Lecture 9

More on Lagrangian Hydrodynamics

1 Curvilinear Pressure Gradient

At first glance, the curvilinear pressure gradient term in the momentum equation seems incorrect. The equation (corrector) is

$$\frac{m_{i+1/2}}{\Delta t^k}\left(u_{i+1/2}^{k+1/2} - u_{i+1/2}^{k-1/2}\right) = -A_{i+1/2}^k\left(P_{i+1}^k - P_i^k\right) \,, \tag{1}$$

We know that the right side of this equation is analytically given by $-\overrightarrow{\nabla} P$. There is a theorem that states that

$$\int \overrightarrow{\nabla} P \, dV = \oint P \overrightarrow{n} \, dA \,, \tag{2}$$

which would lead us to conclude that Eq. (1) should be given by

$$\frac{m_{i+1/2}}{\Delta t^k}\left(u_{i+1/2}^{k+1/2} - u_{i+1/2}^{k-1/2}\right) = -\left(A_{i+1}^k P_{i+1}^k - A_i^k P_i^k\right) \,. \tag{3}$$

However, Eq. (2) does not apply in 1-D spheres. To see this, we note that

$$\int \overrightarrow{\nabla} P \, dV = \int 4\pi r^2 \frac{\partial P}{\partial r} \, dr \,, \tag{4}$$

which is clearly not equivalent to Eq. (2). If we assume that P is piecewise constant within each cell, and integrate Eq. (4) by parts over median mesh cell $i+1/2$, we obtain

$$
\begin{aligned}
\int_{r_i}^{r_{i+1}} 4\pi r^2 \frac{\partial P}{\partial r}\, dr &= 4\pi \left[\int_{r_i}^{r_{i+1}} \frac{\partial P r^2}{\partial r}\, dr - \int_{r_i}^{r_{i+1/2}} 2rP\, dr - \int_{r_{i+1/2}}^{r_{i+1}} 2rP\, dr \right] \\
&= 4\pi \left[r_{i+1}^2 P_{i+1} - r_i^2 P_i - (r_{i+1/2}^2 - r_i^2) P_i - (r_{i+1}^2 - r_{i+1/2}^2) P_{i+1} \right] \\
&= 4\pi r_{i+1/2}^2 (P_{i+1} - P_i)\,, \qquad (5)
\end{aligned}
$$

which is in agreement with Eq. (1)

2 Total Material Energy Conservation

Next we want to demonstrate total material energy conservation for our Lagrangian algorithm. We start by multiplying Eq. (1) by $\left(u_{i+1/2}^{k+1/2} + u_{i+1/2}^{k-1/2}\right)/2 \equiv u_{i+1/2}^k$ to obtain a kinetic energy equation:

$$
\frac{1}{2}\frac{m_{i+1/2}}{\Delta t^k}\left[(u_{i+1/2}^{k+1/2})^2 - (u_{i+1/2}^{k-1/2})^2\right] = -A_{i+1/2}^k \left(P_{i+1}^k - P_i^k\right) u_{i+1/2}^k\,. \qquad (6)
$$

The internal energy equation is

$$
\frac{m_i}{\Delta t^k}\left(e_i^{k+1/2} - e_i^{k-1/2}\right) = -P_i^k \left(A_{i+1/2}^k u_{i+1/2}^k - A_{i-1/2}^k u_{i-1/2}^k\right)\,. \qquad (7)
$$

Summing these two equations over an *interior median-mesh cell* $i+1/2$ (note that only "half" of the two internal energy equations for cells i and $i+1$ are included in the sum), we get a total material energy equation for each interior vertex:

$$\frac{1}{2}\frac{m_{i+1/2}}{\Delta t^k}\left[(u_{i+1/2}^{k+1/2})^2-(u_{i+1/2}^{k-1/2})^2\right]+$$

$$\frac{m_i}{2\Delta t^k}\left(e_i^{k+1/2}-e_i^{k-1/2}\right)+\frac{m_{i+1}}{2\Delta t^k}\left(e_{i+1}^{k+1/2}-e_{i+1}^{k-1/2}\right)=$$

$$-A_{i+1/2}^k P_{i+1}^k u_{i+1/2}^k + A_{i+1/2}^k P_i^k u_{i+1/2}^k - P_i^k A_{i+1/2}^k u_{i+1/2}^k + P_{i+1}^k A_{i+1/2}^k u_{i+1/2}^k = 0\,. \tag{8}$$

