

# Radiative heat transfer solver with fluid motion

Jan I.C. Vermaak<sup>1,2</sup>, Jim E. Morel<sup>1,2</sup>

<sup>1</sup>Center for Large Scale Scientific Simulations, Texas A&M Engineering Experiment Station, College Station, Texas, USA.

<sup>2</sup>Nuclear Engineering Department, Texas A&M University, College Station, Texas, USA.

## Abstract:

Work is work for some, but for some it is play.

**Keywords:** hydrodynamics

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# 1 Definitions

## 1.1 Independent variables

We refer to the following independent variables:

- Position in the cartesian space  $\{x, y, z\}$  is denoted with  $\mathbf{x}$  and each component having units  $[cm]$ .
- Direction,  $\{\varphi, \theta\}$ , is denoted with  $\mathbf{\Omega}$  which takes on the form

$$\mathbf{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \text{ and/or } \mathbf{\Omega} = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix},$$

where  $\varphi$  is the azimuthal-angle and  $\theta$  is the polar-angle, both in spherical coordinates. Commonly,  $\cos \theta$ , is denoted with  $\mu$ . The general dimension of angular phase space is  $[steridian]$ .

- Photon frequency,  $\nu$  in  $[Hertz]$  or  $[s^{-1}]$ .
- Time,  $t$  in  $[s]$ .

## 1.2 Dependent variables

We use the following basic dependent variables:

- The foundation of the dependent unknowns is the **radiation angular intensity**,  $I(\mathbf{x}, \mathbf{\Omega}, \nu, t)$  with units  $[Joule/cm^2-s-steradian-Hz]$ . We often use the corresponding angle-integral of this quantity,  $\phi(\mathbf{x}, \nu, t)$ , and define it as

$$\phi(\mathbf{x}, \nu, t) = \mathcal{E}c = \int_{4\pi} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \quad (1.1)$$

with units  $[Joule/cm^2-s-Hz]$ . Where  $c$  is the speed of light.

- The **radiation energy density**,  $\mathcal{E}$ , is

$$\mathcal{E}(\mathbf{x}, \nu, t) = \frac{\phi}{c} = \frac{1}{c} \int_{4\pi} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \quad (1.2)$$

with units  $[Joule/cm^3-Hz]$ .

- The **radiation energy flux**,  $\mathcal{F}$ , is

$$\mathcal{F}(\mathbf{x}, \nu, t) = \int_{4\pi} \mathbf{\Omega} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \quad (1.3)$$

- **Radiation pressure**,  $\mathcal{P}$ , is

$$\mathcal{P}(\mathbf{x}, \nu, t) = \frac{1}{c} \int_{4\pi} \mathbf{\Omega} \otimes \mathbf{\Omega} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \quad (1.4)$$

and is a tensor.

### 1.3 Blackbody radiation

A blackbody radiation source,  $B(\nu, T)$ , is properly described by **Planck's law**,

$$B(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \quad (1.5)$$

with units [*Joule/cm<sup>2</sup>–s–steradian – Hz*] where  $h$  is Planck's constant and  $k_B$  is the Boltzmann constant.

If we integrate the blackbody source over all angle-space and frequencies then we get the mean radiation intensity from a blackbody at temperature  $T$  as

$$\begin{aligned} \int_0^\infty \int_{4\pi} B(\nu, T) d\Omega d\nu &= \int_0^\infty \int_{4\pi} \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} d\Omega d\nu \\ &= 4\pi \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} d\nu \\ &= acT^4, \end{aligned} \quad (1.6)$$

with units [*Joule/cm<sup>2</sup>–s–steradian*] and where  $a$  is the **blackbody radiation constant** given by

$$a = \frac{8\pi^5 k_B^4}{15h^3 c^3}. \quad (1.7)$$

In both cases this unfortunately is only the intensity. Following Kirchoff's law, which states that the emission and absorption of radiation must be equal in equilibrium, we can determine the **blackbody emission rate**,  $S_{bb}$ , from the absorption rate as

$$S_{bb}(\nu, T) = \rho\kappa(\nu)B(\nu, T), \quad (1.8)$$

with units [*Joule/cm<sup>3</sup>–s–steradian–Hz*] where  $\rho$  is the material density [*g/cm<sup>3</sup>*] and  $\kappa$  is the opacity [*cm<sup>2</sup>/g*]. The combination  $\rho\kappa$  is also equal to the macroscopic absorption cross section  $\sigma_a$ , therefore  $\rho\kappa(\nu) = \sigma_a$ . Data for the opacity of a material is normally available in the form of either the **Rosseland opacity**,  $\kappa_{Rs}$ , or the **Planck opacity**,  $\kappa_{Pl}$ .

## 2 Conservation equations

### 2.1 Conservation equation - Radiative transfer

The basic statement of conservation, is

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)}{\partial t} &= -\boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) - \sigma_t(\mathbf{x}, \nu) I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) \\ &+ \int_0^\infty \int_{4\pi} \frac{\nu}{\nu'} \sigma_s(\mathbf{x}, \nu' \rightarrow \nu, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) I(\mathbf{x}, \boldsymbol{\Omega}', \nu', t) d\nu' d\boldsymbol{\Omega}' \\ &+ \sigma_a(\mathbf{x}, \nu) B(\nu, T(\mathbf{x}, t)) + S \end{aligned} \quad (2.1)$$

where  $S$  is any other sources/sinks of radiation intensity.

### 2.2 Radiative transfer assuming isotropic Thompson scattering

Assuming Thomson-scattering<sup>1</sup> is the only form of scattering, gives

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)}{\partial t} &= -\boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) - \sigma_t(\mathbf{x}, \nu) I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) \\ &+ \frac{\sigma_s(\mathbf{x}, \nu)}{4\pi} c\mathcal{E}(\mathbf{x}, \nu) + \sigma_a(\mathbf{x}, \nu) B(\nu, T(\mathbf{x}, t)) + S \end{aligned} \quad (2.2)$$

where  $S$  is any other sources/sinks of radiation intensity.

Using energy instead of frequency,  $\nu \rightarrow E$ :

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, E, t)}{\partial t} &= -\boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, E, t) - \sigma_t(\mathbf{x}, E) I(\mathbf{x}, \boldsymbol{\Omega}, E, t) \\ &+ \frac{\sigma_s(\mathbf{x}, E)}{4\pi} c\mathcal{E}(\mathbf{x}, E) + \sigma_a(\mathbf{x}, E) B(E, T(\mathbf{x}, t)) + S \end{aligned} \quad (2.3)$$

where  $S$  is any other sources/sinks of radiation intensity.

### 2.3 Radiative transfer with material motion corrections

Applying relativistic corrections for a material in motion, we can derive (e.g., see NUEN 627 lecture 4) the laboratory-frame transport equation

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, E, t)}{\partial t} &= -\boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, E, t) - \left( \frac{E_0}{E} \right) \sigma_t(\mathbf{x}, E_0) I(\mathbf{x}, \boldsymbol{\Omega}, E, t) \\ &+ \left( \frac{E}{E_0} \right)^2 \frac{\sigma_s(\mathbf{x}, E)}{4\pi} \int_{4\pi} \left( \frac{E_0}{E'} \right) I(\mathbf{x}, \boldsymbol{\Omega}', E', t) d\boldsymbol{\Omega}' + \left( \frac{E}{E_0} \right)^2 \sigma_a(\mathbf{x}, E_0) B(E_0, T(\mathbf{x}, t)) + S, \end{aligned} \quad (2.4)$$

where

$$E_0 = E \gamma \left( 1 - \boldsymbol{\Omega} \cdot \frac{\mathbf{u}}{c} \right) \quad (2.5)$$

$$\gamma = \left[ 1 - \left( \frac{\|\mathbf{u}\|}{c} \right)^2 \right]^{-\frac{1}{2}} \quad (2.6)$$

$$\frac{E_0}{E'} = \gamma \left( 1 - \boldsymbol{\Omega}' \cdot \frac{\mathbf{u}}{c} \right) \quad (2.7)$$

$$E' = E \frac{1 - \boldsymbol{\Omega} \cdot \frac{\mathbf{u}}{c}}{1 - \boldsymbol{\Omega}' \cdot \frac{\mathbf{u}}{c}} \quad (2.8)$$

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<sup>1</sup>Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle. The particle's kinetic energy- as well as the photon's frequency, does not change in such a scattering. The scattering is also isotropic.

## 2.4 Radiative transfer with material velocity dependencies expanded to $\mathcal{O}(v/c)$

Very ugly derivations in NUEN 627 lecture 5 to get to,

$$\begin{aligned}
& \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, E, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, E, t) + \sigma_t(\mathbf{x}, E) I(\mathbf{x}, \boldsymbol{\Omega}, E, t) \\
&= \frac{\sigma_s(\mathbf{x}, E)}{4\pi} \phi(E) + \sigma_a(\mathbf{x}, E) B(E, T(\mathbf{x}, t)) \\
&+ \left[ \left( \sigma_t + E \frac{\partial \sigma_a}{\partial E} \right) I + \frac{\sigma_s}{4\pi} \left( 2\phi - E \frac{\partial \phi}{\partial E} \right) + 2\sigma_a B(E, T) - B(E, T) E \frac{\partial \sigma_a}{\partial E} - \sigma_a E \frac{\partial B(E, T)}{\partial E} \right] \boldsymbol{\Omega} \cdot \frac{\mathbf{u}}{c} \\
&- \frac{\sigma_s}{4\pi} \left( \mathcal{F} - E \frac{\partial \mathcal{F}}{\partial E} \right) \cdot \frac{\mathbf{u}}{c}
\end{aligned} \tag{2.9}$$

### Voodoo magic Grey Radiation Transport equation:

Somehow, determined by integrating over energy

$$\frac{1}{c} \frac{\partial I}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I + \sigma_t(\mathbf{x}) I = \frac{\sigma_s}{4\pi} \phi + \frac{\sigma_a}{4\pi} acT^4 - \frac{\sigma_t}{4\pi} \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi} \mathcal{E} \boldsymbol{\Omega} \cdot \mathbf{u} \tag{2.10}$$

### Radiation energy equation:

Obtained by integrating the transport equation over energy and angle

$$\begin{aligned}
\frac{\partial \mathcal{E}(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) &= \int_0^\infty \sigma_a(\mathbf{x}, E) (4\pi B(E, T) - \phi(\mathbf{x}, E, t)) dE \\
&+ \int_0^\infty \left( \sigma_a + E \frac{\partial \sigma_a}{\partial E} - \sigma_s(E) \right) \mathcal{F} \cdot \frac{\mathbf{u}}{c} dE
\end{aligned} \tag{2.11}$$

