Stochastic Simulation: Error Control and Sensitivity

Lecture 3 – Concentration, CLT, Variance Reduction, and Error Propagation

Sebastian Müller

Lecture 3

Why Error Control Matters

- Quantify uncertainty of simulation outputs (means, tails, quantiles).
- Plan sample sizes to meet accuracy targets (half-width, confidence).
- Reduce Monte Carlo noise per unit cost (variance reduction).
- Propagate estimation error of inputs (e.g., λ, μ) to outputs.

Markov and Chebyshev

Theorem

Markov (for nonnegative X): $\mathbb{P}(X \geq t) \leq \dfrac{\mathbb{E}[X]}{t}$.

Theorem

Chebyshev (finite variance): $\mathbb{P}\big(|X - \mathbb{E}[X]| \geq \varepsilon\big) \leq \dfrac{\mathrm{Var}(X)}{\varepsilon^2}$.

- Valid for any n; conservative but assumption-light.
- For sample mean $\bar{X}_n$: $\mathbb{P}\big(|\bar{X}_n - \mu| \geq \varepsilon\big) \leq \dfrac{\sigma^2}{n\,\varepsilon^2}$.

Bernstein: Bounded and Unbounded

Theorem

Bounded case ($|X_i - \mu| \leq b$ i.i.d., variance σ^2):

$$\mathbb{P}\big(|\bar{X}_n - \mu| \geq \varepsilon\big) \leq 2\,\exp\left(-\,\frac{n\,\varepsilon^2}{2\sigma^2 + (2/3)\,b\,\varepsilon}\right).$$

Theorem

Unbounded (sub-exponential; MGF bound): if $\mathbb{E}\,e^{\lambda(X_i-\mu)} \leq \exp(\lambda^2\nu^2/2)$ for $|\lambda| < 1/b$, then

$$\mathbb{P}\big(|\bar{X}_n - \mu| \geq \varepsilon\big) \leq 2\,\exp\Big(-\,\tfrac{n}{2}\,\min\{\varepsilon^2/\nu^2,\ \varepsilon/b\}\Big).$$

CLT and Confidence Intervals

Theorem

CLT: $\dfrac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \Rightarrow \mathcal{N}(0,1)$.

- CI (known σ): $\bar{X} \pm z_{1-\alpha/2}\,\dfrac{\sigma}{\sqrt{n}}$. Unknown σ: replace by s; use t_{n-1} for finite n.
- Target half-width ε: $n \geq \left(\dfrac{z_{1-\alpha/2}\,\sigma}{\varepsilon}\right)^2$. Relative error: $n \geq \left(\dfrac{z\,\sigma}{r|\mu|}\right)^2$.
- Sequential stopping: continue until $s\,z/\sqrt{n} \leq \varepsilon$ (with max n).

Note

Berry–Esseen:
$\sup_x \left|\mathbb{P}(\sqrt{n}(\bar{X}-\mu)/\sigma \leq x) - \Phi(x)\right| \leq C\,\rho_3/(\sigma^3\sqrt{n})$, $C \approx 0.56$.

Choosing Between CLT and Concentration

- CLT-based CIs: tighter but asymptotic; validate via pilot runs or Berry–Esseen.
- Concentration (Chebyshev/Bernstein): non-asymptotic guarantees; may be conservative.
- Practice: start with pilot, use CLT for sizing; report conservative bounds alongside.

Exponential Example: Tails and Means

- $X \sim \mathrm{Exp}(\lambda)$: compare empirical tail $\mathbb{P}(X \geq t)$ vs. Markov bound $\mathbb{E}[X]/t$.
- For sample mean $\bar{X}_n$: compare empirical $\mathbb{P}(|\bar{X}_n - 1/\lambda| \geq \varepsilon)$ to Chebyshev and sub-exp Bernstein.
- See notebook for code and plots.

3-State Markov Chain: Convergence

Theorem

Irreducible, aperiodic chain with transition matrix P has a unique stationary distribution π with $\pi^\top P = \pi^\top$.

- Simulate trajectory and track $\|\hat{\pi}_T - \pi\|_1$ vs T.
- Use warm-up: discard initial transient before estimating steady-state measures.

Antithetic Variates

Idea

Use negatively correlated pairs $(U, 1-U)$ for $U \sim \mathrm{Unif}(0,1)$ and average estimates.

- Example integral $I = \int_0^1 e^{-x} dx$: estimator $\frac{1}{2}\left(e^{-U} + e^{-(1-U)}\right)$ lowers variance.

Control Variates

Idea

With control g s.t. $\mathbb{E}[g] = \gamma$ known, use $\hat{I}_{cv} = \bar{f} - \beta(\bar{g} - \gamma)$ with $\beta^* = \mathrm{Cov}(f, g)/\,\mathrm{Var}(g)$.

- For $f(U) = e^{-U}$, take $g(U) = U$, $\gamma = 1/2$. Estimate β from pilot samples.
- Report variance ratios vs. naive; verify unbiasedness.

Sensitivity Near Criticality

- $L = \dfrac{\rho}{1-\rho}$ with $\rho = \lambda/\mu$; derivatives blow up as $\rho \to 1$.
- Example: $\partial L/\partial\lambda = \dfrac{1}{\mu(1-\lambda/\mu)^2}$, large for high utilisation.
- Implication: small estimation errors create large performance swings; report intervals, not points.

Practical Checklist

- State assumptions (IID? bounded? sub-exponential? mixing?).
- Use pilot runs for s and other paramters; plan n via CLT; report conservative Bernstein bounds.
- Validate: CI coverage via repeated runs; inspect warm-up, autocorrelation.
- Prefer variance reduction; reuse RNG seeds for fair comparisons.