→ Discrete Maths deals with the objects which have distinct values and can be counted.

→ List of topics : set
function & relation
graph & tree
boolean algebra

→ Appⁿ of Discrete maths :
1. Mathematical modeling of the problem
2. Security and Encryption
3. Algorithm Analysis
4. Proof techniques

1. Set & Proposition

→ Proposition – Any statement that can hold true or false is called proposition.

→ set is a collection of distinct discrete objects

→ Membership of the set :
– It is denoted by $\in$

ex $R = \{1,2,3\}$

$1 \in R$, $2 \in R$, $7 \notin R$

$R = \{1, \{2,3\}\}$

$2 \notin R$, $\{2,3\} \in R$

DATE:
PAGE:

→ ~~Prop~~ **Subset :** P is a subset of Q if every element of P is also element of Q.

ex. $P \subseteq Q$

→ **Proper subset :** P is proper subset of Q if P is subset of Q and $P \neq Q$.

ex. $P = \{1, 2\}$, $Q = \{1, 2, 3\}$

Here, $P \subseteq Q$ and $Q \neq P$.

→ **Set Properties :**

- For any set P, P is subset of P. (set is subset of itself).

- The empty set $\emptyset$ is subset of any set. However, it may not be element of every set.

ex $A = \{1, 2, 3\}$

$\emptyset \subseteq A$ but $\emptyset \notin A$

→ No. of elements in a set is called Cardinality of the set and it is denoted by '|A|'.

ex $\{\{\}\}$ → cardinality is 1 because it contains 1 element.

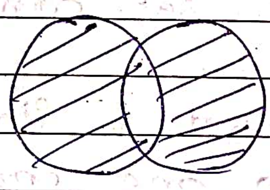
→ Set operations :

① Union → Finding all elements of two sets.

② Intersection - Finding common elements betⁿ two sets.

③ Difference - (A−B) is set of all elements of set A which are not in B.

④ Symmetric difference - $(A \oplus B)$ it includes all elements of A & B but common elements should be skiped.

A  B.

→ Properties related to set cardinality :

① $|P \cup Q| \leq |P| + |Q|$

② $|P \cap Q| \leq \min(|P|, |Q|)$

③ $|P \oplus Q| = |P| + |Q| - 2|P \cap Q|$

④ $|P - Q| \geq |P| - |Q|$

Principle of Inclusion and Exclusion:

$|P \cup Q| = |P| + |Q| - |P \cap Q|$

$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |Q \cap R| - |P \cap R| + |P \cap Q \cap R|$

One to One correspondence :

One to one correspondence bet^n two sets exists if all elements of first set can be uniquely paired with all elements of second set.

- cardinality of a set can be finite or infinite ~~a set~~

- finite set : A set is finite if its elements have one to one correspondence with some other set whose cardinality is $K \in N$

ex. $\{1, 2, 3, 4\}$ and $\{10, 20, 30, 40\}$

$1 \to 10$, $2 \to 20$, $3 \to 30$, $4 \to 40$.

- Infinite set : It can be countable or uncountable.

- Countable infinite - If set has one to one correspondance bet^n its elements and elements of N. then it is called countable infinite

eg. $X = \{3, 6, 9, 12, ---\}$

$N = \{1, 2, 3, ---\}$

eg. Find the no. of integers which are divisible by any number 2, 3, 5 or 7 in the range 1 to 250.

$\rightarrow$ No. of integers that are divisible by $2 = \left\lfloor \frac{250}{2} \right\rfloor = 125$

$3 = \left\lfloor \frac{250}{3} \right\rfloor = 83$

$\cancel{4 = \left\lfloor \frac{250}{4} \right\rfloor = 62}$, $5 = \left\lfloor \frac{250}{5} \right\rfloor = 50$

$7 = \left\lfloor \frac{250}{7} \right\rfloor = 35$

Intersection $2 \& 3 = \left\lfloor \frac{250}{6} \right\rfloor = 41$

$3 \& 5 = \left\lfloor \frac{250}{15} \right\rfloor = 16$, $5 \& 7 = \left\lfloor \frac{250}{35} \right\rfloor = 7$

$2 \& 5 = \left\lfloor \frac{250}{10} \right\rfloor = 25$, $2 \& 7 = \left\lfloor \frac{250}{14} \right\rfloor = 17$

$3 \& 7 = \left\lfloor \frac{250}{21} \right\rfloor = 11$.

$2 \& 3 \& 5 = 8$, $3 \& 5 \& 7 = 2$, $2 \& 5 \& 7 = 3$

$2 \& 3 \& 7 = 5$, $2 \& 3 \& 5 \& 7 = 1$

$|2 \cup 3 \cup 5 \cup 7| = 125 + 83 + 50 + 35 - 41 - 16 - 7$
$- 25 - 17 - 11 + 8 + 2 + 3 + 5 - 1$

$= \cancel{159}\ \boxed{193}$

$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2|$
$- |A_2 \cap A_3| - |A_3 \cap A_4| - |A_1 \cap A_3| - |A_1 \cap A_4|$
$- |A_2 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_3 \cap A_4|$
$+ |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$

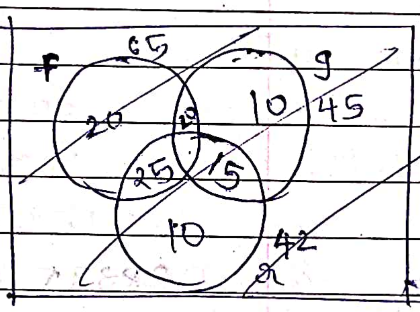
DATE:
PAGE:

Eg. 100 out of 120 students study at least one of the languages. 65 students study french, 45 - german, 42 → Russian,
20 → french + german, 25 → french + Russian,
15 → german + Russian

(i) Find students studying only two languages but not third one
(ii) Find students who studies only one lang
(iii) Find stu. who studies all lang.

→ $|A_1| = 65$, $|A_2| = 45$, $|A_3| = 42$
$|A_1 \cap A_2| = 20$, $|A_1 \cap A_3| = 25$, $|A_2 \cap A_3| = 15$.
$|A_1 \cup A_2 \cup A_3| = 100$

(i) french + german not Russian
$|A_1 \cap A_2| - |A_1 \cap A_2 \cap A_3|$
$= 20 - 8 = 12$



(ii) Only french
$= |A_1| - |A_1 \cap A_2| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
$= 65 - 20 - 25 + 8 = \boxed{28}$

(iii) $|A_1 \cap A_2 \cap A_3| = ?$

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$\therefore 100 = 65 + 45 + 42 - 20 - 25 - 15 + |A_1 \cap A_2 \cap A_3|$

$\therefore \boxed{|A_1 \cap A_2 \cap A_3| = 8}$

DATE:
PAGE:

- Uncountable infinite :- there is no correspondence between its elements

⟹ Diagonal Argument (Cantor's diagonal arg)

	Red	Black	Yellow
A	yes	No	yes
B	No	No	Yes
C	Yes	Yes	Yes
New → D	No	Yes	No.

$m \times n$ grid unsymmetric

The diagonal Argument states that a certain object is not one of the given objects using a fact that it is different from each of the given objects in at least one way.

→ Proof for all real no.'s between 0 & 1 are uncountable infinite set.

- let's assume set of all no.'s betn 0 & 1 is countable infinite.

so there should be one to one correspondence betn its elements and N.

$$\begin{array}{l} 0.a_1\ a_2\ a_3 \ldots a_n \\ 0.b_1\ b_2\ b_3 \ldots b_n \\ 0.c_1\ c_2\ c_3 \ldots c_n \\ \vdots \end{array}$$

DATE:
PAGE:

- Suppose the list of all no.s bet^n 0 & 1 as above.
- let's assume one no. $0.x_1 x_2 \ldots x_n$ such that x_1 is different than a_1, x_2 is different than a_2 and so on.
- So, this new no. is different than all existing no. in given list. so the list is incomplete and there is no one to one correspondence with a set of Natural no. N.

So, our assumption is incorrect.
Hence the set of real no. bet^n 0 & 1 is uncountable infinite

⇒ <u>Principle of Mathematical Induction</u> : (PMI)

- For a given statement which involves natural no. n if we can show that,

1. Statement is true for $n = n_0$ (Basis)
2. Assume that the statement is true for $n = k$ (Hypothesis) and prove that a statement is true for $n = k+1$. (Induction step).
3. Then the statement is true for all natural no. $n \geq n_0$

eg: given the coins of value 3 & 5. Prove that any amount of 8 or more can be made using the available coins.

$8 = 3 + 5$ case-I

$9 = 3 + 3 + 3$ case-II

$10 = 5 + 5$ case-III

$11 = 3 + 3 + 5$

$12 = 3 + 3 + 3 + 3$

① Basis of Induction

- let Base amount is 8 and since $8 = 5 + 3$, so basis is proved.

② Induction Hypothesis

- It is possible to make any amount k using coins of 3 & 5.

- Case-I If amount k is made using 3 & 5 then we can replace 5 with two 3's to get k+1 amount.

case-II If amount k is made using only 3's then we can replace 3 3's with 2 5's to get k+1 amount.

case-III If amount k is made using only 5's then we can replace 5 with two 3's to get k+1 amount.

So, according to above cases if hypothesis is true, then we can make amount k+1 using the coins of 3 & 5.

So, the statement is proved for all the numbers.

eg. Prove that $1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$; $(n \geq 1)$

→ Basis

L.H.S. $= 1^2 = 1$
$(n=1)$

RHS $= \frac{1(1+1)(2+1)}{6} = 1$

So, Basis is proved

Hypothesis

Assume the hypothesis $1^2+2^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$

we need to prove that
$1^2+2^2+\dots+(k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$

Now,
$$\begin{aligned} LHS &= 1^2+2^2+\dots+(k+1)^2 \\ &= 1^2+2^2+\dots+k^2+(k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \end{aligned}$$
($\because$ From Hypothesis)

DATE:
PAGE:

$$= (K+1)\left[\frac{(2K+1)(K) + 6(K+1)}{6}\right]$$

$$= (K+1)\left[\frac{2K^2+7K+6}{6}\right]$$

$$= \frac{(K+1)\,(2K+3)(K+2)}{6}$$

$$= \frac{(K+1)\,(K+2)\,(2(K+1)+1)}{6}$$

$= R.H.S.$

So, statement is true for $n = K+1$.
Hence the statement is true for all $n \geq 1$.

eg Prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for $n \geq 1$.

→ Basis $n=1$. $LHS = \frac{1}{1\cdot 2} = \frac{1}{2}$

$RHS = \frac{1}{1+1} = \frac{1}{2}$

So, Basis is proved.

→ Hypothesis Assume the Hypothesis

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{K(K+1)} = \frac{K}{K+1}$$

we need to prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(K+1)(K+2)} = \frac{K+1}{K+2}$$

DATE:
PAGE:

Now, LHS

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(K+1)(K+2)}$$

$$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{K(K+1)} + \frac{1}{(K+1)(K+2)}$$

$$= \frac{K}{K+1} + \frac{1}{(K+1)(K+2)}$$ (∵ From Hypothesis)

$$= \frac{1}{(K+1)}\left[\frac{K(K+2)+1}{(K+2)}\right]$$

$$= \frac{(K^2+2K+1)}{(K+1)(K+2)} = \frac{(K+1)^2}{(K+1)(K+2)}$$

$$= \frac{K+1}{K+2} = RHS$$

So, Statement is true for $n = K+1$

Hence, the statement is true for all $n \geq 1$.

eg. Show that for $n \geq 1$,
$n^3 + 2n$ is divisibe by 3.

→ Basis $n=1$
$n^3 + 2n = 3$ is divisible by 3

Hypothesis for $n=k$ Assume that
k^3+2k is divisible by 3.

need to prove that $(K+1)^3 + 2(K+1)$ is divisible by 3

DATE:
PAGE:

$(k+1)^3 + 2(k+1)$

$= k^3 + 1 + 3k^2 + 3k + 2k + 2$

$= \underline{k^3 + 2k} + 3k^2 + 3k + 3$

$= 3(k^2+k+1) + k^3 + 2k$ (from hypothesis k^3+2k is divisible by

$= 3m + k^3 + 2k$

since k^3+2k is divisible by 3 as per hypothesis and the other quantity is also divisible by 3. so the induction is proved.

eg Prove that $n^4 - 4n^2$ is divisible by 3. for $n \geq 2$.

→ Basis $n = 2$

LHS $= 16 - 16 = 0$ is divisible by 3.

Hypothesis for $n = k$ Assume that $k^4 - 4k^2$ is divisible by 3.

need to prove that $(k+1)^4 - 4(k+1)^2$ is divisible by 3

$k^2(k^2-4)$

$= (k+1)^4 - 4(k+1)^2$

$= (k+1)^2 [(k+1)^2 - 4]$

$= (k+1)^2 [k^2 + 2k - 3]$

$= (k+1)^2 (k+3)(k-1)$

$= (k^2+2k+1)(k^2+2k-3)$

$= k^4 + 2k^3 - 3k^2 + 2k^3 + 4k^2 - 6k + k^2 + 2k - 3$

$= \underline{k^4} + 4k^3 - \underline{4k^2} + 4k^2 + 2k^2 - 3 - 4k$

$= 4k^3 + 6k^2 - 3 + (k^4 - 4k^2) - 4k$

$$= \cancel{3\left(\frac{4}{3}k^3 + 2k^2\right) - 3 + (k^4 - 4k^2)}$$

$$= (k^4 - 4k^2) + (4k^3 - 4k) + (6k^2 - 3)$$

$$= 4(k+1)k(k-1) + (k^4 - 4k^2) + (6k^2 - 3)$$

Second quantity is divisible by 3 as per hypothesis. first quantity is divisible by 3 because it is product of three consecutive number and third quantity is of form 3(m) that is also divisible by 3.

