Assignment 1

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Exercise 6

We must make some case distinctions in this question

$\mathbf{a = 0}$

When $a = 0$, $y = bx$, so the range of $y = (0, b)$ if $b > 0$, otherwise, it is $(b, 0)$

$\mathbf{b = 0}$

When $b = 0$, $y = a - ax$, so the range of $y = (0, a)$ if $a > 0$, otherwise, it is $(a, 0)$

a and b $\neq 0$

- When $a > 0$ and $b = 0$, $y = a - ax$, and since x is in the range of $(0, 1)$, then y must be in the range of $(0, a)$, as

  When $x = 0$

$$y = a - a(0)$$
$$y = a$$

  When $x = 1$

$$y = a - a(1)$$
$$y = a - a$$
$$y = 0$$

  But since we have already said that $a > 0$, then the range of y must be $(0, a)$

- When $a > 0$ and $b > 0$, $y = a + bx - ax$, and since x is in the range of $(0, 1)$, then if we substitute in values for x, we can see that

  $x = 0$

  $$y = a + b(0) - a(0)$$
  $$y = a$$

  $x = 1$

  $$y = a + b(1) - a(1)$$
  $$y = a - a + b$$
  $$y = b$$

  So, if $a > b$, then the range of $y = (b, a)$, and this will also hold for when $b < 0$, and if $b > a$, then the range of $y = (b, a)$, and this will also hold for when $a < 0$

  If $a = b$, then the range will be $y = (a, a) = (b, b)$, which means that the function will just be a straight line on a graph

Exercise 7

To prove that a function is odd, we must show that $-f(x) = f(-x)\ \forall x \in D$, and to prove that a function is even, we must show that $f(x) = f(-x)\ \forall x \in D$

a)

$f(-x)$ should be equal to this if it is even

$$f(x) = 3x - x^3$$

$f(-x)$ should be equal to this if it is odd

$$\begin{aligned} -f(x) &= -(3x - x^3) \\ &= -3x + x^3 \\ &= x^3 - 3x \end{aligned}$$

Here we test if either it true

$$\begin{aligned} f(-x) &= 3(-x) - (-x)^3 \\ &= -3x - (-x^3) \\ &= -3x + x^3 \\ &= x^3 - 3x \end{aligned}$$

Therefore, $f(x) = 3x - x^3$ is an **odd** function

b)

$f(-x)$ should be equal to this if it is even

$$f(x) = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$$

$f(-x)$ should be equal to this if it is odd

$$\begin{aligned} -f(x) &= -\left(\sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}\right) \\ &= -\sqrt[3]{(1-x)^2} - \sqrt[3]{(1+x)^2} \end{aligned}$$

Here we test if either are true

$$\begin{aligned} f(-x) &= \sqrt[3]{(1-(-x))^2} + \sqrt[3]{(1+(-x))^2} \\ &= \sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2} \\ &\text{Rearrange} \\ &= \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2} \end{aligned}$$

Therefore, $f(x) = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$ is an **even** function

Exercise 8

a) The domain($D(f)$) of this function must be $\left[-\sqrt{7}, \sqrt{7}\right]$, as if $x = 0$, then the function is just $\sqrt{7} + 1$, but x can never be greater than $\sqrt{7}$, or less than $-\sqrt{7}$, as then we would have the square root of a negative number, which is not possible in the real plane, but x can be equal to $\pm\sqrt{7}$, as $\sqrt{0} = 0$.

   The range($R(f)$) of the function must be $\left[1, 1 + \sqrt{7}\right]$, as when $x = 0$, $y = \sqrt{7} + 1$, but when $x = \pm\sqrt{7}$, then $y = 1$

b) The domain($D(f)$) of this function is $\mathbb{R} - 0$, as you can never divide by 0, but apart from that any real number can be used

   The range($R(f)$) of this function is $(0, \infty)$, as $\lim_{x \to 0} \frac{1}{|x|} = \infty$, because as x gets smaller y will get bigger, but since $\forall x \in D(f) \quad x \neq 0$, it will never reach ∞, and $\lim_{x \to \infty} \frac{1}{|x|} = 0$, and since the absolute value will ensure there is always a positive number as the denominator, therefore the function will never tend towards $-\infty$, so $R(f) = (0, \infty)$

Exercise 9

1. In order to compute the inverse of a function $f^{-1}(x)$, you have to express the function in terms of x, and then change the variables, so from the function $y = \frac{ax+b}{cx+d}$

$$\begin{aligned}
y &= \frac{ax + b}{cx + d} \\
y(cx + d) &= ax + b \\
cxy + dy &= ax + b \\
cxy - ax &= b - dy \\
x(cy - a) &= b - dy \\
x &= \frac{b - dy}{cy - a} \\
y &= \frac{b - dx}{cx - a}
\end{aligned}$$

2.

