Contents

Chapter 1

The computation of my life

1.1 A very special Rolling

We want to say something about dice rolling. Consider the roll of a N-sided dice that we roll **with explosion**. This means that on a roll of N, we roll again and add the result to the previous roll. This differ from a standard roll from the fact that the potential result is theoretically unbounded (although the probability decreases as the result increases). We denote this roll as dN^*.

If we then look at the probability of rolling n on such a dice, we get

$$\mathbb{P}_N(n) = \mathbb{P}\left(\text{a roll of } dN^* \text{ gives } n\right) = \tag{1.1}$$

$$= \begin{cases} 0 & \text{if } n \text{ is a multiple of } 6 \\ \frac{1}{N} & \text{if } 1 \leq n \leq N-1 \\ \frac{1}{N^2} & \text{if } N+1 \leq n < 2N-1 \\ \vdots & \\ \frac{1}{N^{k+1}} & \text{if } kN+1 \leq n < (k+1)\,N-1 \\ \vdots & \end{cases} \tag{1.2}$$

$$= \sum_{k=0}^{+\infty} \frac{1}{N^{k+1}} \chi_{[kN+1,(k+1)N-1]}(n). \tag{1.3}$$

We want to understand what is the expected value as well as the variance for $\mathbb{P}_N$. To do that we need to remind ourselves of other computations.

1.1.1 Geometric series

Consider a real number $|x| < 1$, and the following sum:

$$S_R(x) = 1 + x + x^2 + \ldots + x^R. \tag{1.4}$$

If $x \neq 1$ (ensured by the fact that $|x| < 1$), we have that

$$S_R(x) = \frac{1 - x^R}{1 - x}. \tag{1.5}$$

Using that $|x| < 1$, this means that as R grows, $|x|^R$ goes to zero. So, if we perform the limit to R to infinity we get

$$\sum_{\ell=0}^{+\infty} x^\ell = \lim_{R\to+\infty} S_R(x) = \lim_{R\to+\infty} \frac{1 - x^R}{1 - x} = \frac{1}{1 - x}. \tag{1.6}$$

We want to apply this to $X := x^{-1}$. In this case the condition becomes $|X| > 1$, and in this case we get the formula:

$$\sum_{\ell=0}^{+\infty} \frac{1}{X^\ell} = \frac{X}{X - 1}. \tag{1.7}$$

We can also consider the sum starting from 1 and get

$$\sum_{\ell=1}^{+\infty} \frac{1}{X^\ell} = \frac{1}{X - 1}. \tag{1.8}$$

1.1.2 First and second moments of the geometric series

To compute the first two moments of the geometric series (1.8), we differentiate respect to X both sides. Differentiating the left side we get

$$\partial_X \left(\sum_{\ell=1}^{+\infty} \frac{1}{X^\ell} \right) = \sum_{\ell=1}^{+\infty} \frac{-\ell}{X^{\ell+1}} = -\sum_{\ell=2}^{+\infty} \frac{\ell - 1}{X^\ell} = -\sum_{\ell=1}^{+\infty} \frac{\ell - 1}{X^\ell} \tag{1.9}$$

$$= -\sum_{\ell=1}^{+\infty} \frac{\ell}{X^\ell} + \sum_{\ell=1}^{+\infty} \frac{1}{X^\ell} = -\sum_{\ell=1}^{+\infty} \frac{\ell}{X^\ell} + \frac{1}{X - 1}. \tag{1.10}$$

We substitute now the definition in (1.8) to get

$$\sum_{\ell=1}^{+\infty} \frac{\ell}{X^\ell} = -\partial_X \left(\frac{1}{X-1} \right) + \frac{1}{X-1} \tag{1.11}$$

$$= \frac{1}{(X-1)^2} + \frac{1}{X-1} = \frac{X}{(X-1)^2} \tag{1.12}$$

If we differentiate again, the left term of (1.12) gives us

$$\partial_X \left(\sum_{\ell=1}^{+\infty} \frac{\ell}{X^\ell} \right) = \sum_{\ell=1}^{+\infty} \frac{-\ell^2}{X^{\ell+1}} = -\sum_{\ell=2}^{+\infty} \frac{(\ell-1)^2}{X^\ell} = -\sum_{\ell=1}^{+\infty} \frac{(\ell-1)^2}{X^\ell} \tag{1.13}$$

$$= -\sum_{\ell=1}^{+\infty} \frac{\ell^2}{X^\ell} + 2\sum_{\ell=1}^{+\infty} \frac{\ell}{X^\ell} - \sum_{\ell=1}^{+\infty} \frac{1}{X^\ell} \tag{1.14}$$

$$= -\sum_{\ell=1}^{+\infty} \frac{\ell^2}{X^\ell} + 2\left(\frac{1}{(X-1)^2} + \frac{1}{X-1} \right) - \frac{1}{X-1} \tag{1.15}$$

$$= -\sum_{\ell=1}^{+\infty} \frac{\ell^2}{X^\ell} + \frac{2}{(X-1)^2} + \frac{1}{X-1} \tag{1.16}$$