Hence the induction is proved.

eg. Show that, any integer composed of 3^n identical digits is divisibe by 3^n.

→ Bases : $n = 1$

$3^1 = 3$ is divisible by 3.

ex. 111 is divisible by 3.

for $n=1$ all no.'s with identical 3 digits are divisible by 3. such 9 no.s are possible

Hypothesis :

Any no. with 3^k identical digits is divisible by 3^k. prove that, if a no. contains 3^{k+1} identical digits then it is divisible by 3^{k+1}.

let's assume a number with 3^{k+1} identical digits.
Now, assume that number is product of x and y.
where x is a number containing 3^k identical numbers

$$3^{k+1} = x \cdot y$$

(ex. 222 222 222 $\Rightarrow$ $x = 222$
$y = 1001001$)

x is divisible by 3^k by hypothesis.
Because sum of digits in y is equal to 3 so
y is divisible by 3. So the assumed no. is ~~also~~
divisible by 3^{k+1}

Eg Show that any integer n, $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

→ Basis $n = 1$

$$(11)^3 + 12^3 = 1331 + 1728$$
$$= 3059 \text{ is divisible by } 133.$$

Hypothesis for any integer $n = k$
$(11)^{k+2} + (12)^{2k+1}$ is divisible by 133.

we need to prove that $(11)^{k+3} + (12)^{2(k+1)+1}$
is divisible by 133.

$$= (11)^{k+3} + (12)^{2(k+1)+1}$$
$$= (11)^{k+2} \cdot (11) + (12)^{2k+3}$$
$$= (11)^{k+2} \cdot (11) + (12)^{2k+1} (12)^2$$
$$= (11)^{k+2} \cdot (11) + (12)^{2k+1} (133 + 11)$$

$= (11)(11)^{K+2} + (11)(12)^{2K+1} + (133)(12)^{2K+1}$

$= (11)\left[11^{K+2} + 12^{2K+1}\right] + 133\,(12)^{2K+1}$

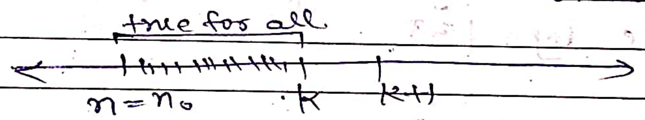
$= 11\,(133y) + 133(\cancel{12}x)$ $\quad$ ($\because$ From Hypothesis)

$= 133\,(11y + x)$

$= 133 \cdot m$ is divisible by 133.

Eg. $\Rightarrow$ <u>Strong Mathematical Induction.</u>

- In strong Mathematical Ind^n the hypothesis is assumed to be true for $n_0 \leq n \leq K$ other details are same as normal ind^n.



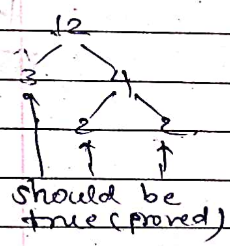
(assumes more)

Ex. Show that any positive integer $n \geq 2$ is either prime no. or product of two prime numbers.

$\rightarrow$ Basis of Induction $n = 2$

since 2 is prime number, basis is true.

<u>Hypothesis</u> (Strong induction)



The statement is true for any n such that ~~two~~ $2 \leq n \leq K$.

Induction step:

Assume a no. K+1

case-I (K+1) is prime
then the statement is true

case-II If (K+1) is not prime, then it is can be always expressed as product of two no.s P and q, such that [p,q ≤ K]
$= p.q \quad [p, q \leq K]$

So As per hypothesis p & q are either prime no.s or product of prime no.s. So the no: K+1 is also either prime or product of prime no.s.

Ex. Show that $2^n > n^3$ for $n \geq 10$.

→ Basis $n = 10$
LHS $= 2^{10} = 1024 > 10^3$ = RHS.

Hypothesis
statement is true for any k such that
$n = k$, $2^k > k^3$
need to prove for $2^{k+1} > (k+1)^3$
$$\underline{2^k \cdot 2} > \cancel{k^3 + 1 + 3k^2 + 3k}$$

Now, $2^k > k^3$
$2^{k+1} > 2k^3$
if we prove $2k^3 > (k+1)^3$

DATE:
PAGE:

$\Rightarrow \quad 2K^3 > K^3 + 3K^2 + 3K + 1$

$\Rightarrow \quad K^3 > 3K^2 + 3K + 1$

$\Rightarrow \quad 1 > \frac{3}{K} + \frac{3}{K^2} + \frac{1}{K^3}$ ($\because$ Divide by K^3)

put $K = 10$

$1 > \frac{3}{10} + \frac{3}{10^2} + \frac{1}{10^3}$

$1 > 0.333$

for $K = 10$ statement is true, $1 > 0.333$

So, for all values $K \geq 10$ the statement is true.

$2^{k+1} > 2K^3$ ——— (1)

$2K^3 > (K+1)^3$ ——— (2)

By transitivity, $2^{K+1} > (K+1)^3$ is also proved.

Proposition

- The use of proposition logic is to convert natural language into mathematical statement

- Proposition is a declarative sentence - either true or false.

- Some propositions are always true. A proposition which is always true - Tautology.

- A proposition which is always false - Contradiction

- Evaluation of proposition means assigning true or false value to the proposition.

- Two propositions can be combined.

- Two or more propositions can be combined using connectives.

- Connectives:

(1) Disjunction ($p \vee q$)
(2) Conjunction ($p \wedge q$)

p	q	$p \vee q$	$p \wedge q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

(3) Negation ($\sim p$)

p	$\tilde{p}$
T	F
F	T

(4) Exclusive-OR ($p \oplus q$)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

DATE:
PAGE:

→ A proposition which is obtained from the combination of the other propositions is called "Compound proposition."

→ A proposition which is not a combination is called "Atomic proposition".

p	q	$(p \wedge q) \wedge \tilde{p}$
T	T	F
T	F	F
F	T	F
F	F	F

→ Compound proposition are of two types:
(i) Conditional $(p \rightarrow q)$
(ii) Biconditional $(p \leftrightarrow q)$

(i) Conditional statement is defined as $p \rightarrow q$ (If p then q)

(ii) Biconditional statement is defined as $p \leftrightarrow q$ (p if and only if q) $[(p \rightarrow q) \wedge (q \rightarrow p)]$

→ <u>Necessary</u> & <u>Sufficient</u> <u>Cond^n</u>:

p is necessary for q means if q is true p must be true but p is true that doesn't mean q is true.

<u>ex</u>. p = a person is 18 years old.
q = a person is president of india.

Sufficient Cond^n :

p is sufficient for q means that if p is true we can always conclude that q is also true

ex. p = person has voter card.
q = person is 18 years old.

- p can be both sufficient and necessary cond^n for q.

ex. p = person is born in some country.
q = person is citizen of that country
- p is sufficient cond^n for q.

ex p = no. is multiple of 9.
q = no. is multiple of 3.
- p is sufficient cond^n for q.

ex. p = two lines are parallel
q = two lines are not intersacting.
- p is sufficient & necessary for q. $(p \leftrightarrow q)$

p	q	$p \to q$	$p \leftrightarrow q$ (X-NOR)
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

DATE:
PAGE:

Ex. An island has two tribes of natives. Any native from the first tribe always tells the truth, while any native from the other tribe always lies. You arrive at the island and ask a native if there is gold on the island. He answers, "There is gold on the island if and only if I always tell the truth". Which tribe is he from? Is there gold on the island?

→ p = he always tells the truth
q = there is gold on the island.

Thus, his answer is $p \leftrightarrow q$

Suppose that ~~his answer to a~~ he always tells the truth. that is, p is true. Therefore, q must be true. that is his answer to our question must be true.

$\therefore$ $p \leftrightarrow q$ is true

Suppose he always lies, that is p is false. Also, his answer to our question is a lie, which means that $p \leftrightarrow q$ is false. Consequently, q must be true.
(F, T = F)

Thus, in both cases we can conclude that there is gold on the island, although the native could have been from either tribe.

Ex. p = the food is good
q = the service is good
r = the rating is 3 star

(1) Either the food is good or service is good or both. $\longrightarrow$ $p \vee q$

(2) Either the food is good or service is good but not both $\longrightarrow$ $p \oplus q$

(3) Food is good but the service is poor. $\longrightarrow$ $p \wedge \tilde{q}$

(4) It is not the case that food is good and rating is 3 star $\longrightarrow$ $(\overline{p \wedge r})$

(5) If both food and service is good then rating is 3 star. $\longrightarrow$ $(p \wedge q) \longrightarrow r$

(6) It is not true that ~~st~~ 3 stare always means good food and good service
$[r \longrightarrow (p \wedge q)]'$

$\longrightarrow$ For proposition $p \rightarrow q$, the proposition $q \rightarrow p$ is called converse.

$\longrightarrow$ The proposition $\bar{q} \rightarrow \bar{p}$ is called contra positive.

$\longrightarrow$ The proposition $\bar{p} \rightarrow \bar{q}$ is called inverse.

Que. Find truth values of all above

p	q	$p \to q$	$q \to p$	$\bar{p} \to \bar{q}$	$\bar{q} \to \bar{p}$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

$p \to q$ & $\bar{q} \to \bar{p}$ is same.

$q \to p$ & $\bar{p} \to \bar{q}$ is same.

Ex. Give the converse, contrapositive and inverse of the foll^n implication : "If it rains today, I will go to college tomorrow."

→ Converse : If I will go to college tomorrow, then it would have rained today.

Contrapositive : If I do not go to college tomorrow, then it will not have rained today.

Inverse : If it does not rain today, then I will not go to college tomorrow

3. Relation & Function

- **Relation:**

→ In a set, the elements are sometimes related with each other, such elements have some comman factor betⁿ them.

$S = \{1, 2, 3, 4, 5, 6\}$

$R = \{(2, 4, 6), (1, 3, 5)\}$ ← called relation.

→ Define cartesian Product:

$A = \{a, b\}$, $B = \{c, d\}$

$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$

$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$

→ Binary relation from set A to B is subset of cartesian product $A \times B$.

$A = \{1, 2\}$ $B = \{1, 3\}$

$A \times B = \{(1,1), (1,3), (2,1), (2,3)\}$

If Relation is defined as "less than" then,

$R = \{(1,3), (2,3)\}$

→ aRb – a is related with b

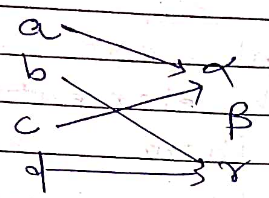
→ Relation is always ordered pair.

→ Tabular Form of Relation:

$R = \{(a, \alpha), (c, \alpha), (b, \gamma), (c, \gamma)\}$

	α	B	γ
a	✓		–
b			✓
c	✓		✓
d			

⟹ Graphical form of Relation



→ Just like a set relation has foll^{n} operations
$R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_1 \oplus R_2$

→ Ternary relation can be defined as subset of Cartesian product of $(A \times B)$ and C
eg. $R = \{((a,\alpha),1), ((a,\alpha),2) \ldots\}$

• Properties of Binary Relation

(1) ~~Reflexibility~~ Reflexivity
- Relation R is defined on set A. for $a \in A$, if $(a,a) \in R$ then R is called Reflexive Relation.

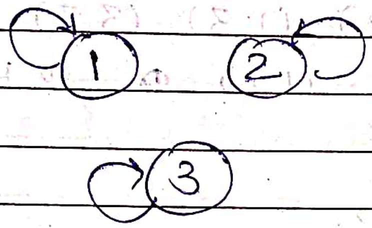
ex $A = \{1,2,3\}$
$R = \{(1,1),(2,2),(2,3),(3,3)\}$ is the example of reflexive relation.

Because, $(1,1),(2,2),(3,3) \in R$

- In the relation matrix, Reflexivity is indicated by diagonal cells.

R	1	2	3
1	—		
2		—	
3			—

– In the directed graph, self loop represent the reflexivity.

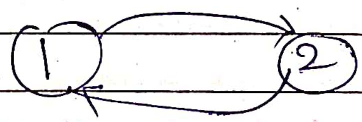


(2) Symmetric Relation

– Relation R is defined on set A, for each pair $(a,b) \in R$ if $(b,a) \in R$ then relation R is symmetric.

ex: $R = \{(1,2),(2,1)\}$ $R = \{(a,b) \mid a=b\}$

– In the relation graph, the cycle indicates the symmetric relation.



(3) Anti-symmetric Relation

– R is relation on A, if $(a,b) \in R$ and $(b,a) \in R$ implies ($\Rightarrow$) $a=b$ then, R is Anti-symmetric Relation.

– Anti-symmetric $\neq$ Not symmetric

DATE:
PAGE:

ex. $R = \{(1,1), (1,2), (1,3), (3,3), (2,1) (2,2)\}$
$A = \{1,2,3\}$

R is Reflexive
R is not symmetric [(1,3)]
R is not anti-symmetric

For make it Anti-symmetric:
$R = \{(1,1), (1,2), (1,3), (3,3), (2,2)\}$
(Remove either (1,2) or (2,1))

Ex. $A = \{a,b\}$
$R = \{(a,a), (b,b)\}$

R is Reflexive, Symmetric, Anti-symmetric

Ex. $A = \{a,b\}$
$R = \{(a,b), (a,c), (c,a)\}$
[(a,c) & (c,a)]

R is not Reflexive, symmetric, Anti-symmetric

Q.1 Is every Reflexive Relation is Anti-symmetric.