Exercise 10

a)

$$\lim_{x \to 2} \frac{x-2}{x^2+x-6}$$

This will be done using the **simplify** method

Factorize the denominator

$$\frac{x-2}{(x-2)(x+3)}$$

Cancel the $(x-2)$ on both sides of the fraction

$$\frac{1}{x+3}$$
$$\frac{1}{2+3}$$
$$\lim_{x \to 2} \frac{x-2}{x^2+x-6} = \frac{1}{5}$$

b) **Squeeze** theorem:

Since we have $-1 \leq cos\left(\frac{1}{x}\right) \leq 1$, then we must have that $|x| \leq -|x|cos\left(\frac{1}{x}\right) \leq |x|$.

Now, we know that the $\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x|$, therefore $\lim_{x \to 0} |x|cos\left(\frac{1}{x}\right) = 0$

c)

$$\lim_{x \to 1} \frac{x^2+4x+3}{x^2+x-2}$$

I will use the **simplify** method:

$$\frac{x^2+4x+3}{x^2+x-2}$$

$$\frac{(x-3)(x-1)}{(x+2)(x-1)} \qquad \text{Factorize}$$

$$\frac{x-3}{x+2} \qquad \text{Cancel the } (x-1)$$

$$\frac{1-3}{1+3} \qquad \text{Substitute } x=1$$

$$\frac{-2}{4}$$

$$-\frac{1}{2}$$

$$\lim_{x\to 1}\frac{x^2+4x+3}{x^2+x-2} = -\frac{1}{2}$$

d)

$$\lim_{x\to 0}\frac{\sqrt{x^2+x+1}-\sqrt{x^2+1}}{x}$$

I will use the **rationalize** method here:

$$\frac{\sqrt{x^2+x+1}-\sqrt{x^2+1}}{x} \times \frac{\sqrt{x^2+x+1}+\sqrt{x^2+1}}{\sqrt{x^2+x+1}+\sqrt{x^2+1}}$$

$$\frac{x^2+x+1-x^2-1}{x\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}$$

$$\frac{x}{x\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}$$

$$\frac{1}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}$$

Substitute $x = 0$

$$\frac{1}{\sqrt{0^2+0+1}+\sqrt{0^2+1}}$$

$$\frac{1}{\sqrt{1}+\sqrt{1}}$$

$$\frac{1}{1+1}$$

$$\frac{1}{2}$$

So,

$$\lim_{x\to 0}\frac{\sqrt{x^2+x+1}-\sqrt{x^2+1}}{x}=\frac{1}{2}$$

Exercise 11

First, we should find what $f^{-1}(x)$ is

$$\begin{aligned} y &= \frac{x}{2x+3} \\ y(2x+3) &= x \\ 2xy+3y &= x \\ 3y &= x-2xy \\ 3y &= x(1-2y) \\ \frac{3y}{1-2y} &= x \\ f^{-1}(x) &= \frac{3x}{1-2x} \end{aligned}$$

So, to find a and b, we must find a line that connects the points $(-2, -\frac{6}{5})$, as this is the last point on the left that $g(x) = f^{-1}(x)$, and $(3, -\frac{9}{5})$.

First, we must find the slope of the line

$$\begin{aligned}\frac{dy}{dx} &= \frac{-\frac{6}{5}+\frac{9}{5}}{-2-3}\\ &= \frac{\frac{3}{5}}{-5}\\ &= -\frac{3}{25}\end{aligned}$$

Because the equation of the line was give as $ax + b$, we know now that $a = -\frac{3}{25}$, so we can solve for b

$$\begin{aligned}-\frac{3}{25}\cdot(-2)+b &= -\frac{6}{5}\\ \frac{6}{25}+b &= -\frac{6}{5}\\ b &= -\frac{6}{5}-\frac{6}{25}\\ b &= -\frac{36}{25}\end{aligned}$$

So, we can now see that

$$\frac{b}{a} = \frac{-36}{-3} = 12$$