \geq 0$. Assume that $\tilde{F} = \sum_{l=\tilde{k}_j}^{\infty} \pi_1^*(w_j^l dw_j) \wedge \tilde{F}_{j,l}$ on $V_{z_j} \times Y$ for any j according to Lemma 3.4, where $F_{j,l}$ is a holomorphic $(n-1,0)$ form on Y for any j,l and $F_{j,\tilde{k}_j} \not\equiv 0$. Since $(\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$, according to Lemma 3.1, we have $\tilde{k}_j = k_j$ and $a_j F_j = \tilde{F}_{j,k_j}$ for any $j \in \{1, 2, \ldots, m\}$. Then Lemma 3.10 shows that

$$\frac{G(t)}{\int_t^{+\infty} c(s)e^{-s}ds} = \frac{\int_{\{\psi < -t\}} |\tilde{F}|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-2u_1(z_j)}}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2}$$

for sufficiently large t, where $d_j = \lim_{z \to z_j} (g_1/w_j^{k_j})(z)$ for any j. And we have

$$\frac{e^{-2u_1(z_j)}}{|d_j|^2} = \frac{e^{-\alpha_j}}{c_\beta(z_j)^{2(k_j+1)}},$$

which implies that

$$\frac{G(t)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}$$

for sufficiently large t. As $G(h^{-1}(r))$ is linear with respect to r, according to inequality (1.3), we have

$$G(t) = \left(\int_t^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}$$

for any $t \geq 0$. Especially, for $t = 0$, we get that equality $\left(\int_0^{+\infty} c(s)e^{-s}ds \right)$ $\cdot \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} = \inf \left\{ \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n, 0) \right.$ form on M such that $(\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0 \Big\}$ holds.

The proof of Theorem 1.8 is done. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

8.1. **Proof of Remark 1.9.** Now we prove Remark 1.9.

Proof of Remark 1.9. Follow the notation in the above subsection. As $G(h^{-1}(r))$ is linear with respect to r and

$$G(0) = \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2},$$

it follows from Remark 1.5 that

$$c_0 \pi_1^* \left(g_1 P_* \left(f_{u_1} \left(\prod_{l=1}^m f_{z_l} \right) \left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right) \right) \wedge \pi_2^*(F_Y)$$

is the unique holomorphic $(n, 0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

Note that

$$f_u := f_{u_1} \frac{P^*(g_2)}{\prod_{j=1}^m f_{z_j}^{k_j}}$$

on Δ and $gg_2 = g_1$. Then we obtain that

$$c_0 \pi_1^* \left(g P_* \left(f_u \left(\prod_{l=1}^m f_{z_l}^{k_l+1} \right) \left(\sum_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}} \right) \right) \right) \wedge \pi_2^*(F_Y)$$

is the unique holomorphic $(n,0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

$\square$

9. PROOFS OF THEOREM 1.10

In this section, we give the proofs of Theorem 1.10.

Proof of Theorem 1.10. Using the Weierstrass Theorem on open Riemann surfaces (see [15]) and the Siu's Decomposition Theorem, we have

$$\varphi_1 + \psi_1 = 2 \log |g_0| + 2u_0,$$

where g_0 is a holomorphic function on Ω and u_0 is a subharmonic function on Ω such that $v(dd^c u_0, z) \in [0, 1)$ for any $z \in \Omega$. Note that $\mathrm{ord}_{z_j} g_0 = k_j + 1$ and

$$e^{2u_0(z_j)} \lim_{z \to z_j} \left| \frac{g_0}{w_j^{k_j+1}}(z) \right|^2 = e^{\alpha_j} c_\beta(z_j)^{2(k_j+1)}. \tag{9.1}$$

Using Lemma 3.8, we obtain that there exists a holomorphic $(n,0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^\infty \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j+1) c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

In the following, we prove that the equality $\left(\int_0^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^\infty \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j+1) c_\beta(z_j)^{2(k_j+1)}}$
$\cdot \int_Y |F_j|^2 e^{-\varphi_2} = \inf \left\{ \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic } (n,0) \text{ form on } M \text{ such that } (\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)} \text{ for any } (z_j, y) \in Z_0 \right\}$ can not hold. In the following, we assume that the equality holds and get a contradiction.

