MATH 180: Homework #4

Due on May 11, 2016 at 12:59pm

Professor Igor Pak Lec 1

Naiyan Xu

Problem 1

Exc 1 in 6.3: Prove that the bound $|E| \le 2|V|-4$ for triangle-free planar graphs is the best possible in general. That is, for infinitely many n construct examples of triangle-free planar graphs with n vertices and 2n-4 edges.

Solution

Proof. Let $|F|$ be the number of faces, $|E|$ be the number of edges, and $|V|$ be the number of vertices. If there are no triangles then each face is enclosed by at least 4 edges, $4|F| \le 2|E|$. By Euler's formula, $|V| - |E| + |E|/2 \ge 2$, so $|E| \le 2|V| - 4$. □

Problem 2

Exc 3 in 6.3: Prove that a planar graph in which each vertex has degree at least 5 must have at least 12 vertices.

Solution

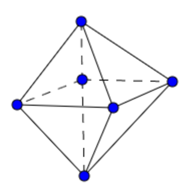
Proof. Since each vertex has degree at least 5, we have $2|E| \ge 5|V|$. By proppsition 6.3.3, we know $|E| \le 3|V| - 6$. Combine those two equations we get $6|V| - 12 \ge 2|E| \ge 5|V| \Rightarrow |V| \ge 12$. □

Problem 3

Exc 6 in 6.3: Find an example, other than the Platonic solids, of a convex polytope in the 3-dimensional space such that all faces are congruent copies of the same regular convex polygon. Can you list all possible examples?

Solution

All polyhedrons composed by two pyramids with the same polygon base. Such as,



Problem 4

Exc 1 in 6.4: Prove that $\chi(G) \le 1 + max\{deg_G(x) : x \in V\}$ holds for every (finite) graph G=(V, E).

Solution

Proof. By Induction.
Base: The graph with just one vertex has maximum degree 0 and can be colored with one color.
Induction: Suppose that any graph with $\le$ n vertices and maximum vertex degree $\le$ D can be colored with D + 1 colors. Let G be a graph with n + 1 vertices and maximum vertex degree D. Remove some vertex v (and its edges) from G to create a smaller graph G'. The maximum vertex degree of G' is no larger than D, because removing a vertex cant increase the degree. So, by the inductive hypothesis, G' can be colored

Problem 4 continued on next page. . .

with D + 1 colors. Because v has at most D neighbors, its neighbors are only using D of the available colors, leaving a spare color that we can assign to v. The coloring of G' can be extended to a coloring of G with D + 1 colors. $\square$

Problem 5

Exc 3 in 6.4: Call a graph G outerplanar if a drawing of G exists in which the boundary of one of the faces contains all the vertices of G (we can always assume the outer face has this property). Prove that every outerplanar graph has chromatic number at most 3.

Solution

Claim: every outerplanar graph contains a vertex of degree at most 2.

Proof. Suppose we have an outerplanar graph G with n vertices. Keep adding edges to G as long as it remains outerplanar. Once we cannot add any more edges, its easy to see that the outer face is bounded by a cycle that contains all the vertices. Also, all internal faces are triangles. Consider an edge (v_i, v_j), which does not belong to the outer cycle, for which min$\{|i-j|, n-|i-j|\}$ is minimum. Clearly, we must have that the minimum is equal to 2, and the vertex that lies between v_i and v_j on the outer cycle is of degree 2. $\square$

Proof. By induction.
Base case: The graph with just one vertex is outerplanar and has chromatic number at most 3.
Inductive step: Assume the theorem holds for n vertices and let G_{n+1} be an outerplanar graph with n+1 vertices. There must exist a vertex v in G_{n+1} with degree at most 2. Removing v and all its incident edges leaves a subgraph G_n with n vertices. Since G_{n+1} could be drawn with its vertices on a circle and its edges drawn as straight lines without intersections, any subgraph can also be drawn in such a way and so G_n is also an outerplanar graph. Since G_n has chromatic number at most 3. We can color all the vertices in G_{n+1} other than v using only 3 colors and since $deg_{(v)} \leq 2$ we may color it a color different than the vertices adjacent to it using only 3 colors. So the theorem holds for an outerplanar graph with n+1 vertices. $\square$

Problem 6

Let G be a planar graph containing no K_3 as a subgraph. Prove $\chi(G) \leq 4$. (A difficult theorem due to Grotsch asserts that, actually, $\chi(G) \leq 3$ holds for all planar triangle-free graphs.)

Solution

Claim: G has a vertex of degree three or less.

Proof. By contradiction. From problem 1 we have proved that $|E| \leq 2|V| - 4$, so $|E|/2 + 2 \leq |V|$. If all vertices have degree greater than 4, we have $4|V|/2 < |E|$. Contradiction. $\square$

Proof. Take a vertex that has degree less or equal to 3. Color it and its neighbors with different colors A,B,C,D. Then delete this vertex and the edges insident on it, call the new graph G'. With a colored G', there is always a way to color the deleted vertex. Also G' still has no triangles, so it also has a vertex of degree three or less. Apply the same steps above we can always color the graph with one vertex deleted. $\square$

Problem 7

Let $P \subset R^3$ be a convex polytope. Denote by c(P) the sum of the number of triangular faces and the number of vertices of degree 3. Prove that c(P) ≥ 8.

Solution

Let v be the number of vertices, e be the number of edges and f be the number of faces. Let v_0 be the number of vertices with degree 3, and v_1 be the number of vertices with degree greater than 3. So $v = v_0 + v_1$. By Handshake theorem,

$$\begin{aligned} 2e &\geq 3v_0 + 4v_1 \\ &\geq 4v_0 + 4v_1 - v_0 \end{aligned}$$

Let f_0 be the number of triangular faces, and f_1 be the number of other faces. So $f = f_0 + f_1$. Similarly,

$$\begin{aligned} 2e &\geq 3f_0 + 4f_1 \\ &\geq 4f_0 + 4f_1 - f_0 \end{aligned}$$

add them up and apply the Euler's formula, we have

$$\begin{aligned} 4e &\geq 4f_0 + 4f_1 - f_0 + 4v_0 + 4v_1 - v_0 \\ &\geq 4f + 4f - (f_0 + v_0) \end{aligned}$$

Finally we have $c(P) = f_0 + v_0 \geq 4(v - e + f) = 8$

Problem 8

Let G be a connected planar graph such that G* has a Hamiltonian cycle. Prove that $\chi(G) \leq 4$.

Solution

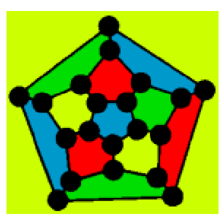
Problem 9

Let D and G be the dodecahedron and icosahedron graphs, respectively. Find $\chi(D)$ and $\chi(G)$.

Solution

$\chi(G) = 4$

This is easier to see by looking at the face coloring of the dodecahedron (since the dodecahedron is dual to the icosahedron, the face coloring of the dodecahedron is equivalent to the vertex coloring of the icosahedron). The center face of the dodecahedron is one color and the surrounding faces cannot be colored in less than 3 colors, therefore the chromatic number is 4 for the icosahedron.



$\chi(D) = 3$
Since we can face color the icosahedron with 3 colors, we can vertex color the dodecahedron with 3 colors. The dodecahedron requires at least 3 colors since it is not bipartite.