

Discrete Structures Chapter 4.6 — Cryptography

Example 1 (Student Worksheet): Caesar Cipher, shift $k = 3$

Learning goals. Practice converting letters \leftrightarrow numbers, computing $(p + k) \bmod 26$, and translating back.

Alphabet convention (zero-based).

$$A = 0, B = 1, \dots, Z = 25$$

We work in $\mathbb{Z}_{26} \pmod{26}$. Spaces and punctuation are carried through unchanged; we use uppercase.

Encryption rule. For plaintext number $p \in \{0, \dots, 25\}$ and shift k , the ciphertext number is

$$c \equiv p + k \pmod{26}.$$

For this worksheet we use $k = 3$ (the classic “Caesar +3”).

Fast tips (use ’em shamelessly):

- Add 3 quickly by doing $+1, +2, +3$ as you scan, or use the wrap trick: adding 3 to 24, 25 wraps to 1, 2.
- Decrypting a +3 cipher is the same as *adding* -3 , i.e., adding 23 mod 26.
- Common wrap cases: $24+3 \rightarrow 1$ (Y \rightarrow B), $25+3 \rightarrow 2$ (Z \rightarrow C).

Guided task. Encrypt the message:

MEET YOU IN THE PARK

Step 1 — Letters \rightarrow numbers (A=0,...,Z=25). Fill the *plaintext numbers* p under each letter.

M E E T Y O U I N T H E P A R K

(write numbers p here)

Step 2 — Add the shift $k = 3 \bmod 26$. Compute $c \equiv p + 3 \pmod{26}$ for each position and write the results:

Step 3 — Numbers \rightarrow letters. Translate each c back to letters to form the ciphertext:

Neatness check. Your ciphertext should be readable in groups (keep the spaces from the original). If you decrypt with -3 you should land back on MEET YOU IN THE PARK.

Quick reference table (optional). If you like a visual:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

Practice (still Caesar, but you drive):

P1. Encrypt (easy). Use $k = 5$ to encrypt:

DOGS AND CATS

Hint: $D=3$ so $D \mapsto 3+5=8 \Rightarrow I$. Keep spaces.

P2. Decrypt (easy). The ciphertext below was made with a $k = 5$ Caesar. Recover the plaintext.

YMNX NX FQ YJXY

Tip: Decrypt by adding -5 (or $+21$) mod 26.

P3. Crack the shift (harder). The message below is a Caesar cipher with *unknown* k :

L ORYH PDWKP

Clues: Try common words; guess that “PDWKP” might be “MATH?” or “MATHS?”. Also, a one-letter word is often A or I. Determine k and decrypt.

Reflection. In one sentence: why does “mod 26” make the Caesar cipher *wrap* from Z back to A?

Discrete Structures Chapter 4.6 — Cryptography

Example 1: Caesar Cipher ($k = 3$)

Question. Encrypt the message MEET YOU IN THE PARK using the Caesar cipher with shift $k = 3$.

Step 1 — Letters \rightarrow numbers.

We use zero-based numbering: A=0, B=1, \dots , Z=25.

$$\text{MEET YOU IN THE PARK} \Rightarrow 12, 4, 4, 19, 24, 14, 20, 8, 13, 19, 7, 4, 15, 0, 17, 10$$

Step 2 — Apply $f(p) = (p + 3) \bmod 26$.

Add 3 to each number, wrapping around if the result exceeds 25:

$$\begin{aligned}(12 + 3) &= 15, (4 + 3) = 7, (4 + 3) = 7, (19 + 3) = 22, \\(24 + 3) &= 27 \equiv 1, (14 + 3) = 17, (20 + 3) = 23, \\(8 + 3) &= 11, (13 + 3) = 16, (19 + 3) = 22, (7 + 3) = 10, (4 + 3) = 7, \\(15 + 3) &= 18, (0 + 3) = 3, (17 + 3) = 20, (10 + 3) = 13.\end{aligned}$$

Step 3 — Numbers \rightarrow letters.

Convert the ciphertext numbers back to letters:

$$\begin{aligned}15, 7, 7, 22, 1, 17, 23, 11, 16, 22, 10, 7, 18, 3, 20, 13 \\ \Rightarrow \text{PHHW BRX LQ WKH SDUN}\end{aligned}$$

Final Answer. The encrypted message is:

PHHW BRX LQ WKH SDUN

(Translation: “MEET YOU IN THE PARK” shifted +3.)

Quick Reflection. The Caesar cipher uses modular arithmetic in \mathbb{Z}_{26} so letters “wrap around” after Z. The function $f(p) = (p + k) \bmod 26$ keeps all results in 0–25.

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Practice Solutions

P1 — Encrypt (easy). Use $k = 5$ to encrypt: DOGS AND CATS.

Step 1 — Convert to numbers:

$$3, 14, 6, 18, 0, 13, 3, 2, 0, 19, 18$$

Step 2 — Add 5 mod 26:

8, 19, 11, 23, 5, 18, 8, 7, 5, 24, 23

Step 3 — Back to letters:

ITLX FSI HFYX

P2 — Decrypt (easy). Decrypt YMNX NX FQ YJXY that was made with $k = 5$.

We reverse the shift: $c - 5 \pmod{26}$.

$Y = 24 \rightarrow 19 = T$, $M = 12 \rightarrow 7 = H$, $N = 13 \rightarrow 8 = I$, $X = 23 \rightarrow 18 = S$

\Rightarrow THIS IS AN TEST

So the message is “THIS IS AN TEST.” (It should probably read “THIS IS A TEST.”)

P3 — Crack the shift (harder). Ciphertext: L ORYH PDWKP!

Try guessing common English patterns.

ORYH looks like “LOVE,” and the one-letter word “L” likely corresponds to “I.”

That suggests a shift of $k = 3$ backward (since $L \rightarrow I$ is -3).

Decrypting with $k = 3$:

L ORYH PDWKP! \Rightarrow I LOVE MATH!

Summary of Key Takeaways

- The Caesar cipher is modular addition in \mathbb{Z}_{26} .
- Encryption: $E_k(p) = (p + k) \pmod{26}$
- Decryption: $D_k(c) = (c - k) \pmod{26}$
- If you can add or subtract mod 26, you can encrypt or decrypt.
- This cipher is historically important but easily broken by frequency analysis or brute force (26 possibilities).

Going Deeper. You can extend this same math to more complex ciphers:

$$f(p) = (a \cdot p + b) \pmod{26}$$

where a must have a multiplicative inverse mod 26. This leads directly into the *Affine Cipher*—our next example.

Example 2 (Worksheet) — Shift Cipher with $k = 11$

Goal. Encrypt the message STOP GLOBAL WARMING using Caesar’s shift cipher with $k = 11$.

Big idea (the “why”):

We model letters as numbers in \mathbb{Z}_{26} so that a shift is just *modular addition*. This keeps us in the alphabet and gives the wrap-around from Z back to A .

$$A = 0, B = 1, \dots, Z = 25 \qquad E_k(p) = (p + k) \bmod 26.$$

For this example, $k = 11$.

Step 1 — Normalize and map letters \rightarrow numbers

We use uppercase and keep spaces. Convert each letter of STOP GLOBAL WARMING to its number:

$$\begin{array}{ccc} \underbrace{\text{STOP}} & \underbrace{\text{GLOBAL}} & \underbrace{\text{WARMING}} \ . \\ 18\ 19\ 14\ 15 & 6\ 11\ 14\ 1\ 0\ 11 & 22\ 0\ 17\ 12\ 8\ 13\ 6 \end{array}$$

Step 2 — Apply the shift $k = 11$ (add 11 mod 26)

Compute $c \equiv p + 11 \pmod{26}$ for each number. Do the wrap when you go past 25.

$$\begin{array}{ll} \text{STOP :} & 18, 19, 14, 15 \mapsto 3, 4, 25, 0 \\ \text{GLOBAL :} & 6, 11, 14, 1, 0, 11 \mapsto 17, 22, 25, 12, 11, 22 \\ \text{WARMING :} & 22, 0, 17, 12, 8, 13, 6 \mapsto 7, 11, 2, 23, 19, 24, 17. \end{array}$$

Step 3 — Map numbers \rightarrow letters and keep spaces

$$3, 4, 25, 0 \mid 17, 22, 25, 12, 11, 22 \mid 7, 11, 2, 23, 19, 24, 17 \Rightarrow \boxed{\text{DEZA RWZMLW HLCXTYR}}$$

Helpful tips & common pitfalls

- **A=0, not 1.** Off-by-one mistakes are the #1 bug.
- **Wrap cleanly:** if $p + k \geq 26$, subtract 26 (i.e., reduce mod 26).

