Lect. 1: Probability

History & Law: কোন কিছুর relayability কতটুকু

Mosteller - Wallace

Fermate, Pascal, Newton - Probability theory কে establish করে গেছে

statistics is the logic of uncertainty

Sample Space: The set of all possible outcomes of an experiment.

Flip a coin twice

HH
HT
TH
TT

Event: A subset of the sample space

event

Naive defination of probability: $P(A)$

$$P(A) = \frac{\#\text{ of favorable outcomes}}{\#\text{ of possible outcomes}}$$

assumes all outcomes are equally likely

Counting: Multiplication rule:

If one experiment has n_1 outcomes and for each outcome of 1st experiment there are n_2 possible outcomes, for the 2nd experiment . . , n_r outcomes for the rth experiment, then there are $n_1 n_2 \ldots n_r$ overall possible outcomes

$3 \; 2 = 6$

$2 \; 3 = 6$

Example: Probability of a full house in poker with a 5 card hand.

Full house: Three cards of one kind
Two cards of another kind

suit - 4
kind - 13

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$P(A) = \frac{13 \cdot \binom{4}{3} \cdot 12 \binom{4}{2}}{\binom{52}{5}}$$

Lecture 2. Counting & Story proofs

Sampling table: Choose k objects out of n

	Order matters	Order doesn't matter
Replace ($\checkmark$)	n^k	$\binom{n+k-1}{k}$
No replace ($\times$)	${}^nP_k$	$\binom{n}{k}$

$${}^nP_k = \frac{n!}{(n-k)!}$$

pick k times from n objects where order doesn't matter

$k=0$. $\binom{n-1}{0} = 1$

$k=1$ $\binom{n}{1} = n$

$n=2$ $\binom{k+1}{k} = k+1$

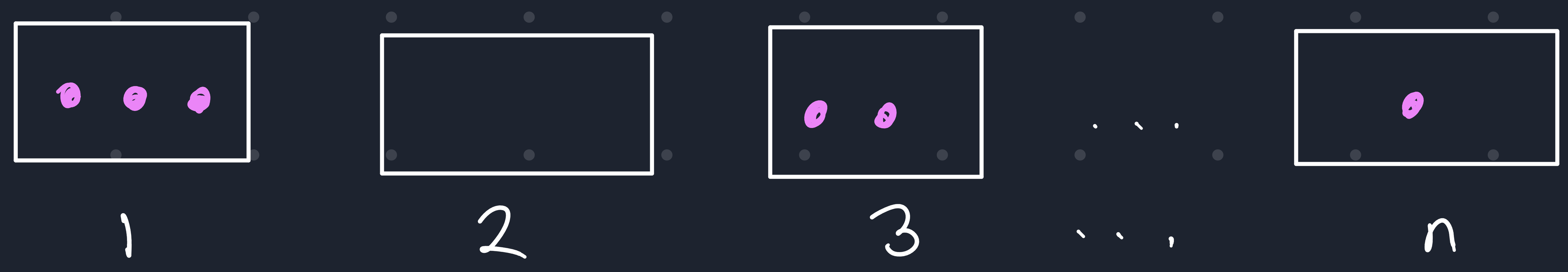
7 বার

possible choices

0, 1, ... 7

# of choice 8 $(k+1)$

Equiv: How many ways to put k indistinguishible objects in n distinguishible boxes?



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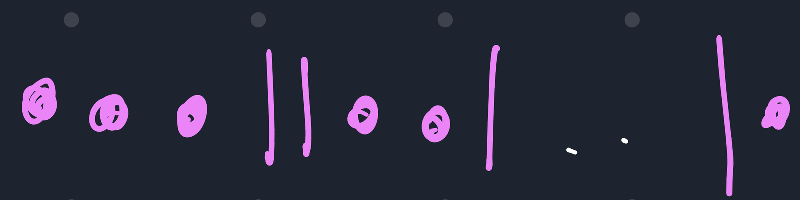


k সংখ্যক dot

$n-1$ " box boundary

total object $n+k-1$

$\therefore \binom{n+k-1}{k}$

Story proofs : (I) $\binom{n}{k} = \binom{n}{n-k}$

(II) $n\binom{n-1}{k-1} = k\binom{n}{k}$

শুরুতে captain select করে তার rest of the team তৈরী করা or

শুরুতে team তৈরী করে তার পর captain choose করা ।

Pick k people out of n, designate a captain.