As a consequence, the second to last term in (1.12) we get

$$\sum_{\ell=1}^{+\infty} \frac{\ell^2}{X^\ell} = -\partial_X \left(\frac{1}{(X-1)^2} + \frac{1}{X-1} \right) + \frac{2}{(X-1)^2} + \frac{1}{X-1} \tag{1.17}$$

$$= -\left(-\frac{2}{(X-1)^3} - \frac{1}{(X-1)^2} \right) + \frac{2}{(X-1)^2} + \frac{1}{X-1} \tag{1.18}$$

$$= \frac{2}{(X-1)^3} + \frac{3}{(X-1)^2} + \frac{1}{X-1} = \frac{X(X+1)}{(X-1)^3}. \tag{1.19}$$

1.1.3 Finite sums

One can easily find also that the following two finite sums are true (which will be useful later):

$$\sum_{\ell=1}^{R} 1 = R, \tag{1.20}$$

$$\sum_{\ell=1}^{R} \ell = \frac{R(R+1)}{2}, \tag{1.21}$$

$$\sum_{\ell=1}^{R} \ell^2 = \frac{R(R+1)(2R+1)}{6}. \tag{1.22}$$

1.2 Expected value

To calculate the expected value, we proceed with an explicit calculation:

$$\mathbb{E}_N = \sum_{n=1}^{+\infty} n\mathbb{P}_N(n) = \sum_{n=1}^{+\infty}\sum_{k=0}^{+\infty} \frac{n}{N^{k+1}} \chi_{[kN+1,(k+1)N-1]}(n) \tag{1.23}$$

$$= \sum_{m=0}^{+\infty}\sum_{j=1}^{N-1}\sum_{k=0}^{+\infty} \frac{mN+j}{N^{k+1}} \chi_{[kN+1,(k+1)N-1]}(mN+j). \tag{1.24}$$

Given that j is between 1 and $N-1$, $\chi_{[kN+1,(k+1)N-1]}(mN+j)$ is 0 if and only if $m=k$, otherwise is 1. As a consequence we get:

$$\mathbb{E}_N = \sum_{j=1}^{N-1}\sum_{k=0}^{+\infty} \frac{kN+j}{N^{k+1}} = \sum_{j=1}^{N-1}\left(\sum_{k=0}^{+\infty}\frac{kN}{N^{k+1}} + \sum_{k=0}^{+\infty}\frac{j}{N^{k+1}}\right) \tag{1.25}$$

$$= \sum_{j=1}^{N-1}\left(\sum_{k=1}^{+\infty}\frac{k}{N^k} + j\sum_{k=1}^{+\infty}\frac{1}{N^k}\right) = \sum_{j=1}^{N-1}\left(\frac{N}{(N-1)^2} + \frac{j}{N-1}\right) \tag{1.26}$$

$$= \frac{N}{N-1} + \frac{N}{2} = \frac{N(N+1)}{2(N-1)}. \tag{1.27}$$

For example, we get that the expected value for an exploding $d6^*$ is $21/5 = 4,2$, slightly more than the one of a normal $d6$ (which is $3,5$).

We now look at the dispersion, computed through the variance:

$$\sigma_N = \sum_{n=1}^{+\infty} \left(n - \mathbb{E}_N\right)^2 \mathbb{P}_N\left(n\right) = \sum_{n=1}^{+\infty} n^2 \mathbb{P}_N\left(n\right) - \mathbb{E}_N \tag{1.28}$$

$$= \sum_{j=1}^{N-1} \sum_{k=0}^{+\infty} \frac{\left(kN + j\right)^2}{N^{k+1}} - \mathbb{E}_N \tag{1.29}$$

$$= \sum_{j=1}^{N-1} \left(\sum_{k=1}^{+\infty} \frac{k^2}{N^{k-1}} + 2j \sum_{k=1}^{+\infty} \frac{k}{N^k} + j^2 \sum_{k=0}^{+\infty} \frac{1}{N^{k+1}} \right) - \mathbb{E}_N \tag{1.30}$$

$$= \sum_{j=1}^{N-1} \left(\sum_{k=0}^{+\infty} \frac{\left(k+1\right)^2}{N^k} + 2j \left(\frac{1}{N-1} + \frac{1}{\left(N-1\right)^2} \right) + j^2 \sum_{k=1}^{+\infty} \frac{1}{N^k} \right) \tag{1.31}$$

$$- \mathbb{E}_N \tag{1.32}$$

$$= \sum_{j=1}^{N-1} \left(\sum_{k=1}^{+\infty} \frac{k^2}{N^k} + 2 \sum_{k=1}^{+\infty} \frac{k}{N^k} + \sum_{k=0}^{+\infty} \frac{1}{N^k} \right. \tag{1.33}$$

$$\left. + 2j \left(\frac{1}{N-1} + \frac{1}{\left(N-1\right)^2} \right) + j^2 \sum_{k=1}^{+\infty} \frac{1}{N^k} \right) - \mathbb{E}_N \tag{1.34}$$

$$= \sum_{j=1}^{N-1} \left(\frac{N\left(N+1\right)}{\left(N-1\right)^3} + 2 \frac{N}{\left(N-1\right)^2} + \frac{N}{N-1} \right. \tag{1.35}$$

$$\left. + 2j \left(\frac{1}{N-1} + \frac{1}{\left(N-1\right)^2} \right) + \frac{j^2}{N-1} \right) - \mathbb{E}_N \tag{1.36}$$

$$= \frac{2N\left(N+1\right)}{\left(N-1\right)^2} + \frac{N^2}{N-1} + \frac{N\left(2N-1\right)}{6} - \mathbb{E}_N \tag{1.37}$$