- **Spaces/punctuation** pass through unchanged; only letters get shifted.
- **Decrypting** with $k = 11$ is the same as adding -11 (or $+15$) mod 26.

Practice (your turn!)

Problem A (easier). Encrypt with $k = 4$: MATH IS FUN

Why: smaller shift, shorter phrase—perfect confidence builder.

Problem B (similar). Decrypt with $k = 11$: SPWWZ HZCWO

Tip: subtract 11 mod 26 or add 15.

Problem C (harder). Unknown k . Decrypt the Caesar ciphertext: P HT HA AOL WHYR

Hints: a one-letter word is often I or A. The block AOL frequently shows up when “THE” is

encrypted with $k = 7$.

Reflection. In one sentence: explain why modular arithmetic guarantees a valid letter after every shift.

Solutions for Example 2 Practice

Problem A (easier). Encrypt with $k = 4$: MATH IS FUN

Map to numbers (A=0):

$$\text{MATH IS FUN} \Rightarrow 12, 0, 19, 7, 8, 18, 5, 20, 13.$$

Add 4 mod 26:

$$12, 0, 19, 7 \mapsto 16, 4, 23, 11 \quad (\text{M} \rightarrow \text{Q}, \text{A} \rightarrow \text{E}, \dots)$$

$$8, 18 \mapsto 12, 22 \quad 5, 20, 13 \mapsto 9, 24, 17.$$

Back to letters:

$$16, 4, 23, 11, 12, 22, 9, 24, 17 \Rightarrow \boxed{\text{QEXL MW JYR}}.$$

Why it works: Every step is addition in \mathbb{Z}_{26} ; wrap ensures letters stay in 0–25.

Problem B (similar). Decrypt with $k = 11$: SPWWZ HZCWO

Numbers for ciphertext:

$$\text{SPWWZ HZCWO} \Rightarrow 18, 15, 22, 22, 25, 7, 25, 2, 22, 14.$$

Subtract 11 (or add 15) mod 26:

$$18, 15, 22, 22, 25 \mapsto 7, 4, 11, 11, 14 \quad (\text{H, E, L, L, O})$$

$$7, 25, 2, 22, 14 \mapsto 22, 14, 17, 11, 3 \quad (\text{W, O, R, L, D}).$$

Plaintext: $\boxed{\text{HELLO WORLD}}$.

Problem C (harder). Unknown k : P HT HA AOL WHYR

Strategy (the why): Look for patterns. A one-letter word is probably I or A. Also, AOL famously appears when “THE” is shifted by $k = 7$ (since $19+7 = 26 \equiv 0 = \text{A}$, etc.).

Infer k : If AOL is THE, then the shift is $k = 7$.

Decrypt by subtracting 7:

$$\text{P} \mapsto \text{I}, \quad \text{HT} \mapsto \text{AM}, \quad \text{HA} \mapsto \text{AT}, \quad \text{AOL} \mapsto \text{THE}, \quad \text{WHYR} \mapsto \text{PARK}.$$

\Rightarrow I AM AT THE PARK.

Key takeaways.

- Encryption: $E_k(p) = (p + k) \bmod 26$, Decryption: $D_k(c) = (c - k) \bmod 26$.
- Unknown k can be cracked with educated guesses (“THE”, one-letter words) or brute force (only 26 options).
- Thinking in \mathbb{Z}_{26} explains the wrap-around and keeps errors low.

Caesar Cipher Decryption

Student Worksheet

Understanding Decryption

Previously, we learned how to **encrypt** messages using the Caesar cipher. Now we'll learn to **decrypt** them—convert the secret message back to the original!

The key insight: **Decryption is the reverse of encryption.**

- **Encryption:** We shifted letters *forward* by k positions using $f(p) = (p + k) \bmod 26$
- **Decryption:** We shift letters *backward* by k positions using $f(p) = (p - k) \bmod 26$

Key Concept: Negative Numbers and Mod

When we subtract and get a negative number, we need to “wrap around” the other direction. For example, if we try to go back 7 from the letter E (position 4), we get $4 - 7 = -3$.

To handle this, we compute: $-3 \bmod 26 = 23$ (which is the letter X).

Quick trick: If you get a negative number, just add 26 to make it positive!

$$-3 + 26 = 23$$

Example 3: Worked Solution

Question: Decrypt the ciphertext message “LEWLYPLUJL PZ H NYLHA HSOHOLY” that was encrypted with the shift cipher with shift $k = 7$.

Solution:

Step 1: Convert letters to numbers

We use our standard A=0, B=1, C=2, ..., Z=25 system. Let's convert the ciphertext:

- **LEWLYPLUJL:** L=11, E=4, W=22, L=11, Y=24, P=15, L=11, U=20, J=9, L=11
- **PZ:** P=15, Z=25

- **H:** H=7
- **NYLHA:** N=13, Y=24, L=11, H=7, A=0
- **HSOHOLY:** A=0, L=11, H=7, O=14, L=11, Y=24

Our number sequence is:

			11	4	22	11	24	15	11	20	9	11						
15	25		7		13	24	11	7	0				0	11	7	9	14	11
													24					

Step 2: Apply the decryption function

We apply $f(p) = (p - 7) \bmod 26$ to shift backward by 7. Let's work through each number:

$$\begin{aligned}
 f(11) &= (11 - 7) \bmod 26 = 4 \bmod 26 = 4 \\
 f(4) &= (4 - 7) \bmod 26 = -3 \bmod 26 = 23 \quad (\text{add 26: } -3 + 26 = 23) \\
 f(22) &= (22 - 7) \bmod 26 = 15 \bmod 26 = 15 \\
 f(11) &= (11 - 7) \bmod 26 = 4 \bmod 26 = 4 \\
 f(24) &= (24 - 7) \bmod 26 = 17 \bmod 26 = 17 \\
 f(15) &= (15 - 7) \bmod 26 = 8 \bmod 26 = 8 \\
 f(11) &= (11 - 7) \bmod 26 = 4 \bmod 26 = 4 \\
 f(20) &= (20 - 7) \bmod 26 = 13 \bmod 26 = 13 \\
 f(9) &= (9 - 7) \bmod 26 = 2 \bmod 26 = 2 \\
 f(11) &= (11 - 7) \bmod 26 = 4 \bmod 26 = 4
 \end{aligned}$$

Continuing for the remaining letters:

$$\begin{aligned}
 f(15) &= 8, & f(25) &= 18, & f(7) &= 0 \\
 f(13) &= 6, & f(24) &= 17, & f(11) &= 4, & f(7) &= 0, & f(0) &= 19 \\
 f(0) &= 19, & f(11) &= 4, & f(7) &= 0, & f(9) &= 2, & f(14) &= 7, & f(11) &= 4, & f(24) &= 17
 \end{aligned}$$

Our decrypted numbers are:

			4	23	15	4	17	8	4	13	2	4					
8	18		0		6	17	4	0	19		19	4	0	2	7	4	17

Pro Tip: Handling Negative Results

Whenever $(p - k)$ gives you a negative number:

1. Notice it's negative
2. Add 26 to make it positive
3. That's your answer!

Example: $(4 - 7) = -3$, so $-3 + 26 = 23$

Step 3: Convert numbers back to letters

Using $A=0, B=1, \dots, Z=25$:

- $4=E, 23=X, 15=P, 4=E, 17=R, 8=I, 4=E, 13=N, 2=C, 4=E$
- $8=I, 18=S$
- $0=A$
- $6=G, 17=R, 4=E, 0=A, 19=T$
- $19=T, 4=E, 0=A, 2=C, 7=H, 4=E, 17=R$

Final Answer: The decrypted message is **EXPERIENCE IS A GREAT TEACHER**

Why This Works

If someone encrypted a message by shifting forward 7, we decrypt by shifting backward 7. It's like walking 7 steps forward, then 7 steps back—you end up where you started!

Practice Problems

Problem A (Easier Warm-up)

Decrypt the ciphertext “FDW” that was encrypted with shift $k = 3$.