According to the above discussions (replacing ψ_1 by $\psi_1 + t$, replacing $c(\cdot)$ by $c(\cdot + t)$ and replacing Ω by $\{\psi_1 < -t\}$), for any $t \ge 0$, there exists a holomorphic $(n,0)$ form F_t on $\{\psi < -t\}$ such that $(F_t - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j, y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) \le \left(\int_t^{+\infty} c(s) e^{-s} ds \right) \sum_{j=1}^\infty \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j+1) c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \tag{9.2}$$

According to the above discussions, there exists a holomorphic $(n,0)$ form $\tilde{F}_1 \neq 0$ on M such that $(\tilde{F}_1 - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |\tilde{F}_1|^2 e^{-\pi_1^*(\varphi_1 + \psi_1 - 2\sum_{j=1}^{\infty}(k_j+1)G_\Omega(\cdot, z_j)) - \pi_2^*(\varphi_2)} c\left(\pi_1^*(-2\sum_{j=1}^{\infty}(k_j+1)G_\Omega(\cdot, z_j))\right)$$

$$\leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{\infty} \frac{2\pi|a_j|^2 e^{-\alpha_j}}{(k_j+1)c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

(9.3)

As $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, $\psi_1 \leq 2\sum_{j=1}^{\infty}(k_j+1)G_\Omega(\cdot, z_j)$ and $\left(\int_0^{+\infty} c(s)e^{-s}ds\right)$ $\cdot \sum_{j=1}^{\infty} \frac{2\pi|a_j|^2 e^{-\alpha_j}}{(k_j+1)c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} = \inf\left\{\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \text{ is a holomorphic}\right.$ $(n,0)$ form on M such that $(\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0\}$, we have

$$\int_M |\tilde{F}_1|^2 e^{-\varphi} c(-\psi)$$

$$= \int_M |\tilde{F}_1|^2 e^{-\pi_1^*(\varphi_1 + \psi_1 - 2\sum_{j=1}^{\infty}(k_j+1)G_\Omega(\cdot, z_j)) - \pi_2^*(\varphi_2)} c\left(\pi_1^*(-2\sum_{j=1}^{\infty}(k_j+1)G_\Omega(\cdot, z_j))\right).$$

As $\frac{1}{2}v(dd^c\psi_1, z_j) = k_j + 1 > 0$, $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$, we have $\psi_1 = 2\sum_{j=1}^{\infty}(k_j+1)G_\Omega(\cdot, z_j)$ according to Lemma 2.13.

As $e^{-\varphi_1}c(-\psi_1) = e^{-\varphi_1-\psi_1}e^{\psi_1}c(-\psi_1)$, $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, and φ_2 is a plurisubharmonic function on Y, $e^{-\varphi}c(-\psi)$ has locally positive lower bound on $M \setminus Z_0$. Thus $G(h^{-1}(r))$ is concave with respect to r. According to the definition of $G(t)$ and inequality (9.2), we have

$$\frac{G(t)}{\int_t^{+\infty} c(s)e^{-s}ds} \leq \sum_{j=1}^{\infty} \frac{2\pi|a_j|^2 e^{-\alpha_j}}{(k_j+1)c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} = \frac{G(0)}{\int_0^{+\infty} c(s)e^{-s}ds} \quad (9.4)$$

for any $t \geq 0$. Then $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$.

Note that $\sum_{j \to +\infty}(k_j + 1) = +\infty$, which contradicts to Proposition 1.6. Thus we have that $\left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{\infty} \frac{2\pi|a_j|^2 e^{-\alpha_j}}{(k_j+1)c_\beta(z_j)^{2(k_j+1)}} \cdot \int_Y |F_j|^2 e^{-\varphi_2} > \inf\left\{\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \right.$ $\tilde{F}$ is a holomorphic $(n,0)$ form on M such that $(\tilde{F} - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0\}$, which implies that there exists a holomorphic $(n,0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) < \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{\infty} \frac{2\pi|a_j|^2 e^{-\alpha_j}}{(k_j+1)c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}.$$

$\square$

10. Proofs of Theorem 1.13, Remark 1.14, Theorem 1.16 and Remark 1.17

In this section, we prove Theorem 1.13, Remark 1.14, Theorem 1.16 and Remark 1.17.

