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We revisit the issue of finding proper boundary conditions for the field equations describing incompressible flow problems, for quantities like pressure or vorticity, which often do not have immediately obvious "physical" boundary conditions. Most of the issues are discussed for the example of a primitive-variables formulation of the incompressible Navier-Stokes equations in the form of momentum equations plus the pressure Poisson equation. However, analogous problems also exist in other formulations, some of which are briefly reviewed as well. This review article cites 95 references.
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1 Introduction

There is an interesting controversy that comes up time and again in the literature on the numerical solution of the Navier-Stokes equations for incompressible flow when it comes to the proper definition of certain kinds of boundary conditions. This debate arises most often in connection with the problem of finding boundary conditions for pressure for a Poisson equation, which we are dealing with, in detail, below. For example, in their book on computational fluid mechanics, Peyret and Taylor [1] identify the specification of pressure boundary conditions as a "primary difficulty," and in his review article Ferziger [2], calls the problem of posing proper pressure boundary conditions an "open question." What is particularly surprising about this issue is not so much that one can sometimes find contributions to that topic in the literature that are erroneous, but that such errors still persist in some of the more recent references and have even found their way into current textbooks.

A main motivation for this paper is to present a critical review of the literature on boundary conditions for incompressible viscous flow, in order to recall both what is known about the subject and also some things that seem to have been forgotten or have remained in relative obscurity. We will see that all but hidden among this material are also some quite interesting and maybe unexpected conclusions about the nature of certain solutions to incompressible Navier-Stokes problems.

The structure of this paper is as follows: For later reference, we start by presenting the fundamental equations that describe viscous, incompressible flow. The difficulty of finding proper and computationally practical boundary conditions for some of these equations will be illustrated. The associated complications were driving the development of various alternative representations of the fundamental equations, in a quest for formulations that would allow for a straightforward implementation of boundary conditions. We will therefore present a few of the more important formulations as measured by their frequency of use in methods for the numerical solution of incompressible viscous flow problems. After having set the stage in this way, we will review what we consider important contributions in the literature to the question of how to pose proper boundary conditions for the various formulations. In this review part of our paper, certain types of boundary conditions will be labeled as—and proven to be—improper, the fact that they have often seen widespread use in the computational fluid dynamics community notwithstanding. In particular, one often finds attempts to simply write down the governing differential equations at the boundaries in order to generate missing boundary conditions, which results in formally ill-posed problems. We present a pair of examples that demonstrates that and why such formulations can lead to wrong results. In order to not deviate too far from the format of a review article, some auxiliary material has been relegated to Appendix A of this paper. Additionally, Appendix B contains some algebraic details for one of the examples mentioned above.

It is important to emphasize at this point that there were others before us who have recognized the crucial role that boundary conditions play in certain formulations of the incompressible Navier-Stokes equations. We refer the reader, in particular, to the excellent book by Quartapelle [3], which, despite its innocuous title, focuses heavily on exactly these issues. On the other hand, some of the results that we are presenting in Appendix A are new, and to the best of this author's knowledge, the formulation of the pressure gradient at the wall using the interior limit rather than the boundary values of the velocity field proposed in Sec. 3.2.2 seems to be original, too.

2 Fundamental Equations for Incompressible Flow

2.1 Velocity-Based Formulations. As the most straightforward form of the fundamental equations for incompressible flow, we start by introducing the set of equations that express conservation of mass and momentum, leading to a formulation of the Navier-Stokes equations in what is known as "primitive variables." The differential equations describing incompressible flow problems are then given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = - \nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u} \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

where $x \in \Omega$ is the vector of Cartesian coordinates, $\boldsymbol{u}$ is the velocity vector, p the pressure divided by the constant density ρ of the fluid, and Ω is the spatial domain of interest, given as an open subset of $\mathbb{R}^3$, which can be bounded or unbounded in general. Re denotes the Reynolds number. The unsteady problem described by (1) and (2) is parabolic in time, and the corresponding steady-state problem is of elliptic character in space. Thus, for a complete description of a particular problem, the above equations need to be complemented to describe an initial/boundary value problem

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(IBVP). One way to do so is by specifying an initial condition $\boldsymbol{u}_0$, which we assume to be C^∞ in all of Ω,

$$\boldsymbol{u}(\boldsymbol{x},t_0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \text{with } \boldsymbol{\nabla} \cdot \boldsymbol{u}_0 = 0, \quad \boldsymbol{x} \in \Omega \tag{3}$$

and by giving boundary conditions

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_\Gamma(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Gamma \tag{4}$$

satisfying the global mass conservation constraint

$$\oint_\Gamma \boldsymbol{\eta} \cdot \boldsymbol{u}_\Gamma d\sigma = 0 \tag{5}$$

on the boundary $\Gamma = \partial\Omega$ of the domain, where $\boldsymbol{\eta}$ is a vector normal to the boundary. We note that there are other, more general ways to pose boundary conditions for incompressible Navier-Stokes problems. In particular, it is possible to replace boundary conditions specifying the full velocity vector by conditions that specify the pressure plus a projection of the velocity vector in the plane tangential to the boundary, see [4] and the book by Quartapelle (Chap. 5.7 in [3]) for details. In the following, we will restrict ourselves to simple velocity boundary conditions as given in (4).

One of the interesting features of the set of equations (1) and (2) appears in the form of the pressure term on the right-hand side of (1). What is somewhat remarkable about this term is that it contains a variable, the pressure p, for which no explicit equation is given. In fact, p appears as a Lagrange multiplier in these equations, whose major function it is to ensure that the velocity field $\boldsymbol{u}$ remains incompressible, satisfying (2) at all times. We will see later, in much more detail, that the pressure in incompressible flow is indeed a most unusual quantity, which exhibits some quite surprising properties. For later, the reader should keep in mind the fact that the pressure in an incompressible flow is not a thermodynamic quantity and only partially corresponds to the physical intuition we ordinarily associate with thermodynamic pressure.

Because of the somewhat unusual way in which the pressure appears in the system of partial differential equations (1) and (2), in computational methods for the solution of incompressible Navier-Stokes problems, often an alternative set of differential equations is used. The motivation is to obtain an explicit equation for the pressure as part of the set of equations to be solved. This is done in the hope of breaking up the intricate coupling between pressure and velocities that exists in the original set of equations. This coupling usually leads to numerical algorithms that require the solution of large simultaneous systems of equations involving all velocity components and the pressure as unknowns. It is clear that the numerical equations resulting from a Navier-Stokes problem will, in general, be coupled through the nonlinear terms in the equations, but this nonlinear coupling is routinely eliminated in the substeps of many numerical algorithms. This is true, in particular, for the large class of time-explicit or semi-explicit methods. Thus, the objective is to devise a formulation of the fundamental equations that lends itself to the construction of a numerical scheme that decouples the equations for the unknowns. This program leads in a straightforward way to what is known as the pressure-Poisson equation formulation (or PPE formulation) of the Navier-Stokes equations. To find it, we take the divergence of (1) and substitute in (2) to obtain the Poisson equation

$$\boldsymbol{\nabla}^2 p = -\boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}). \tag{6}$$

We then regard (1) and (6), rather than (1) and (2), as the differential equations describing the problem of incompressible flow. As before, the set of these two differential equations must be complemented by initial and boundary conditions. It is clear that we can, and must, still use (3)–(5) for our boundary conditions, but now we need an additional condition since the order of our system of equations has increased by one when we replaced the first-order equation (2) by the second-order equation (6). As we will see in more detail below, there are a variety of ways to obtain such an additional condition that lead to mathematically equivalent formulations.

However, these formulations can differ greatly in the computational effort they require for their numerical solution. In order to allow for an efficient numerical solution, it is highly desirable to construct a so-called decoupled formulation or split formulation [3] of the differential equations, such that, ideally, each of the individual partial differential equations of the system is accompanied by "its own" boundary conditions. If this is not possible, then the equations for the three velocity components and the pressure will be coupled through their boundary conditions, which would defeat the purpose of splitting off a separate equation for the pressure. For the case of our PPE formulation, we would therefore like to find boundary conditions for the pressure in the Poisson equation (6). The problem we now face is that while velocity boundary conditions are usually easy to obtain and their physical meaning is intuitively clear, this is not the case for pressure boundary conditions. We will see below that finding a proper way to pose such boundary conditions is far from trivial and presents a major obstacle in the way of working with a PPE formulation of the Navier-Stokes equations.

2.2 Vorticity-Based Formulations. Since the problem of finding appropriate boundary conditions for the pressure in primitive-variables formulations of the Navier-Stokes equations was realized early on, many researchers have sought alternative formulations that would allow them to get rid of the pressure. An obvious method to eliminate the pressure is to take the curl of the momentum equations, which results in formulations of the Navier-Stokes equations that are based on vorticity. This strategy leads to the vorticity-transport equation,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \frac{1}{\text{Re}}\boldsymbol{\nabla}^2\boldsymbol{\omega} \tag{7}$$

where $\boldsymbol{\omega}$ is the vorticity vector defined by

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}. \tag{8}$$

Obviously, (7) is not sufficient to specify a solution. We need at least one more set of partial differential equations that describe the velocity vector $\boldsymbol{u}$. One such set of equations can be found by taking the curl of the vorticity definition (8)

$$\boldsymbol{\nabla}^2 \boldsymbol{u} = -\boldsymbol{\nabla} \times \boldsymbol{\omega} \tag{9}$$

where we made use of the identity

$$\boldsymbol{\nabla}^2 \boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{u}) \tag{10}$$

and we used the fact that $\boldsymbol{u}$ is solenoidal. We note in passing that instead of using (9) it is possible to complement (7) with (2) and (8) [5–8]. In this formulation, now the task is to solve incompressible flow problems by finding solutions to (7) and (9), subject to appropriate initial and boundary conditions. It is again clear that our velocity boundary conditions (4) are not sufficient to uniquely specify the solution to a problem specified by (7) and (9). We also need initial conditions for the vorticity that appears in the parabolic vorticity-transport equation (7). These are straightforward to obtain simply by taking the curl of the initial condition (3) for the velocity. However, since we have again increased the order of our system, we need additional conditions in order to determine a unique solution. Just like for the PPE formulation above, in order to obtain a formulation that lends itself to efficient numerical solution, we would like to find a set of boundary conditions for the vorticity in (7), which are much less obvious.

Another vorticity-based formulation of the fundamental equations that has enjoyed significant popularity is obtained by choosing vorticity and a stream function as primary variables. Because of the difficulty of using this approach in a three-dimensional setting, this formulation is almost exclusively used in the treatment of two-dimensional problems (but see [9] for a discussion of an extension of the stream function approach to three dimensions). In that case, the governing set of differential equations is

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = \frac{1}{\mathrm{Re}} \boldsymbol{\nabla}^2 \omega \tag{11}$$

$$\boldsymbol{\nabla}^2 \psi = \omega \tag{12}$$

where, in Cartesian coordinates, the stream function ψ is defined via

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{13}$$

where ω is the single component of vorticity that can appear in two-dimensional flow, and u and v are the x and y components of the velocity vector $\boldsymbol{u}$, respectively. In this case, we can derive boundary-conditions for derivatives of the stream function ψ from the velocity boundary conditions (4) using (13). Just as in the case of the vorticity-transport formulation above, however, we would like to find boundary conditions for vorticity, which is a difficult problem, as we will see.

3 Strategies for the Computational Solution of Incompressible Flow Problems

In Sec. 2, we have described four alternative sets of partial differential equations describing the motion of an incompressible fluid. The sets of differential equations were:

- primitive-variables formulation, constituted by (1) and (2)
- pressure Poisson equation formulation, (1) and (6)
- vorticity-transport formulation, (7) and (9)
- vorticity-stream function formulation, (11) and (12)

The formulations we have chosen to introduce above are the ones that are most frequently used as the bases for the solution of flow problems. To find solutions to these sets of equations, in almost all cases, a numerical method must be developed that allows us to find approximate solutions. The task of developing a viable numerical method for the solution of the fundamental equations of viscous incompressible fluid dynamics can be split up into two subproblems:

1. We need to develop a numerical discretization that allows us to transform the infinite-dimensional problem of finding a solution to a set of differential equations into a finite (albeit usually high-dimensional) problem that can be solved in finite time and with finite resources.

2. We need to find and numerically express the boundary conditions of the problem in such a way that the solution we obtain will, in fact, represent the solution to an incompressible Navier-Stokes problem.

In the literature, the focus in developing numerical methods has often been very heavily on the first problem, whereas the second one has sometimes only received cursory attention. However, a proper method for posing and implementing boundary conditions for quantities like pressure or vorticity is absolutely crucial. The four formulations we have given above are equivalent *only* if proper boundary conditions are implemented. It is because of the crucial role that boundary conditions play in many formulations of Navier-Stokes problems that this paper will focus exclusively on these issues.

3.1 Formulation in Primitive Variables: Fractional Step Methods. If we simply write down the equations of conservation of momentum and mass, we obtain with the so-called primitive variables formulation of the Navier-Stokes equations the most straightforward mathematical representation of incompressible viscous flow, given by (1)–(5). The first attempts to develop computational methods for the solution of incompressible flow problems were correspondingly based on this formulation. What distinguishes this representation from all other formulations is that it is the only one that requires boundary conditions only for quantities that have an immediate physical meaning and that are part of the specification of the physical problem to be solved. In most cases, one simply specifies the velocity vector everywhere on the boundary, but more complex boundary conditions involving combinations of velocity and pressure as discussed in [4,10] are possible. Thus, among the four different forms of the equations for incompressible flow we listed above, this formulation is also unique in that it is the only one of the four that allows a straightforward implementation of boundary conditions.

So why are we including this formulation in a paper focusing on the difficulties of finding correct boundary conditions for incompressible flow problems? The answer is that boundary condition issues still arise in one of the most popular strategies, known as the *fractional step method*. Before we describe this method and the difficulties it may present in more detail below, we hasten to add that fractional step methods are not the only possible choice for the computational solution of (1)–(5). It is possible to directly discretize (1) and (2) to obtain a fully coupled system of equations for the velocities and pressure. Although this approach is not unusual in the finite element community [11–13], it is only rarely used in conjunction with other numerical discretizations. This reluctance is due to the fact that the pressure couples with the other unknowns of the numerical scheme in such a way that instead of a compact diagonal or block-diagonal structure of the systems of equations that is typical for, e.g., finite difference methods, one obtains matrices of a more complex structure, leading to more cumbersome and slower solution algorithms. For this reason, outside of the finite element community only few codes solve a coupled system of equations for pressure and velocities, but see [14,15] for exceptions to this rule.

