Lecture 16

Flux-Limited Diffusion Theory

1 Introduction

Flux limited diffusion theory (FLDT) represents an attempt to correct the deficiency of diffusion theory in the streaming limit. The central theme of the approximation is that when the macroscopic cross section σ_t, is small with respect to the relative intensity gradient, $\overrightarrow{\nabla}\phi/\phi$, the flux limits to

$$\overrightarrow{F} = \phi \frac{\overrightarrow{\nabla}\phi}{|\overrightarrow{\nabla}\phi|}, \tag{1}$$

whereas when the macroscopic cross section is large with respect to the relative intensity gradient, the flux limits to the standard Fick's law:

$$\overrightarrow{F} = -\frac{1}{3\sigma_t}\overrightarrow{\nabla}\phi\,. \tag{2}$$

There are in infinite number of ways to achieve this limiting behavior. One of the first flux limited-diffusion theories was that of Wilson:

$$\overrightarrow{F} = -\frac{\phi}{3\sigma_t\phi + |\overrightarrow{\nabla}\phi|}\overrightarrow{\nabla}\phi\,. \tag{3}$$

While popular, it was soon recognized that this theory limited too strongly. This can be demonstrated via asymptotics. For simplicity we consider the neutronics equation rather than the radiative transfer equations. Applying the usual asymptotic scaling to the neutronics diffusion equation with Wilson flux limiting, we get

$$-\epsilon\overrightarrow{\nabla}\cdot\frac{\phi}{3\sigma_t\phi+\epsilon|\overrightarrow{\nabla}\phi|}\overrightarrow{\nabla}\phi+\epsilon\sigma_a\phi=\epsilon Q\,. \tag{4}$$

Expanding the flux limited diffusion coefficient about $\epsilon = 0$, we get

$$\frac{\phi}{3\sigma_t\phi+\epsilon|\overrightarrow{\nabla}\phi|}=\frac{1}{3\sigma_t}-\epsilon|\overrightarrow{\nabla}\phi|/\left(9\sigma_t^2\phi\right)+O(\epsilon^2)\,, \tag{5}$$

Substituting from Eq. (5) into Eq. (4), and dividing by ϵ, we get

$$-\overrightarrow{\nabla}\cdot\left[\frac{1}{3\sigma_t}-\epsilon|\overrightarrow{\nabla}\phi|/\left(9\sigma_t^2\phi\right)+O(\epsilon^2)\right]\overrightarrow{\nabla}\phi+\sigma_a\phi=Q\,. \tag{6}$$

Expanding ϕ in a power series in ϵ, substituting that expansion into Eq. (6), and collecting leading-order terms, we get

$$-\overrightarrow{\nabla}\cdot\frac{1}{3\sigma_t}\overrightarrow{\nabla}\phi^{(0)}+\sigma_a\phi^{(0)}=Q\,, \tag{7}$$

which is the standard diffusion equation. Thus Wilson's theory preserves the diffusion limit to leading order. Collecting the O(1) terms, we get

$$-\overrightarrow{\nabla}\cdot\left[\frac{1}{3\sigma_t}\overrightarrow{\nabla}\phi^{(1)}-\frac{|\overrightarrow{\nabla}\phi^{(0)}|}{9\sigma_t^2\phi^{(0)}}\overrightarrow{\nabla}\phi^{0)}\right]+\sigma_a\phi^{(1)}=0\,. \tag{8}$$

The correct equation is

$$-\overrightarrow{\nabla}\cdot\frac{1}{3\sigma_t}\overrightarrow{\nabla}\phi^{(1)}+\sigma_a\phi^{(1)}=0\,. \tag{9}$$

Note that if we multiply Eq. (9) by ϵ and add it to Eq. (7), we get

$$-\overrightarrow{\nabla}\cdot\frac{1}{3\sigma_t}\overrightarrow{\nabla}\left(\phi^{(0)}+\epsilon\phi^{(1)}\right)+\sigma_a\left(\phi^{(0)}+\epsilon\phi^{(1)}\right)=Q\,, \tag{10}$$

which shows that the first-order asymptotic solution satisfies the diffusion equation. Thus Wilson's theory does not preserve the diffusion limit to first order. This explains why it limits too strongly. It departs from diffusion theory while that theory is still valid.

An alternative theory that preserves the diffusion limit through first order is the Larsen limiter:

$$\overrightarrow{F}=-\frac{\phi}{\sqrt{9\sigma_t^2\phi^2+|\overrightarrow{\nabla}\phi|^2}}\overrightarrow{\nabla}\phi\,, \tag{11}$$

2 Discretization

Discretization of FLDT is straightforward for the case of the linear-continuous finite element method. For instance, using the Larsen limiter the interior mesh equation becomes

$$\begin{aligned}-\frac{\hat{D}_{i+1/2}}{h_{i+1/2}}\left(\phi_{i+1}-\phi_i\right)+\frac{\hat{D}_{i-1/2}}{h_{i-1/2}}\left(\phi_i-\phi_{i-1}\right)+\sigma_{a,i+1/2}\left[\frac{2}{3}\phi_i+\frac{1}{3}\phi_{i+1}\right]h_{i+1/2}+\\ \sigma_{a,i-1/2}\left[\frac{2}{3}\phi_i+\frac{1}{3}\phi_{i-1}\right]h_{i-1/2}=\left[\frac{2}{3}Q_i+\frac{1}{3}Q_{i+1}\right]h_{i+1/2}+\left[\frac{2}{3}Q_i+\frac{1}{3}Q_{i-1}\right]h_{i-1/2}\,,\end{aligned} \tag{12}$$

where

$$\hat{D}_{i+1/2}=\left[9\sigma_{i+1/2}^2+2\frac{\phi_{i+1}-\phi_i}{\left(\phi_{i+1}+\phi_i\right)h_{i+1/2}}\right]^{-1/2}\,. \tag{13}$$