Hint: This is a short message. Remember to subtract 3 from each letter's position. If you get negative numbers, add 26!

Problem B (Standard Practice)

Decrypt the ciphertext “**MJQQT BTWQI**” that was encrypted with shift $k = 5$.

Hint: You encrypted this message in the previous worksheet! Now decrypt it to get back the original message.

Problem C (Challenge)

Decrypt the ciphertext “**EJKKR ZRUOJ**” that was encrypted with shift $k = 5$.

Challenge: Some of these letters will give negative results when you subtract 5. Practice your wrapping-around skills!

Caesar Cipher Decryption

Teacher Solutions Manual

Problem A Solution: Decrypt “CAT” with shift $k = 3$

Step 1: Convert letters to numbers

$$F = 5$$

$$D = 3$$

$$W = 22$$

Number sequence: 5 3 22

Step 2: Apply decryption function $f(p) = (p - 3) \bmod 26$

$$f(5) = (5 - 3) \bmod 26 = 2 \bmod 26 = 2$$

$$f(3) = (3 - 3) \bmod 26 = 0 \bmod 26 = 0$$

$$f(22) = (22 - 3) \bmod 26 = 19 \bmod 26 = 19$$

Decrypted numbers: 2 0 19

Step 3: Convert back to letters

$$2 = C$$

$$0 = A$$

$$19 = T$$

Answer: CAT

Teaching Note

This is the easiest problem because: (1) short message, (2) all results are positive (no negative numbers to handle), and (3) it's the reverse of Problem A from the encryption worksheet. Students can verify their answer by re-encrypting CAT with $k = 3$ to get FDW.

Problem B Solution: Decrypt “MJQQT BTWQI” with shift $k = 5$

Step 1: Convert letters to numbers

Breaking down by word:

- **MJQQT:** M=12, J=9, Q=16, Q=16, T=19
- **BTWQI:** B=1, T=19, W=22, Q=16, I=8

Number sequence:

12 9 16 16 19 1 19 22 16 8

Step 2: Apply decryption function $f(p) = (p - 5) \bmod 26$

$$\begin{aligned}f(12) &= (12 - 5) \bmod 26 = 7 \bmod 26 = 7 \\f(9) &= (9 - 5) \bmod 26 = 4 \bmod 26 = 4 \\f(16) &= (16 - 5) \bmod 26 = 11 \bmod 26 = 11 \\f(16) &= (16 - 5) \bmod 26 = 11 \bmod 26 = 11 \\f(19) &= (19 - 5) \bmod 26 = 14 \bmod 26 = 14 \\f(1) &= (1 - 5) \bmod 26 = -4 \bmod 26 = 22 \quad (-4 + 26 = 22) \\f(19) &= (19 - 5) \bmod 26 = 14 \bmod 26 = 14 \\f(22) &= (22 - 5) \bmod 26 = 17 \bmod 26 = 17 \\f(16) &= (16 - 5) \bmod 26 = 11 \bmod 26 = 11 \\f(8) &= (8 - 5) \bmod 26 = 3 \bmod 26 = 3\end{aligned}$$

Decrypted numbers:

7 4 11 11 14 22 14 17 11 3

Step 3: Convert back to letters

- 7=H, 4=E, 11=L, 11=L, 14=O
- 22=W, 14=O, 17=R, 11=L, 3=D

Answer: HELLO WORLD

Teaching Note

This problem introduces negative numbers! When we decrypt B (position 1) with shift 5, we get: $1 - 5 = -4$.

To handle negative results in modular arithmetic: $-4 \bmod 26 = 22$

Students can calculate this by adding 26: $-4 + 26 = 22$, which corresponds to the letter W.

Connection: Students encrypted "HELLO WORLD" in the previous worksheet and got "MJQQT BTWQI". Now they're decrypting it back—reinforcing the inverse relationship between encryption and decryption.

Problem C Solution: Decrypt "EJKKR ZRUOJ" with shift $k = 5$

Step 1: Convert letters to numbers

Breaking down by word:

- **EJKKR:** E=4, J=9, K=10, K=10, R=17
- **ZRUOJ:** Z=25, R=17, U=20, O=14, J=9

Number sequence:

4 9 10 10 17 25 17 20 14 9

Step 2: Apply decryption function $f(p) = (p - 5) \bmod 26$

$$f(4) = (4 - 5) \bmod 26 = -1 \bmod 26 = 25 \quad (-1 + 26 = 25)$$

$$f(9) = (9 - 5) \bmod 26 = 4 \bmod 26 = 4$$

$$f(10) = (10 - 5) \bmod 26 = 5 \bmod 26 = 5$$

$$f(10) = (10 - 5) \bmod 26 = 5 \bmod 26 = 5$$

$$f(17) = (17 - 5) \bmod 26 = 12 \bmod 26 = 12$$

$$f(25) = (25 - 5) \bmod 26 = 20 \bmod 26 = 20$$

$$f(17) = (17 - 5) \bmod 26 = 12 \bmod 26 = 12$$

$$f(20) = (20 - 5) \bmod 26 = 15 \bmod 26 = 15$$

$$f(14) = (14 - 5) \bmod 26 = 9 \bmod 26 = 9$$

$$f(9) = (9 - 5) \bmod 26 = 4 \bmod 26 = 4$$

Decrypted numbers:

25 4 5 5 12 20 12 15 9 4

Step 3: Convert back to letters

- 25=Z, 4=E, 5=F, 5=F, 12=M
- 20=U, 12=M, 15=P, 9=I, 4=E

Answer: ZEFFM UMPIE

Teaching Note

This is the *challenge* problem because it starts with E (position 4), which requires wrapping around when decrypted.

When we compute $f(4) = (4 - 5) = -1$, we need to wrap around to the *end* of the alphabet:

$$-1 \bmod 26 = 25 \text{ (the letter Z)}$$

Students can think of it this way: going back 1 from A brings you to Z (the last letter). Mathematically: $-1 + 26 = 25$

Multiple negative cases: This problem is harder because it has multiple instances where students need to handle negative results, giving them more practice with this crucial concept.

Pattern recognition: Students might notice that letters early in the alphabet (A, B, C, D, E) will always produce negative results when the shift is larger than their position number.

Common Student Errors to Watch For

1. **Forgetting to handle negative numbers:** Students might write $4 - 5 = -1$ and stop there, not realizing they need to add 26. Watch for students who leave negative numbers in their final answer.
2. **Adding instead of subtracting:** Some students confuse encryption and decryption, using $(p + k)$ instead of $(p - k)$.
3. **Incorrect negative arithmetic:** Students might compute $-4 + 26$ incorrectly. Emphasize: start at 26, count backward 4.
4. **Off-by-one errors with A=0:** Remind students that A=0, not A=1. When they decrypt to position 0, that's the letter A.

5. **Not checking their work:** Students can verify decryption by re-encrypting their answer with the same shift—they should get back the original ciphertext.

Extension Activity

Have students encrypt a message with one shift value, then decrypt it with the same shift value to verify they get back the original message. This reinforces the inverse relationship:

$$\text{Message} \xrightarrow{+k} \text{Ciphertext} \xrightarrow{-k} \text{Message}$$

Example 4 — Affine Cipher Warm-Up

Goal. Determine which letter replaces K when the encryption function

$$f(p) = (7p + 3) \bmod 26$$

is used.

Big idea (the why):

The affine cipher multiplies the plaintext value by a “stretch” factor and then shifts it. It combines multiplication and addition inside modular arithmetic.

$$\text{Encryption: } E(p) = (ap + b) \bmod 26 \qquad \text{Decryption: } D(c) = a^{-1}(c - b) \bmod 26.$$

The constants a and b are keys. a must be coprime to 26 so that a^{-1} exists.

Step 1 — Convert letter K to a number

$$K \rightarrow 10$$

Step 2 — Apply the function $f(p) = (7p + 3) \bmod 26$

$$f(10) = (7 \cdot 10 + 3) \bmod 26 = 73 \bmod 26 = 21.$$

Step 3 — Convert number 21 back to a letter

$$21 \rightarrow V$$

Result: K is encrypted as V .

Why it works:

Multiplying by 7 mixes up the order of letters more effectively than a simple shift, yet because 7 and 26 are coprime, every letter still maps to exactly one output.