The fractional step method, also known as the "projection method," or "fractional step projection method," avoids this problem by using an approach that is known as "operator splitting" [16]. The method was first introduced in 1968–1969 for the solution of incompressible Navier-Stokes problems by Chorin [17–19] and, independently and based on a more rigorous foundation, by Temam [20–23]. The basic approach taken by this method consists of first calculating an intermediate solution $\boldsymbol{u}^{n+1^*}$ for the time step $n+1$ that is based on the momentum equation (1), but with the pressure term omitted. Thus, in a simple implementation of a fractional step method, one would find a solution to the boundary-value problem

$$\frac{\boldsymbol{u}^{n+1^*} - \boldsymbol{u}^n}{\Delta t} = -(\boldsymbol{u}^n \cdot \boldsymbol{\nabla})\boldsymbol{u}^n + \frac{1}{\mathrm{Re}} \boldsymbol{\nabla}^2 \boldsymbol{u}^n, \quad \boldsymbol{x} \in \Omega \tag{14}$$

$$\boldsymbol{u}^{n+1^*}|_\Gamma = \boldsymbol{u}_\Gamma(\boldsymbol{x}, t_{n+1}), \quad \boldsymbol{x} \in \Gamma. \tag{15}$$

The intermediate velocity field $\boldsymbol{u}^{n+1^*}$ obtained this way will, in general, not be divergence-free. The next substep of the fractional step method is designed to restore a divergence-free velocity field by introducing a "projector" Π that is specified by

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1^*}}{\Delta t} = -\boldsymbol{\nabla} \Pi^{n+1}, \quad \boldsymbol{x} \in \Omega, \tag{16}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}^{n+1} = 0, \quad \boldsymbol{\eta} \cdot \boldsymbol{u}^{n+1}|_\Gamma = \boldsymbol{\eta} \cdot \boldsymbol{u}_\Gamma(\boldsymbol{x}, t_{n+1}), \quad \boldsymbol{x} \in \Gamma. \tag{17}$$

Three properties of this method are worth noting at this point:

1. The projector Π is not identical with the pressure p that appears in (1) and differs from the latter in a systematic manner [24,23]. The term Π is a numerical quantity that accounts for the difference between the divergence-free solution $\boldsymbol{u}^{n+1}$ and the intermediate velocity field $\boldsymbol{u}^{n+1}$ over one time step of a fractional step scheme. Often, Π is referred to as a "pseudopressure," and sometimes it is even simply referred to—improperly, in this author's opinion—as the "pressure."

2. In the second substep of the fractional step method, one cannot enforce the full velocity boundary condition, instead, only

the wall-normal component of the velocity boundary condition can be specified. One way to see this is to recognize that (16) and (17) is of the same form as a potential flow problem. Formally, the decomposition into the two sets of equations (14) and (15) and (16) and (17) that the fractional step method is based on follows from the orthogonal decomposition theorem by Ladyzhenskaya [25] (see also [26]). Note that this also means that $\boldsymbol{u}^{n+1}$ will usually not satisfy the tangential velocity boundary condition exactly.

3. The fractional step method is not well suited for the solution of stationary problems: If a steady-state solution exists, the value given by the fractional step method depends on the time step Δt that is chosen.

The reader may have noted that (16) and (17) still do not directly allow us to determine $\boldsymbol{u}^{n+1}$, since they contain the new unknown Π^{n+1}. However, similar to the way in which (6) was derived, we can obtain a Poisson equation for Π by taking the divergence of (16) and making use of (17), which gives the Poisson equation

$$\nabla^2\Pi^{n+1} = \frac{\nabla \cdot \boldsymbol{u}^{n+1^*}}{\Delta t}. \tag{18}$$

By considering the boundary condition (15) for the first half step and (17) for the second half-step, both of which specify that the wall-normal component of the velocity at the boundary must not change, we find that the proper boundary condition for this Poisson equation is given by

$$\boldsymbol{\eta} \cdot \nabla\Pi^{n+1}|_\Gamma = 0. \tag{19}$$

Again, and in anticipation of what will be said later, we stress the fact that Π in this method is an artificial, numerical quantity and should not be confused with the pressure p in (1) [24,23]. In particular, the question of what boundary conditions are to be used for (6) is different, and has a different answer, from the question for the boundary condition for (18). This is obvious from (19): The wall-normal gradient of the true pressure in incompressible flow is nonzero, in general, as will become clear below (see Sec. 3.2.2).

The first practical computational solutions of incompressible flow problems using this approach have been presented by Chorin [18], in which he in fact, did not solve the Poisson problem (18) explicitly but rather used an iteration scheme that allows him to find Π without explicitly specifying boundary conditions for this quantity. The difference between the quantity Π and the physical pressure p does not always seem to have been understood in the first applications of fractional step methods. Thus, Chorin does not seem to have realized the difference, and Alfrink [27] attempts to implement more "physically meaningful" boundary conditions for the (nonphysical) Π. A similar misconception also appears in a paper by Gustafson and Halasi [28].

In contrast, more detailed analyses of the properties of fractional step methods appear in the paper by Orszag et al. [29], and Marcus [30] develops a variant of the fractional step method that uses three substeps in conjunction with a capacitance-matrix method that actually calculates the correct pressure p rather than Π. In addition, Kim and Moin [24] seem to be the first to explicitly distinguish the quantity Π from the pressure p. They also suggest improved velocity boundary conditions, giving second-order accuracy in time for the velocity field. A version of the same method with improved accuracy and better efficiency is reported by Le and Moin [31], who present a time-stepping scheme that solves the Poisson equation only in one of the three Runge-Kutta substeps. It is noteworthy that as late as 1991, Temam sees the need to publish a paper [23] clarifying the relation between Π and p, and significant advances in the theoretical understanding of fractional step methods have been made in the early 1990s—more than 25 years after they had been introduced—by contributions of Dukowicz and Dvinsky [32] and Perot [33] (see also [34]). These authors stress the fact that fractional step methods must be analyzed from a discrete point of view and that not taking into account that fundamental aspect of the method leads to various misunderstandings. An in-depth analysis of the accuracy of fractional step methods can be found in the papers by E and Liu [35,36], and by Strikwerda and Lee [37]. Focusing on the fractional step method by Kim and Moin [24], the latter demonstrate that the pressure in any projection method can at best reach first-order accuracy in time. A comparison of three alternative versions of fractional step methods that use three different types of methods for the solution of the pseudo-pressure equation is presented by Armfield and Street [38].

3.2 Pressure-Poisson Equation Formulation. As an alternative to the solution of the set of partial differential equations given by the momentum and continuity equations, (1) and (2), one can try to solve an equivalent formulation of the flow equations given by the momentum equation (1) in conjunction with the Poisson equation for the pressure (6). While this alternative formulation is perfectly legitimate, it is important to note that the resulting "PPE formulation" of the incompressible Navier-Stokes equations is equivalent to the set of Eqs. (1) and (2) with initial and boundary conditions given by (3)–(5), *if and only if* proper boundary conditions for the pressure Poisson equation (6) are specified. In particular, as we both formally prove and demonstrate with a numerical example below, it is illegal to write down the momentum equation (1) taken at the boundary and derive a pressure boundary condition from it by simply projecting the result on the wall-normal coordinate η.

In fact, as we will show below, such a procedure would be fully analogous to the attempt of solving the simple linear ordinary differential equation (ODE) $y'(x)=y(x), x\geqslant 0$, by supplying the equation $y'(0)=y(0)$ as an "initial condition." Unfortunately, and somewhat surprisingly, while the flaw of this type of an approach is universally acknowledged for the case of the ODE above, the exact same practice is often vigorously defended in the more complex case of the solution of incompressible flow problems.

Many numerical schemes that are based either on finite-difference approximations or on spectral methods, in fact, do use (6) rather than the continuity equation (2) and are thus based on momentum balance and the pressure Poisson equation rather than momentum and mass balances. In Sec. 2, we have referred to such formulations as PPE formulations of the Navier-Stokes equations. As in the case of the system (1)–(5), we need to specify initial conditions for the velocity in all of Ω and boundary conditions for the velocity vector, everywhere on the boundary Γ. In addition, however, for a PPE formulation, which represents a system of partial differential equations that is of higher order than the original system (1)–(5), we now need an additional equation to specify a unique solution. One straightforward—and desirable—approach to provide such an equation is to pose boundary conditions for the pressure everywhere on Γ (see Secs. 3.2.4, 3.2.6 for alternative approaches).

The exploration of this problem of finding appropriate boundary conditions for the pressure forms the core of the present paper. As mentioned in our introduction, in addressing this question we are dealing with a highly controversial issue. In order to do so, and to fully support our particular judgments as to the propriety or lack thereof of certain approaches, the following sections will deviate somewhat from the standard format of a review article. Drawing on some results that are described in more detail in Appendix A, we will first present an in-depth study of the issue at hand, including a pair of theorems and two examples that illustrate them, in the following Secs. 3.2.1–3.2.5, before we return to our review of the literature in Sec. 3.2.6.

3.2.1 On "Indirect" Boundary Conditions. The problem we now face is that although velocity boundary conditions are usually easy to obtain and their physical meaning is intuitively clear, this is not the case for pressure boundary conditions. As we will show below, pressure boundary conditions, in fact, can only be obtained indirectly, in such a way that certain additional velocity boundary conditions are met.

We mentioned in Sec. 2.1 that one of the primary functions of the pressure field in incompressible flow is to make sure that the flow will satisfy the continuity equation 2. In order to see more clearly how this is accomplished, let us first take a closer look at what the combination of (1) and (6) really tells us about the divergence of the velocity field.

If we take (6) and insert it into the divergence of (1), we obtain

$$\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{u} = \frac{1}{\mathrm{Re}}\nabla^2(\nabla \cdot \boldsymbol{u}) \tag{20}$$

in other words, the combination of (1) and (6) does *not* enforce the incompressibility of the velocity field. All these equations do is specify that the divergence of the velocity field satisfies a heat equation and is a harmonic function for the steady-state case. From the extremum theorems for harmonic functions [39], which tell us that harmonic functions assume their extremal values on the boundary Γ, we can then conclude that the divergence of our velocity field will be zero throughout our domain if we start with a divergence-free initial condition, and *if and only if* we make sure that the divergence is zero on the boundary. We can thus formulate as follows:

THEOREM 1. *The solution of the incompressible Navier-Stokes problem formulated using the momentum equation* (1) *and a pressure Poisson equation* (6) *is equivalent to the solution of the problem* (1) and (2) *with identical initial and boundary conditions for the velocity, if and only if the boundary conditions for the pressure Poisson equation are chosen such that* $\nabla \cdot \boldsymbol{u}=0$ *holds on the boundary*.

Proof. The proof follows directly from the above remarks. ■

We first note that, of course, the condition $\nabla \cdot \boldsymbol{u}=0$ on Γ by itself already constitutes a perfectly good boundary condition for the system of Eq. (1) and (3)–(6). However, for the practical reasons outlined above, we really would like to find a boundary condition that is expressed as an equation for the pressure on the boundary. In this context, Theorem 1 simply means that we must choose this boundary condition in such a way that the divergence of the velocity field is zero at the boundary. This is an example of what we call an "indirect boundary condition": The boundary condition for a certain dependent variable (pressure in this case) must be chosen such that some boundary condition for a different dependent variable ($\nabla \cdot \boldsymbol{u}=0$ on Γ, a boundary condition for $\boldsymbol{u}$) can be met. Another way to look at this is to say that the velocity field needs to meet boundary condition both on the velocity itself as well as on the velocity divergence. The second-order partial differential equation (PDE) (1) will, in general, not be able to satisfy this overspecified problem; therefore, a pressure field must be found such that the unique solution of (1) satisfies the additional constraint on the velocity divergence. If this is not explicitly enforced, then the resulting velocity field will, in general, not be a solution of the incompressible Navier-Stokes equations.

3.2.2 How Not to Define Pressure Boundary Conditions. Following the discussion above, we can now frame the problem of finding proper boundary conditions for pressure in a more succinct form: What we have to do is find proper boundary conditions for the pressure such that the velocity divergence becomes zero at the boundaries. In response to that problem, the following "solution" can be found in a surprising number of places in the literature: We project (1) on the normal vector $\boldsymbol{\eta}$ at the boundaries and insert the given velocity boundary conditions (e.g., $\boldsymbol{u}=0$ at a solid "no-slip" boundary) to obtain a Neumann condition of the form $\partial p/\partial\eta=$RHS, where η is the coordinate normal to the boundary, and the right-hand side (RHS) depends on the boundary conditions for $\boldsymbol{u}$. Thus this boundary condition can be written as

$$\frac{\partial p}{\partial \eta}(\boldsymbol{x},t) = \boldsymbol{\eta} \cdot \left[-\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\mathrm{Re}}\nabla^2 \boldsymbol{u} \right](\boldsymbol{x},t), \quad \boldsymbol{x} \in \Gamma. \tag{21}$$

If we take the common case of a solid, stationary wall with no-slip boundary conditions for the velocity, this condition takes the particularly simple form

$$\frac{\partial p}{\partial \eta}(\boldsymbol{x},t) = \boldsymbol{\eta} \cdot \frac{1}{\mathrm{Re}}\nabla^2 \boldsymbol{u}(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Gamma. \tag{22}$$

Unfortunately, there are a number of problems with the boundary condition (21):

- First, as becomes clear from the proof of Lemma 2 in Appendix A.2, the projection on the wall normal $\boldsymbol{\eta}$ is not an obvious step. There is a priori no reason why the equation should be projected in that particular direction. We will come back to this issue in our example in Sec. 3.2.5.
- Second, in the general unsteady case, where both $\boldsymbol{u}$ and $\partial \boldsymbol{u}/\partial t$ may be discontinuous at Γ, this boundary condition is incorrect (see Corollary 2 in Appendix A.2 and the discussion below).
- Third, and the above discussion suggests this already, if we follow the procedure for obtaining the pressure at the boundary as proposed above, then the resulting set of differential equations plus boundary conditions represents an *ill-posed* problem. In particular, we can see that the divergence-free condition, $\nabla \cdot \boldsymbol{u}=0$, appears nowhere in our set of equations. We will show this rigorously below, but, in principle, the major error that was committed is easy to spot: To close a differential problem, the boundary conditions need to provide some *additional* information (that is not already contained in the field equations), whereas the "trick" that is described above simply specifies that the differential equation that needs to be satisfied in the field also be met at the boundary, which is trivial and does *not* provide the required additional information, $\nabla \cdot \boldsymbol{u}_{wall}=0$. Thus, indeed, the above choice of pressure boundary conditions is analogous to trying to solve the ODE problem $y'(x)=y(x), x \geqslant 0$, with an initial condition of $y'(0)=y(0)$.

