GENERAL ASSEMBLY

MARKOV CHAIN MONTE CARLO (MCMC) METHODS

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DSI+

LEARNING OBJECTIVES

- Understand and be able to describe how MCMC works.
- Identify situations where MCMC is beneficial.

OPENING: BAYES' THEOREM

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- What happens when we don't have conjugacy?
- If we can't find a "closed-form" solution, can we at least approximate one?

MCMC METHODS

MCMC COMPONENTS

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  - Monte Carlo Simulations
  - Markov Chains
  - Acceptance-Rejection Sampling

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MONTE CARLO

- Monte Carlo methods allow us to approximate a complicated system with a statistical sample.

- This is a way of generating random numbers.

$$\theta_t \sim N(0.5, \sigma)$$

- We can use this to solve a number of problems – like approximating the value of π!

MONTE CARLO

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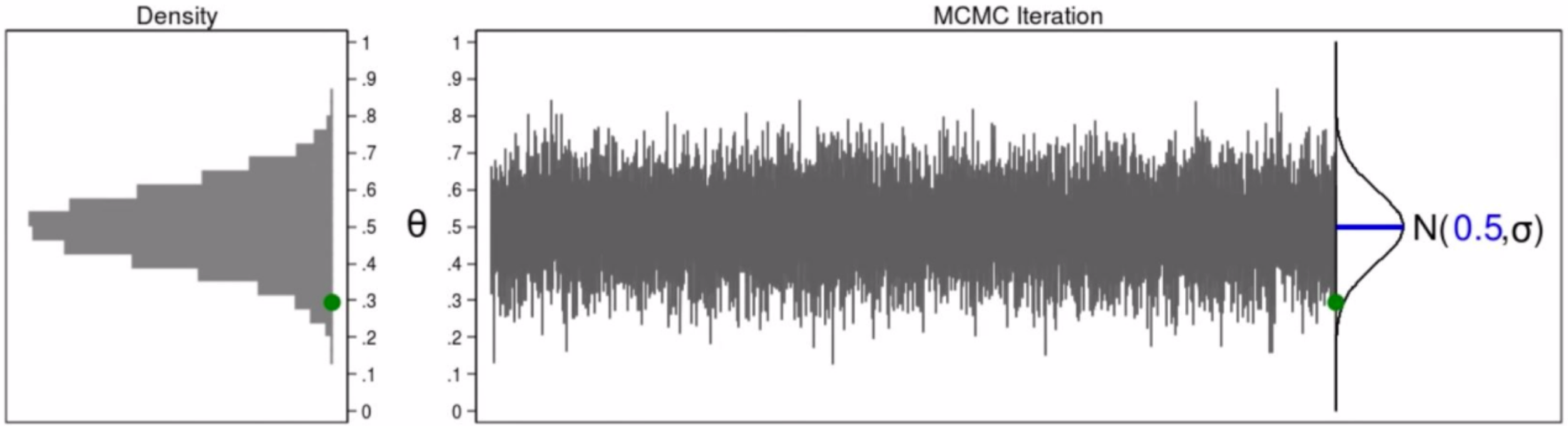

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MARKOV CHAINS

- The Markov property:

$$\mathbb{P}(X_{n+1} = x_i | X_n = x_j) = \mathbb{P}(X_{n+1} = x_i | X_n = x_j, X_{n-1} = x_k, \ldots, X_1 = x_z)$$

  - Knowing what happens at time n tells me everything I can learn about time $n + 1$; knowing what happened at time $n - 1$, $n - 2$, and so on doesn't provide me with any additional information.

MARKOV CHAINS: EXAMPLES

- Genetics
- Monopoly
- Baseball
- Google's PageRank Algorithm

MARKOV CHAINS: LONG-RUN BEHAVIOR

- Let's look at the long-run behavior of Markov chains.