$A = \{1,2,3\}$
$R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

R is Reflexive but not Anti-symmetric
because (1,2) and also (2,1) is present.

(4) Transitivity

- R is relation defined on set A, if $(a,b) \in R$ and $(b,c) \in R$ ~~then~~ implies $(a,c) \in R$ then the relation is transitivity for every pair.

ex $R = \{(a,a), (a,b), (a,c), (b,c)\}$.

R is transitivity

$R = \{(a,b) \mid a > b\}$ is transitivity.

→ **Closure of Relation**

(1) Reflexive closure

The Reflexive closure of Relation R is smallest Reflexive Relation which contains R.

ex $R = \{(1,1), (2,2) (2,3)\}$ defined on set $A = \{1,2,3\}$

$R_R^* = \{(1,1), (2,2), (2,3) (3,3)\}$

Reflexive closure

cannot remove any pair.

$$R_R^* = R \cup \{(a,a) \mid a \in A\}$$

(2) Symmetric Closure

It is smallest symmetric Relation which contains R.

ex. $R = \{(0,1), (1,2), (2,1)\}$
$A = \{0,1,2\}$

$R_s^* = \{(0,1), (1,2), (2,1), (1,0)\}$

$R_s^* = R \cup \{(b,a) \mid (a,b) \in R\}$

(3) Transitive Closure

Transitive closure is the smallest transitive Relation that contains R.

ex. $A = \{1,2,3\}$
$R = \{(1,1), (2,3), (3,1)\}$

$R_t^* = \{(1,1), (2,3), (3,1), (2,1)\}$

In general, $R_t^* = R \cup \{(a,c) \mid (a,b) \in R \;\&\; (b,c) \in R\}$

Ex. $R = \{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)\}$
$A = \{1,2,3,4\}$

$R_t^* = \{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3),$
$(2,4), (3,3), (4,4)\}$

Warshall's Algorithm to find the transitive closure.

Warshall's Algorithm is used to find transitive closure of the relation R.

Step :

(1) If the set on which the relation is defined contains 'n' elements then create a matrix of ~~m~~ n columns 2 rows

first row, is for columns (C)
second row, is for row (R)

	I	II	III
C			
R			

(2) In the ith cell of the column section write the elements which are present in the particular column of Relation matrix.

In the ith cell of the row section, write the elements which are present in the particular row of Relation matrix.

(3) For each column, perform the cartesian product of the elements in the first & second row repeat this for all columns.
Whichever pairs are result of cartesian product they will be included in the transitive closure.

Ex. $A = \{1, 2, 3, 4\}$

$R = \{(2,1), (2,3), (3,1), (3,4), (4,1)(4,3)\}$

→ Relation Matrix :

	1	2	3	4
1	0	0	0	0
2	1	0	1	0
3	1	0	0	1
4	1	0	1	0

	I	II	III	IV
C	$\{2,3,4\}$	ϕ	$\{2,4\}$	$\{3,3\}$
R	ϕ	$\{1,3\}$	$\{1,4\}$	$\{1,3\}$
cross product	ϕ	ϕ	$\{(2,1),(2,4)$ $(4,1),(4,4)\}$	$\{(3,1),$ $(3,3)\}$

Add into the original Matrix

	1	2	3	4
1	0	0	0	0
2	1	0	1	1
3	1	0	1	1
4	1	0	1	1

$R_t^{*} = \{(2,1), (2,3), (2,4), (3,1), (3,3), (3,4),$
$(4,1), (4,3), (4,4)\}$

Ex: Find a transitive closure using Warshall's Algo.

$A = \{a, b, c, d, e\}$

$R = \{(a,c), (b,d), (c,a), (d,b), (e,d)\}$

→ Relation Matrix :

	a	b	c	d	e
a	0	0	1	0	0
b	0	0	0	1	0
c	1	0	0	0	0
d	0	1	0	0	0
e	0	0	0	1	0

	I	II	III	IV	V
C	{c}	{d}	{a}	{b, e}	ϕ
R	{c}	{d}	{a}	{b}	{d}
	{(c,c)}	{(d,d)}	{(a,a)}	{(b,b), (e,b)}	ϕ

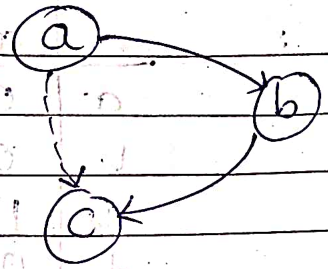
$R =$

	a	b	c	d	e
a	1	0	1	0	0
b	0	1	0	1	0
c	1	0	1	0	0
d	0	1	0	1	0
e	0	1	0	1	0

$R_t^* = \{(a,a), (a,c), (b,b), (b,d), (c,a), (c,c), (d,b), (d,d), (e,b), (e,d)\}$

→ transitive closure indicates the reachability in the graph.

$R = \{(a,b), (b,c)\}$



Graph says, that we dont have path from a to c directly.

If we apply transitive closure then we can get path from a to c via b.

ex. Find the smallest relation containing the relation $R = \{(1,2), (1,4), (3,3), (4,1)\}$ that is Reflexive and transitive.

→ $R_R^* = \{(1,2), (1,4), (3,3), (4,1), (1,1), (2,2), (4,4)\}$

$R_t^* = \{(1,2), (1,4), (3,3), (4,1), (1,1), (4,4), (4,2)\}$

$R = R_R^* \cup R_t^*$

$= \{(1,2), (1,4), (3,3), (4,1), (1,1), (2,2), (4,4), (4,2)\}$

DATE:
PAGE:

ex. let A, Be a set have 10 elements. How many diff binary relation on A is possible?
How many of them are reflexive?
How many of them are symmetric?

ex let R be a binary relation on set of all ~~possible~~ (+ve) integers such that R = {(a,b) | a−b is odd positive integer}
Find the nature of the relation.

→ No Reflexive &
No Symmetric
No transitive [(10,5) − (5,2) = (10,2) which is not odd]

Exam

Ex. Prove that for any positive integer n>1

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

→ Basis : n=2

$$L.H.S. = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + 0.707 = \boxed{1.707}$$

$$R.H.S. = \sqrt{2} = 1.41$$

Here, 1.707 > 1.41 hence state is true for n=2

Hypothesis : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} > \sqrt{K}$;

$$\rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \sqrt{K} + \frac{1}{\sqrt{K+1}} \quad \text{---(1)}$$

Here, $K+1 > K$

$\Rightarrow \sqrt{K+1} > \sqrt{K}$

$\Rightarrow 1 > \dfrac{\sqrt{K}}{\sqrt{K+1}}$

from —(1)

$> \dfrac{K}{\sqrt{K+1}} + \dfrac{1}{\sqrt{K+1}}$

$> \dfrac{K+1}{\sqrt{K+1}} > \sqrt{K+1}$

Hence, proved.

- **Equivalence Relation**

:- It is a binary relation that is Reflexive, symmetric & transitive.

Ex. $A = \{a, b, c, d, e, f\}$

	a	b	c	d	e	f
a	✓	✓				
b	✓	✓				
c			✓			
d				✓	✓	✓
e				✓	✓	✓
f				✓	✓	✓

General eqⁿ $R = \{(a, b) \mid a = b\}$

DATE:
PAGE:

- **Set Partitions**

$S = \{1, 2, 3, 4\}$

- set can be partitioned into non-empty subsets such that every element is included in exactly one subset.

- union of all subsets will be original set and intersection will be null ($\emptyset$).

ex. $S = \{1, 2, 3\}$

		partitions.
partitions	$\Rightarrow \{1, 2\}, \{3\}$	$\{\overline{12}, \overline{3}\}$
	$\Rightarrow \{1\}, \{2, 3\}$	$\{\overline{1}, \overline{23}\}$
	$\Rightarrow \{2\}, \{1, 3\}$	$\{\overline{2}, \overline{13}\}$
	$\Rightarrow \{1\}, \{2\}, \{3\}$	$\{\overline{1}, \overline{2}, \overline{3}\}$

Notation for set partition

ex $\{1, 2\}, \{3\} \rightarrow \{\overline{12}, \overline{3}\}$

→ From the equivalence relation on set A, we can define the partition of A, so that every two elements in a block are related and every two elements in different blocks are not related. This partition is called partition induced by equivalence relation.

ex. $S = \{1, 2, 3\}$

$\Rightarrow \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$\{\{1, 2\}, \{3\}\} \Longrightarrow \{\overline{12}, \overline{3}\}$

- 1 & 2 are related with each other and 3 is not related with either 1 or 2.

Similarly, from the partitions of the set we can define equivalence relation.

ex. partition $= \{\overline{12}, \overline{3}\}$

$R = \{(1,1), (2,2), (1,2), (2,1), (3,3)\}$

ex. P is a partition induced by equivalence relation $P = \{\{a,b,c\}, \{d\}, \{e\}\}$

list all pairs in equivalence relation R

$R = \{(a,a), (b,b), (c,c), (a,b), (a,c), (b,c)$
$(b,a), (c,a), (c,b), (d,d), (e,e)\}$

- Equivalence Class

An equivalence class of x, where x is an element, is defined as $[x]$ such that

$x \rightarrow [x]$

$[x] = \{y \mid y \in A, (x,y) \in R\}$

↑ Equivalence class

ex. let $A = \{1,2,3,4,5\}$

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2),$
$(2,1), (4,5), (5,4)\}$

$[1] = \{1,2\}$] C_1
$[2] = \{2,1\}$]
$[3] = \{3\}$] C_2
$[4] = \{4,5\}$] C_3
$[5] = \{5,4\}$]

3 classes,

$C_1 = \{1,2\}$

$C_2 = \{3\}$

$C_3 = \{4,5\}$

DATE:
PAGE:

partition $= \{\{1,2\},\{3\},\{4,5\}\}$

Que. If set A contains n elements, How many minimum & maximum equivalence class can be possible?

Ans. Maximum $= n$

Minimum $=$

- Congruence Modulo Relation ($\equiv$)

Two integers ~~A~~ a & ~~B~~ b are congruent modulo if and only if (iff) they have the same ~~reminder~~ remainder when divided by some number m,

$$a \equiv b \pmod{m}$$

a is congruent to b modulo m.

ex $16 \equiv 13 \pmod{3}$

$\frac{16}{3}$ modulo $= 1$ & $\frac{13}{3}$ modulo $= 1$

this means that m divides $\boxed{(a-b)}$

ex $29 \equiv 8 \pmod{7}$

→ Show that the relation congruence modulo m over the set of positive numbers is an equivalence relation.

DATE:
PAGE:

→ Proof for congruence modulo m is an equivalence relation.

(1) Reflexive relation
- since m divides zero we can say that
$a \equiv a \pmod m$.

(2) Symmetric relation
- Assume $a \equiv b \pmod m$.
so, $(a-b) = k \cdot m$
$\Rightarrow (b-a) = -k \cdot m$
this result means that $(b-a)$ is also divisible by m. so. $b \equiv a \pmod m$

(3) Transitivity relation
- Assume $a \equiv b \pmod m$ & $b \equiv c \pmod m$
$\boxed{a-b = k.m}$ ← (1) $\boxed{b-c = l.m}$ ← (2)

Add (1) & (2)
$a - c = km + lm$
$a - c = (k+l)m$
this result means that $(a-c)$ is divisible by m. so, $a \equiv c \pmod m$.

Que A = {1,2,3} How many diff eq. relation are possible?
(1) {(1,1), (2,2), (3,3)}
(2) {(1,1), (2,2), (3,3), (1,2), (2,1)}
(3) {(1,1), (2,2), (3,3), (1,3), (3,1)}
(4) {(1,1), (2,2), (3,3), (2,3), (3,2)}
(5) {(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2)}

→ Partial Order

- A relation is partial order if it is ~~a re~~
(a) reflexive, (b) anti-symmetric (c) transitive.

eg $(x,y) \in R \mid x \geq y$.
$A = \{1,2,3\}$
Partial $= \{(1,1);(2,2),(3,3),(3,1),(2,1),(3,2)\}$

eg. $(a,b) \Rightarrow a$ divides b

eg relation $\subseteq$ is partial order

let R be a relation
$R = \{(a,b) \mid a \subseteq b\}$ it defines on powerset ~~(set of sets)~~

H.W.
→ $S = \{1,2,3\}$ R is defined on power set of S.

→ Partial order set (POSET)

- if R is a partial order relation defined on set S.
- set S along with relation R is called partially ordered set and denoted as (S,R) or $(S,\leq)$ ← notation

eg. consider set $S = \{1,2,3,5,6,10,15,30\}$
construct a partial order set in $(S,/)$.
relation

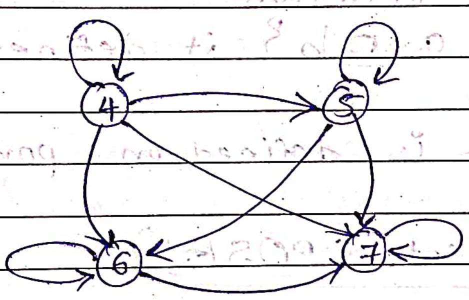
→ $R = \{(1,1),(2,2),(3,3) \ldots (30,30),$
$(1,2),(1,3),(1,5) \ldots$
$(2,6),(2,10),(2,30),(3,6),(3,15),(3,30).$

→ elements a & b are called comparable if $(a,b) \in R$ otherwise they are called non-comparable.

