

Stat. Inf. II: Assignment 1

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Reading:

- a. Read the remaining Sections in Chapter 9: 9.3, 9.5 and 9.6. And, read the final section about errors and power on the R handout for inference about a proportion.

Problem #1

Prelude.

We know that, for $x_1, x_2, \dots, x_n \sim N(\mu, \sigma^2)$, the following is true:

$$\bar{X} \sim N(\mu, \sigma/\sqrt{n}),$$

which leads to

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Typically, we don't have access to population standard deviation σ , hence we substitute it by sample standard deviation $s = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n-1}}$. Then, the test statistic becomes

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{n-1},$$

where t -distribution is

- like $N(0, 1)$, symmetric, bell-shaped and centered at 0,
- but has heavier tails (see <https://istats.shinyapps.io/tdist/> for demo).

By **mathematical definition**, random variable T has a t -distribution with df degrees of freedom if

$$T = \frac{Z}{\sqrt{X_{df}^2/df}} \sim t_{df},$$

where $Z \sim N(0, 1)$, $X_{df}^2 \sim \chi_{df}^2$ (for definition of χ_{df}^2 , please see the https://en.wikipedia.org/wiki/Chi-squared_distribution or the last set of slides from previous semester).

By **Cochran's theorem**, the following is true:

$$\mathbf{z}' A \mathbf{z} \sim \chi_{rank(A)}^2,$$

where

- $\mathbf{z} = (z_1, z_2, \dots, z_n)'$, with z_1, z_2, \dots, z_n being independent standard normal random variables (in short, $z_1, z_2, \dots, z_n \sim_{ind} N(0, 1)$).
- A is a symmetric, idempotent matrix,

Actual problem. Piece-by-piece, we will proceed to show that, for $x_1, \dots, x_n \sim N(\mu, \sigma^2)$, the test statistic $T = \frac{\bar{X} - \mu}{s/\sqrt{n}}$ follows the **mathematical definition** of t_{n-1} distribution, as in

Main statement:

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{Z}{\sqrt{X_{n-1}^2/(n-1)}} \sim t_{n-1},$$

where $Z \sim N(0, 1)$, $X_{n-1}^2 \sim \chi_{n-1}^2$.

1. Show that

$$\bar{X} - \mu = \frac{\sigma}{\sqrt{n}}Z, \quad Z \sim N(0, 1)$$

Solution 1.

The χ^2 distribution is the square of independent standard normal distribution,

$$\chi_{df}^2 = \sum_i^{df} Z_i^2.$$

Also note that the sample mean \bar{X} has follows a normal distribution with mean μ and standard deviation σ ,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Finally, note that Z is given by

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}.$$

Thus by simple algebra we arrive at the desired conclusion.

$$\bar{X} - \mu = \frac{\sigma}{\sqrt{n}}Z.$$

2. Show that

$$\sum_i (x_i - \bar{x})^2 = \sigma^2 \sum_i (z_i - \bar{z})^2,$$

where $z_1, z_2, \dots, z_n \sim_{ind} N(0, 1)$.

Solution 2.

We first recall that

$$X_i \sim N(\mu, \sigma^2).$$

and thus,

$$z_i = \frac{x_i - \mu}{\sigma} \sim N(0, 1).$$

By simple algebra we can deduce that

$$x_i = z_i \sigma + \mu.$$

Consider $\sum_i (x_i - \bar{x})^2$. By part 1, we know that $\bar{x} = \frac{\sigma z}{\sqrt{n}} + \mu$.

Thus,

$$\sum_i (x_i - \bar{x})^2 = \sum_i [(z_i \sigma + \mu) - (\frac{\sigma z}{\sqrt{n}} + \mu)],$$

as required.

3. Applying the basic “row-by-column” matrix multiplication, proceed to calculate $\mathbf{z}' A_{n \times n} = (z_1, z_2, \dots, z_n) \times A_{n \times n}$, where

$$A = \mathbf{I}_{n \times n} - \frac{1}{n} \mathbf{1}_{n \times n}$$

$$\mathbf{I}_{n \times n} = \text{diag}(\underbrace{1, 1, \dots, 1}_n) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad \mathbf{1}_{n \times n} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ \dots & & & & & \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

Solution 3.