MARKOV CHAINS: LONG-RUN BEHAVIOR

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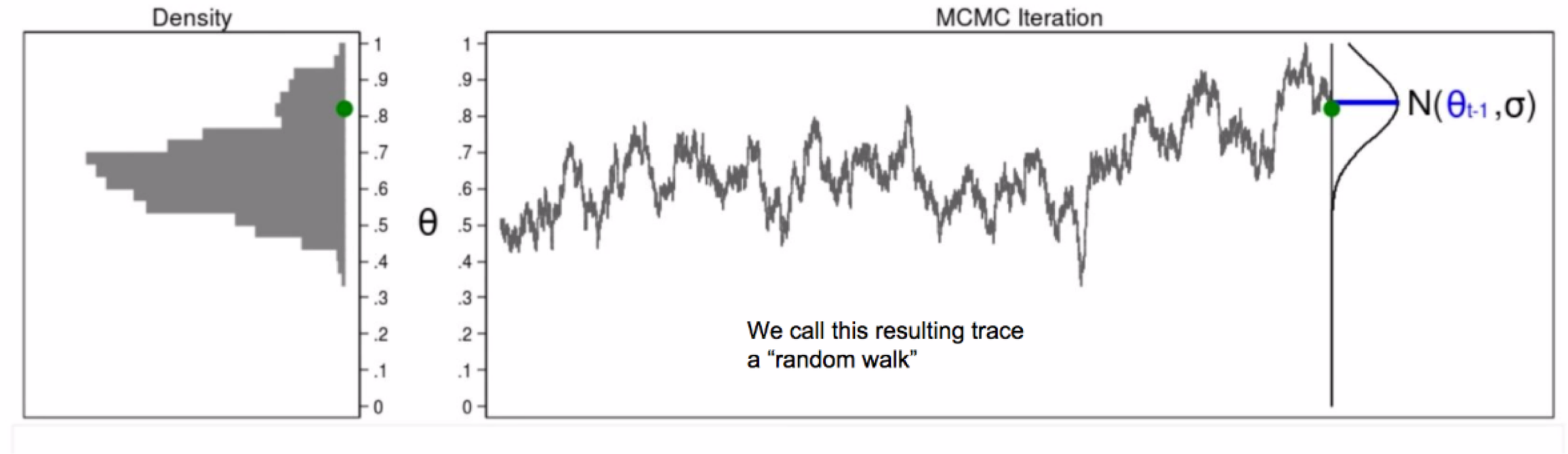
MARKOV CHAINS

- We'll apply ideas from Markov chains here.

$$\theta_t \sim N(\theta_{t-1}, \sigma)$$



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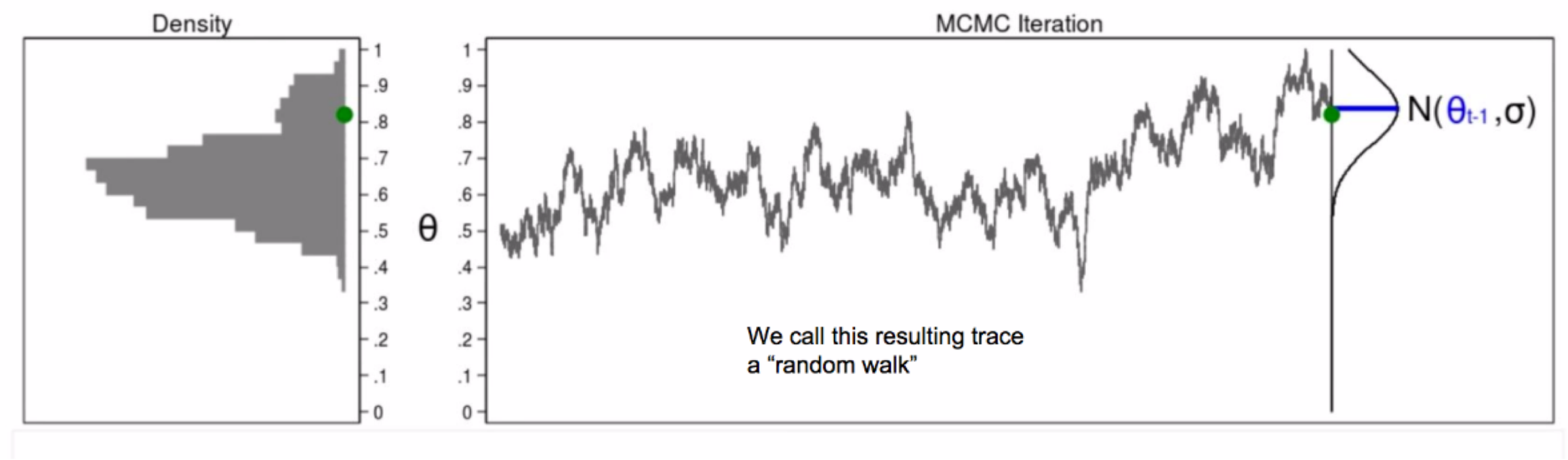
- We'll use a specific algorithm called the Metropolis-Hastings algorithm to accept/reject.

METROPOLIS-HASTINGS ALGORITHM

- Thus far, we've been generating all of these random numbers but have no idea if the numbers we're generating are headed in the right direction.



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- The Metropolis-Hastings algorithm allows us to, at each step, identify whether we are getting "hotter" or "colder."
- Recall that we're simulating $\theta_t \sim N(\theta_{t-1}, \sigma)$.
- Let's not move so quickly and call this θ_t. Let's call it $\theta_{proposal}$ instead.

$$\theta_{proposal} \sim N(\theta_{t-1}, \sigma)$$

METROPOLIS-HASTINGS ALGORITHM

- Well, how do we find θ_t?

