

Radiative heat transfer solver with fluid motion

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Abstract:

Work is work for some, but for some it is play.

Keywords: hydrodynamics

Contents

1	Definitions	2
1.1	Independent variables	2
1.2	Dependent variables	2
1.3	Blackbody radiation	3
2	Conservation equations	4
2.1	Conservation equation - Radiative transfer	4
2.2	Radiative transfer assuming isotropic Thompson scattering	4
2.3	Radiative transfer with material motion corrections	4
2.4	Radiative transfer with material velocity dependencies expanded to $\mathcal{O}(v/c)$	5
2.5	Grey Radiative Transfer	5
2.6	Grey Diffusion Approximation	6
2.7	Conservation equation for fluid flow	7
2.8	The set of Radiation Hydrodynamics Grey Diffusion Equations	8
3	Definitions	8
4	Finite Volume Spatial Discretization	9
4.1	Hydrodynamic and Radiation-energy advection	9
4.2	Density and momentum updates	10
4.3	Energy equations	10
4.3.1	The diffusion term	10
4.3.2	The kinetic energy term	11
5	Temporal scheme A - Implicit Euler Predictor, Crank-Nicolson Corrector	12
5.1	Predictor phase	12
5.2	Corrector phase	12
5.3	General energy equations, Predictor and Corrector phase, with θ factors	13
5.4	Using the energy related algebra for both the predictor and the corrector	16
6	Mixed finite element	17
6.1	Auxiliary notation and variables for \mathcal{F}_0	17
6.2	Using the auxiliary notation in the primary equation	19
6.3	Auxiliary equations	19
7	Temporal scheme B - Phase A: Implicit Euler Predictor, Crank-Nicolson Corrector, Phase B: Implicit Euler Predictor, Weighted θ's Corrector	20
7.1	Predictor phase A	20
7.2	Corrector phase A	20
7.3	Predictor phase B	21
7.4	Corrector phase B	21

A	Angular integration identities	22
B	Boundary and initial conditions for radiation hydrodynamic problems	23
B.1	Hydrodynamics only	23
B.2	Hydrodynamics with radiation energy	24
C	Roderigues's formula	27

1 Definitions

1.1 Independent variables

We refer to the following independent variables:

- Position in the cartesian space $\{x, y, z\}$ is denoted with \mathbf{x} and each component having units $[cm]$.
- Direction, $\{\varphi, \theta\}$, is denoted with $\mathbf{\Omega}$ which takes on the form

$$\mathbf{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \text{ and/or } \mathbf{\Omega} = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix},$$

where φ is the azimuthal-angle and θ is the polar-angle, both in spherical coordinates. Commonly, $\cos \theta$, is denoted with μ . The general dimension of angular phase space is $[steridian]$.

- Photon frequency, ν in $[Hertz]$ or $[s^{-1}]$.
- Time, t in $[s]$.

1.2 Dependent variables

We use the following basic dependent variables:

- The foundation of the dependent unknowns is the **radiation angular intensity**, $I(\mathbf{x}, \mathbf{\Omega}, \nu, t)$ with units $[Joule/cm^2-s-steradian-Hz]$. We often use the corresponding angle-integral of this quantity, $\phi(\mathbf{x}, \nu, t)$, and define it as

$$\phi(\mathbf{x}, \nu, t) = \mathcal{E}c = \int_{4\pi} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \quad (1.1)$$

with units $[Joule/cm^2-s-Hz]$. Where c is the speed of light.

- The **radiation energy density**, \mathcal{E} , is

$$\mathcal{E}(\mathbf{x}, \nu, t) = \frac{\phi}{c} = \frac{1}{c} \int_{4\pi} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \quad (1.2)$$

with units $[Joule/cm^3-Hz]$.

- The **radiation energy flux**, \mathcal{F} , is

$$\mathcal{F}(\mathbf{x}, \nu, t) = \int_{4\pi} \mathbf{\Omega} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \quad (1.3)$$

- **Radiation pressure**, \mathcal{P} , is

$$\mathcal{P}(\mathbf{x}, \nu, t) = \frac{1}{c} \int_{4\pi} \mathbf{\Omega} \otimes \mathbf{\Omega} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \quad (1.4)$$

and is a tensor.

1.3 Blackbody radiation

A blackbody radiation source, $B(\nu, T)$, is properly described by **Planck's law**,

$$B(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \quad (1.5)$$

with units [*Joule/cm²–s–steradian – Hz*] where h is Planck's constant and k_B is the Boltzmann constant.

If we integrate the blackbody source over all angle-space and frequencies then we get the mean radiation intensity from a blackbody at temperature T as

$$\begin{aligned} \int_0^\infty \int_{4\pi} B(\nu, T) d\Omega d\nu &= \int_0^\infty \int_{4\pi} \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} d\Omega d\nu \\ &= 4\pi \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} d\nu \\ &= acT^4, \end{aligned} \quad (1.6)$$

with units [*Joule/cm²–s–steradian*] and where a is the **blackbody radiation constant** given by

$$a = \frac{8\pi^5 k_B^4}{15h^3 c^3}. \quad (1.7)$$

In both cases this unfortunately is only the intensity. Following Kirchoff's law, which states that the emission and absorption of radiation must be equal in equilibrium, we can determine the **blackbody emission rate**, S_{bb} , from the absorption rate as

$$S_{bb}(\nu, T) = \rho\kappa(\nu)B(\nu, T), \quad (1.8)$$

with units [*Joule/cm³–s–steradian–Hz*] where ρ is the material density [*g/cm³*] and κ is the opacity [*cm²/g*]. The combination $\rho\kappa$ is also equal to the macroscopic absorption cross section σ_a , therefore $\rho\kappa(\nu) = \sigma_a$. Data for the opacity of a material is normally available in the form of either the **Rosseland opacity**, κ_{Rs} , or the **Planck opacity**, κ_{Pl} .

2 Conservation equations

2.1 Conservation equation - Radiative transfer

The basic statement of conservation, is

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)}{\partial t} &= -\boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) - \sigma_t(\mathbf{x}, \nu) I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) \\ &+ \int_0^\infty \int_{4\pi} \frac{\nu}{\nu'} \sigma_s(\mathbf{x}, \nu' \rightarrow \nu, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) I(\mathbf{x}, \boldsymbol{\Omega}', \nu', t) d\nu' d\boldsymbol{\Omega}' \\ &+ \sigma_a(\mathbf{x}, \nu) B(\nu, T(\mathbf{x}, t)) + S \end{aligned} \quad (2.1)$$

where S is any other sources/sinks of radiation intensity.

2.2 Radiative transfer assuming isotropic Thompson scattering

Assuming Thomson-scattering¹ is the only form of scattering, gives

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)}{\partial t} &= -\boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) - \sigma_t(\mathbf{x}, \nu) I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) \\ &+ \frac{\sigma_s(\mathbf{x}, \nu)}{4\pi} c\mathcal{E}(\mathbf{x}, \nu) + \sigma_a(\mathbf{x}, \nu) B(\nu, T(\mathbf{x}, t)) + S \end{aligned} \quad (2.2)$$

where S is any other sources/sinks of radiation intensity.

Using energy instead of frequency, $\nu \rightarrow E$:

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, E, t)}{\partial t} &= -\boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, E, t) - \sigma_t(\mathbf{x}, E) I(\mathbf{x}, \boldsymbol{\Omega}, E, t) \\ &+ \frac{\sigma_s(\mathbf{x}, E)}{4\pi} c\mathcal{E}(\mathbf{x}, E) + \sigma_a(\mathbf{x}, E) B(E, T(\mathbf{x}, t)) + S \end{aligned} \quad (2.3)$$

where S is any other sources/sinks of radiation intensity.

2.3 Radiative transfer with material motion corrections

Applying relativistic corrections for a material in motion, we can derive

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, E, t)}{\partial t} &= -\boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, E, t) - \left(\frac{E_0}{E}\right) \sigma_t(\mathbf{x}, E_0) I(\mathbf{x}, \boldsymbol{\Omega}, E, t) \\ &+ \left(\frac{E}{E_0}\right)^2 \frac{\sigma_s(\mathbf{x}, E)}{4\pi} \int_{4\pi} \left(\frac{E_0}{E'}\right) I(\mathbf{x}, \boldsymbol{\Omega}', E', t) d\boldsymbol{\Omega}' + \left(\frac{E}{E_0}\right)^2 \sigma_a(\mathbf{x}, E_0) B(E_0, T(\mathbf{x}, t)) + S, \end{aligned} \quad (2.4)$$

where

$$E_0 = E\gamma \left(1 - \boldsymbol{\Omega} \cdot \frac{\mathbf{u}}{c}\right) \quad (2.5)$$

$$\gamma = \left[1 - \left(\frac{\|\mathbf{u}\|}{c}\right)^2\right]^{-\frac{1}{2}} \quad (2.6)$$

$$\frac{E_0}{E'} = \gamma \left(1 - \boldsymbol{\Omega}' \cdot \frac{\mathbf{u}}{c}\right) \quad (2.7)$$

$$E' = E \frac{1 - \boldsymbol{\Omega} \cdot \frac{\mathbf{u}}{c}}{1 - \boldsymbol{\Omega}' \cdot \frac{\mathbf{u}}{c}} \quad (2.8)$$

¹Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle. The particle's kinetic energy- as well as the photon's frequency, does not change in such a scattering. The scattering is also isotropic.