(III) $\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$ [Vandermonde's identity]

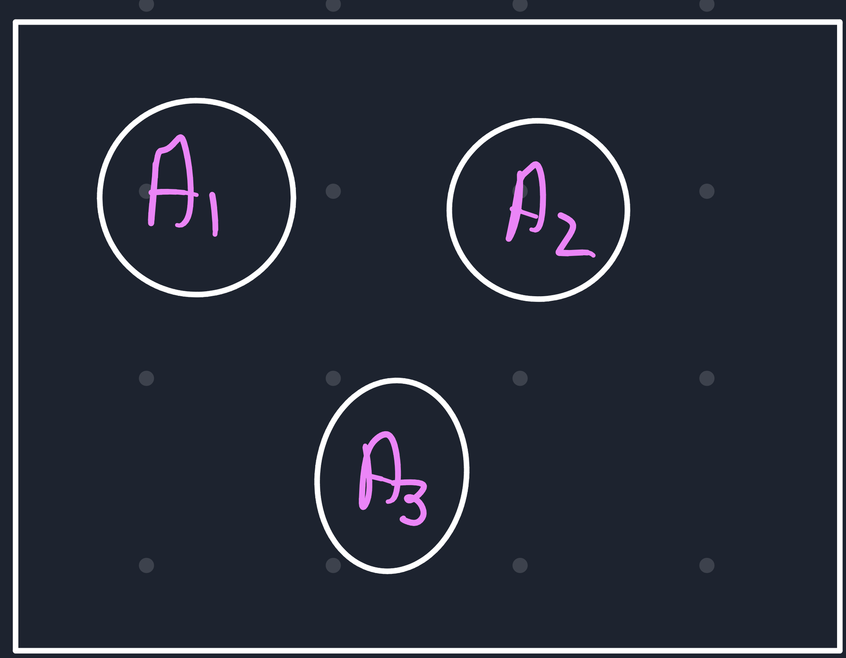
Class এ m সংখ্যক male, n সংখ্যক female

k সংখ্যক student choose করব .

Non naive defination of probability: A probability space consists of S and P, where S is a sample space and P is a function which takes an event $A \subseteq S$ as input and gives $P(A) \in [0,1]$ such that

(I) $P(\emptyset) = 0$, $P(S) = 1$

(II) $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$ if A_i are disjoint



Lecture 3: Birthday Problem

k people find the probability that two people have the same birthday

assume non-leap year
if $k > 365$, $p = 1$ [By pigeonhole principal]

Birthays are equally likely, assume independent birthday.

