NVEN 627

Lecture 14

Numerical Fluxes from Riemann Solutions

Discontinuous finite-element methods and variations on such methods are very popular for solving the Euler equations. Such schemes are generally cell-centered and conservative over each cell. One must explicitly define the fluxes at cell interfaces because such fluxes are located at a discontinuity in the solution. Riemann solutions can be used to do this. This is a popular, but by no means unique technique for doing this. S. K. Godunov was the first person to suggest using Riemann solvers in this way (1959). Whether using exact Riemann solutions or approximate Riemann solutions this approach is often referred to as Godunov's method.

To illustrate this approach, we consider a scalar conservation equation:

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0. \qquad (1)$$

Integrating Eq (1) over the space-time cell, $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \otimes [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}]$, we get

$$(u_i^{n+\frac{1}{2}} - u_i^{n-\frac{1}{2}})\Delta x_i + (F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}})\Delta t^n = 0, \qquad (2)$$

where $u_i^{n+\frac{1}{2}}$ denotes the spatially-averaged value of u at $t = t^{n+\frac{1}{2}}$, and $F^n_{i+\frac{1}{2}}$ represents the time-averaged flux at $x = x_{i+\frac{1}{2}}$. At this point, we have made no approximations. Next we assume a piecewise-constant spatial approximation for

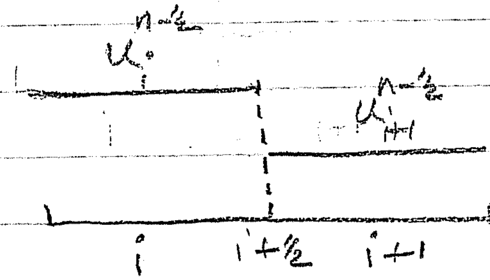
$$u(x) = u_i \quad , \quad x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) . \qquad (3)$$

Note that at this point, our approximation is defined only on the spatial cell interiors. Thus the approximation for $u_i^{n\pm\frac{1}{2}}$ is obvious, but we need to somehow define $F_{i+\frac{1}{2}}$ in terms of u_i and u_{i+1}. The simplest way to do is the central difference method:

$$F_{i+\frac{1}{2}} = \frac{1}{2}(F_i + F_{i+1}). \qquad (4)$$

However, this method can lead to highly

oscillatory solutions. Thus we need to find a better technique. Godunov's method is an excellent alternative. The idea is simple. At $t^{n-\frac{1}{2}}$, we have the following initial condition:



Note that if we extended the left and right values to $-\infty$ and $+\infty$, respectively, we would have initial conditions corresponding to the Riemann problem. Even though these initial values do not extend to infinity, the solution to this problem will still apply for times sufficiently small that the solutions associated with each cell interface do not interfere with one another. In general, let Δx_{min} represent the smallest cell width, then we must choose Δt such that $S\Delta t < \Delta x/2$, where S is the maximum signal speed. For the case of the Euler equations, one can generally assume that $S = a + V$, where "a" is the speed of sound and V is the flow speed. With this time step restriction we can set $F_{i+\frac{1}{2}}$ to that associated with the Reimann solution, i.e.,

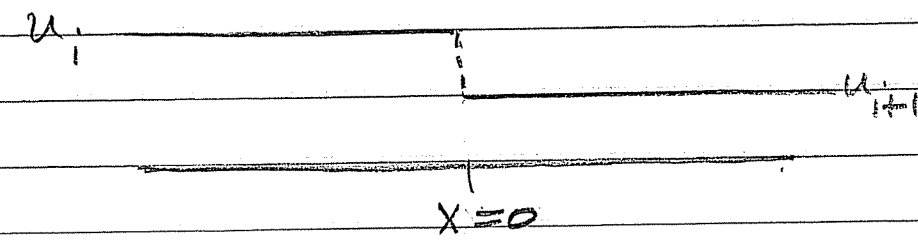
$$F_{i+\frac{1}{2}} = F(u_i, u_{i+1}). \tag{5}$$

Remember that the Reimann solution is a function only of x/t. Thus the solution at $x=0$ for $t>0$ is constant.

