MODELO MATEMÁTICO:

$$\rho(T)c_p(T)\frac{\partial T(x,y,t)}{\partial t} = \nabla \cdot (k(T)\nabla T(x,y,t)) + g(T) \tag{1}$$

$$\frac{\partial T(0,y,t)}{\partial x} = \frac{\partial T(x,0,t)}{\partial y} = 0 \tag{2}$$

$$T(L_x,y,t) = T(x,L_y,t) = T_a \tag{3}$$

$$T(x,y,0) = T_0 \tag{4}$$

$$-k\frac{\partial T(x,y,t)}{\partial x} = h(T_\infty - T(x,y,t)) \qquad [a,b]\times[c,d] \tag{5}$$

Fazendo k, ρc_p e g constantes:

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + g \tag{6}$$

$$\frac{\rho c_p}{k}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{g}{k} \tag{7}$$

Fazendo $\alpha = \dfrac{k}{\rho c_p}$:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{g}{k} \tag{8}$$

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T + \frac{g\alpha}{k} \tag{9}$$

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{g\alpha}{k} \tag{10}$$

Usamos uma aproximação de primeira ordem para a derivada temporal:

$$\left.\frac{\partial T}{\partial t}\right|_{i,j}^{n+1} = \frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} \tag{11}$$

E uma aproximação de segunda ordem para a derivada espacial:

$$\left.\frac{\partial T}{\partial x}\right|_{i,j}^{n+1} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{\Delta x^2} \tag{12}$$

$$\left.\frac{\partial T}{\partial y}\right|_{i,j}^{n+1} = \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{\Delta y^2} \tag{13}$$

Assim, a equação 10 se torna:

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \alpha\left(\frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{\Delta x^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{\Delta y^2}\right) + \frac{g\alpha}{k} \tag{14}$$

Considerando uma malha uniforme constante ($\Delta x = \Delta y$) :

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \alpha\left(\frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{\Delta x^2}\right) + \frac{g\alpha}{k} \tag{15}$$

$$T_{i,j}^{n+1} - T_{i,j}^{n} = \frac{\alpha \Delta t}{\Delta x^2}\left(T_{i-1,j}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j-1}^{n+1} + T_{i,j+1}^{n+1} - 4T_{i,j}^{n+1}\right) + \frac{g\alpha}{k} \tag{16}$$

Fazendo $r = \frac{\alpha \Delta t}{\Delta x^2}$ e $\lambda = \frac{g\alpha}{k}$:

$$T_{i,j}^{n+1} - T_{i,j}^{n} = r\left(T_{i-1,j}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j-1}^{n+1} + T_{i,j+1}^{n+1} - 4T_{i,j}^{n+1}\right) + \lambda \tag{17}$$

Isolando no lado esquerdo os termos em n + 1 :

$$-r\left(T_{i-1,j}^{n+1} + T_{i+1,j}^{n+1} + T_{i,j-1}^{n+1} + T_{i,j+1}^{n+1}\right) + (1+4r)T_{i,j}^{n+1} = T_{i,j}^{n} + \lambda \tag{18}$$

2. DISCRETIZAÇÃO EM NÓS ESPECIAIS:

2.1 Contorno externo superior e externo direito

Em x = L_x e em y = L_y, a temperatura é prescrita.

$$T_{0,j} = T_a \qquad j = 0, 1, ..., M \tag{19}$$

$$T_{i,M} = T_a \qquad i = 0, 1, ..., N \tag{20}$$

2.2 Contorno externo inferior e externo esquerdo

Em x = 0 (j = 0), o fluxo é nulo.

$$\left.\frac{\partial T}{\partial x}\right|_{j=0} = 0 \tag{21}$$

Utilizaremos uma aproximação avançada de segunda ordem, em três pontos:

$$\left.\frac{\partial T}{\partial x}\right|_{j=0}^{n+1} = \frac{-3T_{i,j}^{n+1} + 4T_{i+1,j}^{n+1} - T_{i+2,j}^{n+1}}{2\Delta x} \tag{22}$$

$$\frac{-3T_{i,j}^{n+1} + 4T_{i+1,j}^{n+1} - T_{i+2,j}^{n+1}}{2\Delta x} = 0 \tag{23}$$

$$-3T_{i,j}^{n+1} + 4T_{i+1,j}^{n+1} - T_{i+2,j}^{n+1} = 0 \qquad\qquad j = 0 \quad i = 1, 2, ..., N \tag{24}$$

De maneira similar, para y = 0 (i = N), também não há troca de calor.