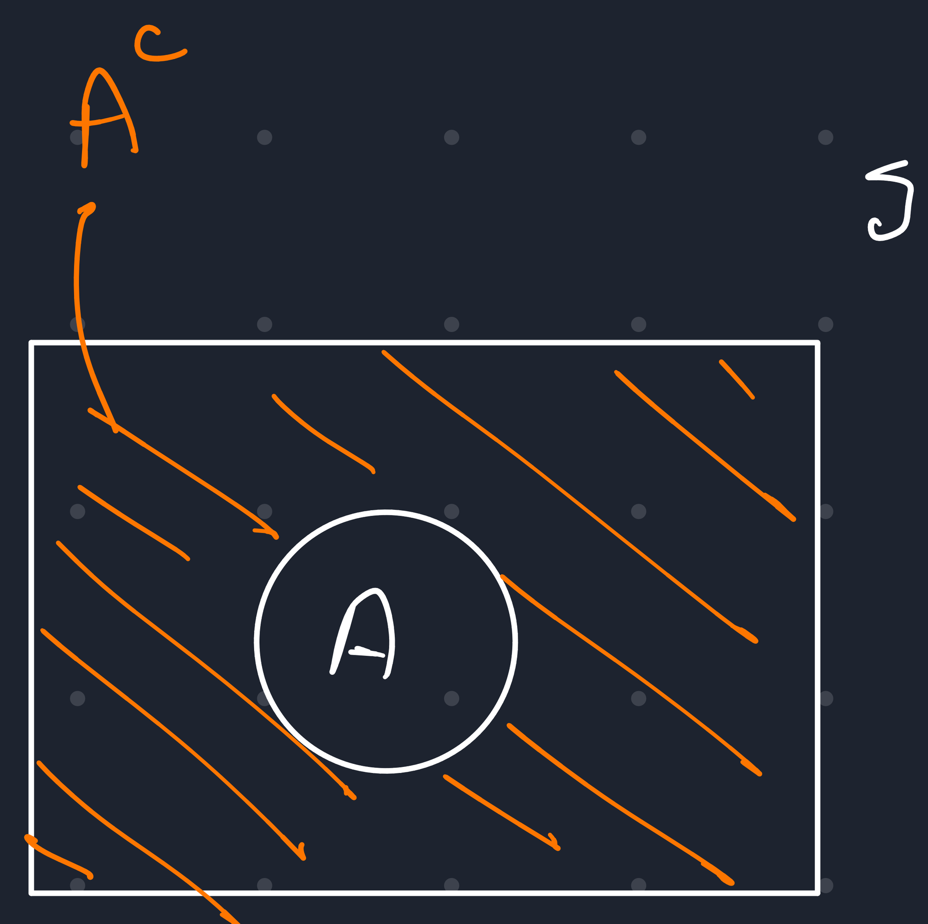
Let $k \leq 365$;

$$P(\text{no match}) = \frac{365 \cdot 364 \cdots (365-k+1)}{365^k}$$

$$P(\text{match}) = \begin{cases} 50.7\% & ; k=23 \\ 97\% & ; k=50 \\ 99.999\% & ; k=100 \end{cases}$$

Property of Probability:

① $P(A^c) = 1 - P(A)$



$$1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$$

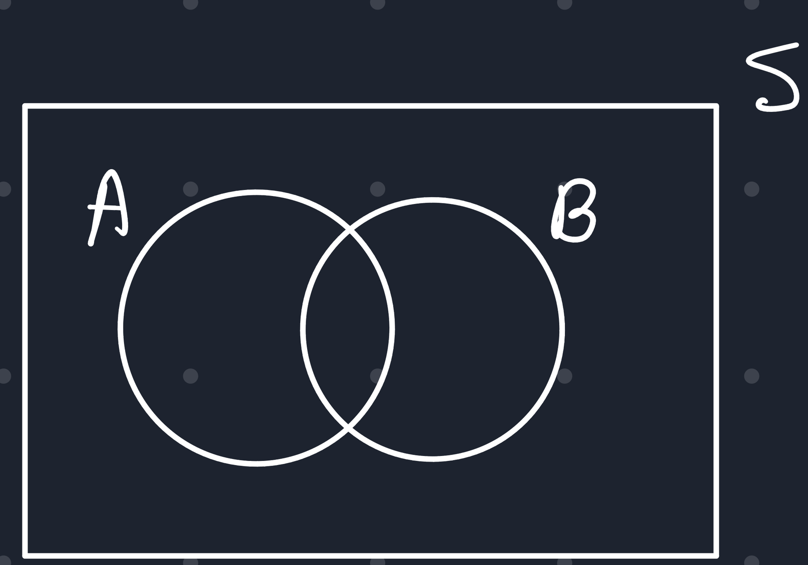
$$\Rightarrow P(A^c) = 1 - P(A)$$

(ii) If $A \subseteq B$, then $P(A) \leq P(B)$

Proof: $B = A \cup (B \cap A^c)$

$$P(B) = P(A) + P(B \cap A^c) \geq P(A)$$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



