MATH 180: Homework #2

Due on April 13, 2016 at 12:59pm

Professor Igor Pak Lec 1

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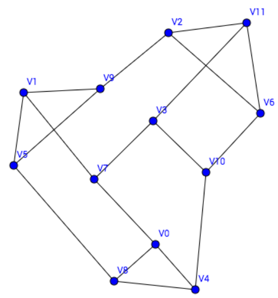
Problem 1

Exc 14 in 4.3: Find a 3-regular asymmetric graph.



Solution

Suppose G' is an arbitrary automorphism of G.

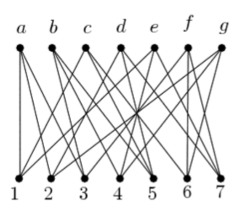
Since G has three triangles $T_1 = \{2, 6, 11\}$, $T_2 = \{1, 5, 9\}$, $T_3 = \{0, 4, 8\}$, and $d(T_3, T_1) = 2$, $d(T_3, T_2) = 1$, $d(T_1, T_2) = 1$, so G' should map T_2 to T_2.

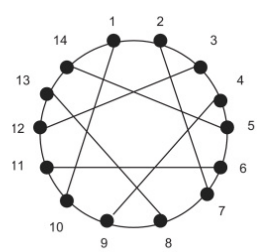
$d(1, T_1) = d(1, T_3) = 1$, $d(9, T_1) = 1 = d(5, T_3)$, $d(9, T_3) = 2 = d(5, T_1)$. So G' has fixed G'(1), G'(5) and G'(9). Then G'(7) is also fixed.

Since $d(7, T_1) = 2$, $d(7, T_3) = 1$, G' has fixed T_1 and T_3 mapping to G. So G'(10) is fixed, so are the last two vertices.

Problem 2

Exc 15(s) in 4.3: Show that the following graphs are isomorphic:





Solution

Call the left figure G and right figure H. First the number of vertices with degrees and edges are the same. Then since they are both bipartite, we now find a subgrapgh (cycle) with 14 vertices in G, which is {1, a, 2, g, 4, b, 3, f, 6, c, 5, d, 7, e, 1}. Then by mapping other edges in H to G, we can find a bijection relation G $\mapsto$ H:

$3 \mapsto 1, f \mapsto 2, 6 \mapsto 3, c \mapsto 4, 5 \mapsto 5, d \mapsto 6, 7 \mapsto 7, e \mapsto 8, 1 \mapsto 9, a \mapsto 10, 2 \mapsto 11, g \mapsto 12, 4 \mapsto 13, b \mapsto 14.$

Problem 3

Exc 7 in 4.4: A Hamiltonian cycle in a graph G is a cycle containing all vertices of G. This notion may seem quite similar to an Eulerian tour but it turns out that it is much more difficult to handle. For instance, no efficient algorithm is known for deciding whether a graph has a Hamiltonian cycle or not. This and the next two exercises are a microscopic introduction to this notion (another nice result is mentioned in Exercise 5.3.3).

(a) Decide which of the graphs drawn in Fig. 6.1 has a Hamiltonian cycle. Try to prove your claims!

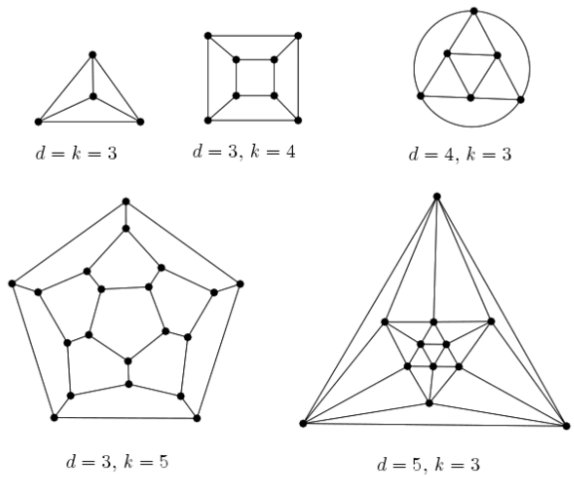
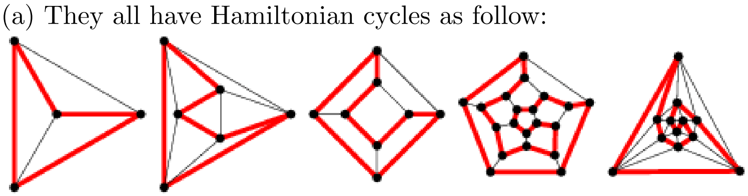


Fig. 6.1 Graphs of the Platonic solids.

