Worksheet: Decimal Expansions from Binary Numbers

Part A: Worked Example

Example: Find the decimal expansion of the binary integer $(0101011111)_2$.

Step 1: Write digits with place values.

$$(0101011111)_2 = 0 \cdot 2^9 + 1 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$

Step 2: Simplify.

$$= 0 + 256 + 0 + 64 + 0 + 16 + 8 + 4 + 2 + 1$$

Step 3: Add.

$$= 351$$

So, $(0101011111)_2 = (351)_{10}$.

Part B: Easier Practice

Convert the binary number $(1011)_2$ into decimal. Show all steps.

Answer: $(1011)_2 = (11)_{10}$.

Part C: Harder Practice

Convert the binary number $(11011010101)_2$ into decimal. Show all steps.

Part D: Reflection

Why do we write binary expansions as sums of powers of 2? How does this compare to base 10?

Worksheet: Decimal Expansion from Octal (Example 2)

Goal

Convert an octal (base 8) numeral into its decimal (base 10) value using place value.

Part A — Worked Example (detailed)

Problem. What is the decimal expansion of $(7016)_8$?

Idea. In base 8, each position is a power of 8: from right to left $8^0, 8^1, 8^2, 8^3, \ldots$. For digits $d_3d_2d_1d_0$ we have:

$$(d_3d_2d_1d_0)_8 = d_3 \cdot 8^3 + d_2 \cdot 8^2 + d_1 \cdot 8^1 + d_0 \cdot 8^0.$$

Step 1: Label digits and place values.

$$(7016)_8 = 7 \cdot 8^3 + 0 \cdot 8^2 + 1 \cdot 8^1 + 6 \cdot 8^0.$$

Step 2: Evaluate powers of 8.

$$8^3 = 512, \quad 8^2 = 64, \quad 8^1 = 8, \quad 8^0 = 1.$$

Step 3: Multiply digits by powers.

$$7 \cdot 512 = 3584, \quad 0 \cdot 64 = 0, \quad 1 \cdot 8 = 8, \quad 6 \cdot 1 = 6.$$

Step 4: Add the contributions.

$$3584 + 0 + 8 + 6 = 3598.$$

Conclusion. $\boxed{(7016)_8 = (3598)_{10}}$

(Optional) Horner's Method (left-to-right accumulate).

$$(((7) \cdot 8 + 0) \cdot 8 + 1) \cdot 8 + 6 = (56 \cdot 8 + 1) \cdot 8 + 6 = (449) \cdot 8 + 6 = 3598.$$

Same answer, fewer big numbers along the way.

Part B — Easier Practice

Convert $(52)_8$ to decimal. Show all steps (place values, multiply, add).

Work:

$$(52)_8 = \underline{\hspace{3cm}} \cdot 8^1 + \underline{\hspace{3cm}} \cdot 8^0$$

$$= \underline{\hspace{5cm}} + \underline{\hspace{5cm}} = \underline{\hspace{5cm}}$$

Part C — Harder Practice

Convert $(574321)_8$ to decimal. *Hint:* write powers $8^5, 8^4, 8^3, 8^2, 8^1, 8^0$ first.

Work setup:

$$(574321)_8 = 5 \cdot 8^5 + 7 \cdot 8^4 + 4 \cdot 8^3 + 3 \cdot 8^2 + 2 \cdot 8^1 + 1 \cdot 8^0$$

$8^5 =$ _________, $8^4 =$ _________, $8^3 =$ _________, $8^2 =$ _________, $8^1 =$ _________, $8^0 =$ _________

Quick Self-Check

Why does the method above look exactly like the base-10 method, except with 8 instead of 10?

Worksheet: Decimal Expansion from Hexadecimal (Example 3)

Part A — Worked Example (detailed)

Problem. What is the decimal expansion of $(FACE)_{16}$?

