Assignment 1 Solution - CSE330 Summer 2022

Saturday, June 11, 2022 8:42 PM

Q1

a Lecture Notes Form : $0.d_1d_2d_3d_4$ where $d_1=1$ for only Lecture Notes Form.

Normalized Form : $1.d_1d_2d_3d_4$

Denormalized Form : $0.1d_1d_2d_3d_4$

Number of exponents = 5, $m=4$ implies $d_1d_2d_3d_4$

Lecture Notes Convention	Denormalized Form	Normalized Form
$0.d_1d_2d_3d_4$ (Note: $d_1=1$)	$0.1d_1d_2d_3d_4$	$1.d_1d_2d_3d_4$
0.1000	0.10000	1.0000
0.1001	0.10001	1.0001
0.1010	0.10010	1.0010
0.1100	0.10011	1.0011
0.1101	0.10100	1.0100
0.1110	0.10101	1.0101
0.1111	0.10110	1.0110
	0.10111	1.0111
	0.11000	1.1000
	0.11001	1.1001
	0.11010	1.1010
	0.11011	1.1011
	0.11100	1.1100
	0.11101	1.1101
	0.11110	1.1110
	0.11111	1.1111

Lecture Notes Convention: 8 binary combinations $\therefore 8\times5=\underline{\underline{40}}$ numbers

Denormalized Form: total 16 combinations $\therefore 16\times5=\underline{\underline{80}}$ numbers

Normalized Form: total 16 combinations $\therefore 16\times5=\underline{\underline{80}}$ numbers

b

	Lecture Notes Form	Denormalized	Normalized
Smallest Positive Number	$(\underbrace{0.1000}_{\text{Smallest fraction}})_2\times2^{\overset{\text{smallest exponent}}{-2}}=\frac{1}{8}$	$(0.10000)_2\times2^{-2}=\frac{1}{8}$	$(1.0000)_2\times2^{-2}=\frac{1}{4}$
Largest Number	$(\underbrace{0.1111}_{\text{Largest fraction}})_2\times2^{\overset{\text{largest exponent}}{2}}=\frac{15}{4}$	$(0.11111)_2\times2^{2}=\frac{31}{8}$	$(1.1111)_2\times2^{2}=\frac{31}{4}$

c

$e\in[0,2047], e\in\mathbb{Z}$

Positive minimum $= (\overset{0.1d_1\dots d_{52}}{0.10\dots0})_2\times2^{1-1022}=(0.1)_2\times2^{-1021}$

maximum $= (\overset{0.1d_1\dots d_{52}}{0.11\dots1})_2\times2^{2046-1022}=(0.11\dots1)_2\times2^{1024}$

$e\in[0,2047]$, $e\in\mathbb{Z}$

d bias= 500

Positive minimum $= (\overset{0.1d_1\dots d_{52}}{0.10\dots0})_2\times2^{1-500}=(0.1)_2\times2^{-499}$

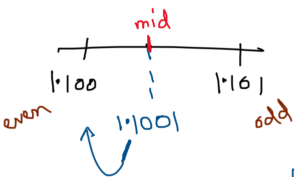
$$\text{maximum} = \overset{0.1d_1\ldots d_{52}}{(0.11\ldots1)_2} \times 2^{2046-500} = (0.11\ldots1)_2 \times 2^{1546}$$

$\boxed{Q2}$

$\boxed{a}$ $(6.25)_{10} = (110.01)_2 \times 2^0$

$\hookrightarrow \overset{1.d_1d_2d_3x}{(1.1001)_2} \times 2^2$

We can not keep this 1 since $m=3$.

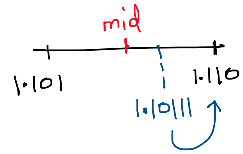


$\therefore fl[6.25] = (1.100)_2 \times 2^2$

The fraction goes left because it is equidistant from 1·100 and 1·101 ∴ take the even number.