From Eq. (8) we can sum over all vertices to see that we have total energy conservation on an infinite mesh (we are neglecting boundary terms). To see what happens on a finite mesh, let's assume a pressure boundary condition on the left:

$$\frac{1}{2}\frac{m_{1/2}}{\Delta t^k}\left[(u_{1/2}^{k+1/2})^2-(u_{1/2}^{k-1/2})^2\right]=-A_{1/2}^k\left(P_1^k-P_{1/2}^k\right)u_{1/2}^k\,. \tag{9}$$

With this boundary condition, Eq. (8) at the left boundary becomes

$$\frac{1}{2}\frac{m_{1/2}}{\Delta t^k}\left[(u_{1/2}^{k+1/2})^2-(u_{1/2}^{k-1/2})^2\right]+\frac{m_1}{2\Delta t^k}\left(e_1^{k+1/2}-e_1^{k-1/2}\right)=$$

$$-A_{1/2}^k P_1^k u_{1/2}^k + A_{1/2}^k P_{1/2}^k u_{1/2}^k + P_1^k A_{1/2}^k u_{1/2}^k = A_{1/2}^k P_{1/2}^k u_{1/2}^k\,. \tag{10}$$

Including the contribution from the right pressure boundary condition, and summing over all median-mesh cells, we get

$$\frac{\partial}{\partial t}\left[\sum_{i=0}^{N}\frac{1}{2}\frac{m_{i+1/2}}{\Delta t^k}\left[(u_{i+1/2}^{k+1/2})^2-(u_{i+1/2}^{k-1/2})^2\right]\right]+$$

$$\frac{\partial}{\partial t}\left[\sum_{i=1}^{N}\frac{m_i}{\Delta t^k}\left(e_i^{k+1/2}-e_i^{k-1/2}\right)\right]=$$

$$- \left(A^k_{N+1/2} P^k_{N+1/2} u^k_{N+1/2} - A^k_{1/2} P^k_{1/2} u^k_{1/2} \right) \, . \tag{11}$$

The right side of Eq. (11) represents the net change in total energy due to the surface pressures, i.e.,

$$\frac{\partial}{\partial t} \left[\int \left(\rho \frac{1}{2} u^2 + \rho e \right) dV \right] = - \int \overrightarrow{\nabla} \cdot (P \overrightarrow{u}) \, dV = - \oint P \overrightarrow{u} \cdot \overrightarrow{n} \, dA \, . \tag{12}$$

Of course, if $r_{1/2} = 0$, then $A_{1/2} = 0$ and the only contribution comes from the right boundary.

If the boundary velocities are specified, there is no equation for $u_{1/2}$ and Eq. (8) at the left boundary becomes

$$\frac{m_1}{2\Delta t^k} \left(e_1^{k+1/2} - e_1^{k-1/2} \right) = P^k_1 A^k_{1/2} u^k_{1/2} \, . \tag{13}$$

If we add the kinetic energy contribution from $u^k_{1/2}$ to both sides of Eq. (13), we get

$$\frac{m_1}{2\Delta t^k} \left(e_1^{k+1/2} - e_1^{k-1/2} \right) + \frac{1}{2} \frac{m_{1/2}}{\Delta t^k} \left[(u_{1/2}^{k+1/2})^2 - (u_{1/2}^{k-1/2})^2 \right] =$$

$$P^k_1 A^k_{1/2} u^k_{1/2} + \frac{1}{2} \frac{m_{1/2}}{\Delta t^k} \left[(u_{1/2}^{k+1/2})^2 - (u_{1/2}^{k-1/2})^2 \right] \, . \tag{14}$$

Proceeding analogously at the right boundary we get

$$\frac{m_N}{2\Delta t^k} \left(e_N^{k+1/2} - e_N^{k-1/2} \right) + \frac{1}{2} \frac{m_{N+1/2}}{\Delta t^k} \left[(u_{N+1/2}^{k+1/2})^2 - (u_{N+1/2}^{k-1/2})^2 \right] =$$

$$P^k_N A^k_{N+1/2} u^k_{1/2} + \frac{1}{2} \frac{m_{1/2}}{\Delta t^k} \left[(u_{N+1/2}^{k+1/2})^2 - (u_{N+1/2}^{k-1/2})^2 \right] \, . \tag{15}$$

