1 Preliminaries

We consider a binary system where the primary is located at the origin of our coordinate system, and the secondary is treated as a point mass orbiting in the Cartesian xy plane with coordinates

$$\mathbf{r}_2 = r_2 \begin{pmatrix} \cos\phi_2 \\ \sin\phi_2 \\ 0 \end{pmatrix}. \tag{1}$$

The azimuthal coordinate ϕ_2 of the secondary can be expressed as

$$\phi_2 = \upsilon + \phi_{\mathrm{p}}; \tag{2}$$

here, the true anomaly υ is the angle between the secondary and the periastron direction, while ϕ_{p} is the angle between the periastron direction and the x axis[1]. The spherical radial coordinate r_2 of the secondary depends on υ via

$$r_2(\upsilon) = \frac{a(1-e^2)}{1+e\cos\upsilon}, \tag{3}$$

describing an elliptical orbit with semi-major axis a and eccentricity e.

To figure out *where* on this orbit the secondary will be found at some time t, we first evaluate the mean anomaly

$$M = \frac{2\pi}{P_{\mathrm{orb}}}(t - t_{\mathrm{p}}) = \Omega_{\mathrm{orb}}(t - t_{\mathrm{p}}), \tag{4}$$

where P_{orb} is the orbital period, $\Omega_{\mathrm{orb}} \equiv 2\pi/P_{\mathrm{orb}}$ the corresponding orbital angular frequency, and t_{p} the time of periastron passage when $\upsilon = 0$. Then, we evaluate the eccentric anomaly E by solving Kepler's equation

$$E - e\sin E = M \tag{5}$$

Finally, the secondary's true anomaly follows from

$$\tan\frac{\upsilon}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \tag{6}$$

In the present analysis, we'll not need most of these equations — but it's good to at least state them for the sake of completeness.

2 Gravitational Potential of the Secondary

2.1 Laplace Expansion

We now develop an expression for the gravitational potential of the secondary, as experienced by the primary. Most generally, this potential at point $\mathbf{r}$ can be written as

$$\Phi_2(\mathbf{r};t) = -\int \frac{G\rho_2(\mathbf{r}';t)}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}\tau' \tag{7}$$

where $\rho_2(\mathbf{r}';t)$ is the mass distribution of the secondary. For a point mass M_2 located at $\mathbf{r}_2$, this reduces to

$$\Phi_2(\mathbf{r};t) = -\frac{GM_2}{|\mathbf{r}-\mathbf{r}_2|}. \tag{8}$$

Using simple trigonometry, we re-write this as

$$\Phi_2(\mathbf{r};t) = -\frac{GM_2}{r_2}\frac{1}{\sqrt{1-2h\cos\chi+h^2}} \tag{9}$$

[1] This is related to the orbit's argument of periastron ω via $\phi_{\mathrm{p}} = \omega + \text{const.}$; a similar relation applies to the longitude of periastron ϖ.

where $h \equiv r/r_2$ and χ is the angle between $\mathbf{r}$ and $\mathbf{r}_2$ such that $\mathbf{r} \cdot \mathbf{r}_2 = r\, r_2 \cos\chi$. The second term on the right-hand side can be recognized as the generating function for the Legendre polynomials, such that — if $h < 1$ — we can write

$$\frac{1}{\sqrt{1 - 2h\cos\chi + h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(\cos\chi). \tag{10}$$

where P_ℓ is the Legendre polynomial of degree ℓ. Hence, the potential becomes (after a little rearranging)

$$\Phi_2(\mathbf{r}; t) = -\frac{GM_2}{r} \sum_{\ell=0}^{\infty} \left(\frac{r}{r_2}\right)^{\ell+1} P_\ell(\cos\chi). \tag{11}$$

2.2 Spherical Harmonic Expansion

The next step is to expand the $P_\ell(\cos\chi)$ term in the above expression using the spherical harmonic addition theorem, which states that

$$P_\ell(\cos\chi) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)\; Y_\ell^{m*}(\theta_2,\phi_2), \tag{12}$$

where Y_ℓ^m is the spherical harmonic of harmonic degree ℓ and azimuthal order m, (θ,ϕ) are the spherical-polar colatitude and azimuth coordinates corresponding to position vector $\mathbf{r}$, and (θ_2,ϕ_2) are the coordinates corresponding to position vector $\mathbf{r}_2$ (i.e., the coordinates of the point-mass secondary). For the most part we don't need to worry about the precise definition we adopt for the spherical harmonics, but for the sake of completeness (and also the comparison against other work, in Appendix A), let's follow Arfken et al. (2013) and adopt

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_\ell^m(\cos\theta) \mathrm{e}^{\mathrm{i}m\phi}, \tag{13}$$

together with

$$P_\ell^m(x) = (-1)^m \left(1 - x^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_\ell \tag{14}$$

for the associated Legendre functions P_ℓ^m; note the $(-1)^m$ Condon-Shortley phase factor included in the latter. Because

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x), \tag{15}$$

(e.g., Arfken et al., 2013, eqn. 15.81), we can also write the spherical harmonics as

$$Y_\ell^m(\theta,\phi) = (-1)^{(|m|-m)/2} \sqrt{\frac{2\ell+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\, P_\ell^{|m|}(\cos\theta) \mathrm{e}^{\mathrm{i}m\phi}, \tag{16}$$

where the factor in front ensures that the Condon-Shortley phase factor is respected. From this latter expression, we can easily confirm that

$$Y_\ell^{m*}(\theta,\phi) = (-1)^m Y_\ell^{-m}(\theta,\phi) \tag{17}$$

As required by the addition theorem (12), the spherical harmonics are orthonormal over the sphere:

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi) Y_{\ell'}^{m'*}(\theta,\phi)\, \mathrm{d}\theta\, \mathrm{d}\phi = \delta_{\ell,\ell'}\delta_{m,m'}. \tag{18}$$

With the expansion (12), setting $\theta_2 = \pi/2$, and taking ϕ_2 from eqn. (2), the potential of the secondary becomes

$$\Phi_2(\mathbf{r}; t) = -\frac{GM_2}{r} \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \left(\frac{r}{r_2}\right)^{\ell+1} \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)\; Y_\ell^{m*}(\pi/2, \upsilon + \phi_\mathrm{p}). \tag{19}$$

It's useful to extract the explicit υ dependence of the final term; we can do this by noting that

$$Y_\ell^m(\theta,\phi) = Y_\ell^m(\theta,0)\,\mathrm{e}^{\mathrm{i}m\phi} \tag{20}$$

and so the potential becomes

$$\Phi_2(\mathbf{r};t) = -\frac{GM_2}{r}\sum_{\ell=0}^{\infty}\frac{4\pi}{2\ell+1}\left(\frac{r}{r_2}\right)^{\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\phi)\; Y_\ell^{m*}(\pi/2,0)\,\mathrm{e}^{-\mathrm{i}m(\upsilon+\phi_{\mathrm{p}})}. \tag{21}$$

2.3 Fourier Expansion

In the expression above, the time t does not appear explicitly on the right-hand side, but we know that both r_2 and υ are time-dependent, repeating on the orbital period P_{orb}. Let

$$F^{\ell,m}(t) = \left(\frac{r}{r_2}\right)^{\ell+1}\mathrm{e}^{-\mathrm{i}m\upsilon} \tag{22}$$

be a function that gathers together the time-dependent parts of the term in the potential associated with indices ℓ and m. Then, since $F^{\ell,m}(t)$ is periodic with an angular frequency Ω_{orb}, we can write it as a Fourier series:

$$F^{\ell,m}(t) = \left(\frac{r}{a}\right)^{\ell+1}\sum_{k=-\infty}^{\infty} X_{-k}^{-(\ell+1),-m}\,\mathrm{e}^{-\mathrm{i}kM}, \tag{23}$$

where M is the mean anomaly introduced in equation (4), and the so-called Hansen coefficients $X_k^{n,m}$ are defined by

$$\left(\frac{r_2}{a}\right)^{n}\mathrm{e}^{\mathrm{i}m\upsilon} = \sum_{k=-\infty}^{\infty} X_k^{n,m}\,\mathrm{e}^{\mathrm{i}kM} \tag{24}$$

(see, e.g., eqn. 1 of Hughes, 1981). Evaluation of the Hansen coefficients is discussed in Appendix B.1.

Combining eqns. (21, (24) and (22), the potential of the secondary becomes

$$\Phi_2(\mathbf{r};t) = -\frac{GM_2}{r}\sum_{\ell=0}^{\infty}\frac{4\pi}{2\ell+1}\left(\frac{r}{a}\right)^{\ell+1}\sum_{m=-\ell}^{\ell}\sum_{k=-\infty}^{\infty} Y_\ell^m(\theta,\phi)\; Y_\ell^{m*}(\pi/2,0)\; X_{-k}^{-(\ell+1),-m}\;\mathrm{e}^{-\mathrm{i}(kM+m\phi_{\mathrm{p}})}. \tag{25}$$

To aid in subsequent comparison against other work (e.g., Willems et al., 2010), this can also be written as

$$\Phi_2(\mathbf{r};t) = -\epsilon_{\mathrm{tide}}\frac{GM_1}{R_1}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{k=-\infty}^{\infty} c_{\ell,m,k}\;\left(\frac{r}{R_1}\right)^{\ell}\;Y_\ell^m(\theta,\phi)\;\mathrm{e}^{\mathrm{i}-(kM+m\phi_{\mathrm{p}})}, \tag{26}$$

where M_1 and R_1 are the mass and photospheric radius[2] of the primary, respectively, while

$$c_{\ell,m,k} \equiv \frac{4\pi}{2\ell+1}\left(\frac{R_1}{a}\right)^{\ell-2} Y_\ell^{m*}(\pi/2,0)\; X_{-k}^{-(\ell+1),-m}, \tag{27}$$

and

$$\epsilon_{\mathrm{tide}} \equiv \left(\frac{R_1}{a}\right)^{3}\frac{M_2}{M_1} \tag{28}$$

is a dimensionless parameter that measures the overall strength of the tidal forcing. The $c_{\ell,m,k}$ coefficients obey the relation

$$c_{\ell,-m,-k} = (-1)^m c_{\ell,m,k}, \tag{29}$$

where we've used eqn. (17) and the fact that $X_k^{n,m} = X_{-k}^{n,-m}$.

