Heath-Jarrow-Morton Model

for

Pricing Of Interest Rates and Credit Derivatives

Introduction

- Framework to predict Forward Interest Rates using an existing term structure of interest rates.
- Created by David Heath , Robert A. Jarrow, and Andrew Morton during the late 1980s.
- Now these predicted Interest Rates are used to calculate the prices of securities affected by interest rate movements, including securities such as bonds and options.
- By being able to price securities, investors can engage in arbitrage opportunities to earn a riskless profit if there are differences between the price of the security in the market and the price of the security calculated based on the Heath-Jarrow-Morton Model.

Derivation

- We will use the traditional risk-neutral world for the derivation of HJM result
- The Stochastic Process for a Zero-Coupon Bond has the form -

$$dP(t,T) = r(t)P(t,T)\,dt + v(t,T,\Omega_t)P(t,T)\,dz(t)$$

Derivation

- Because a bond's price volatility declines to zero at maturity, we must have :-

$$v(t, t, \Omega_t) = 0$$

- R1 and R2 -> Zero Rates for Maturities T1 and T2 respectively.
- RF->Forward interest rate for the period of time between T1 and T2

$$R_F = \frac{R_2T_2 - R_1T_1}{T_2 - T_1}$$

Derivation

This Equation can be written as -

$$R_F = R_2 + (R_2 - R_1)\frac{T_1}{T_2 - T_1}$$

Now, to obtain Instantaneous Forward Rate we take limits as T2 approaching to T1 and taking their common value to be T ->

$$R_F = R + T\frac{\partial R}{\partial T}$$

Derivation

- Because ->

$$P(0, T) = e^{-RT}$$

- The Forward Rate is of the form ->

$$R_F = -\frac{\partial}{\partial T} \ln P(0, T)$$

Derivation

- From Previous Equations, the Zero Coupon Bonds can be related to Forward Rates by ->

$$f(t, T_1, T_2) = \frac{\ln[P(t, T_1)] - \ln[P(t, T_2)]}{T_2 - T_1}$$

- From the First Equation ->

$$d\ln[P(t, T_1)] = \left[r(t) - \frac{v(t, T_1, \Omega_t)^2}{2}\right] dt + v(t, T_1, \Omega_t)\, dz(t)$$

$$d\ln[P(t, T_2)] = \left[r(t) - \frac{v(t, T_2, \Omega_t)^2}{2}\right] dt + v(t, T_2, \Omega_t)\, dz(t)$$

Derivation

- On Solving the Equations ->

$$df(t,T_1,T_2) = \frac{v(t,T_2,\Omega_t)^2 - v(t,T_1,\Omega_t)^2}{2(T_2 - T_1)}\,dt + \frac{v(t,T_1,\Omega_t) - v(t,T_2,\Omega_t)}{T_2 - T_1}\,dz(t)$$

- Now Putting T1 = T and T2 = T + dT and dt->0, we get:-

$$dF(t,T) = v(t,T,\Omega_t)\,v_T(t,T,\Omega_t)\,dt - v_T(t,T,\Omega_t)\,dz(t)$$

DERIVATION

- Integrating Vt from t to T we get :-

$$v(t, T, \Omega_t) = \int_t^T v_\tau(t, \tau, \Omega_t)\, d\tau$$

- If m(t,T,Ω) and s(t,T,Ω) are the instantaneous drift and standard deviation of F(t,T), so that

$$dF(t, T) = m(t, T, \Omega_t)\, dt + s(t, T, \Omega_t)\, dz$$

Derivation

- So from the previous equation it follows that ->

$$m(t, T, \Omega_t) = s(t, T, \Omega_t) \int_t^T s(t, \tau, \Omega_t)\, d\tau$$

- This is the HJM result.

Extension to Several Factors

- The HJM result can be extended to the situation where there are several independent factors. Suppose

$$dF(t, T) = m(t, T, \Omega_t)\, dt + \sum_k s_k(t, T, \Omega_t)\, dz_k$$

- Then a similar analysis shows that ->

$$m(t, T, \Omega_t) = \sum_k s_k(t, T, \Omega_t) \int_t^T s_k(t, \tau, \Omega_t)\, d\tau$$

Finding Volatility Functions

- For this we use PCA
- First we calculate the covariance matrix which contains the covariance of prices of each pair of different tenors.
- Then, we calculate eigen values and corresponding eigen vectors from the covariance matrix.
- According to PCA theory, the highest three eigen values represent most of the variations in our data

Finding Volatility Functions

- The Volatility Matrix is Given by ->

$$\bar{v}(\tau j) = \sqrt{\lambda_i}(v_i)j$$

- Next we find drift from the previously derived HJM result ->

$$m(t, T, \Omega_t) = \sum_k s_k(t, T, \Omega_t) \int_t^T s_k(t, \tau, \Omega_t)\, d\tau$$

- For integration we find a function which fits the volatilities.

Simulating Forward Curves

- We simulate by ->

$$f(t + dt) = f(t) + d\bar{f}$$

$$d\bar{f} = m(t)dt + \sum(v_i * \phi * \sqrt{dt}) + \frac{dF}{d\tau}dt$$