### Radiation momentum equation:

Obtained by first multiplying by  $\frac{1}{c} \boldsymbol{\Omega}$ , then integrating over all directions and energies,

$$\begin{aligned}
\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot \mathcal{P} &= - \int_0^\infty \frac{\sigma_t}{c} \mathcal{F} dE \\
&+ \int_0^\infty (\sigma_s \phi + \sigma_a 4\pi B(E, T)) \frac{\mathbf{u}}{c^2} dE \\
&+ \int_0^\infty \left( \sigma_a + E \frac{\partial \sigma_a}{\partial E} + \sigma_s \right) \mathcal{P} \cdot \frac{\mathbf{u}}{c} dE
\end{aligned} \tag{2.12}$$

## 2.5 Grey Radiative Transfer

$$\begin{aligned}
& \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, t) + \sigma_t(\mathbf{x}) I(\mathbf{x}, \boldsymbol{\Omega}, t) \\
&= \frac{\sigma_s}{4\pi} \phi + \frac{\sigma_a}{4\pi} acT^4 \\
&+ \left[ \sigma_t I + \frac{\sigma_s}{4\pi} 2\phi + 2\sigma_a \frac{1}{4\pi} acT^4 - \sigma_a E \frac{\partial B(E, T)}{\partial E} \right] \boldsymbol{\Omega} \cdot \frac{\mathbf{u}}{c} \\
&- \frac{\sigma_s}{4\pi} \mathcal{F} \cdot \frac{\mathbf{u}}{c}
\end{aligned} \tag{2.13}$$

### Radiation energy equation:

Obtained by integrating Eq. (2.13) over energy and angle

$$\begin{aligned}
\frac{\partial \mathcal{E}(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) &= \sigma_a c (aT^4 - \mathcal{E}) + (\sigma_a - \sigma_s) \mathcal{F} \cdot \frac{\mathbf{u}}{c} \\
&= \sigma_a c (aT^4 - \mathcal{E}_0) - \sigma_t \mathcal{F} \cdot \frac{\mathbf{u}}{c}
\end{aligned} \tag{2.14}$$

### Radiation momentum equation:

Obtained by first multiplying Eq. (2.13) by  $\frac{1}{c}\mathbf{\Omega}$ , then integrating over all directions and energies,

$$\begin{aligned}
\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot \mathcal{P} &= -\frac{\sigma_t}{c} \mathcal{F} + (\sigma_s c \mathcal{E} + \sigma_a a c T^4) \frac{\mathbf{u}}{c^2} + \sigma_t \mathcal{P} \cdot \frac{\mathbf{u}}{c} \\
&= -\frac{\sigma_t}{c} \mathcal{F} + ((\sigma_a + \sigma_s - \sigma_a) \mathcal{E} + \sigma_a a T^4) \frac{\mathbf{u}}{c} + \sigma_t \mathcal{P} \cdot \frac{\mathbf{u}}{c} \\
&= \frac{1}{c} \left[ -\sigma_t \mathcal{F} + ((\sigma_t - \sigma_a) \mathcal{E} + \sigma_a a T^4) \mathbf{u} + \sigma_t \mathcal{P} \cdot \mathbf{u} \right] \\
&= -\frac{1}{c} \left[ \sigma_t \mathcal{F} - ((\sigma_t - \sigma_a) \mathcal{E} + \sigma_a a T^4) \mathbf{u} - \sigma_t \mathcal{P} \cdot \mathbf{u} \right] \\
\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot \mathcal{P} &= -\frac{\sigma_t}{c} \mathcal{F}_0 + \sigma_a c (a T^4 - \mathcal{E}) \frac{\mathbf{u}}{c^2}
\end{aligned} \tag{2.15}$$

## 2.6 Grey Diffusion Approximation

Approximating the angular dependence of  $I(\mathbf{\Omega})$  with a  $P_1$  spherical harmonic expansion, such that the entries of  $\mathcal{P}$  are given by

$$(\mathcal{P})_{i,j} = \frac{1}{3} \mathcal{E} \delta_{i,j}, \tag{2.16}$$

the radiation energy equation is unaffected but the radiation momentum equation changes. We repeat the radiation energy equation below, and the altered radiation moment equations:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) = \sigma_a c (a T^4 - \mathcal{E}) + (\sigma_a - \sigma_s) \mathcal{F} \cdot \frac{\mathbf{u}}{c}, \tag{2.17}$$

$$\frac{1}{3} \nabla \mathcal{E} = -\frac{\sigma_t}{c} \mathcal{F} + (\sigma_s c \mathcal{E} + \sigma_a a c T^4) \frac{\mathbf{u}}{c^2} + \sigma_t \frac{1}{3} \mathcal{E} \frac{\mathbf{u}}{c}. \tag{2.18}$$

Useful transformations:

$$\mathcal{E}_0 = \mathcal{E} - \frac{2}{c^2} \mathcal{F} \cdot \mathbf{u} \tag{2.19a}$$

$$\mathcal{E} = \mathcal{E}_0 + \frac{2}{c^2} \mathcal{F}_0 \cdot \mathbf{u} \tag{2.19b}$$

$$\mathcal{F}_0 = \mathcal{F} - (\mathcal{E} \mathbf{u} + \mathcal{P} \cdot \mathbf{u}) \tag{2.19c}$$

$$\mathcal{F} = \mathcal{F}_0 + (\mathcal{E}_0 \mathbf{u} + \mathcal{P}_0 \cdot \mathbf{u}) \tag{2.19d}$$

$$\mathcal{P}_0 = \mathcal{P} - \frac{2}{c^2} \mathbf{u} \otimes \mathcal{F} \tag{2.19e}$$

$$\mathcal{P} = \mathcal{P}_0 + \frac{2}{c^2} \mathbf{u} \otimes \mathcal{F}_0 \tag{2.19f}$$

With the  $P_1$  approximation

$$\mathcal{F}_0 = \mathcal{F} - \frac{4}{3} \mathcal{E} \mathbf{u} \tag{2.19g}$$

$$\mathcal{F} = \mathcal{F}_0 + \frac{4}{3} \mathcal{E} \mathbf{u} \tag{2.19h}$$

Applying these transformations the radiation energy- and moment equation can be expressed as

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) = \sigma_a c (a T^4 - \mathcal{E}_0) - \sigma_t \mathcal{F} \cdot \frac{\mathbf{u}}{c}, \tag{2.20}$$

$$\frac{1}{3} \nabla \mathcal{E} = -\frac{\sigma_t}{c} \mathcal{F}_0 + \sigma_a c (a T^4 - \mathcal{E}) \frac{\mathbf{u}}{c^2}. \tag{2.21}$$

Several simplifications to these equations are made. Firstly arriving at the expression for the radiation energy equation,

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) = \sigma_a c (aT^4 - \mathcal{E}) - \sigma_t \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c}, \quad (2.22)$$

then the radiation momentum equation,

$$\frac{1}{3} \nabla \mathcal{E} = -\frac{\sigma_t}{c} \mathcal{F}_0 \quad (2.23)$$

from which we can get expression for  $\mathcal{F}_0$  and  $\mathcal{F}$  in terms of  $\mathcal{E}$  as

$$\mathcal{F}_0 = -\frac{c}{3\sigma_t} \nabla \mathcal{E} \quad (2.24)$$

and

$$\begin{aligned} \frac{1}{3} \nabla \mathcal{E} &= -\frac{\sigma_t}{c} \left( \mathcal{F} - \frac{4}{3} \mathcal{E} \mathbf{u} \right) \\ \therefore \mathcal{F} &= -\frac{c}{3\sigma_t} \nabla \mathcal{E} + \frac{4}{3} \mathcal{E} \mathbf{u}. \end{aligned} \quad (2.25)$$

These expressions for  $\mathcal{F}_0$  and  $\mathcal{F}$  are both then inserted into the radiation energy equation as follows

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) &= \sigma_a c (aT^4 - \mathcal{E}) - \sigma_t \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c} \\ \rightarrow \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left( -\frac{c}{3\sigma_t} \nabla \mathcal{E} + \frac{4}{3} \mathcal{E} \mathbf{u} \right) &= \sigma_a c (aT^4 - \mathcal{E}) - \sigma_t \left( -\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) \cdot \frac{\mathbf{u}}{c} \\ \rightarrow \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left( -\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) &= \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \end{aligned} \quad (2.26)$$

Arriving at a **diffusion form** of the **radiation energy equation**,

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left( -\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \quad (2.27)$$

## 2.7 Conservation equation for fluid flow

The governing equations we consider here are the Euler equations defined as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (2.28)$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \{ \rho \mathbf{u} \otimes \mathbf{u} \} + \nabla p = \mathbf{f}, \quad (2.29)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{u}] = q \quad (2.30)$$

where  $\rho$  is the fluid density,  $\mathbf{u} = [u_x, u_y, u_z] = [u, v, w]$  is the fluid velocity in cartesian coordinates,  $p$  is the fluid pressure,  $E$  is the material energy-density comprising kinetic energy-density,  $\frac{1}{2} \rho \|\mathbf{u}\|^2$ , and internal energy-density,  $\rho e$ , such that  $E = \frac{1}{2} \rho \|\mathbf{u}\|^2 + \rho e$ , where  $e$  is the specific internal energy. The values  $q$  and  $\mathbf{f}$  are abstractly used here as energy- and moment- sources/sinks, respectively.

The ideal gas law provides the closure relation

$$p = (\gamma - 1) \rho e \quad (2.31)$$

where  $\gamma$  is the ratio of the constant-pressure specific heat,  $c_p$ , to the constant-volume specific heat,  $c_v$ , i.e.,  $\gamma = \frac{c_p}{c_v}$ , and is a material property.



Coupling terms:

$$\begin{aligned}\mathbf{f} &= \frac{\sigma_t}{c} \mathcal{F}_0 \\ &= -\frac{1}{3} \nabla \mathcal{E}\end{aligned}\tag{2.32}$$

and

$$\begin{aligned}q &= -\left(\sigma_a c(aT^4 - \mathcal{E}) - \sigma_t \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c}\right) \\ &= \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}\end{aligned}\tag{2.33}$$

### 3 Solver A - Radiation Hydrodynamics Grey Diffusion

The set of Radiation Hydrodynamics Grey Diffusion Equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (3.1a)$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \nabla p = -\frac{1}{3} \nabla \mathcal{E}, \quad (3.1b)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{u}] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \quad (3.1c)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left( -\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \quad (3.1d)$$

where

$$E = \frac{1}{2} \rho \|\mathbf{u}\|^2 + \rho e, \quad (3.1e)$$

$$p = (\gamma - 1) \rho e, \quad (3.1f)$$

$$T = \frac{1}{C_v} e \quad (3.1g)$$

$$\sigma_t(T) = \sigma_s(T) + \sigma_a(T) \quad (3.1h)$$

$$\sigma_s(T) = \rho \kappa_s(T) \quad (3.1i)$$

$$\sigma_a(T) = \rho \kappa_a(T) \quad (3.1j)$$

#### 3.1 Definitions

First we define the following terms

- The radiation emission and absorption, the radiation momentum source, and the radiation energy source

$$S_{ea} = \sigma_a c (aT^4 - \mathcal{E}) \quad (3.2a)$$

$$\mathbf{S}_{rp} = \frac{1}{3} \nabla \mathcal{E} \quad (3.2b)$$

$$S_{re} = S_{ea} + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \quad (3.2c)$$

- The conserved hydrodynamic variables,  $\mathbf{U}$ , and associated hydrodynamic flux,  $\mathcal{F}^H$ ,

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho \mathbf{u} \\ E \end{bmatrix} \quad \mathcal{F}^H = \begin{bmatrix} \rho u \\ \rho u u + p \\ \rho u v \\ \rho u w \\ (E + p)u \end{bmatrix} \quad (3.2d)$$

- The stationary reference frame radiation energy flux

$$\mathcal{F}_0 = -\frac{c}{3\sigma_t} \nabla \mathcal{E} \quad (3.2e)$$

Next, we use these terms to define a more condensed version of the RHGD equations.

