Nonparametric Regression

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These all assume a model!

By contrast...

Nonparametric regression makes no model assumptions (i.e. no parameters to estimate)

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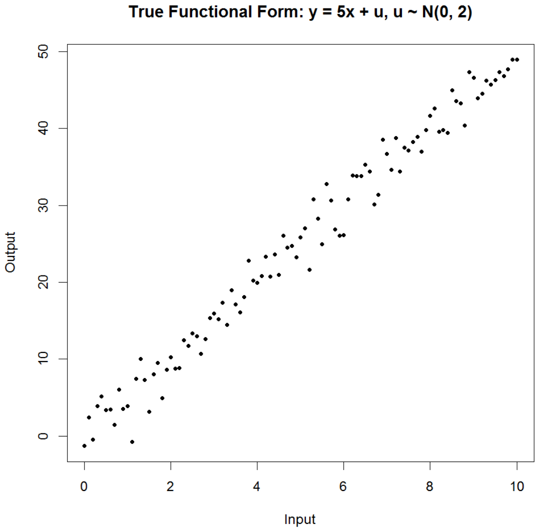
- Parametric estimation: data $\Rightarrow$ parameters $\Rightarrow$ estimation
- Nonparametric estimation: data $\Rightarrow$ estimation

Parametric regression works great sometimes...



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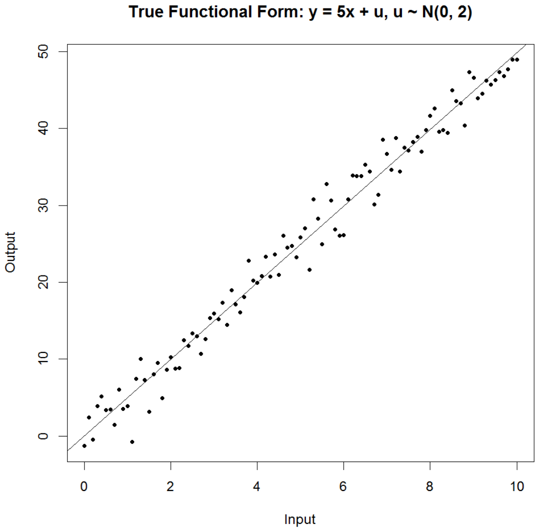

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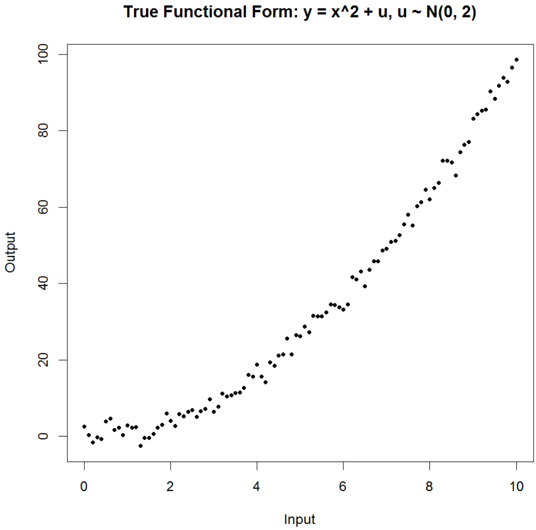

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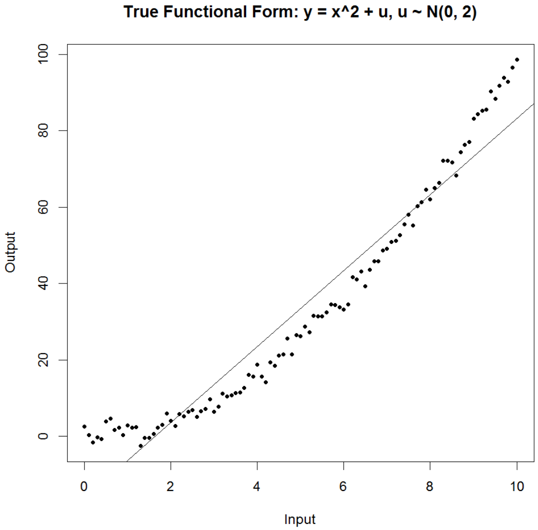

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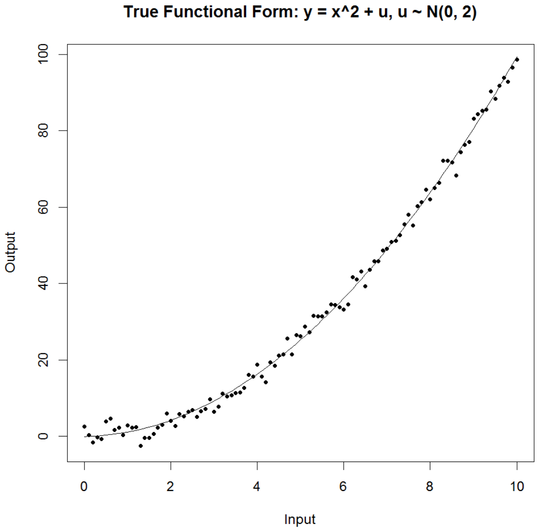