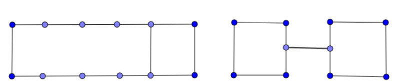
(b) Construct two connected graphs with the same score, one with and one without a Hamiltonian cycle.

Solution

(a) They all have Hamiltonian cycles as follow:



(b) Let the one without Hamiltonian cycyle to have an (or more) edge such that removing it leaves the graph not connected.



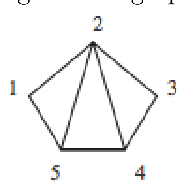
Problem 4

Exc 8 in 4.4: For a graph G, let L(G) denote the so-called line graph of G, given by $L(G) = (E, \{e, e'\} : e, e' \in E(G), e \cap e' \neq \emptyset)$. Decide whether the following is true for every graph G:
(a) G is connected if and only if L(G) is connected.
(b) G is Eulerian if and only if L(G) has a Hamiltonian cycle (see Exercise 7 for a definition).

Solution
(a).FALSE
First assume G is connected, then it contains a path connecting any two of its edges, which translates into a path in L(G) containing any two of the vertices of L(G). However, a graph G that has some isolated vertices, and is therefore disconnected, may nevertheless have a connected line graph.
(b).FALSE
If a graph G has an Euler cycle, that is, if G is connected and has an even number of edges at each vertex, then the line graph of G is Hamiltonian. However, not all Hamiltonian cycles in line graphs come from Euler cycles in this way; for instance, the line graph of a Hamiltonian graph G is itself Hamiltonian, regardless of whether G is also Eulerian.

Problem 5

I. Use any computer algebra system or website wolframalpha.com to compute the number of walks $1 \to 3$ of length 37 in graph below. Include printout of your computation.



Solution
First write down the adjacency matrix, then raise it to 37^{th} power.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} \dots & 24,736,965,591,154,472 & \dots \\ \vdots & \ddots & \vdots \\ \dots & \dots & \dots \end{pmatrix}$$

There are 24,736,965,591,154,472 walks $1 \to 3$ of length 37 in the graph above.

Problem 6

II. Find the number of walks of length 2n in $K_{n,n}$, from $1 \to 1'$. Same question for paths.

Solution

Number of walks: n^{2n-1}. Suppose we start from vertex 1, then we have n choices {1', ..., n'} to make the length of walk to be 1. The graph is bipartite so each time we have n choices moving from one part to the other. For the last step, since we have to end with vertex 1', we have only one choice. So the total number of possible walks is n^{2n-1}.

Problem 6 continued on next page. . .

Number of paths: 0. In order to have a path with length 2n, we need at least 2n+1 vertices. But $K_{n,n}$ only has 2n vertices.

Problem 7

III. Let r = 7. Compute the number of subgraphs of Kr,r isomorphic to the following graphs

Solution
a) P3: 7*7*6*2 = 588
b) P5: 7*7*6*6*5*2 = 17640
c) C3: 7*7*6*2/6 = 96
d) C4: $\binom{7}{2}\binom{7}{2} * 2!/2 = 441$
e) C6: $\binom{7}{3}\binom{7}{3} * 3! * 2!/2 = 7350$
f) K3,1: 2 * $\binom{7}{3}\binom{7}{1}$ = 490
g) K3,2: 2 * $\binom{7}{3}\binom{7}{2}$ = 1470
h) K3,3: 2 * $\binom{7}{3}\binom{7}{3}$ = 1225
i) O9: $\binom{14}{9}$ = 2002
j) P14: C14 * 7 *2 = 25401600
k) C14: $\binom{7}{7}\binom{7}{7} * 7! * 6!/2 = 1814400$

Problem 8

IV. Find all graphs (up to isomorphism) with score (3, 3, 3, 3, 3, 3, 6). Prove that no other such graphs exist. Same question for graphs with score (3, 3, 3, 3, 3, 3).