[2]Note that the photosphere is not necessarily coincident with the nominal outer boundary of the star.

2.4 Partial Potentials

Equation (26) can be expressed compactly as

$$\Phi_2(\mathbf{r};t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \Phi_2^{\ell,m,k}(\mathbf{r};t) \tag{30}$$

where we introduce the partial potentials (one for each $\{\ell, m, k\}$ combination) as

$$\Phi_2^{\ell,m,k}(\mathbf{r};t) = -\epsilon_{\rm tide} \frac{GM_1}{R_1} c_{\ell,m,k} \left(\frac{r}{R_1}\right)^{\ell} Y_\ell^m(\theta,\phi)\, \mathrm{e}^{-\mathrm{i}(kM+m\phi_{\rm p})} \tag{31}$$

Note that although the secondary potential is a purely-real quantity (see, e.g., eqn. 11), the individual partial potentials are complex quantities. To evaluate the overall response of the primary to the secondary's gravitational field, we calculate the response to each partial potential *separately*, and then (assuming we can neglect any non-linear effects) combine the responses via a simple superposition.

3 Tidal Response

3.1 Hydrodynamical Equations

To determine how the primary star responds to the gravitational forcing by a single partial potential (31), we start with the hydrodynamical equations. These comprise the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{32}$$

the momentum conservation equation including the forcing potential

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi - \nabla \Phi_2^{\ell,m,k}, \tag{33}$$

and the (internal) energy conservation equation

$$\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = \frac{Q}{T}, \tag{34}$$

where S is the specific entropy and Q is the specific rate of heating or cooling (due e.g., to energy generation, flux divergence, dissipation, etc; we defer a detailed discussion of Q until later). These equations are augmented by Poisson's equation for the (self-gravity) potential

$$\nabla^2 \Phi = 4\pi G \rho. \tag{35}$$

By construction, the forcing potential automatically satisfies Laplace's equation:

$$\nabla^2 \Phi_2^{\ell,m,k} = 0. \tag{36}$$

3.2 Equilibrium State

In the absence of the forcing potential, we assume the star is in a state of hydrostatic and thermal equilibrium, but we allow for the fact that it rotates about the polar axis with an angular velocity

$$\boldsymbol{\Omega}(\mathbf{r}) = \Omega \left[\cos\theta\, \mathbf{e}_r - \sin\theta\, \mathbf{e}_\theta\right] \tag{37}$$

(here, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit basis vectors in the radial and polar directions, and we'll use the azimuthal basis vector $\mathbf{e}_\phi$ later on). The equilibrium velocity field in the star follows as

$$\mathbf{v}_0 = \boldsymbol{\Omega} \times \mathbf{r}. \tag{38}$$

(here and subsequently, the subscript 0 indicates the equilibrium state). As discussed by Unno et al. (1989, their §32), this velocity field is solenoidal,

$$\nabla \cdot \mathbf{v}_0 = 0, \tag{39}$$

and has a directional derivative for some scalar f of

$$\mathbf{v}_0 \cdot \nabla f = \Omega \frac{\partial f}{\partial \phi}. \tag{40}$$

Using these results, and setting partial time derivatives to zero, the continuity equation (32) is trivially satisfied. The momentum equation becomes

$$\nabla P_0 = -\rho_0 \nabla \Phi_0 - \rho_0 \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{r}, \tag{41}$$

which is the equation of hydrostatic equilibrium with a correction for the centrifugal force. Finally, the energy equation (34) becomes

$$Q_0 = 0, \tag{42}$$

which is the equation of thermal equilibrium.

3.3 Linearized Equations

To solve equations (32–36), we regard $\Phi_2^{\ell,m,k}$ as a small perturbation to the static equilibrium state described above. We then write the perturbed state as a sum of equilibrium quantities and corresponding Eulerian (fixed position) perturbations, so that

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0 + \mathbf{v}' \\ \rho &= \rho_0 + \rho', \\ P &= P_0 + P', \\ S &= S_0 + S', \\ T &= T_0 + T', \\ Q &= Q_0 + Q' \\ \Phi &= \Phi_0 + \Phi' \end{aligned} \tag{43}$$

Substituting these expressions into the hydrodynamical equations (32–35), subtracting the hydrostatic and thermal equilibrium conditions (41,42), and discarding terms of second or higher-order in perturbed quantities, leads to the linearized equations governing the perturbations,

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right) \rho' + \nabla \cdot (\rho_0 \mathbf{v}') = 0 \tag{44}$$

$$\left[\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right) v_i'\right] \mathbf{e}_i + 2\boldsymbol{\Omega} \times \mathbf{v}' + (\mathbf{v}' \cdot \nabla \Omega)\, r \sin\theta\, \mathbf{e}_\phi = -\frac{1}{\rho_0} \nabla P' + \frac{\rho'}{\rho_0^2} \nabla P_0 - \nabla \Psi', \tag{45}$$

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}\right) S' + \mathbf{v}' \cdot \nabla S_0 = \frac{Q'}{T_0}, \tag{46}$$

$$\nabla^2 \Psi' = 4\pi G \rho'. \tag{47}$$

where we combine the self-gravity and forcing potentials into a single, 'total' potential perturbation

$$\Psi' \equiv \Phi' + \Phi_2^{\ell,m,k}. \tag{48}$$

3.4 Solution Forms

We now consider the form of the solutions to the linearized equations (44–47). For simplicity, let's assume that Ω is independent of position, corresponding to uniform rotation; and let's neglect the $2\boldsymbol{\Omega} \times \mathbf{v}'$ Coriolis term in the momentum equation (however, we'll return to it later on). The forcing potential $\Phi_2^{\ell,m,k}$ has an angular dependence proportional to Y_ℓ^m, and a time dependence proportional to $\mathrm{e}^{\mathrm{i}k\Omega_{\rm orb}t}$. Therefore, we assume trial solutions of the form

$$
\begin{aligned}
\xi_r(r,\theta,\phi;t) &= \sqrt{4\pi}\,\tilde{\xi}_r(r)\,Y_\ell^m(\theta,\phi)\,\mathrm{e}^{-\mathrm{i}(\sigma(t-t_{\rm p})+m\phi_{\rm p}]}, \\
\boldsymbol{\xi}_{\rm h}(r,\theta,\phi;t) &= \sqrt{4\pi}\,\tilde{\xi}_h(r)\,r\nabla_{\rm h}Y_\ell^m(\theta,\phi)\,\mathrm{e}^{-\mathrm{i}[\sigma(t-t_{\rm p})+m\phi_{\rm p}]}, \\
f'(r,\theta,\phi;t) &= \sqrt{4\pi}\,\tilde{f}'(r)\,Y_\ell^m(\theta,\phi)\,\mathrm{e}^{-\mathrm{i}[\sigma(t-t_{\rm p})+m\phi_{\rm p}]}.
\end{aligned}
\tag{49}
$$

Here,

$$
\sigma \equiv k\Omega_{\rm orb} \tag{50}
$$

is the forcing frequency in the inertial reference frame. Likewise, f stands for any perturbable scalar; ξ_r is the radial component of the displacement perturbation vector $\boldsymbol{\xi}$; $\boldsymbol{\xi}_{\rm h}$ is the corresponding horizontal (polar and azimuthal) part of this vector; and $\nabla_{\rm h}$ is the horizontal part of the spherical-polar gradient operator. The velocity perturbation follows from $\boldsymbol{\xi}$ as

$$
\mathbf{v}' = \left[\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)\xi_i\right]\mathbf{e}_i \tag{51}
$$

(this comes from eqn. 32.18 of Unno et al., 1989, with $\nabla\Omega$ set to zero). The factor of $\sqrt{4\pi}$ in the above expressions allows us to interpret, e.g., $\tilde{\xi}_r$ as the rms average of ξ_r over the sphere.

An important aside here: comparing the solution forms (49) against Townsend et al. (2018), we note that the former lack the Re[] operatores appearing in the latter. This is because we're forcing with complex-valued partial potentials (cf. eqn. 31), and so we must allow for complex-valued solutions. Of course, when we combine the solutions associated with each partial potential, the resulting perturbations will be purely real.

Substituting the perturbations (49) into the linearized equations, the continuity equation (44) becomes

$$
\frac{\delta\tilde{\rho}}{\rho_0} + \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\tilde{\xi}_r\right) - \frac{\ell(\ell+1)}{r}\,\tilde{\xi}_h = 0. \tag{52}
$$

The momentum equation (45) becomes

$$
\sigma_{\rm c}^2\,\tilde{\xi}_r = \frac{1}{\rho_0}\frac{\mathrm{d}\tilde{P}'}{\mathrm{d}r} + \frac{\tilde{\rho}'}{\rho_0}g_0 + \frac{\mathrm{d}\tilde{\Psi}'}{\mathrm{d}r} \tag{53}
$$

in the radial direction (where $g_0 \equiv \rho_0\,\mathrm{d}\Phi_0/\mathrm{d}r$ is the equilibrium scalar gravity), and

$$
\sigma_{\rm c}^2\,\tilde{\xi}_h = \frac{1}{r}\left(\frac{\tilde{P}'}{\rho_0} + \tilde{\Psi}'\right), \tag{54}
$$

in the horizontal direction. The energy equation (46) becomes

$$
-\mathrm{i}\sigma_{\rm c}\,\delta\tilde{S} = \frac{Q'}{T_0}; \tag{55}
$$

and finally, Poisson's equation (47) becomes

$$
\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\tilde{\Psi}'}{\mathrm{d}r}\right) - \frac{\ell(\ell+1)}{r^2}\tilde{\Psi}' = 4\pi G\tilde{\rho}'. \tag{56}
$$

In these expressions,

$$
\sigma_{\rm c} \equiv \sigma - m\Omega = k\Omega_{\rm orb} - m\Omega \tag{57}
$$

represents the forcing frequency as measured in the frame co-rotating with the primary. We use δf to indicate the Lagrangian perturbation associated with the Eulerian perturbation f'; in the linear analysis, the two are related by

$$
\delta f = f' + (\boldsymbol{\xi}\cdot\nabla)f_0. \tag{58}
$$

The forced oscillation equations (52–56) are essentially identical to the differential equations governing free oscillation (cf. Townsend et al., 2018), save that the potential perturbation $\tilde{\Phi}'$ has been replaced by the total potential perturbation $\tilde{\Psi}'$.