2.4 Radiative transfer with material velocity dependencies expanded to $\mathcal{O}(v/c)$

$$\begin{aligned}
& \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, E, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, E, t) + \sigma_t(\mathbf{x}, E) I(\mathbf{x}, \boldsymbol{\Omega}, E, t) \\
&= \frac{\sigma_s(\mathbf{x}, E)}{4\pi} \phi(E) + \sigma_a(\mathbf{x}, E) B(E, T(\mathbf{x}, t)) \\
&+ \left[\left(\sigma_t + E \frac{\partial \sigma_a}{\partial E} \right) I + \frac{\sigma_s}{4\pi} \left(2\phi - E \frac{\partial \phi}{\partial E} \right) + 2\sigma_a B(E, T) - B(E, T) \frac{\partial \sigma_a}{\partial E} - \sigma_a E \frac{\partial B(E, T)}{\partial E} \right] \boldsymbol{\Omega} \cdot \frac{\mathbf{u}}{c} \\
&- \frac{\sigma_s}{4\pi} \left(\mathcal{F} - E \frac{\partial \mathcal{F}}{\partial E} \right) \cdot \frac{\mathbf{u}}{c}
\end{aligned} \tag{2.9}$$

Radiation energy equation:

Obtained by integrating the transport equation over energy and angle

$$\begin{aligned}
\frac{\partial \mathcal{E}(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) &= \int_0^\infty \sigma_a(\mathbf{x}, E) (4\pi B(E, T) - \phi(\mathbf{x}, E, t)) dE \\
&+ \int_0^\infty \left(\sigma_a + E \frac{\partial \sigma_a}{\partial E} - \sigma_s(E) \right) \mathcal{F} \cdot \frac{\mathbf{u}}{c} dE
\end{aligned} \tag{2.10}$$

Radiation momentum equation:

Obtained by first multiplying by $\frac{1}{c} \boldsymbol{\Omega}$, then integrating over all directions and energies,

$$\begin{aligned}
\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot \mathcal{P} &= - \int_0^\infty \frac{\sigma_t}{c} \mathcal{F} dE \\
&+ \int_0^\infty \left(\sigma_s \phi + \sigma_a 4\pi B(E, T) \right) \frac{\mathbf{u}}{c^2} dE \\
&+ \int_0^\infty \left(\sigma_a + E \frac{\partial \sigma_a}{\partial E} + \sigma_s \right) \mathcal{P} \cdot \frac{\mathbf{u}}{c} dE
\end{aligned} \tag{2.11}$$

2.5 Grey Radiative Transfer

$$\begin{aligned}
& \frac{1}{c} \frac{\partial I(\mathbf{x}, \boldsymbol{\Omega}, t)}{\partial t} + \boldsymbol{\Omega} \cdot \nabla I(\mathbf{x}, \boldsymbol{\Omega}, t) + \sigma_t(\mathbf{x}) I(\mathbf{x}, \boldsymbol{\Omega}, t) \\
&= \frac{\sigma_s}{4\pi} \phi + \frac{\sigma_a}{4\pi} a c T^4 \\
&+ \left[\sigma_t I + \frac{\sigma_s}{4\pi} 2\phi + 2\sigma_a \frac{1}{4\pi} a c T^4 - \sigma_a E \frac{\partial B(E, T)}{\partial E} \right] \boldsymbol{\Omega} \cdot \frac{\mathbf{u}}{c} \\
&- \frac{\sigma_s}{4\pi} \mathcal{F} \cdot \frac{\mathbf{u}}{c}
\end{aligned} \tag{2.12}$$

Radiation energy equation:

Obtained by integrating Eq. (2.12) over energy and angle

$$\frac{\partial \mathcal{E}(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) = \sigma_a c (a T^4 - \mathcal{E}) + (\sigma_a - \sigma_s) \mathcal{F} \cdot \frac{\mathbf{u}}{c} \tag{2.13}$$

Radiation momentum equation:

Obtained by first multiplying Eq. (2.12) by $\frac{1}{c} \boldsymbol{\Omega}$, then integrating over all directions and energies,

$$\frac{1}{c^2} \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot \mathcal{P} = - \frac{\sigma_t}{c} \mathcal{F} + (\sigma_s c \mathcal{E} + \sigma_a a c T^4) \frac{\mathbf{u}}{c^2} + \sigma_t \mathcal{P} \cdot \frac{\mathbf{u}}{c} \tag{2.14}$$

2.6 Grey Diffusion Approximation

Approximating the angular dependence of $I(\mathbf{\Omega})$ with a P_1 spherical harmonic expansion, such that the entries of \mathcal{P} are given by

$$(\mathcal{P})_{i,j} = \frac{1}{3}\mathcal{E}\delta_{i,j}, \quad (2.15)$$

the radiation energy equation is unaffected but the radiation momentum equation changes. We repeat the radiation energy equation below, and the altered radiation moment equations:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) = \sigma_a c (aT^4 - \mathcal{E}) + (\sigma_a - \sigma_s) \mathcal{F} \cdot \frac{\mathbf{u}}{c}, \quad (2.16)$$

$$\frac{1}{3} \nabla \mathcal{E} = -\frac{\sigma_t}{c} \mathcal{F} + (\sigma_s c \mathcal{E} + \sigma_a a c T^4) \frac{\mathbf{u}}{c^2} + \sigma_t \frac{1}{3} \mathcal{E} \frac{\mathbf{u}}{c}. \quad (2.17)$$

Useful transformations:

$$\mathcal{E}_0 = \mathcal{E} - \frac{2}{c^2} \mathcal{F} \cdot \mathbf{u} \quad (2.18a)$$

$$\mathcal{E} = \mathcal{E}_0 + \frac{2}{c^2} \mathcal{F}_0 \cdot \mathbf{u} \quad (2.18b)$$

$$\mathcal{F}_0 = \mathcal{F} - (\mathcal{E} \mathbf{u} + \mathcal{P} \cdot \mathbf{u}) \quad (2.18c)$$

$$\mathcal{F} = \mathcal{F}_0 + (\mathcal{E}_0 \mathbf{u} + \mathcal{P}_0 \cdot \mathbf{u}) \quad (2.18d)$$

$$\mathcal{P}_0 = \mathcal{P} - \frac{2}{c^2} \mathbf{u} \otimes \mathcal{F} \quad (2.18e)$$

$$\mathcal{P} = \mathcal{P}_0 + \frac{2}{c^2} \mathbf{u} \otimes \mathcal{F}_0 \quad (2.18f)$$

With the P_1 approximation

$$\mathcal{F}_0 = \mathcal{F} - \frac{4}{3} \mathcal{E} \mathbf{u} \quad (2.18g)$$

$$\mathcal{F} = \mathcal{F}_0 + \frac{4}{3} \mathcal{E} \mathbf{u} \quad (2.18h)$$

Applying these transformations the radiation energy- and moment equation can be expressed as

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) = \sigma_a c (aT^4 - \mathcal{E}_0) - \sigma_t \mathcal{F} \cdot \frac{\mathbf{u}}{c}, \quad (2.19)$$

$$\frac{1}{3} \nabla \mathcal{E} = -\frac{\sigma_t}{c} \mathcal{F}_0 + \sigma_a c (aT^4 - \mathcal{E}) \frac{\mathbf{u}}{c^2}. \quad (2.20)$$

Several simplifications to these equations are made. Firstly arriving at the expression for the radiation energy equation,

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) = \sigma_a c (aT^4 - \mathcal{E}) - \sigma_t \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c}, \quad (2.21)$$

then the radiation momentum equation,

$$\frac{1}{3} \nabla \mathcal{E} = -\frac{\sigma_t}{c} \mathcal{F}_0 \quad (2.22)$$

from which we can get expression for \mathcal{F}_0 and \mathcal{F} in terms of \mathcal{E} as

$$\mathcal{F}_0 = -\frac{c}{3\sigma_t} \nabla \mathcal{E} \quad (2.23)$$

and

$$\begin{aligned} \frac{1}{3} \nabla \mathcal{E} &= -\frac{\sigma_t}{c} \left(\mathcal{F} - \frac{4}{3} \mathcal{E} \mathbf{u} \right) \\ \therefore \mathcal{F} &= -\frac{c}{3\sigma_t} \nabla \mathcal{E} + \frac{4}{3} \mathcal{E} \mathbf{u}. \end{aligned} \quad (2.24)$$

These expressions for \mathcal{F}_0 and \mathcal{F} are both then inserted into the radiation energy equation as follows

$$\begin{aligned}
& \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{x}, t) = \sigma_a c (aT^4 - \mathcal{E}) - \sigma_t \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c} \\
& \rightarrow \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(-\frac{c}{3\sigma_t} \nabla \mathcal{E} + \frac{4}{3} \mathcal{E} \mathbf{u} \right) = \sigma_a c (aT^4 - \mathcal{E}) - \sigma_t \left(-\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) \cdot \frac{\mathbf{u}}{c} \\
& \rightarrow \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(-\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}.
\end{aligned} \tag{2.25}$$

Arriving at a **diffusion form** of the **radiation energy equation**,

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(-\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \tag{2.26}$$

2.7 Conservation equation for fluid flow

The governing equations we consider here are the Euler equations defined as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.27}$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \nabla p = \mathbf{f}, \tag{2.28}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{u}] = q \tag{2.29}$$

where ρ is the fluid density, $\mathbf{u} = [u_x, u_y, u_z] = [u, v, w]$ is the fluid velocity in cartesian coordinates, p is the fluid pressure, E is the material energy-density comprising kinetic energy-density, $\frac{1}{2} \rho ||\mathbf{u}||^2$, and internal energy-density, ρe , such that $E = \frac{1}{2} \rho ||\mathbf{u}||^2 + \rho e$, where e is the specific internal energy. The values q and \mathbf{f} are abstractly used here as energy- and moment- sources/sinks, respectively.