$$\left.\frac{\partial T}{\partial y}\right|_{i=N} = 0 \tag{25}$$

Utilizaremos derivada recuada (pois temos apenas as informações dos nós anteriores):

$$\left.\frac{\partial T}{\partial y}\right|_{i=N}^{n+1} = \frac{T_{N,j-2}^{n+1} - 4T_{N,j-1}^{n+1} + 3T_{N,j}^{n+1}}{2\Delta y} \tag{26}$$

$$\frac{T_{N,j-2}^{n+1} - 4T_{N,j-1}^{n+1} + 3T_{N,j}^{n+1}}{2\Delta y} = 0 \tag{27}$$

$$T_{N,j-2}^{n+1} - 4T_{N,j-1}^{n+1} + 3T_{N,j}^{n+1} = 0 \qquad\qquad i = N \quad j = 1, 2, ..., M-1 \tag{28}$$

2.3 Contorno interno superior e interno inferior

Consideremos que $T_{p,q}$ é o vértice superior esquerdo da região interna Ω em contato com o meio sujeito à temperatura T_f e w é o número de nós necessários para preencher as arestas de comprimento d, tal que:

$$w = \frac{d}{\Delta x} + 1 \tag{29}$$

Para troca térmica por convecção, temos:

$$k\frac{\partial T}{\partial x|y} = h(T_\infty - T(x, y, t)) \tag{30}$$

Na aresta superior (i = p) ocorre a troca térmica por convecção no sentido do eixo y. Utilizaremos a derivada recuada:

$$k\left(\frac{T_{i-2,j}^{n+1} - 4T_{i-1,j}^{n+1} + 3T_{i,j}^{n+1}}{2\Delta y}\right) = h(T_f - T_{i,j}^{n+1}) \tag{31}$$

$$k\left(T_{i-2,j}^{n+1} - 4T_{i-1,j}^{n+1} + 3T_{i,j}^{n+1}\right) = 2h\Delta y T_f - 2h\Delta y T_{i,j}^{n+1} \tag{32}$$

$$T_{i-2,j}^{n+1} - 4T_{i-1,j}^{n+1} + 3T_{i,j}^{n+1} = \frac{2h\Delta y}{k}T_f - \frac{2h\Delta y}{k}T_{i,j}^{n+1} \tag{33}$$

Fazendo $\gamma = \dfrac{2h\Delta y}{k}$:

$$T_{i-2,j}^{n+1} - 4T_{i-1,j}^{n+1} + 3T_{i,j}^{n+1} = \gamma T_f - \gamma T_{i,j}^{n+1} \tag{34}$$

$$T_{i-2,j}^{n+1} - 4T_{i-1,j}^{n+1} + (3+\gamma)T_{i,j}^{n+1} = \gamma T_f \quad i = p \quad q \leq j \leq q + w \tag{35}$$

De maneira similar, podemos expressar a discretização na aresta inferior (i = p + w) onde também ocorre troca térmica por convecção no sentido do eixo y. Utilizaremos, desta vez, diferenças avançadas:

$$k\left(\frac{-3T_{i,j}^{n+1} + 4T_{i+1,j}^{n+1} - T_{i+2,j}^{n+1}}{2\Delta y}\right) = h(T_f - T_{i,j}^{n+1}) \tag{36}$$

$$k\left(-3T_{i,j}^{n+1} + 4T_{i+1,j}^{n+1} - T_{i+2,j}^{n+1}\right) = 2h\Delta y T_f - 2h\Delta y T_{i,j}^{n+1} \tag{37}$$

$$-3T_{i,j}^{n+1} + 4T_{i+1,j}^{n+1} - T_{i+2,j}^{n+1} = \frac{2h\Delta y}{k}T_f - \frac{2h\Delta y}{k}T_{i,j}^{n+1} \tag{38}$$

$$-3T_{i,j}^{n+1} + 4T_{i+1,j}^{n+1} - T_{i+2,j}^{n+1} = \gamma T_f - \gamma T_{i,j}^{n+1} \tag{39}$$

$$(\gamma - 3)T_{i,j}^{n+1} + 4T_{i+1,j}^{n+1} - T_{i+2,j}^{n+1} = \gamma T_f \quad i = p + w \quad q \leq j \leq q + w \tag{40}$$

