Section 4.2 — Example 9 Worksheet
Counting Bit Additions in Algorithm 2

Problem (from Example 9)

How many *additions of bits* are required by **Algorithm 2 (Addition of Integers)** to add two integers with at most n bits in their binary representations?

Key idea

At each bit position j (from right to left), Algorithm 2 forms the three-bit sum

$$a_j + b_j + c \quad (c \in \{0, 1\} \text{ is the incoming carry}),$$

then outputs the sum bit s_j and the next carry c'. Computing $a_j + b_j + c$ can be done with *at most two* one-bit additions:

1. add the pair $a_j + b_j$ (one bit-addition);
2. add the carry: $(a_j + b_j) + c$ (one more bit-addition, sometimes trivial if $c = 0$ and no carry occurs).

Hence the total number of bit additions is *less than* $2n$; in particular it is $\leq 2n$ and therefore the running time is $O(n)$.

Worked examples (Algorithm 2, step-by-step)

A. Two-bit example

Add $a = (11)_2$ and $b = (01)_2$.

We process bits from right (least significant) to left (most significant). Let c be the incoming carry; initially $c = 0$.

j	a_j	b_j	c (in)	s_j	c (out)
0	1	1	0	0	1
1	1	0	1	0	1

$s_2 = c_{\text{final}} = 1$

Thus the result is $s = (1\,00)_2 = (100)_2$. Bit-addition count: at most 2 per column $\Rightarrow$ at most 4 total.

B. Four-bit example

Add $a = (1011)_2$ and $b = (0110)_2$.

j	a_j	b_j	c (in)	s_j	c (out)
0	1	0	0	1	0
1	1	1	0	0	1
2	0	1	1	0	1
3	1	0	1	0	1
$s_4 = c_{\text{final}} = 1$					

Hence $a + b = (1\,0001)_2 = (10001)_2$. Again, at most two bit additions per column $\Rightarrow$ at most $2 \cdot 4 = 8$ bit additions.

Why "less than $2n$" and $O(n)$?

Some columns may not *need* to add the carry (e.g., $c = 0$ and $a_j + b_j < 2$); practical implementations can skip that second bit-addition. Therefore the # of bit additions is $< 2n$ and certainly $\leq 2n$. As a function of n, that is linear time: $O(n)$.

Your turn

Use Algorithm 2. Show your table of columns j, a_j, b_j, c (in), s_j, c (out).

Two easier

1. $(0101)_2 + (0011)_2$
2. $(1001)_2 + (0001)_2$

One harder

3. $(11101101)_2 + (10111011)_2$