The ideal gas law provides the closure relation

$$p = (\gamma - 1) \rho e \tag{2.30}$$

where γ is the ratio of the constant-pressure specific heat, c_p , to the constant-volume specific heat, c_v , i.e., $\gamma = \frac{c_p}{c_v}$, and is a material property.

Coupling terms:

$$\begin{aligned}
\mathbf{f} &= \frac{\sigma_t}{c} \mathcal{F}_0 \\
&= -\frac{1}{3} \nabla \mathcal{E}
\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}
q &= -\left(\sigma_a c (aT^4 - \mathcal{E}) - \sigma_t \mathcal{F}_0 \cdot \frac{\mathbf{u}}{c} \right) \\
&= \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}
\end{aligned} \tag{2.32}$$

2.8 The set of Radiation Hydrodynamics Grey Diffusion Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (2.33a)$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \nabla p = -\frac{1}{3} \nabla \mathcal{E}, \quad (2.33b)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{u}] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \quad (2.33c)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(-\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla (\mathcal{E} \mathbf{u}) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \quad (2.33d)$$

where

$$E = \frac{1}{2} \rho \|\mathbf{u}\|^2 + \rho e, \quad (2.33e)$$

$$p = (\gamma - 1) \rho e, \quad (2.33f)$$

$$T = \frac{1}{C_v} e \quad (2.33g)$$

$$\sigma_t(T) = \sigma_s(T) + \sigma_a(T) \quad (2.33h)$$

$$\sigma_s(T) = \rho \kappa_s(T) \quad (2.33i)$$

$$\sigma_a(T) = \rho \kappa_a(T) \quad (2.33j)$$

3 Definitions

First we define the following terms

- The radiation emission and absorption, the radiation momentum source, and the radiation energy source

$$S_{ea} = \sigma_a c (aT^4 - \mathcal{E}) \quad (3.1a)$$

$$\mathbf{S}_{rp} = \frac{1}{3} \nabla \mathcal{E} \quad (3.1b)$$

$$S_{re} = S_{ea} + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \quad (3.1c)$$

- The conserved hydrodynamic variables, \mathbf{U} , and associated hydrodynamic flux, \mathcal{F}^H ,

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho \mathbf{u} \\ E \end{bmatrix} \quad \mathcal{F}^H = \begin{bmatrix} \rho u \\ \rho u u + p \\ \rho u v \\ \rho u w \\ (E + p)u \end{bmatrix} \quad (3.1d)$$

- The stationary reference frame radiation energy flux

$$\mathcal{F}_0 = -\frac{c}{3\sigma_t} \nabla \mathcal{E} \quad (3.1e)$$

Next, we use these terms to define a more condensed version of the RHGD equations.

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \quad (3.2)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}_0 + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = S_{ea} + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \quad (3.3)$$

4 Finite Volume Spatial Discretization

To apply a finite volume spatial discretization we integrate our time-discretized equations over the volume, V_c , of cell c , and afterwards divide by V_c . This leaves all the terms containing τ unchanged. In this process we develop the following terms:

4.1 Hydrodynamic and Radiation-energy advection

$$\frac{1}{V_c} \int_{V_c} \nabla \cdot \mathcal{F}^H(\mathbf{U}) dV = \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot \mathcal{F}^H(\mathbf{U}_f) \quad (4.1)$$

$$\frac{1}{V_c} \int_{V_c} \left(\frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right) dV = \frac{1}{V_c} \sum_f \frac{4}{3} \mathbf{A}_f \cdot (\mathcal{E} \mathbf{u})_f \quad (4.2)$$

The face values are reconstructed from gradients in both the predictor and corrector phases. In the corrector-phase the hydrodynamic flux, \mathcal{F}^H , is used in its earlier defined form, whilst in the corrector-phase the flux is determined by an approximate Riemann-solver, i.e., the HLLC Riemann solver.

Predictor phases:

For the predictor phase we have the following:

$$\nabla \cdot \mathcal{F}^H(\mathbf{U}) \mapsto \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot \mathcal{F}^H(\mathbf{U}_f) \quad (4.3)$$

$$\left(\frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right) \mapsto \frac{1}{V_c} \sum_f \frac{4}{3} \mathbf{A}_f \cdot (\mathcal{E} \mathbf{u})_f \quad (4.4)$$

$$\mathbf{U}_f = \mathbf{U}_c + (\mathbf{x}_f - \mathbf{x}_c) \cdot \{\nabla \mathbf{U}\}_c \quad (4.5)$$

$$\mathcal{E}_f = \mathcal{E}_c + (\mathbf{x}_f - \mathbf{x}_c) \cdot \{\nabla \mathcal{E}\}_c \quad (4.6)$$

Corrector phases:

For the corrector phase we have the following:

$$\nabla \cdot \mathcal{F}^H(\mathbf{U}) \mapsto \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot \mathbf{F}^{*hllc}(\mathbf{U}_f) \quad (4.7)$$

$$\left(\frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right) \mapsto \frac{1}{V_c} \sum_f \frac{4}{3} \mathbf{A}_f \cdot (\mathcal{E} \mathbf{u})_{upw} \quad (4.8)$$

where

$$\mathbf{U}_f = \mathbf{U}_c + (\mathbf{x}_f - \mathbf{x}_c) \cdot \{\nabla \mathbf{U}\}_c \quad (4.9)$$

$$(\mathcal{E} \mathbf{u})_{upw} = \begin{cases} (\mathcal{E} \mathbf{u})_{c,f}, & \text{if } \mathbf{u}_{c,f} \cdot \mathbf{n}_f > 0 \text{ and } \mathbf{u}_{cn,f} \cdot \mathbf{n}_f > 0 & \rightarrow | \rightarrow \\ (\mathcal{E} \mathbf{u})_{cn,f}, & \text{if } \mathbf{u}_{c,f} \cdot \mathbf{n}_f < 0 \text{ and } \mathbf{u}_{cn,f} \cdot \mathbf{n}_f < 0 & \leftarrow | \leftarrow \\ (\mathcal{E} \mathbf{u})_{cn,f} + (\mathcal{E} \mathbf{u})_{c,f}, & \text{if } \mathbf{u}_{c,f} \cdot \mathbf{n}_f > 0 \text{ and } \mathbf{u}_{cn,f} \cdot \mathbf{n}_f < 0 & \rightarrow | \leftarrow \\ 0, & \text{if } \mathbf{u}_{c,f} \cdot \mathbf{n}_f < 0 \text{ and } \mathbf{u}_{cn,f} \cdot \mathbf{n}_f > 0 & \leftarrow | \rightarrow \end{cases} \quad (4.10)$$

$$\mathcal{E}_{c,f} = \mathcal{E}_c + (\mathbf{x}_f - \mathbf{x}_c) \cdot \{\nabla \mathcal{E}\}_c \quad (4.11)$$

4.2 Density and momentum updates

We apply the same process as before:

$$-\frac{1}{V_c} \int_{V_c} \mathbf{S}_{rp} dV = -\frac{1}{V_c} \sum_f \frac{1}{3} \mathbf{A}_f \mathcal{E}_f, \quad (4.12)$$

however, here we want \mathcal{E}_f to satisfy the following relationship

$$\frac{D_c}{\|\mathbf{x}_{cf}\|} (\mathcal{E}_f - \mathcal{E}_c) = \frac{D_{cn}}{\|\mathbf{x}_{fcn}\|} (\mathcal{E}_{cn} - \mathcal{E}_f) \quad (4.13)$$

where

$$D_c = -\frac{c}{3\sigma_{t,c}} \quad (4.14)$$

and where \mathbf{x}_{cf} is the vector from cell c 's centroid to the face centroid, \mathbf{x}_{fcn} is the vector from the face centroid to cell cn 's centroid (where cell cn is the neighbor to c at face f). The norm $\|\cdot\|$ refers to the L_2 norm.

