Basic derivations of Radiation-Hydrodynamics

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Abstract:
Work is work for some, but for some it is play.

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1 Definitions

1.1 Independent variables

We refer to the following independent variables:

- Position in the cartesian space $\{x, y, z\}$ is denoted with $\mathbf{x}$ and each component having units $[cm]$.
- Direction, $\{\varphi, \theta\}$, is denoted with $\mathbf{\Omega}$ which takes on the form

$$\mathbf{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \text{ and/or } \mathbf{\Omega} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix},$$

  where φ is the azimuthal-angle and θ is the polar-angle, both in spherical coordinates. Commonly, $\cos\theta$, is denoted with μ. The general dimension of angular phase space is $[steridian]$.
- Photon frequency, ν in $[Hertz]$ or $[s^{-1}]$.
- Time, t in $[s]$.

1.2 Dependent variables

We use the following basic dependent variables:

- The foundation of the dependent unknowns is the **radiation angular intensity**, $I(\mathbf{x}, \mathbf{\Omega}, \nu, t)$ with units $[Joule/cm^2{-}s{-}steradian{-}Hz]$. We often use the corresponding angle-integral of this quantity, $\phi(\mathbf{x}, \nu, t)$, and define it as

$$\phi(\mathbf{x}, \nu, t) = \mathcal{E}c = \int_{4\pi} I(\mathbf{x}, \mathbf{\Omega}, \nu, t)\ d\mathbf{\Omega} \tag{1}$$

  with units $[Joule/cm^2{-}s{-}Hz]$.
- The **radiation energy density**, $\mathcal{E}$, is

$$\mathcal{E}(\mathbf{x}, \nu, t) = \frac{\phi}{c} = \frac{1}{c}\int_{4\pi} I(\mathbf{x}, \mathbf{\Omega}, \nu, t)\ d\mathbf{\Omega} \tag{2}$$

  with units $[Joule/cm^3{-}Hz]$.

1.3 Blackbody radiation

A blackbody source, $B(\nu, T)$, is properly described by Planck's law,

$$B(\nu, T) = \frac{1}{4\pi}\frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{3}$$

with units $[Joule/cm^2{-}s{-}steridian - Hz]$ where h is Planck's constant and k_B is the Boltzmann constant.

If we integrate the blackbody source over all angle-space and frequencies the we get the mean radiation intensity from a blackbody at temperature T as

$$\begin{aligned} \int_0^\infty \int_{4\pi} B(\nu, T)\ d\mathbf{\Omega} d\nu &= \int_0^\infty \int_{4\pi} \frac{1}{4\pi}\frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1}\ d\mathbf{\Omega} d\nu \\ &= acT^4, \end{aligned} \tag{4}$$

with units $[Joule/cm^2{-}s{-}steridian]$ and where a is the **blackbody radiation constant** given by

$$a = \frac{8\pi^5 k_B^4}{15h^3c^3}. \tag{5}$$

In both cases this unfortunately is only the intensity. Following Kirchoff's law, which states that the emission and absorption of radiation must be equal in equilibrium, we can determine the **blackbody emission rate**, S_{bb}, from the absorption rate as

$$S_{bb}(\nu, T) = \rho\kappa(\nu)B(\nu, T), \tag{6}$$

with units $[Joule/cm^3{-}s{-}steridian{-}Hz]$ where ρ is the material density $[g/cm^3]$ and κ is the opacity $[cm^2/g]$. Data for the opacity of a material is normally available in the form of either the **Rosseland opacity**, κ_{Rs}, or the **Planck opacity**, κ_{Pl}.

2 Conservation equation - Electromagnetic Radiation

The basic statement of conservation, without hydrodynamics, is

$$\begin{aligned} \frac{1}{c}\frac{\partial I(\mathbf{x}, \mathbf{\Omega}, \nu, t)}{\partial t} &= -\mathbf{\Omega} \cdot \mathbf{\nabla} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) - \sigma_t(\mathbf{x}, \nu) I(\mathbf{x}, \mathbf{\Omega}, \nu, t) \\ &+ \int_0^\infty \int_{4\pi} \frac{\nu}{\nu'}\sigma_s(\mathbf{x}, \nu' {\to} \nu, \mathbf{\Omega}' {\cdot} \mathbf{\Omega}) I(\mathbf{x}, \mathbf{\Omega}', \nu, t) d\nu' d\mathbf{\Omega}' + S \end{aligned} \tag{7}$$

References

[1] Lewis E.E., Miller W.F., *Computational Methods of Neutron Transport*, JohnWiley & Sons, 1984

A First appendix

Put “Lazy reader stuff here”.