Solution:

1) a) $f(x) = 6e^{-5x}$

$x_0 = 0.2$

$h = 0.5$

Using formula of central difference,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$= \frac{f(0.7) - f(-0.3)}{2 \times 0.5}$$

$$= \frac{6e^{-5\times0.7} - 6e^{-5\times(-0.3)}}{1}$$

$$= 0.181184 - 26.8901$$

$$= -26.7089$$

(Ans)

(b) using forward difference,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \frac{f(0.7) - f(0.2)}{0.5}$$

$$= \frac{6e^{-5\times0.7} - 6e^{-5\times0.2}}{0.5}$$

$$= \frac{0.181184 - 2.20728}{0.5}$$

$$= -4.05219$$

(Ans.)

(c)

h	forward difference $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$ here, $x_0 = 2$	touncation error real derivative, $f'(2) = -30e^{-5x}$ $= -30e^{-5\times2}$ $= -1.36199\times10^{-3}$ $\therefore$ error $= (-1.36199\times10^{-3} - \text{forward})$
1	-2.70564×10^{-4}	-1.09143×10^{-3}
0.1	-1.07181×10^{-3}	-2.90180×10^{-4}
0.01	-1.32851×10^{-3}	-3.34800×10^{-5}
0.0001	-1.36166×10^{-3}	~~-3.3×10^{-7}~~ -3.30000×10^{-7}

h	central difference $f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$, $x_0 = 2$	truncation error: error = $(-1.36199 \times 10^{-3}$ - central)
1	-0.0202130	0.0188510
0.1	-1.41946×10^{-3}	5.74700×10^{-5}
0.01	-1.362557×10^{-3}	5.80000×10^{-7}
0.0001	-1.36199×10^{-3}	0.000000

d) $$D_h^{(1)} = \frac{2^2 \cdot D_{h/2} - D_h}{2^2 - 1}$$ (where $x_0 = 0.2$)

we know, $$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

$$D_{h/2} = \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{2 \times \frac{h}{2}}$$

$$\therefore D_{0.5} = \frac{f(0.2+0.5) - f(0.2-0.5)}{2 \times 0.5}$$

$$= \frac{6[e^{-5 \times 0.7} - e^{-5 \times -0.3}]}{1}$$

$$= -26.7089 \approx -26.7090$$

$h = 0.5$

$\frac{h}{2} = 0.25$

$$\therefore D_{0.25} = \frac{f(0.2+0.25) - f(0.2-0.25)}{2 \times 0.25}$$

$$= \frac{6\left[e^{-5 \times 0.45} - e^{-5 \times -0.05}\right]}{0.5}$$

$$= -14.1435$$

$$\therefore D_{0.5}^{(1)} = \frac{4 \cdot D_{0.25} - D_{0.5}}{4-1}$$

$$= \frac{4(-14.1435) - (-26.7090)}{3}$$

$$= -9.95500$$

(Ans)

Real derivative at $x_0 = 0.2$, $f'(0.2) = -30e^{-5 \times 0.2}$

$$= -11.0364$$

$\therefore$ truncation error = real derivative − approximated derivative

$$= -11.0364 + 9.95500$$

$$= -1.08140$$

(Ans)

~x~

Solution:

4) a) $f(x) = x^3 - x^2 - 9x + 9 = 0$

$\Rightarrow x^2(x-1) - 9(x-1) = 0$

$\Rightarrow (x-1)(x^2-9) = 0$

$\therefore x_k = -3, 1, 3 \longrightarrow$ actual/exact roots

(Ans.)

b) $x^3 - x^2 - 9x + 9 = 0$

1st choice:

$9x = x^3 - x^2 + 9$

$x = \frac{1}{9}(x^3 - x^2 + 9) = g(x)$

2nd choice:

$x(x^2 - x - 9) = -9$

$x = \dfrac{-9}{x^2 - x - 9} = g(x)$

3rd choice:

$x = x + x^3 - x^2 - 9x + 9$

$x = x^3 - x^2 - 8x + 9 = g(x)$

(c) convergence rate/ratio, $\lambda = |g'(x_*)|$

$$= \left| \frac{dg}{dx} \Big|_{x=x_*} \right|$$

For 1st case:

$$g(x) = \frac{1}{9}(x^3 - x^2 + 9)$$

$$\Rightarrow g'(x) = \frac{1}{9}(3x^2 - 2x)$$

$$\therefore \lambda = |g'(x_*)| = \begin{cases} \frac{1}{9} (<1) \text{ for } x_* = 1 \text{ (Linear convergence)} \\ \frac{33}{9} (>1) \text{ for } x_* = -3 \text{ (divergence)} \\ \frac{21}{9} (>1) \text{ for } x_* = 3 \text{ (divergence)} \end{cases}$$

$\therefore g(x)$ is converging to $x_* = 1$ for 1st case.

for 2nd case:

$$g(x) = \frac{-9}{x^2-x-9}$$

$$\therefore g'(x) = \frac{9(2x-1)}{(x^2-x-9)^2}$$

$$\lambda = |g'(x_*)| = \begin{cases} \frac{9}{81} \ (<1) & \text{for } x_* = 1 \ \text{(Linear convergen} \\ \frac{63}{9} \ (>1) & \text{for } x_* = -3 \ \text{(divergence)} \\ \frac{45}{9} = 5 \ (>1) & \text{for } x_* = 3 \ \text{(divergence)} \end{cases}$$

$\therefore g(x)$ is converging to $x_* = 1$ for 2nd case.

for 3rd case:

$$g(x) = x^3 - x^2 - 8x + 9$$

$$g'(x) = 3x^2 - 2x - 8$$

$$\lambda = |g(x_*)| = \left.\begin{cases} 7 \ (>1) & \text{for } x_* = 1 \\ 25 \ (>1) & \text{for } x_* = -3 \\ 13 \ (>1) & \text{for } x_* = 3 \end{cases}\right\} \rightarrow \text{divergence}$$

as we need $\lambda < 1$ for convergence,

we don't have any converging point in ~~~~ 3rd case.