Solving the above relationship for \mathcal{E}_f we first set

$$k_c = \frac{D_c}{\|\mathbf{x}_{cf}\|}, \quad k_{cn} = \frac{D_{cn}}{\|\mathbf{x}_{fcn}\|}$$

then get

$$\begin{aligned} k_c \mathcal{E}_f - k_c \mathcal{E}_c &= k_{cn} \mathcal{E}_{cn} - k_{cn} \mathcal{E}_f \\ \rightarrow (k_c + k_{cn}) \mathcal{E}_f &= k_{cn} \mathcal{E}_{cn} + k_c \mathcal{E}_c \\ \therefore \mathcal{E}_f &= \frac{k_{cn} \mathcal{E}_{cn} + k_c \mathcal{E}_c}{k_c + k_{cn}}. \end{aligned} \quad (4.15)$$

Predictor and corrector phases:

We do the same for both,

$$-\mathbf{S}_{rp} \mapsto -\frac{1}{V_c} \sum_f \frac{1}{3} \mathbf{A}_f \mathcal{E}_f \quad (4.16)$$

4.3 Energy equations

Only two terms require special consideration here. They are: the divergence of the co-moving frame radiation energy flux, and the kinetic energy terms source terms,

$$\begin{aligned} \frac{1}{V_c} \int_{V_c} \nabla \cdot \mathcal{F}_0 dV &= \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot (\mathcal{F}_0)_f \\ \frac{1}{V_c} \int_{V_c} \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} dV &= \frac{1}{V_c} \sum_f \frac{1}{3} \mathbf{A}_f \cdot (\mathcal{E} \mathbf{u})_f. \end{aligned} \quad (4.17)$$

4.3.1 The diffusion term

Considering the \mathcal{F}_0 -term first, we apply Gauss' divergence theorem to get

$$\nabla \cdot \mathcal{F}_0 \mapsto \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot (\mathcal{F}_0)_f. \quad (4.18)$$

For $(\mathcal{F}_0)_f$ we have

$$(\mathcal{F}_0)_f = -\frac{c}{3\sigma_{tf}} (\nabla \mathcal{E})_f. \quad (4.19)$$

Now define

$$D_f = -\frac{c}{3\sigma_{tf}}. \quad (4.20)$$

To find D_f we seek the equivalence:

$$D_f \frac{\mathcal{E}_{cn} - \mathcal{E}_c}{\|\mathbf{x}_{cn} - \mathbf{x}_c\|} = D_c \frac{\mathcal{E}_f - \mathcal{E}_c}{\|\mathbf{x}_f - \mathbf{x}_c\|} = D_{cn} \frac{\mathcal{E}_{cn} - \mathcal{E}_f}{\|\mathbf{x}_{cn} - \mathbf{x}_f\|} \quad (4.21)$$

Now let us define

$$\begin{aligned} k_c &= \frac{D_c}{\|\mathbf{x}_f - \mathbf{x}_c\|} \\ k_{cn} &= \frac{D_{cn}}{\|\mathbf{x}_{cn} - \mathbf{x}_f\|} \end{aligned} \quad (4.22)$$

Now

$$\begin{aligned} k_c(\mathcal{E}_f - \mathcal{E}_c) &= k_{cn}(\mathcal{E}_{cn} - \mathcal{E}_f) \\ (k_c + k_{cn})\mathcal{E}_f &= k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c \\ \therefore \mathcal{E}_f &= \frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}} \end{aligned} \quad (4.23)$$

Now we choose any of the right-two terms in the three way equality and plug the expression for \mathcal{E}_f ,

$$\begin{aligned} &k_c(\mathcal{E}_f - \mathcal{E}_c) \\ &= k_c \left(\frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}} - \mathcal{E}_c \right) \\ &= k_c \left(\frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c - k_c\mathcal{E}_c - k_{cn}\mathcal{E}_c}{k_c + k_{cn}} \right) \\ \therefore D_f \frac{\mathcal{E}_{cn} - \mathcal{E}_c}{\|\mathbf{x}_{cn} - \mathbf{x}_c\|} &= \frac{k_c k_{cn}}{k_c + k_{cn}} (\mathcal{E}_{cn} - \mathcal{E}_c) \\ \therefore D_f &= \frac{k_c k_{cn}}{k_c + k_{cn}} \|\mathbf{x}_{cn} - \mathbf{x}_c\| \end{aligned} \quad (4.24)$$

From the earlier expression for $(\mathcal{F}_0)_f$, we can write

$$(\mathcal{F}_0)_f = D_f (\mathcal{E}_{cn} - \mathcal{E}_c) \frac{\mathbf{x}_{cn} - \mathbf{x}_c}{\|\mathbf{x}_{cn} - \mathbf{x}_c\|^2} \quad (4.25)$$

for which we can define

$$\mathbf{k}_f = D_f \frac{\mathbf{x}_{cn} - \mathbf{x}_c}{\|\mathbf{x}_{cn} - \mathbf{x}_c\|^2} \quad (4.26)$$

such that we finally arrive at

$$(\mathcal{F}_0)_f = \mathbf{k}_f (\mathcal{E}_{cn} - \mathcal{E}_c). \quad (4.27)$$

4.3.2 The kinetic energy term

For the kinetic energy source terms, we similarly have

$$\left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^n \mapsto \frac{1}{V_c} \sum_f \frac{1}{3} \mathbf{A}_f \cdot (\mathcal{E}_f^n \mathbf{u}_f^n) \quad (4.28)$$

where we use the reconstructed values as in the Hydrodynamic and radiation-energy advection portion.

5 Temporal scheme A - Implicit Euler Predictor, Crank-Nicolson Corrector

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \quad (5.1a)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}_0 + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = S_{re}. \quad (5.1b)$$

5.1 Predictor phase

$$\tau = \frac{1}{\frac{1}{2}\Delta t}$$

$$\tau(\mathbf{U}^{n*} - \mathbf{U}^n) + \nabla \cdot \mathcal{F}^H(\mathbf{U}^n) = \mathbf{0} \quad (5.2a)$$

$$\tau \left(\begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{1}{2}} - \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n*} \right) = \begin{bmatrix} 0 \\ -\frac{1}{3} \nabla \mathcal{E} \end{bmatrix}^n \quad (5.2b)$$

$$\tau(\mathcal{E}^{n*} - \mathcal{E}^n) + \left(\frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right)^n = 0 \quad (5.2c)$$

$$\tau(E^{n+\frac{1}{2}} - E^{n*}) = -\theta_1 S_{ea}^{n+\frac{1}{2}} - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^n \quad (5.2d)$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^{n*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+\frac{1}{2}} + \theta_2 \nabla \cdot \mathcal{F}_0^n = \theta_1 S_{ea}^{n+\frac{1}{2}} + \theta_2 S_{ea}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^n \quad (5.2e)$$

For S_{ea} and \mathcal{F}_0 both at $n + \frac{1}{2}$:

$$\sigma^{n+\frac{1}{2}} = \rho^{n+\frac{1}{2}} \kappa(T^n) \quad (5.2f)$$

$$T^{4,n+\frac{1}{2}} = T^{4,n*} + \frac{4T^{3,n*}}{C_v} (e^{n+\frac{1}{2}} - e^{n*}) \quad (5.2g)$$

5.2 Corrector phase

$$\tau = \frac{1}{\Delta t}$$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^n) + \nabla \cdot \mathcal{F}^H(\mathbf{U}^{n+\frac{1}{2}}) = \mathbf{0} \quad (5.3a)$$

$$\tau \left(\begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+1} - \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{1}{2}*} \right) = \begin{bmatrix} 0 \\ -\frac{1}{3} \nabla \mathcal{E} \end{bmatrix}^{n+\frac{1}{2}} \quad (5.3b)$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}*} - \mathcal{E}^n) + \left(\frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right)^{n+\frac{1}{2}} = 0 \quad (5.3c)$$

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 S_{ea}^{n+1} - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (5.3d)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = \theta_1 S_{ea}^{n+1} + \theta_2 S_{re}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (5.3e)$$

For S_{ea} and \mathcal{F}_0 both at $n + 1$:

$$\sigma^{n+1} = \rho^{n+1} \kappa(T^{n+\frac{1}{2}}) \quad (5.3f)$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v} (e^{n+1} - e^{n+\frac{1}{2}*}) \quad (5.3g)$$

5.3 General energy equations, Predictor and Corrector phase, with θ factors

Time integration scheme A uses **implicit Euler** for the predictor phase and **Crank-Nicolson** in the corrector phase. Both these schemes can be represented with a general θ -scheme where we define:

$$\begin{aligned}\theta_1 &\in [0, 1] \\ \theta_2 &= 1 - \theta_1.\end{aligned}\tag{5.4}$$

For implicit Euler, $\theta_1 = 1$, $\theta_2 = 0$, and for Crank-Nicolson, $\theta_1 = \theta_2 = \frac{1}{2}$. With these factors defined we can repeat the energy equations and apply a series of manipulations. First we attempt to segregate known terms from all unknown terms. Thereafter we eliminate the internal energy, e , from the two sets of equations to get a single formulation for the radiation energy, \mathcal{E} . The latter formulation forms a diffusion system that needs to be assembled and solved for \mathcal{E} .

