Master in High Performance Computing

Continuum mechanics and fluid-structure interaction problems: mathematical modelling and numerical approximation

Vector Spaces, co-variance, and notations

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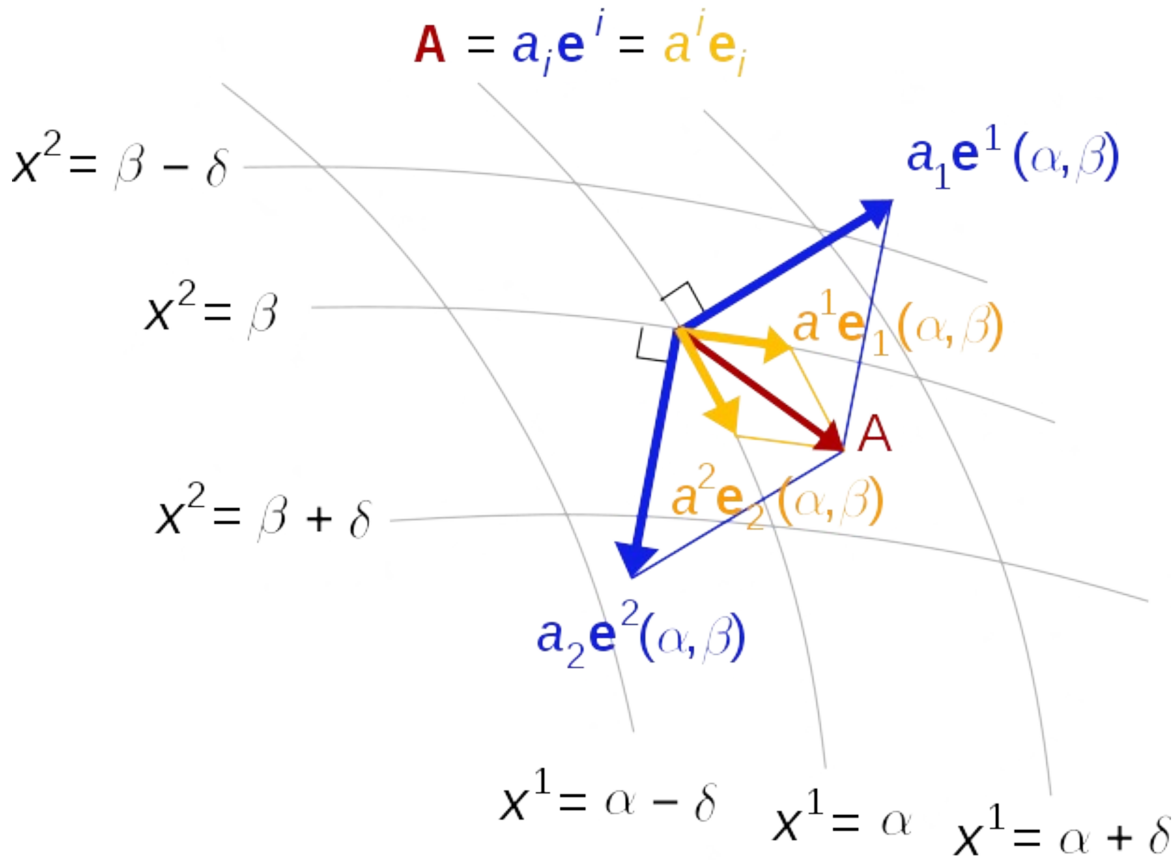


Basic Principles, notations, and definitions

- Real Vector Spaces (on the Real Numbers)

  $(V, +, 0, \mathbb{R}) : \underline{u}, \underline{v}, \alpha, \beta \quad \alpha u + \beta v \in V \quad \forall \underline{u}, \underline{v} \in V \; \forall \alpha, \beta \in \mathbb{R}$

- Banach Spaces

  V.S. + a norm, complete w.r.t. $\|\cdot\|$

- Hilbert Spaces.

  B.S. where the norm is induce by $(\cdot, \cdot) : \|u\|^2 = (u, u)$

TODAY : Finite Dim. Vector Spaces

$$V = \text{span}\{e_i\}_{i=1}^{n} \qquad \dim(V) = n \qquad e_i \text{ are lin. ind.}$$

$$\forall v \in V \quad \exists! \; \boxed{\{v^i\}_{i=1}^{n}} \quad \text{s.t.} \quad \underline{v} = \sum_{i=1}^{n} v^i \underline{e_i}$$

$v^i \in \mathbb{R}$, are called contra-variant coefficients

Einstein Summation Convention

i) Diagonally repeated indices imply summation

$$\underline{v} = \sum_{i=1}^{n} v^i \underline{e_i} = \boxed{v^i \underline{e_i}}$$

ii) All indices in an expression must be balanced
- diagonally if they appear on the same side
- at the same level if they appear on opp. sides.

$$\underline{e}_\alpha = F_\alpha{}^i \underline{E}_i \qquad \text{all indices appear twice}$$

iii) Free indices mean : take the whole set $\quad v_i \equiv \{v_i\}_{i=1}^{n}$

$V^* := \mathcal{L}(V, \mathbb{R})$ dual space of V

elements of V^* are called co-vectors

Given a basis function set: $\underline{e}_i \in V$ the functions in V^*

$$\boxed{\langle \bar{e}^i, v \rangle = \bar{e}^i(\underline{v}) := v^i} \qquad \forall\, \underline{v} \in V$$

are a lin. independent set of functions in V^*, called the canonical dual basis.