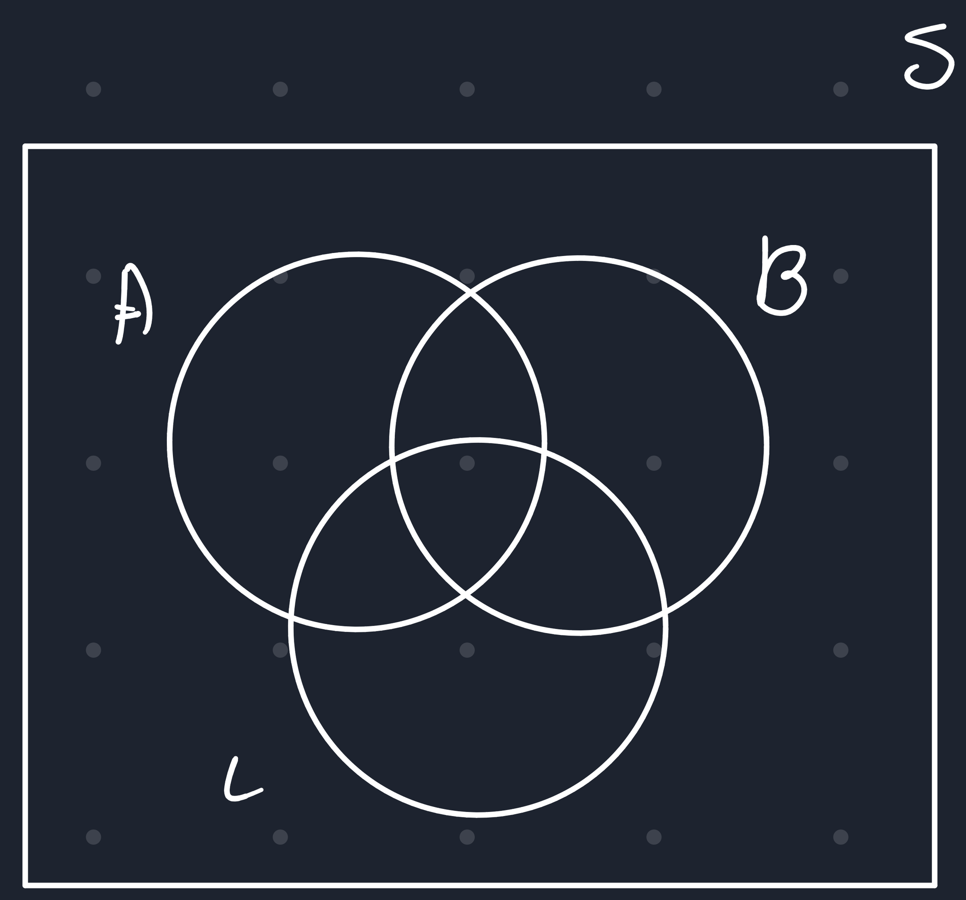
Proof: $P(A \cup B) = P(A \cup (B \cap A^c))$

$$= P(A) + P(B \cap A^c)$$

We need to proof $P(B) - P(A \cap B) = P(B \cap A^c)$

$\Rightarrow P(B) = \underbrace{P(B \cap A) + P(B \cap A^c)}_{\text{Disjoint union}}$, which is true.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$



Principle of inclusion & exclusion.

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

deMontmort's problem (1718); matching problem

$1 \to n$ sorted card

suffle $\downarrow$

যেকোন card previous position এ

n cards : 1, 2, 3 . . . n A_j be the event j^{th} card matches.

$$P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$P(A_j) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

← 1 টি card কে fix করে বাকিগুলো free to arrange

n সংখ্যক card কে n! উপায়ে arrange করতে পারি

$$P(A_1 \cap A_2) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$P(A_1 \cap \dots \cap A_k) = \frac{(n-k)!}{n!} = \frac{1}{n(n-1)(n-2)\dots(n-k+1)}$$

Lecture 4

n cards: 1, 2, 3 ., n Let A_j be the event that

j-th card matches.

$$\frac{1}{1} \quad \frac{2}{2} \quad \frac{3}{3} \quad \frac{4}{4} \quad \frac{5}{5}$$