Before providing a proof for the ill-posedness of the above procedure, we need to correct the second of the problems mentioned above. So let us look at a situation where the velocity field and/or its time derivative are discontinuous at the boundary. As discussed in Appendix A.2, this case arises commonly when we start our calculation from some initial condition: At $t=t_0$, $\partial \boldsymbol{u}/\partial t$ is usually, and $\boldsymbol{u}$ is often, discontinuous at the boundary (see also Sec. 3.2.5). Thus the question is, which value of the velocity and its time derivative should we insert into (21) in order to find the value of the normal pressure gradient: Should we use the values given by the velocity boundary conditions, $\boldsymbol{u}(\boldsymbol{x} \in \Gamma)$ and $(\partial \boldsymbol{u}/\partial t)(\boldsymbol{x} \in \Gamma)$ as suggested above, or should we use $\lim_{\boldsymbol{x}\to\Gamma}(\boldsymbol{u})$ and $\lim_{\boldsymbol{x}\to\Gamma}(\partial \boldsymbol{u}/\partial t)$? From the discussion in Sec. A.2, we can conclude that the correct answer to this question is the latter choice. There are two ways to see this: First, from Corollary 2 in Appendix A it follows that (21), in general, does not have a solution at $t=t_0$ (or, indeed, whenever $\partial \boldsymbol{u}/\partial t$ and/or $\boldsymbol{u}$ are discontinuous at the boundary). Second, since the pressure field follows from the solution of a Poisson equation, the pressure itself will be more regular than the right-hand side of that equation. In particular, if we assume that the given velocity field is C^∞ in Ω, then the pressure field will be C^∞ in $\bar{\Omega}=\Omega \cup \Gamma$. It follows that the pressure gradient that should be used in (21) is equal to the interior limit of the right-hand side of (21), and not equal to its boundary value. We will demonstrate that this is so with an example below (first example in Sec. 3.2.5).

Now we return to the issue of ill-posedness. We make the above statement on that subject more rigorous by formulating the following

THEOREM 2. *The IBVP defined for* $\boldsymbol{x} \in \Omega$ *by the field equations*

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u} \tag{23}$$

$$\nabla^2 p = -\nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \tag{24}$$

with velocity boundary conditions

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_\Gamma(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Gamma \tag{25}$$

satisfying the global mass conservation constraint

$$\oint_\Gamma \boldsymbol{\eta} \cdot \boldsymbol{u}_\Gamma d\sigma = 0 \tag{26}$$

and with pressure boundary conditions derived from taking the normal component of (23) *at the boundary*,

$$\frac{\partial p}{\partial \boldsymbol{\eta}}(\boldsymbol{x},t) = \boldsymbol{\eta} \cdot \lim_{\boldsymbol{x}^* \to \Gamma} \left[-\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u} \right] (\boldsymbol{x}^*, t), \quad x \in \Gamma, \tag{27}$$

with an initial condition specified by

$$\boldsymbol{u}(\boldsymbol{x},t_0) = \boldsymbol{u}_0(\boldsymbol{x}) \tag{28}$$

where $\boldsymbol{u}_0$ *is divergence-free and* $\boldsymbol{\eta} \cdot \boldsymbol{u}_0 = \boldsymbol{\eta} \cdot \boldsymbol{u}\Gamma$ *on the boundary*, *is ill-posed and admits an infinite number of solutions*.

Proof. To prove the theorem, we consider a modified IBVP that replaces the pressure boundary condition (27) with a Dirichlet condition,

$$p(\boldsymbol{x},t) = p_\Gamma(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Gamma \tag{29}$$

where p_Γ is completely arbitrary and does not depend on $\boldsymbol{u}$. In this case, we have an independent boundary condition for (24), and the resulting IBVP is well-posed and admits unique solutions for any given p_Γ. Furthermore, it is clear that the solution will, in general, be different for different choices of p_Γ. To conclude our proof we only need to observe that any such solution will also identically satisfy the set of Eqs. (23)–(28).

A number of remarks on the above theorem are in order:

- For generality, we have formulated the theorem for the unsteady Navier-Stokes equation. It is clear that this statement also covers the steady equations. Mathematically, one can easily see how the above proof carries through, *mutatis mutandis*, for the steady case as well.
- The formulation of the boundary condition (27) in terms of the interior limit of the right-hand side of that equation was also introduced for generality, in order to cover the case of discontinuous fields at the boundary. If we have velocity fields (and time derivatives) that are continuous on the closed domain, the limit in (27) can simply be omitted. This will usually be the case for solutions of the steady Navier-Stokes equations.

Now, at this point we could stop and state that everything that needs to be said about using a field equation to derive boundary conditions for a problem has been said: such a procedure generally results in ill-posedness. However, there are a couple of more subtle points to consider, among other things because we also should discuss "amended" forms of the above equations that have been suggested. For example, it was suggested to use a so-called consistent Poisson equation [40,12] for the pressure instead of (6), which is obtained by leaving the velocity-divergence term in the Poisson equation, giving

$$\nabla^2 p = -\nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \frac{1}{\mathrm{Re}} \nabla^2 (\nabla \cdot \boldsymbol{u}). \tag{30}$$

The advantage of this modification seems to that if we substitute (30) into the divergence of (1) just as we did above for the case of (6), then, instead of the second-order PDE (20) we obtain

$$\frac{\partial}{\partial t} \nabla \cdot \boldsymbol{u} = 0 \tag{31}$$

which is a much simpler equation. Thus, now one might conclude that if we start from a solenoidal velocity field as an initial condition (which is something we have to do anyway for reasons of consistency), then the divergence of the velocity field should stay zero at all times. To invalidate this argument, it is sufficient to observe that, in general, $\nabla \cdot \boldsymbol{u}$ can be discontinuous (a close inspection of the examples given in Sec. 3.2.5 reveals that, indeed, $\nabla \cdot \boldsymbol{u}$ is discontinuous when improper boundary conditions are used, also see Lemma 2. Thus, for $t > t_0$, $\nabla \cdot \boldsymbol{u}$ can be nonzero (and constant in time). At this point, one is then in a situation where (1), (30) do not represent a solution of the incompressible Navier-Stokes equations anymore since (30) is equivalent to (6) only if $\nabla \cdot \boldsymbol{u} = 0$.

More importantly, we can see that the proof of Theorem 2 is not at all affected by the change in (30): Just as for Theorem 2, we find that *any* solution that follows from an arbitrary pressure boundary condition would satisfy the proposed set of equations, demonstrating ill-posedness. Thus, we can formulate the following corollary to the above theorem:

COROLLARY 1. *The IBVP that is obtained by substituting* (30) *for* (24) *in the system* (23)–(28) *is ill-posed and admits an infinite number of solutions*.

Proof. The proof is the same as the one for the above theorem; any solution of the modified problem, with arbitrary Dirichlet boundary conditions for the pressure, will satisfy the above IBVP. ■

3.2.3 How to Obtain Proper Boundary Conditions for Pressure. We do not want to leave our readers completely without an answer to the question of how to pose *correct* pressure boundary conditions. In order to give one such answer, we outline the fundamentals of the influence matrix method [41,42] as one of several possible methods to obtain proper pressure boundary conditions. An alternative but equivalent approach is described in Chap. 5 of Quartapelle's book [3]. For the influence matrix method, we assume that we want to prescribe Dirichlet conditions for the pressure at the boundary

$$p(x) = p_\Gamma, \quad \boldsymbol{x} \in \Gamma \tag{32}$$

which we will apply to the Poisson problem

$$\nabla^2 p = \mathcal{R} \tag{33}$$

where $\mathcal{R} = -\nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})$. We now assume that we know the associated Green's function G of that problem, which will satisfy

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{\xi}), \quad \boldsymbol{x}, \boldsymbol{\xi} \in \Omega, \quad G = 0 \text{ on } \Gamma. \tag{34}$$

Of course, finding this Green's function is not always trivial, but we will see that it is not necessary to have an explicit expression for G. With G given, the solution to the Poisson problem can be written as the integral

$$p(\boldsymbol{x}) = \int_\Omega \mathcal{R}(\boldsymbol{\xi}) G(\boldsymbol{x}, \boldsymbol{\xi}) \mathrm{d}\Omega_\xi + \oint_\Gamma p_\Gamma(\boldsymbol{\xi}) \frac{\partial G}{\partial \eta_\xi}(\boldsymbol{x}, \boldsymbol{\xi}) \mathrm{d}\sigma_\xi. \tag{35}$$

Thus, the pressure gradient that appears in our Navier-Stokes equations may be written as

$$\nabla p(\boldsymbol{x}) = \int_\Omega \mathcal{R}(\boldsymbol{\xi})\nabla_x G(\boldsymbol{x},\boldsymbol{\xi})\mathrm{d}\Omega_\xi + \oint_\Gamma p_\Gamma(\boldsymbol{\xi})\frac{\partial}{\partial \eta_\xi}\nabla_x(\boldsymbol{x},\boldsymbol{\xi})\mathrm{d}\sigma_\xi. \tag{36}$$

We can insert this pressure gradient into the momentum equation (1), written in the form $\partial \boldsymbol{u}/\partial t = \mathrm{RHS}$ to obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u} + \int_\Omega \mathcal{R}(\boldsymbol{\xi})\nabla_x G(\boldsymbol{x},\boldsymbol{\xi})\mathrm{d}\Omega_\xi - \oint_\Gamma p_\Gamma(\boldsymbol{\xi})\frac{\partial}{\partial \eta_\xi}\nabla_x G(\boldsymbol{x},\boldsymbol{\xi})\mathrm{d}\sigma_\xi. \tag{37}$$

Now, what we were looking for was an equation that would give us pressure boundary conditions such that the velocity field remains solenoidal. In order to obtain this, we can take the divergence of the above equation and require that the divergence be zero on the boundary

$$\nabla\cdot\frac{\partial \boldsymbol{u}}{\partial t} = 0, \quad \boldsymbol{x}\in\Gamma \tag{38}$$

which gives

$$\nabla\cdot\oint_\Gamma p_\Gamma(\boldsymbol{\xi})\frac{\partial}{\partial \eta_\xi}\nabla_x G(\boldsymbol{x},\boldsymbol{\xi})\mathrm{d}\sigma_\xi = \nabla\cdot\left\{\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \int_\Omega \mathcal{R}(\boldsymbol{\xi})\nabla_x G(\boldsymbol{x},\boldsymbol{\xi})\mathrm{d}\Omega_\xi\right\}, \quad \boldsymbol{x}\in\Gamma. \tag{39}$$

The above equation allows us to determine the boundary values of the pressure we are looking for, up to an arbitrary constant, of course. The Dirichlet conditions suggested by (39) are only determined modulo a constant. For example, by substituting (34) into (39), one can show that any additive constant in p_Γ will translate into a pressure that is also changed by an additive constant. Although (39) may look somewhat forbidding, the important point is that it establishes a linear relation between the pressure and the velocity divergence at the boundary. Thus, in a preprocessing step of a numerical scheme, one can determine this linear relation (which, after discretization, can be described by a so-called influence matrix) and then use this relation to compute pressure boundary conditions such that the divergence is zero at the boundaries. In order to do this, no explicit knowledge of the Green's function is required.

We should also mention two variants of an influence-matrix method that are outlined in [43]. Instead of using (38), for their first variant (Eq. (27) of [43]) the authors use $\partial\boldsymbol{u}/\partial t = \partial\boldsymbol{u}_\Gamma/\partial t$, which, as we have seen above, is inconsistent at $t=t_0$, and results in a trivial identity for $t>0$ because it is already enforced by the boundary condition. For the second variant (Eq. (29) of [43]), (21) is used. In both cases, one would end up with a set of equations that does not require that the solution be solenoidal.

3.2.3.1 Numerical implementation. For completeness, we will briefly outline a possible implementation of the above idea in a numerical method. For details of specific implementations, see, e.g., [42,44–46].

We define a vector containing the boundary values of the pressure (in some suitable discretization) by $\vec{p}_\Gamma$ (so the elements of $\vec{p}_\Gamma$ will be the values of the pressure at a series of grid points on the boundary for a finite-difference or finite-volume method, or the coefficients of a series representation for the pressure at the boundary for a spectral method). Analogously, we define a vector $\vec{\mathcal{D}}_\Gamma$ containing the values of the divergence of $\boldsymbol{u}$ at the boundary. In a preprocessing step, starting from a divergence-free initial condition, we consecutively set the pressure at the boundary equal to the basis vectors $\vec{p}_\Gamma$, given by $p^i_j = \delta_{ij}$, with δ_{ij} the Kronecker-δ (thus, one of the boundary pressures will be equal to one; all the others are zero), perform one time step of our numerical scheme, and then compute the resulting vector $\vec{\mathcal{D}}^i_\Gamma$ holding the values of the divergence of the resulting velocity field at the boundary. We then assemble a matrix $\mathcal{I}^{-1}$ by interpreting each of the $\vec{\mathcal{D}}^i_\Gamma$ as a row of that matrix, thus $\mathcal{I}^{-1}_{ij} = \mathcal{D}^i_j$. Because of the linearity of (39), we know that the divergence at the boundary for an arbitrary discretized pressure boundary condition given by some vector $\vec{p}_\Gamma$ can be computed from

$$\vec{\mathcal{D}}_\Gamma = \mathcal{I}^{-1}\vec{p}^{\,T}_\Gamma. \tag{40}$$

Conversely, this knowledge can then be exploited in a numerical simulation as follows: Starting from the initial condition, we perform one time step, numerically integrating (1) and (6), with given boundary conditions for the velocity and some arbitrary Dirichlet conditions for pressure. At the end of this time step, the resulting velocity field will not be divergence-free, in general. We then compute the divergence at the boundary, which gives us a vector $\vec{\mathcal{D}}_\Gamma$. Multiplying this vector with the inverse of the above matrix, $\mathcal{I}$, which is just our *influence matrix*, gives us correction to our original pressure boundary condition. Subtracting this from the original arbitrary pressure boundary condition gives the correct boundary values for pressure. Finally, we have to repeat our time step, now with the correct pressure boundary condition, to obtain a divergence-free velocity field at the end of the time step. Implementations of this approach usually achieve velocity fields that are divergence-free to machine accuracy.

3.2.4 On the Character of Pressure in Incompressible Flow. Because we feel it is a particularly enlightening exercise, we want to revisit the problem of determining a solution for the pressure in incompressible Navier-Stokes problem one last time, this time from a slightly different and more formal perspective. We emphasize that the material presented here closely follows the ideas in Chap. 6 of the book by Quartapelle [3] and thus owes everything to the work by this author.