- **Graphical representation of POSET**

eg $A = \{4,5,6,7\}$
$R = \leq$

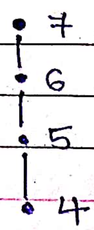
→ $R = \{(4,4), (4,5), (4,6), (4,7), (5,5), (5,6), (5,7), (6,6), (6,7), (7,7)\}$



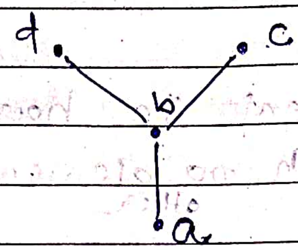
can be converted into Hasse Diagram

→ To convert ~~he~~ directed graph into hasse diagram:
① Remove all self loops.
② Remove transitive edges.
③ Remove directed edges & use undirected edges.
④ Instead of nodes use '•' to represent element

Hasse diagram ⟹ (bottom-top)



eg

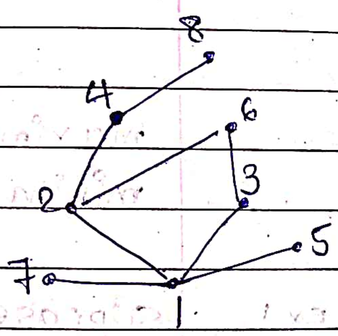


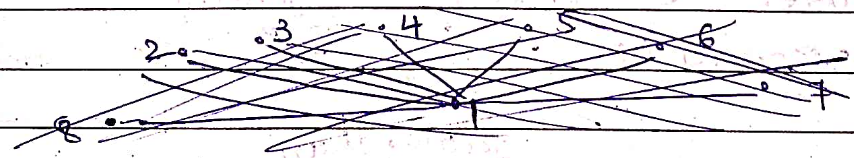
$\{(a,b), (b,d), (b,c), (a,d), (a,c)\}$

Que construct hasse diagram for given poset
$R = \{(a,b) \mid a \text{ divides } b\}$
on set
$S = \{1,2,3,4,5,6,7,8\}$

→ $R = \{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8),$
poset → $(2,4), (2,6), (2,8), (3,6), (4,8),$ self loops $\}$

hasse diagram :





→ Maximal Element

It is an element of poset which is not related to any other element.

eg In above example, 8, 6, ~~4~~, 5, 7 ~~is~~ are maximal elements

DATE:
PAGE:

Minimal Elements:

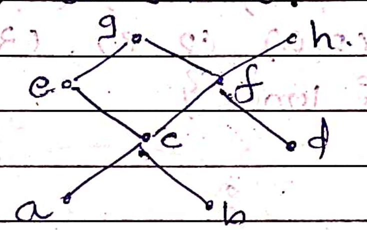
The lowest element in hasse diagram or the element with which no other elements are related.

eg No other ~~n~~ element is there. (x, 1).
1 is minimal element.

Upper bound:

Consider B as a subset of partial order set A. An element $x \in A$ is called upper bound of B if $y \leq x$, for y in B.

eg.



maximal elements = g, h
minimal elements = a, b, d.

ex 1 suppose, B = {e, c} (given)

(i) UpperBound (B) = {g, e}
greater than e&c

h is not related with e so it shouldn't add.

(ii) LowerBound (B) = {a, b, c}

ex.2 B = {c, f, d}

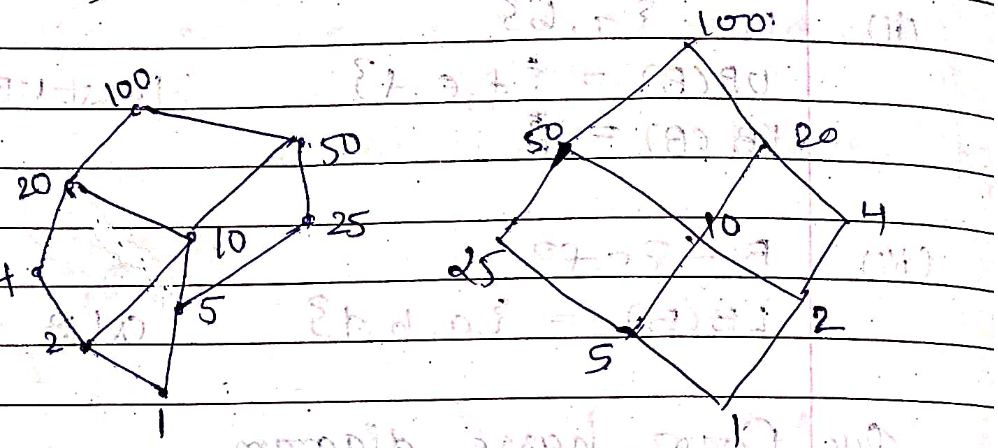
(i) UB (B) = {g, h, f}

LB (B) = ~~{a, b, c, d}~~ ϕ

Ex. Draw hasse diagram for given ~~ol latai~~ relation R is device relation defined on set S

$S = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$

→



(i) $B = \{5, 10\}$

$UB(B) = \{20, 50, 100, 10\}$

$LB(B) = \{1, 5\}$

(ii) $B = \{5, 10, 2, 4\}$

$UB(B) = \{20, 100\}$

$LB(B) = \{1\}$

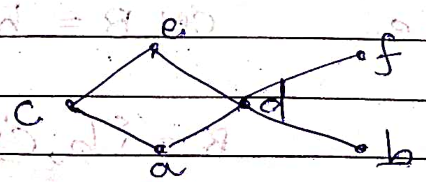
→ Least upper bound / (Supreme, join point)

It is the min. element in the upper bound

→ Greatest lower bound / (~~Grea~~ infimum, meet

It is the greatest ele. in the lower bound.

eg



(i) $B = \{c, d\}$

UB(B) = {e}

LB(B) = {a}

(ii) B = {a, b}

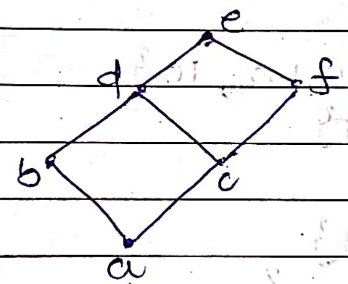
UB(B) = {d, e, f} Least UB = d

LB(B) = ϕ

(iii) B = {e, f}

LB(B) = {a, b, d} GLB = d

Que Draw hasse diagram.



(i) B = {a, e, f}

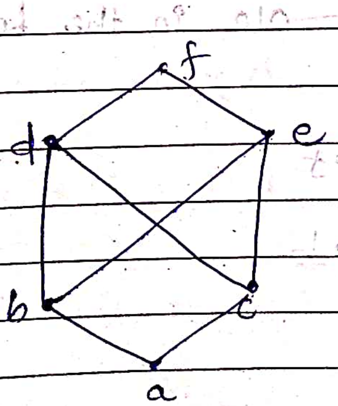
LB(B) = {a} LUB = e

UB(B) = {e} GLB = a

(ii) B = {d, c}

LB(B) = {a, c} GLB = c, LUB = d

Que



B = {d, e}

GLB = b, c , LUB = d, e

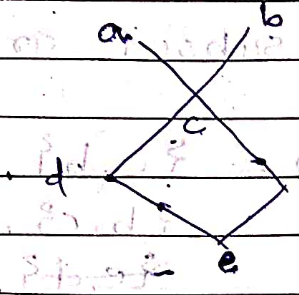
B = {b, c}

GLB = a , LUB = b, c

• **Lattice**

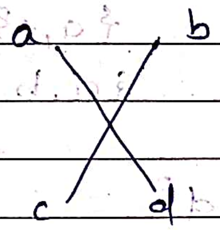
→ A POSET is called Lattice if for every two elements in the set we can find GLB and LUB $\boxed{\text{unique}}$

Eg



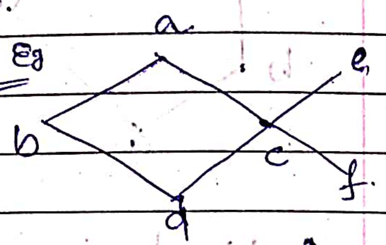
Whether this is lattice or not?
→ No.
Because No UB for a&b.

Eg



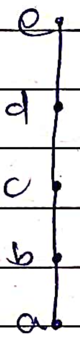
→ No

Eg



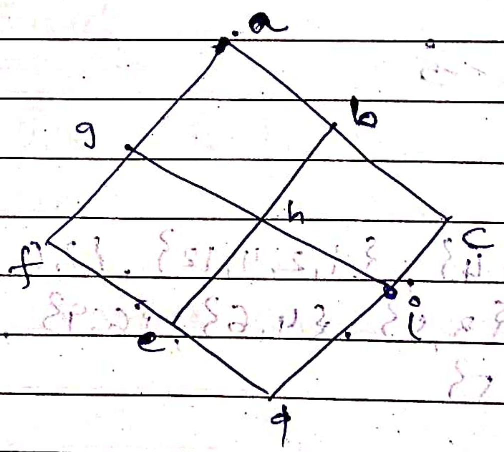
→ No

Eg



→ Yes. because for every two element we can find GLB & LUB.

Eg



→ ~~Yes~~ No

~~for~~ ~~(g,b)~~ (f,c)

LB = ~~e, i, d~~ g, b, a

~~GLB~~ LUB = g, b

Not valid

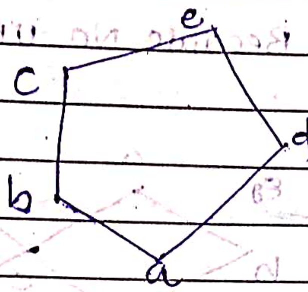
It should be unique

DATE:
PAGE:

Chain and Anti-chain

let (S, R) is a POSET. A subset of S is called chain if every two elements in the subset are related and anti-chain if No two elements in the subset are related.

Eg

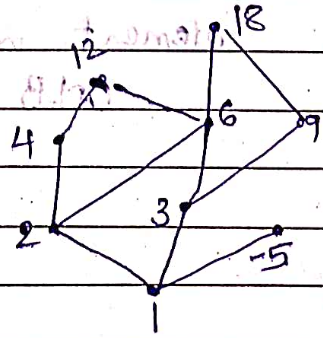


Chain = {a, b}, {b, c}, {c, e}, ~~{c, d}~~, {a, c}, {a, e}, {b, c, e}, {a, b, c, e}

Anti-chain = {c, d}, {b, d}

Eg

S = {1, 2, 3, 4, 5, 6, 9, 12, 18}
R = divides relation



Chain = {1, 4}, {1, 2, 4}, {1, 2, 4, 12}, {2, 6, 18}
Anti-chain = {2, 5}, {3, 5}, {4, 6}, {6, 9}, {4, 5, 6}
{4, 6, 9, 5}

longest chain & anti-chain length = 4

single element can be considered as chain or antichain.

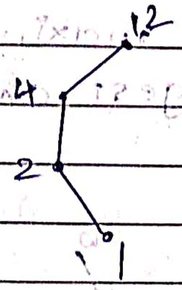
DATE:
PAGE:

(Relation)

- Totally Ordered Set

A POSET (S, R) is called totally ordered if S is a chain.

eg.

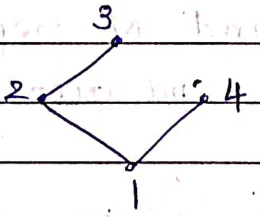


$S = \{1, 2, 4, 12\}$
$R =$ divides
$\longrightarrow$ Totally Ordered set

Theorem : 1

$\longrightarrow$ Let (S, R) be a POSET if the length of longest chain in S is ~~on~~ 'n' then the elements of S can be partitioned into n disjoint anti chains.

eg



$n = 3$
3 diff. antichains (disjoint)
$\{3, 4\}, \{2\}, \{1\}$

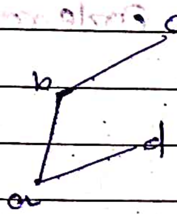
Proof

(i) Basis of Induction:
let's assume $n = 1$ (single elements)
if $n = 1$ then we can find one antichain which contains all the elements of set.

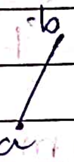
eg a · b antichain = $\{a, b\}$

(ii) Induction Hypothesis
Assume that the statement is true when the length of longest chain is $(n-1)$

(iii) Induction step
let's assume M is set of maximal elements so the set P−M has ~~so~~ the longest chain of n−1 elements.



$M = \{c, d\}$
$n = 3$



P−M
$n = 2$
$= n-1$

So according to hypothesis the theorem holds for POSET P−M.

Now, $P = (P-M) \cup M$ and M represents set of maximal elements and also anti-chain.

$P = (P-M) \cup M$
$= \underbrace{n-1}_{\text{anti chain}} + \underbrace{1}_{\text{already anti-chain}} = n.$

So, P can be partitioned into $(n-1)+1$ disjoint antichains. So theorem is true.

→ From this theorem,

Result: If a partial order set is containing $mn+1$ no. of elements then either there is an antichain of $m+1$ element or there is a chain of $n+1$ elements.

<u>Eg</u> $mn+1 = 8$

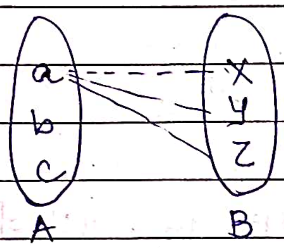
$mn = 7$

$m = 1, \quad n = 7$

<u>Functions</u>

→ A binary relation from $A \to B$ is called <u>function</u> if for every element in A there is an element in B such that (a,b) is in R.

