L2 Mathématiques

Algèbre Linéaire
Combinatoire et Probabilités discrètes
Analyse approfondie
Diagonalisation
Séries et intégrales généralisées
Fonctions de deux variables
Séries de Fourier

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Chapitre 1

1.1 Random Examples

Définition 1.1.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say
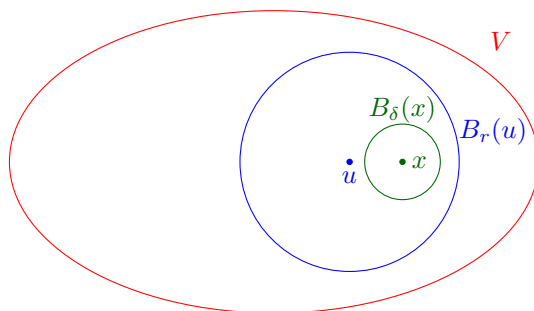$$\lim_{n \to \infty} s_n = s$$
where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$
$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 1

Is the set $x-\text{axis}\backslash\{\text{Origin}\}$ a closed set

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Démonstration: By openness of V, $x \in B_r(u) \subset V$

Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leqslant d(u, x) + d(x, y) < d + \delta < r$$

$\square$

Corollaire 1.1.1

By the result of the proof, we can then show...

Lemme 1.1.1

Suppose $\vec{v_1}, \ldots, \vec{v_n} \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

> **Proposition 1.1.1**
> $1 + 1 = 2$.

1.2 Random

> **Dã©finition 1.2.1: Normed Linear Space and Norm $\| \cdot \|$**
>
> Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\| \cdot \| \; V \to \mathbb{R}_{\geqslant 0}$ satisfying
> **①** $\|x\| = 0 \iff x = 0 \; \forall \; x \in V$
> **②** $\|\lambda x\| = |\lambda| \|x\| \; \forall \; \lambda \in \mathbb{R}(\text{or } \mathbb{C}), \; x \in V$
> **③** $\|x + y\| \leqslant \|x\| + \|y\| \; \forall \; x, y \in V$ (Triangle Inequality/Subadditivity)
> And V is called a normed linear space.
> ● Same definition works with V a vector space over $\mathbb{C}$ (again $\| \cdot \| \to \mathbb{R}_{\geqslant 0}$) where **②** becomes $\|\lambda x\| = |\lambda| \|x\|$
> $\forall \; \lambda \in \mathbb{C}, \; x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

> **Exemple 1.2.1 ($p-$Norm)**
> $V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$
>
> $$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p \right)^{\frac{1}{p}}$$
>
> (In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\| \cdot \|_\infty$) : $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

> **Question 2**
>
> Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property ③ for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2$$

$$\implies \sum_i (x_i^2 + 2x_i y_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2$$

$$\implies \left[\sum_i x_i y_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]$$

So in other words prove $\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$ where

$$\langle x, y \rangle = \sum_i x_i y_i$$

> **Note:-**
>
> — $\|x\|^2 = \langle x, x \rangle$
>
> — $\langle x, y \rangle = \langle y, x \rangle$
>
> — $\langle \cdot, \cdot \rangle$ is $\mathbb{R}-$linear in each slot i.e.
>
> $$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$
>
> Here in $\langle x, y \rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\
&= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\
&= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle
\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y \rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x \rangle \geqslant 0$

1.3   Algorithms