Step 1: Recall place values in base 16. Each digit corresponds to 16^k:

$$(d_3d_2d_1d_0)_{16} = d_3 \cdot 16^3 + d_2 \cdot 16^2 + d_1 \cdot 16^1 + d_0 \cdot 16^0.$$

Step 2: Translate hex digits to decimal digits.

$$F = 15,\ A = 10,\ C = 12,\ E = 14.$$

Step 3: Substitute digits.

$$(FACE)_{16} = 15 \cdot 16^3 + 10 \cdot 16^2 + 12 \cdot 16^1 + 14 \cdot 16^0.$$

Step 4: Compute powers of 16.

$$16^3 = 4096, \quad 16^2 = 256, \quad 16^1 = 16, \quad 16^0 = 1.$$

Step 5: Multiply out.

$$15 \cdot 4096 = 61{,}440, \quad 10 \cdot 256 = 2{,}560, \quad 12 \cdot 16 = 192, \quad 14 \cdot 1 = 14.$$

Step 6: Add contributions.

$$61{,}440 + 2{,}560 + 192 + 14 = 64{,}206.$$

Answer: $\boxed{(FACE)_{16} = (64206)_{10}}$

Part B — Easier Practice

Convert $(2B)_{16}$ to decimal. Show all steps.

Part C — Easier Practice

Convert $(7F)_{16}$ to decimal. Show all steps.

Part D — Harder Practice

Convert $(BEEF)_{16}$ to decimal. Show all steps.

Reflection

How is converting from hexadecimal to decimal similar to converting from octal or binary?

Worksheet: Octal Expansion (Example 4)

Part A — Worked Example

Problem. Find the octal expansion of $(12345)_{10}$.

Step 1: Recall. To convert from decimal to base 8, divide repeatedly by 8 and record the remainders. The remainders (read bottom-to-top) give the digits.

Step 2: Divide 12345 **by** 8**.**

$$12345 = 8 \cdot 1543 + 1.$$

Remainder $= 1$, quotient $= 1543$.

Step 3: Divide 1543 **by** 8**.**

$$1543 = 8 \cdot 192 + 7.$$

Remainder $= 7$, quotient $= 192$.

Step 4: Divide 192 **by** 8**.**

$$192 = 8 \cdot 24 + 0.$$

Remainder $= 0$, quotient $= 24$.

Step 5: Divide 24 **by** 8**.**

$$24 = 8 \cdot 3 + 0.$$

Remainder $= 0$, quotient $= 3$.

Step 6: Divide 3 **by** 8**.**

$$3 = 8 \cdot 0 + 3.$$

Remainder $= 3$, quotient $= 0$. Stop here.

Step 7: Collect remainders. Reading bottom-to-top: $3, 0, 0, 7, 1$.

$$(12345)_{10} = (30071)_8.$$

$\boxed{(12345)_{10} = (30071)_8}$

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Part B — Easier Practice Problems

1. Convert $(25)_{10}$ into octal. 2. Convert $(64)_{10}$ into octal. 3. Convert $(255)_{10}$ into octal. Show all division steps!

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Part C — Harder Challenge

Convert $(54321)_{10}$ into octal.

Hint: Write each division step clearly and check by converting back.

Worksheet: Hexadecimal Expansion (Example 5)

Part A — Worked Example

Problem. Find the hexadecimal expansion of $(177130)_{10}$.

Step 1: Recall. To convert from decimal to base 16, divide repeatedly by 16 and record the remainders. The remainders (read bottom-to-top) give the digits.

Step 2: Divide 177130 **by** 16**.**

$$177130 = 16 \cdot 11070 + 10.$$

Remainder $= 10$ (which corresponds to A in hexadecimal).

Step 3: Divide 11070 **by** 16**.**

$$11070 = 16 \cdot 691 + 14.$$

Remainder $= 14$ (which corresponds to E).

Step 4: Divide 691 **by** 16**.**

$$691 = 16 \cdot 43 + 3.$$

Remainder $= 3$.

Step 5: Divide 43 **by** 16**.**

$$43 = 16 \cdot 2 + 11.$$

Remainder $= 11$ (which corresponds to B).