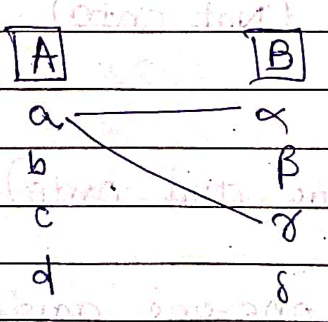
<u>Ex</u>



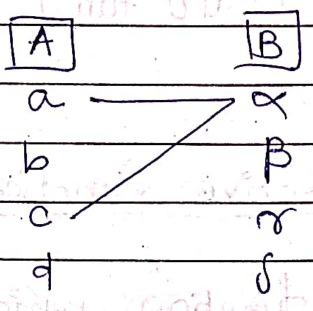
$R = \{(a,x), (b,y), (c,z)\}$ is called function of $A \times B$.

– fuction is special kind of relation.

→ In a <u>general</u> function one element of A never points to more than one element of B.
However reverse scenario is allowed. (if more than one element from A is pointing to ~~ete~~ same element of B is allowed.)



(Not allowed)



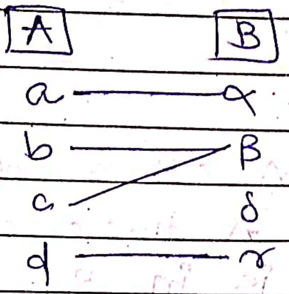
(Allowed)

DATE:
PAGE:

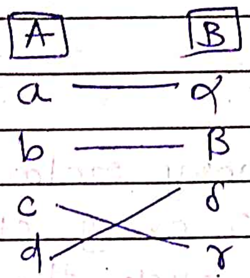
• One-One (Injective) function:

A function from $A \to B$ is called one-one function if no two elements of A have the same image.

Eg



(Not one-one)

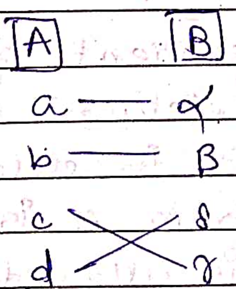


(one-one function)

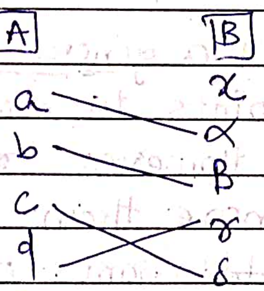
• Onto (Surjective) function:

A function from $A \to B$ is called Onto if every element of B is the image of one or more elements of A.

Eg



(onto funᶜ)



(Not onto)

Not valid there should be corresponding element in A.

• Bijective function: (One-one and onto)

A function which is both one-one and onto is called Bijective function.

DATE:
PAGE:

(i) → To show any function is one-one we have to assume $f(\alpha) = f(\beta)$ and prove that $\alpha = \beta$.

Eg.1 A function is defined on R. $f : R \to R$. $f(x) = 2x+3$. Prove that this function is one-one

→ $f(x) = 2x+3$
$f(1) = 5$, $f(2) = 7$.

Assume, $f(\alpha) = f(\beta)$
$\Rightarrow 2\alpha+3 = 2\beta+3$
$\Rightarrow 2(\alpha-\beta) = 0$
$\Rightarrow \boxed{\alpha = \beta}$

Hence, ~~it~~ ~~is~~ this fun^c is one to one function.

Eg.2 $f : R \to R$
$f(x) = x^2$. Determine one-one or not?

→ Assume, $f(\alpha) = f(\beta)$
$\Rightarrow \alpha^2 = \beta^2$
$\Rightarrow \boxed{\alpha = \pm\beta}$

Hence, it is not one-one function.

(ii) → To verify whether the function is onto or not check that for every y in codomain there must exist some x in the domain such that $f(x) = y$

$f : A \to B$ (B is codomain of A)

DATE:
PAGE:

Eg.3 $f : N \rightarrow R$
$f(x) = 2x + 3$

→ let $f(x) = y \Rightarrow 2x + 3 = y$
$$\boxed{x = \frac{y-3}{2}}$$

Since $y \in R$, $\left(\frac{y-3}{2}\right) \in R$

So, for every y it will not be possible to find corresponding $x \in N$.

Eg.4 $f : R^+ \rightarrow N$, $f(x) = \lfloor x \rfloor$

→ ~~let $f(\alpha) = f(\beta)$~~ Given fun^n is not one-one but ~~it~~ is onto function.

Graph

→ Graph : Graph is a collection of nodes & edges. where nodes are called vertices and edges are connection between the vertices.

→ node can be any object.

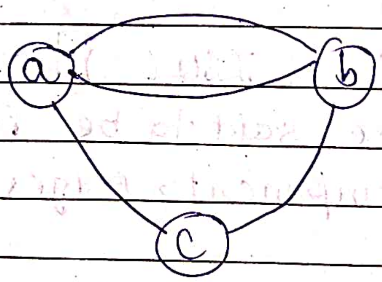
→ Graph
- ① Directed graph
  - always represent one way relation
- ② Undirected graph
  - It is two way relation.

→ Graph is mathematically represented as (V, E) where V is set of vertices and E is binary relation if the graph is directed. However E is also set of multisets if the graph is undirected.

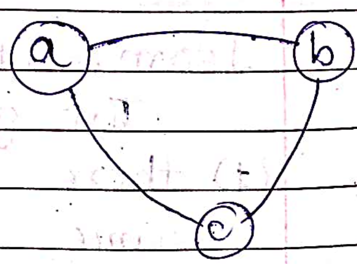
(i) Simple Graph.

A simple graph does not have more than two edges betn any two vertices and it has no self loops.

Eg.



(Not simple graph)

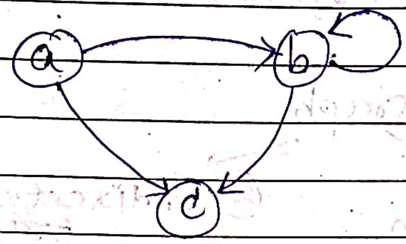


(Simple graph)

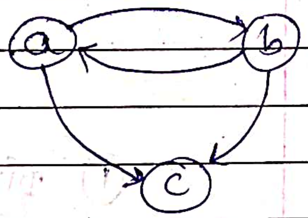
(ii) Asymmetric Directed Graph.

A directed graph that has at most one directed edge bet^n pair of vertices is called Asymmetric digraph. (directed graph)

Eg.



(Asymmetric)



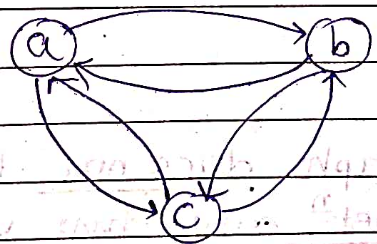
(Not Asymmetric)

- self loops are allowed.

(iii) Symmetric directed graph

In this kind of graph for every edge a→b, there is also an edge from b→a.

Eg



(Symmetric digraph)

(iv) Isomorphic Graph [ISO( )]

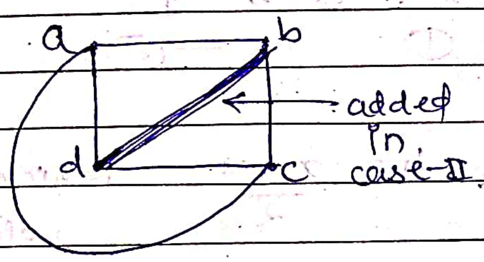
Two graphs are said to be isomorphic if

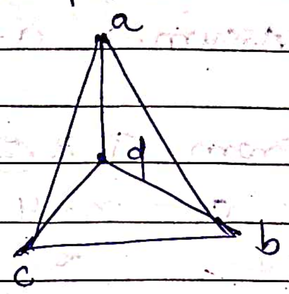
(1) there no. of components (edges & vertices) are same

(2) edge connectivity is retained.

DATE:
PAGE:

Eg.   Graph-1   Graph-2





Case-I   Nodes = 5   Nodes = 5
edges = 5   edges = 6

Hence Not Isomorphic.

Case-II   Nodes = 5   Nodes = 5
edges = 6   edges = 6

Hence Isomorphic

* Continue in function.

Eg $f : Z \times Z \to Z \times Z$
~~f(A, B)~~ $f(a, b) = (a+b, a-b)$

→ for one to one,
$f(a,b) = f(c,d)$ // Assume
$\Rightarrow (a+b, a-b) = (c+d, c-d)$
$\Rightarrow \boxed{(a+b) = c+d}$ — (1)   and $\boxed{a-b = c-d}$ — (2)

from (1) & (2),
$a = c$, $b = d$; $a, b \in Z$
(add both)   (sub both)
So, function is one-one.

For onto,

Assume, $a+b=x$, $a-b=y$
—(1) —(2)

From (1) & (2)

$a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$ which does not belongs to Z.

So, a, b does not always belong to Z

So, function is not onto function.

Eg

A - |m| ^{3} B - |n| ^{2}

a x

b y

c

How many functions from A to B can be possible?

- a has 2 options
  b has 2 options
  c has 2 options

$2\times2\times2 = \boxed{8}$

Generalized function : $\boxed{n^m}$ ∴ $2^3 = 8$.

Eg

A	B
a	1
b	2
c	3
d	4
e	5
	6
	7

How many fun$^{\underline{c}}$s from A to B are possible which are one-one

- a has 7 options
  b has 6 options
  c has 5 options
  d has 4 options
  e has 3 options.

$m=5$, $n=7$

$= 7\times6\times5\times4\times3$

- Generalized formula : $\boxed{{}^nP_m} = \frac{n!}{(n-m)!}$

where $n \geq m$.

Eg.

A	B
a	x
b	y
c	
$m=3$	$n=2$

How many fun^c s from A to B are possible which are onto function

$$= \boxed{\sum_{r=1}^{n} (-1)^{n-r} \cdot {}^nC_r \cdot r^m}$$

$$= \sum_{r=1}^{2} (-1)^{2-r} \cdot {}^2C_r \cdot r^3$$

- **Pigeonhole Principle**

Container : □ □ □ $\quad i = \left\lceil \frac{4}{3} \right\rceil$

Items : △ △ △ △

|Container| < |Items|

→ Pigeonhole principle ~~says~~ states that if n items are put in m containers $(n > m)$, then at least one container must contain more than one item.

let D and R are finite ~~states~~ sets where $|D| > |R|$ then any function $f : D \to R$ there exist elements $d_1, d_2, \ldots, d_i$ such that $f[d_1] = f[d_2] = \ldots = f[d_i]$.

and $i = \left\lceil \frac{|D|}{|R|} \right\rceil$.

DATE:
PAGE:

i = no. of elements which will be mapped into container
(at least)

Eg 50 objects are to be coloured with 7 diff. colours. then at least how many objects have the same colors.

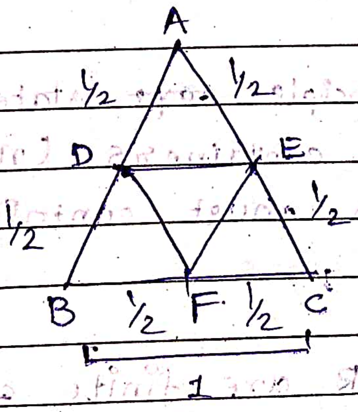
→ Here, $|D| = 50$, $|R| = 7$

$\therefore$ let i be No. of elements which will ~~be~~ belongs to same box.

$$\therefore i = \left\lceil \frac{|D|}{|R|} \right\rceil = \left\lceil \frac{50}{7} \right\rceil = \boxed{8}$$

eg. Consider an equilateral triangle whose size are equal to 1 unit. ~~Now, show~~ that if any 5 points are chosen either lying on or inside the triangle. then prove that ~~any two points~~ then two of them must be no more than 0.5 units apart.

→



Now, the original triangle is divided into 4 equilateral triangles.

Here, 5 points and 4 triangles.

∴ According to pigeonhole principle,

5 points are to be accomodated in 4 triangles. so,

at least $\lceil 5/4 \rceil = 2$ points must belong to same triangle and distance bet^n those 2 points are always less than 0.5($\frac{1}{2}$).

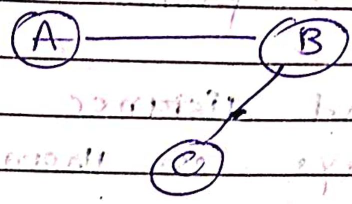
Eg. Single day 1 game minimum. no more than 132 games. 77 days are there.

Prove that there is a period of consecutive days the player will exactly play 21 games.

Graph Continue

→ **Path :** A path is sequence of vertices connected by edges such that each pair of consecutive vertices is connected by an edge.

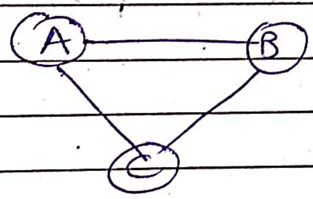
Eg



path : AB, BC, ABC.

→ **Circuit :** Circuit is a path in which terminal vertex is connected with initial vertex.

Eg

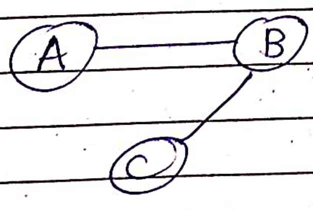


Circuit : ABCA
(come back from where you start)

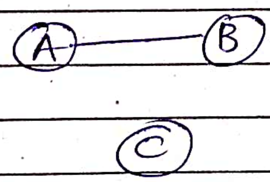
→ **Connected & Disconnected Graph :**

The graph is connected if there is a path bet^n every two vertices, otherwise it is disconnected graph.

Eg.