3.5 Boundary Conditions

The boundary conditions applied to the forced oscillation equations (52–56) are the same as for free oscillations, with one exception that we discuss here. On grounds of physicality, we require that the total potential and its gradient (the total gravity) remain continuous across the perturbed surface of the primary. These requirements can be expressed as

$$(\delta\Psi)_{\rm in} = (\delta\Psi)_{\rm ex} \tag{59}$$

and

$$[\delta(\nabla\Psi)]_{\rm in} = [\delta(\nabla\Psi)]_{\rm ex}\,, \tag{60}$$

where the subscripts 'in' ('ex') indicate evaluation just interior (exterior) to the perturbed surface.

Starting with the potential continuity requirement (59), we convert the Lagrangian perturbations to Eulerian ones to yield

$$(\Psi' + g_0\,\xi_r)_{\rm in} = (\Psi' + g_0\,\xi_r)_{\rm ex} \tag{61}$$

Since both ξ_r and g_0 are continuous across the boundary, this reduces to

$$(\Psi')_{\rm in} = (\Psi')_{\rm ex} \tag{62}$$

Turning now to the gravity continuity requirement (60), we likewise convert to Eulerian perturbations to yield

$$[\nabla\Psi' + (\boldsymbol{\xi}\cdot\nabla)\,g_0\,\mathbf{e}_r]_{\rm in} = [\nabla\Psi' + (\boldsymbol{\xi}\cdot\nabla)\,g_0\,\mathbf{e}_r]_{\rm ex}\,. \tag{63}$$

With a little bit of vector calculus, the directional derivatives appearing in this expression are expanded out to yield

$$\left[\nabla\Psi' + \xi_r\frac{{\rm d}g_0}{{\rm d}r}\,\mathbf{e}_r + \xi_\theta\frac{g_0}{r}\,\mathbf{e}_\theta + \xi_\phi\frac{g_0}{r}\,\mathbf{e}_\phi\right]_{\rm in} = \left[\nabla\Psi' + \xi_r\frac{{\rm d}g_0}{{\rm d}r}\,\mathbf{e}_r + \xi_\theta\frac{g_0}{r}\,\mathbf{e}_\theta + \xi_\phi\frac{g_0}{r}\,\mathbf{e}_\phi\right]_{\rm ex}. \tag{64}$$

From the horizontal parts of this equation, and the properties of the spherical harmonics, we see that ξ_θ and ξ_ϕ must be continuous across the boundary. For the remaining radial part, we use Poisson's equation to write

$$\frac{{\rm d}g_0}{{\rm d}r} = 4\pi r^2 G\rho_0 - \frac{2g_0}{r}, \tag{65}$$

and so the continuity requirement becomes

$$\left[\frac{\partial\Psi'}{\partial r} + \xi_r\left(4\pi G\rho_0 - \frac{2g_0}{r}\right)\right]_{\rm in} = \left[\frac{\partial\Psi'}{\partial r} + \xi_r\left(4\pi G\rho_0 - \frac{2g_0}{r}\right)\right]_{\rm ex} \tag{66}$$

Once again making use of the continuity of ξ_r and g_0, this simplifies to

$$\left[\frac{\partial\Psi'}{\partial r} + 4\pi G\rho_0\,\xi_r\right]_{\rm in} = \left[\frac{\partial\Psi'}{\partial r} + 4\pi G\rho_0\,\xi_r\right]_{\rm ex} \tag{67}$$

We now convert the pair (62,67) of continuity conditions into a single boundary condition for the total potential perturbation. The key step here is to note that outside the primary the potential perturbation Φ' is a solution of Laplace's equation that vanishes as $r \to \infty$; hence, we can write

$$\Phi' = (\Phi')_{\rm ex}\left(\frac{r}{r_{\rm b}}\right)^{-(\ell+1)}, \tag{68}$$

where $r_{\rm b}$ is the boundary radius (which does not have to equal the photosphere radius R_1). Likewise, the forcing partial potential (both inside and outside the primary) is a solution of Laplace's equation that vanishes as $r \to 0$; hence, we can write

$$\Phi_2^{\ell,m,k} = \left(\Phi_2^{\ell,m,k}\right)_{\rm ex}\left(\frac{r}{r_{\rm b}}\right)^{\ell}. \tag{69}$$

Combining these two expressions allows us to evaluate the radial derivative of Ψ' outside the boundary:

$$\left(\frac{\partial \Psi'}{\partial r}\right)_{\rm ex} = -\frac{\ell+1}{r_{\rm b}} (\Psi')_{\rm ex} + \frac{2\ell+1}{r_{\rm b}} \left(\Phi_2^{\ell,m,k}\right)_{\rm ex} \tag{70}$$

Applying the potential continuity condition (62), this can also be written

$$\left(\frac{\partial \Psi'}{\partial r}\right)_{\rm ex} = -\frac{\ell+1}{r_{\rm b}} (\Psi')_{\rm in} + \frac{2\ell+1}{r_{\rm b}} \left(\Phi_2^{\ell,m,k}\right)_{\rm in} \tag{71}$$

(where we've also used the continuity of $\Phi_2^{\ell,m,k}$ across the boundary). Finally, we substitute this into the gravity continuity condition (67) to yield

$$\left(\frac{\partial \Psi'}{\partial r} + \frac{\ell+1}{r_{\rm b}}\Psi + 4\pi G \rho_0\, \xi_r\right)_{\rm in} = \frac{2\ell+1}{r_{\rm b}} \left(\Phi_2^{\ell,m,k}\right)_{\rm in} \tag{72}$$

where we've assumed ρ_0 vanishes outside the boundary. This is the boundary condition we apply to the total potential perturbation inside the star. Expressed in terms of tilded quantities, and dropping the 'in' subscripts, it becomes

$$\frac{{\rm d}\tilde{\Psi}'}{{\rm d}r} + \frac{\ell+1}{r_{\rm b}}\tilde{\Psi}' + 4\pi G\rho_0\,\tilde{\xi}_r = \frac{2\ell+1}{r_{\rm b}}\tilde{\Phi}_2^{\ell,m,k}(r_{\rm b}) \tag{73}$$

where

$$\tilde{\Phi}_2^{\ell,m,k}(r_{\rm b}) = -\frac{\epsilon_{\rm tide}}{\sqrt{4\pi}}\frac{GM_1}{R_1} c_{\ell,m,k} \left(\frac{r_{\rm b}}{R_1}\right)^{\ell} \tag{74}$$

3.6 Static Tides

In some situations, the co-rotating frequency $\sigma_{\rm c}$ can be zero. This is the case, for instance, for forcing by the $k=0$ partial potentials when $\Omega = 0$; or by forcing by the $k=-m$ partial potentials when $\Omega = \Omega$ (synchronized rotation). For such *static* tides, the oscillation equations (53,54) associated with momentum conservation become

$$\frac{1}{\rho_0}\frac{{\rm d}\tilde{P}'}{{\rm d}r} = -\frac{\tilde{\rho}'}{\rho_0} g_0 - \frac{{\rm d}\tilde{\Psi}'}{{\rm d}r}, \tag{75}$$

and

$$\frac{\tilde{P}'}{\rho_0} = -\tilde{\Psi}'. \tag{76}$$

These equations describe how the star re-establishes hydrostatic equilibrium in response to a time-independent forcing potential. Using the latter to eliminate $\tilde{\Psi}'$ from the former, we find

$$\tilde{\rho}' \frac{{\rm d}P_0}{{\rm d}r} = \tilde{P}' \frac{{\rm d}\rho_0}{{\rm d}r}. \tag{77}$$

From the symmetry of this expression with respect to pressure and density, it likewise follows that

$$\delta\tilde{\rho}\frac{{\rm d}P_0}{{\rm d}r} = \delta\tilde{P}\frac{{\rm d}\rho_0}{{\rm d}r}. \tag{78}$$

To proceed further, we need to consider the oscillation equation (55) associated with energy conservation. At first glance, it seems when considering static tides that we should set $\sigma_{\rm c} = 0$ in this equation, implying that $Q' = 0$ even when $\delta\tilde{S} \neq 0$. However, as Willems et al. (2010) argue in their section 4, we can expect the timescale $\tau = 2\pi/\sigma_{\rm c}$ of a static tide to be *long* (effectively, infinite) when compared to the star's dynamical timescale $\tau_{\rm dyn}$, but *short* compared to the local thermal timescale $\tau_{\rm thm}$. Of course, this hierarchy cannot hold everywhere, since $\tau_{\rm thm}$ becomes small in the very outermost layers of the star where then density becomes negligible. However, beneath these layers (i.e., throughout the bulk of the stellar interior) the fact that $\tau \ll \tau_{\rm thm}$ means the tidal response will be almost adiabatic, with $\delta\tilde{S} \approx 0$.

In this adiabatic limit, the Lagrangian pressure and density perturbations are related by

$$\frac{\delta\tilde{\rho}}{\rho_0} = \frac{1}{\Gamma_1}\frac{\delta\tilde{P}}{P_0}, \tag{79}$$

where

$$\Gamma_1 \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S \tag{80}$$

is the first adiabatic exponent. Combining this with equation (78) to eliminate the density perturbation, we find that

$$N^2 \frac{\delta\tilde{\rho}}{\rho_0} = 0, \tag{81}$$

where we introduce the Brunt-Väisäla frequency N via

$$N^2 = \frac{g_0}{r}\left(\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln P_0}{\mathrm{d}\ln r} - \frac{\mathrm{d}\ln \rho_0}{\mathrm{d}\ln r}\right). \tag{82}$$

Outside of convection zones, $N^2 \neq 0$ and so we arrive at the result

$$\frac{\delta\tilde{\rho}}{\rho_0} = 0; \tag{83}$$

that is, the tidal response is incompressible. An immediate corollary, via the adiabatic condition (79), is that $\delta\tilde{P}/P_0$ is also zero.