Practice (your turn!)

Problem A (easier). Using $f(p) = (3p + 1) \bmod 26$, find what letter replaces C. *Hint:*

$C = 2$.

Problem B (similar). Using $f(p) = (5p + 7) \bmod 26$, find what letter replaces H. *Hint:*

compute carefully, mod 26.

Problem C (harder). Encrypt the word DOG using $f(p) = (11p + 8) \bmod 26$. Write each

step clearly: letter \rightarrow number \rightarrow formula \rightarrow result \rightarrow letter.

Reflection. Why must a be coprime with 26 for this cipher to be reversible?

Solutions — Example 4 Affine Cipher

Example Walk-Through

$K \rightarrow 10, f(p) = (7p + 3) \bmod 26.$

$$f(10) = (7 \cdot 10 + 3) \bmod 26 = 73 \bmod 26 = 21.$$

21 corresponds to V . $\boxed{K \rightarrow V}$

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Problem A

$C = 2.$

$$f(2) = (3 \cdot 2 + 1) \bmod 26 = 7.$$

$7 \rightarrow H$. $\boxed{C \rightarrow H}$

Problem B

$H = 7.$

$$f(7) = (5 \cdot 7 + 7) \bmod 26 = 42 \bmod 26 = 16.$$

$16 \rightarrow Q$. $\boxed{H \rightarrow Q}$

Problem C

Encrypt **DOG** with $f(p) = (11p + 8) \bmod 26.$

$$\begin{aligned} D &= 3 &\Rightarrow (11 \cdot 3 + 8) \bmod 26 &= 41 \bmod 26 = 15 \rightarrow P \\ O &= 14 &\Rightarrow (11 \cdot 14 + 8) \bmod 26 &= 162 \bmod 26 = 6 \rightarrow G \\ G &= 6 &\Rightarrow (11 \cdot 6 + 8) \bmod 26 &= 74 \bmod 26 = 22 \rightarrow W \end{aligned}$$

$\boxed{\text{DOG} \rightarrow \text{PGW}}$

Reflection Answer

If a shares a factor with 26, then some letters collapse to the same output (no unique inverse), making decryption impossible. Only when $\gcd(a, 26) = 1$ does the cipher remain bijective and reversible.

Example 5 (Worksheet) — Cracking a Shift Cipher by Frequency

Problem. We intercepted the ciphertext ZNK KGXRE HOXJ MKZY ZNK CUXS produced by a shift cipher. What was the original plaintext?

Why this works

In English text, some letters appear more often (E, T, A, O, I, N). A shift cipher preserves *relative* frequencies, just moves them around the alphabet. If a letter occurs most often in the ciphertext, it likely corresponds to one of the most common plaintext letters. Hypothesize a mapping, compute the shift k , and test by decrypting.

Step 1 — Count letter frequencies

Ignore spaces/punctuation and count:

K	Z	X	N	G	R	E	H	O	J	M	Y	C	U
S													
4	3	3	2	1	1	1	1	1	1	1	1	1	1
1													

The most frequent letter is K.

Step 2 — Form a hypothesis

In normal English, E is often the most frequent letter. Hypothesize that E (which is 4 with $A=0$) was shifted to K (which is 10). Then the encryption used

$$k \equiv 10 - 4 \equiv 6 \pmod{26}.$$

So decryption should be $p \equiv c - 6 \pmod{26}$.

Step 3 — Test by decrypting

Try a few letters to check the hypothesis:

$$Z \ (25) \mapsto 25 - 6 = 19 \Rightarrow T, \quad N \ (13) \mapsto 7 \Rightarrow H, \quad K \ (10) \mapsto 4 \Rightarrow E.$$

The first three letters become THE, which is promising. Decrypt the whole string with $k = 6$.

Step 4 — Conclusion

Full decryption yields:

THE EARLY BIRD GETS THE WORM

Because this makes excellent English, our hypothesis $k = 6$ is accepted.

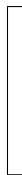
Tips, tricks, and pitfalls

- **A=0** convention: $E = 4$, $K = 10$. Off-by-one mistakes derail the shift quickly.
 - **Test, then trust.** A frequency guess is just a hypothesis; always decrypt a chunk to confirm.
 - **One-letter words** in ciphertext often map to A or I; common bigrams like TH, HE, TO are great anchors.
 - **Decrypt rule:** $p \equiv c - k \pmod{26}$. Negative values? Add 26.
-

Practice — Your Turn

Problem A (easier). Decrypt the ciphertext URYB JBEYQ given it was made with a shift $k = 13$.

Hint: subtract 13 from each letter mod 26.



Problem B (similar). The ciphertext below was made with an *unknown* shift:

ZHOFRPH WR FODVV

Find k and the plaintext.

Hints: the block WR might be TO, or FODVV looks like CLASS.



Problem C (harder). The ciphertext was produced by a shift cipher with *unknown* k :

YMJ VZNHP GWTBS KTC OZRUX TAJW YMJ QFED ITL

Determine k using a smart guess (look for a repeated common word), then decrypt the whole

message.

Reflection. Briefly explain why frequency analysis defeats a shift cipher but not a one-time

pad.

Solutions — Example 5 Practice

Problem A (easier)

Ciphertext: URYYB JBEYQ; shift $k = 13$ (ROT13).

Decrypt by $p \equiv c - 13 \pmod{26}$ (or apply ROT13 again):

HELLO WORLD

Problem B (similar)

Ciphertext: ZHOFRPH WR FODVV, unknown k .

Guess that WR is TO. Then $W = 22$ should map to $T = 19$, so $k \equiv 22 - 19 \equiv 3$ and decryption uses $p \equiv c - 3 \pmod{26}$. Check also that FODVV becomes CLASS:

$$F(5) \rightarrow C(2), O(14) \rightarrow L(11), D(3) \rightarrow A(0), V(21) \rightarrow S(18), V(21) \rightarrow S(18).$$

Hence $k = 3$ and

WELCOME TO CLASS

Problem C (harder)

Ciphertext: YMJ VZNHP GWTBS KTC OZRUX TAJW YMJ QFED ITL.

The trigram YMJ repeats and often corresponds to THE. If so,

$$Y(24) \rightarrow T(19) \Rightarrow k \equiv 24 - 19 \equiv 5, \quad \text{so decrypt with } p \equiv c - 5 \pmod{26}.$$

Applying $k = 5$ across the text yields:

THE QUICK BROWN FOX JUMPS OVER THE LAZY DOG

Takeaways.

- Shift ciphers preserve frequency shape; a good guess (E, T, A, O) usually cracks k .
- Decryption rule: $p \equiv c - k \pmod{26}$; verify the guess by reading for sensible English.
- Longer texts make frequency clues stronger; short texts can be ambiguous, so test multiple hypotheses.

Example 6 (Worksheet) — Transposition Cipher with a Permutation

Cipher rule (why it's cool). A *transposition* cipher keeps the letters but shuffles their *positions*. We split plaintext into blocks of 4 and apply the permutation

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}.$$

That is: 1st→3rd, 2nd→1st, 3rd→4th, 4th→2nd. (So for plaintext block $p_1p_2p_3p_4$ the ciphertext block is $c_1c_2c_3c_4 = p_2p_4p_1p_3$.)

(a) Encrypt PIRATE ATTACK

Step 1 — Normalize and block (remove spaces, then group 4).

$$\text{PIRATEATTACK} \Rightarrow \text{PIRA TEAT TACK}.$$

Step 2 — Apply σ to each block.

$$\text{PIRA} : p_1 = P, p_2 = I, p_3 = R, p_4 = A \Rightarrow c = p_2p_4p_1p_3 = \text{IAPR},$$

$$\text{TEAT} : p = \text{T, E, A, T} \Rightarrow c = \text{E T T A},$$

$$\text{TACK} : p = \text{T, A, C, K} \Rightarrow c = \text{A K T C}.$$

Ciphertext: IAPR ET TA AKTC.

(b) Decrypt SWUE TRAE OEHS

To undo the shuffle, use σ^{-1} :

$$\sigma^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix} \quad (\text{so } c_1 \rightarrow p_2, c_2 \rightarrow p_4, c_3 \rightarrow p_1, c_4 \rightarrow p_3).$$

Block and apply σ^{-1} :

$$\text{SWUE} \rightarrow \text{USEW}, \quad \text{TRAE} \rightarrow \text{ATER}, \quad \text{OEHS} \rightarrow \text{HOSE}.$$

Plaintext (grouped): USE WATER HOSE.