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \quad (3.3)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}_0 + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = S_{ea} + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \quad (3.4)$$

## 3.2 Finite Volume Spatial Discretization

To apply a finite volume spatial discretization we integrate our time-discretized equations over the volume,  $V_c$ , of cell  $c$ , and afterwards divide by  $V_c$ . This leaves all the terms containing  $\tau$  unchanged. In this process we develop the following terms:

### 3.2.1 Hydrodynamic and Radiation-energy advection

$$\frac{1}{V_c} \int_{V_c} \nabla \cdot \mathcal{F}^H(\mathbf{U}) dV = \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot \mathcal{F}^H(\mathbf{U}_f) \quad (3.5)$$

$$\frac{1}{V_c} \int_{V_c} \left( \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right) dV = \frac{1}{V_c} \sum_f \frac{4}{3} \mathbf{A}_f \cdot (\mathcal{E} \mathbf{u})_f \quad (3.6)$$

The face values are reconstructed from gradients in both the predictor and corrector phases. In the corrector-phase the hydrodynamic flux,  $\mathcal{F}^H$ , is used in its earlier defined form, whilst in the corrector-phase the flux is determined by an approximate Riemann-solver, i.e., the HLLC Riemann solver.

#### Predictor phases:

For the predictor phase we have the following:

$$\nabla \cdot \mathcal{F}^H(\mathbf{U}) \mapsto \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot \mathcal{F}^H(\mathbf{U}_f) \quad (3.7)$$

$$\left( \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right) \mapsto \frac{1}{V_c} \sum_f \frac{4}{3} \mathbf{A}_f \cdot (\mathcal{E} \mathbf{u})_f \quad (3.8)$$

$$\mathbf{U}_f = \mathbf{U}_c + (\mathbf{x}_f - \mathbf{x}_c) \cdot \{\nabla \mathbf{U}\}_c \quad (3.9)$$

$$\mathcal{E}_f = \mathcal{E}_c + (\mathbf{x}_f - \mathbf{x}_c) \cdot \{\nabla \mathcal{E}\}_c \quad (3.10)$$

#### Corrector phases:

For the corrector phase we have the following:

$$\nabla \cdot \mathcal{F}^H(\mathbf{U}) \mapsto \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot \mathbf{F}^{*hllc}(\mathbf{U}_f) \quad (3.11)$$

$$\left( \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right) \mapsto \frac{1}{V_c} \sum_f \frac{4}{3} \mathbf{A}_f \cdot (\mathcal{E} \mathbf{u})_{upw} \quad (3.12)$$

where

$$\mathbf{U}_f = \mathbf{U}_c + (\mathbf{x}_f - \mathbf{x}_c) \cdot \{\nabla \mathbf{U}\}_c \quad (3.13)$$

$$(\mathcal{E} \mathbf{u})_{upw} = \begin{cases} (\mathcal{E} \mathbf{u})_{c,f}, & \text{if } \mathbf{u}_{c,f} \cdot \mathbf{n}_f > 0 \text{ and } \mathbf{u}_{cn,f} \cdot \mathbf{n}_f > 0 & \rightarrow | \rightarrow \\ (\mathcal{E} \mathbf{u})_{cn,f}, & \text{if } \mathbf{u}_{c,f} \cdot \mathbf{n}_f < 0 \text{ and } \mathbf{u}_{cn,f} \cdot \mathbf{n}_f < 0 & \leftarrow | \leftarrow \\ (\mathcal{E} \mathbf{u})_{cn,f} + (\mathcal{E} \mathbf{u})_{c,f}, & \text{if } \mathbf{u}_{c,f} \cdot \mathbf{n}_f > 0 \text{ and } \mathbf{u}_{cn,f} \cdot \mathbf{n}_f < 0 & \rightarrow | \leftarrow \\ 0, & \text{if } \mathbf{u}_{c,f} \cdot \mathbf{n}_f < 0 \text{ and } \mathbf{u}_{cn,f} \cdot \mathbf{n}_f > 0 & \leftarrow | \rightarrow \end{cases} \quad (3.14)$$

$$\mathcal{E}_{c,f} = \mathcal{E}_c + (\mathbf{x}_f - \mathbf{x}_c) \cdot \{\nabla \mathcal{E}\}_c \quad (3.15)$$

### 3.2.2 Density and momentum updates

We apply the same process as before:

$$-\frac{1}{V_c} \int_{V_c} \mathbf{S}_{rp} dV = -\frac{1}{V_c} \sum_f \frac{1}{3} \mathbf{A}_f \mathcal{E}_f, \quad (3.16)$$

however, here we want  $\mathcal{E}_f$  to satisfy the following relationship

$$\frac{D_c}{\|\mathbf{x}_{cf}\|} (\mathcal{E}_f - \mathcal{E}_c) = \frac{D_{cn}}{\|\mathbf{x}_{fcn}\|} (\mathcal{E}_{cn} - \mathcal{E}_f) \quad (3.17)$$

where

$$D_c = -\frac{c}{3\sigma_{t,c}} \quad (3.18)$$

and where  $\mathbf{x}_{cf}$  is the vector from cell  $c$ 's centroid to the face centroid,  $\mathbf{x}_{fcn}$  is the vector from the face centroid to cell  $cn$ 's centroid (where cell  $cn$  is the neighbor to  $c$  at face  $f$ ). The norm  $\|\cdot\|$  refers to the  $L_2$  norm.

Solving the above relationship for  $\mathcal{E}_f$  we first set

$$k_c = \frac{D_c}{\|\mathbf{x}_{cf}\|}, \quad k_{cn} = \frac{D_{cn}}{\|\mathbf{x}_{fcn}\|}$$

then get

$$\begin{aligned} k_c \mathcal{E}_f - k_c \mathcal{E}_c &= k_{cn} \mathcal{E}_{cn} - k_{cn} \mathcal{E}_f \\ \rightarrow (k_c + k_{cn}) \mathcal{E}_f &= k_{cn} \mathcal{E}_{cn} + k_c \mathcal{E}_c \\ \therefore \mathcal{E}_f &= \frac{k_{cn} \mathcal{E}_{cn} + k_c \mathcal{E}_c}{k_c + k_{cn}}. \end{aligned} \quad (3.19)$$

### Predictor and corrector phases:

We do the same for both,

$$-\mathbf{S}_{rp} \mapsto -\frac{1}{V_c} \sum_f \frac{1}{3} \mathbf{A}_f \mathcal{E}_f \quad (3.20)$$

### 3.2.3 Energy equations

Only two terms require special consideration here. They are: the divergence of the co-moving frame radiation energy flux, and the kinetic energy terms source terms,

$$\begin{aligned} \frac{1}{V_c} \int_{V_c} \nabla \cdot \mathcal{F}_0 dV &= \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot (\mathcal{F}_0)_f \\ \frac{1}{V_c} \int_{V_c} \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} dV &= \frac{1}{V_c} \sum_f \frac{1}{3} \mathbf{A}_f \cdot (\mathcal{E} \mathbf{u})_f. \end{aligned} \quad (3.21)$$

#### 3.2.3.1 The diffusion term

Considering the  $\mathcal{F}_0$ -term first, we apply Gauss' divergence theorem to get

$$\nabla \cdot \mathcal{F}_0 \mapsto \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot (\mathcal{F}_0)_f. \quad (3.22)$$

For  $(\mathcal{F}_0)_f$  we have

$$(\mathcal{F}_0)_f = -\frac{c}{3\sigma_{tf}} (\nabla \mathcal{E})_f. \quad (3.23)$$

Now define

$$D_f = -\frac{c}{3\sigma_{tf}}. \quad (3.24)$$

To find  $D_f$  we seek the equivalence:

$$D_f \frac{\mathcal{E}_{cn} - \mathcal{E}_c}{\|\mathbf{x}_{cn} - \mathbf{x}_c\|} = D_c \frac{\mathcal{E}_f - \mathcal{E}_c}{\|\mathbf{x}_f - \mathbf{x}_c\|} = D_{cn} \frac{\mathcal{E}_{cn} - \mathcal{E}_f}{\|\mathbf{x}_{cn} - \mathbf{x}_f\|} \quad (3.25)$$

Now let us define

$$\begin{aligned} k_c &= \frac{D_c}{\|\mathbf{x}_f - \mathbf{x}_c\|} \\ k_{cn} &= \frac{D_{cn}}{\|\mathbf{x}_{cn} - \mathbf{x}_f\|} \end{aligned} \quad (3.26)$$

Now

$$\begin{aligned} k_c(\mathcal{E}_f - \mathcal{E}_c) &= k_{cn}(\mathcal{E}_{cn} - \mathcal{E}_f) \\ (k_c + k_{cn})\mathcal{E}_f &= k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c \\ \therefore \mathcal{E}_f &= \frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}} \end{aligned} \quad (3.27)$$

Now we choose any of the right-two terms in the three way equality and plug the expression for  $\mathcal{E}_f$ ,

$$\begin{aligned} &k_c(\mathcal{E}_f - \mathcal{E}_c) \\ &= k_c \left( \frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}} - \mathcal{E}_c \right) \\ &= k_c \left( \frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c - k_c\mathcal{E}_c - k_{cn}\mathcal{E}_c}{k_c + k_{cn}} \right) \\ \therefore D_f \frac{\mathcal{E}_{cn} - \mathcal{E}_c}{\|\mathbf{x}_{cn} - \mathbf{x}_c\|} &= \frac{k_c k_{cn}}{k_c + k_{cn}} (\mathcal{E}_{cn} - \mathcal{E}_c) \\ \therefore D_f &= \frac{k_c k_{cn}}{k_c + k_{cn}} \|\mathbf{x}_{cn} - \mathbf{x}_c\| \end{aligned} \quad (3.28)$$

From the earlier expression for  $(\mathcal{F}_0)_f$ , we can write

$$(\mathcal{F}_0)_f = D_f (\mathcal{E}_{cn} - \mathcal{E}_c) \frac{\mathbf{x}_{cn} - \mathbf{x}_c}{\|\mathbf{x}_{cn} - \mathbf{x}_c\|^2} \quad (3.29)$$

for which we can define

$$\mathbf{k}_f = D_f \frac{\mathbf{x}_{cn} - \mathbf{x}_c}{\|\mathbf{x}_{cn} - \mathbf{x}_c\|^2} \quad (3.30)$$

such that we finally arrive at

$$(\mathcal{F}_0)_f = \mathbf{k}_f (\mathcal{E}_{cn} - \mathcal{E}_c). \quad (3.31)$$

### 3.2.3.2 The kinetic energy term

For the kinetic energy source terms, we similarly have

$$\left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^n \mapsto \frac{1}{V_c} \sum_f \frac{1}{3} \mathbf{A}_f \cdot (\mathcal{E}_f^n \mathbf{u}_f^n) \quad (3.32)$$

where we use the reconstructed values as in the Hydrodynamic and radiation-energy advection portion.