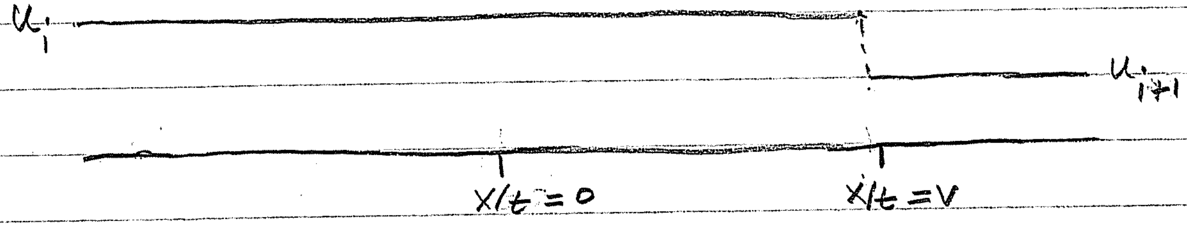
Let us illustrate this approach for the simple case of linear advection:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0, \tag{6}$$

where we assume without loss of generality that $v>0$. The initial condition is given at $t=0$ by



The solution as a function of x/t is given by



Thus, following (5), we get

$$F_{i+\frac{1}{2}} = F(u_i, u_{i+1}) = V u_i . \qquad (7)$$

Substituting from (7) into (2) yields

$$(u_i^{n+\frac{1}{2}} - u_i^{n-\frac{1}{2}}) \Delta x_i + V(u_i^{n-\frac{1}{2}} - u_{i-1}^{n-\frac{1}{2}}) \Delta t^n = 0 \qquad (8)$$

This is called upwinding, i.e., one defines the flux at an interface in terms of the "upstream" value of the interface. This is a common technique in transport theory, and with a piecewise-constant trial space results in "step" differencing. However, the transport community did not arrive at upwinding via the Riemann problem. It was simply discovered independently. Upwinding is trivial when the unknown has a unique direction of flow, but is much more complicated when an unknown has no unique direction of flow, e.g., pressure

In this case either a Riemann solver or an approximate Riemann solver is usually used. In the case of a linear hyperbolic system, one can upwind the characteristic variables and then transform back to to the physical variables to obtain the proper upwinding. To demonstrate the technique, consider the P-1 equations (neglecting interactions):

$$\frac{1}{v}\frac{\partial \phi}{\partial t} + \frac{\partial J}{\partial x} = 0, \tag{9a}$$

$$\frac{1}{v}\frac{\partial J}{\partial t} + \frac{1}{3}\frac{\partial \phi}{\partial x} = 0, \tag{9b}$$

or equivalently,

$$\frac{1}{v}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} \phi \\ J \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1/3 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} \phi \\ J \end{pmatrix} = 0. \tag{10}$$

The eigenvalues of $\begin{pmatrix} 0 & 1 \\ 1/3 & 0 \end{pmatrix}$ are $\lambda = \pm\frac{1}{\sqrt{3}}$,

and the eigenvectors for $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$, respectively, are

$$\begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{3}} \end{pmatrix}. \tag{11}$$

Thus the R-matrix is

$$R = \begin{pmatrix} 1 & 1 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}. \qquad (12)$$

The inverse of this matrix is

$$R^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}. \qquad (13)$$

So the characteristic variables are

$$\vec{w} = R^{-1}\vec{u}, \qquad (14)$$

or

$$w_1 = \tfrac{1}{2}\phi + \tfrac{\sqrt{3}}{2}J, \qquad (14a)$$

$$w_2 = \tfrac{1}{2}\phi - \tfrac{\sqrt{3}}{2}J. \qquad (14b)$$

Note that these are just the two discrete angular fluxes associated with

The S_2 approximation with Gauss quadrature.

$$\psi^{+} = \psi\left(\tfrac{1}{\sqrt{3}}\right) = w_1 \qquad (15a)$$

$$\psi^{-} = \psi\left(-\tfrac{1}{\sqrt{3}}\right) = w_2. \qquad (15b)$$

This should not be surprising since the S_2 equations with Gauss quadrature are known to be equivalent to the P-1 equations (in 1-D slabs). The inverse relationships corresponding to (14a) and (14b) are

$$\phi = w_1 + w_2 , \qquad (16a)$$

$$J = \tfrac{1}{\sqrt{3}}(w_1 - w_2). \qquad (16b)$$

To upwind ϕ and J, we first upwind w_1 and w_2:

$$w_1(x_{i+\frac{1}{2}}) = w_1(x_i) \qquad (17a)$$

$$w_2(x_{i+\frac{1}{2}}) = w_2(x_{i+1}) \qquad (17b)$$

Next we substitute from Eqs. (14) into Eqs. (17):

$$W_1(x_{i+\frac{1}{2}}) = \frac{1}{2}\phi_i + \frac{\sqrt{3}}{2}J_i , \qquad (18a)$$

$$W_2(x_{i+\frac{1}{2}}) = \frac{1}{2}\phi_{i+1} - \frac{\sqrt{3}}{2}J_{i+1} . \qquad (18b)$$

