Quick slides on RMC

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Basics, sampling from a CDF

Given a non-normalized probability distribution, $p^*(\mathbf{x})$, defined over a composite domain $X = V\mathbf{\Omega}$, we can determine a normalized probability distribution, $p(\mathbf{x})$, with

$$p(\mathbf{x}) = \frac{p^*(\mathbf{x})}{\int_X p^*(\mathbf{x})d\mathbf{x}}. \tag{1}$$

We can then determine a CDF from

$$c(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} p(\mathbf{x})d\mathbf{x} \tag{2}$$

where $\mathbf{x}_0$ is some starting domain point.

We can then determine an inverse CDF, $\mathbf{x}_i = c^{-1}(\theta)$, $\theta \in [0,1], \theta \in \mathbb{R}$, from which we can compute the average probability over the sub-domain (or bin), $Y \subset X$,

$$p_{Y,avg} = \frac{1}{N} \sum_{i=1}^{N} \begin{cases} 1, & \text{if } \mathbf{x}_i \in Y \\ 0, & \text{if } \mathbf{x}_i \notin Y \end{cases} \tag{3}$$

This value integrated over the portion of the composite domain, A, can be written as

$$\int_{\mathbf{\Omega}} p_{Y,avg} d\mathbf{\Omega} = \left(\int_{\mathbf{\Omega}} d\mathbf{\Omega} \right) \frac{1}{N} \sum_{i=1}^{N} \begin{cases} 1, & \text{if } \mathbf{x}_i \in Y \\ 0, & \text{if } \mathbf{x}_i \notin Y \end{cases} \tag{4}$$

Finally:

$$\int_{\mathbf{\Omega}} p^*_{Y,avg} d\mathbf{\Omega} = \left(\int_X p^*(\mathbf{x}) d\mathbf{x} \right) \int_{\mathbf{\Omega}} p_{Y,avg} d\mathbf{\Omega} \tag{5}$$

Monte Carlo integration

We seek $\int_X p^*(\mathbf{x})d\mathbf{x}$ via Monte Carlo integration we can be determined simply from a uniform sampling in Y

$$\int_X p^*(\mathbf{x})d\mathbf{x} = \sum_Y \int_Y p^*(\mathbf{x})d\mathbf{x} \tag{6}$$

$$\int_Y p^*(\mathbf{x})d\mathbf{x} = \left(\int_{V_Y} .dV \int_{\boldsymbol{\Omega}} .d\boldsymbol{\Omega} \right) \frac{1}{N_Y} \sum_{i=1}^{N_Y} p^*(\mathbf{x}_i) \tag{7}$$

Residual

Our residual is given by

$$r^* = r^*_{interior} + r^*_{surface} \tag{8}$$

where we lumped surface sources into $r^*_{surface}$. These surface sources present themselves through discontinuities between the present cell flux, ϕ^P, and the neighboring flux, ϕ^N. The latter can be a neighboring cell or a boundary.

Interior residual

The interior residual is then given by:

$$r^*_{interior} = \frac{1}{4\pi}\left(q - \sigma_a\phi - \mathbf{\Omega}\cdot\nabla\phi\right) \tag{9}$$

from which we can determine the phase-space integration as

$$\begin{aligned}
\int_X r^*_{interior}.d\mathbf{x} &= \int_V \int_{\mathbf{\Omega}} \frac{1}{4\pi}\left(q - \sigma_a\phi - \mathbf{\Omega}\cdot\nabla\phi\right).d\mathbf{\Omega}.dV \\
&= \sum_Y \int_{V_Y} \int_{\mathbf{\Omega}} r^*_{Y,interior}.d\mathbf{\Omega}.dV \\
&= \sum_Y \left(\int_{V_Y} .dV \int_{\mathbf{\Omega}} .d\mathbf{\Omega}\right) r^*_{Y,interior,avg}
\end{aligned} \tag{10}$$

Via Monte Carlo:

$$r^*_{Y,interior,avg} = \frac{1}{N_Y}\sum_{i=1}^{N_Y} r^*_{Y,interior}(\mathbf{x}_i) \tag{11}$$

Surface residual

The surface residual can be presented as

$$\int_S \int_{2\pi} \int_{-1}^{0} \mu \, r^*_{surface}.d\mu.d\varphi.dA = \int_S \int_{2\pi} \int_{-1}^{0} \mu \frac{1}{4\pi}(\phi^N - \phi^P).d\mu.d\varphi.dA \tag{12}$$

$$\begin{aligned} \therefore \int_X r^*_{surface}.d\mathbf{x} &= \sum_Y \left(\int_{S_Y} .dS \int_{2\pi} \int_{-1}^{0} \mu.d\mu.d\varphi \right) r^*_{Y,surface,avg} \\ &= \sum_Y \sum_f \left(A_f \pi \right) r^*_{Y,f,avg} \end{aligned} \tag{13}$$

Via Monte Carlo:

$$r^*_{Y,f,avg} = \frac{1}{N_f} \sum_{i=1}^{N_f} r^*_{Y,f}(\mathbf{x}_i) \tag{14}$$

Sampling strategy

Sample the CDF, first interior vs surface:

- If interior, sample cell ($\rightarrow$ *cell Y*), sample position randomly in cell, sample direction randomly:

$$r^*_{i,interior} = \frac{1}{4\pi}\left(q - \sigma_a \phi - \mathbf{\Omega} \cdot \nabla \phi\right) \tag{15}$$

  Particle weight: $w = \frac{r^*_{i,interior}}{r^*_{Y,interior,avg}}$

- If surface, sample face (f), sample random position on face, sample direction using cosine-law:

$$r^*_{i,surface} = \frac{1}{4\pi}(\phi^N - \phi^P) \tag{16}$$

  Particle weight: $w = \frac{r^*_{i,surface}}{r^*_{Y,surface,avg}}$

Global normalization:

$$\int_X r^*.d\mathbf{x} = \int_X r^*_{interior}.d\mathbf{x} + \int_X r^*_{surf}.d\mathbf{x} \tag{17}$$