### 3.3 Temporal scheme - Implicit Euler Predictor, Crank-Nicolson Corrector

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \quad (3.33a)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}_0 + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = S_{re}. \quad (3.33b)$$

#### 3.3.1 Predictor phase

$$\tau = \frac{1}{\frac{1}{2}\Delta t}$$

$$\tau(\mathbf{U}^{n*} - \mathbf{U}^n) + \nabla \cdot \mathcal{F}^H(\mathbf{U}^n) = \mathbf{0} \quad (3.34a)$$

$$\tau \left( \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{1}{2}} - \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n*} \right) = \begin{bmatrix} 0 \\ -\frac{1}{3} \nabla \mathcal{E} \end{bmatrix}^n \quad (3.34b)$$

$$\tau(\mathcal{E}^{n*} - \mathcal{E}^n) + \left( \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right)^n = 0 \quad (3.34c)$$

$$\tau(E^{n+\frac{1}{2}} - E^{n*}) = -\theta_1 S_{ea}^{n+\frac{1}{2}} - \theta_2 S_{ea}^n - \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^n \quad (3.34d)$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^{n*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+\frac{1}{2}} + \theta_2 \nabla \cdot \mathcal{F}_0^n = \theta_1 S_{ea}^{n+\frac{1}{2}} + \theta_2 S_{ea}^n + \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^n \quad (3.34e)$$

For  $S_{ea}$  and  $\mathcal{F}_0$  both at  $n + \frac{1}{2}$ :

$$\sigma^{n+\frac{1}{2}} = \rho^{n+\frac{1}{2}} \kappa(T^n) \quad (3.34f)$$

$$T^{4,n+\frac{1}{2}} = T^{4,n*} + \frac{4T^{3,n*}}{C_v} (e^{n+\frac{1}{2}} - e^{n*}) \quad (3.34g)$$

#### 3.3.2 Corrector phase

$$\tau = \frac{1}{\Delta t}$$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^n) + \nabla \cdot \mathcal{F}^H(\mathbf{U}^{n+\frac{1}{2}}) = \mathbf{0} \quad (3.35a)$$

$$\tau \left( \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+1} - \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{1}{2}*} \right) = \begin{bmatrix} 0 \\ -\frac{1}{3} \nabla \mathcal{E} \end{bmatrix}^{n+\frac{1}{2}} \quad (3.35b)$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}*} - \mathcal{E}^n) + \left( \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right)^{n+\frac{1}{2}} = 0 \quad (3.35c)$$

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 S_{ea}^{n+1} - \theta_2 S_{ea}^n - \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (3.35d)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = \theta_1 S_{ea}^{n+1} + \theta_2 S_{ea}^n + \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (3.35e)$$

For  $S_{ea}$  and  $\mathcal{F}_0$  both at  $n + 1$ :

$$\sigma^{n+1} = \rho^{n+1} \kappa(T^{n+\frac{1}{2}}) \quad (3.35f)$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v} (e^{n+1} - e^{n+\frac{1}{2}*}) \quad (3.35g)$$

### 3.3.3 General energy equations, Predictor and Corrector phase, with $\theta$ factors

Time integration scheme A uses **implicit Euler** for the predictor phase and **Crank-Nicolson** in the corrector phase. Both these schemes can be represented with a general  $\theta$ -scheme where we define:

$$\begin{aligned}\theta_1 &\in [0, 1] \\ \theta_2 &= 1 - \theta_1.\end{aligned}\tag{3.36}$$

For implicit Euler,  $\theta_1 = 1$ ,  $\theta_2 = 0$ , and for Crank-Nicolson,  $\theta_1 = \theta_2 = \frac{1}{2}$ . With these factors defined we can repeat the energy equations and apply a series of manipulations. First we attempt to segregate known terms from all unknown terms. Thereafter we eliminate the internal energy,  $e$ , from the two sets of equations to get a single formulation for the radiation energy,  $\mathcal{E}$ . The latter formulation forms a diffusion system that needs to be assembled and solved for  $\mathcal{E}$ .

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 \sigma_a^{n+1} c \left( aT^{4,n+1} - \mathcal{E}^{n+1} \right) - \theta_2 S_{ea}^n - \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{3.37a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = \theta_1 \sigma_a^{n+1} c \left( aT^{4,n+1} - \mathcal{E}^{n+1} \right) + \theta_2 S_{re}^n + \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{3.37b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v} (e^{n+1} - e^{n+\frac{1}{2}*})\tag{3.37c}$$

Define:

$$\begin{aligned}k_1 &= \theta_1 \sigma_a^{n+1} c \\ k_2 &= \frac{4T^{3,n+\frac{1}{2}*}}{C_v}\end{aligned}\tag{3.38}$$

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 \left( aT^{4,n+1} - \mathcal{E}^{n+1} \right) - \theta_2 S_{ea}^n - \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{3.39a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = k_1 \left( aT^{4,n+1} - \mathcal{E}^{n+1} \right) + \theta_2 S_{re}^n + \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{3.39b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2 (e^{n+1} - e^{n+\frac{1}{2}*})\tag{3.39c}$$

ungroup right-hand side elements by multiplying out terms within parentheses,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a T^{4,n+1} + k_1 \mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{3.40a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = k_1 a T^{4,n+1} - k_1 \mathcal{E}^{n+1} + \theta_2 S_{re}^n + \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{3.40b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2 (e^{n+1} - e^{n+\frac{1}{2}*})\tag{3.40c}$$

now plug in the temperature equation into both the energy equations,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a (T^{4,n+\frac{1}{2}*} + k_2 (e^{n+1} - e^{n+\frac{1}{2}*})) + k_1 \mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{3.41a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = k_1 a (T^{4,n+\frac{1}{2}*} + k_2 (e^{n+1} - e^{n+\frac{1}{2}*})) - k_1 \mathcal{E}^{n+1} + \theta_2 S_{re}^n + \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (3.41b)$$

ungroup elements on the both the right-hand sides,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a T^{4,n+\frac{1}{2}*} - k_1 a k_2 e^{n+1} + k_1 a k_2 e^{n+\frac{1}{2}*} + k_1 \mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (3.42a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = k_1 a T^{4,n+\frac{1}{2}*} + k_1 a k_2 e^{n+1} - k_1 a k_2 e^{n+\frac{1}{2}*} - k_1 \mathcal{E}^{n+1} + \theta_2 S_{re}^n + \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (3.42b)$$

Define:

$$\begin{aligned} k_3 &= -k_1 a T^{4,n+\frac{1}{2}*} + k_1 a k_2 e^{n+\frac{1}{2}*} - \theta_2 S_{ea}^n - \left( \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \\ k_4 &= -k_1 a k_2 \end{aligned} \quad (3.43)$$

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \quad (3.44a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \quad (3.44b)$$

Note:

$$E^{n+1} = \left( \frac{1}{2} \rho ||\mathbf{u}||^2 \right)^{n+1} + \rho^{n+1} e^{n+1} \quad (3.45)$$

which gives,

$$\tau \left( \left( \frac{1}{2} \rho ||\mathbf{u}||^2 \right)^{n+1} + \rho^{n+1} e^{n+1} - E^{n+\frac{1}{2}*} \right) = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \quad (3.46a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \quad (3.46b)$$

ungroup the material energy in the first equation,

$$\tau \left( \frac{1}{2} \rho ||\mathbf{u}||^2 \right)^{n+1} + \tau \rho^{n+1} e^{n+1} - \tau E^{n+\frac{1}{2}*} = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \quad (3.47a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \quad (3.47b)$$

and isolate the internal energy in the first equation,

$$(\tau \rho^{n+1} - k_4) e^{n+1} = k_1 \mathcal{E}^{n+1} + k_3 - \tau \left( \frac{1}{2} \rho ||\mathbf{u}||^2 \right)^{n+1} + \tau E^{n+\frac{1}{2}*} \quad (3.48a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \quad (3.48b)$$



Define:

$$\begin{aligned} k_5 &= \frac{k_1}{\tau \rho^{n+1} - k_4} \\ k_6 &= \frac{k_3 - \tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau E^{n+\frac{1}{2}*}}{\tau \rho^{n+1} - k_4} \end{aligned} \quad (3.49)$$

and plug these constants into the first equation above,

$$e^{n+1} = k_5 \mathcal{E}^{n+1} + k_6 \quad (3.50a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = -k_1 \mathcal{E}^{n+1} - k_3 - k_4 e^{n+1} \quad (3.50b)$$

now plug the first equation into the second,

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = -k_1 \mathcal{E}^{n+1} - k_3 - k_4 k_5 \mathcal{E}^{n+1} - k_4 k_6 \quad (3.51a)$$

now collect all the  $\mathcal{E}^{n+1}$  terms on the left-hand side,

$$(\tau + k_1 + k_4 k_5) \mathcal{E}^{n+1} + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} = -k_3 - k_4 k_6 + \tau \mathcal{E}^{n+\frac{1}{2}*} - \theta_2 \nabla \cdot \mathcal{F}_0^n \quad (3.52a)$$

Recall:

$$\nabla \cdot \mathcal{F}_0 \mapsto \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot (\mathcal{F}_0)_f \quad (3.53)$$

and

$$(\mathcal{F}_0)_f = \mathbf{k}_f (\mathcal{E}_{cn} - \mathcal{E}_c) \quad (3.54)$$

which gives the system,

$$(\tau + k_1 + k_4 k_5) \mathcal{E}^{n+1} + \frac{\theta_1}{V_c} \sum_f \mathbf{A}_f \cdot \mathbf{k}_f^{n+1} (\mathcal{E}_{cn}^{n+1} - \mathcal{E}_c^{n+1}) = -k_3 - k_4 k_6 + \tau \mathcal{E}^{n+\frac{1}{2}*} - \frac{\theta_2}{V_c} \sum_f \mathbf{A}_f \cdot \mathbf{k}_f^n (\mathcal{E}_{cn}^n - \mathcal{E}_c^n) \quad (3.55a)$$

This system is SPD and in one dimension forms a tridiagonal system.