2.4 Contorno interno esquerdo e direito

Na aresta esquerda (j = q) ocorre a troca térmica por convecção no sentido do eixo x. Utilizaremos uma aproximação recuada de segunda ordem, com pontos no interior do domínio, para representar a derivada primeira presente condição de contorno de terceiro tipo:

$$k\left(\frac{T_{i,j-2}^{n+1}-4T_{i,j-1}^{n+1}+3T_{i,j}^{n+1}}{2\Delta x}\right)=h(T_f-T_{i,j}^{n+1}) \tag{41}$$

$$k\left(T_{i,j-2}^{n+1}-4T_{i,j-1}^{n+1}+3T_{i,j}^{n+1}\right)=2h\Delta xT_f-2h\Delta xT_{i,j}^{n+1} \tag{42}$$

$$T_{i,j-2}^{n+1}-4T_{i,j-1}^{n+1}+3T_{i,j}^{n+1}=\frac{2h\Delta x}{k}T_f-\frac{2h\Delta x}{k}T_{i,j}^{n+1} \tag{43}$$

Fazendo $\gamma=\dfrac{2h\Delta x}{k}$:

$$T_{i,j-2}^{n+1}-4T_{i,j-1}^{n+1}+3T_{i,j}^{n+1}=\gamma T_f-\gamma T_{i,j}^{n+1} \tag{44}$$

$$T_{i,j-2}^{n+1}-4T_{i,j-1}^{n+1}+(3+\gamma)T_{i,j}^{n+1}=\gamma T_f \quad j=q \quad p\leq i\leq p+w \tag{45}$$

De maneira similar, podemos expressar a discretização na aresta direita (j = q + w) onde também ocorre troca térmica por convecção no sentido do eixo x. Aqui utilizamos diferenças finitas avançadas:

$$k\left(\frac{-3T_{i,j}^{n+1}+4T_{i,j+1}^{n+1}-T_{i,j+2}^{n+1}}{2\Delta y}\right)=h(T_f-T_{i,j}^{n+1}) \tag{46}$$

$$k\left(-3T_{i,j}^{n+1}+4T_{i,j+1}^{n+1}-T_{i,j+2}^{n+1}\right)=2h\Delta yT_f-2h\Delta yT_{i,j}^{n+1} \tag{47}$$

$$-3T_{i,j}^{n+1}+4T_{i,j+1}^{n+1}-T_{i,j+2}^{n+1}=\frac{2h\Delta y}{k}T_f-\frac{2h\Delta y}{k}T_{i,j}^{n+1} \tag{48}$$

$$-3T_{i,j}^{n+1}+4T_{i,j+1}^{n+1}-T_{i,j+2}^{n+1}=\gamma T_f-\gamma T_{i,j}^{n+1} \tag{49}$$

$$(\gamma-3)T_{i,j}^{n+1}+4T_{i,j+1}^{n+1}-T_{i,j+2}^{n+1}=\gamma T_f \quad j=q+w \quad p\leq i\leq p+w \tag{50}$$

RESUMO:

$$\begin{array}{ccccc}
T_{0,j} & = & T_a & i=0 & j=0,1,...,M \\
T_{i,M} & = & T_a & j=M & i=0,1,...,N \\
-3T_{i,j}^{n+1}+4T_{i+1,j}^{n+1}-T_{i+2,j}^{n+1} & = & 0 & j=0 & i=1,2,...,N \\
T_{N,j-2}^{n+1}-4T_{N,j-1}^{n+1}+3T_{N,j}^{n+1} & = & 0 & i=N & j=1,2,...,M-1 \\
T_{i-2,j}^{n+1}-4T_{i-1,j}^{n+1}+(3+\gamma)T_{i,j}^{n+1} & = & \gamma T_f & i=p & q\leq j\leq q+w \\
(\gamma-3)T_{i,j}^{n+1}+4T_{i+1,j}^{n+1}-T_{i+2,j}^{n+1} & = & \gamma T_f & i=p+w & q\leq j\leq q+w \\
T_{i,j-2}^{n+1}-4T_{i,j-1}^{n+1}+(3+\gamma)T_{i,j}^{n+1} & = & \gamma T_f & j=q & p\leq i\leq p+w \\
(\gamma-3)T_{i,j}^{n+1}+4T_{i,j+1}^{n+1}-T_{i,j+2}^{n+1} & = & \gamma T_f & j=q+w & p\leq i\leq p+w \\
-r(T_{i-1,j}^{n+1}+T_{i+1,j}^{n+1}+T_{i,j-1}^{n+1}+T_{i,j+1}^{n+1})+(1+4r)T_{i,j}^{n+1} & = & T_{i,j}^{n}+\lambda & \text{caso contrário} &
\end{array}$$