The case inside convection zones, where N^2 is very close to zero, is trickier to handle. Then, we have to take into account how close to zero it is, versus how close to zero the entropy perturbation is. Let's just set this issue aside for now, and assume that the Lagrangian density and pressure perturbations vanish, both outside *and* inside convection zones. This implies

$$\tilde{P}' = -\frac{\mathrm{d}P_0}{\mathrm{d}r}\tilde{\xi}_r \tag{84}$$

and

$$\tilde{\rho}' = -\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\tilde{\xi}_r. \tag{85}$$

Combining the first of these with equation (76), we find

$$\tilde{\xi}_r = -\frac{\tilde{\Psi}'}{g_0}. \tag{86}$$

Then, combining this with Poisson's equation (56) gives

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\tilde{\Psi}'}{\mathrm{d}r}\right) - \frac{\ell(\ell+1)}{r^2}\tilde{\Psi}' = 4\pi G\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\frac{\tilde{\Psi}'}{g_0}. \tag{87}$$

This is the single, second-order differential equation we must solve in order to model static tides.

4 Secular Evolution

4.1 Basic Expressions

The instantaneous rate of change in orbital elements is related to the perturbing function $\mathcal{R}$ (the 1st order Hamiltonian) via

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{2}{\Omega_{\mathrm{orb}}^2 a}\frac{\partial\mathcal{R}}{\partial t_{\mathrm{p}}}, \tag{88}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{1}{\Omega_{\mathrm{orb}}a^2 e}\left[\frac{1-e^2}{\Omega_{\mathrm{orb}}}\frac{\partial\mathcal{R}}{\partial t_{\mathrm{p}}} + (1-e^2)^{1/2}\frac{\partial\mathcal{R}}{\partial\varpi}\right], \tag{89}$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = \frac{1}{\Omega_{\mathrm{orb}}a^2}\frac{(1-e^2)^{1/2}}{e}\frac{\partial\mathcal{R}}{\partial e} \tag{90}$$

(e.g., Sterne, 1960), where $\mathcal{R}$ is the perturbing function, and ϖ is the longitude of periastron (the compound angle defined as the sum of the longitude of the ascending node Ω and the argument of periastron ω). Averaging over one orbit, we can transform these expressions into equivalent ones for the secular rates of change of the orbital elements:

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_{\mathrm{sec}} = -\frac{2}{\Omega_{\mathrm{orb}}^2 a} \left\langle \frac{\partial \mathcal{R}}{\partial t_{\mathrm{p}}} \right\rangle, \tag{91}$$

$$\left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_{\mathrm{sec}} = -\frac{1}{\Omega_{\mathrm{orb}} a^2 e} \left[\frac{1-e^2}{\Omega_{\mathrm{orb}}} \left\langle \frac{\partial \mathcal{R}}{\partial t_{\mathrm{p}}} \right\rangle + (1-e^2)^{1/2} \left\langle \frac{\partial \mathcal{R}}{\partial \varpi} \right\rangle \right], \tag{92}$$

$$\left(\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right)_{\mathrm{sec}} = \frac{1}{\Omega_{\mathrm{orb}} a^2} \frac{(1-e^2)^{1/2}}{e} \left\langle \frac{\partial \mathcal{R}}{\partial e} \right\rangle, \tag{93}$$

where

$$\langle f \rangle \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} f \,\mathrm{d}M = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-e^2)^{3/2}}{(1+e\cos\upsilon)^2} f \,\mathrm{d}\upsilon \tag{94}$$

is the temporal average over one orbit (the second equality follows from eqn. 176).

The perturbing function $\mathcal{R}$ in the above expressions can be written in terms of the perturbation to the self-gravity of the primary *at the location of the secondary*:

$$\mathcal{R} = -\frac{M_1 + M_2}{M_1} \Phi'(r_2, \pi/2, \phi_2; t). \tag{95}$$

4.2 Evaluating the Perturbing Function

To evaluate the perturbing function (95), we expand Φ' as

$$\Phi'(r_2, \pi/2, \phi_2; t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \Phi'_{\ell,m,k}(r_2, \pi/2, \phi_2; t). \tag{96}$$

Here, $\Phi'_{\ell,m,k}$ is the perturbation to the primary's self-gravity potential, arising as a response to tidal forcing by the partial potential $\Phi_2^{\ell,m,k}$. In the preceding sections, we denoted this simply as Φ', but we now need to add indices to distinguish the different responses. Thus, the perturbing function becomes

$$\mathcal{R} = -\frac{M_1 + M_2}{M_1} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \Phi'_{\ell,m,k}(r_2, \pi/2, \phi_2; t). \tag{97}$$

Remembering that $\Phi'_{\ell,m,k}$ satisfies Laplace's equation outside the star, we rewrite this as

$$\mathcal{R} = -\frac{M_1 + M_2}{M_1} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \left(\frac{r_2}{r_{\mathrm{b}}}\right)^{-(\ell+1)} \sqrt{4\pi}\, \tilde{\Phi}'_{\ell,m,k}(r_{\mathrm{b}})\, Y_\ell^m(\pi/2, \phi_2)\, \mathrm{e}^{-\mathrm{i}(kM + m\phi_{\mathrm{p}})}. \tag{98}$$

When we solve the oscillation equations (52–56), we don't find $\tilde{\Phi}'_{\ell,m,k}(r_{\mathrm{b}})$ directly, but can calculate it indirectly from the surface values of the total potential perturbation and the forcing potential (cf. eqn. 74):

$$\tilde{\Phi}'_{\ell,m,k}(r_{\mathrm{b}}) = \tilde{\Psi}'_{\ell,m,k}(r_{\mathrm{b}}) + \frac{\epsilon_{\mathrm{tide}}}{\sqrt{4\pi}} \frac{GM_1}{R_1} c_{\ell,m,k} \left(\frac{r_{\mathrm{b}}}{R_1}\right)^{\ell}. \tag{99}$$

Therefore, the perturbing function becomes

$$\mathcal{R} = 2\frac{G(M_1 + M_2)}{R_1} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=-\infty}^{\infty} \left(\frac{r_2}{r_{\mathrm{b}}}\right)^{-(\ell+1)} c_{\ell,m,k}\, F_{\ell,m,k}\, Y_\ell^m(\pi/2, \phi_2)\, \mathrm{e}^{-\mathrm{i}(kM + m\phi_{\mathrm{p}})}, \tag{100}$$

where, following Willems et al. (2010), we define

$$F_{\ell,m,k} \equiv -\frac{1}{2} \frac{\sqrt{4\pi}}{\epsilon_{\rm tide} c_{\ell,m,k}} \frac{\tilde{\Phi}'_{\ell,m,k}(r_{\rm b})}{GM_1/R_1} = -\frac{1}{2} \left[\frac{\sqrt{4\pi}}{\epsilon_{\rm tide} c_{\ell,m,k}} \frac{\tilde{\Psi}'_{\ell,m,k}(r_{\rm b})}{GM_1/R_1} + \left(\frac{r_{\rm b}}{R_1} \right)^{\ell} \right]. \tag{101}$$

In the interests of computational efficiency, it is useful to transform the sum over all k in eqn. (100) into one over $k \geq 0$. To do this, we first note that

$$\tilde{\Phi}_2^{\ell,-m,-k} = (-1)^m \left[\tilde{\Phi}_2^{\ell,m,k} \right]^*, \qquad (\sigma_{\rm c})_{\ell,-m,-k} = -(\sigma_{\rm c})_{\ell,m,k}. \tag{102}$$

By taking the complex conjugate of the oscillation equations, we therefore find that

$$f'_{\ell,-m,-k} = (-1)^m \left[f'_{\ell,m,k} \right]^* \tag{103}$$

for any perturbation $f'_{\ell,m,k}$. In particular,

$$\tilde{\Phi}'_{\ell,-m,-k} = (-1)^m \left[\tilde{\Phi}'_{\ell,m,k} \right]^*, \tag{104}$$

and so

$$F_{\ell,-m,-k} = F^*_{\ell,m,k} \tag{105}$$

(note that the -1 term has cancelled). Then, pairing each term $\{\ell, m, k\}$ in eqn. (100) with its $\{\ell, -m, -k\}$ counterpart, we find after some algebra that

$$\mathcal{R} = 4 \frac{G(M_1+M_2)}{R_1} \frac{M_2}{M_1} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{k=0}^{\infty} \left(\frac{R_1}{a} \right)^{\ell+4} \left(\frac{r_{\rm b}}{R_1} \right)^{\ell+1} Y_\ell^m(\pi/2, 0)$$
$$\times \left(\frac{r_2}{a} \right)^{-(\ell+1)} \kappa_{\ell,m,k}\, c_{\ell,m,k} \left| F_{\ell,m,k} \right| \cos\left[m(\phi_2 - \phi_{\rm p}) - kM + \gamma_{\ell,m,k} \right]. \tag{106}$$

Here,

$$\gamma_{\ell,m,k} \equiv \arg(F_{\ell,m,k}) \tag{107}$$

is the phase lag/lead of the tide, and

$$\kappa_{\ell,m,k} = \begin{cases} 1/2 & k = 0, m = 0 \\ 1 & k = 0, 0 \leq m \leq \ell \\ 1 & k > 0, -\ell \leq m \leq \ell \\ 0 & \text{otherwise} \end{cases}. \tag{108}$$

Note that the Re[...] operator does not appear these expressions; all terms are now real.

4.3 Partials of the Perturbing Function

The rates of change given in eqns. (91–93) depend on partial derivatives of the perturbing function with respect to $t_{\rm p}$, ϖ and e; whereas, the expansion (106) for the perturbing function expresses it in terms of r_2, ϕ_2 and t. We can use the chain rule here to help out, but it is important to be clear about which variables are being held constant when taking partial derivatives. Specifically, the partials of $\mathcal{R}$ with respect to the orbital elements ($t_{\rm p}$, ϖ, e and a) are taken with the other orbital elements *and the time t* held constant.