Next we manipulate Eqs. (18) using Eqs. (16) to obtain the desired result:

$$\phi_{i+\frac{1}{2}} = \frac{1}{2}(\phi_i + \phi_{i+1}) - \frac{\sqrt{3}}{2}(J_{i+1} - J_i) , \qquad (19a)$$

$$J_{i+\frac{1}{2}} = \frac{1}{2}(J_i + J_{i+1}) - \frac{1}{2\sqrt{3}}(\phi_{i+1} - \phi_i) . \qquad (19b)$$

Note that Eqs. (19) take the form of central difference plus a "correction" term.

Artificial Viscosity

An alternative to upwinding (or equivalently using a Riemann solver), is the use of an artificial viscosity. We have already seen that if we add even a small amount of diffusion to a hyperbolic conservation system:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} , \qquad (20)$$

The solution even with a discontinuous

initial condition at $t=0$ becomes smooth at $t=0^+$ and remains so. Furthermore as $\epsilon \to 0$ the smooth solution approaches the unperturbed solution. Thus we can "smooth out" shocks to whatever extent we want by adding diffusion. The only restriction is that ϵ cannot really be a constant, because we would converge to the wrong solution, or equivalently the solution to the wrong equation. Rather, we must make ϵ large enough to eliminate the bad behavior associated with central differencing, but we must also make ϵ decrease sufficiently with the cell size "Δx" so that we properly converge to the correct solution as $\Delta x \to 0$.

To get an idea of how to do this, we can actually look at upwinding. Let us return to Eq. (8). This equation can easily be algebraically manipulated into the following form assuming a uniform mesh:

$$\frac{U_i^{n+\frac{1}{2}} - U_i^{n-\frac{1}{2}}}{\Delta t^n} + V\left\{\frac{\frac{1}{2}\left(U_{i+1}^{n-\frac{1}{2}} + U_i^{n-\frac{1}{2}}\right) - \frac{1}{2}\left(U_i^{n-\frac{1}{2}} + U_{i-1}^{n-\frac{1}{2}}\right)}{\Delta x}\right\} =$$

$$V\Delta x\left\{\frac{U_{i+1}^{n-\frac{1}{2}} - 2U_i^{n-\frac{1}{2}} + U_{i-1}^{n-\frac{1}{2}}}{\Delta x^2}\right\}. \qquad (21)$$

Note that Eq (21) takes the form of Eq. (20) with central differencing and a value of "ϵ" that is dependent on "Δx". In particular,

$$\epsilon = V \Delta x. \tag{22}$$

As required $\epsilon \to 0$ as $\Delta x \to 0$. If we do a truncation error analysis, we find that

$$V \left\{ \frac{\frac{1}{2}(u_{i+1}^{n-\frac{1}{2}} + u_i^{n-\frac{1}{2}}) - \frac{1}{2}(u_i^{n-\frac{1}{2}} + u_{i-1}^{n-\frac{1}{2}})}{\Delta x} \right\} - V\Delta x \left\{ \frac{u_{i+1}^{n-\frac{1}{2}} - 2u_i^{n-\frac{1}{2}} + u_{i-1}^{n-\frac{1}{2}}}{\Delta x} \right\} =$$

$$V \frac{\partial u^{n-\frac{1}{2}}}{\partial x} + O(\Delta x). \tag{23}$$

Thus we are led to two conclusions. The first is that upwinding for the simple case we considered is equivalent to central difference plus a diffusion or "artificial viscosity" term. As one might suspect, physical viscosity is represented by a diffusion term. In general upwinding is related to central differencing with an artificial viscosity treatment. The second conclusion (given Eq (22)) is that upwinding for the case of a piecewise-constant trial space is first-order accurate.