L2 Mathématiques

Algèbre Linéaire
Combinatoire et Probabilités discrètes
Analyse approfondie
Diagonalisation
Séries et intégrales généralisées
Fonctions de deux variables
Séries de Fourier

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Chapitre 1

1.1 Random Examples

Définition 1.1.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$$\lim_{n\to\infty} s_n = s$$

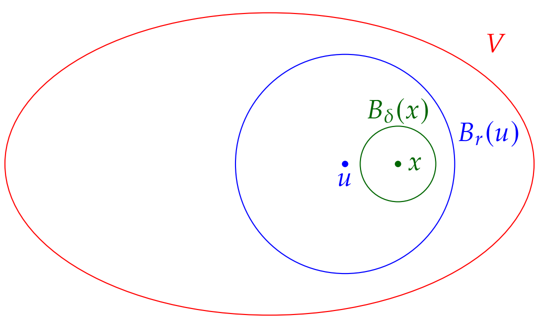
where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 1

Is the set x−axis$\setminus\{$Origin$\}$ a closed set

Démonstration: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leqslant d(u, x) + d(x, y) < d + \delta < r$$

$\square$

Corollaire 1.1.1

By the result of the proof, we can then show...

Lemme 1.1.1

Suppose $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 1.1.1
$1+1=2$.

1.2 Random

Définition 1.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|\ V \longrightarrow \mathbb{R}_{\geqslant 0}$ satisfying

1. $\|x\| = 0 \iff x = 0\ \forall\ x \in V$
2. $\|\lambda x\| = |\lambda|\|x\|\ \forall\ \lambda \in \mathbb{R}$(or $\mathbb{C}$), $x \in V$
3. $\|x+y\| \leqslant \|x\| + \|y\|\ \forall\ x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.
• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \longrightarrow \mathbb{R}_{\geqslant 0}$) where 2 becomes $\|\lambda x\| = |\lambda|\|x\|$ $\forall\ \lambda \in \mathbb{C}$, $x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2+b^2}$

Exemple 1.2.1 ($p-$Norm)
$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p=1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$) : $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

Question 2

Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property 3 for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2$$

$$\implies \sum_i (x_i^2 + 2x_iy_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2$$

$$\implies \left[\sum_i x_iy_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]$$

So in other words prove $\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_iy_i$$

Note:-

— $\|x\|^2 = \langle x, x\rangle$

— $\langle x, y\rangle = \langle y, x\rangle$

— $\langle \cdot, \cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y\rangle = r\langle x, y\rangle + \langle x', y\rangle \text{ and similarly for second slot}$$

Here in $\langle x, y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\
&= \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \\
&= \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle
\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y\rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x\rangle \geqslant 0$

1.3 Algorithms