### 3.3.4 Using the energy related algebra for both the predictor and the corrector

To perform the energy related algebra for the corrector step we need the following inputs:

$\kappa_a^n$	For $\sigma_a^n$ in $S_{ea}^n$
$\kappa_t^n$	For $\sigma_t^n$ in $\nabla \cdot \mathcal{F}_0^n$
$\kappa_a^{n+\frac{1}{2}}$	For $\sigma_a^{n+1}$ in $S_{ea}^{n+1}$
$\kappa_t^{n+\frac{1}{2}}$	For $\sigma_t^{n+1}$ in $\nabla \cdot \mathcal{F}_0^{n+1}$
$C_v$	For the linearization of $T^{4,n+1}$
$\tau$	For the time constant
$\theta_1, \theta_2$	For the time scheme
$\mathbf{U}^n$	For $T, \rho$ in $S_{ea}^n$
$\mathbf{U}^{n+\frac{1}{2}}$	For $\mathbf{u}$ in $\left(\frac{1}{3}\nabla\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathbf{U}^{n+\frac{1}{2}*}$	For $E^{n+\frac{1}{2}*}$ and $e^{n+\frac{1}{2}*}$
$\mathbf{U}_{0,1}^{n+1} = \begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}_{n+1}$	For the kinetic energy in $E^{n+1}$ , and $\rho^{n+1} \rightarrow \sigma_a^{n+1}, \sigma_t^{n+1}$
$\nabla\mathbf{U}^{n+\frac{1}{2}}$	For the reconstructions in $\left(\frac{1}{3}\nabla\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathcal{E}^n$	For $S_{ea}^n$
$\mathcal{E}^{n+\frac{1}{2}}$	For $\mathcal{E}$ in $\left(\frac{1}{3}\nabla\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathcal{E}^{n+\frac{1}{2}*}$	For itself
$\nabla\mathcal{E}^{n+\frac{1}{2}}$	For the reconstructions in $\left(\frac{1}{3}\nabla\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}}$

To following remapping(s) then applies to the predictor:

$\kappa_a^n \rightarrow \kappa_a^n$	$\mathcal{E}^n \rightarrow \mathcal{E}^n$
$\kappa_t^n \rightarrow \kappa_t^n$	$\mathcal{E}^n \rightarrow \mathcal{E}^{n+\frac{1}{2}}$
$\kappa_a^n \rightarrow \kappa_a^{n+\frac{1}{2}}$	$\mathcal{E}^{n*} \rightarrow \mathcal{E}^{n+\frac{1}{2}*}$
$\kappa_t^n \rightarrow \kappa_t^{n+\frac{1}{2}}$	$\nabla\mathcal{E}^n \rightarrow \nabla\mathcal{E}^{n+\frac{1}{2}}$
$\mathbf{U}^n \rightarrow \mathbf{U}^n$	
$\mathbf{U}^n \rightarrow \mathbf{U}^{n+\frac{1}{2}}$	
$\mathbf{U}^{n*} \rightarrow \mathbf{U}^{n+\frac{1}{2}*}$	
$\mathbf{U}^{n+\frac{1}{2}} \rightarrow \mathbf{U}^{n+1}$	
$\nabla\mathbf{U}^n \rightarrow \nabla\mathbf{U}^{n+\frac{1}{2}}$	

## 4 Solver B - Radiation Hydrodynamics Grey Diffusion - Mixed finite element

We now derive a general mixed finite element approach for

$$\nabla \cdot \mathcal{F}_0(\mathbf{x}) = 1, \quad \mathbf{x} \in \mathcal{D} \quad (4.1a)$$

$$\mathcal{F}_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\mathcal{D} \quad (4.1b)$$

where

$$\mathcal{F}_0(\mathbf{x}) = D(\mathbf{x}) \nabla \mathcal{E}(\mathbf{x}). \quad (4.2)$$

### 4.1 Auxiliary notation and variables for $\mathcal{F}_0$

First we discretize  $\mathcal{F}_0$  on  $N_n$  number of nodes per cell  $c$ , using continuous basis functions  $b_j(\mathbf{x})$  such that

$$\mathcal{F}_0(\mathbf{x}) \approx \sum_{j=1}^{N_n} (\mathcal{F}_0)_j b_j(\mathbf{x}), \quad (4.3)$$

whilst keeping the cell-centered representation for  $\mathcal{E}$ . Next we discretize eq. (4.2) by applying a weight function  $b_i(\mathbf{x})$  and integrating over the volume of the cell  $c$ ,

$$\begin{aligned} \int_{V_c} b_i \mathcal{F}_0 dV &= \int_{V_c} b_i D \nabla \mathcal{E} dV \\ \sum_j \left[ \int_{V_c} b_i b_j dV \right] (\mathcal{F}_0)_j &= \int_{V_c} b_i D \nabla \mathcal{E} dV. \end{aligned} \quad (4.4)$$

The integral coefficients on the left-hand side are generally known as the  $ij$  coefficients in the standard finite element mass-matrix, which we shall use in a moment to define a general scheme. The right hand side of the equation requires some treatment. We introduce the values  $\mathcal{E}_{c,j}$  at the cell surface to remedy the discontinuities in the cell-centered  $\mathcal{E}$  and to define auxiliary unknowns for developing the  $\mathcal{F}_0$  in finite element form. With these new variables declared, we next apply integration by parts to the right-hand side,

$$\int_{V_c} b_i D \nabla \mathcal{E} dV = \int_{V_c} D \nabla (b_i \mathcal{E}) dV - \int_{V_c} D \mathcal{E} \nabla b_i dV. \quad (4.5)$$

Next we apply Gauss's divergence theorem on the first term on the right-hand side,

$$\int_{V_c} b_i D \nabla \mathcal{E} dV = \sum_f \int_{S_f} D \mathbf{n}_f b_i \mathcal{E} dA - \int_{V_c} D \mathcal{E} \nabla b_i dV, \quad (4.6)$$

after which we insert  $\mathcal{E}_{c,j}$  in the first term on the right, since they are designated unknowns on the surface of the cell, and  $\mathcal{E}_c$  into the right most term. Since  $\mathcal{E}_c$  is cell-constant within the cell-domain, the  $D$  coefficient is also dependent only on  $\mathcal{E}_c$  and therefore constant within cell  $c$ , hence denoted as  $D_c$ ,

$$\int_{V_c} b_i D \nabla \mathcal{E} dV = \sum_f \sum_j \left[ D_c \mathbf{n}_f \int_{S_f} b_i b_j dA \right] \mathcal{E}_{c,j} - \left[ D_c \int_{V_c} \nabla b_i dV \right] \mathcal{E}_c. \quad (4.7)$$

Putting the developed right- and left-hand sides back together we then get,

$$\sum_j \left[ \int_{V_c} b_i b_j dV \right] (\mathcal{F}_0)_j = \sum_f \sum_j \left[ D_c \mathbf{n}_f \int_{S_f} b_i b_j dA \right] \mathcal{E}_{c,j} - \left[ D_c \int_{V_c} \nabla b_i dV \right] \mathcal{E}_c. \quad (4.8)$$

This equation can be written in more succinct form as

$$\bar{M}_c \bar{\mathbf{F}}_c = \bar{C}_c \mathcal{E}_c \quad (4.9)$$

where the structure still needs to be defined (which follows).  $\bar{M}_c$  is a square block-matrix with block-dimension  $N_n \times N_n$ ,  $\bar{\mathbf{F}}_c$  is a block-vector with block-dimension  $N_n \times 1$ ,  $\bar{C}_c$  is a rectangular block-matrix with block-dimension  $N_n \times (N_f + 1)$ . The vector  $\mathcal{E}_c$  is simply the cell-centered and surface unknowns for cell  $c$ , i.e.,  $\mathcal{E}_c = [\mathcal{E}_c, \mathcal{E}_{n=0}, \dots, \mathcal{E}_{n=N_n-1}]^T$ .

The dimension of the inner blocks of  $\bar{M}_c$ ,  $\bar{\mathbf{F}}_c$  and  $\bar{C}_c$ , depend on the number of dimensions,  $N_d$ , in the problem. For further reference we shall denote dimensions with  $d$  but it generally refers to  $d \in [0, 1, 2] \mapsto [x, y, z]$  and vice-versa.

The block entries of  $\bar{M}$  are small diagonal matrices,

$$(\bar{M})_{ij} = \text{diag}(M_{ij}, \dots, M_{ij})^{N_d \times N_d} \quad (4.10)$$

where  $M_{ij}$  are the elements of the standard finite element mass-matrix for cell  $c$ , i.e.,

$$M_{ij} = \int_V b_i b_j dV. \quad (4.11)$$

The block entries of  $\bar{\mathbf{F}}$  are

$$(\bar{\mathbf{F}})_i = \begin{bmatrix} (\mathcal{F}_0)_{i,x} \\ (\mathcal{F}_0)_{i,y} \\ (\mathcal{F}_0)_{i,z} \end{bmatrix}^{N_d \times 1} \quad (4.12)$$

obviously only up to  $y$  for 2D and only up to  $x$  for 1D. The entries of  $\bar{C}$  are formed as follows. First the structure of  $\bar{C}$  is such that

$$\text{block-row } i \text{ of } \bar{C} = \text{columns } (\mathbf{C}_i^c \quad \mathbf{C}_{i,j=0}^s \quad \dots \quad \mathbf{C}_{i,j=N_n-1}^s)^{N_d \times (N_n+1)}. \quad (4.13)$$

We then define the vectors

$$\begin{aligned} \mathbf{G}_i &= \int_V \nabla b_i dV \\ M_{ij}^f &= \int_{S_f} b_i b_j dA \end{aligned} \quad (4.14)$$

Then,

$$\mathbf{C}_i^c = \begin{bmatrix} D_c(\mathbf{G}_i)_x \\ D_c(\mathbf{G}_i)_y \\ D_c(\mathbf{G}_i)_z \end{bmatrix}^{N_d \times 1} \quad (4.15)$$

and

$$\mathbf{C}_{ij}^s = \begin{bmatrix} \sum_f D_c n_{f,x} M_{ij}^f \\ \sum_f D_c n_{f,y} M_{ij}^f \\ \sum_f D_c n_{f,z} M_{ij}^f \end{bmatrix}^{N_d \times 1} \quad (4.16)$$

With these definitions in-hand we can see that the true dimensions of  $\bar{M}$  is  $N_d N_n \times N_d N_n$ , that of  $\bar{\mathbf{F}}$  is  $N_d N_n \times 1$ , and the true dimensions of  $\bar{C}$  is  $N_d N_n \times (N_n + 1)$ .