Step 6: Divide 2 **by** 16**.**

$$2 = 16 \cdot 0 + 2.$$

Remainder $= 2$, quotient $= 0$. Stop here.

Step 7: Collect remainders. Reading bottom-to-top: $2, B, 3, E, A$.

$$(177130)_{10} = (2B3EA)_{16}.$$

$\boxed{(177130)_{10} = (2B3EA)_{16}}$

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Part B — Easier Practice Problems

1. Convert $(255)_{10}$ into hexadecimal. 2. Convert $(4095)_{10}$ into hexadecimal.

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Part C — Harder Challenge

Convert $(1048575)_{10}$ into hexadecimal. (Hint: $1048575 = 2^{20} - 1$.)

Worksheet: Binary Expansion (Example 6)

Part A — Worked Example

Problem. Find the binary expansion of $(241)_{10}$.

Step 1: Recall. To convert a decimal number to binary (base 2), divide repeatedly by 2 and record the remainders. The remainders (read bottom-to-top) give the binary digits.

Step 2: Divide 241 **by** 2.

$$241 = 2 \cdot 120 + 1$$

Remainder = 1.

Step 3: Divide 120 **by** 2.

$$120 = 2 \cdot 60 + 0$$

Remainder = 0.

Step 4: Divide 60 **by** 2.

$$60 = 2 \cdot 30 + 0$$

Remainder = 0.

Step 5: Divide 30 **by** 2.

$$30 = 2 \cdot 15 + 0$$

Remainder = 0.

Step 6: Divide 15 **by** 2.

$$15 = 2 \cdot 7 + 1$$

Remainder = 1.

Step 7: Divide 7 **by** 2.

$$7 = 2 \cdot 3 + 1$$

Remainder = 1.

Step 8: Divide 3 **by** 2.

$$3 = 2 \cdot 1 + 1$$

Remainder = 1.

Step 9: Divide 1 **by** 2.

$$1 = 2 \cdot 0 + 1$$

Remainder = 1, quotient = 0. Stop here.

Step 10: Collect remainders. Reading bottom-to-top: 11110001.

$$(241)_{10} = (11110001)_2.$$

$\boxed{(241)_{10} = (11110001)_2}$

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Part B — Easier Practice Problems

1. Convert $(13)_{10}$ to binary. 2. Convert $(100)_{10}$ to binary.

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Part C — Harder Challenge

Convert $(1023)_{10}$ into binary. (Hint: $1023 = 2^{10} - 1$.)

Worksheet: Conversions Between Binary, Octal, and Hexadecimal (Example 7)

Part A — Worked Example

Problem. Find the octal and hexadecimal expansions of $(11\ 1110\ 1011\ 1100)_2$ and the binary expansions of $(765)_8$ and $(\text{A8D})_{16}$.

Step 1: Convert $(11\ 1110\ 1011\ 1100)_2$ **to octal.**
- Group binary digits into blocks of 3 (add leading zeros if needed):

$$011,\ 111,\ 010,\ 111,\ 100$$

- Convert each block to octal: $3, 7, 2, 7, 4$.

$$(11\ 1110\ 1011\ 1100)_2 = (37274)_8$$

Step 2: Convert $(11\ 1110\ 1011\ 1100)_2$ **to hexadecimal.**
- Group binary digits into blocks of 4:

$$0011,\ 1110,\ 1011,\ 1100$$

- Convert each block: $3, E, B, C$.

$$(11\ 1110\ 1011\ 1100)_2 = (3EBC)_{16}$$

Step 3: Convert $(765)_8$ **to binary.**
- Each octal digit becomes 3 binary digits:

$$7 = 111,\ 6 = 110,\ 5 = 101$$

$$(765)_8 = (111110101)_2$$

Step 4: Convert $(\textbf{A8D})_{16}$ **to binary.**
- Each hex digit becomes 4 binary digits:

$$A = 1010,\ 8 = 1000,\ D = 1101$$

$$(\text{A8D})_{16} = (101010001101)_2$$

$$\boxed{\text{Final Answers: } (37274)_8,\ (3EBC)_{16},\ (111110101)_2,\ (101010001101)_2}$$

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Part B — Easier Practice Problems

1. Convert $(101101)_2$ to octal. 2. Convert $(47)_8$ to binary.

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Part C — Harder Challenge

Convert $(111011110101)_2$ into both octal and hexadecimal.