10.1. **Proofs of Theorem 1.13 and Remark 1.14.**

Proofs of Theorem 1.13 and Remark 1.14. Let $f_1 = dw \wedge d\tilde{w}_1 \wedge \ldots \wedge d\tilde{w}_{n-1}$ on $V_{z_0} \times U_{y_0}$, and let $f_2 = d\tilde{w}_1 \wedge \ldots \wedge d\tilde{w}_{n-1}$ on U_{y_0}. Let $\psi = \pi_1^*(2G_\Omega(\cdot, z_0))$. It follows from Lemma 3.1, that $(F_1 - F_2, (z_0, y)) \in \mathcal{I}(\psi)_{(z_0, y)}$ if and only if $F_1(z_0, y) = F_2(z_0, y)$ for any $y \in Y$, where F_1 and F_2 are holomorphic functions on $V_{z_0} \times Y$. Let f be a holomorphic $(n-1, 0)$ form on Y satisfying $\int_Y |f|^2 < +\infty$. It follows from Theorem 1.8 that there exists a holomorphic $(n, 0)$ form F on M such that $F|_{\{z_0\} \times Y} = \pi_1^*(dw) \wedge \pi_2^*(f)$ and

$$\int_M |F|^2 \leq \frac{2\pi}{c_\beta(z_0)^2} \int_Y |f|^2.$$

Note that

$$B_Y(y_0) = \frac{2^{n-1}}{\inf\left\{\int_Y |f|^2 : f \in H^0(Y, \mathcal{O}(K_Y)) \,\&\, f(y_0) = f_2(y_0)\right\}}$$

and

$$B_M((z_0, y_0)) = \frac{2^n}{\inf\left\{\int_M |F|^2 : F \in H^0(M, \mathcal{O}(K_M)) \,\&\, F((z_0, y_0)) = f_1((z_0, y_0))\right\}}.$$

Thus, we have

$$c_\beta(z_0)^2 B_Y(y_0) \leq \pi B_M((z_0, y_0)).$$

In the following, we prove the characterization of the holding of the equality $c_\beta(z_0)^2 B_Y(y_0) \leq \pi B_M((z_0, y_0))$.

By the definition of B_Y, there exists a holomorphic $(n-1, 0)$ form f_0 on Y such that $f_0(y_0) = f_2(y_0)$ and

$$B_Y(y_0) = \frac{2^{n-1}}{\int_Y |f_0|^2} > 0.$$

It follows from Theorem 1.8 that there exists a holomorphic $(n, 0)$ form F_0 on M such that $F_0 = \pi_1^*(dw) \wedge \pi_2^*(f_0)$ and

$$\int_M |F_0|^2 \leq \frac{2\pi}{c_\beta(z_0)^2} \int_Y |f_0|^2. \tag{10.1}$$

Firstly, we prove the necessity. Note that

$$B_M((z_0, y_0)) \geq \frac{2^n}{\int_M |\tilde{F}|^2},$$

where $\tilde{F}$ is a holomorphic $(n, 0)$ form on M satisfying $\tilde{F} = \pi_1^*(dw) \wedge \pi_2^*(f_0)$ on $\{z_0\} \times Y$. According to inequality (10.1),

$$c_\beta(z_0)^2 B_Y(y_0) = \pi B_M((z_0, y_0)),$$

and

$$B_Y(y_0) = \frac{2^{n-1}}{\int_Y |f_0|^2},$$

we obtain that

$$\frac{2\pi}{c_\beta(z_0)^2} \int_Y |f_0|^2 = \inf \left\{ \int_M |\tilde{F}|^2 : \tilde{F} \in H^0(M, \mathcal{O}(K_M)) \text{ \& } \tilde{F}|_{\{z_0\} \times Y} = \pi_1^*(dw) \wedge \pi_2^*(f_0) \right\}.$$

It follows from Theorem 1.8 that $\chi_{z_0} = 1$. $\chi_{z_0} = 1$ implies that there exists a holomorphic function $\tilde{f}$ on Ω such that $|\tilde{f}| = e^{G_\Omega(\cdot, z_0)}$, thus Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero (see [50], see also [53] and [34]).