Finally, we repeat here that the vector  $\mathcal{E}$  is

$$\mathcal{E}_c = \begin{bmatrix} \mathcal{E}_c \\ \mathcal{E}_{n=0} \\ \vdots \\ \mathcal{E}_{n=N_n-1} \end{bmatrix}^{(N_n+1) \times 1}. \quad (4.17)$$

To get an expression for all of the nodal  $\mathcal{F}_0$ 's we take the system form of the equation and we invert  $\bar{M}$  to get, in coefficient form, expressions for nodal  $\mathcal{F}_0$ 's,

$$\mathbf{F}_c = \begin{bmatrix} (\mathcal{F}_0)_0 \\ \vdots \\ (\mathcal{F}_0)_{N_n-1} \end{bmatrix} = \bar{M}_c^{-1} \bar{C}_c \mathcal{E} = C_c^* \mathcal{E}_c, \quad (4.18)$$

where  $C_c^* = \bar{M}_c^{-1} \bar{C}_c$ .

With this expression-form of the individual nodal  $\mathcal{F}_0$ 's we need to modify the primary equation, eq. (4.1). Additionally, since we introduced additional variables in the form of the face-based  $\mathcal{E}_f$ 's, we need to define additional equations to close the system. For the primary equations we will simply plug in the expressions for  $\mathcal{F}_0$ , which is detailed in the next subsection. For additional equations we will use the interface between cells to enforce continuity of  $\mathcal{F}_0$  at the face, for each cell of the face.

## 4.2 Using the auxiliary notation in the primary equation

Using this coefficient-form in the primary equations is done by first integrating eq. (4.1) over the volume of cell  $c$ , assuming the coefficient matrix  $\bar{M}^{-1}\bar{C}$  has been developed for cell  $c$ , after which we apply Gauss's divergence theorem,

$$\begin{aligned}\int_{V_c} \nabla \cdot \mathcal{F}_0 dV &= V_c \\ \int_{S_c} \mathbf{n} \cdot \mathcal{F}_0 dA &= V_c \\ \sum_f \left[ \mathbf{n}_f \cdot \int_{S_f} \mathcal{F}_0 dA \right] &= V_c.\end{aligned}\tag{4.19}$$

We now expand  $\mathcal{F}_0$ ,

$$\sum_j \sum_f \left[ \mathbf{n}_f \cdot (\mathcal{F}_0)_j \int_{S_f} b_j dA \right] = V_c,\tag{4.20}$$

define

$$S_{i,f} = \int_{S_f} b_i dA\tag{4.21}$$

$$\sum_j \sum_f \left[ n_{f,x} S_{j,f} (\mathcal{F}_0)_{j,x} + n_{f,y} S_{j,f} (\mathcal{F}_0)_{j,y} + n_{f,z} S_{j,f} (\mathcal{F}_0)_{j,z} \right] = V_c,\tag{4.22}$$

or

$$\sum_j \sum_f \sum_d \left[ n_{f,d} S_{j,f} (\mathcal{F}_0)_{j,d} \right] = V_c,\tag{4.23}$$

where  $d$  denotes dimension such that  $d \in [0, 1, 2] \mapsto [x, y, z]$ , the indices  $(j, d)$  of  $(\mathcal{F}_0)_{j,d}$  maps to a row in  $C^*$ , i.e.,

$$(j, d) \mapsto k : k = N_d j + d,\tag{4.24}$$

from which we get

$$\sum_j \sum_f \sum_d \left[ n_{f,d} S_{j,f} C_{(j,d) \mapsto \text{row } k}^* \cdot \mathcal{E}_c \right] = V_c, \quad \forall c,\tag{4.25}$$

If the indices of  $\mathcal{E}_c$  are then mapped to global system indexes for the corresponding  $\mathcal{E}_c$  and collection of  $\mathcal{E}_f$ 's then the system can be constructed.

## 4.3 Auxiliary equations

For each face-node we now require continuity of flux. This can generally be expressed as

$$\sum_c \sum_f \int_{S_f} \mathbf{n}_f \cdot (\mathcal{F}_0)_j dA = 0\tag{4.26}$$

from which we get

$$\sum_c \sum_f \sum_d \left[ n_{f,d} S_{j,f} C_{(j,d) \mapsto \text{row } k}^* \cdot \mathcal{E}_c \right] = 0, \quad \forall j.\tag{4.27}$$

## 5 Solver C - Radiation Hydrodynamics Grey Radiation with the Variable Eddington Factor (VEF) method

We first repeat eqs. (2.14) and (2.15),

$$\frac{1}{c} \frac{\partial I}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi} \phi + \frac{\sigma_a}{4\pi} acT^4 - \frac{\sigma_t}{4\pi} \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi} \mathcal{E} \boldsymbol{\Omega} \cdot \mathbf{u} \quad (5.1a)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (5.1b)$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \nabla p = \frac{\sigma_t}{c} \mathcal{F}_0, \quad (5.1c)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) + \frac{\sigma_t}{c} \mathcal{F}_0 \cdot \mathbf{u} \quad (5.1d)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F} = \sigma_a c(aT^4 - \mathcal{E}) - \frac{\sigma_t}{c} \mathcal{F}_0 \cdot \mathbf{u}. \quad (5.1e)$$

$$\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot \mathcal{P} = -\frac{\sigma_t}{c} \mathcal{F} + \frac{\sigma_t}{c} (\mathcal{E} \mathbf{u} + \mathcal{P} \cdot \mathbf{u}) \quad (5.1f)$$

where the radiation moment equation has been obtained by dropping the energy exchange terms,

$$\begin{aligned} \frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot \mathcal{P} &= -\frac{\sigma_t}{c} \mathcal{F}_0 + \cancel{\sigma_a c(aT^4 - \mathcal{E})} \frac{\mathbf{u}}{c^2} \\ &= -\frac{\sigma_t}{c} \left[ \mathcal{F} - \mathcal{E} \mathbf{u} - \mathcal{P} \cdot \mathbf{u} \right] \\ &= -\frac{\sigma_t}{c} \mathcal{F} + \frac{\sigma_t}{c} (\mathcal{E} \mathbf{u} + \mathcal{P} \cdot \mathbf{u}). \end{aligned}$$

Now, recall the definition of the radiation pressure tensor,  $\mathcal{P}$ ,

$$\mathcal{P}(\mathbf{x}, \nu, t) = \frac{1}{c} \int_{4\pi} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) d\boldsymbol{\Omega}. \quad (5.2)$$

If we expand the tensor-product we get

$$\mathcal{P} = \frac{1}{c} \int_{4\pi} \begin{bmatrix} \Omega_x \Omega_x & \Omega_x \Omega_y & \Omega_x \Omega_z \\ \Omega_y \Omega_x & \Omega_y \Omega_y & \Omega_y \Omega_z \\ \Omega_z \Omega_x & \Omega_z \Omega_y & \Omega_z \Omega_z \end{bmatrix} I(\boldsymbol{\Omega}) d\boldsymbol{\Omega}. \quad (5.3)$$

The VEF-method involves the approximation

$$\begin{aligned} \mathcal{P} &\approx \{f\} \frac{1}{c} \int_{4\pi} I(\boldsymbol{\Omega}) d\boldsymbol{\Omega} \\ \therefore \mathcal{P} &= \{f\} \mathcal{E} \end{aligned} \quad (5.4)$$

where  $\{f\}$  is the variable Eddington factor computed as an angular-intensity weighted-average such that the entries of the tensor are given by

$$\{f\} : f_{ij} = \frac{\frac{1}{c} \int_{4\pi} \Omega_i \Omega_j I(\boldsymbol{\Omega}) d\boldsymbol{\Omega}}{\frac{1}{c} \int_{4\pi} I(\boldsymbol{\Omega}) d\boldsymbol{\Omega}} \quad i, j \in [x, y, z]. \quad (5.5)$$

Now, rewriting our set of equations we get

$$\frac{1}{c} \frac{\partial I}{\partial t} + \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi} \phi + \frac{\sigma_a}{4\pi} acT^4 - \frac{\sigma_t}{4\pi} \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi} \mathcal{E} \boldsymbol{\Omega} \cdot \mathbf{u} \quad (5.6a)$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \quad (5.6b)$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \boldsymbol{\nabla} p = \frac{\sigma_t}{c} \mathcal{F}_0, \quad (5.6c)$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) + \frac{\sigma_t}{c} \mathcal{F}_0 \cdot \mathbf{u} \quad (5.6d)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F} = \sigma_a c(aT^4 - \mathcal{E}) - \frac{\sigma_t}{c} \mathcal{F}_0 \cdot \mathbf{u}. \quad (5.6e)$$

$$\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \boldsymbol{\nabla} \cdot (\{f\}\mathcal{E}) = -\frac{\sigma_t}{c} \mathcal{F} + \frac{\sigma_t}{c} (\mathcal{E} \mathbf{u} + \{f\}\mathcal{E} \cdot \mathbf{u}) \quad (5.6f)$$

with

$$\mathcal{F}_0 = \mathcal{F} - (\mathcal{E} \mathbf{u} + \{f\}\mathcal{E} \cdot \mathbf{u}) \quad (5.6g)$$

## 5.1 Definitions

We can cast the above equations into the following form

$$\frac{1}{c} \frac{\partial I}{\partial t} + \boldsymbol{\Omega} \cdot \boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi} \phi + \frac{\sigma_a}{4\pi} acT^4 - \frac{\sigma_t}{4\pi} \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi} \mathcal{E} \boldsymbol{\Omega} \cdot \mathbf{u} \quad (5.7a)$$

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \quad (5.7b)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F} = \sigma_a c(aT^4 - \mathcal{E}) - \frac{\sigma_t}{c} \mathcal{F}_0 \cdot \mathbf{u}. \quad (5.7c)$$

$$\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \boldsymbol{\nabla} \cdot (\{f\}\mathcal{E}) = -\frac{\sigma_t}{c} \mathcal{F} + \frac{\sigma_t}{c} (\mathcal{E} \mathbf{u} + \{f\}\mathcal{E} \cdot \mathbf{u}) \quad (5.7d)$$

where

$$\mathbf{S}_{rp} = -\frac{\sigma_t}{c} \mathcal{F}_0 \quad (5.7e)$$

$$S_{ea} = \sigma_a c(aT^4 - \mathcal{E}) \quad (5.7f)$$

$$S_{re} = S_{ea} - \frac{\sigma_t}{c} \mathcal{F}_0 \cdot \mathbf{u} \quad (5.7g)$$

## 5.2 Mixed Finite Element Method

We now turn our attention to just the radiation momentum equation,

$$\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \boldsymbol{\nabla} \cdot (\{f\}\mathcal{E}) = -\frac{\sigma_t}{c} \mathcal{F} + \frac{\sigma_t}{c} (\mathcal{E} \mathbf{u} + \{f\}\mathcal{E} \cdot \mathbf{u}). \quad (5.8)$$

We assume that all the unknowns in this equation have a linear FE representation on a cell, except  $\mathcal{E}$  and  $\mathbf{u}$ .

