Information Retrieval

Exercise Sheet 9

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Exercise 1

Prove: $V = S^{-1} \cdot U^T \cdot A$
Because U is column-orthonormal $U^T \cdot U = I_r$.
Apply $A = U \cdot S \cdot V$:
$V = S^{-1} \cdot U^T \cdot U \cdot S \cdot V \iff V = S^{-1} \cdot I_r \cdot S \cdot V \iff V = V$

Exersice 2

Compute the singular value decomposition

$$A = \begin{pmatrix} 13 & 5 & 5 & 0 & 13 \\ 9 & 15 & 15 & 0 & 9 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 13 & 9 & 0 \\ 5 & 15 & 0 \\ 5 & 15 & 0 \\ 0 & 0 & 20 \\ 13 & 9 & 0 \end{pmatrix}$$

1. Compute the symmetric 3x3 matrix $A \cdot A^T$

$$A \cdot A^T = \begin{pmatrix} 388 & 384 & 0 \\ 384 & 612 & 0 \\ 0 & 0 & 400 \end{pmatrix}$$

2. Compute the Eigenvector decomposition (EVD) of $A \cdot A^T$

Determine eigenvalues λ:
Using characteristic polynomial:
$0 = det(A \cdot A^T - \lambda \cdot I_n)$

$$0 = det \begin{pmatrix} 388 - \lambda & 384 & 0 \\ 384 & 612 - \lambda & 0 \\ 0 & 0 & 400 - \lambda \end{pmatrix}$$

Using Rule of Sarrus:
$0 = (388 - \lambda) \cdot (612 - \lambda) \cdot (400 - \lambda) - 384 \cdot 384 \cdot (400 - \lambda)$
$0 = ((388 - \lambda) \cdot (612 - \lambda) - 384^2) \cdot (400 - \lambda)$
$0 = (\lambda^2 - 1000\lambda + 90000) \cdot (400 - \lambda)$
$0 = (\lambda^2 - 900\lambda - 100\lambda + 90000) \cdot (400 - \lambda)$
$0 = (\lambda - 100) \cdot (\lambda - 900) \cdot (400 - \lambda)$

The eigenvalues are: $\lambda_1 = 100, \lambda_2 = 400, \lambda_3 = 900$. $A \cdot A^T$ ist diagonalizable.

Determine eigenspace vectors v:
$Eig(M; \lambda) = \{v | ker(M - \lambda \cdot I) \cdot v = 0\}$

For $A \cdot A^T$ and $\lambda_1 = 100$:

$$0 = \begin{pmatrix} 388-100 & 384 & 0 \\ 384 & 612-100 & 0 \\ 0 & 0 & 400-100 \end{pmatrix} \cdot v \Rightarrow \begin{pmatrix} 288 & 384 & 0 \\ 96 & 128 & 0 \\ 0 & 0 & 300 \end{pmatrix} \cdot v \Rightarrow \begin{pmatrix} 0 & & 0 \\ 96 & 128 & 0 \\ 0 & 0 & 300 \end{pmatrix}$$

$0 = 96x + 128y \iff \frac{96}{128}x = y \iff \frac{3}{4}x = y$
$0 = 300z \iff z = 0$

Eigenvector for $\lambda_1 = 100$ is $v_1 = \begin{pmatrix} 4 & -3 & 0 \end{pmatrix}^T$.

For $\lambda_2 = 400$:

$$0 = \begin{pmatrix} -12 & 384 & 0 \\ 384 & 212 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ Eigenvector for } \lambda_2 = 400 \text{ is } v_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T.$$

For $\lambda_3 = 900$:

$$0 = \begin{pmatrix} -512 & 384 & 0 \\ 384 & -312 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot v \Rightarrow \begin{pmatrix} -4 & 3 & 0 \\ 48 & -39 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; 0 = -4x + 3y \iff \frac{-4}{-3}x = y$$

Eigenvector for $\lambda_3 = 900$ is $v_3 = \begin{pmatrix} -3 & -4 & 0 \end{pmatrix}^T$.

Determine U, S of SVD:
S is a diagonal matrix with the square-roots of the eigenvalues.

$$S = \begin{pmatrix} 100 & & 0 \\ 0 & 400 & 0 \\ 0 & 0 & 900 \end{pmatrix}$$

The columns of U are the eigenvectors.

$$U_{3,3} = \begin{pmatrix} 4 & 0 & -3 \\ -3 & 0 & -4 \\ 0 & 1 & 0 \end{pmatrix}$$

U is a 3x3-Matrix.

Compute V of the SVD:
$V = S^{-1} \cdot U^T \cdot A$

$$V = \begin{pmatrix} \frac{1}{100} & 0 & 0 \\ 0 & \frac{1}{400} & 0 \\ 0 & 0 & \frac{1}{900} \end{pmatrix} \cdot \begin{pmatrix} 4 & -3 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 13 & 5 & 5 & 0 & 13 \\ 9 & 15 & 15 & 0 & 9 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix} \iff$$

$$V = \begin{pmatrix} \frac{4}{100} & \frac{-3}{100} & 0 \\ 0 & 0 & \frac{1}{400} \\ \frac{-3}{900} & \frac{-4}{900} & 0 \end{pmatrix} \cdot \begin{pmatrix} 13 & 5 & 5 & 0 & 13 \\ 9 & 15 & 15 & 0 & 9 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix} \iff$$

$$V_{3,5} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & \frac{1}{4} \\ 9 & 0 & 0 & \frac{1}{20} & 0 \\ -\frac{3}{4} & -\frac{3}{4} & -\frac{3}{4} & 0 & -\frac{3}{4} \end{pmatrix}$$

Test:

$A = U \cdot S \cdot V$

$$A_{3,5} = \begin{pmatrix} 325 & 125 & 125 & 0 & 325 \\ 225 & 375 & 375 & 0 & 225 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 13 & 5 & 5 & 0 & 13 \\ 9 & 15 & 15 & 0 & 9 \\ 0 & 0 & 0 & 20 & 0 \end{pmatrix}$$