Tips & pitfalls

- **Always block first.** Remove spaces, then group in 4s. If the last block is short, pad (e.g., with X).
 - **Keep “from” vs “to” straight:** here σ says where each *plaintext position* lands in ciphertext.
 - **Decrypt with σ^{-1} :** move each ciphertext position back to the correct plaintext spot.
-

Practice — Your Turn

Use the same permutation $\sigma = [3, 1, 4, 2]$. Work neatly: show the block, show $p_1p_2p_3p_4$, then the rearranged $c_1c_2c_3c_4$.

Problem A (easier). Encrypt MATH NERD. (No padding needed.)

Problem B (similar). Decrypt the ciphertext OEHM OKWR.

Problem C (harder). Encrypt DATA SCIENCE. If needed, *pad the last block with X* to fill 4 letters. Show every block and the final ciphertext.

Reflection. Why does transposition preserve letter frequencies but still hide the message structure?

Solutions — Example 6 Practice (Transposition, $\sigma = [3, 1, 4, 2]$)

Permutation recap. Encryption (per block): $c_1 c_2 c_3 c_4 = p_2 p_4 p_1 p_3$. Decryption uses σ^{-1} : $c_1 \rightarrow p_2$, $c_2 \rightarrow p_4$, $c_3 \rightarrow p_1$, $c_4 \rightarrow p_3$.

Problem A (easier) — Encrypt MATH NERD

Remove space and block: MATHNERD.

MATH : $p = \text{M,A,T,H} \Rightarrow c = \text{A H M T}$,

NERD : $p = \text{N,E,R,D} \Rightarrow c = \text{E D N R}$.

AHMT EDNR

Problem B (similar) — Decrypt OEHM OKWR

Blocks: OEHM OKWR. Use σ^{-1} .

OEHM : $c_1 \rightarrow p_2 = O$, $c_2 \rightarrow p_4 = E$, $c_3 \rightarrow p_1 = H$, $c_4 \rightarrow p_3 = M \Rightarrow \text{HOME}$.

OKWR : $c_1 \rightarrow p_2 = O$, $c_2 \rightarrow p_4 = K$, $c_3 \rightarrow p_1 = W$, $c_4 \rightarrow p_3 = R \Rightarrow \text{WORK}$.

HOME WORK

Problem C (harder) — Encrypt DATA SCIENCE (pad with X)

Normalize: DATASCIENCE (11 letters) \rightarrow pad: DATASCIENCEX. Blocks: DATA SCIE NCEX.

DATA : $p = \text{D,A,T,A} \Rightarrow c = \text{A A D T}$,

SCIE : $p = \text{S,C,I,E} \Rightarrow c = \text{C E S I}$,

NCEX : $p = \text{N,C,E,X} \Rightarrow c = \text{C X N E}$.

AADT CESI CXNE

Key takeaways.

- Transposition ciphers permute positions, not letters—so frequencies are unchanged.
- Always decrypt with the inverse permutation σ^{-1} .
- Padding guarantees all blocks are full; document your padding rule (e.g., use X).

Example 7 (Worksheet) — Shift Ciphers as a Cryptosystem

Goal. Describe the family of shift ciphers in the formal language of a cryptosystem.

The Big Idea: What's a Cryptosystem?

A **cryptosystem** is a mathematical framework describing how messages are encrypted and decrypted. Formally, it's written as a 5-tuple:

$$(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$$

where each symbol represents a part of the encryption ecosystem:

- \mathcal{P} – the set of possible *plaintexts*
- \mathcal{C} – the set of possible *ciphertexts*
- \mathcal{K} – the *keyspace*, all keys that can be used
- \mathcal{E} – the set of *encryption functions*
- \mathcal{D} – the set of *decryption functions*

The golden rule of any cryptosystem is:

$$D_k(E_k(p)) = p \quad \text{for every plaintext } p.$$

That means: decrypting an encrypted message must always give you back the original.

Step 1 — Translate the Language of Letters into Math

Each letter of the alphabet is assigned a number in \mathbb{Z}_{26} (the integers 0–25 mod 26).

$$A = 0, B = 1, \dots, Z = 25$$

A message like HELLO becomes [7, 4, 11, 11, 14].

Step 2 — Define the Shift Cipher Functions

To encrypt, we *add* a fixed key $k \bmod 26$:

$$E_k(p) = (p + k) \bmod 26.$$

To decrypt, we *subtract* the same $k \bmod 26$:

$$D_k(c) = (c - k) \bmod 26.$$

Step 3 — Describe the Family of Shift Ciphers as a Cryptosystem

Putting it all together:

$$\begin{aligned}\mathcal{P} &= \mathcal{C} = \text{all strings of elements in } \mathbb{Z}_{26}, \\ \mathcal{K} &= \mathbb{Z}_{26}, \\ \mathcal{E} &= \{ E_k(p) = (p + k) \bmod 26 \mid k \in \mathbb{Z}_{26} \}, \\ \mathcal{D} &= \{ D_k(c) = (c - k) \bmod 26 \mid k \in \mathbb{Z}_{26} \}.\end{aligned}$$

This means each possible shift k defines one member of the family of shift ciphers.

Step 4 — Check the “Undo” Property

To verify that encryption and decryption work as a matched pair:

$$D_k(E_k(p)) = (p + k - k) \bmod 26 = p.$$

So every message can be perfectly recovered.

Tips & Common Pitfalls

- Don’t confuse the “keyspace” \mathcal{K} with a single key k . The keyspace is the entire set of possible shifts.
- Forgetting to take $\bmod 26$ is a very common mistake.
- A shift cipher is *not secure* — only 26 possible keys! We study it to understand the structure of more complex systems.

Practice — Your Turn!

Problem A (Easier). For a shift cipher with $k = 5$, write down $E_k(p)$ and $D_k(c)$. Explain

in your own words what “mod 26” ensures.

Problem B (Similar). Let $p = 19$ (the letter T) and $k = 7$. Compute $E_k(p)$ and translate

it back into a letter. Then apply D_k to check that you get back T.

Problem C (Harder). Write the complete 5-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$ for a system that works

on uppercase English letters and digits (0–9). What changes?

Reflection. How does writing cryptography in formal notation help us build new systems

in the future?

Example 7 (Solutions) — Shift Ciphers as a Cryptosystem

Goal. Describe the family of shift ciphers as a formal cryptosystem and verify that encryption and decryption are inverses.

Full Walkthrough and Explanation

We want to represent the shift cipher in the five-part framework

$$(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D}).$$

Step 1 — Mapping letters to numbers. Each letter is represented as an integer between 0 and 25:

$$A = 0, B = 1, \dots, Z = 25.$$

This lets us use modular arithmetic instead of alphabet juggling.

Step 2 — Defining encryption and decryption.

$$E_k(p) = (p + k) \bmod 26, \quad D_k(c) = (c - k) \bmod 26.$$

Here k is the key (the amount of shift).

Step 3 — Building the formal 5-tuple.

$$\begin{aligned} \mathcal{P} &= \mathcal{C} = \text{all strings of elements in } \mathbb{Z}_{26}, \\ \mathcal{K} &= \mathbb{Z}_{26}, \\ \mathcal{E} &= \{ E_k(p) = (p + k) \bmod 26 \mid k \in \mathbb{Z}_{26} \}, \\ \mathcal{D} &= \{ D_k(c) = (c - k) \bmod 26 \mid k \in \mathbb{Z}_{26} \}. \end{aligned}$$

Step 4 — Verifying the “undo” property.

$$D_k(E_k(p)) = ((p + k) - k) \bmod 26 = p.$$

So decryption perfectly reverses encryption.

Key Insight. Shift ciphers show how a single idea—addition mod 26—can define a whole family of related ciphers, one for each k in \mathbb{Z}_{26} .