### 5.2.1 The radiation-flux at $n$

One of the first items we will need in any temporal discretization is the old  $\mathcal{F}^n$ . In order to get value of  $\mathcal{F}^n$ , when starting the iterations, we simply use the radiation-momentum equation with no time derivative to get

$$\mathcal{F}^n = -\frac{c}{\sigma_t^n} \nabla \cdot (\{f\}^n \mathcal{E}^n) + (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})^n, \quad \text{if } n = 0. \quad (5.9)$$

This equation, however, still requires a suitable spatial discretization. Applying a linear FEM first requires multiplying by a trial function, then integrating over volume

$$\int_V b_i \left[ \mathcal{F}^n = -\frac{c}{\sigma_t^n} \nabla \cdot (\{f\}^n \mathcal{E}^n) + (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})^n \right] dV \quad (5.10)$$

Let us now consider the terms one-by-one. First are the  $\mathcal{F}$  terms. Since  $\mathcal{F} \approx \sum_j \mathcal{F}_j b_j(\mathbf{x})$  we get

$$\int_V b_i \mathcal{F} dV = \sum_j \mathcal{F}_j \int_V b_i b_j dV \quad (5.11)$$

Second are the divergence terms. First we rewrite

$$\begin{aligned} \int_V b_i \nabla \cdot (\{f\} \mathcal{E}) dV &= \int_S \mathbf{n} \cdot (b_i \{f\} \mathcal{E}) dA - \int_V \{f\} \mathcal{E} \cdot \nabla b_i dV \\ &= \sum_f \int_{S_f} \mathbf{n}_f \cdot (b_i \{f\} \mathcal{E}) dA - \int_V \{f\} \mathcal{E} \cdot \nabla b_i dV \\ &= \sum_j \sum_f \mathbf{n}_f \cdot (\{f\} \mathcal{E})_j \int_{S_f} b_i b_j dA - \sum_j (\{f\} \mathcal{E})_j \cdot \int_V b_j \nabla b_i dV \\ \text{1D} \quad \int_V b_i \nabla \cdot (\{f\} \mathcal{E}) dV &= \sum_f \mathbf{n}_f \cdot (\{f\} \mathcal{E})_f \int_{S_f} b_i dA - (\{f\} \mathcal{E})_c \cdot \int_V \nabla b_i dV \end{aligned} \quad (5.12)$$

Last are the advection terms. If the velocity and radiation-energy are only considered to be cell-constant then we have

$$\int_V b_i (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u}) dV = (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})_c \int_V b_i dV \quad (5.13)$$

Putting all this together, we get

$$\sum_j \mathcal{F}_j^n \int_V b_i b_j dV = -\frac{c}{\sigma_t^n} \left[ \sum_f \mathbf{n}_f \cdot (\{f\} \mathcal{E})_f^n \int_{S_f} b_i dA - (\{f\} \mathcal{E})_c^n \cdot \int_V \nabla b_i dV \right] + (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})_c^n \int_V b_i dV \quad (5.14)$$

which can be written as

$$\bar{M}_c \bar{\mathbf{F}}_c^n = -\frac{c}{\sigma_t} \bar{C}_c (\mathbf{f} \mathcal{E})_c^n \quad (5.15)$$

### 5.2.2 The radiation-flux at $n+1$

We now seek a similar expression for the radiation-flux at timestep  $n+1$ . Our first discretization is a temporal theta-scheme discretization where we lag the advection terms,

$$\begin{aligned} &\frac{1}{c^2 \tau} (\mathcal{F}^{n+1} - \mathcal{F}^n) + \theta_1 \nabla \cdot (\{f\} \mathcal{E})^{n+1} + \theta_2 \nabla \cdot (\{f\} \mathcal{E})^n \\ &= -\frac{\theta_1}{c} \sigma_t^{n+1} \mathcal{F}^{n+1} - \frac{\theta_2}{c} \sigma_t^n \mathcal{F}^n + \frac{\sigma_t^{n+\frac{1}{2}}}{c} (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})^{n+\frac{1}{2}} \end{aligned} \quad (5.16)$$



we then multiply by  $c^2\tau$

$$\begin{aligned} & \mathcal{F}^{n+1} - \mathcal{F}^n + \theta_1 c^2 \tau \nabla \cdot (\{f\} \mathcal{E})^{n+1} + \theta_2 c^2 \tau \nabla \cdot (\{f\} \mathcal{E})^n \\ &= -\frac{\theta_1 c^2 \tau}{c} \sigma_t^{n+1} \mathcal{F}^{n+1} - \frac{\theta_2 c^2 \tau}{c} \sigma_t^n \mathcal{F}^n + \frac{c^2 \tau \sigma_t^{n+\frac{1}{2}}}{c} (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})^{n+\frac{1}{2}}. \end{aligned} \quad (5.17)$$

Next we define,

$$\begin{aligned} a_1 &= \theta_1 c^2 \tau \\ a_2 &= \theta_2 c^2 \tau \\ a_3 &= c \tau \sigma_t^{n+\frac{1}{2}} \end{aligned} \quad (5.18)$$

to get,

$$\begin{aligned} & \mathcal{F}^{n+1} - \mathcal{F}^n + a_1 \nabla \cdot (\{f\} \mathcal{E})^{n+1} + a_2 \tau \nabla \cdot (\{f\} \mathcal{E})^n \\ &= -\frac{a_1}{c} \sigma_t^{n+1} \mathcal{F}^{n+1} - \frac{a_2}{c} \sigma_t^n \mathcal{F}^n + a_3 (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})^{n+\frac{1}{2}}. \end{aligned} \quad (5.19)$$

which we can rearrange as

$$\left(1 + \frac{a_1}{c} \sigma_t^{n+1}\right) \mathcal{F}^{n+1} + a_1 \nabla \cdot (\{f\} \mathcal{E})^{n+1} = \left(1 - \frac{a_2}{c} \sigma_t^n\right) \mathcal{F}^n - a_2 \tau \nabla \cdot (\{f\} \mathcal{E})^n + a_3 (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})^{n+\frac{1}{2}}. \quad (5.20)$$

Next we define

$$\begin{aligned} a_4 &= \frac{a_1}{1 + \frac{a_1}{c} \sigma_t^{n+1}} \\ a_5 &= \frac{1 - \frac{a_2}{c} \sigma_t^n}{1 + \frac{a_1}{c} \sigma_t^{n+1}} \\ a_6 &= \frac{a_2}{1 + \frac{a_1}{c} \sigma_t^{n+1}} \\ a_7 &= \frac{a_3}{1 + \frac{a_1}{c} \sigma_t^{n+1}} \end{aligned} \quad (5.21)$$

to get

$$\mathcal{F}^{n+1} + a_4 \nabla \cdot (\{f\} \mathcal{E})^{n+1} = a_5 \mathcal{F}^n - a_6 \tau \nabla \cdot (\{f\} \mathcal{E})^n + a_7 (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})^{n+\frac{1}{2}}. \quad (5.22)$$

$$\mathcal{F}^{n+1} + a_4 \nabla (f \mathcal{E})^{n+1} = a_5 \mathcal{F}^n - a_6 \tau \nabla (f \mathcal{E})^n + a_7 (1 + f^{n+\frac{1}{2}}) (\mathcal{E} \mathbf{u})^{n+\frac{1}{2}}. \quad (5.23)$$

Now we apply our spatial discretization scheme by multiplying by trial functions, defined as the basis functions on each cell, then integrating over volume

$$\int_V b_i(\mathbf{x}) \left[ \mathcal{F}^{n+1} + a_4 \nabla \cdot (\{f\} \mathcal{E})^{n+1} = a_5 \mathcal{F}^n - a_6 \tau \nabla \cdot (\{f\} \mathcal{E})^n + a_7 (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})^{n+\frac{1}{2}} \right] dV. \quad (5.24)$$

Let us now consider the terms one-by-one. First are the  $\mathcal{F}$  terms. Since  $\mathcal{F} \approx \sum_j \mathcal{F}_j b_j(\mathbf{x})$  we get

$$\int_V b_i \mathcal{F} dV = \sum_j \mathcal{F}_j \int_V b_i b_j dV \quad (5.25)$$

Second are the divergence terms. First we rewrite

$$\begin{aligned} \int_V b_i \nabla \cdot (\{f\} \mathcal{E}) dV &= \int_S \mathbf{n} \cdot (b_i \{f\} \mathcal{E}) dA - \int_V \{f\} \mathcal{E} \cdot \nabla b_i dV \\ &= \sum_f \int_{S_f} \mathbf{n}_f \cdot (b_i \{f\} \mathcal{E}) dA - \int_V \{f\} \mathcal{E} \cdot \nabla b_i dV \\ &= \sum_j \sum_f \mathbf{n}_f \cdot (\{f\} \mathcal{E})_j \int_{S_f} b_i b_j dA - \sum_j (\{f\} \mathcal{E})_j \cdot \int_V b_j \nabla b_i dV \\ \text{1D} \quad \int_V b_i \nabla \cdot (\{f\} \mathcal{E}) dV &= \sum_f \mathbf{n}_f \cdot (\{f\} \mathcal{E})_f \int_{S_f} b_i dA - (\{f\} \mathcal{E})_c \cdot \int_V \nabla b_i dV \end{aligned} \quad (5.26)$$

Last are the advection terms. If the velocity and radiation-energy are only considered to be cell-constant then we have

$$\int_V b_i a_7 (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u}) dV = a_7 (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})_c \int_V b_i dV \quad (5.27)$$

The resulting small system of equations can be written as

$$\bar{M}_c \bar{\mathbf{F}}_c^{n+1} = -a_4 \bar{C}_c \mathcal{E}_c^{n+1} + a_5 \bar{M}_c \bar{\mathbf{F}}_c^n - a_6 \bar{C}_c \mathcal{E}_c^n + a_7 \mathbf{N}_c (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})_c^{n+\frac{1}{2}}, \quad (5.28)$$

where

$$\mathcal{E}_c = \begin{bmatrix} \mathcal{E}_c \\ \mathcal{E}_{n=0} \\ \vdots \\ \mathcal{E}_{n=N_n-1} \end{bmatrix}^{(N_n+1) \times 1}, \quad (5.29)$$

$$\mathbf{F}_c = \begin{bmatrix} \mathcal{F} \\ \vdots \\ \mathcal{F}_{N_n-1} \end{bmatrix} \quad (5.30)$$