$\Big\downarrow$ random suffle

$$\frac{3}{1} \quad \frac{1}{2} \quad \frac{5}{3} \quad \frac{4}{4} \quad \frac{2}{5}$$

$$P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$P(A_j) = \frac{1 . (n-1)(n-2) . \;1}{n(n-1) \ldots \;1}$$

$$= \frac{(n-1)!}{n!}$$

$$= \frac{1}{n}$$

$$P(A_1 \cap A_2) = \frac{1 \, 1 \, (n-2) \ldots 1}{n!} = \frac{(n-2)!}{n!}$$

$$= \frac{1}{n(n-1)}$$

$$P(A_1 \cap A_2 \ldots \cap A_k) = \frac{1 1 \ldots 1 . (n-k)(n-k-1) \ldots 1}{n!}$$

$$= \frac{(n-k)!}{n!} \; ; \binom{n}{k} \text{ such terms}$$

& all are equal

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$+ (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

$$= n\frac{1}{n} - \frac{n(n-1)}{2!}\frac{1}{n(n-1)} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n(n-1)(n-2)} \ldots$$

$$= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \ldots (-1)^{n+1}\frac{1}{n!}$$

$$\approx 1 - e^{-1}$$

$$P(A_1^c \cap A_2^c \ldots \cap A_3^c) \approx \frac{1}{e} = 0.3678$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdot$$

$$1 - e^{-1} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} \cdots$$

Fast matrix multiplication: Strassen's Algorithm

$$\Theta(n^{2.71})$$

Conditional probability:

Independence (defⁿ): Events A, B are independent if $P(A \cap B) = P(A)\,P(B)$

$P(A \cap B)$

A, B, C are independent if
$$\left\{\begin{array}{l} P(A,B) = P(A)\,P(B) \\ P(B,C) = P(B)\,P(C) \\ P(C,A) = P(C)\,P(A) \\ P(A,B,C) = P(A)\,P(B)\,P(C) \end{array}\right.$$

Newton Peps Problem (1693): Have fair dice.

3 events which is more likely.

Ⓐ at least 1 six with 6 dice $P(A) = 1 - \left(\frac{5}{6}\right)^6 = 0.665$

Ⓑ at least 2 sixes with 12 dice

Ⓒ at least 3 sixes with 18 dice

$$P(B) = 1 - \left\{\left(\frac{5}{6}\right)^{12} + 12 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^{11}\right\} = 0.61$$

$$P(C) = 1 - \sum_{k=0}^{2} \binom{18}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{18-k}$$

$$= 0.597$$

Conditional probability: How should we update probability / belief / uncertainty based on new evidence.

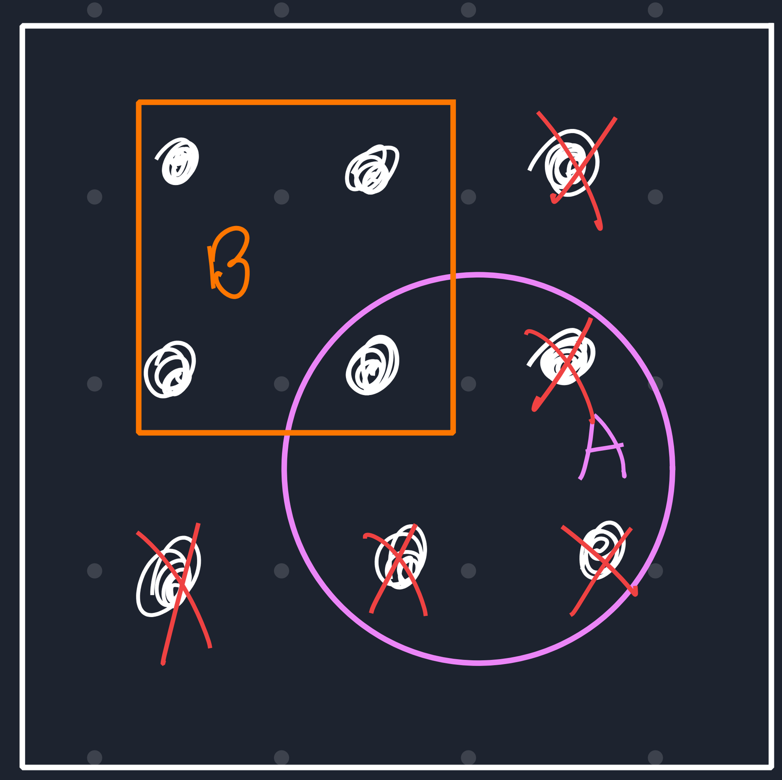
Def^n: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) > 0$

Lec 5: Conditioning cont.