With this in mind, we arrive at the relations

$$\begin{aligned}
\frac{\partial \mathcal{R}}{\partial t_{\rm p}} &= \left(\frac{\partial \mathcal{R}}{\partial r_2} \right)_{\phi_2,t} \left(\frac{\partial r_2}{\partial t_{\rm p}} \right)_{\varpi,e,a} + \left(\frac{\partial \mathcal{R}}{\partial \phi_2} \right)_{r_2,t} \left(\frac{\partial \phi_2}{\partial t_{\rm p}} \right)_{\varpi,e,a} \\
\frac{\partial \mathcal{R}}{\partial \varpi} &= \left(\frac{\partial \mathcal{R}}{\partial r_2} \right)_{\phi_2,t} \left(\frac{\partial r_2}{\partial \varpi} \right)_{t_{\rm p},e,a} + \left(\frac{\partial \mathcal{R}}{\partial \phi_2} \right)_{r_2,t} \left(\frac{\partial \phi_2}{\partial \varpi} \right)_{t_{\rm p},e,a} \\
\frac{\partial \mathcal{R}}{\partial e} &= \left(\frac{\partial \mathcal{R}}{\partial r_2} \right)_{\phi_2,t} \left(\frac{\partial r_2}{\partial e} \right)_{t_{\rm p},\varpi,a} + \left(\frac{\partial \mathcal{R}}{\partial \phi_2} \right)_{r_2,t} \left(\frac{\partial \phi_2}{\partial e} \right)_{t_{\rm p},\varpi,a} \\
\frac{\partial \mathcal{R}}{\partial a} &= \left(\frac{\partial \mathcal{R}}{\partial r_2} \right)_{\phi_2,t} \left(\frac{\partial r_2}{\partial a} \right)_{t_{\rm p}\varpi,e} + \left(\frac{\partial \mathcal{R}}{\partial \phi_2} \right)_{r_2,t} \left(\frac{\partial \phi_2}{\partial a} \right)_{t_{\rm p},\varpi,e}
\end{aligned} \tag{109}$$

The partial derivatives of $\mathcal{R}$ with respect to r_2 and ϕ_2 are easily obtained from eqn. (106):

$$\frac{\partial \mathcal{R}}{\partial r_2} = -4a\,\Omega_{\rm orb}^2\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\,\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\,Y_\ell^m(\pi/2,0)\,\kappa_{\ell,m,k}\,c_{\ell,m,k}\,|F_{\ell,m,k}|\times \\ (\ell+1)\left(\frac{1+e\cos\upsilon}{1-e^2}\right)^{\ell+2}\,\cos\left[m(\phi_2-\phi_{\rm p})-kM+\gamma_{\ell,m,k}\right] \tag{110}$$

and

$$\frac{\partial \mathcal{R}}{\partial \phi_2} = -4a^2\,\Omega_{\rm orb}^2\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\,\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\,Y_\ell^m(\pi/2,0)\,\kappa_{\ell,m,k}\,c_{\ell,m,k}\,|F_{\ell,m,k}|\times \\ m\left(\frac{1+e\cos\upsilon}{1-e^2}\right)^{\ell+1}\,\sin\left[m(\phi_2-\phi_{\rm p})-kM+\gamma_{\ell,m,k}\right], \tag{111}$$

where, as before, the summation is over $\ell\in[0,\infty]$, $m\in[-\ell,\ell]$ and $k\in[0,\infty]$. In deriving these expressions, we've taken advantage of Kepler's third law

$$G(M_1+M_2)=a^3\Omega_{\rm orb}^2, \tag{112}$$

and eliminated r_2/a via equation (3).

The remaining partial derivatives require a little more effort to derive. With the help of *Mathematica*, we can show that

$$\begin{aligned}\left(\frac{\partial r_2}{\partial t_{\rm p}}\right)_{\varpi,e,a} &= -\Omega_{\rm orb}\frac{a\,e\sin\upsilon}{(1-e^2)^{1/2}}, &\left(\frac{\partial r_2}{\partial \varpi}\right)_{t_{\rm p},e,a} &= 0, \\ \left(\frac{\partial r_2}{\partial e}\right)_{t_{\rm p},\varpi,a} &= -a\cos\upsilon, &\left(\frac{\partial r_2}{\partial a}\right)_{t_{\rm p},\varpi,e} &= \frac{1-e^2}{1+e\cos\upsilon},\end{aligned} \tag{113}$$

and

$$\begin{aligned}\left(\frac{\partial \phi_2}{\partial t_{\rm p}}\right)_{\varpi,e,a} &= -\Omega_{\rm orb}\frac{(1+e\cos\upsilon)^2}{(1-e^2)^{3/2}}, &\left(\frac{\partial \phi_2}{\partial \varpi}\right)_{t_{\rm p},e,a} &= 1, \\ \left(\frac{\partial \phi_2}{\partial e}\right)_{t_{\rm p},\varpi,a} &= \frac{(2+e\cos\upsilon)\sin\upsilon}{1-e^2}, &\left(\frac{\partial \phi_2}{\partial a}\right)_{t_{\rm p},\varpi,e} &= 0.\end{aligned} \tag{114}$$

Armed with these expressions, we're now in a position to evaluate the partial derivatives of $\mathcal{R}$ appearing in eqns. (91–93). After a fair bit of algebra, we find the following:

$$\frac{\partial \mathcal{R}}{\partial t_{\rm p}} = 4a^2\,\Omega_{\rm orb}^3\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\,\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\,\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\frac{c_{\ell,m,k}\,Y_\ell^m(\pi/2,0)}{(1-e^2)^{\ell+5/2}}\times \\ \big[(\ell+1)e(1+e\cos\upsilon)^{\ell+2}\,\sin\upsilon\,\cos\left(m\upsilon-kM+\gamma_{\ell,m,k}\right)+ \\ m(1+e\cos\upsilon)^{\ell+3}\,\sin\left(m\upsilon-kM+\gamma_{\ell,m,k}\right)\big], \tag{115}$$

$$\frac{\partial \mathcal{R}}{\partial \varpi} = -4a^2\,\Omega_{\rm orb}^2\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\,\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\,\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\frac{c_{\ell,m,k}\,Y_\ell^m(\pi/2,0)}{(1-e^2)^{\ell+1}}\times \\ \big[m(1+e\cos\upsilon)^{\ell+1}\,\sin\left(m\upsilon-kM+\gamma_{\ell,m,k}\right)\big], \tag{116}$$

$$\frac{\partial \mathcal{R}}{\partial e} = 4a^2\,\Omega_{\rm orb}^2\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\,\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\,\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\frac{c_{\ell,m,k}\,Y_\ell^m(\pi/2,0)}{(1-e^2)^{\ell+2}}\times \\ \big[(\ell+1)(1+e\cos\upsilon)^{\ell+2}\,\cos\upsilon\,\cos\left(m\upsilon-kM+\gamma_{\ell,m,k}\right)- \\ m(1+e\cos\upsilon)^{\ell+1}\,(2+e\cos\upsilon)\,\sin\upsilon\,\sin\left(m\upsilon-kM+\gamma_{\ell,m,k}\right)\big], \tag{117}$$

$$\frac{\partial \mathcal{R}}{\partial a} = -4a\,\Omega_{\rm orb}^2\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\frac{c_{\ell,m,k}\,Y_\ell^m(\pi/2,0)}{(1-e^2)^{\ell+1}}\times \\ \left[(\ell+1)(1+e\cos\upsilon)^{\ell+1}\,\cos\left(m\upsilon - kM + \gamma_{\ell,m,k}\right)\right]. \quad (118)$$

The next step is to calculate averages of these partials over one orbit. Using the second equality in eqn. (94), and making judicious use of trig formulae together with parity relations, we find

$$\left\langle\frac{\partial \mathcal{R}}{\partial t_{\rm p}}\right\rangle = -4a^2\,\Omega_{\rm orb}^3\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\frac{c_{\ell,m,k}\,Y_\ell^m(\pi/2,0)}{(1-e^2)^{\ell+1}}\times \\ \frac{1}{\pi}\left[(\ell+1)e\int_0^\pi(1+e\cos\upsilon)^\ell\,\sin\upsilon\,\sin\left(m\upsilon-kM\right)\,{\rm d}\upsilon - \right. \\ \left. m\int_0^\pi(1+e\cos\upsilon)^{\ell+1}\cos\left(m\upsilon-kM\right)\,{\rm d}\upsilon\right]\sin\gamma_{\ell,m,k}, \quad (119)$$

$$\left\langle\frac{\partial \mathcal{R}}{\partial \varpi}\right\rangle = -4a^2\,\Omega_{\rm orb}^2\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\frac{c_{\ell,m,k}\,Y_\ell^m(\pi/2,0)}{(1-e^2)^{\ell-1/2}}\times \\ \frac{1}{\pi}\left[m\int_0^\pi(1+e\cos\upsilon)^{\ell-1}\,\cos\left(m\upsilon-kM\right)\,{\rm d}\upsilon\right]\sin\gamma_{\ell,m,k}, \quad (120)$$

$$\left\langle\frac{\partial \mathcal{R}}{\partial e}\right\rangle = 4a^2\,\Omega_{\rm orb}^2\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\frac{c_{\ell,m,k}\,Y_\ell^m(\pi/2,0)}{(1-e^2)^{\ell+1/2}}\times \\ \frac{1}{\pi}\left[(\ell+1)\int_0^\pi(1+e\cos\upsilon)^\ell\,\cos\upsilon\,\cos\left(m\upsilon-kM\right)\,{\rm d}\upsilon - \right. \\ \left. m\int_0^\pi(1+e\cos\upsilon)^{\ell-1}\,(2+e\cos\upsilon)\,\sin\upsilon\,\sin\left(m\upsilon-kM\right)\,{\rm d}\upsilon\right]\cos\gamma_{\ell,m,k}, \quad (121)$$

$$\left\langle\frac{\partial \mathcal{R}}{\partial a}\right\rangle = -4a\,\Omega_{\rm orb}^2\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\frac{c_{\ell,m,k}\,Y_\ell^m(\pi/2,0)}{(1-e^2)^{\ell-1/2}}\times \\ \frac{1}{\pi}\left[(\ell+1)\int_0^\pi(1+e\cos\upsilon)^{\ell-1}\,\cos\left(m\upsilon-kM\right)\,{\rm d}\upsilon\right]\cos\gamma_{\ell,m,k}. \quad (122)$$