Next we can multiply with  $\bar{M}_c^{-1}$  to get

$$\bar{\mathbf{F}}_c^{n+1} = -\bar{C}_c^* \mathcal{E}_c^{n+1} + a_5 \bar{\mathbf{F}}_c^n - a_6 \bar{C}_c^* \mathcal{E}_c^n + a_7 \bar{M}_c^{-1} \mathbf{N}_c (\mathcal{E} \mathbf{u} + \{f\} \mathcal{E} \cdot \mathbf{u})_c^{n+\frac{1}{2}}. \quad (5.31)$$

## A Angular integration identities

**Identity A-1**

$$\int_{4\pi} d\Omega = 4\pi.$$

**Identity A-2**

$$\int_{4\pi} \Omega d\Omega = 0.$$

**Identity A-3** Given the known three component vector,  $\mathbf{v}$ ,

$$\int_{4\pi} \Omega \cdot \mathbf{v} d\Omega = 0.$$

**Identity A-4** Given the known three component vector,  $\mathbf{v}$ ,

$$\int_{4\pi} \Omega \cdot \nabla (\Omega \cdot \mathbf{v}) d\Omega = \frac{4\pi}{3} \nabla \cdot \mathbf{v}.$$

**Identity A-5** Given the scalar,  $a$ ,

$$\int_{4\pi} \Omega \left( \Omega \cdot \nabla a \right) d\Omega = \frac{4\pi}{3} \nabla a.$$

**Identity A-6** Given the known three component vector,  $\mathbf{v}$ ,

$$\int_{4\pi} \Omega \left( \Omega \cdot \mathbf{v} \right) d\Omega = \frac{4\pi}{3} \mathbf{v}.$$

**Identity A-7** Given the known three component vector,  $\mathbf{v}$ ,

$$\int_{4\pi} \Omega \left( \Omega \cdot \nabla (\Omega \cdot \mathbf{v}) \right) d\Omega = 0.$$

## B Boundary and initial conditions for radiation hydrodynamic problems

In a one dimensional simulation we can simulate steady-state shocks by setting the appropriate pre- and post-shock conditions. Pre-shock conditions will be denoted with a subscript  $L$  whereas post-shock conditions will be denoted with a subscript  $R$ .

### B.1 Hydrodynamics only

With no radiation energy present we wish to have  $\mathcal{F}_L^H = \mathcal{F}_R^H$ , therefore

$$\begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}_R. \quad (\text{B.1})$$

Here we have three equations but 4 unknowns, i.e.,  $\rho$ ,  $u$ ,  $p$  and  $e$ . Fortunately, we can express both  $e$  and  $p$  in terms of temperature since

$$p = (\gamma - 1)\rho e$$

and

$$e = C_v T.$$

Therefore,

$$p = (\gamma - 1)\rho C_v T$$

and

$$\begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \rho C_v T u + p u \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \rho C_v T u + p u \end{bmatrix}_R. \quad (\text{B.2})$$

When the left state is known then we can frame these equations as seeking the non-linear solution of

$$\mathbf{F} \begin{pmatrix} \rho_R \\ T_R \\ u_R \end{pmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \gamma \rho C_v T u \end{bmatrix}_R - \mathcal{F}_L^H = \mathbf{0} \quad (\text{B.3})$$

or simply

$$\mathbf{F}(\mathbf{x}) = \mathcal{F}^H(\mathbf{x}_R) - \mathcal{F}_L^H = \mathbf{0}. \quad (\text{B.4})$$

to which Newton-iteration can be applied in the form

$$\mathbf{x}_R^{\ell+1} = \mathbf{x}_R^\ell - J^{-1}(\mathbf{x}_R^\ell) \mathbf{F}(\mathbf{x}_R^\ell)$$

where the Jacobian matrix,  $J$ , is given by

$$J = \begin{bmatrix} u & 0 & \rho \\ u^2 + (\gamma - 1)C_v T & (\gamma - 1)\rho C_v & 2\rho u \\ \frac{1}{2}u^3 + \gamma C_v T u & \gamma \rho C_v u & \frac{3}{2}\rho u^2 + \gamma \rho C_v T \end{bmatrix} \quad (\text{B.5})$$

**Note:** The initial guess,  $\mathbf{x}^0$  cannot be the same as  $\mathbf{x}_L$  since the iteration will terminate immediately. Generally the values need to be perturbed sufficiently such that  $\rho_R > \rho_L$ ,  $T_R > T_L$  and  $u_R < a_R$  where  $a$  is the sound-speed.

## B.2 Hydrodynamics with radiation energy

With radiation energy present we are concerned with the following set of equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (\text{B.6a})$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \nabla p = -\frac{1}{3} \nabla \mathcal{E}, \quad (\text{B.6b})$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{u}] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \quad (\text{B.6c})$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left( -\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla (\mathcal{E} \mathbf{u}) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \quad (\text{B.6d})$$

which for a steady-steady, one dimensional simulation becomes

$$\nabla \cdot (\rho u) = 0 \quad (\text{B.7a})$$

$$\nabla \cdot (\rho u^2) + \nabla p = -\frac{1}{3} \nabla \mathcal{E}, \quad (\text{B.7b})$$

$$\nabla \cdot [(E + p)u] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot u \quad (\text{B.7c})$$

$$\nabla \cdot \left( -\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla (\mathcal{E} u) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} u. \quad (\text{B.7d})$$

Additionally, far away from the interface the co-moving frame radiation flux,  $\mathcal{F}_0$ , is zero, therefore

$$\mathcal{F}_0 = -\frac{c}{3\sigma_t} \nabla \mathcal{E} = 0$$

and the equation set becomes

$$\nabla \cdot (\rho u) = 0 \quad (\text{B.8a})$$

$$\nabla \cdot (\rho u^2) + \nabla p = -\frac{1}{3} \nabla \mathcal{E}, \quad (\text{B.8b})$$

$$\nabla \cdot [(E + p)u] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot u \quad (\text{B.8c})$$

$$\frac{4}{3} \nabla (\mathcal{E} u) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} u. \quad (\text{B.8d})$$

Now, adding the last equation to the third, we get

$$\nabla \cdot (\rho u) = 0 \quad (\text{B.9a})$$

$$\nabla \cdot (\rho u^2) + \nabla p + \frac{1}{3} \nabla \mathcal{E} = 0, \quad (\text{B.9b})$$

$$\nabla \cdot [(E + p)u] + \frac{4}{3} \nabla (\mathcal{E} u) = 0. \quad (\text{B.9c})$$

We now express internal energy,  $e$ , the pressure,  $p$ , and the radiation energy,  $\mathcal{E}$ , in terms of temperature

$$\nabla \cdot (\rho u) = 0 \quad (\text{B.10a})$$

$$\nabla \cdot (\rho u^2) + \nabla ((\gamma - 1) \rho C_v T) + \frac{1}{3} \nabla aT^4 = 0, \quad (\text{B.10b})$$

$$\nabla \cdot \left[ \left( \frac{1}{2} \rho u^2 + \rho C_v T + (\gamma - 1) \rho C_v T \right) u \right] + \frac{4}{3} \nabla (aT^4 u) = 0. \quad (\text{B.10c})$$

Finally we integrate this equation set over the entire domain to get

$$\left[ \begin{array}{c} \rho u \\ \rho u^2 + (\gamma - 1) \rho C_v T + \frac{1}{3} aT^4 \\ \frac{1}{2} \rho u^3 + \gamma \rho C_v T u + \frac{4}{3} aT^4 u \end{array} \right]_L = \left[ \begin{array}{c} \rho u \\ \rho u^2 + (\gamma - 1) \rho C_v T + \frac{1}{3} aT^4 \\ \frac{1}{2} \rho u^3 + \gamma \rho C_v T u + \frac{4}{3} aT^4 u \end{array} \right]_R \quad (\text{B.11a})$$

Similar to the previous case we can now define

$$\mathbf{F} \begin{pmatrix} \rho_R \\ T_R \\ u_R \end{pmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T + \frac{1}{3}aT^4 \\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u + \frac{4}{3}aT^4 u \end{bmatrix}_R - \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T + \frac{1}{3}aT^4 \\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u + \frac{4}{3}aT^4 u \end{bmatrix}_L = \mathbf{0}, \quad (\text{B.12})$$

where the subscript  $L$  quantities are all known. Applying Newton-iteration to this equation again is

$$\mathbf{x}_R^{\ell+1} = \mathbf{x}_R^\ell - J^{-1}(\mathbf{x}_R^\ell)\mathbf{F}(\mathbf{x}_R^\ell)$$

where the Jacobian matrix,  $J$  is given by

$$J = \begin{bmatrix} u & 0 & \rho \\ u^2 + (\gamma - 1)C_v T & (\gamma - 1)\rho C_v + \frac{4}{3}aT^3 & 2\rho u \\ \frac{1}{2}u^3 + \gamma C_v T u & \gamma\rho C_v u + \frac{16}{3}aT^3 u & \frac{3}{2}\rho u^2 + \gamma\rho C_v T + \frac{4}{3}aT^4 \end{bmatrix} \quad (\text{B.13})$$

Example mach 3 conditions,  $C_v = 0.14472799784454$  and  $\gamma = \frac{5}{3}$ :

```
[0]  rho0  1.00000000e+00 u0    3.80431331e-01 T0    1.00000000e-01 e0    1.44727998e-02 radE0 1.37223549e-06
[0]  rho1  3.00185103e+00 u1    1.26732249e-01 T1    3.66260705e-01 e1    5.30081785e-02 radE1 2.46939153e-04
```

**Note:** The initial guess,  $\mathbf{x}^0$  cannot be the same as  $\mathbf{x}_L$  since the iteration will terminate immediately. Generally the values need to be perturbed sufficiently such that  $\rho_R > \rho_L$ ,  $T_R > T_L$ ,  $u_R < a_R$  where  $a$  is the sound-speed, and  $\mathcal{E}_R > \mathcal{E}_L$ .

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## C Roderigues's formula

Roderigues' formula for the rotation of a vector  $\mathbf{v}$  about a unit vector  $\mathbf{a}$  with right-hand rule

$$\mathbf{v}_{rotated} = \cos \theta \mathbf{v} + (\mathbf{a} \cdot \mathbf{v})(1 - \cos \theta) \mathbf{a} + \sin \theta (\mathbf{a} \times \mathbf{v}) \quad (\text{C.1})$$

In matrix form

$$\mathbf{v}_{rotated} = A \mathbf{v} \quad (\text{C.2})$$

where

$$A = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \quad (\text{C.3})$$

and

$$R = I + \sin \theta A + (1 - \cos \theta) A^2 \quad (\text{C.4})$$