Common pitfalls

- Students sometimes treat “keyspace” \mathcal{K} as just one value instead of the full set of possible k .
 - Forgetting the modulus (especially when $p + k > 25$) leads to wrong letters.
 - Because there are only 26 possible k , a brute-force attack breaks the cipher immediately—this motivates more sophisticated systems.
-

Practice Problem Solutions

Problem A (Easier). Given $k = 5$:

$$E_k(p) = (p + 5) \bmod 26, \quad D_k(c) = (c - 5) \bmod 26.$$

“Mod 26” guarantees we stay inside the alphabet—after Z (25), we wrap around to A (0).

—

Problem B (Similar). Let $p = 19$ (the letter T) and $k = 7$.

$$E_k(p) = (19 + 7) \bmod 26 = 0 \Rightarrow A.$$

Encrypting T gives A. Decrypting:

$$D_k(0) = (0 - 7) \bmod 26 = 19 \Rightarrow T.$$

We return to the original plaintext, confirming correctness.

—

Problem C (Harder). If we expand the system to include digits 0–9, we now have 36 symbols. So the modulus becomes 36 and each component adjusts:

$$\mathcal{P} = \mathcal{C} = \text{all strings of elements in } \mathbb{Z}_{36},$$

$$\mathcal{K} = \mathbb{Z}_{36},$$

$$\mathcal{E} = \{E_k(p) = (p + k) \bmod 36\},$$

$$\mathcal{D} = \{D_k(c) = (c - k) \bmod 36\}.$$

The idea is identical—just a larger alphabet!

—

Reflection Answer. Writing cryptography formally gives us a reusable structure: we can swap in new alphabets, key spaces, or modular groups and instantly define new families of ciphers. It’s mathematics as blueprint—one small idea, infinitely extendable.

Example 8 (Worksheet) — Encrypting with the RSA Cryptosystem

Goal. Encrypt the message STOP using the RSA cryptosystem with key $(n, e) = (2537, 13)$.

Background idea

RSA is a **public key cryptosystem**. Anyone can use the public key (n, e) to encrypt, but only the private key (involving d) can decrypt. Each letter is first turned into a number (A=00, B=01, ..., Z=25), grouped into blocks that fit under n , and then encrypted using

$$c \equiv m^e \pmod{n}.$$

Step 1 — Convert letters to numbers

We map STOP as:

$$S \ T \ O \ P \Rightarrow 18 \ 19 \ 14 \ 15.$$

Group into four-digit blocks:

$$1819 \quad 1415$$

(because $2525 < 2537 < 252525$, so 4 digits per block fits safely).

Step 2 — Apply RSA encryption

For each block m , compute

$$c \equiv m^{13} \pmod{2537}.$$

You can use fast modular exponentiation (successive squaring) to simplify:

$$1819^{13} \pmod{2537} = 2081, \quad 1415^{13} \pmod{2537} = 2182.$$

Hence, the ciphertext is:

$$\boxed{2081 \ 2182}.$$

Step 3 — Interpretation

We transmit 2081 2182. Only someone with the private key d (that satisfies $ed \equiv 1 \pmod{(p-1)(q-1)}$) can decrypt the message.

Tips & tricks

- **Why 13?** — Because $\gcd(13, (p-1)(q-1)) = 1$, ensuring encryption is reversible.
 - **Always check block size.** m must be smaller than n .
 - **Decryption uses the inverse of e** — It “undoes” the exponentiation by modular arithmetic symmetry.
 - **RSA loves primes.** Choosing p, q large keeps n hard to factor.
-

Practice — Your Turn

Problem A (easier). Encrypt G0 using RSA with $(n, e) = (2537, 13)$. Hint: Convert G0
→ 06014 → use 4-digit block 0601, compute $c \equiv m^{13} \pmod{2537}$.

Problem B (similar). Encrypt HELP using RSA with $(n, e) = (2537, 13)$. Show all modular
exponentiation steps clearly.

Problem C (challenge). Encrypt SAVE THE PLANET using RSA with $(n, e) = (2537, 13)$.
Break your message into 4-digit blocks and compute each ciphertext block. (Hint: spaces
can be ignored or replaced by 26.)

Reflection. In one or two sentences, explain *why* RSA’s security depends on factoring large
primes.

Example 8 — Encrypting with the RSA Cryptosystem

Given: Key $(n, e) = (2537, 13)$, message STOP.

Step 1 — Convert to numerical form

$$S \ T \ O \ P \rightarrow 18 \ 19 \ 14 \ 15$$

Group into blocks of four digits (since $2525 < 2537 < 252525$):

$$1819 \quad 1415.$$

Step 2 — Apply RSA encryption

RSA uses:

$$c \equiv m^{13} \pmod{2537}.$$

Compute each block using fast modular exponentiation (square-and-multiply):

$$\begin{aligned} 1819^{13} &\equiv ((1819^2 \bmod 2537)^6 \times 1819) \bmod 2537 \\ &\equiv (1390^6 \times 1819) \bmod 2537 \\ &\equiv 2081, \\ 1415^{13} &\equiv 2182. \end{aligned}$$

Encrypted message: 2081 2182.

Verification (optional)

If we compute the private key d satisfying

$$ed \equiv 1 \pmod{(p-1)(q-1)} = 1 \pmod{42 \times 58 = 2436},$$

we find $d = 937$. Decryption would be $m \equiv c^{937} \pmod{2537}$, recovering the original STOP.

Why it works

Because of Euler's theorem:

$$m^{\varphi(n)} \equiv 1 \pmod{n}, \quad \text{where } \varphi(n) = (p-1)(q-1),$$

and since $ed \equiv 1 \pmod{\varphi(n)}$, we have:

$$(m^e)^d \equiv m^{ed} \equiv m \pmod{n}.$$

That's the mathematical backbone of RSA.

Common pitfalls

- Forgetting that $A \neq 00$ (not 1). Off-by-one ruins all blocks.
 - Using too large a block size (must keep $m < n$).
 - Mismanaging modular exponentiation — use reduction at each step!
-

Practice Solutions

Problem A. Encrypt GO with (2537, 13). $G O \rightarrow 06\ 14 \Rightarrow 0614$. Compute $0614^{13} \bmod 2537 = 1097$. 1097

Problem B. Encrypt HELP with (2537, 13). $H\ E\ L\ P \rightarrow 07\ 04\ 11\ 15 \Rightarrow 0704, 1115$.

$$0704^{13} \bmod 2537 = 1954, \quad 1115^{13} \bmod 2537 = 1730.$$

1954 1730

Problem C. Encrypt SAVE THE PLANET. Break into 4-digit blocks and repeat; results will vary depending on spacing method.

$$\boxed{\text{Each block encrypted via}} c_i = m_i^{13} \pmod{2537}.$$

(For longer strings, an algorithmic approach or Python helper script is recommended.)

Historical Spotlight — Clifford Cocks, the Hidden Father of RSA

Clifford Cocks (born 1950, Cheshire, England) quietly invented what we now call the **RSA cryptosystem** in 1973 — three years before Rivest, Shamir, and Adleman. Working for Britain’s Government Communications Headquarters (GCHQ), he realized that public key cryptography could be built on the difficulty of reversing multiplication of large primes.

However, his discovery remained classified until 1997. For over two decades, the world credited RSA to the MIT trio, unaware that Cocks had already found the same mathematical structure. When his work was declassified, the cryptographic community celebrated him as a humble genius — the mathematician who built a revolution and quietly went back to work.

Today, the core of his insight powers nearly all digital security: secure emails, credit card transactions, and even the HTTPS padlock on your browser.

Fun fact: Cocks also developed early ideas for identity-based encryption — letting your name or email serve as part of a public key!

Teacher Reflection. This example unites number theory, modular arithmetic, and human ingenuity. Encourage students to appreciate both the mathematics *and* the people who dared to imagine a new kind of secrecy — one where publishing your key could actually make you safer.

Example 9 (Worksheet) — RSA Decryption (work the process, not just the answer)

Problem. We receive the ciphertext blocks 0981 0461 produced by the RSA cryptosystem from Example 8. The public key was $(n, e) = (2537, 13)$ with $n = 43 \cdot 59$. Decrypt the message.

The big idea (the why)

RSA works in blocks. If a block c was encrypted with $c \equiv m^e \pmod{n}$, then anyone who knows the *private* exponent d (the inverse of $e \pmod{\phi(n)}$) can recover the plaintext block via

$$m \equiv c^d \pmod{n}.$$

This is fast thanks to *repeated squaring*. After recovering each numeric block m , translate back to letters using two digits per letter: $A = 00, \dots, Z = 25$. Leading zeros matter!