4.4 Secular Rates of Change

We're now in a position to put everything together. Combining equations (91–93) with (119–122), we find

$$\left(\frac{{\rm d}a}{{\rm d}t}\right)_{\rm sec} = 4a\,\Omega_{\rm orb}\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\,\sin\gamma_{\ell,m,k}\,G^{(2)}_{\ell,m,k}, \quad (123)$$

$$\left(\frac{{\rm d}e}{{\rm d}t}\right)_{\rm sec} = 4\Omega_{\rm orb}\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\,\sin\gamma_{\ell,m,k}\,G^{(3)}_{\ell,m,k}, \quad (124)$$

$$\left(\frac{{\rm d}\varpi}{{\rm d}t}\right)_{\rm sec} = 4\Omega_{\rm orb}\,\frac{M_2}{M_1}\sum_{\ell,m,k}\left(\frac{R_1}{a}\right)^{\ell+3}\left(\frac{r_{\rm b}}{R_1}\right)^{\ell+1}\kappa_{\ell,m,k}\,|F_{\ell,m,k}|\,\cos\gamma_{\ell,m,k}G^{(1)}_{\ell,m,k}. \quad (125)$$

Here we define

$$G^{(1)}_{\ell,m,k} \equiv \frac{c_{\ell,m,k}\, Y^m_\ell(\pi/2,0)}{e(1-e^2)^\ell} \times \frac{1}{\pi}\left[(\ell+1)\int_0^\pi (1+e\cos\upsilon)^\ell \cos(m\upsilon - kM)\cos\upsilon\,\mathrm{d}\upsilon - m\int_0^\pi (1+e\cos\upsilon)^{\ell-1}(2+e\cos\upsilon)\sin\upsilon\sin(m\upsilon-kM)\;\mathrm{d}\upsilon\right], \quad (126)$$

$$G^{(2)}_{\ell,m,k} \equiv 2\frac{c_{\ell,m,k}\, Y^m_\ell(\pi/2,0)}{(1-e^2)^{\ell+1}} \times \frac{1}{\pi}\left[(\ell+1)e\int_0^\pi (1+e\cos\upsilon)^\ell\;\sin\upsilon\;\sin(m\upsilon-kM)\;\mathrm{d}\upsilon - m\int_0^\pi (1+e\cos\upsilon)^{\ell+1}\cos(m\upsilon-kM)\;\mathrm{d}\upsilon\right], \quad (127)$$

and

$$G^{(3)}_{\ell,m,k} \equiv \frac{c_{\ell,m,k}\, Y^m_\ell(\pi/2,0)}{e(1-e^2)^\ell} \times \frac{1}{\pi}\left[(\ell+1)e\int_0^\pi (1+e\cos\upsilon)^\ell\;\sin\upsilon\;\sin(m\upsilon-kM)\;\mathrm{d}\upsilon - m\int_0^\pi (1+e\cos\upsilon)^{\ell-1}\left\{(1+e\cos\upsilon)^2-(1-e^2)\right\}\cos(m\upsilon-kM)\;\mathrm{d}\upsilon\right]; \quad (128)$$

these are equivalent to the expressions given by Smeyers et al. (1998) and Willems et al. (2010).

5 Stellar Torque

In this section, let's derive the local and global torque on the star arising from the companion. Some of the formalism is already developed in Townsend et al. (2018), and we'll rely on that extensively.

5.1 Transport Equation

We start from eqn. (6) of Townsend et al. (2018) to write the angular momentum transport equation as

$$\frac{\partial\tau}{\partial r} = -\frac{\partial}{\partial r}\left(r^2\rho\int_0^{2\pi}\int_0^\pi r\sin\theta\, v'_\phi\, v'_r\;\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\right) - r^2\int_0^{2\pi}\int_0^\pi \rho'\frac{\partial\Psi'}{\partial\phi}\;\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \quad (129)$$

Here,

$$\frac{\partial\tau}{\partial r} = 4\pi r^2\frac{\partial j}{\partial t}, \quad (130)$$

is the local *instantaneous* differential torque, being the rate of change of angular momentum per unit radius; while

$$j \equiv \frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi r\sin\theta\rho v_\phi\;\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi r^2\sin^2\theta\rho\Omega\;\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \quad (131)$$

is the angle-averaged angular momentum per unit volume. In these expressions, we've explicitly written averages over θ and ϕ rather than using overbars and angle brackets, as the latter have already been used to denote secular averages.

The integrand in the second term on the right-hand side of the transport equation must be evaluated via a double sum:

$$\int_0^{2\pi}\int_0^{\pi} \rho' \frac{\partial \Psi'}{\partial \phi} \,\sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi = 4\pi \int_0^{2\pi}\int_0^{\pi} \sum_{\ell,m,k,\ell',m',k'} \left\{ \tilde{\rho}'_{\ell,m,k} Y_\ell^m \,\mathrm{e}^{-\mathrm{i}[\sigma_{\ell,m,k}(t-t_\mathrm{p})+m\phi_\mathrm{p}]} \right\} \times \\ \left\{ \tilde{\Psi}'_{\ell',m',k'} \frac{\partial Y_{\ell'}^{m'}}{\partial \phi} \mathrm{e}^{-\mathrm{i}[\sigma_{\ell',m',k'}(t-t_\mathrm{p})+m'\phi_\mathrm{p}]} \right\} \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi, \quad (132)$$

where the sums extend over $k, k' \in [-\infty, \infty]$. Making use of the orthogonality of the spherical harmonics, together with eqn. (103), this simplifies to

$$\int_0^{2\pi}\int_0^{\pi} \rho' \frac{\partial \Psi'}{\partial \phi} \,\sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi = -\sum_{\ell,m,k,k'} 4\pi \,\mathrm{i}m \,\tilde{\rho}'_{\ell,m,k} \,\tilde{\Psi}'^{*}_{\ell,m,k'} \,\mathrm{e}^{-\mathrm{i}(\sigma_{\ell,m,k}-\sigma_{\ell,m,k'})(t-t_\mathrm{p})} \quad (133)$$

A similar process applied to the integrand in the first term yields

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta \, v'_\phi \, v'_r \,\sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi = -\sum_{\ell,m,k,k'} 4\pi \,\mathrm{i}m \,\sigma_{\ell,m,k} \,\tilde{\xi}_{r,\ell,m,k} \,\sigma_{\ell,m,k'} \,\tilde{\xi}^{*}_{h,\ell,m,k'} \,\mathrm{e}^{-\mathrm{i}(\sigma_{\ell,m,k}-\sigma_{\ell,m,k'})(t-t_\mathrm{p})} \quad (134)$$

In these two expressions, the remaining double sum over k and k' can be collapsed by averaging over one orbital period. This leads ultimately for an expression for the time-averaged differential torque,

$$\left\langle \frac{\partial \tau}{\partial r} \right\rangle = \sum_{\ell,m,k} 4\pi \,\mathrm{i}m \left[\sigma_{\ell,m,k}^2 \frac{\partial}{\partial r} \left(r^3 \rho \,\tilde{\xi}_{r,\ell,m,k} \,\tilde{\xi}^{*}_{h,\ell,m,k} \right) + r^2 \,\tilde{\rho}'_{\ell,m,k} \,\tilde{\Phi}'^{*}_{\ell,m,k} \right]. \quad (135)$$

We can view this torque as a combination of partial time-averaged differential torques, each arising from a corresponding partial potential:

$$\left\langle \frac{\partial \tau}{\partial r} \right\rangle = \sum_{\ell,m,k} \left\langle \frac{\partial \tau}{\partial r} \right\rangle_{\ell,m,k} \quad (136)$$

where

$$\left\langle \frac{\partial \tau}{\partial r} \right\rangle_{\ell,m,k} = 4\pi \,\mathrm{i}m \left[\sigma_{\ell,m,k}^2 \frac{\partial}{\partial r} \left(r^3 \rho \,\tilde{\xi}_{r,\ell,m,k} \,\tilde{\xi}^{*}_{h,\ell,m,k} \right) + r^2 \,\tilde{\rho}'_{\ell,m,k} \,\tilde{\Phi}'^{*}_{\ell,m,k} \right] \quad (137)$$

5.2 Global Torque

To obtain an expression for the global (net) torque on the star, we integrate eqn. (137) from the center of the star to (just outside) the outer boundary. The first (r-derivative) term in the brackets integrates to zero because the density vanishes at the boundary. In the second term, we use Poisson's equation (56) to eliminate the $\tilde{\rho}'_{\ell,m,k}$ perturbations. The result is a time-averaged global torque

$$\langle \tau \rangle = \sum_{\ell,m,k} \langle \tau \rangle_{\ell,m,k} \quad (138)$$

where

$$\langle \tau \rangle_{\ell,m,k} = \int_0^{r_\mathrm{b}} \left\langle \frac{\partial \tau}{\partial r} \right\rangle_{\ell,m,k} \mathrm{d}r = \frac{\mathrm{i}m}{G} \int_0^{r_\mathrm{b}} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \tilde{\Psi}_{\ell,m,k}}{\partial r} \right) \tilde{\Psi}'^{*}_{\ell,m,k} \,\mathrm{d}r. \quad (139)$$

is a partial time-averaged global torque. Integrating by parts, the latter becomes

$$\langle \tau \rangle_{\ell,m,k} = \frac{\mathrm{i}m}{G} \left[r^2 \frac{\partial \tilde{\Psi}_{\ell,m,k}}{\partial r} \,\tilde{\Psi}'^{*}_{\ell,m,k} \right]_\mathrm{ex}. \quad (140)$$

(where, as before, the 'ex' subscript indicates evaluation just exterior to the boundary). We can use eqn. (71) to eliminate the potential derivative, obtaining

$$\langle\tau\rangle_{\ell,m,k} = \frac{\mathrm{i}m}{G} \left[(2\ell+1) r\, \tilde{\Phi}_2^{\ell,m,k} \tilde{\Psi}'^{*}_{\ell,m,k}\right]_{\mathrm{in}}. \tag{141}$$

(note the switch from 'ex' to 'in'). After a bit of tedious algebra, this can also be expressed as

$$\langle\tau\rangle_{\ell,m,k} = 2\mathrm{i}\Omega_{\mathrm{orb}} \sqrt{\frac{GM_1^2 M_2^2}{M_1+M_2}} \frac{M_2}{M_1} a^{1/2} \left(\frac{R_1}{a}\right)^{\ell+3} \left(\frac{r_{\mathrm{b}}}{R_1}\right)^{\ell+1} F^{*}_{\ell,m,k}\, G^{(4)}_{\ell,m,k}, \tag{142}$$

where

$$G^{4}_{\ell,m,k} \equiv m \frac{2\ell+1}{4\pi} \left(\frac{R_1}{a}\right)^{-\ell+2} c^2_{\ell,m,k}. \tag{143}$$

Combining the above expressions, obtain the final result for the time-averaged global torque,

$$\tau = 4\Omega_{\mathrm{orb}} \sqrt{\frac{GM_1^2 M_2^2}{M_1+M_2}} \frac{M_2}{M_1} a^{1/2} \sum_{\ell,m,k} \left(\frac{R_1}{a}\right)^{\ell+3} \left(\frac{r_{\mathrm{b}}}{R_1}\right)^{\ell+1} \kappa_{\ell,m,k}\, |F_{\ell,m,k}|\, \sin\gamma_{\ell,m,k}\, G^{(4)}_{\ell,m,k}, \tag{144}$$

where $\kappa_{\ell,m,k}$ is the same as before (eqn. 108), and the summations are now over $\ell \in [0,\infty]$, $m \in [-\ell,\ell]$ and $k \in [0,\infty]$.