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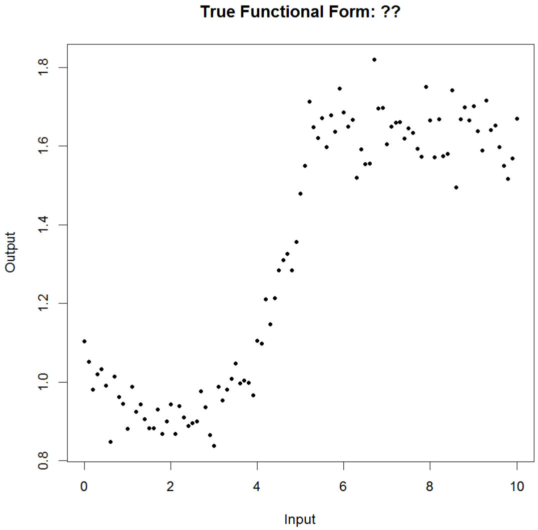

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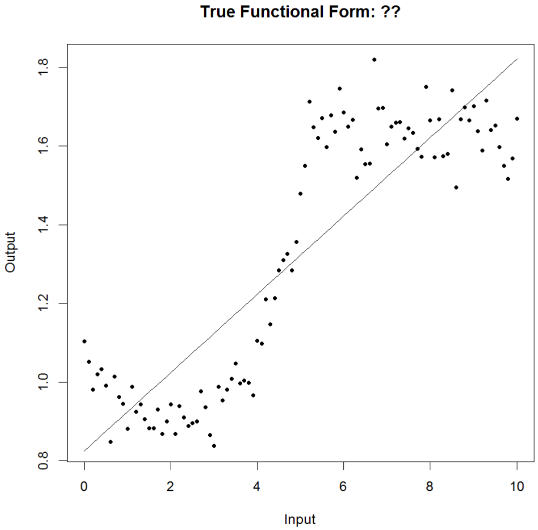

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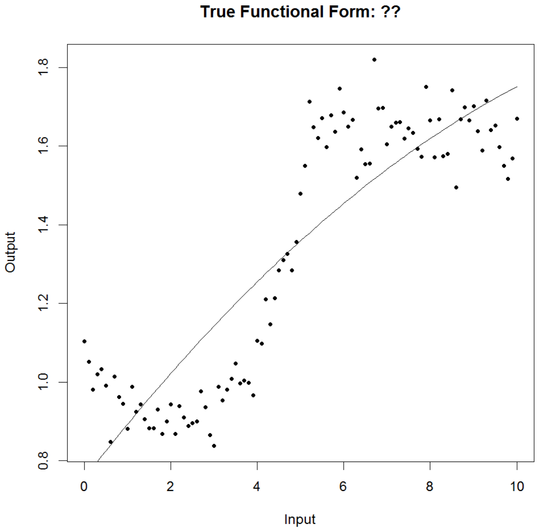

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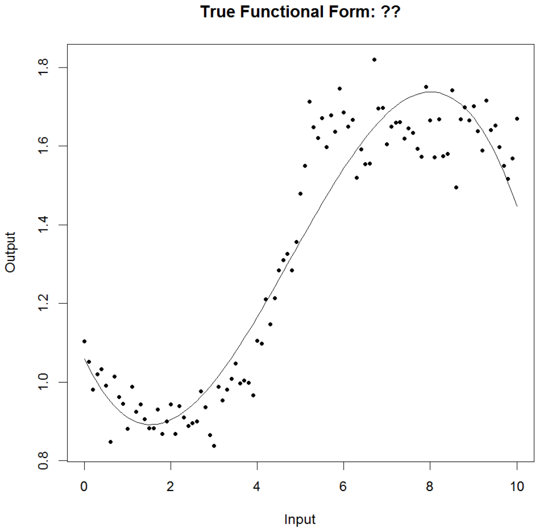

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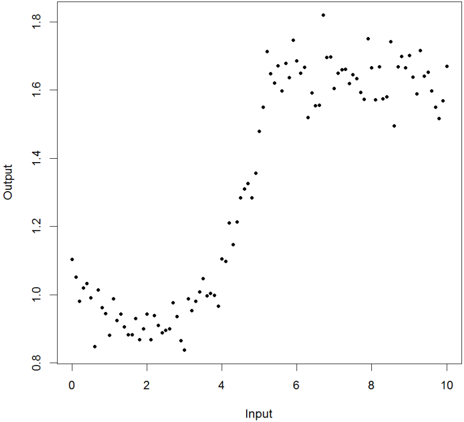
$$1.02^{\min\{0.5x^3-1.5x^2+1.1x-6\sqrt{x}+3,\ 25\}+u},\ u \sim \mathcal{N}(0,2)$$