$(6.875)_{10} = (110.111)_2 \times 2^0$

$\hookrightarrow (1.10111)_2 \times 2^2$



$\therefore fl[6.875] = (1.110)_2 \times 2^2$

$\boxed{b}$ $\delta_1 = \left|\dfrac{6.25 - fl[6.25]}{6.25}\right| = \dfrac{1}{25}$

$\delta_2 = \left|\dfrac{6.875 - fl[6.875]}{6.875}\right| = \dfrac{1}{55}$

$\boxed{c}$ Denormalized Form

$(6.25)_{10} = (110.01)_2 \times 2^0 \longrightarrow \overset{1d_1d_2d_3X}{(0.11001)_2} \times 2^3$

$(6.875)_{10} = (110.111)_2 \times 2^0 \longrightarrow \overset{1d_1d_2d_3XX}{(0.110111)_2} \times 2^3$

$e = 3$

Our system only supports $-1 \leq e \leq 2$ ∴ Not representable in denormalized form

$\boxed{d}$ Lecture Notes $\rightarrow \frac{1}{2}\beta^{1-m} = \frac{1}{2}\cdot 2^{1-3} = \frac{1}{8}$

Denormalized, Normalized $\} \rightarrow \frac{1}{2}\beta^{-m} = \frac{1}{2}\cdot 2^{-3} = \frac{1}{16}$

3(a)

$x_0 = -0.5 \quad f[x_0] = 1.87$

$$f[x_0, x_1] = \frac{2.20 - 1.87}{0 + 0.5} = 0.66$$

$x_1 = 0 \quad f[x_1] = 2.20$

$$f[x_0, x_1, x_2] = \frac{0.48 - 0.66}{0.5 + 0.5}$$

$$f[x_1, x_2] = \frac{2.44 - 2.20}{0.5 - 0} = 0.48$$

$$= -0.18$$

$x_2 = 0.5 \quad f[x_2] = 2.44$

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= 1.87 + 0.66(x + 0.5) - 0.18(x + 0.5)(x - 0)$$

$$= 1.87 + 0.66x + 0.33 - 0.18(x^2 + 0.5x)$$

$$= 1.87 + 0.66x + 0.33 - 0.18x^2 - 0.09x$$

$$= 2.2 + 0.57x - 0.18x^2$$

3(b)

$$f(x) = \ln(5x+9)$$

$$f'(x) = \frac{1}{5x+9}(5) = 5(5x+9)^{-1}$$

$$f''(x) = -5(5x+9)^{-2}(5) = -25(5x+9)^{-2}$$

$$f(0) = \ln(9)$$

$$f'(0) = \frac{5}{9}$$

$$f''(0) = -\frac{25}{81}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$= \ln(9) + \frac{5}{9}x + \frac{\left(-\frac{25}{81}\right)}{2!}x^2$$

$$= \ln(9) + \frac{5}{9}x - \frac{25}{162}x^2$$

$$= 2.20 + 0.56x - 0.15x^2$$

(c) No, they should not match (exactly). In part (a), we have used polynomial interpolation to find the equation of a ~~funct~~ polynomial which fits through the given nodes. However, in part (b), we have used the taylor series expansion to ~~find the approxi~~ approximate the equation of a given (known) function.

(4) (a) Yes, both the equations should match.

(b)
$$\begin{bmatrix} 1 & 0 & 0^2 \\ 1 & 3 & 3^2 \\ 1 & 6 & 6^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9.5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 9 \\ 1 & 6 & 36 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9.5 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{2}{3} & -\frac{1}{6} \\ \frac{1}{18} & -\frac{1}{9} & \frac{1}{18} \end{bmatrix} \begin{bmatrix} 5 \\ 9.5 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \\ -0.5 \end{bmatrix}$$

$$P_2(x) = a_0 + a_1 x^1 + a_2 x^2$$

$$= 5 + 3x + (-0.5)x^2$$

$$= 5 + 3x - 0.5x^2$$

④

(c) $l_0(x) = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \dfrac{(x-3)(x-6)}{(0-3)(0-6)} = \dfrac{1}{18}(x^2-9x+18)$

$l_1(x) = \dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \dfrac{(x-0)(x-6)}{(3-0)(3-6)} = \dfrac{-1}{9}(x^2-6x)$

$l_2(x) = \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \dfrac{(x-0)(x-3)}{(6-0)(6-3)} = \dfrac{1}{18}(x^2-3x)$