We reconsider the solution of an incompressible flow problem described by the following set of equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p - \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}, \quad \boldsymbol{x}\in\Omega \tag{41}$$

$$\nabla^2 p = -\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}), \quad \boldsymbol{x}\in\Omega \tag{42}$$

$$\boldsymbol{u}(\boldsymbol{x},t_0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \text{with } \nabla\cdot\boldsymbol{u}_0 = 0, \quad \boldsymbol{x}\in\Omega \tag{43}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_\Gamma(\boldsymbol{x},t), \quad \boldsymbol{x}\in\Gamma \tag{44}$$

$$\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}\in\Gamma \tag{45}$$

$$\oint_\Gamma \boldsymbol{\eta}\cdot\boldsymbol{u}_\Gamma d\sigma = 0 \tag{46}$$

with $t\in(t,T]$. The above is our familiar PPE formulation, with proper boundary conditions given as boundary values of the velocity vector and its divergence. Because no explicit boundary conditions for pressure are given, this form of the equations couples the PDEs (41) and (42) closely together, which makes for cumbersome solution procedures when one is aiming for a nonfractional step scheme. Thus, we would like to find an explicit expression that allows us to specify the pressure at the boundary. The influence matrix method described above represents one approach to this problem.

Another solution, however, is derived in Chap. 6.3 of [3]. Let us use the notation $\mathcal{N}[-(1/\mathrm{Re})\nabla^2-(\partial/\partial t)]$ to refer to the space of vector fields $\boldsymbol{\mathcal{X}}(\boldsymbol{x},t;T)$, which are solutions to the antidiffusion equation

$$\left(-\frac{1}{\mathrm{Re}}\nabla^2-\frac{\partial}{\partial t}\right)\boldsymbol{\mathcal{X}}(\boldsymbol{x},t;T)=0 \tag{47}$$

with boundary conditions given by

$$\boldsymbol{n}\times\boldsymbol{\chi}=0,\quad \boldsymbol{x}\in\Gamma,\quad t\in(0,T),\quad \text{and}\quad \boldsymbol{\chi}=0,\quad \boldsymbol{x}\in\Omega,\quad t=T. \tag{48}$$

Note that (47) and (48) do not have unique solutions, but rather specify a space of functions of a dimensionality that is equal to the "number" of boundary points. Given this space, Quartapelle shows that (45) can be replaced by a condition of the form

$$\begin{aligned}-\int_0^t\int_\Omega(\boldsymbol{\nabla}p+\boldsymbol{u}\cdot\nabla\boldsymbol{u})\mathrm{d}\Omega\mathrm{d}t' &= \frac{1}{\mathrm{Re}}\int_0^t\oint_\Gamma(\boldsymbol{n}\times\boldsymbol{u}_\Gamma\cdot\boldsymbol{\nabla}\times\boldsymbol{\chi}+\boldsymbol{n}\cdot\boldsymbol{u}_\Gamma\boldsymbol{\nabla}\cdot\boldsymbol{\chi})\mathrm{d}\sigma\mathrm{d}t' \\ &\quad-\int_\Omega\boldsymbol{u}_0\cdot\boldsymbol{\chi}\bigg|_{t'=0}\mathrm{d}\Omega\;\forall\,\chi\in\mathcal{N}\left(-\frac{1}{\mathrm{Re}}\nabla^2-\frac{\partial}{\partial t'}\right). \end{aligned}\tag{49}$$

Now, (49) does not represent a particularly convenient equation that anyone might be eager to implement in a numerical scheme. However, it gives some interesting insights into the nature of the pressure solution in incompressible flow. What this equation shows very clearly is that—despite the fact that the pressure Poisson equation itself is an innocuous-looking steady equation—because of the way that the pressure is linked to the incompressibility constraint, the pressure appears as a *spatiotemporal functional* of the velocity field. A conclusion that immediately follows is that it is not possible to formulate an instantaneous elliptic problem that would give the instantaneous pressure of an unsteady Navier-Stokes problem. In other words, unless additional constraints are introduced, one can set up a pressure problem only for a time interval, and not for a single time instant, in accordance with our result given as Lemma 4. Note that, in the case of influence-matrix methods, the interval that is being treated is the single time step that is used to set up the influence matrix. In our example in Sec. 3.2.5, we will demonstrate that this means that the concept of an initial pressure field in incompressible viscous flows with no-slip boundaries is invalid.

As a final point, it may be interesting to note that the form of (49) has a remarkable consequence for the case of the Stokes equation, where the convective term in the first integral of (49) is absent. Note that with the nonlinear term gone, there are no terms in that equation that depend on the value of the velocity field in the interior of the domain Ω. As a consequence, for a Stokes problem, the pressure field can be determined for all times, completely independent of the velocity field in the interior of the domain, given only velocity boundary conditions.

3.2.5 Two Simple Examples. In the following, we would like to illustrate the above formal results with two examples. Both of the examples have the advantage that the mathematical problems involved can be solved exactly, in closed form (albeit involving infinite sums in the second case), so that the question of numerical accuracy of the solutions presented is immaterial.

In our first example, we will consider a solution of the full unsteady Navier-Stokes equations and will try to find $\partial\boldsymbol{u}/\partial t$ at $t=0$. This example serves a twofold purpose: First, it will provide a demonstration of why the pressure boundary condition needs to be formulated in terms of the interior limit of flow quantities, as discussed above. Second, it will be shown that no unique solution can be obtained when one relies on (27) as a boundary condition for pressure.

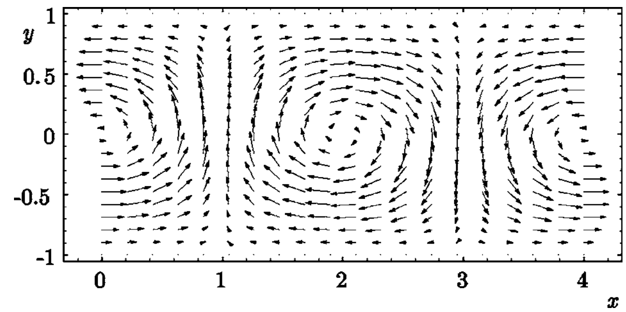

Fig. 1 Initial velocity field calculated from the stream function (50) with $A_\lambda=2.64244$, $\lambda=0.349911$

For our second example, we will look at a solution of the unsteady Stokes equations. In this case, we will explicitly construct a solution to the driven-cavity problem that is completely wrong, yet it exactly satisfies each of Eq. (23)–(28) (for the case of Re $\to 0$), illustrating that no unique solution exists for pressure boundary conditions of the form (27).

3.2.5.1 Navier-Stokes flow in a periodic channel. We first want to look at an example involving the solution of the unsteady Navier-Stokes equations. In order to render things manageable, we will restrict ourselves to the basic task of trying to find the initial time derivative of the velocity field, $\partial\boldsymbol{u}/\partial t$ at $t=0$, for a particularly simple situation, which will allow us to perform all calculations in closed form.

We consider the case of two-dimensional flow in a channel with plane walls at $y=-1$ and $y=1$, periodic in a domain $0<x\leq 4$. To construct an initial condition, we use a *Chandrasekhar-Reid function* [47] as our stream function Ψ

$$\Psi(x,y)=\cos\left(\frac{\pi}{2}x\right)\left[\cos(\lambda y)+A_\lambda\cosh\left(\frac{\pi}{2}y\right)\right]. \tag{50}$$

This problem was also considered, in a different context, in [48]. We want to find A_λ, λ such that the resulting velocity field satisfies no-slip boundary conditions at the wall of our channel. There is an infinite set of solutions for A_λ and λ that have this property. We choose the simplest one of these (the one with the smallest λ), which gives $A_\lambda=0.349911$, $\lambda=2.64244$. This choice of parameters therefore gives us an initial condition that is both divergence-free (by virtue of being computed from a stream function) and that satisfies our no-slip conditions at the channel wall. In addition, the velocity field is given in terms of simple elementary functions and is, thus, C^∞ everywhere in our domain. The resulting velocity field, which consists of a series of counterrotating vortices, is shown in Fig. 1.

We now want to solve the following problem: Given the velocity field $\boldsymbol{u}_0$ as defined by the stream function above,

$$\boldsymbol{u}_0(x,y)=\begin{pmatrix}u_0(x,y)\\ v_0(x,y)\end{pmatrix}=\begin{pmatrix}\cos\left(\frac{\pi}{2}x\right)\left[\frac{\pi}{2}A_\lambda\sinh\left(\frac{\pi}{2}y\right)-\lambda\sin(\lambda y)\right]\\ \frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)\left[\cos(\lambda y)+A_\lambda\cosh\left(\frac{\pi}{2}y\right)\right]\end{pmatrix} \tag{51}$$

find the field $\partial\boldsymbol{u}/\partial t$ at $t=0$, subject to Eqs. (23)–(28), where we set Re$=1$ for simplicity. The approach to complete this task is obvious: All we need to do is find the pressure gradient at $t=0$ and insert this together with the velocity field into (23), which will then allow us to calculate the local acceleration of the flow. Thus, the only difficulty we face is to solve the Poisson equation for pressure (24). Thus, we will try to do just that and see where this takes us.

Let us first take one step back and look at the equation

$$\nabla p(\boldsymbol{x},t) = \frac{1}{\mathrm{Re}}\nabla^2 \boldsymbol{u}(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Gamma \tag{52}$$

from which (22) is derived by projecting on the wall normal $\boldsymbol{\eta}$. Equation (52) is a vector equality, and there is a priori no reason to prefer any particular coordinate to project this on in order to generate a scalar boundary condition for the pressure. In fact, if indeed there is a pressure field that satisfies both (52) and the pressure Poisson equation (24), then no matter on what coordinate we project (52), the resulting projected equation will be satisfied as well. We will show that (24) with boundary conditions (52) does not have a solution.

In order to see this, let us check what we obtain if we use two different choices for our projection. As our first choice, we will use (22), which is obtained by projecting on the wall-normal coordinate, and will denote the resulting pressure field by p_{Neu} since it is calculated from a Neumann boundary condition. In addition, we will calculate the pressure one obtains when projecting on the wall-tangential coordinate $\boldsymbol{\tau}$,

$$\frac{\partial p_{\mathrm{Dir}}}{\partial \tau}(\boldsymbol{x},t) = \boldsymbol{\tau} \cdot \frac{1}{\mathrm{Re}}\nabla^2 \boldsymbol{u}(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Gamma \tag{53}$$

with $\boldsymbol{\tau}$ a unit vector tangential to the wall, which we denote by p_{Dir} because it follows from a Dirichlet condition that one obtains after integrating (53) along the wall. Because of the periodicity in the x coordinate, the pressure Poisson equation can be reduced to a few decoupled second-order ODEs, which can be solved in closed form. As the complete solution is somewhat unwieldy, we relegate the presentation of its algebraic form to Appendix B.

Figure 2 shows the pressure fields and Fig. 3 shows the acceleration fields for the two different boundary conditions. Looking at these figures, we can make a number of important observations: First of all, although the respective pressure solutions look superficially similar, Fig. 4 demonstrates that they are not identical. Thus, the two pressure boundary conditions (22) and (53) are *not* equivalent. Also note that because of the linearity of the equation governing the pressure, we can calculate the pressure field that would correspond to a projection of (52) in the direction of any unit vector $\boldsymbol{\zeta}$: the resulting pressure field is obtained from the linear superposition

$$\tilde{p} = \zeta_x p_{\mathrm{Dir}} + \zeta_y p_{\mathrm{Neu}} \tag{54}$$

where ζ_x and ζ_y denote the x and y components of $\boldsymbol{\zeta}$, respectively. We have thus found that, depending on how we project (52), we can obtain an infinite number of different pressure fields. So, clearly, there must be something wrong with (52): In agreement with Corollary 2 (the corollaries and lemmas are given in Appendix A), there is no solution of the Poisson equation (24) that satisfies (52) at $t=0$.

The second observation we can make is closely connected to the question of what it is that went wrong above: From Figs. 3(*a*) and 3(*d*), we can see that there exists no solution for the pressure that would lead to a $\partial \boldsymbol{u}/\partial t$ that could simultaneously satisfy both boundary conditions for the velocity at the wall, $\partial u/\partial t=0$ and $\partial v/\partial t=0$. In other words, with the initial and boundary conditions we have chosen, it is impossible to have a continuous local acceleration field at the wall at $t=0$, confirming Lemma 2. Also note that since we have found that $\partial \boldsymbol{u}/\partial t$ is C^∞ and finite in the open domain, this observation also confirms our Lemma 3: the velocity field assumes the initial condition continuously as $t\to 0$, but the acceleration field is discontinuous at $t=0$. Hence, the pressure field cannot be continuous at $t=0$. Furthermore, we find that if we had chosen to use the formally correct boundary condition (27) rather than (21), so that the finite values of the velocity derivative at the wall were included, then we would have found that we can indeed satisfy (27). Thus, while (27) does have solutions (alas, an infinite number of them), (21), in general, has none. This observation demonstrates that (21) is wrong: to find a correct pressure boundary condition, we need to use the interior limits of the flow data and not the boundary values.