Summing over all but the first and last median-mesh cells, and then adding Eqs. (14) and (15) to the sum, we obtain the following total material energy conservation equation:

$$\sum_{i=0}^{N} \frac{1}{2} \frac{m_{i+1/2}}{\Delta t^k} \left[(u_{i+1/2}^{k+1/2})^2 - (u_{i+1/2}^{k-1/2})^2 \right] +$$

$$\sum_{i=1}^{N} \frac{m_i}{\Delta t^k} \left(e_i^{k+1/2} - e_i^{k-1/2} \right) =$$

$$\frac{1}{2} \frac{m_{1/2}}{\Delta t^k} \left[(u_{1/2}^{k+1/2})^2 - (u_{1/2}^{k-1/2})^2 \right] + \frac{1}{2} \frac{m_{N+1/2}}{\Delta t^k} \left[(u_{N+1/2}^{k+1/2})^2 - (u_{N+1/2}^{k-1/2})^2 \right] . \tag{16}$$

In Eq. (11) we assumed two pressure boundaries, and in Eq. (16) we assumed two boundaries with velocities specified. Of course, one could have one pressure boundary and one boundary with a specified velocity. In the total energy equation, only the boundary pressure or the boundary kinetic energy appears on the right side for a given boundary, depending upon the boundary condition. In the general case, one obtains

$$\sum_{i=0}^{N} \frac{1}{2} \frac{m_{i+1/2}}{\Delta t^k} \left[(u_{i+1/2}^{k+1/2})^2 - (u_{i+1/2}^{k-1/2})^2 \right] +$$

$$\sum_{i=1}^{N} \frac{m_i}{\Delta t^k} \left(e_i^{k+1/2} - e_i^{k-1/2} \right) =$$

$$\gamma_L \frac{1}{2} \frac{m_{1/2}}{\Delta t^k} \left[(u_{1/2}^{k+1/2})^2 - (u_{1/2}^{k-1/2})^2 \right] + \gamma_R \frac{1}{2} \frac{m_{N+1/2}}{\Delta t^k} \left[(u_{N+1/2}^{k+1/2})^2 - (u_{N+1/2}^{k-1/2})^2 \right] -$$

$$\left[(1-\gamma_R) A_{N+1/2}^k P_{N+1/2}^k u_{N+1/2}^k - (1-\gamma_L) A_{1/2}^k P_{1/2}^k u_{1/2}^k \right] , \tag{17}$$

where $\gamma = 0$ for a pressure boundary and $\gamma = 1$ for a specified velocity boundary.

If we include radiation, we obtain the following total (radiation plus material) energy equation:

$$
\begin{aligned}
&\left[\sum_{i=0}^{N}\frac{1}{2}\frac{m_{i+1/2}}{\Delta t^k}\left[(u_{i+1/2}^{k+1/2})^2-(u_{i+1/2}^{k-1/2})^2\right]\right]+\\
&\left[\sum_{i=1}^{N}\frac{m_i}{\Delta t^k}\left(e_i^{k+1/2}-e_i^{k-1/2}\right)\right]+\\
&\sum_{i=1}^{N}\left[\frac{m_i}{\rho_i^{k+1/2}}\mathcal{E}_i^{k+1/2}-\frac{m_i}{\rho_i^{k-1/2}}\mathcal{E}_i^{k-1/2}\right]=\\
&\gamma_L\frac{1}{2}\frac{m_{1/2}}{\Delta t^k}\left[(u_{1/2}^{k+1/2})^2-(u_{1/2}^{k-1/2})^2\right]+\gamma_R\frac{1}{2}\frac{m_{N+1/2}}{\Delta t^k}\left[(u_{N+1/2}^{k+1/2})^2-(u_{N+1/2}^{k-1/2})^2\right]-\\
&\left[(1-\gamma_R)A_{N+1/2}^k P_{N+1/2}^k u_{N+1/2}^k-(1-\gamma_L)A_{1/2}^k P_{1/2}^k u_{1/2}^k\right]-\\
&\left[A_{N+1/2}^k\mathcal{F}_{N+1/2}-A_{1/2}^k\mathcal{F}_{1/2}\right]-\\
&\left[A_{N+1/2}^k\frac{1}{3}\mathcal{E}_{N+1/2}^k u_{N+1/2}^k-A_{1/2}^k\frac{1}{3}\mathcal{E}_{1/2}^k u_{1/2}^k\right]. \qquad (18)
\end{aligned}
$$

To compute a conservation parameter, it is useful at non-reflective boundaries to break the net radiative flux into its incoming and outgoing components. The details are left to the reader. If you integrate Eq. (18) over all time steps, you obtain the global conservation expressions given in Lecture 8.