Example 8: Binary Addition

We want to add the binary numbers:

$$a = (1110)_2 \quad \text{and} \quad b = (1011)_2$$

Worked Example

Step 1: Add the rightmost digits: $0 + 1 = 1$. No carry. Step 2: Next column: $1 + 1 = 0$ with carry 1. Step 3: Next column: $1 + 0 + 1 = 0$ with carry 1. Step 4: Leftmost column: $1 + 1 + 1 = 1$ with carry 1.

Thus, the result is:

$$(1110)_2 + (1011)_2 = (11001)_2$$

Practice Problems

1. Easier: Add $(101)_2$ and $(11)_2$.

2. Easier: Add $(111)_2$ and $(1)_2$.

3. Harder: Add $(11011)_2$ and $(11101)_2$.

Section 4.2 — Example 9 Worksheet
Counting Bit Additions in Algorithm 2

Problem (from Example 9)

How many *additions of bits* are required by **Algorithm 2 (Addition of Integers)** to add two integers with at most n bits in their binary representations?

Key idea

At each bit position j (from right to left), Algorithm 2 forms the three-bit sum

$$a_j + b_j + c \quad (c \in \{0,1\} \text{ is the incoming carry}),$$

then outputs the sum bit s_j and the next carry c'. Computing $a_j + b_j + c$ can be done with *at most two* one-bit additions:

1. add the pair $a_j + b_j$ (one bit-addition);
2. add the carry: $(a_j + b_j) + c$ (one more bit-addition, sometimes trivial if $c = 0$ and no carry occurs).

Hence the total number of bit additions is *less than* $2n$; in particular it is $\leq 2n$ and therefore the running time is $O(n)$.

Worked examples (Algorithm 2, step-by-step)

A. Two-bit example

Add $a = (11)_2$ and $b = (01)_2$.

We process bits from right (least significant) to left (most significant). Let c be the incoming carry; initially $c = 0$.

j	a_j	b_j	c (in)	s_j	c (out)
0	1	1	0	0	1
1	1	0	1	0	1

$s_2 = c_{\text{final}} = 1$

Thus the result is $s = (1\,00)_2 = (100)_2$. Bit-addition count: at most 2 per column $\Rightarrow$ at most 4 total.

B. Four-bit example

Add $a = (1011)_2$ and $b = (0110)_2$.

j	a_j	b_j	c (in)	s_j	c (out)
0	1	0	0	1	0
1	1	1	0	0	1
2	0	1	1	0	1
3	1	0	1	0	1

$s_4 = c_{\text{final}} = 1$

Hence $a + b = (1\,0001)_2 = (10001)_2$. Again, at most two bit additions per column $\Rightarrow$ at most $2 \cdot 4 = 8$ bit additions.

Why "less than $2n$" and $O(n)$?

Some columns may not *need* to add the carry (e.g., $c = 0$ and $a_j + b_j < 2$); practical implementations can skip that second bit-addition. Therefore the # of bit additions is $< 2n$ and certainly $\leq 2n$. As a function of n, that is linear time: $O(n)$.

Your turn

Use Algorithm 2. Show your table of columns j, a_j, b_j, c (in), s_j, c (out).

Two easier

1. $(0101)_2 + (0011)_2$
2. $(1001)_2 + (0001)_2$

One harder

3. $(11101101)_2 + (10111011)_2$

Ch 4.2 — Example 10 Worksheet
Binary Multiplication via Algorithm 3

Discrete Structures

Goal

Multiply two positive integers given in binary using **Algorithm 3 (Multiplication of Integers)**:

- For each bit b_j of the multiplier b, form a partial product c_j:

$$c_j = \begin{cases} a \text{ shifted left } j \text{ places}, & \text{if } b_j = 1, \\ (0)_2, & \text{if } b_j = 0. \end{cases}$$

- Add all partial products using **Algorithm 2 (binary addition)**.