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 \sigma_a^{n+1} c \left(aT^{4,n+1} - \mathcal{E}^{n+1} \right) - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{5.5a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = \theta_1 \sigma_a^{n+1} c \left(aT^{4,n+1} - \mathcal{E}^{n+1} \right) + \theta_2 S_{re}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{5.5b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v} (e^{n+1} - e^{n+\frac{1}{2}*})\tag{5.5c}$$

Define:

$$\begin{aligned}k_1 &= \theta_1 \sigma_a^{n+1} c \\ k_2 &= \frac{4T^{3,n+\frac{1}{2}*}}{C_v}\end{aligned}\tag{5.6}$$

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 \left(aT^{4,n+1} - \mathcal{E}^{n+1} \right) - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{5.7a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = k_1 \left(aT^{4,n+1} - \mathcal{E}^{n+1} \right) + \theta_2 S_{re}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{5.7b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2 (e^{n+1} - e^{n+\frac{1}{2}*})\tag{5.7c}$$

ungroup right-hand side elements by multiplying out terms within parentheses,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a T^{4,n+1} + k_1 \mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{5.8a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = k_1 a T^{4,n+1} - k_1 \mathcal{E}^{n+1} + \theta_2 S_{re}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{5.8b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2 (e^{n+1} - e^{n+\frac{1}{2}*})\tag{5.8c}$$

now plug in the temperature equation into both the energy equations,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a (T^{4,n+\frac{1}{2}*} + k_2 (e^{n+1} - e^{n+\frac{1}{2}*})) + k_1 \mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}}\tag{5.9a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = k_1 a (T^{4,n+\frac{1}{2}*} + k_2 (e^{n+1} - e^{n+\frac{1}{2}*})) - k_1 \mathcal{E}^{n+1} + \theta_2 S_{re}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (5.9b)$$

ungroup elements on the both the right-hand sides,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a T^{4,n+\frac{1}{2}*} - k_1 a k_2 e^{n+1} + k_1 a k_2 e^{n+\frac{1}{2}*} + k_1 \mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (5.10a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = k_1 a T^{4,n+\frac{1}{2}*} + k_1 a k_2 e^{n+1} - k_1 a k_2 e^{n+\frac{1}{2}*} - k_1 \mathcal{E}^{n+1} + \theta_2 S_{re}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (5.10b)$$

Define:

$$\begin{aligned} k_3 &= -k_1 a T^{4,n+\frac{1}{2}*} + k_1 a k_2 e^{n+\frac{1}{2}*} - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \\ k_4 &= -k_1 a k_2 \end{aligned} \quad (5.11)$$

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \quad (5.12a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \quad (5.12b)$$

Note:

$$E^{n+1} = \left(\frac{1}{2} \rho ||\mathbf{u}||^2 \right)^{n+1} + \rho^{n+1} e^{n+1} \quad (5.13)$$

which gives,

$$\tau \left(\left(\frac{1}{2} \rho ||\mathbf{u}||^2 \right)^{n+1} + \rho^{n+1} e^{n+1} - E^{n+\frac{1}{2}*} \right) = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \quad (5.14a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \nabla \cdot (\theta_1 \mathcal{F}_0^{n+1} + \theta_2 \mathcal{F}_0^n) = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \quad (5.14b)$$

ungroup the material energy in the first equation,

$$\tau \left(\frac{1}{2} \rho ||\mathbf{u}||^2 \right)^{n+1} + \tau \rho^{n+1} e^{n+1} - \tau E^{n+\frac{1}{2}*} = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \quad (5.15a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \quad (5.15b)$$

and isolate the internal energy in the first equation,

$$(\tau \rho^{n+1} - k_4) e^{n+1} = k_1 \mathcal{E}^{n+1} + k_3 - \tau \left(\frac{1}{2} \rho ||\mathbf{u}||^2 \right)^{n+1} + \tau E^{n+\frac{1}{2}*} \quad (5.16a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \quad (5.16b)$$

Define:

$$\begin{aligned} k_5 &= \frac{k_1}{\tau \rho^{n+1} - k_4} \\ k_6 &= \frac{k_3 - \tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau E^{n+\frac{1}{2}*}}{\tau \rho^{n+1} - k_4} \end{aligned} \quad (5.17)$$

and plug these constants into the first equation above,

$$e^{n+1} = k_5 \mathcal{E}^{n+1} + k_6 \quad (5.18a)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = -k_1 \mathcal{E}^{n+1} - k_3 - k_4 e^{n+1} \quad (5.18b)$$

now plug the first equation into the second,

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^n = -k_1 \mathcal{E}^{n+1} - k_3 - k_4 k_5 \mathcal{E}^{n+1} - k_4 k_6 \quad (5.19a)$$

now collect all the \mathcal{E}^{n+1} terms on the left-hand side,

$$(\tau + k_1 + k_4 k_5) \mathcal{E}^{n+1} + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} = -k_3 - k_4 k_6 + \tau \mathcal{E}^{n+\frac{1}{2}*} - \theta_2 \nabla \cdot \mathcal{F}_0^n \quad (5.20a)$$

Recall:

$$\nabla \cdot \mathcal{F}_0 \mapsto \frac{1}{V_c} \sum_f \mathbf{A}_f \cdot (\mathcal{F}_0)_f \quad (5.21)$$

and

$$(\mathcal{F}_0)_f = \mathbf{k}_f (\mathcal{E}_{cn} - \mathcal{E}_c) \quad (5.22)$$

which gives the system,

$$(\tau + k_1 + k_4 k_5) \mathcal{E}^{n+1} + \frac{\theta_1}{V_c} \sum_f \mathbf{A}_f \cdot \mathbf{k}_f^{n+1} (\mathcal{E}_{cn}^{n+1} - \mathcal{E}_c^{n+1}) = -k_3 - k_4 k_6 + \tau \mathcal{E}^{n+\frac{1}{2}*} - \frac{\theta_2}{V_c} \sum_f \mathbf{A}_f \cdot \mathbf{k}_f^n (\mathcal{E}_{cn}^n - \mathcal{E}_c^n) \quad (5.23a)$$

This system is SPD and in one dimension forms a tridiagonal system.

5.4 Using the energy related algebra for both the predictor and the corrector

To perform the energy related algebra for the corrector step we need the following inputs:

κ_a^n	For σ_a^n in S_{ea}^n
κ_t^n	For σ_t^n in $\nabla \cdot \mathcal{F}_0^n$
$\kappa_a^{n+\frac{1}{2}}$	For σ_a^{n+1} in S_{ea}^{n+1}
$\kappa_t^{n+\frac{1}{2}}$	For σ_t^{n+1} in $\nabla \cdot \mathcal{F}_0^{n+1}$
C_v	For the linearization of $T^{4,n+1}$
τ	For the time constant
θ_1, θ_2	For the time scheme
\mathbf{U}^n	For T, ρ in S_{ea}^n
$\mathbf{U}^{n+\frac{1}{2}}$	For \mathbf{u} in $\left(\frac{1}{3}\nabla\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathbf{U}^{n+\frac{1}{2}*}$	For $E^{n+\frac{1}{2}*}$ and $e^{n+\frac{1}{2}*}$
$\mathbf{U}_{0,1}^{n+1} = \begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}_{n+1}$	For the kinetic energy in E^{n+1} , and $\rho^{n+1} \rightarrow \sigma_a^{n+1}, \sigma_t^{n+1}$
$\nabla\mathbf{U}^{n+\frac{1}{2}}$	For the reconstructions in $\left(\frac{1}{3}\nabla\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}}$
\mathcal{E}^n	For S_{ea}^n
$\mathcal{E}^{n+\frac{1}{2}}$	For \mathcal{E} in $\left(\frac{1}{3}\nabla\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathcal{E}^{n+\frac{1}{2}*}$	For itself
$\nabla\mathcal{E}^{n+\frac{1}{2}}$	For the reconstructions in $\left(\frac{1}{3}\nabla\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}}$

To following remapping(s) then applies to the predictor:

$\kappa_a^n \rightarrow \kappa_a^n$	$\mathcal{E}^n \rightarrow \mathcal{E}^n$
$\kappa_t^n \rightarrow \kappa_t^n$	$\mathcal{E}^n \rightarrow \mathcal{E}^{n+\frac{1}{2}}$
$\kappa_a^n \rightarrow \kappa_a^{n+\frac{1}{2}}$	$\mathcal{E}^{n*} \rightarrow \mathcal{E}^{n+\frac{1}{2}*}$
$\kappa_t^n \rightarrow \kappa_t^{n+\frac{1}{2}}$	$\nabla\mathcal{E}^n \rightarrow \nabla\mathcal{E}^{n+\frac{1}{2}}$
$\mathbf{U}^n \rightarrow \mathbf{U}^n$	
$\mathbf{U}^n \rightarrow \mathbf{U}^{n+\frac{1}{2}}$	
$\mathbf{U}^{n*} \rightarrow \mathbf{U}^{n+\frac{1}{2}*}$	
$\mathbf{U}^{n+\frac{1}{2}} \rightarrow \mathbf{U}^{n+1}$	
$\nabla\mathbf{U}^n \rightarrow \nabla\mathbf{U}^{n+\frac{1}{2}}$	

6 Mixed finite element

We now derive a general mixed finite element approach for

$$\nabla \cdot \mathcal{F}_0(\mathbf{x}) = 1, \quad \mathbf{x} \in \mathcal{D} \quad (6.1a)$$

$$\mathcal{F}_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\mathcal{D} \quad (6.1b)$$

where

$$\mathcal{F}_0(\mathbf{x}) = D(\mathbf{x}) \nabla \mathcal{E}(\mathbf{x}). \quad (6.2)$$

6.1 Auxiliary notation and variables for \mathcal{F}_0

First we discretize \mathcal{F}_0 on N_n number of nodes per cell c , using continuous basis functions $b_j(\mathbf{x})$ such that

$$\mathcal{F}_0(\mathbf{x}) \approx \sum_{j=1}^{N_n} (\mathcal{F}_0)_j b_j(\mathbf{x}), \quad (6.3)$$

whilst keeping the cell-centered representation for \mathcal{E} . Next we discretize eq. (6.2) by applying a weight function $b_i(\mathbf{x})$ and integrating over the volume of the cell c ,

$$\begin{aligned} \int_{V_c} b_i \mathcal{F}_0 dV &= \int_{V_c} b_i D \nabla \mathcal{E} dV \\ \sum_j \left[\int_{V_c} b_i b_j dV \right] (\mathcal{F}_0)_j &= \int_{V_c} b_i D \nabla \mathcal{E} dV. \end{aligned} \quad (6.4)$$