$$\forall\, \bar{\omega} \in V^* \quad \exists! \{\omega_i\}_{i=1}^{n} \quad \text{s.t.} \quad \bar{\omega} = \omega_i \bar{e}^i$$

Two sets of basis: $\underline{e}_i \in V$ $\qquad V = \text{span}\{\underline{e}_i\}_{i=1}^{n}$

$V^{**} = V$ $\qquad \bar{e}^i \in V^*$ $\qquad V^* = \text{span}\{\bar{e}^i\}_{i=1}^{n}$

$$\underline{v} = \bar{e}^i(v)\, \underline{e}_i \qquad \forall\, \underline{v} \in V$$

$$\bar{\omega} = \underline{e}_i(\omega)\, \bar{e}^i \qquad \forall\, \omega \in V^* \qquad (V^{**} \equiv V)$$

$$\bar{e}^i(\underline{e}_J) = \delta^i{}_J = \begin{cases} 1 & i = J \\ 0 & i \neq J \end{cases}$$ Kronecker delta

co-variance, contra variance?

$$\boxed{\begin{aligned} \underline{e}_i &= F_i{}^{\alpha}\, \underline{E}_\alpha \\ v^\alpha &= v^i F_i{}^{\alpha} \\ v^i &= F^i{}_\alpha\, v^\alpha \\ \bar{e}^i &= F^i{}_\alpha\, \bar{E}^\alpha \end{aligned}}$$

$$V \ni \underline{v} = v^\alpha \underline{E}_\alpha = v^i \underline{e}_i = v^i F_i{}^{\alpha} \underline{E}_\alpha$$

in matrix form $F^i{}_\alpha = (F^{-T})^i{}_\alpha$

Linear operators between U, V vector spaces

$\mathcal{L}(U,V)$: $U=V \Rightarrow$ Tensors
: $U \neq V \Rightarrow$ Two-point tensors.

Tensor product : $\underline{v} \otimes \bar{\omega}$ $\quad \bar{\omega} \in U^*, \ \underline{v} \in V$

$$(\underline{v} \otimes \bar{\omega})(\underline{u}) := \langle \bar{\omega}, \underline{u} \rangle \, \underline{v}$$

$$\underline{v} \otimes \bar{\omega} \in \mathcal{L}(U,V) = V \otimes U^*$$

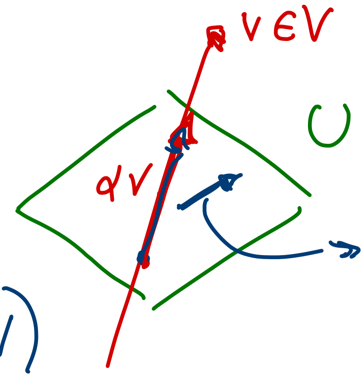
Non-commutative, bilinear

$$(\underline{v} \times \bar{\omega}) : U \longrightarrow V$$

$$u \longrightarrow \langle \bar{\omega}, u \rangle \, \underline{v}$$

$u \in U$ as $u^i \underline{e}_i$

$v \in V$ as $v^\alpha \underline{E}_\alpha$



$\alpha = \omega(u)$

Representation theorem $\forall T \in \mathcal{L}(U,V)$

$$\exists \{T^\alpha{}_i\}_{i=1, \alpha=1}^{n,n} \quad \text{s.t.} \quad T = T^\alpha{}_i \underline{E}_\alpha \otimes \bar{e}^i$$

$$u = u^i \underline{e}_i \qquad v = v^\alpha \underline{E}_\alpha$$

$$\underline{E}^\alpha(T(u^i \underline{e}_i)) \underline{E}_\alpha = T^\alpha{}_i u^i \underline{E}_\alpha$$

Transpose Operator $\underline{T}^T$.

$$T : U \longrightarrow V$$
$$T^T : V^* \longrightarrow U^*$$

$$_{V^*}\langle \bar{\gamma}, T(\underline{u}) \rangle_V = {}_{U^*}\langle T^T(\bar{\gamma}), u \rangle_U \quad \forall u, \bar{\gamma} \in U, V^*$$

$$(\underline{v} \otimes \bar{\omega})^T = (\bar{\omega} \otimes \underline{v})$$

$$\bar{\omega} \otimes \underline{v}(\bar{\gamma}) = \langle \underline{v}, \bar{\gamma} \rangle \bar{\omega}$$

$$T = T^\alpha{}_i \, \underline{E}_\alpha \otimes \bar{e}^i \qquad T^T = T^\alpha{}_i \, \bar{e}^i \otimes \underline{E}_\alpha$$

$$(T^T)_i{}^\alpha \, \bar{e}^i \otimes \underline{E}_\alpha \Rightarrow (T^T)_i{}^\alpha = T^\alpha{}_i$$

$$\bar{\omega}(\underline{u}) = \omega_i \bar{e}^i(u^J \underline{e}_J) = \omega_i u^J \bar{e}^i(\underline{e}_J)$$
$$= \omega_i u^J \delta^i{}_J$$
$$= \omega_i u^i \in \mathbb{R}$$

(Skew) Symmetric tensors?

ONLY possible for $\mathcal{L}(V, V^*)$

$A^T = A$ symmetric

$A^T = -A$ skew-symmetric

$$A : V \longrightarrow V^*$$
$$\equiv$$
$$A^T : V^{**} \longrightarrow V^*$$

Some tensors <u>cannot</u> be symmetric.

$F \in \mathcal{L}(\underline{U}, \underline{V})$ $\quad F^T \in \mathcal{L}(V^*, U^*)$

Two point tensor

"Geometrical language of Continuum Mechanics"

On spatial and material covariant balance laws in elasticity

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