It follows by the linear algebra operations that,

$$\begin{aligned} \mathbf{z}' A &= \mathbf{z}' (\mathbf{I} - \frac{1}{n} \mathbf{1}) \\ &= \mathbf{z}' \mathbf{I} - \frac{1}{n} \mathbf{z}' \mathbf{1} \\ &= \mathbf{z}' - (\frac{1}{n} \sum_i z_i, \frac{1}{n} \sum_i z_i, \dots, \frac{1}{n} \sum_i z_i)_{1 \times n} \\ &= \mathbf{z}' - (\bar{z}, \bar{z}, \dots, \bar{z})_{1 \times n} \\ &= \mathbf{z}' - \bar{\mathbf{z}} \end{aligned}$$

Which is the $1 \times n$ vector,

$$= (z_i - \bar{z})_i^n.$$

4. Using your result from part 3, show that $(\mathbf{z}' A_{n \times n})(\mathbf{z}' A_{n \times n})' = \sum_i (z_i - \bar{z})^2$.

Solution 4.

We have already shown that $\mathbf{z}'A_{n \times n} = (z_1 - \bar{z}, z_2 - \bar{z}, \dots, z_n - \bar{z})_{1 \times n}$.

It follows that

$$\begin{aligned} (\mathbf{z}'A_{n \times n})(\mathbf{z}'A_{n \times n})' &= (z_1 - \bar{z}, z_2 - \bar{z}, \dots, z_n - \bar{z})_{1 \times n} \times (z_1 - \bar{z}, z_2 - \bar{z}, \dots, z_n - \bar{z})_{n \times 1} \\ &= (z_1 - \bar{z})^2 + (z_2 - \bar{z})^2 + \dots + (z_n - \bar{z})^2 \\ &= \sum_i^n (z_i - \bar{z})^2, \end{aligned}$$

as we wished to show.

5. For matrix $A_{n \times n}$ as defined in part 3, show that it is **idempotent**, as in

$$A_{n \times n} \times A_{n \times n} = A_{n \times n},$$

hence, combined with part 4, leading to the fact that

$$\sum_i (z_i - \bar{z})^2 = (\mathbf{z}'A_{n \times n})(\mathbf{z}'A_{n \times n})' = \mathbf{z}'A_{n \times n}A_{n \times n}'\mathbf{z} = \mathbf{z}'A_{n \times n}\mathbf{z}$$

Solution 5.

Recall that $A_{n \times n} = \mathbf{I}_{n \times n} - \frac{1}{n}\mathbf{1}_{n \times n}$.

Thus

$$\begin{aligned} A_{n \times n} \times A_{n \times n} &= (\mathbf{I} - \frac{1}{n}\mathbf{1})(\mathbf{I} - \frac{1}{n}\mathbf{1}) \\ &= (\mathbf{I}^2 - \frac{2}{n}\mathbf{1} + \frac{1}{n^2}\mathbf{1}^2). \end{aligned}$$

Here, we note that $\mathbf{1}^2 = \mathbf{n}$, where $\mathbf{n} \in \mathbb{R}^{n \times n}$ such that every element $\times_{(i,j)} = n$, for all (i, j) .

Thus

$$\begin{aligned} &(\mathbf{I}^2 - \frac{2}{n}\mathbf{1} + \frac{1}{n^2}\mathbf{1}^2) \\ &= (\mathbf{I}^2 - \frac{2}{n}\mathbf{1} + \frac{1}{n^2}\mathbf{n}) \\ &= (\mathbf{I}^2 - \frac{2}{n}\mathbf{1} + \frac{1}{n}\mathbf{1}) \\ &= (\mathbf{I}^2 - \frac{1}{n}\mathbf{1}), \end{aligned}$$

as desired.

6. Matrix $A_{n \times n}$ as defined in part 3, has rank of $n - 1$. Why not n ?

Solution 6.