The integral coefficients on the left-hand side are generally known as the ij coefficients in the standard finite element mass-matrix, which we shall use in a moment to define a general scheme. The right hand side of the equation requires some treatment. We introduce the values $\mathcal{E}_{c,j}$ at the cell surface to remedy the discontinuities in the cell-centered \mathcal{E} and to define auxiliary unknowns for developing the \mathcal{F}_0 in finite element form. With these new variables declared, we next apply integration by parts to the right-hand side,

$$\int_{V_c} b_i D \nabla \mathcal{E} dV = \int_{V_c} D \nabla (b_i \mathcal{E}) dV - \int_{V_c} D \mathcal{E} \nabla b_i dV. \quad (6.5)$$

Next we apply Gauss's divergence theorem on the first term on the right-hand side,

$$\int_{V_c} b_i D \nabla \mathcal{E} dV = \sum_f \int_{S_f} D \mathbf{n}_f b_i \mathcal{E} dA - \int_{V_c} D \mathcal{E} \nabla b_i dV, \quad (6.6)$$

after which we insert $\mathcal{E}_{c,j}$ in the first term on the right, since they are designated unknowns, on the surface of the cell, and \mathcal{E}_c from the right most term, since \mathcal{E}_c is cell-constant within the cell-domain, the D coefficient is also dependent only on \mathcal{E}_c and therefore constant within cell c , hence denoted as D_c ,

$$\int_{V_c} b_i D \nabla \mathcal{E} dV = \sum_f \sum_j \left[D_c \mathbf{n}_f \int_{S_f} b_i b_j dA \right] \mathcal{E}_{c,j} - \left[D_c \int_{V_c} \nabla b_i dV \right] \mathcal{E}_c. \quad (6.7)$$

Putting the developed right- and left-hand sides back together we then get,

$$\sum_j \left[\int_{V_c} b_i b_j dV \right] (\mathcal{F}_0)_j = \sum_f \sum_j \left[D_c \mathbf{n}_f \int_{S_f} b_i b_j dA \right] \mathcal{E}_{c,j} - \left[D_c \int_{V_c} \nabla b_i dV \right] \mathcal{E}_c. \quad (6.8)$$

This equation can be written in more succinct form as

$$\bar{M}_c \bar{\mathbf{F}}_c = \bar{C}_c \mathcal{E}_c \quad (6.9)$$

where the structure still needs to be defined (which follows). \bar{M}_c is a square block-matrix with block-dimension $N_n \times N_n$, $\bar{\mathbf{F}}_c$ is a block-vector with block-dimension $N_n \times 1$, \bar{C}_c is a rectangular block-matrix with block-dimension $N_n \times (N_f + 1)$. The vector \mathcal{E}_c is simply the cell-centered and surface unknowns for cell c , i.e., $\mathcal{E}_c = [\mathcal{E}_c, \mathcal{E}_{n=0}, \dots, \mathcal{E}_{n=N_n-1}]^T$.

The dimension of the inner blocks of \bar{M}_c , $\bar{\mathbf{F}}_c$ and \bar{C}_c , depend on the number of dimensions, N_d , in the problem. For further reference we shall denote dimensions with d but it generally refers to $d \in [0, 1, 2] \mapsto [x, y, z]$ and vice-versa.

The block entries of \bar{M} are small diagonal matrices,

$$(\bar{M})_{ij} = \text{diag}(M_{ij}, \dots, M_{ij})^{N_d \times N_d} \quad (6.10)$$

where M_{ij} are the elements of the standard finite element mass-matrix for cell c , i.e.,

$$M_{ij} = \int_V b_i b_j dV. \quad (6.11)$$

The block entries of $\bar{\mathbf{F}}$ are

$$(\bar{\mathbf{F}})_i = \begin{bmatrix} (\mathcal{F}_0)_{i,x} \\ (\mathcal{F}_0)_{i,y} \\ (\mathcal{F}_0)_{i,z} \end{bmatrix}^{N_d \times 1} \quad (6.12)$$

obviously only up to y for 2D and only up to x for 1D. The entries of \bar{C} are formed as follows. First the structure of \bar{C} is such that

$$\text{block-row } i \text{ of } \bar{C} = \text{columns } (\mathbf{C}_i^c \quad \mathbf{C}_{i,j=0}^s \quad \dots \quad \mathbf{C}_{i,j=N_n-1}^s)^{N_d \times (N_n+1)}. \quad (6.13)$$

We then define the vectors

$$\begin{aligned} \mathbf{G}_i &= \int_V \nabla b_i dV \\ M_{ij}^f &= \int_{S_f} b_i b_j dA \end{aligned} \quad (6.14)$$

Then,

$$\mathbf{C}_i^c = \begin{bmatrix} D_c(\mathbf{G}_i)_x \\ D_c(\mathbf{G}_i)_y \\ D_c(\mathbf{G}_i)_z \end{bmatrix}^{N_d \times 1} \quad (6.15)$$

and

$$\mathbf{C}_{ij}^s = \begin{bmatrix} \sum_f D_c n_{f,x} M_{ij}^f \\ \sum_f D_c n_{f,y} M_{ij}^f \\ \sum_f D_c n_{f,z} M_{ij}^f \end{bmatrix}^{N_d \times 1} \quad (6.16)$$

With these definitions in-hand we can see that the true dimensions of \bar{M} is $N_d N_n \times N_d N_n$, that of $\bar{\mathbf{F}}$ is $N_d N_n \times 1$, and the true dimensions of \bar{C} is $N_d N_n \times (N_n + 1)$.

Finally, we repeat here that the vector \mathcal{E} is

$$\mathcal{E}_c = \begin{bmatrix} \mathcal{E}_c \\ \mathcal{E}_{n=0} \\ \vdots \\ \mathcal{E}_{n=N_n-1} \end{bmatrix}^{(N_n+1) \times 1}. \quad (6.17)$$

To get an expression for all of the nodal \mathcal{F}_0 's we take the system form of the equation and we invert \bar{M} to get, in coefficient form, expressions for nodal \mathcal{F}_0 's,

$$\mathbf{F}_c = \begin{bmatrix} (\mathcal{F}_0)_0 \\ \vdots \\ (\mathcal{F}_0)_{N_n-1} \end{bmatrix} = \bar{M}_c^{-1} \bar{C}_c \mathcal{E} = C_c^* \mathcal{E}_c, \quad (6.18)$$

where $C_c^* = \bar{M}_c^{-1} \bar{C}_c$.

With this expression-form of the individual nodal \mathcal{F}_0 's we need to modify the primary equation, eq. (6.1). Additionally, since we introduced additional variables in the form of the face-based \mathcal{E}_f 's, we need to define additional equations to close the system. For the primary equations we will simply plug in the expressions for \mathcal{F}_0 , which is detailed in the next subsection. For additional equations we will use the interface between cells to enforce continuity of \mathcal{F}_0 at the face, for each cell of the face.

6.2 Using the auxiliary notation in the primary equation

Using this coefficient-form in the primary equations is done by first integrating eq. (6.1) over the volume of cell c , assuming the coefficient matrix $\bar{M}^{-1}\bar{C}$ has been developed for cell c , after which we apply Gauss's divergence theorem,

$$\begin{aligned} \int_{V_c} \nabla \cdot \mathcal{F}_0 dV &= V_c \\ \int_{S_c} \mathbf{n} \cdot \mathcal{F}_0 dA &= V_c \\ \sum_f \left[\mathbf{n}_f \cdot \int_{S_f} \mathcal{F}_0 dA \right] &= V_c. \end{aligned} \tag{6.19}$$

We now expand \mathcal{F}_0 ,

$$\sum_j \sum_f \left[\mathbf{n}_f \cdot (\mathcal{F}_0)_j \int_{S_f} b_j dA \right] = V_c, \tag{6.20}$$

define

$$S_{i,f} = \int_{S_f} b_i dA \tag{6.21}$$

$$\sum_j \sum_f \left[n_{f,x} S_{j,f} (\mathcal{F}_0)_{j,x} + n_{f,y} S_{j,f} (\mathcal{F}_0)_{j,y} + n_{f,z} S_{j,f} (\mathcal{F}_0)_{j,z} \right] = V_c, \tag{6.22}$$

or

$$\sum_j \sum_f \sum_d \left[n_{f,d} S_{j,f} (\mathcal{F}_0)_{j,d} \right] = V_c, \tag{6.23}$$

where d denotes dimension such that $d \in [0, 1, 2] \mapsto [x, y, z]$, the indices (j, d) of $(\mathcal{F}_0)_{j,d}$ maps to a row in C^* , i.e.,

$$(j, d) \mapsto k : k = N_d j + d, \tag{6.24}$$

from which we get

$$\sum_j \sum_f \sum_d \left[n_{f,d} S_{j,f} C_{(j,d) \mapsto \text{row } k}^* \cdot \mathcal{E}_c \right] = V_c, \tag{6.25}$$

If the indices of \mathcal{E}_c are then mapped to global system indexes for the corresponding \mathcal{E}_c and collection of \mathcal{E}_f 's then the system can be constructed.

