# Division in Number Theory

## Definition of Division

If a and b are integers with $b \neq 0$, we say that b divides a if there is an integer c such that $a = bc$. In this case, we write $b \mid a$, and say 'b divides a'. For example, 3 divides 12 since $12 = 3 \times 4$.

## Example 1

Determine whether $3 \mid 7$ and whether $3 \mid 12$.

> Solution: 3 does not divide 7, since $7 \div 3$ is not an integer. However, 3 divides 12, since $12 \div 3 = 4$ is an integer.

Easier Example: Does 2 divide 8?

> Solution: Yes, since $8 \div 2 = 4$.

Challenging Example: Does 7 divide 100?

> Solution: No, since $100 \div 7 = 14.2857...$, not an integer.

## Example 2

Let $n = 4$. The positive integers divisible by 4 are all integers of the form $4k$, where k is a positive integer: 4, 8, 12, 16, ...

Easier Example: List the positive integers divisible by 2 up to 20.

> Solution: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

Challenging Example: List the positive integers divisible by 9 up to 100.

> Solution: 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99.

## Theorem 1

Let a, b, and c be integers, where $a \neq 0$. (i) If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$. (ii) If $a \mid b$, then $a \mid (b \times m)$ for all integers m. (iii) If $a \mid b$ and $b \mid c$, then $a \mid c$.

Easier Example: Show that if 2 divides 6 and 2 divides 8, then 2 divides $(6 + 8)$.

> Solution: $6 + 8 = 14$, and $14 \div 2 = 7$ is an integer. So $2 \mid 14$.

Challenging Example: If $5 \mid 20$ and $5 \mid 35$, prove $5 \mid (20 + 35)$.

> Solution: $20 + 35 = 55$, and $55 \div 5 = 11$, so $5 \mid 55$.

## Corollary 1

If a, b, and c are integers, with $a \neq 0$, such that $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ whenever m and n are integers.

Easier Example: If $3 \mid 6$ and $3 \mid 9$, prove $3 \mid (2\times6 + 1\times9)$.

Solution: $2\times6 + 9 = 21$, and $21 \div 3 = 7$, so $3 \mid 21$.

Challenging Example: If $4 \mid 12$ and $4 \mid 20$, prove $4 \mid (3\times12 + 2\times20)$.

Solution: $3\times12 + 2\times20 = 36 + 40 = 76$. Since $76 \div 4 = 19$, $4 \mid 76$.