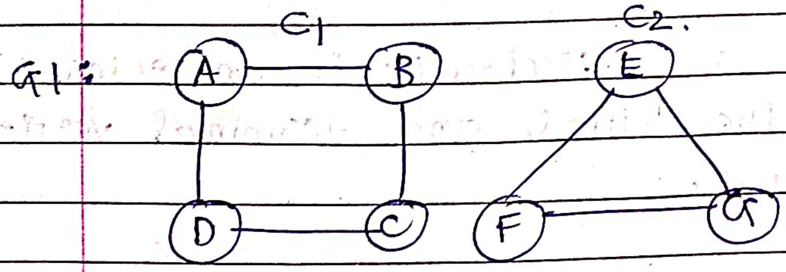


Connected graph



Disconnected graph.

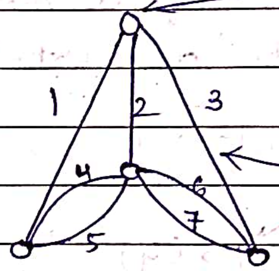
DATE:
PAGE:



Disconnected graph.

- In disconnected graph, it contains two or more components. Each component is a connec graph.

→ Eulerian Path

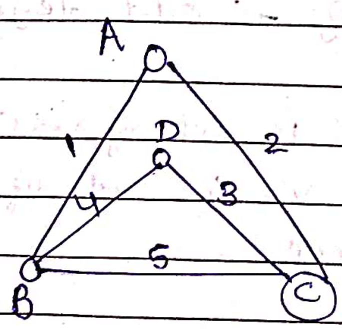


← No euler path

No Eulerian Path on which he can walk
↳ given a graph, path from which he can't repeat the path.

- Euler path is a path that uses every edge in a graph exactly once.

Eg.

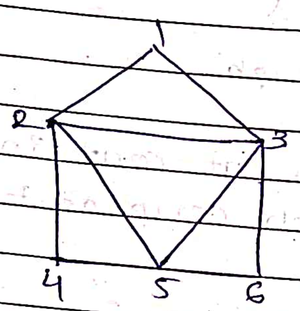


it is euler path.

DATE:
PAGE:

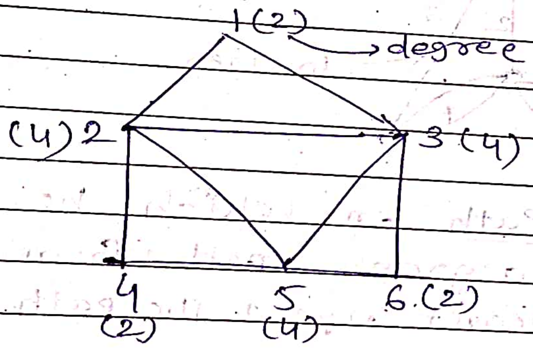
→ **Euler circuit** : Euler circuit is an Euler path where the initial and terminal vertex are same.

Eg



~~2135~~
4213253654 is euler circuit

→ **Degree** : degree of a vertex is no. of edges connected to a vertex.



Hand shaking theorem.

$$\sum_{i=1}^{n} \text{degree}(V_i) = 2 \times \#\text{ of edges}$$
$$= 2 \times 9$$
$$= 18.$$

sult : In an undirected graph, it always has even no. of vertices of ~~at~~ odd degree.

Proof : let V_1 is set of vertices having even degree & V_2 is set of vertices having odd degree.

$$\sum_{i=1}^{n} \deg(V_i) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

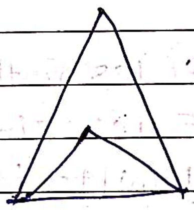
~~⇒ 2.e = even no. +~~

Now, first quantity is even no. and the second quantity also must be even no. and for this condition the no. of vertices in set V2 must be even no.

→ Theorem-II

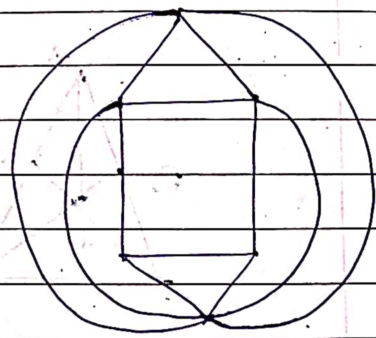
- A graph will contain euler path if it contains at most two vertices having odd degree

- Graph will contain euler circuit if all vertices have even degree.



odd degree = 2.

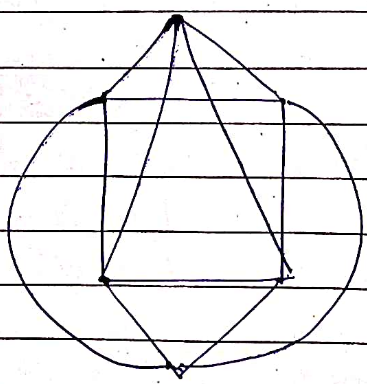
∴ euler path. is exist



euler path.

odd degree = 2.

Res



euler circuit

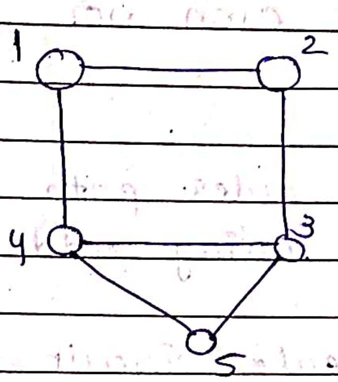
∴ also euler path.

Date:
Page:

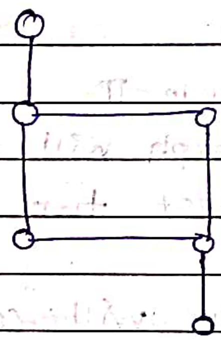
→ Hamiltonian

– Hamiltonian Path :

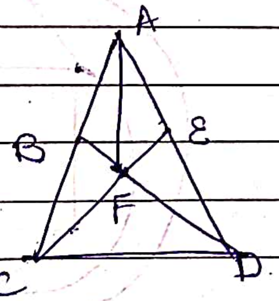
A hamiltonian path is a path which passes through all vertex of the graph.



Hamiltonian path



No hamiltonian path.



Euler path — X
Euler circuit — X
Hamiltonian path — ✓
Hamiltonian circuit — ✓
(ABCDFEA)

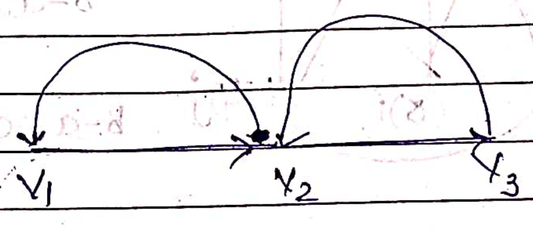
Sessional - III

~~Graph Tree.~~

Theorem 1 A directed graph contains an Eulerian circuit iff it is connected $\boxed{\text{and}}$ the incomming degree of every vertex is equal to outgoing degree.

Eulerian circuit.

Eg



Euler circuit : $V_1 - V_2 - V_3 - V_2 - V_1$

$In(V_1) = 1$ $In(V_2) = 2$

$Out(V_1) = 1$ $Out(V_2) = 2$

$In(V_3) = 1$

$Out(V_3) = 1$

$\therefore$ Theorem is true for above graph. Hence it contains Eulerian circuit.

Theorem 2 A directed graph contains an Eulerian path iff it is connected $\boxed{\text{and}}$ the indegree of each vertex is equal to outdegree with possible exceptions of two vertex. For these vertices, there is a difference of 1 in the indegree and out degree

eulerian path.

Eg. $V_1 \rightarrow V_2 \rightarrow V_3$

$In(V_1) = 0$, $Out(V_1) = 1$

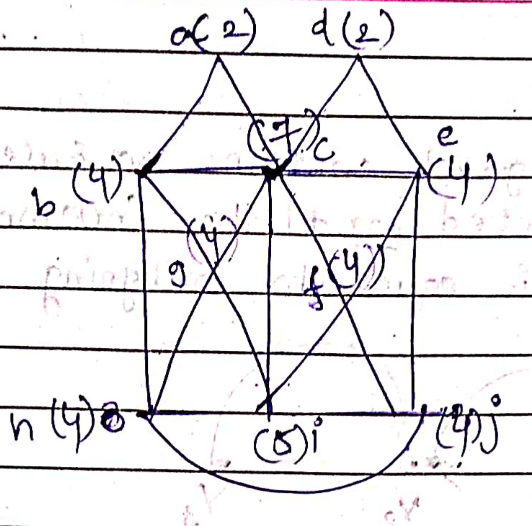
$In(V_2) = 1$, $In(V_2) = 1$ $\leftarrow$ same

$In(V_3) = 1$; $Out(V_3) = 0$.

for v_1 & v_2 Indegree & outdegree are diff but there difference is 1. so it contains Eulerian path.

DATE:
PAGE:

Eg



Euler path — ✓
Euler circuit — ✗
Hamiltonian path — ✓

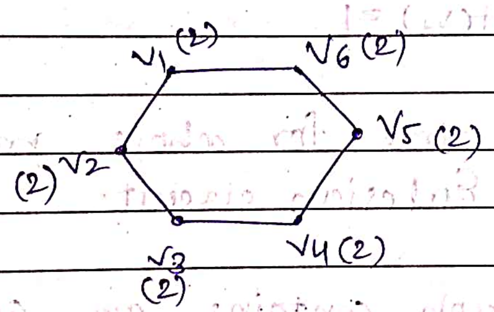
b-a-c-d-e-f-j-i-g-h — HP

b-a-c-d-e-f-j-i-g-h-b → HC.

Theorem 3. G is graph with n vertices, if the sum of degrees of each pair of vertices is $(n-1)$ or larger then there exist a Hamiltonian path.

↳ this is only sufficient condⁿ

Eg



The above theorem is sufficient condⁿ for detection of Hamiltonian path.
It means that even if the condⁿ is not satisfied there may be still hamiltonian path.

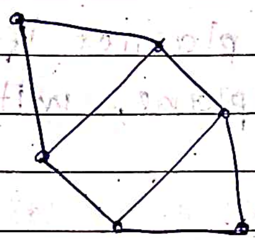
→ For finding Hamiltonian path,
we only have sufficient condⁿ but not be sufficient & necessary. this is the problem.

DATE:
PAGE:

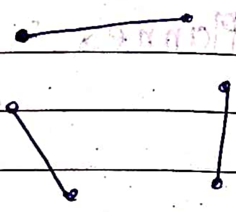
- Factors of a graph

K factor of a graph is defined as a spanning subgraph such that degree of each vertices is K.

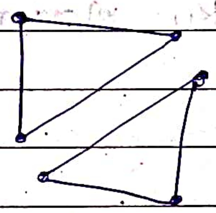
eg



graph



1 factor graph

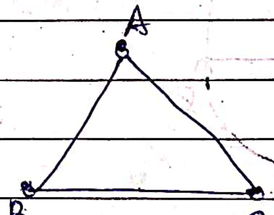


2 factor graph

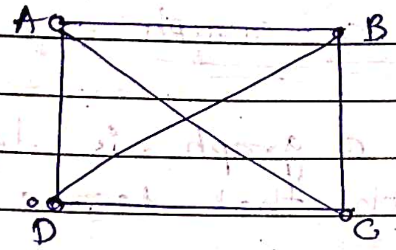
- Complete graph (Kn)

A graph with n vertices is called complete if there is a connectivity betn every pair of vertices.

eg K_3.

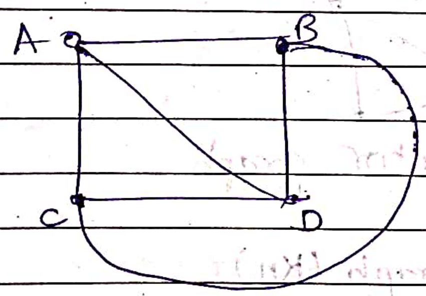


K_4

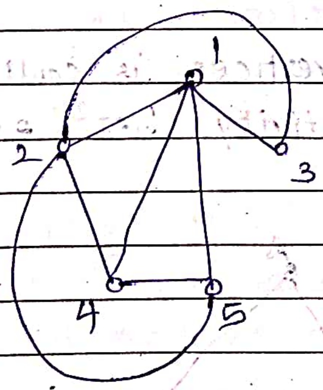


- **Planner :** A graph is planner if it can be drawn on a plane, without crossing of edges.

eg In previous example, K_4 is not planner graph but K_3 is planner graph.
We can make K_4 planner graph.

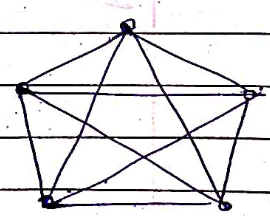


eg K_5

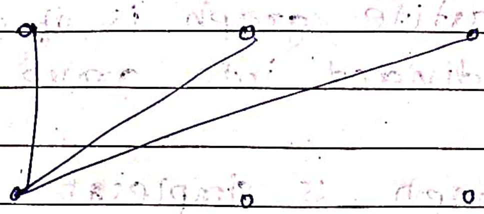


K_5 is non planner graph. because we can't connect 4 & 3.

K_5 :



K6 :



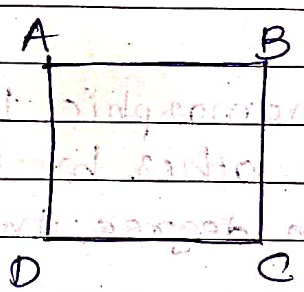
K6 is non planner

- starting from K5 all other are non-planner

- ## Bipeartite Graph

A bipartite graph is a graph whose vertices can be divided into two independent sets U & V such that every edge in the graph connects the vertex from U → V or V → U.

eg

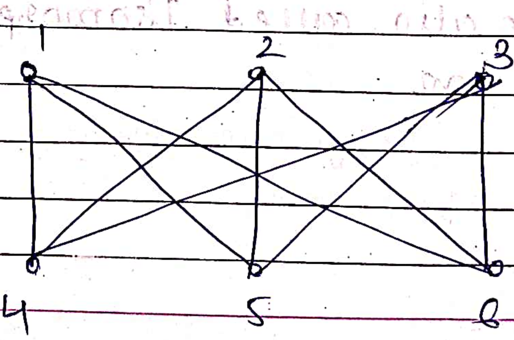


Here, U = {A, C}
V = {B, D}

- ## Complete Bipartite Graph

In Complete Bipartite Graph, every vertex of set U is connected to every vertex of set V

eg.