We first consider $A_{n \times n}$.

$$A_{n \times n} = \begin{pmatrix} (1 - \frac{1}{n}) & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ -\frac{1}{n} & (1 - \frac{1}{n}) & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ \cdots & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & (1 - \frac{1}{n}) \end{pmatrix}$$

$$= \frac{1}{n} \begin{pmatrix} (n-1) & 1 & 1 & \cdots & 1 & 1 \\ 1 & (n-1) & 1 & \cdots & 1 & 1 \\ \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & (n-1) \end{pmatrix}.$$

Choose any row, such as the first row, and subtract all other rows from it. The resulting row is a vector $(0, 0, \dots, 0)1 \times n$. Since this row can be expressed as a linear combination of the other rows, the matrix is not linearly independent. Thus it cannot have a rank n .

7. Combine the parts 2, 5 and 6 with **Cochran's theorem** (see page 1) to show that

$$s/\sqrt{n} = \sqrt{\sum_i (x_i - \bar{x})^2 / (n-1)} \times \frac{1}{\sqrt{n}} = \sqrt{\sigma^2 X_{n-1}^2 / (n-1)} \times \frac{1}{\sqrt{n}}, \quad X_{n-1}^2 \sim \chi_{n-1}^2$$

Solution 7.

We have, by definition, that

$$s/\sqrt{n} = \sqrt{\sum_i (x_i - \bar{x})^2 / (n-1)} \times \frac{1}{\sqrt{n}}.$$

Consider the numerator $\sum_i (x_i - \bar{x})^2$. We showed in part 2 that

$$\sum_i (x_i - \bar{x})^2 = \sigma^2 \sum_i (z_i - \bar{z})^2.$$

By part 5 we can rewrite the left-hand side as

$$\sum_i (x_i - \bar{x})^2 = \sigma^2 \mathbf{z}' A \mathbf{z}.$$

And by Cochran's Theorem we can rewrite it as

$$\sum_i (x_i - \bar{x})^2 = \sigma^2 \chi_{n-1}^2.$$

Thus, we can conclude that

$$s/\sqrt{n} = \sqrt{\sum_i (x_i - \bar{x})^2 / (n-1)} \times \frac{1}{\sqrt{n}} = \sqrt{\sigma^2 X_{n-1}^2 / (n-1)} \times \frac{1}{\sqrt{n}}, \quad X_{n-1}^2 \sim \chi_{n-1}^2,$$

as required.

Solution 8.

8. Combine parts 2 and 7 to prove the **main statement**.

In part 1 we showed that

$$\bar{X} - \mu = \frac{\sigma}{\sqrt{n}}Z.$$

And in part 7 we showed that,

$$\frac{s}{\sqrt{n}} = \sqrt{\sigma^2 X_{n-1}^2 / (n-1)} \times \frac{1}{\sqrt{n}}.$$

Thus,

$$\begin{aligned} \frac{\bar{X} - \mu}{s/\sqrt{n}} &= \frac{\frac{\sigma}{\sqrt{n}}Z}{\frac{\sigma}{\sqrt{n}}\sqrt{\sigma^2 X_{n-1}^2 / (n-1)}} \\ &= \frac{Z}{\sqrt{\sigma^2 X_{n-1}^2 / (n-1)}}, \end{aligned}$$

as required.

Note: Some extra details could be found here (see p. 1-2 of the main post): <https://stats.stackexchange.com/questions/306937/quadratic-form-and-chi-squared-distribution>.

Problem 2:

1. Write a *prop.sample.size()* function that will output the sample size needed for a one-sample proportion test to achieve

- a desired margin of error (argument #1)
- for a given confidence level (argument #2)

in the “worst-case scenario” (as was explained in class). What was meant by the “worst-case scenario”?

Solution 1.

We recall that

$$m = z_{1-\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

We wish to compute the desired sample size given margin of error m and confidence level $1 - \frac{\alpha}{2}$.