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2. Nonparametric multiplicative regression
3. Regression trees

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Say you want to predict a response ($\hat{y}$) at a given point (x)

- Linear regression
  - Use available data $(x_1, y_1), (x_2, y_2), (x_3, y_3)$... to estimate $\hat{\beta}_0$ and $\hat{\beta}_1$
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- Kernel regression
  - Calculate a weight for each data point: w_1 for (x_1, y_1), w_2 for (x_2, y_2), w_3 for (x_3, y_3), etc.
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  - The weight is computed by the Kernel

Heights versus Weight

Height (in)	Weight (lb)
68	140
70	150
72	165
74	170

Now say we want to estimate the weight of someone who is 71 inches.

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Linear regression: $\hat{\beta}_0 = -216.5, \hat{\beta}_1 = 5.25$
Estimation: $\hat{y} = -216.5 + 5.25 \cdot 71 = 156.25$

Heights versus Weight

Take a made-up kernel: $K(x_1, x_2) = \frac{1}{|x_1 - x_2|}$

$x = 71$ inches, $\hat{y} =?$

Height (in)	Weight (lb)	Original Weight	Normalized Weight
68	140	0.333	0.125
70	150	1	0.375
72	165	1	0.375
74	170	0.333	0.125

$\hat{y} = 0.125 \cdot 140 + 0.375 \cdot 150 + 0.375 \cdot 165 + 0.125 \cdot 170$

$\hat{y} = 156.875$

Theory

Nadaraya-Watson Kernel Estimator

$$\boxed{\hat{y} = \widehat{m}_h(x) = \frac{\sum_{i=1}^{n} K_h(x - X_i) Y_i}{\sum_{i=1}^{n} K_h(x - X_i)}}$$

Popular Kernels

- Uniform
- Triangular
- Normal
- Epanechnikov

Kernels: Some Details

Turns out they are a little more complicated than just weights...

- They have a parameter (h) called bandwidth
- They can be multiplied when we have several predictors (e.g. height and gender):

$$K_h(x - X_i) = K_h^1(x^1 - X_i^1)K_h^2(x^2 - X_i^2)$$

Returning to Original Plot

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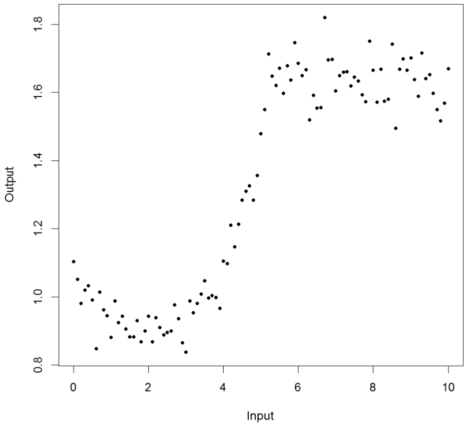

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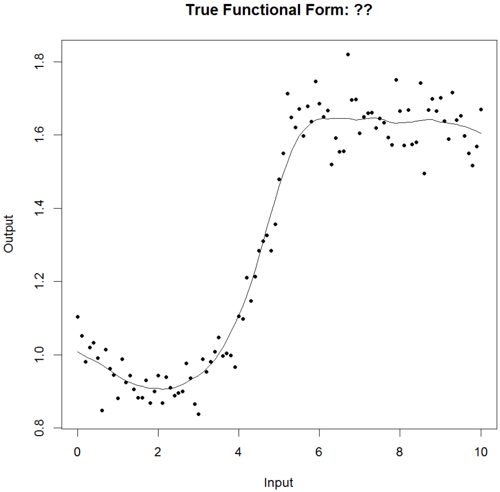
Using an Epanechnikov Kernel...



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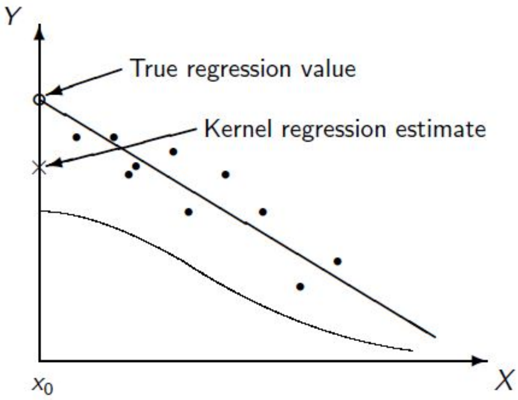

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- No parameters to analyze
- Requires lots of data for accurate prediction
- Still requires some assumptions...