Division in Number Theory – Teacher's Solution Guide

Definition of Division

If a and b are integers with $b \neq 0$, we say that b divides a if there is an integer c such that $a = bc$. Notation: $b \mid a$. Example: $3 \mid 12$ because $12 = 3 \times 4$.

> Teaching Note: Reinforce that 'divides' does not mean 'fractional division'; it specifically requires the result to be an integer.

Example 1 – Solutions

Problem: Determine whether $3 \mid 7$ and whether $3 \mid 12$.

> Solution: $7 \div 3 = 2.333...$, not an integer → 3 does not divide 7.
>
> $12 \div 3 = 4$, which is an integer → 3 divides 12.

Easier Example: Does 2 divide 8?

> Solution: Yes, $8 \div 2 = 4$, an integer. Therefore $2 \mid 8$.

Challenging Example: Does 7 divide 100?

> Solution: $100 \div 7 = 14$ remainder 2, so 7 does not divide 100.
>
> Teaching Note: Highlight the role of the remainder to connect with modular arithmetic.

Example 2 – Solutions

Problem: Let n = 4. Positive integers divisible by 4: 4, 8, 12, 16, ...

Easier Example: List integers divisible by 2 up to 20 → 2, 4, 6, ..., 20.

Challenging Example: List integers divisible by 9 up to 100 → 9, 18, 27, ..., 99.

Teaching Note: Good chance to illustrate arithmetic sequences.

Theorem 1 – Solutions

Theorem 1: If a | b and a | c, then a | (b + c). Also: if a | b, then a | (b×m) for all integers m. If a | b and b | c, then a | c.

Easier Example: If 2 | 6 and 2 | 8, then 2 | (6 + 8).

Solution: 6 + 8 = 14. Since 14 ÷ 2 = 7, integer, so 2 | 14.

Challenging Example: If 5 | 20 and 5 | 35, prove 5 | (20 + 35).

Solution: 20 + 35 = 55. 55 ÷ 5 = 11, integer, so 5 | 55.

Teaching Note: Emphasize closure properties of divisibility under addition.

Corollary 1 – Solutions

If a, b, and c are integers with $a \neq 0$, and $a \mid b$, $a \mid c$, then $a \mid (mb + nc)$ for any integers m, n.

Easier Example: If $3 \mid 6$ and $3 \mid 9$, prove $3 \mid (2\times6 + 1\times9)$.

Solution: $2\times6 + 9 = 21$. $21 \div 3 = 7$, integer, so $3 \mid 21$.

Challenging Example: If $4 \mid 12$ and $4 \mid 20$, prove $4 \mid (3\times12 + 2\times20)$.

Solution: $3\times12 + 2\times20 = 36 + 40 = 76$. $76 \div 4 = 19$, integer, so $4 \mid 76$.

Teaching Note: Use this to show how linear combinations are divisible.