Step 1 — Compute the private exponent d

$$\phi(n) = (p-1)(q-1) = 42 \cdot 58 = 2436, \quad \text{find } d \text{ with } 13d \equiv 1 \pmod{2436}.$$

Extended Euclid gives $d = 937$ (indeed $13 \cdot 937 = 12181 = 1 + 5 \cdot 2436$).

Step 2 — Decrypt each block with repeated squaring

Block 1: $c = 0981 \Rightarrow c = 981$.

$$m \equiv 981^{937} \pmod{2537}, \quad 937 = 512 + 256 + 128 + 32 + 8 + 1.$$

Squares mod 2537:

power	1	2	4	8	16	32	64	128	256
512									
981 ^{power} mod 2537	981	838	1922	1325	450	441	322	2472	1688
293									

Multiply only the needed entries (powers 1, 8, 32, 128, 256, 512):

$$\begin{aligned}
 r &\leftarrow 1 \\
 r \cdot 981 &\equiv 981 \\
 r \cdot 1325 &\equiv 981 \cdot 1325 \equiv 1717 \\
 r \cdot 441 &\equiv 1717 \cdot 441 \equiv 1251 \\
 r \cdot 2472 &\equiv 1251 \cdot 2472 \equiv 282 \\
 r \cdot 1688 &\equiv 282 \cdot 1688 \equiv 1292 \\
 r \cdot 293 &\equiv 1292 \cdot 293 \equiv \boxed{704}
 \end{aligned}$$

So $m_1 = 0704 \Rightarrow 07\ 04 = \text{H E}$.

Block 2: $c = 0461 \Rightarrow c = 461$.

$$m \equiv 461^{937} \pmod{2537}, \quad 937 = 512 + 256 + 128 + 32 + 8 + 1.$$

Squares mod 2537:

power 512	1	2	4	8	16	32	64	128	256
$461^{\text{power}} \pmod{2537}$	461	1950	2074	1261	1959	1737	676	316	913
1433									

Multiply the needed entries (powers 1, 8, 32, 128, 256, 512):

$$\begin{aligned}
 r &\leftarrow 1 \\
 r \cdot 461 &\equiv 461 \\
 r \cdot 1261 &\equiv 461 \cdot 1261 \equiv 1327 \\
 r \cdot 1737 &\equiv 1327 \cdot 1737 \equiv 1559 \\
 r \cdot 316 &\equiv 1559 \cdot 316 \equiv 1122 \\
 r \cdot 913 &\equiv 1122 \cdot 913 \equiv 82 \\
 r \cdot 1433 &\equiv 82 \cdot 1433 \equiv \boxed{1115}
 \end{aligned}$$

So $m_2 = 1115 \Rightarrow 11\ 15 = \text{L P}$.

Step 3 — Read the plaintext

Blocks 0704 1115 translate to HELP.

Tips, tricks, and common pitfalls

- Keep the two-digit mapping straight: $A = 00, \dots, J = 09, \dots, Z = 25$. Leading zeros are part of the block!

- Choose the block size so that each four-digit block m is $< n$. Here $2N = 4$ works because $2525 < 2537 < 252525$.
- When doing repeated squaring, build a small table of c^1, c^2, c^4, \dots and then multiply only the powers that add up to d .
- Arithmetic gets easier if you reduce *often*. Every product should be brought back modulo n immediately.

Practice — Your Turn (use $n = 2537$, $e = 13$, $d = 937$)

Use the same key as above. Show your exponentiation steps and *keep* leading zeros when converting back to letters.

Problem A (easier). Decrypt the single block 2081. What two letters do you get? *Hint:*

compute $2081^{937} \bmod 2537$ and then split the result as -- --.

Problem B (similar). Decrypt the two-block ciphertext 2081 2182. *Reminder:* convert

each four-digit block separately, then map back to letters.

Problem C (harder). The ciphertext 0981 0724 1774 was made with the same key.

- Decrypt all three blocks.
- Translate to letters. If the last block ends with a padding letter X, circle it.

Reflection. In one or two sentences, explain why knowing e and n does *not* make decryption

easy, but knowing d does.



Core facts for this example

Public key: $(n, e) = (2537, 13)$ with $n = 43 \cdot 59$.

Euler totient: $\phi(n) = (p-1)(q-1) = 42 \cdot 58 = 2436$.

Private exponent d is the inverse of e modulo $\phi(n)$:

$$13d \equiv 1 \pmod{2436} \Rightarrow d = 937.$$

Decryption works blockwise: for each ciphertext block c ,

$$m \equiv c^d \pmod{n} \quad \text{and then map } m \text{ back to letters with } A = 00, \dots, Z = 25.$$

Textbook Example 9 — Full decryption

Ciphertext: 0981 0461.

Block 1: $c = 0981 \Rightarrow 981$

We use repeated squaring (mod 2537) and write $d = 937 = 512 + 256 + 128 + 32 + 8 + 1$.

power	1	2	4	8	16	32	64	128	256
512									
$981^{\text{power}} \bmod 2537$	981	838	1922	1325	450	441	322	2472	1688
293									

Multiply only the needed entries (powers 1, 8, 32, 128, 256, 512), reducing after each step:

$$981 \cdot 1325 \cdot 441 \cdot 2472 \cdot 1688 \cdot 293 \equiv \boxed{704} \pmod{2537}.$$

So $m_1 = 0704 \Rightarrow 07 \ 04 = \text{H E}$.

Block 2: $c = 0461 \Rightarrow 461$

Again with $d = 937 = 512 + 256 + 128 + 32 + 8 + 1$:

power	1	2	4	8	16	32	64	128	256
512									
$461^{\text{power}} \bmod 2537$	461	1950	2074	1261	1959	1737	676	316	913
1433									

Multiply the needed entries (powers 1, 8, 32, 128, 256, 512):

$$461 \cdot 1261 \cdot 1737 \cdot 316 \cdot 913 \cdot 1433 \equiv \boxed{1115} \pmod{2537}.$$

So $m_2 = 1115 \Rightarrow 11 \ 15 = \text{L P}$.

Plaintext: $\boxed{\text{HELP}}$.

Checks and teaching notes. Emphasize (i) mapping is two digits per letter with leading zeros preserved; (ii) block size 4 works because $2525 < n = 2537 < 252525$; (iii) reduce after *every* multiplication to keep numbers small.

Practice Solutions

Problem A (easier)

Prompt. Decrypt the single block 2081.

Work. Compute $m \equiv 2081^{937} \pmod{2537}$. (Repeated squaring or any correct modular-pow tool is fine.) One clean path gives

$$2081^{937} \equiv \boxed{1819} \pmod{2537}.$$

Split to letters: $18 \rightarrow \text{S}$, $19 \rightarrow \text{T}$.

Answer: $\boxed{\text{ST}}$.

Problem B (similar)

Prompt. Decrypt the two blocks 2081 2182.

Work. From part A, $2081^{937} \equiv 1819 \Rightarrow \text{ST}$. Similarly,

$$2182^{937} \equiv \boxed{1415} \pmod{2537} \Rightarrow 14 \ 15 = \text{O P}.$$

Answer: $\boxed{\text{STOP}}$.

Problem C (harder)

Prompt. Decrypt 0981 0724 1774. Same key.

Work. Blockwise decryption:

$$0981^{937} \equiv \boxed{0704} \pmod{2537} \Rightarrow \text{HE},$$

$$0724^{937} \equiv \boxed{1111} \pmod{2537} \Rightarrow \text{LL},$$

$$1774^{937} \equiv \boxed{1423} \pmod{2537} \Rightarrow \text{OX}.$$

Answer: HELLOX (final X is padding to complete a two-letter block).

Coach's notes.

- When a message length is odd (in letters), a padding letter (commonly X) is appended so every numeric string splits cleanly into four-digit blocks.
- If students' intermediate residues differ, check two things: (1) their exponent decomposition of 937 and (2) that they reduced modulo 2537 after *every* multiply and square.

Quick reference: letter map (A=00,...,Z=25).

$\{00, 01, \dots, 09\} \rightarrow \{A, B, \dots, J\}$, $10 \rightarrow K$, $11 \rightarrow L$, \dots , $25 \rightarrow Z$.