6.3 Auxiliary equations

For each face we now require that $(\mathcal{F}_0)_c = (\mathcal{F}_0)_{cn}$ at each node, where c and cn are the cell indices on either side of the face. Before we can do this we need to establish the convention of face ownership. For this we assign a cell ownership of a face based on the indices of the cells sharing the face. Suppose a face is shared by cell c and cn ; cell c owns that face if $c < cn$. Now for each node j part of the owned face we require that

$$\begin{aligned} (\mathcal{F}_0)_{c,j} &= (\mathcal{F}_0)_{cn,j^*} \\ (\mathcal{F}_0)_{c,j} - (\mathcal{F}_0)_{cn,j^*} &= 0 \end{aligned} \tag{6.26}$$

where $(\mathcal{F}_0)_{c,j}$ is the flux at the j -th node of cell c and node j is on the shared face. $(\mathcal{F}_0)_{cn,j^*}$ is the flux at the j^* -th node of cell cn , and where j^* maps to the same system index as j .

7 Temporal scheme B - Phase A: Implicit Euler Predictor, Crank-Nicolson Corrector, Phase B: Implicit Euler Predictor, Weighted θ 's Corrector

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \quad (7.1a)$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F}_0 + \frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) = S_{re}. \quad (7.1b)$$

7.1 Predictor phase A

$$\tau = \frac{1}{\frac{1}{4}\Delta t}$$

$$\tau(\mathbf{U}^{n*} - \mathbf{U}^n) + \nabla \cdot \mathcal{F}^H(\mathbf{U}^n) = \mathbf{0} \quad (7.2a)$$

$$\tau \left(\begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{1}{4}} - \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n*} \right) = \begin{bmatrix} 0 \\ -\frac{1}{3} \nabla \mathcal{E} \end{bmatrix}^n \quad (7.2b)$$

$$\tau(\mathcal{E}^{n*} - \mathcal{E}^n) + \left(\frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right)^n = 0 \quad (7.2c)$$

$$\tau(E^{n+\frac{1}{4}} - E^{n*}) = -\theta_1 S_{ea}^{n+\frac{1}{4}} - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^n \quad (7.2d)$$

$$\tau(\mathcal{E}^{n+\frac{1}{4}} - \mathcal{E}^{n*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+\frac{1}{4}} + \theta_2 \nabla \cdot \mathcal{F}_0^n = \theta_1 S_{ea}^{n+\frac{1}{4}} + \theta_2 S_{ea}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^n \quad (7.2e)$$

For S_{ea} and \mathcal{F}_0 both at $n + \frac{1}{4}$:

$$\sigma^{n+\frac{1}{4}} = \rho^{n+\frac{1}{4}} \kappa(T^n) \quad (7.2f)$$

$$T^{4,n+\frac{1}{4}} = T^{4,n*} + \frac{4T^{3,n*}}{C_v} (e^{n+\frac{1}{4}} - e^{n*}) \quad (7.2g)$$

7.2 Corrector phase A

$$\tau = \frac{1}{\frac{1}{2}\Delta t}$$

$$\tau(\mathbf{U}^{n+\frac{1}{4}*} - \mathbf{U}^n) + \nabla \cdot \mathcal{F}^H(\mathbf{U}^{n+\frac{1}{4}}) = \mathbf{0} \quad (7.3a)$$

$$\tau \left(\begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{1}{2}} - \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{1}{4}*} \right) = \begin{bmatrix} 0 \\ -\frac{1}{3} \nabla \mathcal{E} \end{bmatrix}^{n+\frac{1}{4}} \quad (7.3b)$$

$$\tau(\mathcal{E}^{n+\frac{1}{4}*} - \mathcal{E}^n) + \left(\frac{4}{3} \nabla \cdot (\mathcal{E} \mathbf{u}) \right)^{n+\frac{1}{4}} = 0 \quad (7.3c)$$

$$\tau(E^{n+\frac{1}{2}} - E^{n+\frac{1}{4}*}) = -\theta_1 S_{ea}^{n+\frac{1}{2}} - \theta_2 S_{ea}^n - \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{4}} \quad (7.3d)$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^{n+\frac{1}{4}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+\frac{1}{2}} + \theta_2 \nabla \cdot \mathcal{F}_0^n = \theta_1 S_{ea}^{n+\frac{1}{2}} + \theta_2 S_{re}^n + \left(\frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{4}} \quad (7.3e)$$

For S_{ea} and \mathcal{F}_0 both at $n + \frac{1}{2}$:

$$\sigma^{n+\frac{1}{2}} = \rho^{n+\frac{1}{2}} \kappa(T^{n+\frac{1}{4}}) \quad (7.3f)$$

$$T^{4,n+\frac{1}{2}} = T^{4,n+\frac{1}{4}*} + \frac{4T^{3,n+\frac{1}{4}*}}{C_v} (e^{n+\frac{1}{2}} - e^{n+\frac{1}{4}*}) \quad (7.3g)$$

7.3 Predictor phase B

$$\tau = \frac{1}{\frac{1}{4}\Delta t}$$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^{n+\frac{1}{2}}) + \nabla \cdot \mathcal{F}^H(\mathbf{U}^{n+\frac{1}{2}}) = \mathbf{0} \quad (7.4a)$$

$$\tau \left(\begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{3}{4}} - \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{1}{2}*} \right) = \begin{bmatrix} 0 \\ -\frac{1}{3}\nabla \mathcal{E} \end{bmatrix}^{n+\frac{1}{2}} \quad (7.4b)$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}*} - \mathcal{E}^{n+\frac{1}{2}}) + \left(\frac{4}{3}\nabla \cdot (\mathcal{E}\mathbf{u}) \right)^{n+\frac{1}{2}} = 0 \quad (7.4c)$$

$$\tau(E^{n+\frac{3}{4}} - E^{n+\frac{1}{2}*}) = -\theta_1 S_{ea}^{n+\frac{3}{4}} - \theta_2 S_{ea}^{n+\frac{1}{2}} - \left(\frac{1}{3}\nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (7.4d)$$

$$\tau(\mathcal{E}^{n+\frac{3}{4}} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+\frac{3}{4}} + \theta_2 \nabla \cdot \mathcal{F}_0^{n+\frac{1}{2}} = \theta_1 S_{ea}^{n+\frac{3}{4}} + \theta_2 S_{ea}^{n+\frac{1}{2}} + \left(\frac{1}{3}\nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{1}{2}} \quad (7.4e)$$

For S_{ea} and \mathcal{F}_0 both at $n + \frac{3}{4}$:

$$\sigma^{n+\frac{3}{4}} = \rho^{n+\frac{3}{4}} \kappa(T^{n+\frac{1}{2}}) \quad (7.4f)$$

$$T^{4,n+\frac{3}{4}} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+\frac{3}{4}} - e^{n+\frac{1}{2}*}) \quad (7.4g)$$

7.4 Corrector phase B

$$\tau = \frac{1}{\frac{1}{2}\Delta t}$$

$$\tau(\mathbf{U}^{n+\frac{3}{4}*} - \mathbf{U}^{n+\frac{1}{2}}) + \nabla \cdot \mathcal{F}^H(\mathbf{U}^{n+\frac{3}{4}}) = \mathbf{0} \quad (7.5a)$$

$$\tau \left(\begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+1} - \begin{bmatrix} \rho \\ \rho \mathbf{u} \end{bmatrix}^{n+\frac{3}{4}*} \right) = \begin{bmatrix} 0 \\ -\frac{1}{3}\nabla \mathcal{E} \end{bmatrix}^{n+\frac{3}{4}} \quad (7.5b)$$

$$\tau(\mathcal{E}^{n+\frac{3}{4}*} - \mathcal{E}^{n+\frac{1}{2}}) + \left(\frac{4}{3}\nabla \cdot (\mathcal{E}\mathbf{u}) \right)^{n+\frac{3}{4}} = 0 \quad (7.5c)$$

$$\tau(E^{n+1} - E^{n+\frac{3}{4}*}) = -\theta_1 S_{ea}^{n+1} - \theta_2 S_{ea}^{n+\frac{1}{2}} - \theta_3 S_{ea}^n - \left(\frac{1}{3}\nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{3}{4}} \quad (7.5d)$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{3}{4}*}) + \theta_1 \nabla \cdot \mathcal{F}_0^{n+1} + \theta_2 \nabla \cdot \mathcal{F}_0^{n+\frac{1}{2}} + \theta_3 \nabla \cdot \mathcal{F}_0^n = \theta_1 S_{ea}^{n+1} + \theta_2 S_{re}^{n+\frac{1}{2}} + \theta_3 S_{ea}^n + \left(\frac{1}{3}\nabla \mathcal{E} \cdot \mathbf{u} \right)^{n+\frac{3}{4}} \quad (7.5e)$$

For S_{ea} and \mathcal{F}_0 both at $n + 1$:

$$\sigma^{n+1} = \rho^{n+1} \kappa(T^{n+\frac{3}{4}}) \quad (7.5f)$$

$$T^{4,n+1} = T^{4,n+\frac{3}{4}*} + \frac{4T^{3,n+\frac{3}{4}*}}{C_v}(e^{n+1} - e^{n+\frac{3}{4}*}) \quad (7.5g)$$

A Angular integration identities

Identity A-1

$$\int_{4\pi} d\Omega = 4\pi.$$

Identity A-2

$$\int_{4\pi} \Omega d\Omega = 0.$$

Identity A-3 Given the known three component vector, \mathbf{v} ,

$$\int_{4\pi} \Omega \cdot \mathbf{v} d\Omega = 0.$$

Identity A-4 Given the known three component vector, \mathbf{v} ,

$$\int_{4\pi} \Omega \cdot \nabla (\Omega \cdot \mathbf{v}) d\Omega = \frac{4\pi}{3} \nabla \cdot \mathbf{v}.$$

Identity A-5 Given the scalar, a ,

$$\int_{4\pi} \Omega \left(\Omega \cdot \nabla a \right) d\Omega = \frac{4\pi}{3} \nabla a.$$

Identity A-6 Given the known three component vector, \mathbf{v} ,

$$\int_{4\pi} \Omega \left(\Omega \cdot \mathbf{v} \right) d\Omega = \frac{4\pi}{3} \mathbf{v}.$$

Identity A-7 Given the known three component vector, \mathbf{v} ,

$$\int_{4\pi} \Omega \left(\Omega \cdot \nabla (\Omega \cdot \mathbf{v}) \right) d\Omega = 0.$$

B Boundary and initial conditions for radiation hydrodynamic problems

In a one dimensional simulation we can simulate steady-state shocks by setting the appropriate pre- and post-shock conditions. Pre-shock conditions will be denoted with a subscript L whereas post-shock conditions will be denoted with a subscript R .