(Ans.)

(d) $\epsilon_M = 10^{-3}$ (max error bound)

$x_0 = 0$

$g(x) = \frac{1}{9}(x^3 - x^2 + 9)$

K=0 $g(0) = 1.000$

K=1 $g(1.000) = 1.000 \Rightarrow x_1 - x_0 \simeq 0.000 \; (< 10^{-3})$

$\therefore$ 1.000 is the fixed point of $g(x)$ and it is also the root of $f(x)$.

(Ans.)

OR, $g(x) = \frac{-9}{x^2 - x - 9}$

$g(0) = 1.000$

$g(1.000) = 1.000 \longrightarrow$ fixed point of $g(x)$ and root of $f(x)$.

$\boxed{\therefore x_k = 1.000}$

—x—

Solution:

2.(b) $D_h = \dfrac{f(x+h) - f(x-h)}{2h}$

$$f(x+h) = f(x) + f'(x).h + \frac{f^{(2)}(x)}{2!}h^2 + \frac{f^{(3)}(x)}{3!}.h^3 + \frac{f^{(4)}(x)}{4!}.h^4 + \frac{f^{(5)}(x)}{5!}.h^5 + \cdots$$

$$f(x-h) = f(x) - f^{(1)}(x).h + \frac{f^{(2)}(x)}{2!}.h^2 - \frac{f^{(3)}(x)}{3!}.h^3 + \frac{f^{(4)}(x)}{4!}.h^4 - \frac{f^{(5)}(x)}{5!}.h^5 + \cdots$$

$$\therefore D_h = \frac{1}{2h}\left[2f^{(1)}(x).h + \frac{2f^{(3)}(x)}{3!}.h^3 + \frac{2f^{(5)}(x)}{5!}.h^5 + O(h^7)\right]$$

$$D_h = f^{(1)}(x) + \frac{f^{(3)}(x)}{3!}.h^2 + \frac{f^{(5)}(x)}{5!}.h^4 + O(h^6)$$

$$\therefore D_{h/3} = f^{(1)}(x) + \frac{f^{(3)}(x)}{3!}.\left(\frac{h}{3}\right)^2 + \frac{f^{(5)}(x)}{5!}.\left(\frac{h}{3}\right)^4 + O(h^6)$$

$$\Rightarrow 3^2.D_{h/3} = 3^2 f^{(1)}(x) + \frac{f^{(3)}(x)}{3!}.\frac{h^2}{1} + \frac{1}{9}.\frac{f^{(5)}(x)}{5!}.h^4 + O(h^6)$$

$$\Rightarrow 3^2 . D_{h/3} - D_h = f'(x)\left(3^2-1\right) - \frac{8}{9} \cdot \frac{f^{(5)}(x)}{5!} \cdot h^4 + O(h$$

$$\Rightarrow \frac{3^2 . D_{h/3} - D_h}{3^2-1} = f'(x) - \frac{1}{9} \cdot \frac{f^{(5)}(x)}{5!} \cdot h^4 + O(h^6)$$

$$\Rightarrow D_h^{(1)} = f'(x) - \frac{1}{9} \cdot \frac{f^{(5)}(x)}{5!} \cdot h^4 + O(h^6).$$

(Ans)

—×—

3

$f(x) = x^3 - 2x^2 - 11x + 12$

(a)

$$n \geq \frac{\log|b-a| - \log|\epsilon|}{\log 2} - 1$$

$$\geq \frac{\log|8.25 - 1.78| - \log|1\times10^{-3}|}{\log 2} - 1$$

$n \geq 11.65955$

(b) Iteration-1 $[1.78, 8.25]$

$$x_m = \frac{1.78 + 8.25}{2} = 5.015$$

$f(x_m) \times f(x_l) = 32.66 \times -8.277 = -ve$

Iteration 2 : $[1.78, 5.015]$

$$x_m = \frac{1.78 + 5.015}{2} = 3.3975$$

$f(3.3975) \times f(1.78) = -9.241 \times -8.272$

$= +ve$

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Iteration 3 $[3.3975, 5.015]$

$x_m = 4.20625$

$f(x_l)\,f(x_m) = -9.2411 \times 4.76541 = -ve$

$[3.3975, 4.20625]$

$\boxed{\text{Iteration} = 4}$ $[3.3975, 4.20625]$

$x_m = 3.801875$

$f(x_l) \times f(x_m) = -9.2411 \times -3.775$

$= +ve$

$[3.801875, 4.20625]$

Iteration 5 $[3.801875, 4.20625]$

$x_m = 4.0040625$

$f(x_l)\,f(x_m) = -3.775 \times 0.085422$

$= -ve$

$[\cancel{3.801875}\ 3.801875, 4.0040625]$

Solution

5. $f(x) = x^2 e^{-x} - 0.6$

$f'(x) = 2xe^{-x} - x^2 e^{-x}$

$x_0 = 0.2$

$\delta = 10^{-4}$

$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$ → formula

K	x_K	$f(x_K)$	is $\lvert f(x_K)\rvert < \delta$?
0	0.2	−0.56725	NO
1	2.12456	−0.06067	NO
2	0.20579	−0.56553	NO
3	2.08740	−0.05966	NO
4	−0.54973	−0.07635	NO
5	−0.58116	3.929×10^{-3}	No
6	−0.57970	2.191×10^{-5}	Yes ✓

$\therefore x_* = -0.5797$

(Ans)