



# Stochastic Simulation: Error Control and Sensitivity

Lecture 3 – Concentration, CLT, Variance Reduction, and Error Propagation

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Lecture 3



# Why Error Control Matters

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- ▶ Quantify uncertainty of simulation outputs (means, tails, quantiles).
- ▶ Plan sample sizes to meet accuracy targets (half-width, confidence).
- ▶ Reduce Monte Carlo noise per unit cost (variance reduction).
- ▶ Propagate estimation error of inputs (e.g.,  $\lambda, \mu$ ) to outputs.

# Markov and Chebyshev

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## Theorem

Markov (for nonnegative  $X$ ):  $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$ .

## Theorem

Chebyshev (finite variance):  $\mathbb{P}(|X - \mathbb{E}[X]| \geq \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2}$ .

- ▶ Valid for any  $n$ ; conservative but assumption-light.
- ▶ For sample mean  $\bar{X}_n$ :  $\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$ .

# Bernstein: Bounded and Unbounded

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## Theorem

Bounded case ( $|X_i - \mu| \leq b$  i.i.d., variance  $\sigma^2$ ):

$$\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \leq 2 \exp\left(-\frac{n\varepsilon^2}{2\sigma^2 + (2/3)b\varepsilon}\right).$$

## Theorem

Unbounded (sub-exponential; MGF bound): if

$\mathbb{E} e^{\lambda(X_i - \mu)} \leq \exp(\lambda^2 \nu^2 / 2)$  for  $|\lambda| < 1/b$ , then

$$\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \leq 2 \exp\left(-\frac{n}{2} \min\{\varepsilon^2 / \nu^2, \varepsilon / b\}\right).$$

# CLT and Confidence Intervals

## Theorem

CLT:  $\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \Rightarrow \mathcal{N}(0, 1)$ .

- ▶ CI (known  $\sigma$ ):  $\bar{X} \pm z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$ . Unknown  $\sigma$ : replace by  $s$ ; use  $t_{n-1}$  for finite  $n$ .
- ▶ Target half-width  $\varepsilon$ :  $n \geq \left( \frac{z_{1-\alpha/2} \sigma}{\varepsilon} \right)^2$ . Relative error:  
$$n \geq \left( \frac{z \sigma}{r|\mu|} \right)^2.$$
- ▶ Sequential stopping: continue until  $s z / \sqrt{n} \leq \varepsilon$  (with max  $n$ ).

## Note

Berry–Esseen:

$$\sup_x |\mathbb{P}(\sqrt{n}(\bar{X} - \mu)/\sigma \leq x) - \Phi(x)| \leq C \rho_3/(\sigma^3 \sqrt{n}), \quad C \approx 0.56.$$

# Choosing Between CLT and Concentration

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- ▶ CLT-based CIs: tighter but asymptotic; validate via pilot runs or Berry–Esseen.
- ▶ Concentration (Chebyshev/Bernstein): non-asymptotic guarantees; may be conservative.
- ▶ Practice: start with pilot, use CLT for sizing; report conservative bounds alongside.

# Exponential Example: Tails and Means

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- ▶  $X \sim \text{Exp}(\lambda)$ : compare empirical tail  $\mathbb{P}(X \geq t)$  vs. Markov bound  $\mathbb{E}[X]/t$ .
- ▶ For sample mean  $\bar{X}_n$ : compare empirical  $\mathbb{P}(|\bar{X}_n - 1/\lambda| \geq \varepsilon)$  to Chebyshev and sub-exp Bernstein.
- ▶ See notebook for code and plots.

## 3-State Markov Chain: Convergence

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### Definition

Irreducible, aperiodic chain with transition matrix  $P$  has a unique stationary distribution  $\pi$  with  $\pi^\top P = \pi^\top$ .

- ▶ Simulate trajectory and track  $\|\hat{\pi}_T - \pi\|_1$  vs  $T$ .
- ▶ Use warm-up: discard initial transient before estimating steady-state measures.



# Antithetic Variates

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## Definition

Use negatively correlated pairs  $(U, 1 - U)$  for  $U \sim \text{Unif}(0, 1)$  and average estimates.

- ▶ Example integral  $I = \int_0^1 e^{-x} dx$ : estimator  $\frac{1}{2}(e^{-U} + e^{-(1-U)})$  lowers variance.

# Control Variates

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## Definition

With control  $g$  s.t.  $\mathbb{E}[g] = \gamma$  known, use  $\hat{l}_{cv} = \bar{f} - \beta(\bar{g} - \gamma)$  with  $\beta^* = \text{Cov}(f, g) / \text{Var}(g)$ .

- ▶ For  $f(U) = e^{-U}$ , take  $g(U) = U$ ,  $\gamma = 1/2$ . Estimate  $\beta$  from pilot samples.
- ▶ Report variance ratios vs. naive; verify unbiasedness.

# Sensitivity Near Criticality

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- ▶  $L = \frac{\rho}{1 - \rho}$  with  $\rho = \lambda/\mu$ ; derivatives blow up as  $\rho \rightarrow 1$ .
- ▶ Example:  $\partial L / \partial \lambda = \frac{1}{\mu(1 - \rho)^2}$ , large for high utilisation.
- ▶ Implication: small estimation errors create large performance swings; report intervals, not points.

# Practical Checklist

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- ▶ State assumptions (IID? bounded? sub-exponential? mixing?).
- ▶ Use pilot runs for  $s$  and other parameters; plan  $n$  via CLT; report conservative Bernstein bounds.
- ▶ Validate: CI coverage via repeated runs; inspect warm-up, autocorrelation.
- ▶ Prefer variance reduction; reuse RNG seeds for fair comparisons.