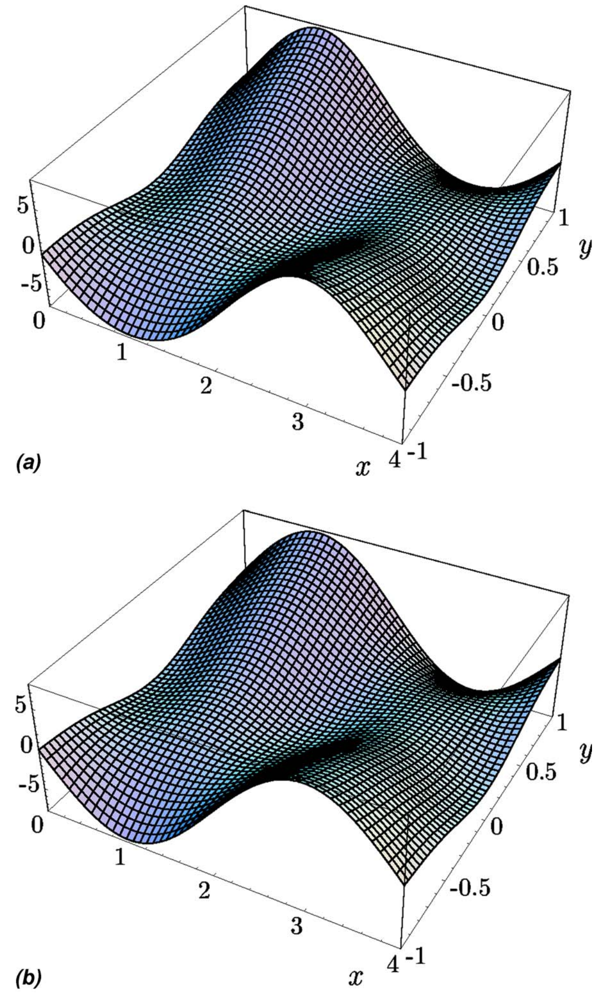

Fig. 2 Pressure fields or Neumann and Dirichlet conditions, respectively: (*a*) $p_{\mathrm{Neu}}(x,y)$ and (*b*) $p_{\mathrm{Dir}}(x,y)$

At this point now the ill-posedness of the set of Eqs. (23)–(28) comes into focus: In order for us to set up a correct boundary condition for the pressure in the form of a prescription of the pressure gradient at the wall, we have to know $\lim_{\boldsymbol{x}\to\Gamma}\partial \boldsymbol{u}/\partial t$. Unfortunately, this function depends on a pressure field that we have not determined yet. Thus, without any additional information, there is no way for us to decide among the infinite number of possible pressure fields $\tilde{p}$ given by (54), each of which corresponds to the choice of making the local acceleration of the flow, in one particular direction vanish at the wall. So we see that the problem given by (23)–(28) is ill-posed and does not allow us to determine a unique solution. There is an infinite number of possible pressure fields that satisfy this set of equations, in agreement with Lemma 4.

Thus, to sum up the lesson from this example, we can say two things:

1. When determining the value of the pressure gradient at the wall, the interior limit of the quantities appearing in the momentum equation must be used. Simply inserting the boundary values of the velocities and associated derivatives in (21) or (52) is inconsistent and renders the resulting boundary condition arbitrary in the sense that it

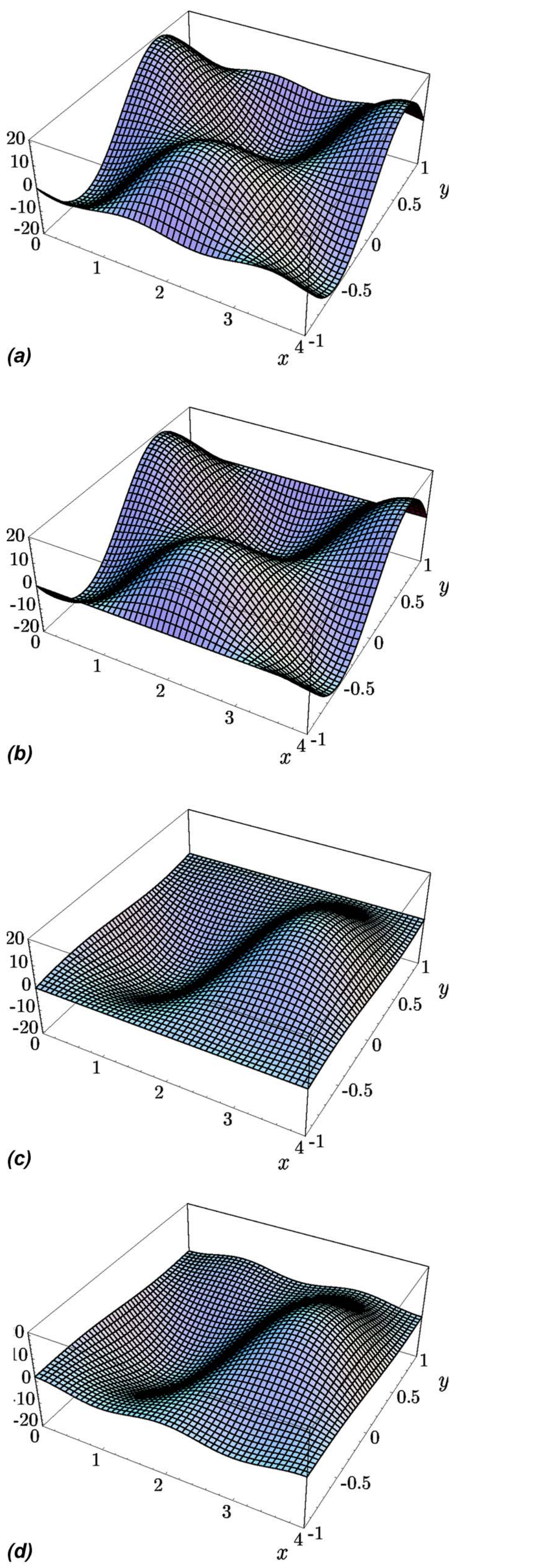

Fig. 3 Acceleration fields for Neumann and Dirichlet conditions, respectively: (*a*) $\partial u_{\mathrm{Neu}}/\partial t(x,y)$, (*b*) $\partial u_{\mathrm{Dir}}/\partial t(x,y)$, (c) $\partial v_{\mathrm{Neu}}/\partial t(x,y)$, and (d) $\partial v_{\mathrm{Dir}}/\partial t(x,y)$. Note that $\partial u_{\mathrm{Neu}}/\partial t$ and $\partial v_{\mathrm{Dir}}/\partial t$ do not vanish at the wall.

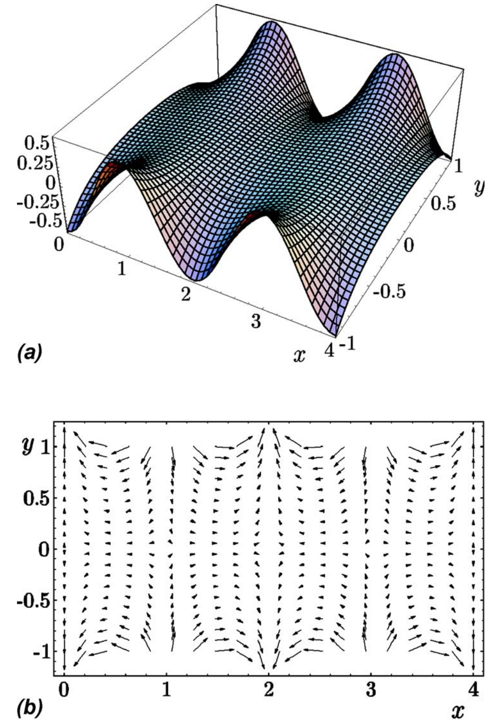

Fig. 4 Discrepancy between the pressures and velocity derivatives for the two choices of boundary conditions: (*a*) $p_{\mathrm{Neu}} - p_{\mathrm{Dir}}$ and (*b*) $\partial u_{\mathrm{Neu}}/\partial t - \partial u_{\mathrm{Dir}}/\partial t$

cannot be justified, neither mathematically nor physically.

2. Once we have fixed this flaw by using the correct representation of the pressure gradient at the boundary as in (27), we are left with an ill-posed problem, where any of an infinite number of possible pressure gradients at the wall can solve our problem.

In addition, the following observations and comments are pertinent:

- The reader should be careful not to confuse the presence of a finite $\partial \boldsymbol{u}/\partial t$ at the wall at an instant $t=0$ with the presence of "slip" or "flow through the wall." From the velocity boundary conditions of the problem it is clear that $\partial \boldsymbol{u}/\partial t=0$ at the wall for $t>0$, so our solution will satisfy the proper boundary conditions on velocity. In other words, $\partial \boldsymbol{u}/\partial t$ is discontinuous on the boundary $t=0$ of our spatiotemporal domain, $\bar{\Omega}\times[0,\infty)$, but it is continuous on $\bar{\Omega}\times(0,\infty)$. A similar point is also made in [49].
- The necessity to use the interior limits of the flow quantities rather than the boundary values to compute the pressure gradient at the wall generally arises not only for $t=0$, but also whenever the time derivative of the velocity boundary condition is discontinuous.
- Finally, although it is true that the problem to find $\partial \boldsymbol{u}/\partial t$ satisfying (21), (23)–(26), and (28) does have a unique solution (albeit one that is arbitrary in that it does not uniquely follow from the original Navier-Stokes equations, as we have shown), this does not mean that the associated IBVP has a unique solution. Specifically, if we start with any arbitrary boundary condition for pressure, then the resulting solution $\boldsymbol{u}^*$ will also satisfy (21) for all $t>0$ (see Theorem 2). Of course, that solution (more precisely $\lim_{\boldsymbol{x}\to\Gamma}\partial \boldsymbol{u}^*/\partial t$) will not satisfy (21) at t

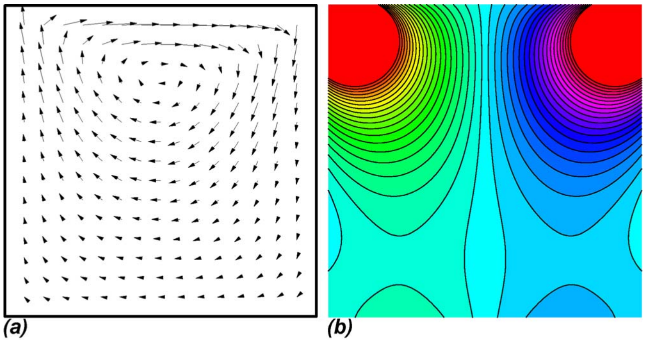

Fig. 5 Stokes flow in a driven cavity. Note that the pressure is singular in the upper corners for this flow: (*a*) velocity field and (*b*) pressure field

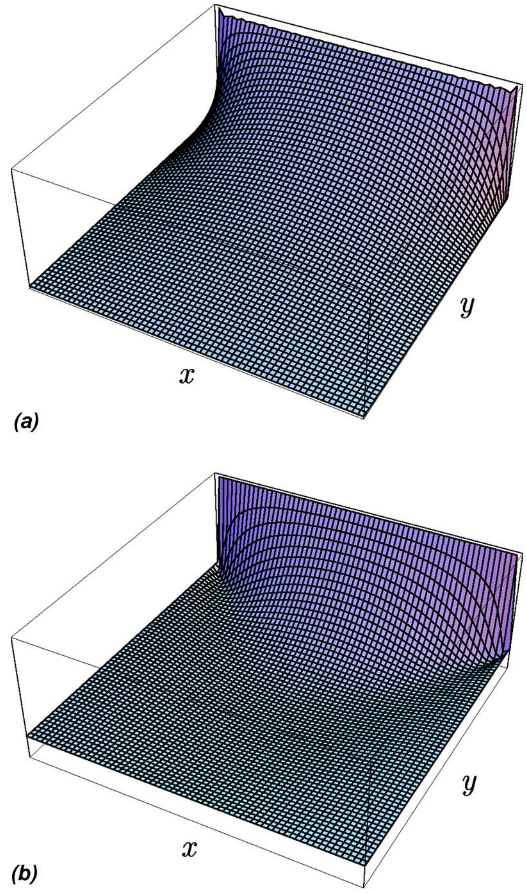

Fig. 6 Distribution of the *x* component of velocity, without and with pressure: (*a*) *u* field for $p \equiv 0$ and (*b*) Correct *u* field

$=0$, but we have already seen that boundary discontinuities of $\partial \boldsymbol{u}/\partial t$ at $t=0$ are not at all unusual, and that in any case the proper boundary condition one should require is not (21) but (27), which *will* be satisfied by $\boldsymbol{u}^*$.

Now, a persistent reader might argue that, "Well, yes, there obviously is some sort of problem at $t=0$, but why not dismiss this as a somewhat pathological but ultimately marginal problem, somehow related to our inability to start with "proper" initial conditions? After all, is it not true that the problem discussed above disappears if we can (somehow) make sure that all quantities, including the time derivatives, are continuous at the wall? So, does that not mean that for $t>0$, the Neumann boundary condition (22) for the pressure is still correct?" Rather than engaging in a long-winded discussion of what is wrong with that argument, or somewhat condescendingly referring the reader to our Theorem 2, we will now present our second example in the spirit of "an example is worth a thousand proofs."

3.2.5.2 Stokes flow in a driven cavity. We will demonstrate the correctness of our central Theorem 2 by revisiting a problem that was also studied in [12]. In that paper, the problem of two-dimensional Stokes flow in a driven cavity is considered, which is described by the following equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla p = \nabla^2 \boldsymbol{u} \tag{55}$$

$$\nabla^2 p = 0. \tag{56}$$

These equations are to be solved in a rectangular domain $\Omega = (0,1)\times(0,1)$, with boundary conditions for the velocities as

$$u(x,1,t) = 1, \quad u(x,0,t) = u(0,y,t) = u(1,y,t) = 0$$

$$v(x,y,t) = 0, \quad (x,y) \in \partial\Omega. \tag{57}$$

As initial conditions we use $\boldsymbol{u}(\boldsymbol{x})=0$. The asymptotic solution (for $t\to\infty$) of this problem is shown in Fig. 5.

In [43] it is claimed that "…the proper boundary condition for the pressure Poisson equation is simply the Neumann boundary condition obtained by applying the normal component of the momentum equation on $\partial\Omega$…." Although our Theorem 2 already shows that this cannot be true, we want to examine this claim a little bit more closely to see why precisely it fails.

To do so, we will consider the solution of the driven cavity problem that we obtain if we simply use $p=0$ on $\partial\Omega$ as our pressure boundary condition, which is not correct but has the advantage of admitting a simple formal solution to the above problem. With homogeneous pressure boundary conditions, the solution to the pressure Poisson equation is simply $p\equiv 0$ everywhere in Ω, and the momentum equation then separates into $\partial u/\partial t = \nabla^2 u$ and $\partial v/\partial t = \nabla^2 v$. The latter of these again has homogeneous boundary conditions; we obtain $v\equiv 0$. The solution for the heat equation we have for the u component can be obtained by separation of variables[1] and reads

$$u(x,y,t) = -u_\infty(x,y) + \sum_{mn=1}^{\infty} a_{mn}\sin(m\pi x)\sin(n\pi y)e^{-\pi^2(m^2+n^2)t} \tag{58}$$

where the coefficients a_{mn} follow from the initial condition

$$a_{mn} = 4\int_0^1\int_0^1 -u_\infty(x,y)\sin(m\pi x)\sin(n\pi y)dxdy \tag{59}$$

and u_∞ is the asymptotic steady-state solution

$$u_\infty(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh(n\pi y) \tag{60}$$

with coefficients b_n given by

$$b_n = 2\int_0^1 \frac{\sin(n\pi x)}{\sinh(n\pi)}dx. \tag{61}$$

The resulting velocity field at $t=\infty$ is shown in Fig. 6 together with the correct solution for $u(x,y)$, and it is immediately clear

[1]We note in passing that this problem is also discussed by Gresho in [12], who gives the asymptotic solution as $u=y$, which is obviously wrong since it does not satisfy the velocity boundary conditions.

that this velocity field is not divergence-free, and obviously has very little in common with the correct Stokes flow in the driven cavity. Of course, we would not have expected that to be the case, since we have chosen arbitrary pressure boundary conditions. However, what is more important here is to realize that the above wrong solution indeed does meet the boundary condition that was suggested in [12,43], e.g., for the steady state, we have

$$\frac{\partial p}{\partial \eta} = \boldsymbol{\eta} \cdot \nabla^2 \boldsymbol{u} = 0. \tag{62}$$

In fact, as we have proven above *any* solution of Eqs. (55) and (56), no matter what boundary condition for the pressure is chosen, will satisfy the boundary condition (62). In other words, the problem given by Eqs. (55) and (56), with boundary conditions (57) and (62), is ill posed and does not have a unique solution.