Secondly, we prove the sufficiency. As Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero, we have $\chi_{z_0} = 1$. We prove

$$c_\beta(z_0)^2 B_Y(y_0) = \pi B_M((z_0, y_0))$$

by contradiction: if not, there exists a holomorphic $(n, 0)$ form $\tilde{F}_0$ on M such that $\tilde{F}_0((z_0, y_0)) = f_1((z_0, y_0))$ and

$$\int_M |\tilde{F}_0|^2 < \frac{2\pi}{c_\beta(z_0)^2} \int_Y |f_0|^2. \tag{10.2}$$

There exists a holomorphic $(n-1, 0)$ form $\tilde{f}_0$ on Y such that $\tilde{F}_0 = \pi_1^*(dw) \wedge \pi_2^*(\tilde{f}_0)$ on $\{z_0\} \times Y$. Hence $\tilde{f}_0(y_0) = f_2(y_0) = f_0(y_0)$, which implies that $\int_Y |\tilde{f}|^2 \geq \int_Y |f_0|^2$. Combining with inequality 10.2, we have

$$\inf \left\{ \int_M |\tilde{F}|^2 : F \in H^0(M, \mathcal{O}(K_M)) \text{ \& } \tilde{F}|_{\{z_0\} \times Y} = \pi_1^*(dw) \wedge \pi_2^*(\tilde{f}_0) \right\} < \frac{2\pi}{c_\beta(z_0)^2} \int_Y |\tilde{f}_0|^2,$$

which contradicts to Theorem 1.8. Hence

$$c_\beta(z_0)^2 B_Y(y_0) = \pi B_M((z_0, y_0)).$$

Thus, Theorem 1.13 holds.

Note that $B_{\tilde{M}}((z_0, y_0)) \geq B_M((z_0, y_0)) > 0$ and $B_{\tilde{M}}((z_0, y_0)) = B_M((z_0, y_0))$ if and only if $M = \tilde{M}$, thus Theorem 1.13 shows that Remark 1.14 holds.

$\square$

10.2. **Proof of Theorem 1.16 and Remark 1.17.**

Proofs of Theorem 1.16 and Remark 1.17. Let $f_1 = dw \wedge d\tilde{w}_1 \wedge \ldots \wedge dw_{n-1}$ on $V_{z_0} \times U_{y_0}$, and let $f_2 = d\tilde{w}_1 \wedge \ldots \wedge dw_{n-1}$ on U_{y_0}. Let $\psi = \pi_1^*(2G_\Omega(\cdot, z_0))$. It follows from Lemma 3.1, that $(F_1 - F_2, (z_0, y)) \in \mathcal{I}(\psi)_{(z_0, y)}$ if and only if $F_1(z_0, y) = F_2(z_0, y)$ for any $y \in Y$, where F_1 and F_2 are holomorphic functions on $V_{z_0} \times Y$. Let f be a holomorphic $(n-1, 0)$ form on Y satisfying $\int_Y |f|^2 < +\infty$. It follows from Theorem 1.8 that there exists a holomorphic $(n, 0)$ form F on M such that $F|_{\{z_0\} \times Y} = \pi_1^*(dw) \wedge \pi_2^*(f)$ and

$$\int_M |F|^2 \rho \leq \frac{2\pi \rho(z_0)}{c_\beta(z_0)^2} \int_Y |f|^2.$$

Note that

$$B_Y(y_0) = \frac{2^{n-1}}{\inf \left\{ \int_Y |f|^2 : f \in H^0(Y, \mathcal{O}(K_Y)) \text{ \& } f(y_0) = f_2(y_0) \right\}}$$

and

$$B_{M,\rho}((z_0, y_0)) = \frac{2^n}{\inf \left\{ \int_M |F|^2 \rho : F \in H^0(M, \mathcal{O}(K_M)) \text{ \& } F((z_0, y_0)) = f_1((z_0, y_0)) \right\}}.$$

Thus, we have
$$c_\beta(z_0)^2 B_Y(y_0) \le \pi\rho(z_0) B_{M,\rho}((z_0, y_0)).$$

In the following, we prove the characterization of the holding of the equality $c_\beta(z_0)^2 B_Y(y_0) \le \pi\rho(z_0) B_{M,\rho}((z_0, y_0))$.