~~Not~~ Notation is $K_{3,3}$

Here,
Complete Bipartite graph is of 6 vertices and vertices are divided into group of 3 vertices.

eg Which graph is simpleset Non-planner graph?
($K_{3,3}$ or K_5)
↓ ↓
9 edge 10 edges

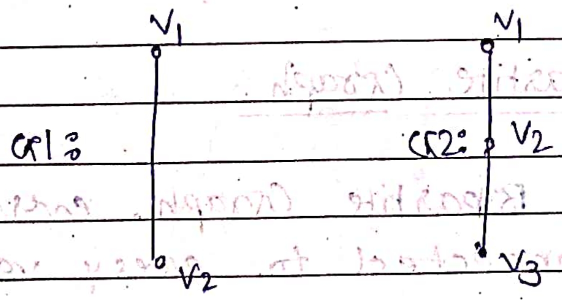
- Based upon no. of edges, we can say $K_{3,3}$ is the simplest non-planner graph.

→ $K_{3,3}$ & K_5 are called Kuratowski graphs.

• Homomorphism

Two graphs are called Homomorphic they can be transformed into each other by insertion or removal of a vertex with degree two.

eg



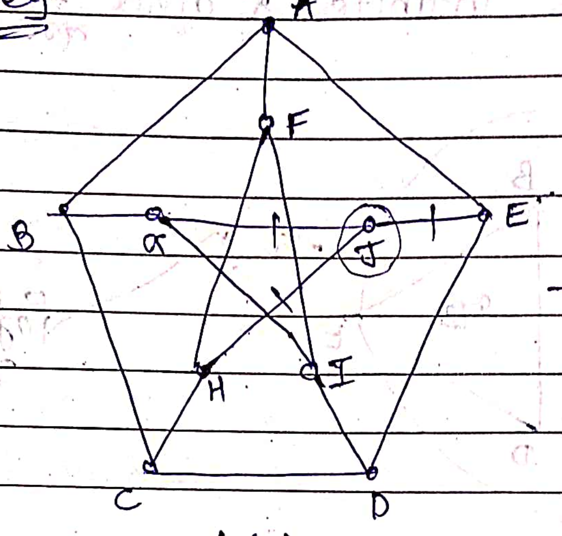
G1 & G2 are called Homomorphic graphs.

They are also called Isomorphic ~~of~~ within degree two.

Kuratowski's Theorem

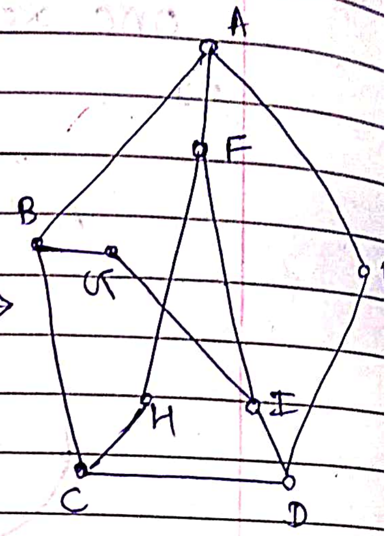
Every non-planner graph has a subgraph that is Homomorphic to K_5 or $K_{3,3}$

eg



(G)

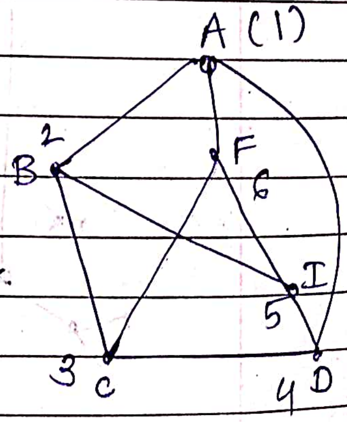
sub graph →



From subgraph, we have to delete 3 vertices of 2 degree to make isomorphic with $K_{3,3}$.

Delete vertices E, a, H
because it has degree 2.

New graph,



(G1)

$U = \{1, 5, 3\}$
$V = \{2, 4, 6\}$

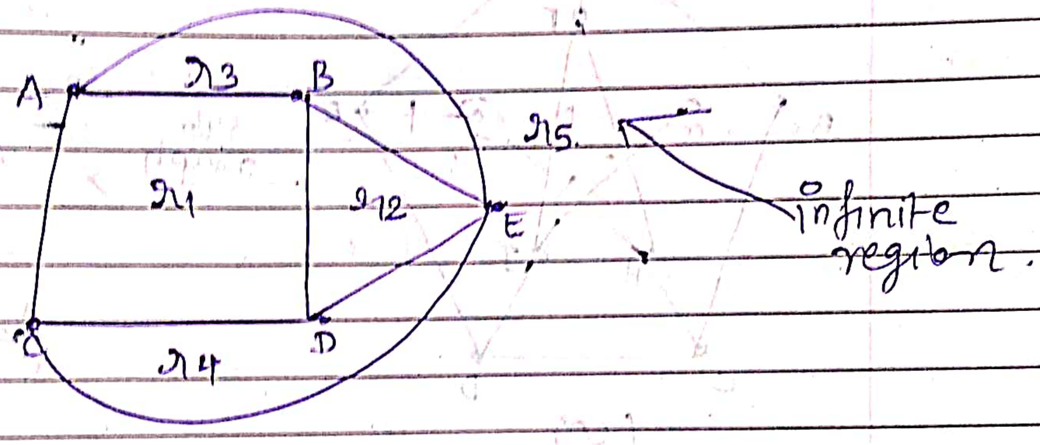
graph is a non-planner graph and it is Homomorphic to $K_{3,3}$.

DATE:
PAGE:

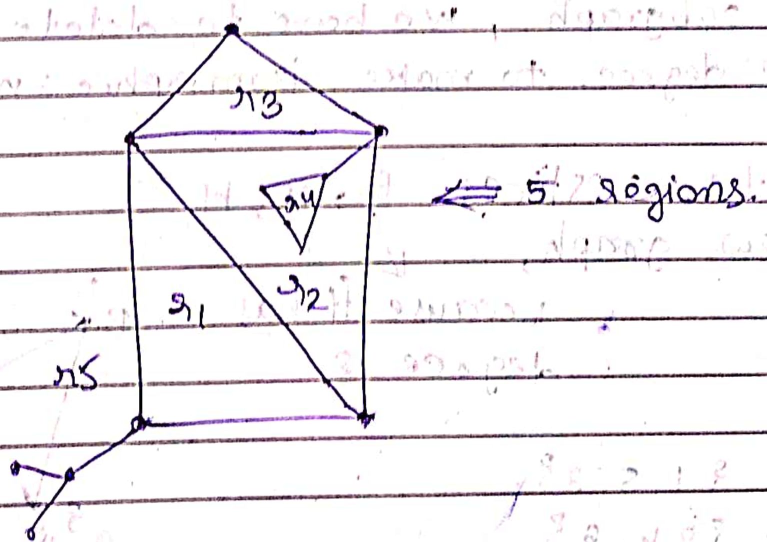
- ## Region OF a Graph

A Region of the planner graph is an area of the plane that is bounded by edges and cannot be further sub divided. Every graph has one region is infinite & other ~~is~~ are finite.

eg



eg



→ If the graph has E edges V vertices & r regions then,

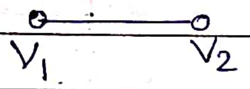
$\boxed{V - E + r = 2}$ ← Euler's formula for planner graph.

Result: In a connected planner graph, with e edges & v vertices

$$\boxed{3v - e \geq 6}$$

- Proof of Euler's formula for planner graph:

step: 1) Basis of Induction:



for given graph,

$e = 1$ (~~Assume~~) Let

$v = 2$

$r = 1$

step: 2 Induction Hypothesis:

Assume that, formula is correct for $e = K$

$$v - e_K + r = 2$$

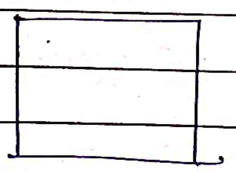
Induction step:

Prove the statement for $e = K+1$

case-I. In this case, Adding a new edge increases the no. of regions by 1. so,

$$V_{K+1} - e_{K+1} + r_{K+1}$$
$$= V_K - (e_K + 1) + r_K + 1$$
$$= V_K - e_K + r_K = 2$$

Eg

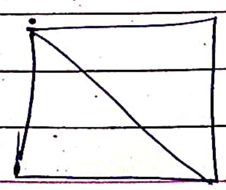


$e = 4$

$v = 4$

$r = 2$

And now,



$e = 5$

$v = 4$

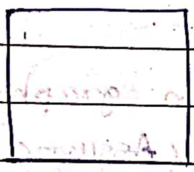
$r = 3$

case-II In this case, adding a new edge increases the no. of vertices by 1. ~~and~~ so,

$$v_{k+1} - e_{k+1} + r_{k+1}$$
$$= v_{k+1} - (e_{k+1}) + r_k$$
$$= v_k - e_k + r_k = 2.$$

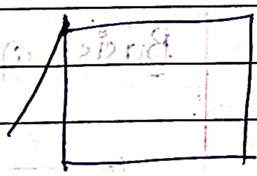
Hence, Hypothesis is proved.

Eg.



$v=4$
$e=4$
$r=2$

And now



$v=5$
$e=5$
$r=2$

Hence, $v_k - e_k + r_k = 2$ is proved.

Tree And Cutset

- Tree is a connected acyclic undirected graph
- Tree is always undirected.
- Tree represents hierarchical relationship betⁿ individual elements.

• **Properties of the Tree:**

(1) There is an unique path betⁿ every two vertices in the tree.

(2) The no of vertices in a tree is one more than the no. of edges. ($\because e = v - 1$)

(3) A tree with two or more vertices has at least two leaves.

proof :- let the no. of total vertices $= n$
and out of that leaves $= K$.
$\therefore$ Non-leaves $= n - K$.

sum of degrees of all vertices $\sum_{i=1}^{n} d(v_i) = K(1) + (n-1) \cdot y$

y is degree of non-leaves.

And $y \geq 2$, so,

$\sum d(v_i) \geq K + (n-1) \cdot 2$ ($\because$ Principle of inequality)

$$\boxed{\sum_{i=1}^{n} (d(v_i)) \geq 2n - K}$$

Now, we know;

$$\sum_{i=1}^{n} d(v_i) = 2e \quad (\because \text{Hand shaking theorem})$$

$$= 2(n-1) \xrightarrow{} \text{according to previous property}$$

So,

$$2(n-1) \geq 2n-k$$

$$2n-2 \geq 2n-k$$

$$\therefore \boxed{k \geq 2}$$

Hence, it is proved.

- **Characteristics of the Tree:**

(1) A graph in which there is an unique path (acyclic) betn every pair of vertices is a tree.

(2) A connected graph with $e = v-1$ edges is a tree

proof : Suppose, a connected graph of v vertices and v-1 edges has a cycle

Now,

to ~~make~~ the remove the cycle we remove one edge from the graph so the no. of edges will be V-2. but this will make the graph disconnected. which is contradiction to the original statement. So, the graph does not have any cycle & it is a tree.

DATE:
PAGE:

(3) A graph with $e = v-1$ that has no circuit is a tree.

- **Rooted Trees**

— A directed graph is said to be directed tree if it becomes a tree when directions are ignored.

— A directed tree is called rooted tree if there is exactly one vertex whose indegree is 0 & all other nodes directly or indirectly originate from that node whose indegree is 0.

— In a directed tree all vertices whose outdegree is non-zero are called branch or internal nodes.

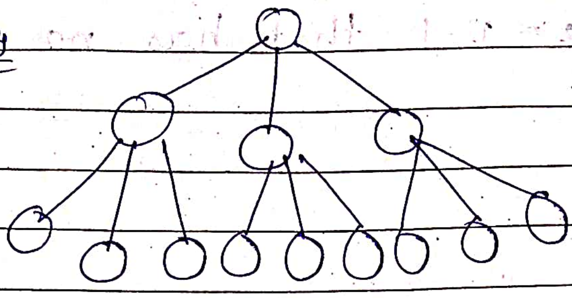
- **Ordered Tree**

An ordered tree is a rooted tree in which the edges originating from each branch is labeled as 1, 2, 3, ... i

→ An ordered tree in which every branch has at most m children is called m-ary tree.

→ An m-ary tree is regular if every brach node has exactly m children.

eg



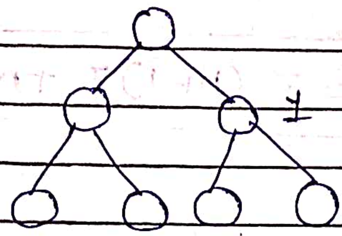
regular
3-ary

- **Path Length :**

Path length in a Rooted tree :

Path length of a vertex is no. of edges in the path from root to vertex.

eg



→ The Height of the tree is maximum of path lengths in a tree.

– 'i' is no. of internal nodes in the tree.
't' is no. of terminal nodes in the tree, then for m-ary tree

$$\boxed{(m-1)\,i = t-1}$$

Que 19 lamps & single electricity outlets. each chord has 4 outlets. How many chords will be required ?