We can also see that the use of a "consistent pressure Poisson equation" does not change anything, since it is easily checked that our solution (60) satisfies the linear version of (30)

$$\nabla^2 p = \frac{1}{\mathrm{Re}} \nabla^2 (\nabla \cdot \boldsymbol{u}) \tag{63}$$

as well. Thus, we find that these "consistent" equations leave us with a problem that is just as ill posed as the original one, in agreement with Corollary 1.

3.2.6 Pertinent Literature. The impropriety of simply using the momentum equations formulated at a boundary to derive pressure boundary conditions was first recognized by Glowinski and Pironneau [50–52] who propose a method for the solution of the incompressible Navier-Stokes equations that is based on a finite element discretization. Their technique employs an additional equation for a scalar velocity potential that can be used to implement correct boundary conditions on the pressure (see also [53,54]).

On the other hand, despite the situation noted above, the use of improper boundary conditions abounds in the literature. Thus, Ghia [40] uses such boundary conditions to generate numerical solutions for the driven-cavity problem. Not surprisingly, he finds that his original numerical scheme diverges and thus fails to give a solution.

In an attempt to remove the problem, he uses a modified pressure Poisson equation

$$\nabla^2 p = -\nabla \cdot \left[(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u} \right] - \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{u} \tag{64}$$

meaning that he retains two terms on the right-hand side of his equation that formally vanish for incompressible flow. It is clear that, formally, for a divergence-free velocity field, the resulting system of Eqs. (1) and (64) is exactly equivalent to the set (1) and (6), but one can see that the above modification will make a difference in a numerical scheme that does not produce a divergence-free solution. Ghia [40] indicates that, in fact, the solutions resulting from his scheme do not satisfy the incompressibility constraint.

In their paper [14], Moin and Kim state that the version of (21) that results from setting the velocity $\boldsymbol{u}$ identically equal to zero

$$\frac{\partial p}{\partial \eta} = \boldsymbol{\eta} \cdot \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in T \tag{65}$$

"is normally used [at solid boundaries] in conjunction with the Poisson equation for pressure." They then continue to demonstrate, rigorously, how this approach leads to a numerical scheme that cannot converge, but they label this problem as a "numerical difficulty," possibly not realizing that the fundamental problem is not a numerical one but follows from the attempt to solve an ill-posed differential equation problem, which does not have a unique solution. It is also worth noting that these authors briefly discuss alternative formulations of pressure boundary conditions that can be derived in the spirit of (21) by projecting the momentum equations in some other direction. In agreement with [55] and with the discussion in Appendix A, they state that such alternative pressure boundary conditions may not produce the same solution as (21). Nevertheless, Moin and Kim then proceed to develop an alternative numerical method that does not rely on the solution of a pressure Poisson equation and thus enables them to generate proper solutions of incompressible flow problems.

A second method for the proper treatment of boundary conditions for the PPE formulation of the Navier-Stokes equations was introduced in a paper by Kleiser and Schumann [42] that appeared in a proceedings volume of the GAMM conference in 1980. The authors develop an influence matrix technique similar to the one we have presented in Sec. 3.2.3. Thus, they derive correct boundary conditions for the pressure based on the fact that there is a linear relation—our Eq. (39)—between the pressure and the divergence of the velocity field at the boundary. It is worth repeating that this method demonstrates an important fundamental property of the pressure: The boundary values of the pressure at any point of the boundary depend on properties of the velocity field *everywhere* on the boundary. Thus, it is fundamentally impossible to find correct pressure boundary conditions that are given in the form of a local equation, such as (21).

Maybe one of the most important contributions to the topic of pressure boundary conditions for PPE formulations of the incompressible Navier-Stokes equations is the paper by Quartapelle and Napolitano [56] in 1986. In their introduction, the authors provide a brief but very lucid discussion of some of the fundamental issues.

- They distinguish between fractional step and "nonfractional step" or "one-step methods," noting the advantage of the latter of satisfying both the momentum and continuity equations, with no-slip boundary conditions, at the same time, whereas the former do not allow satisfaction of the full velocity boundary conditions at the end of any given time step. The authors point out that the disadvantage of the one-step methods is the difficulty of implementing correct boundary conditions for the pressure.
- The authors note that at the time they published their paper, only two correct methods of posing pressure boundary conditions had appeared. One of them is the method by Glowinski and Pironneau [50], the other one is Kleiser and Schumann's influence matrix technique [42].
- The authors observe a lack of attention to the problem of a correct formulation of pressure boundary conditions. We will see that, despite their best efforts, this is a problem that persists even today.

Quartapelle and Napolitano then proceed to derive integral conditions for the pressure at the boundary based on a rigorous analysis of the time-discretized equations. As noted by the authors (and much later by Daube et al. [57]), their method is equivalent to Kleiser and Schumann's influence matrix method, which seems to have inspired their approach.

Despite the progress in understanding that had been achieved by the contributions mentioned above, in 1987 Gresho and Sani perpetuated and greatly contributed to the confusion in this area by publishing a paper [43] focused on the problem of pressure boundary conditions. Unfortunately, this paper contains a few crucial misconceptions, leading the authors to promote the Neumann boundary conditions (21) for the pressure that we have demonstrated to be invalid above. Despite the fact that, in a paper that is quoted by Gresho and Sani, Strikwerda [58] had correctly stated that such boundary conditions lead to an ill-posed problem, the latter authors [43] claim that Strikwerda is mistaken. A couple of points are important:

- While Gresho and Sani [43] demonstrate that the pressure Poisson equation complemented by the boundary conditions (21) is solvable, they do not answer the more important question of whether the combination of (1), (3)–(6), and (21) represents a well-posed problem. Although these authors suggest an "equivalence theorem" implying that it does, we have seen above that this is not correct.
- The authors base part of their conclusions on a peripheral result that can be found in a paper by Heywood and Rannacher [59], who present a theorem (Proposition 2.1 in [59]) stating that the pressure at $t=t_0$ for an unsteady, incompressible Navier-Stokes problem can be calculated from (6) with boundary conditions (21). In Appendix A, we present a proof that this result is incorrect and that, in fact, the pressure at the initial time $t=0$ is *undefined*, in general, for unsteady Navier-Stokes problems. This somewhat surprising fact also follows immediately from Quartapelle's analysis (Chap. 6 in [3], see Sec. 3.2.4). The value of the pressure at any point in space and time does not only depend on the value of the velocity field everywhere in the domain, it also depends on its *history* in time. Our first example in Sec. 3.2.5 clearly demonstrates this somewhat surprising property of the pressure field in incompressible flow.
- The authors discuss the modified form of the pressure Poisson equation used by Ghia [40], given here as (64), which they call a "consistent pressure Poisson equation." Gresho and Sani [43] claim that this manipulation leads to a solution that will satisfy the incompressible Navier-Stokes equation, but we have seen above (Sec. 3.2.2) that this is not the case.

Thus, we can understand why in the mid-1980s, Ferziger [2] can still report that "the boundary conditions to be applied to the pressure at a solid boundary are the subject of some controversy," so much so that some researchers recommended avoiding the use of the PPE formulation altogether [58]. Thus, one might ask, is the situation any more settled now? Unfortunately, the answer to that question is negative: Improper pressure boundary conditions were still presented in review articles in the 1990s [12,60], and they found their way into some of the newest textbooks on computational fluid mechanics [13,61].

The perception of a controversy surrounding the correct use of pressure boundary conditions has also had the effect of prompting many researchers to find alternative formulations that would not include the ill-famed pressure. Unfortunately, as we will see in Sec. 3.3 the decision to turn to vorticity-based formulations does not make the problem of boundary conditions go away. The problems there take a slightly different shape, but they are certainly no less difficult, and sometimes even more so, than in the case of pressure boundary conditions.

3.3 Vorticity-Transport Formulation. An obvious method to eliminate the pressure is to take the curl of the momentum equations, which immediately leads to the vorticity-transport formulation we have introduced above. Before we review the literature on the subject, we first want to look at the form that boundary condition issues take in this representation. We will see that, just as with the PPE formulation, proper boundary conditions are required in order for this representation to be equivalent to the primitive-variables formulation given by (1)–(5).

3.3.1 On Vorticity Boundary Conditions. In this formulation, we start with the vorticity-transport equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{\omega} \tag{66}$$

where $\boldsymbol{\omega}$ is the vorticity vector defined by

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \cdot \boldsymbol{u}. \tag{67}$$

In order to fully specify a solution, we complement (66) with a Poisson equation for the velocities as described in Sec. 2.2,

$$\nabla^2 \boldsymbol{u} = -\nabla \times \boldsymbol{\omega}. \tag{68}$$

We want to look at the associated PDE problem more closely. Specifically, we want to find out what conditions must be satisfied so that a solution to (66) satisfies the incompressible Navier-Stokes equations (1) and (2) with appropriate boundary conditions for $\boldsymbol{u}$ given in a domain Ω. We will see that the problems that appeared in formulations using a pressure Poisson equation will come back to haunt us, and in a more severe form than before.

As discussed in Sec. 2.2, we would like to use boundary conditions for the vorticity in the transport equation (66). In order to see the role of these boundary conditions, as for the case of the primitive-variables formulation, let us find out what Eqs. (66) and (68) tell us about the divergence of the resulting velocity field. This is easily seen by taking the divergence of the velocity Poisson equations (68), which gives

$$\nabla^2 (\nabla \cdot \boldsymbol{u}) = 0. \tag{69}$$

We can see that again the divergence of the velocity field is only determined to be a harmonic function and will, in general, not be identically equal to zero. In order to have $\nabla \cdot \boldsymbol{u} = 0$ everywhere in Ω, we need to specify boundary conditions that ensure that the divergence is zero everywhere on the boundary.

Another interesting question that is specific to the vorticity-transport formulation as given by (66) and (68) is the one of whether $\boldsymbol{\omega}$ is actually the curl of the velocity field $\boldsymbol{u}$ as it should be. If we take the curl of the Poisson equations (68), we find

$$\nabla^2 (\nabla \times \boldsymbol{u} - \boldsymbol{\omega}) = 0 \tag{70}$$

where we have taken $\nabla \cdot \boldsymbol{\omega} = 0$ since this is supposed to be the curl of a divergence-free vector field.[2] Thus, we see that the difference between the curl of the velocity field (the "true" vorticity) and the quantity $\boldsymbol{\omega}$ that is a solution to (66) and (70), is not necessarily zero and, in fact, can be any harmonic function. And again, in order to make sure that $\boldsymbol{\omega}$ is identical with the curl of the velocity field, we need boundary conditions that ensure that this is the case everywhere on the boundary.

Looking at (69) and (70), it seems that we have four conditions that must be met, but there are only three boundary conditions (for the three vorticity components). However, one can easily show that, if (70) is satisfied, then (69) almost follows: (68) and (10) together with (70) imply that $\nabla(\nabla \cdot \boldsymbol{u}) = 0$, so the divergence of the velocity field is constant. Together with a global mass conservation constraint,

$$\oint_\Gamma \boldsymbol{u} \cdot \eta \, d\sigma = 0 \tag{71}$$

the differential continuity equation is satisfied.

Thus, at this point we face a situation analogous to that of a PPE formulation of the Navier-Stokes equations: We want to find certain boundary conditions, now for the vorticity rather than for the pressure, such that the velocity field that follows from (68) (with boundary conditions) satisfies $\nabla \times \boldsymbol{u} = \boldsymbol{\omega}$ on the boundary. Note that the consequences of using incorrect boundary conditions are even more severe in the case we have studied above: If the correct boundary conditions are not explicitly enforced, then one is dealing with a velocity field that is not divergence-free, and with a "vorticity" field that does not represent the curl of the velocity field.

[2] Alternatively, we could assume that we have a numerical scheme that automatically enforces $\nabla \cdot \boldsymbol{\omega} = 0$ by calculating one of the vorticity components from this constraint. For two-dimensional flow, the single component ω_z is always divergence-free.

3.3.2 Pertinent Literature. A vorticity-transport formulation was first used as a basis for a numerical method by Fasel [62] (see also [63–65] for more recent versions of the method) in order to simulate laminar-turbulent transition in boundary layers. With respect to our discussion of boundary conditions, Fasel's approach contains an interesting feature that is worth discussing in a bit more detail. For simplicity, we will restrict our exposition to the two-dimensional method that Fasel had developed in his dissertation [66], which formed the basis of [62].

In the two-dimensional case, we need to calculate the velocity components u and v in the x and y directions, respectively, and there is only one vorticity component $\omega=\omega_z$ parallel to the z-axis. In this case, the vorticity-transport equation is a scalar equation that reads

$$\frac{\partial \omega}{\partial t}+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\frac{1}{\mathrm{Re}}\nabla^2\omega \tag{72}$$

which is complemented by two Poisson equations for the velocity. For reasons to become soon clear, Fasel uses the two-dimensional version of (9) only for the v component of the velocity,

$$\nabla^2 v=-\frac{\partial \omega}{\partial x} \tag{73}$$

but uses the x derivative of the continuity equation (2) in order to derive an equation for the u component,

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 v}{\partial x\,\partial y}. \tag{74}$$

A crucial feature of Fasel's method is that he uses a fully and explicit scheme to integrate (72) in time. Although the boundary conditions to be used for (73) and (74) are straightforward, he now had to deal with the problem of finding a boundary condition for IM in order to be able to integrate the parabolic equation (72). So, what boundary condition does he use? In [62], and most of the later papers describing the method, one finds as the vorticity boundary condition the following surprising relation:

$$\frac{\partial \omega}{\partial x}=-\nabla^2 v(\boldsymbol{x},t), \quad \boldsymbol{x}\in\Gamma. \tag{75}$$

Why is this surprising? Note that this "boundary condition" is simply the v-Poisson equation (73) written down at the boundary. Did we not emphasize that this type of approach of simply writing down a field equation at the boundary in order to generate a boundary condition is invalid and leads to an ill-posed problem? While that is true, and we still most emphatically maintain that the set of Eq. (72)–(75) with velocity boundary conditions (4) and (5) represents an ill-posed problem, it turns out that these are not quite the equations that Fasel is solving. The above equations were indeed the ones Fasel originally tried to solve [67], but he found, not surprisingly, that his scheme would not work, and exhibited an instability. Looking for a way to remove the problem, he found as his solution an approach that can be considered a type of fractional step approach relying on what could be called "boundary-condition splitting" in reference to the "operator-splitting" approach that underlies conventional fractional step methods.