Example 10 — Signing a Message with RSA

Scenario: Alice’s public RSA cryptosystem uses $n = 43 \times 59 = 2537$ and $e = 13$. Her private key is $d = 937$, as computed in Example 9. She wishes to send the message “MEET AT NOON” to her friends so that they are *certain* it came from her.

Goal: Learn how RSA can be used to *sign* a message—proving authenticity, not just secrecy.

Step 1 — Translate the message into numbers

Using the standard letter-number system (A=00, B=01, . . . , Z=25):

$$M = \text{MEET AT NOON} \Rightarrow 1204\ 0419\ 0019\ 1314\ 1413$$

(Verify this translation carefully! It’s essential to the encryption and decryption process.)

Step 2 — Apply Alice’s *private key* to each block

Alice uses her private key $d = 937$ to compute:

$$x^{937} \pmod{2537}$$

for each message block x . This operation produces a “digital signature” that can only be generated with Alice’s private key.

Step 3 — Compute the results (with modular exponentiation)

Using fast modular exponentiation (as in Example 9):

$$1204^{937} \equiv 817 \pmod{2537}$$

$$0419^{937} \equiv 555 \pmod{2537}$$

$$0019^{937} \equiv 1310 \pmod{2537}$$

$$1314^{937} \equiv 2173 \pmod{2537}$$

$$1413^{937} \equiv 1026 \pmod{2537}$$

So, the message Alice sends (in blocks) is:

$$0817\ 0555\ 1310\ 2173\ 1026$$

Step 4 — Verification by the recipient

When her friends receive the message, they apply Alice’s *public key* $e = 13$ to each block:

$$E_{(2537,13)}(c) = c^{13} \pmod{2537}$$

This reverses Alice’s signature and recovers the plaintext. If the recovered message reads “MEET AT NOON,” they know it truly came from Alice.

Key takeaway

Digital signatures use the *private key to sign* and the *public key to verify*. This is the opposite direction from encryption (where public encrypts and private decrypts). It guarantees message authenticity and integrity — no one else could have produced this result.

Practice — Your Turn!

Problem A (Warm-up): Alice’s key is $n = 77$, $e = 13$, and $d = 37$. She wants to sign the message “HI,” represented as 0708. Compute the signature block $c = m^d \pmod{77}$. Then,

verify that $c^e \pmod{77}$ returns 0708.

Problem B (Moderate): Bob uses the same RSA parameters as Example 10: $n = 2537$, $e = 13$, $d = 937$. He signs the message “HELP” (encoded as 0704 1115). Compute m^d

$\pmod{2537}$ for each block, and verify correctness.

Problem C (Challenge): Suppose Eve intercepts Alice’s public key $(2537, 13)$ and one of her signed messages. Why can’t Eve “fake” Alice’s signature without knowing $d = 937$? Use your understanding of modular arithmetic and factorization to explain the barrier to

forgery.

Reflection: Describe in your own words how digital signatures strengthen security compared to regular RSA encryption.

Quick Tips:

- Public key (n, e) — used to verify or encrypt.
- Private key d — used to sign or decrypt.
- Large primes p, q make $n = pq$ hard to factor, ensuring security.
- Modular arithmetic is your shield: it keeps numbers within manageable limits.

Example 10 (Solutions) — RSA Digital Signatures

Quick reminders

- Letter↔number map (used by the text): A→00, B→01, ..., I→08, J→09, ..., Z→25.
- Make fixed-size blocks so each block m satisfies $0 \leq m < n$.
- **Sign** one block: $s \equiv m^d \pmod{n}$. **Verify**: $m' \equiv s^e \pmod{n}$; accept iff $m' = m$.
- **Fast modular exponentiation (square-and-multiply)** is the tool for computing $a^b \bmod n$ efficiently.

Textbook walkthrough (signature for “MEET AT NOON”)

Public key $(n, e) = (2537, 13)$ and private key $d = 937$ (from Ex. 9). Blocks of the message (using A=00,...,Z=25): 1204 0419 0019 1314 1413.

Signer (Alice) computes the signature blocks

$$s_i \equiv m_i^d \pmod{2537}.$$

With fast modular exponentiation (or a calculator), this yields

$$\boxed{0817\ 0555\ 1310\ 2173\ 1026}.$$

Verifier (anyone) checks

$$m'_i \equiv s_i^e \pmod{2537}.$$

Raising each block above to the 13th power modulo 2537 gives back

$$1204\ 0419\ 0019\ 1314\ 1413,$$

which matches Alice’s original blocks, so the signature is *valid*. Because only the holder of d can produce blocks that verify under e , recipients are convinced the message came from Alice.

Practice Solutions

Problem A (easier)

Task. With $(n, e, d) = (77, 13, 37)$, sign the message HI and verify.

Block set-up. Since $n = 77 < 100$, we must use *two-digit* blocks:

$$\text{HI} \longrightarrow 07\ 08 \quad (A = 00, \dots, I = 08).$$

Sign each block: $s \equiv m^{37} \pmod{77}$.

$m = 07$. Write $37 = 32 + 4 + 1$. Square-and-multiply (all mod 77):

$$7^1 = 7, \quad 7^2 = 49, \quad 7^4 = 14, \quad 7^8 = 42, \quad 7^{16} = 70, \quad 7^{32} = 49.$$

So $7^{37} \equiv 7^{32} \cdot 7^4 \cdot 7 \equiv 49 \cdot 14 \cdot 7 \equiv 28 \pmod{77}$.

$m = 08$. Powers (mod 77):

$$8^1 = 8, \quad 8^2 = 64, \quad 8^4 = 15, \quad 8^8 = 71, \quad 8^{16} = 36, \quad 8^{32} = 64.$$

Hence $8^{37} \equiv 8^{32} \cdot 8^4 \cdot 8 \equiv 64 \cdot 15 \cdot 8 \equiv 57 \pmod{77}$.

Signature blocks: 28 57.

Verify: compute $m' \equiv s^{13} \pmod{77}$. One can reuse the tables above or a calculator:

$$28^{13} \equiv 7 \pmod{77} \quad \text{and} \quad 57^{13} \equiv 8 \pmod{77}.$$

Thus we recover $07\ 08 \Rightarrow \text{HI}$. ✓

Problem B (similar)

Task. With $(n, e, d) = (2537, 13, 937)$, sign OK and verify.

Blocks. $n = 2537$ allows 4-digit blocks. $\text{OK} \rightarrow 14\ 10 \Rightarrow m = 1410$.

Sign: $s \equiv 1410^{937} \pmod{2537} =$ 0802.

Verify: $s^{13} \equiv 802^{13} \equiv 1410 \pmod{2537} \Rightarrow$ back to OK. ✓

(Computation notes.) A short binary-exponent table for $1410^{2^k} \pmod{2537}$ plus multiply on 1-bits of 937 (binary = 1110101001_2) reproduces the result efficiently; a CAS or Python also confirms 802.

Problem C (harder)

Task. Using public key $(2537, 13)$, check whether the claimed signature

$0817\ 0555\ 1310\ 2173\ 1026$

matches the message “MEET AT NOON”.

Verification. Raise each s_i to the 13th power mod 2537:

$$0817^{13} \equiv 1204, 0555^{13} \equiv 0419, 1310^{13} \equiv 0019, 2173^{13} \equiv 1314, 1026^{13} \equiv 1413 \pmod{2537}.$$

These are exactly the blocks for “MEET AT NOON,” so the signature is valid. *If even one block failed to match, we would reject the signature immediately.*

Teaching notes, tips, and gotchas (for review)

- **Block sizing matters.** Always choose the largest even number of digits so each block m is $< n$. Small n (like 77) means 2-digit blocks; $n = 2537$ allows 4-digit blocks.
- **Signature vs. encryption.** Sign with the *private* exponent d ; anyone verifies with the *public* exponent e . (Encrypting for secrecy goes the other way.)
- **Square-and-multiply** is your friend: precompute $a^1, a^2, a^4, a^8, \dots \pmod n$ and multiply the powers that correspond to 1-bits of the exponent.
- **Common mistakes:**
 - Mixing the A=0 mapping (00–25) with A=1. Stick to A=00, ..., Z=25 for RSA in this section.
 - Building a block $\geq n$. If that happens, reduce the block size.
 - Forgetting leading zeros when translating back (e.g., 0419 not 419).