→ $m = 4$

$t = 19$

$\therefore (m-1)\,i = t-1$

$\therefore 3i = 18$

$\therefore \boxed{i = 6}$

$\therefore$ 6 chords will be required.

Result: i = no. of internal nodes

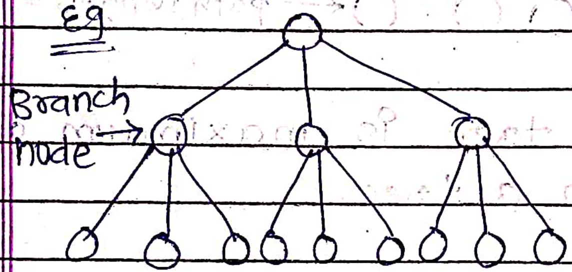
I = sum of Path lengths of all Branch nodes

E = sum of Path lengths of leaves

~~$E = I +$~~

$$\boxed{E = (m-1)I + mi}$$

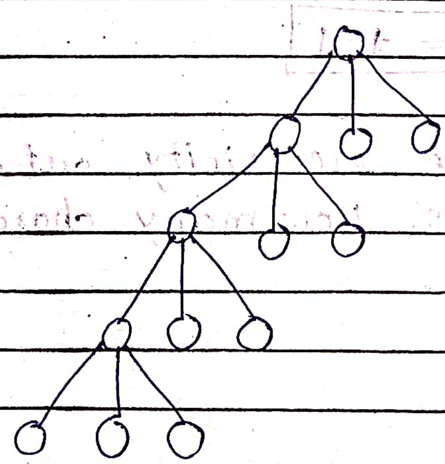
Eg



$I = 3$
$i = 4$

So, $(3-1)3 + 3(4) = \boxed{18}$

Eg.



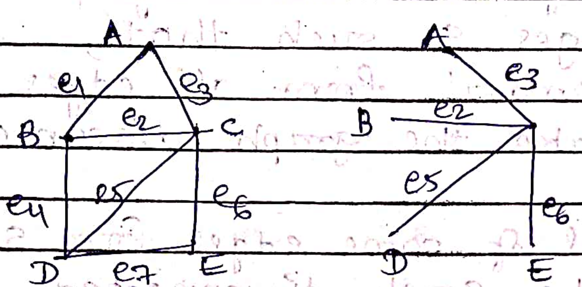
$I = 6$
$i = 4$

So, $(3-1)6 + 3(4)$
$= \boxed{24}$
$= E$
= sum of path lengths of leaves

Date: / /
Page No.

Spanning Tree

- A spanning tree of a connected graph is a spanning subgraph which is a tree.
  ↳ it should contain all vertices.



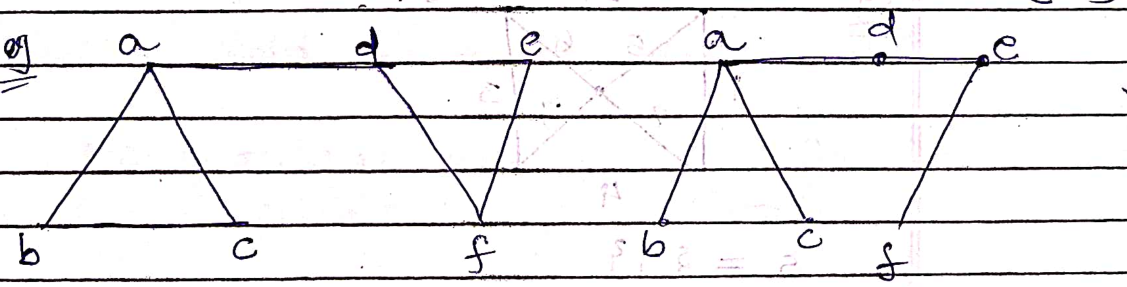
e3, e2, e5, e6
— Branch

- An edge of the tree which is included in the spanning tree is called branch.

- An edge of the graph which is not included in the spanning tree is called Chord.
  eg e1, e4, e7

- A connected graph with e edges and v vertices is converted into a spanning tree ~~if there is a graph~~, then no. of branches will be (V−1)

eg



graph
e=7
V=6

spanning tree
e=5
V=6

Date: / /
Page No.

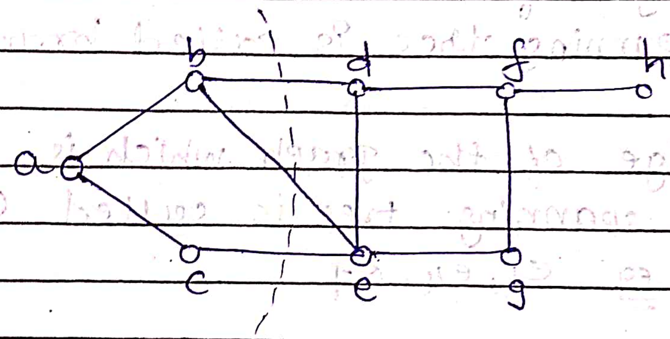
No. of chord will be $\boxed{e-V+1}$

- ## Cut Set

A cut set of the connected graph G is set of edges S such that,
i) the removal from all edges from S will make the graph disconnected.

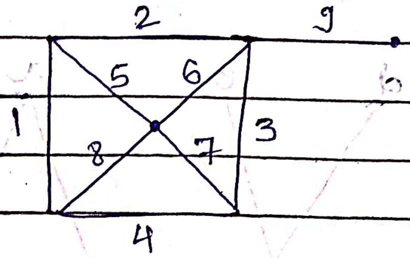
ii) Removal of some edges from S will not make the graph disconnected.

eg



S = {bd, be, ce}
S = {df, eg}

eg



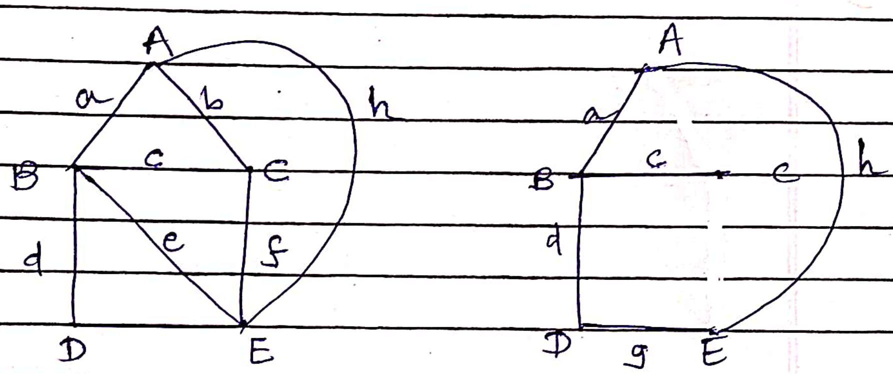
S = {9}
S = {1, 4, 8}
S = {3, 4, 7}
S = {1, 2, 5}
S = {5, 6, 7, 8}

Fundamental Circuit

let T be a spanning tree of connected graph G.

When a chord is added to the spanning tree it will form exactly one circuit. this circuit is called Fundamental Circuit.

eg.



chords = {b, e, f, h}

let h chord. → add it to the spanning tree. it will form cycle.

① A – E – D – B – A ← Fundamental circuit

Similarly, add all chords one by one.

→ No. of fundamental circuit = No. of chords.

② A – B – C – A (b)

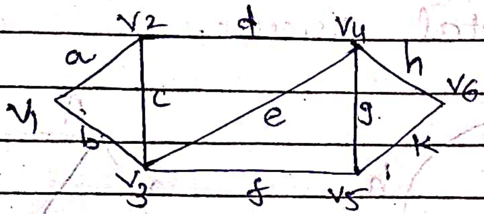
③ B – D – E – B (e)

④ C – E – D – B – C (f)

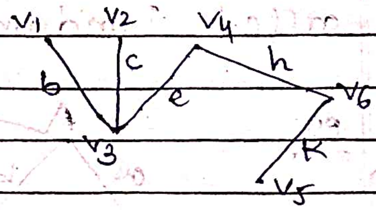
Date: / /
Page No.

Fundamental Cutset

let T be a spanning tree then fundamental cutset is a cutset formed by exactly one branch



graph



spanning tree

branch b = {b, c, e, h, k}

~~fundamental~~
cutset C_1 = {e, d, f} // e
C_2 = {d, a, b} // b
C_3 = {a, c, d} // c
C_4 = {h, g, f} // h
C_5 = {k, g, f} // k

- **Result:**
A circuit and complement of any spanning tree must have at least 1 edge in comman

- **Result:**
A cutset and any spanning tree must have at least 1 edge in comman.

- **Result:**
Every circuit has even no. of edges in comman with every cutset

Date: / /
Page No.

Ex A tree has 2n vertices of degree 1, 3n vertices of degree 2, n vertices of degree 3. Find the no. of vertices & edges in the tree.

→

Ex. A tree has 50 edges. The removal of certain edge from T creates two disjoint trees T1 and T2. No. of vertices in T1 is equal to no. of edges in T2. Find no. of vertices ~~and edges~~ in T1 and T2.

→ $E = 50$

Ans: 25
26

no. of vertices in T1 + no. of vertices in T2 = 51

$\boxed{n_1 + n_2 = 51}$ —— ①

Now, $n_1 = e_2$

$\boxed{n_1 = n_2 - 1}$ —— ②

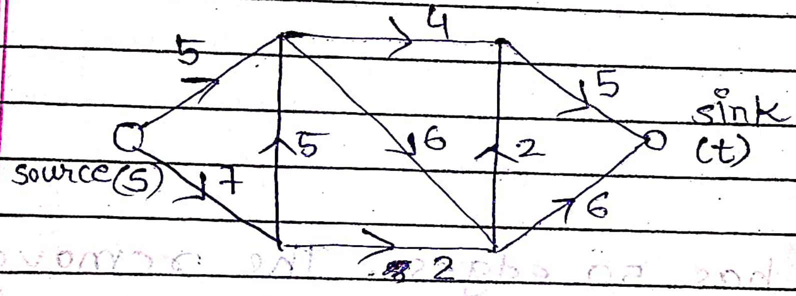
from (1) & (2) $\boxed{n_1 = 25}$ and $\boxed{n_2 = 26}$

Date: / /
Page No.

- ## Transport Network

Transport Network is a weighted directed graph if
i) It is connected and having no loops.
ii) There is only 1 vertex which has no incoming edge.
iii) There is only 1 vertex that has no outgoing edge.
iv) weight of each edge is a non-negative real number.

ex



- The weight of each edge in the transport network is called capacity of the edge.

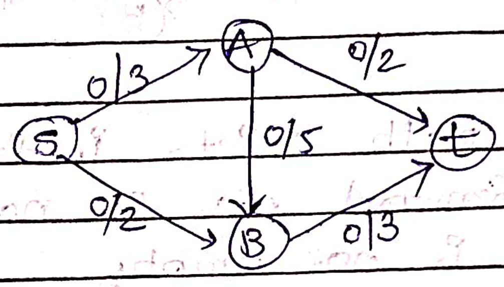
- Flow (ϕ) :
Flow in a transport network is assignment of non-negative numbers to each edge such that
i) For each edge, flow $\leq$ capacity.
ii) For each vertex except source and sink the incoming flow should be equal to outgoing flow.
iii) total outgoing flow at source should be equal to total incoming flow at sink.

→ Maximum flow in Transport Network:

A max. flow is the flow that achieves largest possible value.

⇒ **Ford - Fulkerson's Algorithm**

- Used to find max. flow in given transport network.

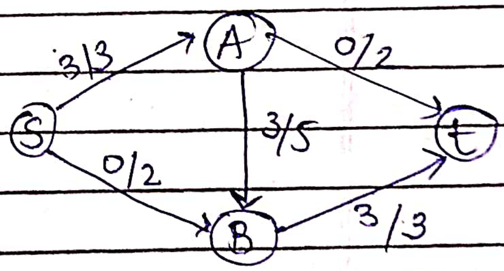


0/3 — flow (current) / capacity

i) S-A-T path ⇒ only we can sent flow of 2

Bottle-neck - max. capacity we can pass through any path.

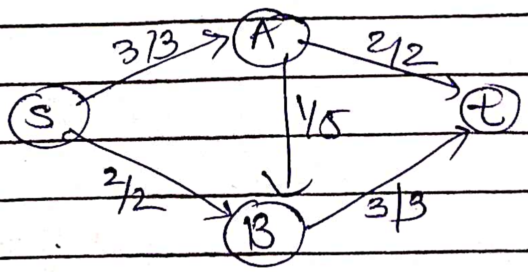
ii) S-A-B-T ⇒ 3



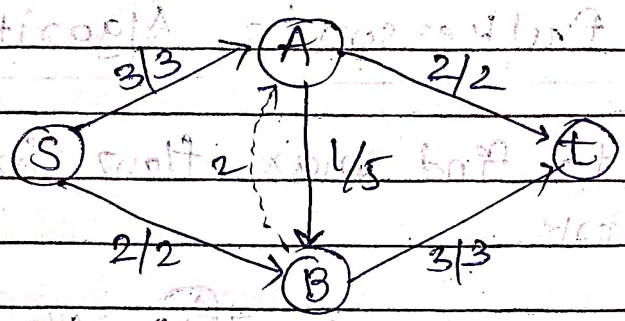
$\boxed{\text{max. flow} = 3}$

But this is not correct

$\boxed{\text{max. flow} = 5}$



In the ford-fulkerson's Algo the reverse edges may be considered if they are not empty and if they lead to the max-flow



The path exists in ford-fulkerson Algo if forward edge is not full and backward edge is not empty