The "boundary-condition-splitting" approach introduced by Fasel works as follows: We start by performing one discrete time step for (72), updating in this time step, however, only the interior values of ω, but not the boundary values. We note that this can only be done consistently in an explicit scheme, since (72) contains boundary-normal derivatives that would require knowledge of the boundary values of ω in a time-implicit integration scheme. In a second step, first the Poisson equation (73) and after that (74) are solved. Note that the x component of the velocity Poisson equation cannot be used in this method, since it contains a y derivative of ω, which would require knowledge of the boundary values of ω. In Fasel's case, and for his particular geometry containing only x derivatives at the wall, these equations again do not require knowledge of the vorticity at the boundary, so that the fact that no vorticity boundary condition has been given yet does not present a problem. Finally, only in the third and last step are the vorticity values at the boundary updated using the values of v computed in the previous step. As a result, the scheme we have just described represents a valid numerical method for the solution of (72)–(74) that is well posed. The scheme does introduce a splitting error the can lead to significant violations of the continuity equation near the boundary, but it does, in principle, represent a valid method for the approximate solution of an incompressible flow problem.

There are a number of other authors who have used a vorticity-transport formulation as the basis of their numerical schemes as well, see [56,68–71,71,72]. Interestingly enough, in many of these studies the issue of posing boundary conditions in such a way that the resulting initial and boundary value problem is, in fact, equivalent to the primitive-variables problem has not been given any attention. The first notable exception to this rule is the work by Daube et al. [57,45] who use an influence matrix similar to that by Kleiser and Schumann [42]. Although this method incurs some additional computational effort, it has the advantage that it can guarantee "exact" enforcement (to machine accuracy) of the continuity equation and vorticity definition. Similar methods have also been developed by Clercx [73] and by Trujillo and Karniadakis [74]. The latter of these, in particular, address an issue with this type of vorticity-transport formulation that leads to the presence of an instability creating a *lower* limit to the admissible size of the time step. By introducing a penalty formulation for the enforcement of the vorticity boundary conditions, they are able to remove this issue (which had led to an artificial limit on the accuracy of this type of vorticity-transport methods).

An interesting approach to the issue of boundary conditions for vorticity-transport formulations is described in a paper by Davies and Carpenter [75]. These authors, apparently unaware of the very similar conditions proposed in Quartapelle's book ([3] Chap. 4), introduce integral constraints on the vorticity that result in a formulation that can be shown to be equivalent to the primitive-variables formulation of the Navier-Stokes problem.

3.4 Vorticity-Streamfunction Formulation. For two-dimensional flow, the vorticity-transport formulation discussed above requires the solution of differential equations for three dependent variables, u, v, and ω, and boundary conditions have to be chosen carefully to ensure that the resulting velocity field meets the continuity constraint. It is possible to simplify the formulation further and immediately satisfy the continuity equation by replacing the velocities in the vorticity-velocity formulation by a stream function. In this way, we have only two remaining unknowns in our system and continuity will be satisfied automatically. If we substitute the stream function into (11), we obtain the set of equations

$$\frac{\partial \omega}{\partial t}+J(\omega,\psi)=\frac{1}{\mathrm{Re}}\nabla^2\omega \tag{76}$$

$$\nabla^2\psi=\omega \tag{77}$$

where $J(\omega,\psi)$ is the Jacobian determinant

$$J(\psi,\omega)=\left|\frac{\partial(\omega,\psi)}{\partial(x,y)}\right|. \tag{78}$$

Again, as in the case of the vorticity-transport formulation, we would like to find proper boundary conditions for the vorticity. And again, as in the formulations we have discussed before, the problem of finding proper boundary conditions for the vorticity has often not been given the attention it deserves. In fact, it seems that there is no other formulation of the incompressible Navier-

Stokes equations that has seen such an extensive use of improper boundary conditions as can be found in the literature on numerical methods for the solution of the vorticity-stream function equations.

This situation is all the more surprising as one of the first numerical solutions based on this formulation, as presented by Dennis and Staniforth [76] in 1971 went to great lengths to derive and apply correct integral boundary conditions for vorticity. Also, Thoman and Szewczyk [77] and Rimon and Cheng [78] had presented time-explicit methods using a boundary-condition-splitting approach quite similar to the one by Fasel that we discussed above. Likewise, Campion-Renson and Crochet [79] describe a finite element procedure capable of properly enforcing the boundary conditions without the necessity of iteration. On the other hand, a significant number of the early numerical solutions used implicit time integration while computing vorticity boundary conditions from field equations, which results in formally ill-posed schemes. Depending on the finite difference approximation used, slightly different formulas for the vorticity boundary condition were derived by Thom [80], Fromm [81], Wilkes [82], Pearson [83], and Woods [84]. Not surprisingly, many of these methods did not meet the expectations set in them, with presumably "higher-order" boundary conditions giving no better, or sometimes worse accuracy than lower-order ones, and an abundance of "stability problems" [55]. Although there are a few attempts at finding alternative ways to implement proper vorticity boundary conditions [85,86], a general understanding of this issue does not seem to be present in the literature up until the 1980s.

In 1981, Quartapelle and Valz-Gris [87] and Quartapelle [88] presented a thorough analysis of the issue of vorticity boundary conditions in which they show, in particular, that in order for the boundary conditions on the velocity to be satisfied, the vorticity must evolve subject to an integral constraint coupling vorticity values everywhere on the boundary. If we consider a no-slip boundary, then the Poisson equation for ψ (77) is supplied with the two boundary conditions

$$\psi\bigg|_{\Gamma} = 0, \qquad \frac{\partial \psi}{\partial \eta}\bigg|_{\Gamma} = 0 \tag{79}$$

which is an overdetermined problem. Quartapelle and Valz-Gris [87] show that this problem has a solution if and only if the vorticity ω is orthogonal (with respect to an L^2 inner product) to the space of harmonic functions in the solution domain. This condition can be translated into a set of integral constraints that has the same dimensionality as the set of boundary points. Thus, either one of the two conditions in (79) together with this set of integral constraints represents a valid system of boundary conditions for the vorticity-stream function formulation.

Although these integral conditions provide a mathematically exact way to determine the vorticity at the boundary, they can be quite cumbersome in practice. Another way to provide proper vorticity boundary conditions is the use of an influence-matrix technique as introduced by Dennis and Quartapelle [68] and by Tuckerman [89]. In a paper in 1989, Anderson [90] proposes a third alternative to obtaining vorticity boundary conditions that is quite similar to the one by Quartapelle and Valz-Gris [87] but is partially inspired by the "vortex-blob" method first introduced by Chorin [91–93]. Again, use of these kinds of techniques makes it clear that the boundary vorticity is a "nonlocal" quantity, which cannot be properly calculated by a local formula of a type going back to the ones used by Thom [80], Fromn [81], or Wilkes [82] This point is also made in the review paper by Gresho [60] who (based on the results by Quartapelle and Napolitano [56] mentioned above) states that "there are no [local] boundary conditions on the vorticity." Finally, in a paper by E and Liu [94] a detailed comparison between different approaches of generating vorticity boundary conditions is given. Although these authors do not discuss the issue of illposedness of mathematical formulations of vorticity boundary conditions that are derived from the field equations, it is interesting to note that they discourage the use of implicit time integration for vorticity-stream function formulations. Instead, they propose an explicit method that amounts to a rediscovery of the boundary-condition-splitting method that had been introduced by Thoman and Szewczyk [77] and Rimon and Cheng [78] almost 30 years earlier. Although only partially related to our topic, we conclude this chapter with a reference to the paper by Morton [95]. Morton's paper is not directly concerned with the question of how to correctly pose vorticity boundary conditions for numerical algorithms, but his discussion of the issue of boundary vorticity from a physical point of view is nevertheless instructive. In his paper, he describes in quite some detail the physical mechanism by which vorticity is created at no-slip boundaries in viscous flows.

4 Discussion

One of the goals of this paper was to reestablish the sometimes disputed fact that one cannot generate boundary conditions for IBVPs by just "recycling" the field equations. We have also shown that it is typical for a number of formulations of the Navier-Stokes equations to require what we call "indirect boundary conditions" in Sec. 3.2.1, meaning that there are more boundary conditions for a first set of quantities (usually the velocities or a stream function) than can be satisfied by a general solution to the related field equation, but not enough (or no) direct boundary conditions for some second set of quantities (pressure, vorticities). This situation can be resolved by, indirectly, specifying boundary conditions for the second set of quantities via an influence matrix or other integral constraints such that the desired boundary condition for the first set of quantities can be met, see, e.g., [3,75,94]. Alternatively, various kinds of splitting methods can be used to avoid the sometimes cumbersome explicit evaluation of constraint equations. Although a detailed analysis of this approach is beyond the scope of this paper, we note that there is some evidence [37] that the convenience of these methods may come at the price of a penalty on accuracy.

As we have seen, in general, any proper and accurate solution of the boundary condition problem results in a "global" method that couples all boundary values of one quantity (e.g., pressure) to all boundary values of another quantity (e.g., velocity divergence). Looking at the data presented in [94] it is, however, interesting to speculate whether, on close inspection of the influence matrix of a given problem, one might be able to derive a local approximation for the boundary conditions that would be much more convenient to handle numerically. We suspect that some of the local boundary conditions presented in [94], although found heuristically rather than through the analysis suggested above, might come very close to that goal.

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Appendix A

A.1 Preliminaries: On Partial Differential Equations. Although we will assume basic familiarity with the concept of a partial differential equation (PDE) in the following, a couple of preparatory remarks are in order. We can write a PDE in a symbolic fashion as

$$\mathcal{F}^n\{\boldsymbol{u}(\boldsymbol{x})\}=0, \quad x\in\Omega \tag{A1}$$

where $\mathcal{F}^n$ is a differential operator of order n, meaning that the highest derivative that appears in $\mathcal{F}^n$ is of order n, and Ω is an open domain in $\mathbb{R}^m$, with $m \leqslant 4$ for the equations in which we are interested. Since that is all we need for our purposes [49], we can assume that functions of $\boldsymbol{u}$ appearing in $\mathcal{F}^n$ are C^∞ in Ω to avoid any unnecessary complications. For the same reason, we will also assume that the boundary $\Gamma=\partial\Omega$ of our domain is "sufficiently smooth," without specifying this any further. Equation (A1) usually has an infinite number of solutions, and a unique solution can be selected by requiring appropriate boundary conditions along the border Γ of the domain, e.g.,

$$\mathcal{B}^k\{\boldsymbol{u}(\boldsymbol{x})\}=u_\Gamma(\boldsymbol{x}), \quad x\in\Gamma \tag{A2}$$

with $\mathcal{B}^k$ another differential operator of order $k\leqslant n-1$. Finally, because this will play a certain role in the following, we want to remind our readers of the following:

DEFINITION 1. *We call a function $\boldsymbol{u}(\boldsymbol{x})$ a solution of the problem given by* (A1) *and* (A2), *if it is at least C^n in Ω and C^k at the boundary, satisfies* (A1) *everywhere in Ω, and* (A2) *everywhere on the boundary $\partial\Omega$.*

As an immediate consequence we note that if the required derivatives exist for $\boldsymbol{x}\in\Gamma$, in which case the derivatives are understood as one-sided derivatives, then the solution will also satisfy (A1) on the boundary.

A.2 Some Statements on Navier-Stokes Initial Conditions. After these preliminary remarks, we can now deal with the specific case of the Navier- Stokes and related equations. We begin by reminding the reader of a number of regularity properties of Navier-Stokes solutions.

In the following, we are considering solutions for a formulation of the incompressible Navier-Stokes-equations in primitive variables. Thus, the problem we are considering is given by (1)–(5). We first restate an important fundamental result from [49], which we repeat without proof as follows:

LEMMA 1. *If the boundary condition $\boldsymbol{u}_\Gamma$ is C^∞ in $\Gamma\times[t_0,T]$, then the solution $\boldsymbol{u}(\boldsymbol{x},t)$ of* (1)–(5) *is C^∞ in the interval $\Omega\times[t_0,T]$ and in $\bar{\Omega}\times[t_0,T]$, with $\bar{\Omega}=\Omega\cup\Gamma$, and T some finite time.*

We now consider $\partial\boldsymbol{u}_0/\partial t(\boldsymbol{x})=\partial\boldsymbol{u}/\partial t(\boldsymbol{x},t_0)$ and $p_0(\boldsymbol{x})=p(\boldsymbol{x},t_0)$, which one might expect to obtain from the solution of (1)–(5) at the initial time $t=t_0$. We can make two interesting statements:

LEMMA 2. *The time derivative of the solution $\boldsymbol{u}(\boldsymbol{x},t)$ of* (1)–(5) *will, in general, be discontinuous at $t=t_0$, such that*

$$\lim_{t\to t_0}\left\|\frac{\partial\boldsymbol{u}(\boldsymbol{x},t_0)}{\partial t}-\frac{\partial\boldsymbol{u}(\boldsymbol{x},t)}{\partial t}\right\|>0 \tag{A3}$$

$$\lim_{x\to\Gamma}\left\|\frac{\partial\boldsymbol{u}_\Gamma(\boldsymbol{x},t_0)}{\partial t}-\frac{\partial\boldsymbol{u}(\boldsymbol{x},t_0)}{\partial t}\right\|>0 \tag{A4}$$

where $\lim_{\boldsymbol{x}\to\Gamma}$ denotes the limit as the boundary is approached from the interior of the domain, and $\|\cdot\|$ is the L^2-norm.

Remark. When we say that a certain relation holds, *in general*, for initial conditions $\boldsymbol{u}_0$ satisfying (3), we mean that the functions $\boldsymbol{u}_0$ for which the relation is not satisfied form a set of measure zero in an appropriate Sobolev space. Also, in the following we will generally assume that the L^2-norm exists whenever we need it. This means that on unbounded domains, we need solutions that decay sufficiently fast in that sense.