Worked Example (from the text)

Find the product of $a = (110)_2$ and $b = (101)_2$.

Step 1 — Identify bits of the multiplier

$b = (101)_2$ has bits (from least significant to most) $b_0 = 1$, $b_1 = 0$, $b_2 = 1$.

Step 2 — Form partial products

Shift $a = (110)_2$ by j places when $b_j = 1$.

$$\begin{aligned} b_0 = 1: &\quad c_0 = (110)_2 \cdot 2^0 = (110)_2. \\ b_1 = 0: &\quad c_1 = (000)_2 \qquad \text{(all zeros)}. \\ b_2 = 1: &\quad c_2 = (110)_2 \cdot 2^2 = (11000)_2. \end{aligned}$$

Step 3 — Add the partial products (Algorithm 2)

Pad with initial zeros to align columns and add:

$$\begin{array}{r} 11000 \\ 00000 \\ +\ 00110 \\ \hline 11110 \end{array} \quad \Rightarrow \quad (11110)_2.$$

Step 4 — Quick decimal check (optional)

$(110)_2 = 6$, $(101)_2 = 5$, and $6 \cdot 5 = 30$, while $(11110)_2 = 16 + 8 + 4 + 2 = 30$. Checks out.

Answer. $(110)_2(101)_2 = (1\,1110)_2$.

Student Practice

Show clear partial products and use Algorithm 2 for each addition.

A. Easier (warm-up)

1) $(101)_2 \times (11)_2$

B. Harder (challenge)

2) $(10011)_2 \times (1011)_2$ (two 1s close together; more carries)
3) $(111010)_2 \times (10111)_2$ (longer; many partials)

Space for work:

Example 12 Worksheet: Fast Modular Exponentiation

Problem: Use Algorithm 5 to find $3^{544} \pmod{645}$.

Step-by-Step Solution

Algorithm 5 works by:

1. Writing the exponent (544) in binary.
2. Iterating through the bits, squaring at each step, and multiplying only when the current bit is 1.
3. Reducing by the modulus after every multiplication.

Binary form of 544: $(1000100000)_2$.
Steps (values of i, x, and *power*):

- $i = 0$: $a_0 = 0$, $x = 1$, $power = 3^2 \pmod{645} = 9$.
- $i = 1$: $a_1 = 0$, $x = 1$, $power = 9^2 \pmod{645} = 81$.
- $i = 2$: $a_2 = 0$, $x = 1$, $power = 81^2 \pmod{645} = 111$.
- $i = 3$: $a_3 = 0$, $x = 1$, $power = 111^2 \pmod{645} = 66$.
- $i = 4$: $a_4 = 0$, $x = 1$, $power = 66^2 \pmod{645} = 486$.
- $i = 5$: $a_5 = 1$, $x = (1 \cdot 486) \pmod{645} = 486$, $power = 486^2 \pmod{645} = 126$.
- $i = 6$: $a_6 = 0$, $x = 486$, $power = 126^2 \pmod{645} = 396$.
- $i = 7$: $a_7 = 0$, $x = 486$, $power = 396^2 \pmod{645} = 81$.
- $i = 8$: $a_8 = 0$, $x = 486$, $power = 81^2 \pmod{645} = 111$.
- $i = 9$: $a_9 = 1$, $x = (486 \cdot 111) \pmod{645} = 36$, $power = 111^2 \pmod{645} = 36$.

Final answer: $3^{544} \pmod{645} = \boxed{36}$.

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Practice Problems

1. Easier: Compute $2^{13} \pmod{19}$ using Algorithm 5.
2. Medium: Compute $7^{45} \pmod{50}$ using Algorithm 5.
3. Harder: Compute $11^{117} \pmod{221}$ using Algorithm 5.