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METROPOLIS-HASTINGS ALGORITHM

- Suppose $\theta_{proposal}$ is 10% likelier than θ_{t-1}, based on the posterior probability.
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- $\alpha_t = \min\{r, 1\}$
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METROPOLIS-HASTINGS ALGORITHM

- Suppose $\theta_{proposal}$ is only 80% as likely as θ_{t-1}, based on the posterior probability.
- $r(\theta_{proposal}, \theta_{t-1}) = \frac{P(\theta_{proposal}|y)}{P(\theta_{t-1}|y)}$
- $\alpha_t = \min\{r, 1\}$
- $u_t \sim Uniform(0,1)$
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METROPOLIS-HASTINGS ALGORITHM

- Wait… so sometimes we pick a worse value?

METROPOLIS-HASTINGS ALGORITHM

- Step 1: Generate $\theta_{proposal} \sim N(\theta_{t-1}, \sigma)$.
- Step 2: Calculate $r(\theta_{proposal}, \theta_{t-1})$, the ratio of posteriors of $\theta_{proposal}$ and θ_{t-1}.
- Step 3: Calculate the acceptance probability $\alpha_t = \min\{r, 1\}$.
- Step 4: Generate u_t from a $Uniform(0,1)$ distribution.
- Step 5: If the generated number u_t is greater than the acceptance probability α_t, then $\theta_t = \theta_{t-1}$. Otherwise, $\theta_t = \theta_{proposal}$.

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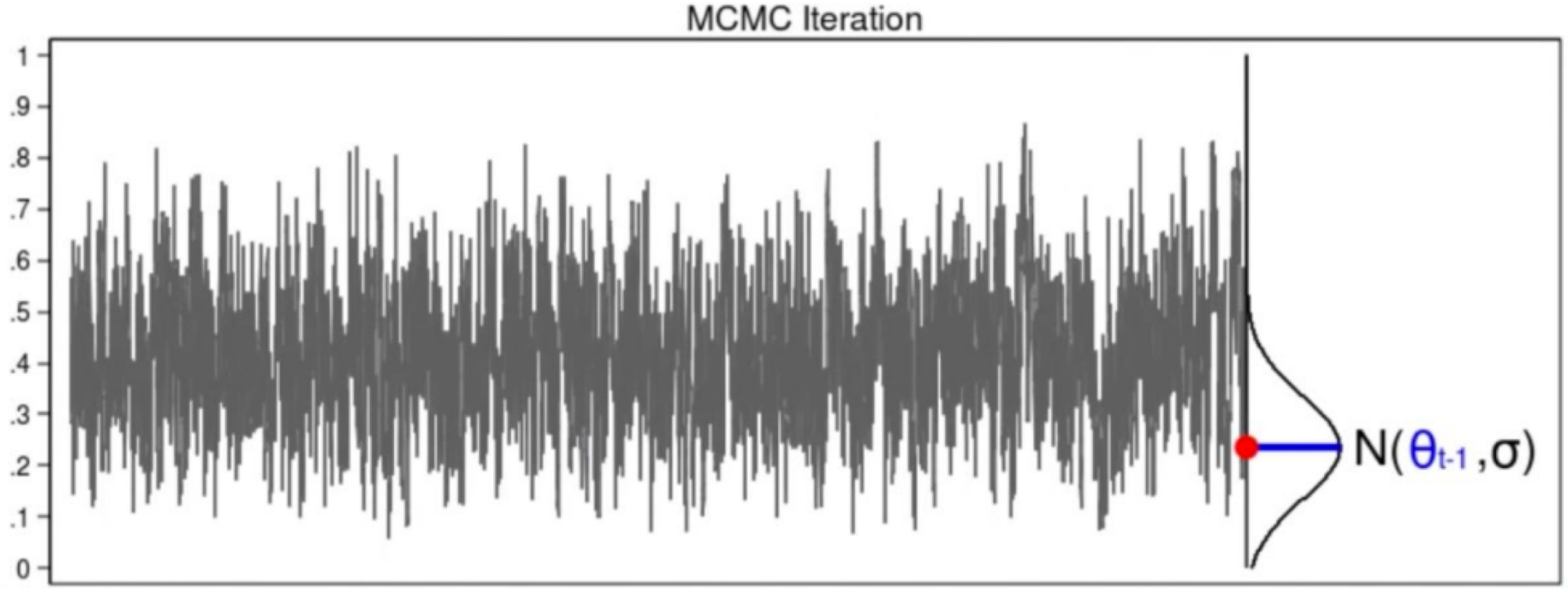

Source: STATA

METROPOLIS-HASTINGS ALGORITHM

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- Once we do this long enough and visually inspect plots to convince ourselves that we have "converged" to the posterior distribution of interest, we usually discard early samples (before we converged), then we store some large n of the later samples.

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- Once we have our sample of size n, we can conduct whatever inference we want.
  - Find the mean.
  - Find the median.
  - Find the 95% 'highest posterior density.'
  - Find the variance.

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- When is MCMC helpful?
- How does MCMC work?