Proof. In order to find the initial acceleration $\partial\boldsymbol{u}(\boldsymbol{x},t_0)/\partial t$, we need to determine the initial pressure p_0. To find it, we derive the *pressure Poisson equation* (PPE) by taking the divergence of (1) and substituting (2),

$$\boldsymbol{\nabla}^2 p=-\boldsymbol{\nabla}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}). \tag{A5}$$

We note that Eq. (A5) (which is the same as (6), repeated here for convenience) will play a central role in much of what follows.

A solution of Eq. (A5) that is unique up to an arbitrary constant could be obtained by projecting the momentum equation (1) onto any vector $\boldsymbol{\zeta}$ to obtain a scalar boundary condition for the pressure gradient

$$\frac{\partial p}{\partial\zeta}(\boldsymbol{x},t_0)=\boldsymbol{\zeta}\cdot\left[-\frac{\partial\boldsymbol{u}_\Gamma}{\partial t}-\boldsymbol{u}_\Gamma\cdot\boldsymbol{\nabla}\boldsymbol{u}+\frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\boldsymbol{u}\right](\boldsymbol{x},t_0), \quad x\in\Gamma. \tag{A6}$$

Using this boundary condition guarantees that we will obtain an acceleration field that satisfies

$$\boldsymbol{\zeta}\cdot\frac{\partial\boldsymbol{u}}{\partial t}(\boldsymbol{x},t_0)=\boldsymbol{\zeta}\cdot\lim_{t\to t_0}\frac{\partial\boldsymbol{u}}{\partial t}(\boldsymbol{x},t_0)=\boldsymbol{\zeta}\cdot\frac{\partial\boldsymbol{u}_\Gamma}{\partial t}(\boldsymbol{x},t_0), \quad x\in\Gamma. \tag{A7}$$

However, if $\boldsymbol{\xi}$ is any vector that is not parallel to $\boldsymbol{\zeta}$, then

$$\boldsymbol{\xi}\cdot\frac{\partial\boldsymbol{u}}{\partial t}(\boldsymbol{x},t_0)=\boldsymbol{\xi}\cdot\lim_{t\to t_0}\frac{\partial\boldsymbol{u}}{\partial t}(\boldsymbol{x},t_0)=\boldsymbol{\xi}\cdot\frac{\partial\boldsymbol{u}_\Gamma}{\partial t}(\boldsymbol{x},t_0), \quad x\in\Gamma \tag{A8}$$

will hold if and only if the initial velocity field $\boldsymbol{u}_0(\boldsymbol{x})$ satisfies the additional, implicit compatibility condition

$$\frac{\partial p}{\partial\xi}(\boldsymbol{x},t_0)=\boldsymbol{\xi}\cdot\left[-\frac{\partial\boldsymbol{u}_\Gamma}{\partial t}-\boldsymbol{u}_\Gamma\cdot\boldsymbol{\nabla}u_0+\frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\boldsymbol{u}_0\right](\boldsymbol{x},t_0), \quad x\in\Gamma. \tag{A9}$$

Since we did not specify such a requirement on our initial condition, (A8) will not be satisfied, in general. ∎

As an immediate corollary we have the following:

COROLLARY 2. *The equation*

$$\boldsymbol{\nabla}p(\boldsymbol{x},t)=\left[-\frac{\partial\boldsymbol{u}_\Gamma}{\partial t}-\boldsymbol{u}_\Gamma\cdot\boldsymbol{\nabla}\boldsymbol{u}+\frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\boldsymbol{u}\right](\boldsymbol{x},t), \quad x\in\Gamma \tag{A10}$$

in general does not have a solution for $t=t_0$. All that can be said of the pressure field at boundary is that it (trivially) satisfies

$$\nabla p(x,t_0)=\lim_{x^*\to\Gamma}\left[-\frac{\partial u}{\partial t}-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{u}\right](x^*,t_0), \quad (x\in\Gamma). \tag{A11}$$

Proof. The proof follows from Lemma 2 and its proof. ∎

As an aside, we note that in the special case where both (A7) and (A8) are satisfied for any two nonparallel vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$, (A10) does have a solution, and the acceleration and pressure fields will be continuous even at $t=t_0$. This will generally be the case when $\boldsymbol{u}_0$ is a solution of the Navier-Stokes equations for $t<t_0$ such that the boundary condition $\boldsymbol{u}_\Gamma$ is continuous at $t=t_0$.

Although the above two statements are not new [49], we follow with two further consequences that seem to have been sometimes overlooked. First, from the discontinuity of the acceleration field at $t=t_0$, we can also conclude that the pressure field must be discontinuous at $t=t_0$.

LEMMA 3. *Any pressure field $p(x,t)$ solving* (1)–(5) *will, in general, be discontinuous* at $t=t_0$, *and*

$$\lim_{t\to t_0}\|\boldsymbol{\nabla}[p(\boldsymbol{x},t)-p(\boldsymbol{x},t_0)]\|>0. \tag{A12}$$

Proof. From Corollary 2, we know that at $t=t_0$, the pressure gradient at the boundary does not satisfy (A10). To conclude the proof we only need to observe that (A10) is satisfied for $t>t_0$, since (A8) is satisfied for any vector ξ if $t>t_0$. Together with Lemma 1, the statement follows. ∎

We note that Heywood and Rannacher [59] have presented an argument (leading to their Proposition 2.1 [59]) which claims that the pressure solution p for $t>0$ of the system (1) and (2), with reasonable initial and boundary conditions, satisfies (A5) and (A6), with $\boldsymbol{\zeta}$ a vector normal to the boundary at $t=t_0$ in the sense that

$$\|\boldsymbol{\nabla}[p(\boldsymbol{x},t)-p_0(\boldsymbol{x})]\| \to 0 \quad \text{as } t \to t_0 \tag{A13}$$

where p_0 is the solution of (1) with boundary conditions given by (2) and both relations evaluated at $t=t_0$. It is clear that this proposition conflicts with the above: Per our Lemma 3, whatever the limit of the Navier-Stokes solution $p(\boldsymbol{x},t)$ for $t \to t_0$ to might be, it cannot, in general, be equal to p_0.

Finally, taking the above statements together we arrive at Lemma 4.

LEMMA 4. *The problem given by* (1)–(5), *in general, does not determine a unique solution for the initial pressure* $p_0(\boldsymbol{x}) = p(\boldsymbol{x},t_0)$, *and it does not give a unique solution for the initial acceleration field* $\partial u(x,t_0)/\partial t$.

Proof. We prove the statement by showing that the initial velocity field $\boldsymbol{u}_0(x)$ given by (3) is not associated with a unique pressure field. Indeed, let us suppose that $\boldsymbol{u}_0$ was found as the solution of some unsteady Navier-Stokes problem. We can then integrate (1)–(5) backward in time for a finite time ΔT (see [59,49]), obtaining a solution $u(\boldsymbol{x},t)$, $\boldsymbol{x} \in \Omega$, $t \in [t_0-\Delta T, t_0]$. We consider two such solutions, $\boldsymbol{u}^{\dagger}$ and $\boldsymbol{u}^{\ddagger}$, satisfying some boundary conditions $\boldsymbol{u}_{\Gamma}^{\dagger}(\boldsymbol{x},t)$ and $\boldsymbol{u}_{\Gamma}^{\ddagger}(\boldsymbol{x},t)$, with $\boldsymbol{u}_{\Gamma}^{\dagger}(\boldsymbol{x},t_0) = \boldsymbol{u}_{\Gamma}^{\ddagger}(\boldsymbol{x},t_0) = u_{\Gamma}(\boldsymbol{x},t_0)$, but with $\partial u_{\Gamma}^{\ddagger}(\boldsymbol{x},t_0)/\partial t$, $\partial \boldsymbol{u}_{\Gamma}(\boldsymbol{x},t_0)/\partial t$ arbitrary and different for the two solutions. Observe that the corresponding pressure gradients $\boldsymbol{\nabla} p^{\dagger}(\boldsymbol{x},t)$, and $\boldsymbol{\nabla} p^{\ddagger}(\boldsymbol{x},t)$ are different at the boundary. Since $\boldsymbol{u}_0$ and the associated pressure follow from an integration of the Navier-Stokes equations forward in time and are therefore C^{∞} in $\bar{\Omega}$, the statement follows. ■

We note that this situation does not affect the question of uniqueness of the solution of (1)–(5). In order for a unique solution to these equations to exist, we only need uniqueness of the pressure and acceleration fields for $t>0$. Note also that the problem of nonuniqueness of the initial pressure field can be removed by giving some information on the initial acceleration of the velocity field at the boundary; what is needed is a scalar equation describing the initial acceleration parallel to any vector $\boldsymbol{\zeta}$.

Appendix B

Although the solution of the equations for Sec. 3.2.5 is quite straightforward, the algebra becomes somewhat tedious. In order to avoid the numerous opportunities to get various signs and factors wrong, the author has used the computer algebra software MATHEMATICA® for the mechanical parts of the solution. While a fully symbolic solution to the equations is thus available, in order to keep the resulting expressions to a reasonable length, below we only show the results for the case of the Chandrasekhar-Reid function with $A_{\lambda}=2.64244$, $\lambda=0.349911$ that was considered in Sec. 3.2.5. In each of the following equations, the terms that differ between the two types of boundary conditions are underlined. The expressions for the pressure field from the Neumann and Dirichlet boundary conditions, respectively, read as follows:

$$\begin{aligned} P_{\text{Neu}} = & -0.692376 - 0.61685\cos(2\lambda y) \\ & - 0.863371\cos(2\lambda y)\cosh\left(\frac{\pi y}{2}\right) - 0.0755258\cosh(\pi y) \\ & + 3.30663\sin\left(\frac{\pi x}{2}\right)\sinh\left(\frac{\pi y}{2}\right) + \cos(\pi x)\left[-1.6701\right. \\ & + 0.0124142\cos(\lambda y)\cosh\left(\frac{\pi y}{2}\right) + \underline{0.0489026\cosh(\pi y)} \\ & \left. + 0.491091\sin(\lambda y)\sinh\left(\frac{\pi y}{2}\right)\right] \end{aligned} \tag{B1}$$

$$\begin{aligned} P_{\text{Dir}} = & -0.692376 - 0.61685\cos(2\lambda y) \\ & - 0.863371\cos(\lambda y)\cosh\left(\frac{\pi y}{2}\right) - 0.0755258\cosh(\pi y) \\ & + 3.30663\sin\left(\frac{\pi x}{2}\right)\sinh\left(\frac{\pi y}{2}\right) + \cos(\pi x)\left[-1.6701\right. \\ & + 0.0124142\cos(\lambda y)\cosh\left(\frac{\pi y}{2}\right) + \underline{0.0997648\cosh(\pi y)} \\ & \left. + 0.491091\sin(\lambda y)\sinh\left(\frac{\pi y}{2}\right)\right]. \end{aligned} \tag{B2}$$

Thus, the difference between the two pressures is simply

$$p_{\text{Neu}} - p_{\text{Dir}} = -0.0508623\cos(\pi x)\cosh(\pi y). \tag{B3}$$

With the above pressure fields, the time derivative of the velocity fields can easily be calculated. We find for the case of Neumann conditions

$$\begin{aligned} \partial u_{\text{Neu}}/\partial t = & \cos\left(\frac{\pi x}{2}\right)\left[24.9708\sin(\lambda y) - 5.19404\sinh\left(\frac{\pi y}{2}\right)\right] \\ & + \cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)\left[2.55968\cos(\lambda y)\cosh\left(\frac{\pi y}{2}\right)\right. \\ & \left. + \underline{0.307264\cosh(\pi y)} - 1.4772\sin(\lambda y)\sinh\left(\frac{\pi y}{2}\right)\right] \end{aligned} \tag{B4}$$

$$\begin{aligned} \partial v_{\text{Neu}}/\partial t = & \sin\left(\frac{\pi x}{2}\right)\left[-14.8439\cos(\lambda y) - 5.19404\cosh\left(\frac{\pi y}{2}\right)\right] \\ & - \cos(\pi x)\left[-0.7386\cosh\left(\frac{\pi y}{2}\right)\sin(\lambda y)\right. \\ & \left. - 1.31718\cos(\lambda y)\sinh\left(\frac{\pi y}{2}\right) - \underline{0.153632\sinh(\pi y)}\right]. \end{aligned} \tag{B5}$$

When using Dirichlet boundary conditions for the pressure, we get

$$\begin{aligned} \partial u_{\text{Dir}}/\partial t = & \cos\left(\frac{\pi x}{2}\right)\left[24.9708\sin(\lambda y) - 5.19404\sinh\left(\frac{\pi y}{2}\right)\right] \\ & + \cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)\left[2.55968\cos(\lambda y)\cosh\left(\frac{\pi y}{2}\right)\right. \\ & \left. + \underline{0.626841\cosh(\pi y)} - 1.4772\sin(\lambda y)\sinh\left(\frac{\pi y}{2}\right)\right] \end{aligned} \tag{B6}$$

$$\begin{aligned} \partial v_{\text{Dir}}/\partial t = & \sin\left(\frac{\pi x}{2}\right)\left[-14.8439\cos(\lambda y) - 5.19404\cosh\left(\frac{\pi y}{2}\right)\right] \\ & - \cos(\pi x)\left[-0.7386\cosh\left(\frac{\pi y}{2}\right)\sin(\lambda y)\right. \\ & \left. - 1.31718\cos(\lambda y)\sinh\left(\frac{\pi y}{2}\right) - \underline{0.31342\sinh(\pi y)}\right]. \end{aligned} \tag{B7}$$

The difference between the two acceleration fields is thus

$$
\begin{pmatrix} \dfrac{\partial(u_{\text{Neu}}-u_{\text{Dir}})}{\partial t} \\ \dfrac{\partial(v_{\text{Neu}}-v_{\text{Dir}})}{\partial t} \end{pmatrix} = 0.159789 \begin{pmatrix} -\sin(\pi x) & \cosh(\pi y) \\ \cos(\pi x) & \sinh(\pi y) \end{pmatrix}. \tag{B8}
$$

Note Added in Proof: "*In a recent conversation with Prof. Temam, the author learned that Temam has discussed the issue of the undefined pressure at* $t=0$ *in his paper* R. Teman, Behaviour at time $t=0$ of the solutions of semi-linear evolution equations, *J. Diff. Equ.*, **17**, 1982, 73–92."

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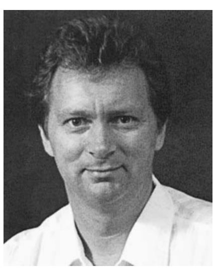

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