By the definition of B_Y, there exists a holomorphic $(n-1, 0)$ form f_0 on Y such that $f_0(y_0) = f_2(y_0)$ and
$$B_Y(y_0) = \frac{2^{n-1}}{\int_Y |f_0|^2} > 0.$$

It follows from Theorem 1.8 that there exists a holomorphic $(n, 0)$ form F_0 on M such that $F_0 = \pi_1^*(dw) \wedge \pi_2^*(f_0)$ and
$$\int_M |F_0|^2 \rho \le \frac{2\pi\rho(z_0)}{c_\beta(z_0)^2} \int_Y |f_0|^2. \tag{10.3}$$

Firstly, we prove the necessity. Note that
$$B_{M,\rho}((z_0, y_0)) \ge \frac{2^n}{\int_M |\tilde{F}|^2 \rho},$$

where $\tilde{F}$ is a holomorphic $(n, 0)$ form on M satisfying $\tilde{F} = \pi_1^*(dw) \wedge \pi_2^*(f_0)$ on $\{z_0\} \times Y$. According to inequality (10.3),
$$c_\beta(z_0)^2 B_Y(y_0) = \pi\rho(z_0) B_{M,\rho}((z_0, y_0)),$$

and
$$B_Y(y_0) = \frac{2^{n-1}}{\int_Y |f_0|^2},$$

we obtain that
$$\frac{2\pi\rho(z_0)}{c_\beta(z_0)^2} \int_Y |f_0|^2 = \inf\left\{\int_M |\tilde{F}|^2 \rho : \tilde{F} \in H^0(M, \mathcal{O}(K_M)) \ \& \ \tilde{F}|_{\{z_0\}\times Y} = \pi_1^*(dw) \wedge \pi_2^*(f_0)\right\}.$$

It follows from Theorem 1.8 that $\chi_{z_0} = \chi_{-u}$.

Secondly, we prove the sufficiency. We prove
$$c_\beta(z_0)^2 B_Y(y_0) \le \pi\rho(z_0) B_{M,\rho}((z_0, y_0))$$

by contradiction: if not, there exists a holomorphic $(n, 0)$ form $\tilde{F}_0$ on M such that $\tilde{F}_0((z_0, y_0)) = f_1((z_0, y_0))$ and
$$\int_M |\tilde{F}_0|^2 \rho < \frac{2\pi\rho(z_0)}{c_\beta(z_0)^2} \int_Y |f_0|^2. \tag{10.4}$$

There exists a holomorphic $(n-1, 0)$ form $\tilde{f}_0$ on Y such that $\tilde{F}_0 = \pi_1^*(dw) \wedge \pi_2^*(\tilde{f}_0)$ on $\{z_0\} \times Y$. Hence $\tilde{f}_0(y_0) = f_2(y_0) = f_0(y_0)$, which implies that $\int_Y |\tilde{f}|^2 \ge \int_Y |f_0|^2$. Combining with inequality 10.4, we have
$$\inf\left\{\int_M |\tilde{F}|^2 \rho : F \in H^0(M, \mathcal{O}(K_M)) \ \& \ \tilde{F}|_{\{z_0\}\times Y} = \pi_1^*(dw) \wedge \pi_2^*(\tilde{f}_0)\right\} < \frac{2\pi\rho(z_0)}{c_\beta(z_0)^2} \int_Y |\tilde{f}_0|^2,$$

which contradicts to Theorem 1.8. Hence
$$c_\beta(z_0)^2 B_Y(y_0) \le \pi\rho(z_0) B_{M,\rho}((z_0, y_0)).$$

Thus, Theorem 1.16 holds.

Note that $B_{\tilde{M},\rho}((z_0, y_0)) \ge B_{M,\rho}((z_0, y_0)) > 0$ and $B_{\tilde{M},\rho}((z_0, y_0)) = B_{M,\rho}((z_0, y_0))$ if and only if $M = \tilde{M}$, thus Theorem 1.16 shows that Remark 1.17 holds.

$\square$

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