Information Retrieval

Exercise Sheet 4

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Exercise 1

Show that the first of *Shannons source coding theorem* $E\ L(X) \geq H(X)$, where $X = \{1, ..., m\}$, $H(X) = -\sum_i^m p_i \cdot \log_2 p_i$ and $E\ L(X) = \sum_i^m p_i \cdot L_i$.

Under the second *central lemma* constraint $\sum_i^m 2^{-L_i} \leq 1 \Rightarrow L_i$ is prefix-free code (PF) with length L_i.
For proofing the upper constraint we need to minimize $E\ L(X)$ and can use the *central lemma*. In that case L_i is a PF.

Using the Lagrange multiplication is $E\ L(X)$ the constraint and $\sum_i^m 2^{-L_i} = s \leq 1 \iff \sum_i^m 2^{-L_i} - s = 0$ the condition. This results in the following equation: $\mathcal{L} = \sum_i^m p_i \cdot L_i - \lambda(\sum_i^m 2^{-L_i} - s)$.

I $\frac{\partial \mathcal{L}}{\partial p_i} = L_i = 0$

II $\frac{\partial \mathcal{L}}{\partial L_i} = p_i + \lambda \cdot 2^{-L_i} = 0 \iff L_i = -\log_2(\frac{-p_i}{\lambda})$

III $\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_i^m 2^{-L_i} - s = 0 \iff s = \sum_i^m 2^{-L_i}$

In I is $L_i = 0$, but a code with no length is not possible with a $i \in X$.

II in III
$s = \sum_i^m 2^{(\log_2 \frac{-p_i}{\lambda})} = \sum_i^m \frac{-p_i}{\lambda} = \frac{1}{-\lambda} \cdot \sum_i^m p_i = \frac{1}{-\lambda} \cdot 1 \iff$
$\lambda = \frac{1}{-s}$

s in II
$L_i = \log_2(\frac{\frac{1}{s}}{p_i}) L_i = -\log_2(s \cdot p_i), \forall i$

Is $L_i = -\log_2(s \cdot p_i)$ a minimum?
For $s \leq 1$ is the maximum value $s = 1$. Let $L_i = 3$. If $s \cdot p_i = 1$ and $p_i = 1$ then all i are equal and the optimal code length must be $L_i = 0$.
$L_i = -\log_2 1 = 0$ So $L_i = 3$ isn't an optimum. $L_i = -\log_2(s \cdot p_i)$ is a minimum.

Show $E\ L(X) \geq H(X)$:
Let $s = 1$ be its maximum value.
$E\ L(X) = -\sum_i^m p_i \cdot \log_2(1 \cdot p_i) = H(X)$
For $s < 1, E\ L(X) < H(X)$ qed

Exersice 2

Show that *Golomb* is *entropy-optimal*.
A code is optimal for $p_i \Rightarrow L_i \leq \log_2 \frac{1}{p_i} + O(1)$, with $p_i = (1-p)^{i-1} \cdot p$, $p < 1$ and $L_i = \lfloor \frac{i}{M} \rfloor + 1 + \lceil \log_2 M \rceil$.
Let $i = 1$ than $p_i = p$ and for all $i > 1$ is $p_i < p$, so $p_i \leq p \iff \frac{1}{p} \leq \frac{1}{p_i}$, $\forall i$.
$M = \lceil \frac{1}{p} \cdot \ln 2 \rceil \leq \lceil \frac{1}{p} \cdot 1 \rceil = M'$
$L_i \leq \lfloor \frac{i}{M'} \rfloor + 1 + \lceil \log_2 M' \rceil \leq \frac{i}{M'} + 1 + \lceil \log_2 M' \rceil \leq \frac{i}{\lceil \frac{1}{p} \rceil} + \log_2(\lceil \frac{1}{p} \rceil) + 1 \leq \frac{i}{\frac{1}{p}+1} \log_2(\frac{1}{p}) + 2$