$P_2(x) = f(x_0)\,l_0(x) + f(x_1)\,l_1(x) + f(x_2)\,l_2(x)$

$= \dfrac{5}{18}(x^2-9x+18) + \left(-\dfrac{9.5}{9}\right)(x^2-6x) + \dfrac{5}{18}(x^2-3x)$

$= \dfrac{5}{18}x^2 - \dfrac{5}{2}x + 5 - \dfrac{19}{18}x^2 + \dfrac{19}{3}x + \dfrac{5}{18}x^2 - \dfrac{15}{18}x$

$= -\dfrac{1}{2}x^2 + 3x + 5$

$= 5 + 3x - 0.5x^2$

Scanned with CamScanner

⑤ $x_0 = 0.1 \quad f(x_0) = -0.620 \quad f'(x_0) = 3.585$

$x_1 = 0.2 \quad f(x_1) = -0.283 \quad f'(x_1) = 3.140$

---

number of nodes $= 2$

$n+1 = 2$

$n = 1$

$$P_{2n+1} = P_3(x) = f(x_0)h_0(x) + f'(x_0)\hat{h}_0(x) + f(x_1)h_1(x) + f'(x_1)\hat{h}_1(x)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 0.2}{0.1 - 0.2} = -10x + 2$$

$$l_0'(x) = -10$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0.1}{0.2 - 0.1} = 10x - 1$$

$$l_1'(x) = 10$$

$$h_k(x) = \left(1 - 2(x - x_k) l_k'(x_k)\right) l_k^2(x)$$

$$\hat{h}_k(x) = (x - x_k) l_k^2(x)$$

$$\begin{aligned} h_0(x) &= 1 - 2(x - x_0) l_0'(x_0) l_0^2(x) \\ &= [1 - 2(x - 0.1)(-10)](-10x + 2)^2 \\ &= [1 + 20(x - 0.1)](-10x + 2)^2 \\ &= (1 + 20x - 2)(100x^2 + 40x + 4) \\ &= (20x - 1)(100x^2 + 40x + 4) \\ &= 2000x^3 - 900x^2 + 120x - 4 \end{aligned}$$

$$\begin{aligned} \hat{h}_0(x) &= (x - x_0) l_0^2(x) \\ &= (x - 0.1)(-10x + 2)^2 \\ &= 100x^3 - 50x^2 + 8x - 0.4 \end{aligned}$$

$$h_1(x) = \left(1 - 2(x - x_1)\, l_1'(x_1)\right) l_1^2(x)$$
$$= [1 - 2(x - 0.2)(10)](10x - 1)^2$$
$$= [1 - 20(x - 0.2)](10x - 1)^2$$
$$= (1 - 20x + 4)(10x - 1)^2$$
$$= (-20x + 5)(100x^2 - 20x + 1)$$
$$= -2000x^3 + 900x^2 - 120x + 5$$

$$\hat{h}_1(x) = (x - x_1)\, l_1^2(x)$$
$$= (x - 0.2)(10x - 1)^2$$
$$= 100x^3 - 40x^2 + 5x - 0.2$$

(b)

$$P_3(x) = f(x_0) h_0(x) + f'(x_0)\hat{h}_0(x) + f(x_1) h_1(x) + f'(x_1)\hat{h}_1(x)$$
$$= -0.620[2000x^3 - 900x^2 + 120x - 4]$$
$$+ 3.585[100x^3 - 50x^2 + 8x - 0.4]$$
$$- 0.283[-2000x^3 + 900x^2 - 120x + 5]$$
$$+ 3.140[100x^3 - 40x^2 + 5x - 0.2]$$

$$= -1240x^3 + 558x^2 - 74.4x + 2.48$$
$$+ 358.5x^3 - 179.25x^2 + 28.68x - 1.434$$
$$+ 566x^3 - 254.7x^2 + 33.96x - 1.415$$
$$+ 314x^3 - 125.6x^2 + 15.7x - 0.628$$

$$P_3(x) = -1.5x^3 - 1.55x^2 + 3.94x - 0.997$$

$$P_3(0.15) = -1.5(0.15)^3 - 1.55(0.15)^2 + 3.94(0.15) - 0.997$$
$$= -0.4459375$$

Scanned with CamScanner

Q6

Tuesday, June 14, 2022 11:06 PM

$\boxed{Q6}$

- We know that $P_n(x) = \sum_{i=0}^{n} a_i n_i(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_n(x-x_0)\cdots(x-x_{n-1})$