Defn: $$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Intution:

S



$P(A|B)$

B ঘটে যাওয়ায় বাকিগুলোর probability 0

$$P(B) = \frac{4}{9}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Normalization

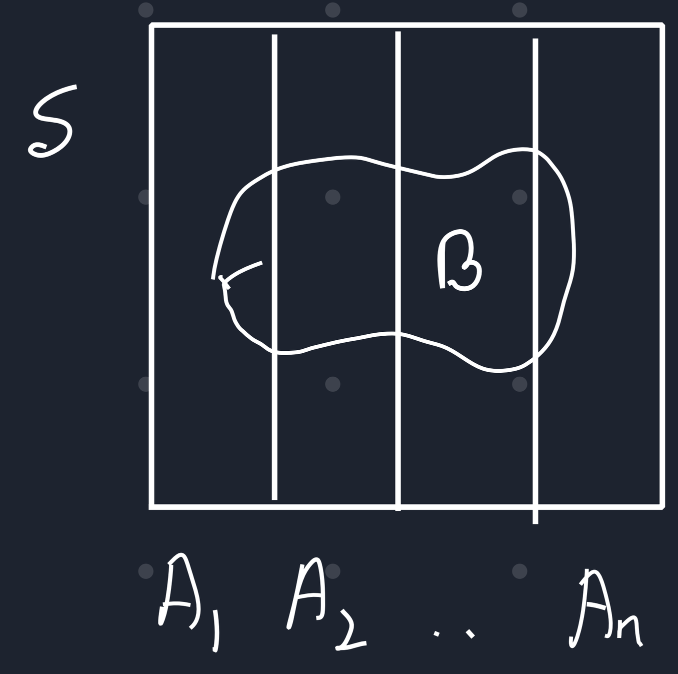
Thm 1: $P(A \cap B) = P(B) . P(A|B) = P(A) . P(B|A)$

Thm 2: $P(A_1, A_2, \ldots, A_n)$

$$= P(A_1) P(A_2|A_1) P(A_3|A_2, A_1) \ldots P(A_n|A_1 \ldots A_{n-1})$$

$\text{Th}^m\ 3:$ $$P(A|B) = \frac{P(A)\,P(B|A)}{P(B)}$$

Thinking conditionally, how to solve a problem



Let $A_1, A_2 \ldots A_n$ be a partition of S

Then

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n)$$

$$= P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + \ldots + P(A_n)\,P(B|A_n)$$

→ Law of total probability

Suits

hearts, diamonds,
clubs, spades

Ex. Get random 2-card hand from standard deck. Find $p(\text{both aces} \mid \text{have ace})$;

$p(\text{both aces} \mid \text{have ace of spades})$

$$\Rightarrow p(\text{both aces} \mid \text{have ace})$$

$$= \frac{p(\text{both aces, have ace})}{p(\text{have ace})}$$

$$= \frac{p(\text{both aces})}{p(\text{have ace})}$$

$$= \frac{\binom{4}{2} \Big/ \binom{52}{2}}{1 - \frac{\binom{48}{2}}{\binom{52}{2}}}$$

$$= \frac{1}{33} = 3.03\%$$

$\boxed{AS}$ $\boxed{AS}$

$P(\text{both aces} \mid \text{have ace of spades})$

$= \frac{3}{51} = \frac{1}{17} = 5.88\%$

Ex A disease affect 1% of population

Suppose test is 95% accurate.

D: patient has disease

T: test positive

$P(D) = 0.01$

$P(D^c) = 0.99$

$P(T \mid D) = 0.95 = P(T^c \mid D^c)$

$P(T^c \mid D) = 0.05 = P(T \mid D^c)$

$$P(D|T) = \frac{P(D\cap T)}{P(T)} = \frac{P(D)\,P(T|D)}{P(T\cap D) + P(T\cap D^c)}$$

$$= \frac{P(D)\,P(T|D)}{P(D)P(T|D) + P(D^c)\,P(T|D^c)}$$

$$= \frac{0.01 \times 0.95}{0.01 \times 0.95 + 0.99 \times 0.05}$$

$$= 0\,16$$