5.3 Local Torques

To evaluate the partial time-averaged local (differential) torques, eqn. (137) is inconvenient due to the radial derivative in the first term on the right-hand side. We can use eqns. (52–54) to rewrite this term. After a fair amount of algebra, we arrive at the result

$$\left\langle\frac{\partial\tau}{\partial r}\right\rangle_{\ell,m,k} = -4\pi\, \mathrm{i}m\, r^2 \left\{\frac{\delta\tilde{\rho}_{\ell,m,k}}{\rho} \tilde{P}'^{*}_{\ell,m,k} + g\tilde{\xi}_{r,\ell,m,k}\tilde{\rho}'^{*}_{\ell,m,k} - \rho\sigma^2_{\ell,m,k}\left[|\tilde{\xi}_{r,\ell,m,k}|^2 + \ell(\ell+1)|\tilde{\xi}_{h,\ell,m,k}|^2\right]\right\} \tag{145}$$

Combining this back with eqn. (135), and with a little further algebra, the time-averaged differential torque is

$$\left\langle\frac{\partial\tau}{\partial r}\right\rangle = 8\pi r^2 \sum_{\ell,m,k} m\, \kappa_{\ell,m,k}\, |B_{\ell,m,k}|\, \sin\beta_{\ell,m,k} \tag{146}$$

where

$$B_{\ell,m,k} = \frac{\delta\tilde{\rho}_{\ell,m,k}}{\rho} \delta P^{*}_{\ell,m,k}, \tag{147}$$

$$\beta_{\ell,m,k} = \arg(B_{\ell,m,k}), \tag{148}$$

and the summations are now over $\ell \in [0,\infty]$, $m \in [-\ell,\ell]$ and $k \in [0,\infty]$.

The quantity $B_{\ell,m,k}$ is closely related to the differental work[3]

$$\frac{\mathrm{d}W_{\ell,m,k}}{\mathrm{d}r} = -4\pi^2 r^2 |B_{\ell,m,k}|\, \beta_{\ell,m,k}, \tag{149}$$

which represents the amount of energy per unit radius extracted or deposited in the star by the $\Phi_2^{\ell,m,k}$ partial potential, over one cycle of length $2\pi/\sigma_{\ell,m,k}$). For forcing by prograde ($m > 0$) partial potentials, the torque is positive in regions of the star where the partial tide is damped ($\mathrm{d}W_{\ell,m,k}/\mathrm{d}r < 0$), and vice versa. This is what we expect.

[3] See, e.g., Castor (1971) and Ando & Osaki (1977); note that the minus sign here arises from our choice of time dependence in equation 49).

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A Comparison to Other Work

A.1 Expansion Coefficents

Willems et al. (2010) define their expansion coefficients as

$$\hat{c}_{\ell,m,k} = \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{|m|}(0) \left(\frac{R_1}{a}\right)^{\ell-2} \frac{1}{(1-e^2)^{\ell-1/2}} \frac{1}{\pi} \int_0^\pi (1+e\cos\upsilon)^{\ell-1} \cos(kM + m\upsilon)\,\mathrm{d}\upsilon \tag{150}$$

Here and in the following, the hat ($\hat{\ }$) is used to distinguish their definitions from ours. Using eqn. (22) of Smeyers et al. (1998), this can rewritten in terms of Hansen coefficients as

$$\hat{c}_{\ell,m,k} = \frac{(\ell-|m|)!}{(\ell+|m|)!} P_\ell^{|m|}(0) \left(\frac{R_1}{a}\right)^{\ell-2} X_k^{-(\ell+1),-m} \tag{151}$$

With our definition (16) of the spherical harmonics, we can expand eqn. (27) to obtain our equivalent expression as

$$c_{\ell,m,k} = (-1)^{(|m|-m)/2} \sqrt{\frac{4\pi}{2\ell+1} \frac{(l-|m|)!}{(l+|m|)!}} P_\ell^{|m|}(0) \left(\frac{R_1}{a}\right)^{\ell-2} X_{-k}^{-(\ell+1),-m}. \tag{152}$$

Hence, we see that

$$c_{\ell,m,k} = (-1)^{(|m|-m)/2} \sqrt{\frac{4\pi}{2\ell+1} \frac{(l+|m|)!}{(l-|m|)!}}\, \hat{c}_{\ell,m,-k} \tag{153}$$

As a corrollary, we note the relationship

$$c_{\ell,m,k}\, Y_\ell^m(\pi/2, 0) = \hat{c}_{\ell,m,-k}\, P_\ell^{|m|}(0). \tag{154}$$

A.2 Partial Potentials

In their eqn. (3), Willems et al. (2010) write the potential of the secondary as a nested sum over ℓ, m and k. The partial potentials in this sum (cf. §2.4) are

$$\hat{\Phi}_2^{\ell,m,k}(\mathbf{r};t) = -\epsilon_{\rm tide}\frac{GM_1}{R_1}\hat{c}_{\ell,m,k}\left(\frac{r}{R_1}\right)^{\ell}\hat{Y}_\ell^m(\theta,\phi)\,\mathrm{e}^{\mathrm{i}kM}, \tag{155}$$

Their spherical harmonics are *unnormalized*,

$$\hat{Y}_\ell^m(\theta,\phi) = P_\ell^{|m|}\mathrm{e}^{\mathrm{i}m\phi}. \tag{156}$$

(this definition is implicitly established by the form of the spherical harmonic addition theorem given in eqns. 14–16 of Polfliet & Smeyers, 1990). Combining these expressions, their partial potentials become

$$\hat{\Phi}_2^{\ell,m,k}(\mathbf{r};t) = -\epsilon_{\rm tide}\frac{GM_1}{R_1}\frac{(\ell-|m|)!}{(\ell+|m|)!}P_\ell^{|m|}(0)\left(\frac{R_1}{a}\right)^{\ell-2}X_k^{-(\ell+1),-m}\left(\frac{r}{R_1}\right)^{\ell}P_\ell^{|m|}(\cos\theta)\,\mathrm{e}^{\mathrm{i}(kM+m\phi)}, \tag{157}$$

Our equivalent expression, obtained by combining eqn. (31) with eqn. (152 and setting $\phi_{\rm p}=0$ (see paragraph after eqn. 2 of Willems et al., 2010), is

$$\Phi_2^{\ell,m,k}(\mathbf{r};t) = -\epsilon_{\rm tide}\frac{GM_1}{R_1}\frac{(\ell-|m|)!}{(\ell+|m|)!}P_\ell^{|m|}(0)\left(\frac{R_1}{a}\right)^{\ell-2}X_{-k}^{-(\ell+1),-m}\left(\frac{r}{R_1}\right)^{\ell}P_\ell^{|m|}(\cos\theta)\,\mathrm{e}^{\mathrm{i}(-kM+m\phi)} \tag{158}$$

We therefore conclude that

$$\Phi_2^{\ell,m,k}(\mathbf{r};t) = \hat{\Phi}_2^{\ell,m,-k}(\mathbf{r};t) \tag{159}$$

A.3 Secular Evolution Coefficients

Smeyers et al. (1998) and Willems et al. (2010) define their $\hat{G}$ coefficients via

$$\begin{aligned}\hat{G}^{(1)}_{\ell,m,k} = \frac{\hat{c}_{\ell,m,k}\,P_\ell^{|m|}(0)}{e(1-e^2)^{\ell}} \times \\ \frac{1}{\pi}\Bigg[(\ell+1)e\int_0^\pi(1+e\cos\upsilon)^{\ell}\,\cos\upsilon\,\cos(m\upsilon+kM)\,\mathrm{d}\upsilon - \\ m\int_0^\pi(1+e\cos\upsilon)^{\ell-1}(2+e\cos\upsilon)\,\sin(m\upsilon+kM)\,\mathrm{d}\upsilon\Bigg],\end{aligned} \tag{160}$$

$$\begin{aligned}\hat{G}^{(2)}_{\ell,m,k} = \frac{2\,\hat{c}_{\ell,m,k}\,P_\ell^{|m|}(0)}{(1-e^2)^{\ell+1}} \times \\ \frac{1}{\pi}\Bigg[(\ell+1)e\int_0^\pi(1+e\cos\upsilon)^{\ell}\,\sin\upsilon\,\sin(m\upsilon+kM)\,\mathrm{d}\upsilon - \\ m\int_0^\pi(1+e\cos\upsilon)^{\ell+1}\cos(m\upsilon+kM)\,\mathrm{d}\upsilon\Bigg],\end{aligned} \tag{161}$$

$$\begin{aligned}\hat{G}^{(3)}_{\ell,m,k} = \frac{\hat{c}_{\ell,m,k}\,P_\ell^{|m|}(0)}{e(1-e^2)^{\ell}} \times \\ \frac{1}{\pi}\Bigg[(\ell+1)e\int_0^\pi(1+e\cos\upsilon)^{\ell}\,\sin\upsilon\,\sin(m\upsilon+kM)\,\mathrm{d}\upsilon - \\ m\int_0^\pi(1+e\cos\upsilon)^{\ell-1}\left\{(1+e\cos\upsilon)^2-(1-e^2)\right\}\cos(m\upsilon+kM)\,\mathrm{d}\upsilon\Bigg],\end{aligned} \tag{162}$$

and

$$\hat{G}^{(4)}_{\ell,m,k} = m\frac{(\ell+|m|)!}{(\ell-|m|)!}\left(\frac{R_1}{a}\right)^{-\ell+2}\hat{c}^2_{\ell,m,k}. \tag{163}$$