Here, $a_0 = f[x_0]$

$a_1 = f[x_0 x_1]$

$a_2 = f[x_0 x_1 x_2]$

$\vdots$

$a_n = f[x_0 x_1 \cdots x_n]$

- Let's look at Eq. (2.16), (2.17) Pg 19.

$*$ $f[x_0] = a_0$ (2.16)

$\therefore P_0(x) = a_0 = f[x_0]$

We know that $f[x_1] = f(x_1)$

$*$ $f[x_0 x_1] = a_1 = \dfrac{f(x_1) - P_0(x_1)}{(x_1 - x_0)} \rightarrow \dfrac{f(x_1) - f[x_0]}{(x_1 - x_0)} \rightarrow \dfrac{f[x_1] - f[x_0]}{(x_1 - x_0)}$ (2.17)

$\therefore P_1(x) = a_0 + a_1(x - x_0) \rightarrow a_0 + \dfrac{f[x_1] - f[x_0]}{(x_1 - x_0)}(x - x_0)$

$\boxed{\text{The general formula to find coefficient } a_i \text{ is } \quad a_i = \dfrac{f(x_i) - P_{i-1}(x_i)}{(x_i - x_0)\cdots(x_i - x_{i-1})}}$

$*$ For our case $f[x_0, x_1, x_2]$ Eq. (2.18)

$$f[x_0 x_1 x_2] = a_2 = \frac{f(x_2) - P_1(x_2)}{(x_2 - x_0)(x_2 - x_1)} \rightarrow \frac{f(x_2) - (a_0 + a_1(x_2 - x_0))}{(x_2 - x_0)(x_2 - x_1)}$$

$$\hookrightarrow \frac{f[x_2] - f[x_0] - f[x_0, x_1](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \rightarrow \frac{f[x_2] - f[x_0] - \left(\dfrac{f[x_1] - f[x_0]}{(x_1 - x_0)}\right) \cdot (x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$\hookrightarrow \frac{f[x_2](x_1 - x_0) - f[x_0](x_1 - x_0) - f[x_1](x_2 - x_0) + f[x_0](x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

I can add the terms because $f[x_1]x_1 - f[x_1]x_1 = 0$

$$\hookrightarrow \frac{\overset{a}{f[x_2]x_1} - \overset{b}{f[x_2]x_0} - \overset{c}{f[x_0]x_1} + \cancel{f[x_0]x_0} - \overset{d}{f[x_1]x_2} + \overset{e}{f[x_1]x_0} + \overset{f}{f[x_0]x_2} - \cancel{f[x_0]x_0} + \underbrace{\overset{g}{f[x_1]x_1} - \overset{h}{f[x_1]x_1}}}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$\hookrightarrow$ factorise with all $f[x_i]$

$$\frac{f[x_2](\overset{a}{x_1} - \overset{b}{x_0}) - f[x_1](\overset{h}{x_1} - \overset{e}{x_0}) - f[x_1](\overset{d}{x_2} - \overset{g}{x_1}) + f[x_0](\overset{f}{x_2} - \overset{c}{x_1})}{(x_2 - x_0)(x_2 - x_1)(x_1 - x_0)}$$

$$\hookrightarrow \frac{(x_1-x_0)(f[x_2]-f[x_1]) - (x_2-x_1)(f[x_1]-f[x_0])}{(x_1-x_0)(x_2-x_1)} \cdot \frac{1}{(x_2-x_0)}$$

$$\hookrightarrow \left(\frac{f[x_2]-f[x_1]}{(x_2-x_1)} - \frac{f[x_1]-f[x_0]}{(x_1-x_0)}\right)\frac{1}{x_2-x_0} = \frac{f[x_1,x_2]-f[x_1,x_0]}{(x_2-x_0)}$$