MATH 180: Homework #5

Due on May 18, 2016 at 12:59pm

Professor Igor Pak Lec 1

Naiyan Xu

Problem 1

Exc 2 in 7.1: A well-known problem about a tourist climbing a mountain also relates to fixed points. A tourist starts climbing a mountain at 6 in the morning. He reaches the summit at 6pm and spends the night there (in a shelter built there especially for that purpose). At 6 the next morning he starts descending along the same trail. He often pauses to contemplate the view, and so he reaches the starting point at 6 in the evening again. Prove that there is a place on the trail that he passed through at the same time on both days.

Solution

Proof. Let F(x) be the function defined as distance from bottom to summit on the first day. Clearly this is a continuous function, since the tourist cannot teleport. Let G(x) be the function defined as distance from summit to bottom on the second day. And it is also continuous. Both of the functions have the same domain and range. Now consider the function F(x) - G(x). This is a composition of continuous functions, along the same domain, and thus is also continuous. Now, it ranges from bottom to summit and must achieve every value in between at some point along the way (by the definition of continuity). □

Problem 2

Exc 3 in 7.1: A building engineer is standing in a freshly finished apartment and holding a floor plan of the same apartment. Prove that some point in the plan is positioned exactly above the point on the apartments floor it corresponds to.

Solution

Proof. In this question, consider them as 2 functions, one transforms to another by scaling and rotating. Clearly the two functions are continuous since there's no hole on the floor. By Brouwers fixed point theorem, every continuous function maps to itself has a fixed point. So there exists a point in the plan is positioned exactly above the point on the apartments floor it corresponds to. □

Problem 3

Exc 1 in 7.2: Let us call a system of subsets of X semiindependent if it contains no three sets A, B, C such that $A \subset B \subset C$. (a) Show by a method similar to the first proof of Sperners theorem that $|N| \leq 2\binom{n}{\lfloor n/2 \rfloor}$,where n = $|X|$. (b) Show that for odd n, the estimate from (a) cannot be improved.

Solution

Problem 4

Exc 7 in 7.2: Let n be a natural number that is not divisible by the square of any integer greater than 1. Determine the maximum possible size of a set of divisors of n such that no divisor in this set divides another.

Solution

Problem 5

Exc 1 in 7.3: Call a graph G outerplanar if a drawing of G exists in which the boundary of one of the faces contains all the vertices of G (we can always assume the outer face has this property). Prove that every outerplanar graph has chromatic number at most 3.

Solution

Problem 6

Exc 2 in 7.3: Let G be a planar graph containing no K_3 as a subgraph. Prove $\chi(G) \leq 4$. (A difficult theorem due to Grotsch asserts that, actually, $\chi(G) \leq 3$ holds for all planar triangle-free graphs.)

Solution

Problem 7

Exc 3 in 7.3: Let $P \subset R^3$ be a convex polytope. Denote by c(P) the sum of the number of triangular faces and the number of vertices of degree 3. Prove that c(P) $\geq$ 8.

Solution