B.1 Hydrodynamics only

With no radiation energy present we wish to have $\mathcal{F}_L^H = \mathcal{F}_R^H$, therefore

$$\begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}_R. \quad (\text{B.1})$$

Here we have three equations but 4 unknowns, i.e., ρ , u , p and e . Fortunately, we can express both e and p in terms of temperature since

$$p = (\gamma - 1)\rho e$$

and

$$e = C_v T.$$

Therefore,

$$p = (\gamma - 1)\rho C_v T$$

and

$$\begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \rho C_v T u + p u \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \rho C_v T u + p u \end{bmatrix}_R. \quad (\text{B.2})$$

When the left state is known then we can frame these equations as seeking the non-linear solution of

$$\mathbf{F} \begin{pmatrix} \rho_R \\ T_R \\ u_R \end{pmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \gamma \rho C_v T u \end{bmatrix}_R - \mathcal{F}_L^H = \mathbf{0} \quad (\text{B.3})$$

or simply

$$\mathbf{F}(\mathbf{x}) = \mathcal{F}^H(\mathbf{x}_R) - \mathcal{F}_L^H = \mathbf{0}. \quad (\text{B.4})$$

to which Newton-iteration can be applied in the form

$$\mathbf{x}_R^{\ell+1} = \mathbf{x}_R^\ell - J^{-1}(\mathbf{x}_R^\ell) \mathbf{F}(\mathbf{x}_R^\ell)$$

where the Jacobian matrix, J , is given by

$$J = \begin{bmatrix} u & 0 & \rho \\ u^2 + (\gamma - 1)C_v T & (\gamma - 1)\rho C_v & 2\rho u \\ \frac{1}{2}u^3 + \gamma C_v T u & \gamma \rho C_v u & \frac{3}{2}\rho u^2 + \gamma \rho C_v T \end{bmatrix} \quad (\text{B.5})$$

Note: The initial guess, \mathbf{x}^0 cannot be the same as \mathbf{x}_L since the iteration will terminate immediately. Generally the values need to be perturbed sufficiently such that $\rho_R > \rho_L$, $T_R > T_L$ and $u_R < a_R$ where a is the sound-speed.

B.2 Hydrodynamics with radiation energy

With radiation energy present we are concerned with the following set of equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (\text{B.6a})$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \nabla p = -\frac{1}{3} \nabla \mathcal{E}, \quad (\text{B.6b})$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{u}] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u} \quad (\text{B.6c})$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(-\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla (\mathcal{E} \mathbf{u}) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} \cdot \mathbf{u}. \quad (\text{B.6d})$$

which for a steady-steady, one dimensional simulation becomes

$$\nabla \cdot (\rho u) = 0 \quad (\text{B.7a})$$

$$\nabla \cdot (\rho u^2) + \nabla p = -\frac{1}{3} \nabla \mathcal{E}, \quad (\text{B.7b})$$

$$\nabla \cdot [(E + p)u] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot u \quad (\text{B.7c})$$

$$\nabla \cdot \left(-\frac{c}{3\sigma_t} \nabla \mathcal{E} \right) + \frac{4}{3} \nabla (\mathcal{E} u) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} u. \quad (\text{B.7d})$$

Additionally, far away from the interface the co-moving frame radiation flux, \mathcal{F}_0 , is zero, therefore

$$\mathcal{F}_0 = -\frac{c}{3\sigma_t} \nabla \mathcal{E} = 0$$

and the equation set becomes

$$\nabla \cdot (\rho u) = 0 \quad (\text{B.8a})$$

$$\nabla \cdot (\rho u^2) + \nabla p = -\frac{1}{3} \nabla \mathcal{E}, \quad (\text{B.8b})$$

$$\nabla \cdot [(E + p)u] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3} \nabla \mathcal{E} \cdot u \quad (\text{B.8c})$$

$$\frac{4}{3} \nabla (\mathcal{E} u) = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \nabla \mathcal{E} u. \quad (\text{B.8d})$$

Now, adding the last equation to the third, we get

$$\nabla \cdot (\rho u) = 0 \quad (\text{B.9a})$$

$$\nabla \cdot (\rho u^2) + \nabla p + \frac{1}{3} \nabla \mathcal{E} = 0, \quad (\text{B.9b})$$

$$\nabla \cdot [(E + p)u] + \frac{4}{3} \nabla (\mathcal{E} u) = 0. \quad (\text{B.9c})$$

We now express internal energy, e , the pressure, p , and the radiation energy, \mathcal{E} , in terms of temperature

$$\nabla \cdot (\rho u) = 0 \quad (\text{B.10a})$$

$$\nabla \cdot (\rho u^2) + \nabla ((\gamma - 1) \rho C_v T) + \frac{1}{3} \nabla aT^4 = 0, \quad (\text{B.10b})$$

$$\nabla \cdot \left[\left(\frac{1}{2} \rho u^2 + \rho C_v T + (\gamma - 1) \rho C_v T \right) u \right] + \frac{4}{3} \nabla (aT^4 u) = 0. \quad (\text{B.10c})$$

Finally we integrate this equation set over the entire domain to get

$$\left[\begin{array}{c} \rho u \\ \rho u^2 + (\gamma - 1) \rho C_v T + \frac{1}{3} aT^4 \\ \frac{1}{2} \rho u^3 + \gamma \rho C_v T u + \frac{4}{3} aT^4 u \end{array} \right]_L = \left[\begin{array}{c} \rho u \\ \rho u^2 + (\gamma - 1) \rho C_v T + \frac{1}{3} aT^4 \\ \frac{1}{2} \rho u^3 + \gamma \rho C_v T u + \frac{4}{3} aT^4 u \end{array} \right]_R \quad (\text{B.11a})$$

Similar to the previous case we can now define

$$\mathbf{F} \begin{pmatrix} \rho_R \\ T_R \\ u_R \end{pmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T + \frac{1}{3}aT^4 \\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u + \frac{4}{3}aT^4 u \end{bmatrix}_R - \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma - 1)\rho C_v T + \frac{1}{3}aT^4 \\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u + \frac{4}{3}aT^4 u \end{bmatrix}_L = \mathbf{0}, \quad (\text{B.12})$$

where the subscript L quantities are all known. Applying Newton-iteration to this equation again is

$$\mathbf{x}_R^{\ell+1} = \mathbf{x}_R^\ell - J^{-1}(\mathbf{x}_R^\ell)\mathbf{F}(\mathbf{x}_R^\ell)$$

where the Jacobian matrix, J is given by

$$J = \begin{bmatrix} u & 0 & \rho \\ u^2 + (\gamma - 1)C_v T & (\gamma - 1)\rho C_v + \frac{4}{3}aT^3 & 2\rho u \\ \frac{1}{2}u^3 + \gamma C_v T u & \gamma\rho C_v u + \frac{16}{3}aT^3 u & \frac{3}{2}\rho u^2 + \gamma\rho C_v T + \frac{4}{3}aT^4 \end{bmatrix} \quad (\text{B.13})$$

Example mach 3 conditions, $C_v = 0.14472799784454$ and $\gamma = \frac{5}{3}$:

```
[0]  rho0  1.00000000e+00 u0    3.80431331e-01 T0    1.00000000e-01 e0    1.44727998e-02 radE0 1.37223549e-06
[0]  rho1  3.00185103e+00 u1    1.26732249e-01 T1    3.66260705e-01 e1    5.30081785e-02 radE1 2.46939153e-04
```

Note: The initial guess, \mathbf{x}^0 cannot be the same as \mathbf{x}_L since the iteration will terminate immediately. Generally the values need to be perturbed sufficiently such that $\rho_R > \rho_L$, $T_R > T_L$, $u_R < a_R$ where a is the sound-speed, and $\mathcal{E}_R > \mathcal{E}_L$.

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C Roderigues's formula

Roderigues' formula for the rotation of a vector \mathbf{v} about a unit vector \mathbf{a} with right-hand rule

$$\mathbf{v}_{rotated} = \cos \theta \mathbf{v} + (\mathbf{a} \cdot \mathbf{v})(1 - \cos \theta) \mathbf{a} + \sin \theta (\mathbf{a} \times \mathbf{v}) \quad (\text{C.1})$$

In matrix form

$$\mathbf{v}_{rotated} = A \mathbf{v} \quad (\text{C.2})$$

where

$$A = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \quad (\text{C.3})$$

and

$$R = I + \sin \theta A + (1 - \cos \theta) A^2 \quad (\text{C.4})$$