

Stochastic Simulation: Error Control and Sensitivity

Lecture 3 – Concentration, CLT, Variance Reduction, and
Error Propagation

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Lecture 3



Why Error Control Matters

- ▶ Quantify uncertainty of simulation outputs (means, tails, quantiles).
- ▶ Plan sample sizes to meet accuracy targets (half-width, confidence).
- ▶ Reduce Monte Carlo noise per unit cost (variance reduction).
- ▶ Propagate estimation error of inputs (e.g., λ, μ) to outputs.

Markov and Chebyshev

Theorem

Markov (for nonnegative X): $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.$

Theorem

Chebyshev (finite variance): $\mathbb{P}(|X - \mathbb{E}[X]| \geq \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2}.$

- ▶ Valid for any n ; conservative but assumption-light.
- ▶ For sample mean \bar{X}_n : $\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n \varepsilon^2}.$

Bernstein: Bounded and Unbounded

Theorem

Bounded case ($|X_i - \mu| \leq b$ i.i.d., variance σ^2):

$$\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \leq 2 \exp\left(-\frac{n\varepsilon^2}{2\sigma^2 + (2/3)b\varepsilon}\right).$$

Theorem

Unbounded (sub-exponential; MGF bound): if

$\mathbb{E} e^{\lambda(X_i - \mu)} \leq \exp(\lambda^2 \nu^2 / 2)$ for $|\lambda| < 1/b$, then

$$\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \leq 2 \exp\left(-\frac{n}{2} \min\{\varepsilon^2/\nu^2, \varepsilon/b\}\right).$$

CLT and Confidence Intervals

Theorem

$$\text{CLT: } \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \Rightarrow \mathcal{N}(0, 1).$$

- ▶ CI (known σ): $\bar{X} \pm z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$. Unknown σ : replace by s ; use t_{n-1} for finite n .
- ▶ Target half-width ε : $n \geq \left(\frac{z_{1-\alpha/2} \sigma}{\varepsilon} \right)^2$. Relative error:
$$n \geq \left(\frac{z \sigma}{r|\mu|} \right)^2.$$
- ▶ Sequential stopping: continue until $s z / \sqrt{n} \leq \varepsilon$ (with max n).

Note

Berry–Esseen:

$$\sup_x |\mathbb{P}(\sqrt{n}(\bar{X} - \mu)/\sigma \leq x) - \Phi(x)| \leq C \rho_3 / (\sigma^3 \sqrt{n}), \quad C \approx 0.56.$$

Choosing Between CLT and Concentration

- ▶ CLT-based CIs: tighter but asymptotic; validate via pilot runs or Berry–Esseen.
- ▶ Concentration (Chebyshev/Bernstein): non-asymptotic guarantees; may be conservative.
- ▶ Practice: start with pilot, use CLT for sizing; report conservative bounds alongside.

Exponential Example: Tails and Means

- ▶ $X \sim \text{Exp}(\lambda)$: compare empirical tail $\mathbb{P}(X \geq t)$ vs. Markov bound $\mathbb{E}[X]/t$.
- ▶ For sample mean \bar{X}_n : compare empirical $\mathbb{P}(|\bar{X}_n - 1/\lambda| \geq \varepsilon)$ to Chebyshev and sub-exp Bernstein.
- ▶ See notebook for code and plots.

3-State Markov Chain: Convergence

Definition

Irreducible, aperiodic chain with transition matrix P has a unique stationary distribution π with $\pi^\top P = \pi^\top$.

- ▶ Simulate trajectory and track $\|\hat{\pi}_T - \pi\|_1$ vs T .
- ▶ Use warm-up: discard initial transient before estimating steady-state measures.

Antithetic Variates

Definition

Use negatively correlated pairs $(U, 1 - U)$ for $U \sim \text{Unif}(0, 1)$ and average estimates.

- ▶ Example integral $I = \int_0^1 e^{-x} dx$: estimator $\frac{1}{2}(e^{-U} + e^{-(1-U)})$ lowers variance.

Control Variates

Definition

With control g s.t. $\mathbb{E}[g] = \gamma$ known, use $\hat{l}_{cv} = \bar{f} - \beta(\bar{g} - \gamma)$ with $\beta^* = \text{Cov}(f, g) / \text{Var}(g)$.

- ▶ For $f(U) = e^{-U}$, take $g(U) = U$, $\gamma = 1/2$. Estimate β from pilot samples.
- ▶ Report variance ratios vs. naive; verify unbiasedness.

Sensitivity Near Criticality

- ▶ $L = \frac{\rho}{1 - \rho}$ with $\rho = \lambda/\mu$; derivatives blow up as $\rho \rightarrow 1$.
- ▶ Example: $\partial L / \partial \lambda = \frac{1}{\mu(1 - \rho)^2}$, large for high utilisation.
- ▶ Implication: small estimation errors create large performance swings; report intervals, not points.

Practical Checklist

- ▶ State assumptions (IID? bounded? sub-exponential? mixing?).
- ▶ Use pilot runs for s and other parameters; plan n via CLT; report conservative Bernstein bounds.
- ▶ Validate: CI coverage via repeated runs; inspect warm-up, autocorrelation.
- ▶ Prefer variance reduction; reuse RNG seeds for fair comparisons.