Using the relationship (154), our equivalent expressions are

$$G^{(1)}_{\ell,m,k} = \frac{\hat{c}_{\ell,m,-k}\,P_\ell^{|m|}(0)}{e(1-e^2)^\ell}\times \frac{1}{\pi}\left[(\ell+1)\int_0^\pi(1+e\cos\upsilon)^\ell\cos(m\upsilon-kM)\cos\upsilon\,\mathrm{d}\upsilon - m\int_0^\pi(1+e\cos\upsilon)^{\ell-1}(2+e\cos\upsilon)\sin\upsilon\sin(m\upsilon-kM)\,\mathrm{d}\upsilon\right], \tag{164}$$

$$G^{(2)}_{\ell,m,k} = \frac{2\,\hat{c}_{\ell,m,-k}\,P_\ell^{|m|}(0)}{(1-e^2)^{\ell+1}}\times \frac{1}{\pi}\left[(\ell+1)e\int_0^\pi(1+e\cos\upsilon)^\ell\,\sin\upsilon\,\sin(m\upsilon-kM)\,\mathrm{d}\upsilon - m\int_0^\pi(1+e\cos\upsilon)^{\ell+1}\cos(m\upsilon-kM)\,\mathrm{d}\upsilon\right], \tag{165}$$

and

$$G^{(3)}_{\ell,m,k} = \frac{\hat{c}_{\ell,m,-k}\,P_\ell^{|m|}(0)}{e(1-e^2)^\ell}\times \frac{1}{\pi}\left[(\ell+1)e\int_0^\pi(1+e\cos\upsilon)^\ell\,\sin\upsilon\,\sin(m\upsilon-kM)\,\mathrm{d}\upsilon - m\int_0^\pi(1+e\cos\upsilon)^{\ell-1}\left\{(1+e\cos\upsilon)^2-(1-e^2)\right\}\cos(m\upsilon-kM)\,\mathrm{d}\upsilon\right]; \tag{166}$$

(note how we've flipped the arguments of the sin functions in the integrands), together with

$$G^{(4)}_{\ell,m,k} = m\frac{(\ell+|m|)!}{(\ell-|m|)!}\left(\frac{R_1}{a}\right)^{-\ell+2}\hat{c}^2_{\ell,m,-k}. \tag{167}$$

Hence, we obtain the result

$$G^{(1)}_{\ell,m,k} = \hat{G}^{(1)}_{\ell,m,-k}, \tag{168}$$
$$G^{(2)}_{\ell,m,k} = \hat{G}^{(2)}_{\ell,m,-k}, \tag{169}$$
$$G^{(3)}_{\ell,m,k} = \hat{G}^{(3)}_{\ell,m,-k}, \tag{170}$$
$$G^{(4)}_{\ell,m,k} = \hat{G}^{(4)}_{\ell,m,-k}. \tag{171}$$

B Evaluating Orbital Integrals

B.1 Hansen Coefficients

The Hansen coefficients are defined by eqn. (24). To evaluate an individual coefficient, we multiply this equation by $\mathrm{e}^{-\mathrm{i}k'M}$ and integrate with respect to the mean anomaly M over one orbit:

$$\int_{-\pi}^{\pi}\left(\frac{r_2}{a}\right)^n \mathrm{e}^{\mathrm{i}(m\upsilon-k'M)}\,\mathrm{d}M = \sum_{k=-\infty}^{\infty}X_k^{n,m}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(k-k')M}\,\mathrm{d}M = \sum_{k=-\infty}^{\infty}X_k^{n,m}\,2\pi\delta_{k,k'}. \tag{172}$$

Hence,

$$X_k^{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r_2}{a}\right)^n \mathrm{e}^{\mathrm{i}(m\upsilon - kM)}\,\mathrm{d}M. \tag{173}$$

Expanding out the complex exponential, this becomes

$$X_k^{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r_2}{a}\right)^n \cos\left(m\upsilon - kM\right)\,\mathrm{d}M + \frac{\mathrm{i}}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r_2}{a}\right)^n \sin\left(m\upsilon - kM\right)\,\mathrm{d}M \tag{174}$$

The integrand in the first term is even, while that in the second term is odd; hence, the second term vanishes, and we're left with

$$X_k^{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r_2}{a}\right)^n \cos\left(m\upsilon - kM\right)\,\mathrm{d}M. \tag{175}$$

To evaluate the integral in this form, it is necessary to solve Kepler's equation (5). However, we can avoid this by transforming to an integral over υ. Noting that

$$\mathrm{d}M = \frac{(1-e^2)^{3/2}}{(1+e\cos\upsilon)^2}\,\mathrm{d}\upsilon, \tag{176}$$

and replacing r_2 using eqn. (3), the integral becomes

$$X_k^{n,m} = \frac{(1-e^2)^{n+3/2}}{2\pi} \int_{-\pi}^{\pi} (1+e\cos\upsilon)^{-(n+2)} \cos\left(m\upsilon - kM\right)\,\mathrm{d}\upsilon. \tag{177}$$

B.2 Secular Evolution Coefficients

The secular evolution coefficients can be expressed in terms of Hansen coefficients:

$$\begin{aligned} G^{(1)}_{\ell,m,k} = c_{\ell,m,k}\, Y_\ell^m(\pi/2,0) \frac{(1-e^2)^{1/2}}{e} \Bigg\{ &\frac{\ell+1}{2} \left[X_{-k}^{-(\ell+2),-m-1} + X_{-k}^{-(\ell+2),-m+1} \right] + \\ &\frac{m}{2} \left[X_{-k}^{-(\ell+2),-m-1} - X_{-k}^{-(\ell+2),-m+1} \right] + \frac{m}{2(1-e^2)} \left[X_{-k}^{-(\ell+1),-m-1} - X_{-k}^{-(\ell+1),-m+1} \right] \Bigg\} \end{aligned} \tag{178}$$

$$\begin{aligned} G^{(2)}_{\ell,m,k} = -2c_{\ell,m,k}\, Y_\ell^m(\pi/2,0) \frac{1}{(1-e^2)^{1/2}} \Bigg\{ &\frac{\ell+1}{2} e \left[X_{-k}^{-(\ell+2),-m-1} - X_{-k}^{-(\ell+2),-m+1} \right] + \\ &m(1-e^2) X_{-k}^{-(\ell+3),-m} \Bigg\} \end{aligned} \tag{179}$$

$$\begin{aligned} G^{(3)}_{\ell,m,k} = -c_{\ell,m,k}\, Y_\ell^m(\pi/2,0) \frac{(1-e^2)^{1/2}}{e} \Bigg\{ &\frac{\ell+1}{2} e \left[X_{-k}^{-(\ell+2),-m-1} - X_{-k}^{-(\ell+2),-m+1} \right] + \\ &m(1-e^2) X_{-k}^{-(\ell+3),-m} - m X_{-k}^{-(\ell+1),-m} \Bigg\} \end{aligned} \tag{180}$$

$$G^{(4)}_{\ell,m,k} = m c_{\ell,m,k}\, Y_\ell^m(\pi/2,0) X_{-k}^{-(\ell+1),-m} \tag{181}$$

Note that these expressions are linked by

$$G^{(4)}_{\ell,m,k} = \frac{e}{(1-e^2)^{1/2}} \left[G^{(3)}_{\ell,m,k} - \frac{1-e^2}{2e} G^{(2)}_{\ell,m,k} \right] \tag{182}$$

(cf. eqn. 65 of Willems et al., 2010).

C Tidal Resonances

Willems et al. (2003) analyze nonadiabatic resonant dynamic tides using a semi-analytic two-time (fast/slow) formalism. Their eqn. (39) gives an expression for the tidal response in the co-rotating frame when the forcing frequency is close to resonance with a single oscillation mode. With our nomenculature, this expression can be written as

$$\boldsymbol{\xi}_{\rm c}(\mathbf{r};t) = \frac{\epsilon_{\rm tide} c_{\ell,m,k} Q_{\ell,N}}{2(\epsilon^2 + C^2 \kappa_{\ell,N}^2/\sigma_{\ell,N}^2)^{1/2}} \boldsymbol{\xi}_{\ell,N}(\mathbf{r}) \mathrm{e}^{-\mathrm{i}(\sigma t + k\Omega_{\rm orb} t_{\rm p} + m\phi_{\rm p})} \tag{183}$$

Here,

$$\epsilon \equiv \frac{\sigma_{\ell,N} - \sigma_{\rm c}}{\sigma_{\ell,N}} \tag{184}$$

is the detuning parameter, representing how close the co-rotating forcing frequency $\sigma_{\rm c}$ is to the free oscillation mode of the (non-rotating) star with harmonic degree ℓ and radial order N. Likewise, $\boldsymbol{\xi}_{\ell,N}$ is the displacement perturbation eigenfunction of the same (free) oscillation mode. The overlap integral $Q_{\ell,N}$ is defined by

$$Q_{\ell,N} = \frac{1}{\sigma_{\ell,N}^2} \frac{\int_0^{R_1} \ell r^{\ell-1} \left[\xi_{r,\ell,N} + (l+1)\xi_{h,\ell,N}\right] \rho r^2 \,\mathrm{d}r}{\int_0^{R_1} \left[\xi_{r,\ell,N}^2 + \ell(\ell+1)\xi_{h,\ell,N}^2\right] \rho r^2 \,\mathrm{d}r} \tag{185}$$

Under the assumption that the density vanishes at the surface, this can also be written as

$$Q_{\ell,N} = \tag{186}$$

Burkart et al. (2012).