Solution
(3, 3, 3, 3, 3, 3, 6)

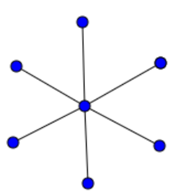


Fig 1

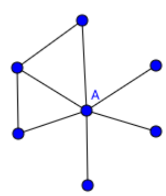


Fig 2

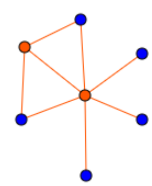


Fig 3

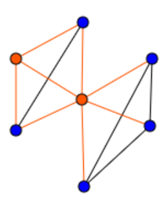


Fig 4

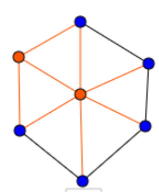


Fig 5

Let the vertex with degree 6 be A, since there are total 6 vertices besides A, A must connect to all of them (see Fig 1). Now pick up one of rest 6 vertices, it must connect to two other vertices in order to have degree

Problem 8 continued on next page. . .

3 (Fig 2). Now the question becomes "Find all graphs (up to isomorphism) with score (1, 1, 2, 2, 2)" (Fig 3). Consider a vertex (call it V_1) with degree 1. It can connect to another vertex with degree either 1 or 2.
Case 1: V_1 connect to a vertex with degree 1. Then the rest of the three vertices with degree 2 only have one choice - to form a triangle. (Fig 4)
Case 2: V_1 connect to a vertex with degree 2. Then the question becomes "Find all graphs (up to isomorphism) with score (1, 1, 2, 2)". Since the vertices with degree 2 must connect to a vertex with degree 1 and a vertex with degree 2, there is only one choice. (Fig 5)
So there are total two graphs with score (3, 3, 3, 3, 3, 3, 6).

(3, 3, 3, 3, 3, 3)

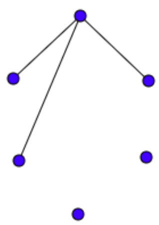


Fig 0

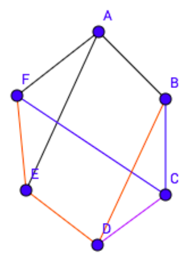


Fig 1

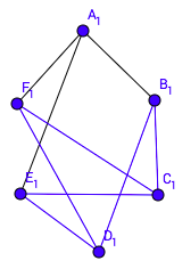


Fig 2

Call the six vertices to be A through F. Without loss of generality, pick an arbitrary vertex A and since its degree is 3 it must connect to other two vertices, call them B, E, F (Fig 0). Now there are two cases:
Case 1: The left two vertices C and D are connected. Then there is only one choice for other vertices (Fig 1).
Case 2: C and D are not connected. Then they both should connect to B, E, and F (Fig 2).

Problem 9

V. Suppose graph G has score (8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 10, 12). Prove that G has a subgraph isomorphic to P_9.

Solution

Claim: Let G be a graph in which all vertices have degree at least k. Then G contains a path of length k.

Proof. : If k = 1 there is an edge, and this is the needed path. Suppose k = n - 1 holds, then by the induction hypothesis G has a path P of length at least n 1. Now consider k = n. We already have a path P with length at least n 1, so assume the length is n - 1. Let the sequence of vertices on the path be $v_1, ..., v_n - 1$. Since he vertex v_1 has at least n adjacent vertices, so there exists a vertex w which is adjacent to v_1, but is not on the path P. Add w to the path P we now have a new path with length n. So the claim holds. □

In this question, k = 8, so G has a subgraph isomorphic to P_9.

Problem 10

Describe all graphs (up to isomorphism) which contain no subgraph isomorphic to P_3.

Solution

All graphs in which all vertices have degree less than 2.