In particular, we wish to account for the “worst-case scenario”, where the standard error is maximized by our choice of \hat{p} .

```
prop.sample.size <- function(m, conf) {  
  alpha = 1-conf  
  n <- (qnorm(1-(alpha/2))^2 * 0.5^2)/(m^2)  
  response <- c("The sample size for a one-sample proportion test with a desired margin of error", m, "  
  return(response)  
}
```

For example, suppose $m = 0.4$, and the given confidence level is 0.995. Then,

```
prop.sample.size(0.4, .995)
```

```
## [1] "The sample size for a one-sample proportion test with a desired margin of error"
## [2] "0.4"
## [3] " and given confidence level"
## [4] "0.995"
## [5] " is "
## [6] "12.3116227759726"
```

2. use your *prop.sample.size()* from part 1 to do exercise 8.50 from the Agresti book.

How many businesses fail? A study is planned to estimate the proportion of businesses started in the year 2006 that failed within five years of their start-up. How large a sample size is needed to guarantee estimating this proportion correct to within

a. 0.10 with probability 0.95?

```
prop.sample.size(0.10, 0.95)
```

```
## [1] "The sample size for a one-sample proportion test with a desired margin of error"
## [2] "0.1"
## [3] " and given confidence level"
## [4] "0.95"
## [5] " is "
## [6] "96.0364705173531"
```

b. 0.05 with probability 0.95?

```
prop.sample.size(0.05, 0.95)
```

```
## [1] "The sample size for a one-sample proportion test with a desired margin of error"
## [2] "0.05"
## [3] " and given confidence level"
## [4] "0.95"
## [5] " is "
## [6] "384.145882069412"
```

c. 0.05 with probability 0.99?

```
prop.sample.size(0.05, 0.99)
```

```
## [1] "The sample size for a one-sample proportion test with a desired margin of error"
## [2] "0.05"
## [3] " and given confidence level"
## [4] "0.99"
## [5] " is "
## [6] "663.489660102121"
```

d. Compare sample sizes for parts a and b, and b and c, and summarize the effects of decreasing the margin of error and increasing the confidence level.

3. Write a *mean.sample.size()* function that will output the sample size needed for a one-sample mean test to achieve

- a desired margin of error (argument #1)
- for a given confidence level (argument #2)
- for a given standard deviation (argument #3)

Proceed to use that function in order to do exercise 8.53 from the Agresti book.

Solution 3

Recall that

$$t_{(n-1), 1-\alpha/2} \cdot \frac{s}{\sqrt{n}} = m,$$

implying,

$$n = \frac{\sigma^2 z^2}{m^2}.$$

This is working under the assumption that we have a population size of at least $n = 30$, since this would make our distribution approach the normal distribution.

Thus we can construct our function.

```
mean.sample.size <- function(m, conf, s) {  
  alpha = 1-conf  
  n <- ((s * qnorm(1-(alpha/2)))/m)^2  
  response <- c("The sample size for a one-sample mean test with a desired margin of error,", m, " given  
  return(response)  
}
```

(8.53) *Income of the Native Americans* How large a sample size do we need to estimate the mean annual income of Native Americans in onondaga County, New York, correct to within \$1000 with probability 0.99? No information is available to us about the standard deviation of their annual income. We guess that nearly all of the incomes fall between \$0 and \$120,000 and that this distribution is approximately bell shaped.

Solution

We are given that $m = 1000$, and the confidence level is 0.99. Although we are not given the standard deviation σ , we recall that we may approximate σ by

$$\sigma \approx \frac{\text{range}}{6},$$

since 99% of the data lies within 3 standard deviations from the average, or 1/6 of the range.

Thus

$$\sigma \approx \frac{120,000}{6} = 20,000.$$

Using the function written above,

```
mean.sample.size(1000, 0.99, 20000)  
  
## [1] "The sample size for a one-sample mean test with a desired margin of error,"  
## [2] "1000"  
## [3] " given confidence level"  
## [4] "0.99"  
## [5] "and standard deviation"  
## [6] "20000"  
## [7] " is "  
## [8] "2653.95864040848"
```


We find that 2165 participants are needed to reach the desired criteria.