1.3 A very special Die

Consider now this configuration: we roll a $d6$; if it rolls more than 3, it counts as a 0, if it rolls a 3, we roll again and add the new result to the value of the dice, potentially rolling again or summing a 0. This new probability is as follows:

$$\mathbb{P}_N^0(n) = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ \frac{1}{6} & \text{if } n = 1,2 \\ \frac{1}{12} & \text{if } n = 3 \\ \frac{1}{36} & \text{if } n = 4,5 \\ \vdots & \\ \frac{1}{2\times 6^k} & \text{if } n = 3k \\ \frac{1}{6^{k+1}} & \text{if } n = 3k+1, 3k+2 \\ \vdots & \end{cases} \tag{1.38}$$

$$= \frac{1}{2}\sum_{k=0}^{+\infty}\frac{1}{6^k}\chi_{3k}(n) + \sum_{k=0}^{+\infty}\frac{1}{6^{k+1}}\left[\chi_{3k+1}(n) + \chi_{3k+2}(n)\right]. \tag{1.39}$$

We can now compute the expected value and we get

$$\mathbb{E}_N^0 = \sum_{n=0}^{+\infty} n\left[\frac{1}{2}\sum_{k=0}^{+\infty}\frac{1}{6^k}\chi_{3k}(n) + \sum_{k=0}^{+\infty}\frac{1}{6^{k+1}}\left[\chi_{3k+1}(n) + \chi_{3k+2}(n)\right]\right] \tag{1.40}$$

$$= \frac{1}{2}\sum_{k=0}^{+\infty}\frac{3k}{6^k} + \sum_{k=0}^{+\infty}\frac{6k+3}{6^{k+1}} = \frac{3}{2}\sum_{k=0}^{+\infty}\frac{k}{6^k} + \sum_{k=0}^{+\infty}\frac{k}{6^k} + 3\sum_{k=0}^{+\infty}\frac{1}{6^{k+1}} \tag{1.41}$$

$$= \frac{5}{2}\sum_{k=1}^{+\infty}\frac{k}{6^k} + 3\sum_{k=1}^{+\infty}\frac{1}{6^k} = \frac{5}{2}\times\frac{6}{25} + 3\times\frac{1}{5} = \frac{6}{5}. \tag{1.42}$$

We then look at the variance of this variable

$$\left(\sigma_N^0\right)^2 = \sum_{n=0}^{+\infty}\left(n - \mathbb{E}_N^0\right)^2\mathbb{P}_N^0(n) = \sum_{n=0}^{+\infty} n^2\mathbb{P}_N^0(n) - \mathbb{E}_N^0. \tag{1.43}$$

We first compute the second moment to get

$$\sum_{n=0}^{+\infty} n^2 \mathbb{P}_N^0(n) = \sum_{n=0}^{+\infty} n^2 \frac{1}{2} \sum_{k=0}^{+\infty} \frac{1}{6^k} \chi_{3k}(n) \tag{1.44}$$

$$+ \sum_{n=0}^{+\infty} n^2 \sum_{k=0}^{+\infty} \frac{1}{6^{k+1}} \left[\chi_{3k+1}(n) + \chi_{3k+2}(n)\right] \tag{1.45}$$

$$= \frac{1}{2} \sum_{k=0}^{+\infty} \frac{9k^2}{6^k} + \sum_{k=0}^{+\infty} \frac{1}{6^{k+1}} \left[(3k+1)^2 + (3k+2)^2\right] \tag{1.46}$$

$$= \frac{9}{2} \sum_{k=1}^{+\infty} \frac{k^2}{6^k} + \sum_{k=0}^{+\infty} \frac{1}{6^{k+1}} \left(18k^2 + 9k + 5\right) \tag{1.47}$$

$$= \frac{9}{2} \sum_{k=1}^{+\infty} \frac{k^2}{6^k} + 3 \sum_{k=1}^{+\infty} \frac{k^2}{6^k} + \frac{3}{2} \sum_{k=1}^{+\infty} \frac{k}{6^k} + \sum_{k=1}^{+\infty} \frac{5}{6^k} \tag{1.48}$$

$$= \frac{15}{2} \times \frac{6 \times 7}{5^3} + \frac{3}{2} \times \frac{6}{5^2} + 5 \times \frac{1}{5} = \frac{97}{25} \tag{1.49}$$

So, we get

$$\sigma_N^0 = \sqrt{\frac{97}{25} - \frac{6}{5}} = \frac{\sqrt{67}}{5} \tag{1.50}$$