Lecture 4

Thermal Radiation Transport with Material Motion

The purpose of this lecture is to

- Introduce the concept of the comoving frame.
- Use the Lorentz tranformation laws to derive tranformations between comoving-frame interaction terms and laboratory-frame interaction terms.
- Derive the laboratory-frame thermal radiation transport equation.

1 The comoving frame

A concept that is invaluable for dealing with material motion is the comoving frame. The comoving frame velocity of a particle at time t and position $\overrightarrow{r}$ is simply the velocity of the particle measured in a local frame that is at rest with respect to the material. Thus a velocity of zero in the comoving frame corresponds to the material velocity in the laboratory frame. We will use the notational convention that all comoving-frame quanities carry a subscript “0”. The comoving-frame intensity, $I_0(t, \overrightarrow{r}, \overrightarrow{\Omega}_0, E_0)$ is the angular intensity at time t and position $\overrightarrow{r}$ in an instantaneous inertial frame moving at the material velocity $\overrightarrow{u}(t, \overrightarrow{r})$. This frame is instantaneous in the sense that an inertial frame must have a constant velocity and the material velocity at a given position is generally changing in time. Furthermore, the material velocity is generally changing at each position. Thus one can think of the comoving frame as a spatial continuum of independent inertial frames that are redefined at each instant of time. It is valid to apply Lorentz transformations between the laboratory frame (a true inertial frame) and the continuum of instantaneous inertial frames that collectively form the comoving frame. Because photon cross sections (indeed all cross sections) are defined with respect to a material at rest, they cannot be directly used to calculate interaction rates in the laboratory frame, but they can be so used in the comoving frame. Thus we can naturally define interaction rates in the comoving frame and use simple physical arguments in conjunction with fundamental Lorentz transformations to obtain corresponding laboratory-frame interaction rates. This procedure will enable us to derive the laboratory-frame thermal radiation transport equation with material motion. It is also possible to derive a thermal radiation transport equation for the comoving-frame angular intensity. However, for a variety of reasons, we believe it is numerically advantageous to work with the laboratory-frame equation rather than the comoving-frame equation. The primary reason is that energy conservation can only be established in the laboratory frame, making it very difficult (but not impossible) to obtain conservative discretizations of the comoving-frame equation. We should note that our opinion is not shared by everyone in the field of computational radiation-hydrodynamics.

2 Fundamental Transformations

We will use the following fundamental transformations.

$$dt_0 = \gamma dt \,, \tag{1}$$

where

$$\gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}} \,, \tag{1a}$$

and

$$\beta = \frac{u}{c} \,. \tag{1b}$$

This is an expression of time dilation, i.e. time slows down in the comoving frame. Under the assumption that the material velocity is directed along $\pm z$, one finds that

$$dx_0 = dx \,, \tag{2a}$$

$$dy_0 = dy \,, \tag{2b}$$

$$dz_0 = dz/\gamma \,. \tag{2c}$$

$$\tag{2d}$$

This is an expression of the Lorentz-Fitzgerald contraction, i.e., the spatial dimension parallel to the direction of velocity shrinks while the other two dimensions remain invariant. Equations (2a) through (2c) imply that

$$dV_0 = dV/\gamma \,. \tag{2e}$$

Equation (2e) is actually valid regardless of the orientation of the material velocity vector. Using Eqs. (1) and (2e), we find that the space-time volume element is Lorentz-invariant:

$$dV_0\, dt_0 = dV\, dt\,. \tag{3a}$$

The comoving-frame and laboratory-frame photon energies are related by

$$E_0 = E\,\gamma \left(1 - \overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right)\,, \tag{4a}$$

and

$$E = E_0\,\gamma \left(1 + \overrightarrow{\Omega}_0 \cdot \overrightarrow{u}/c\right)\,. \tag{4b}$$

Note that the relationship between the photon energies is a function of both direction and energy, but the ratio of the energies is only a function of direction. The comoving-frame and laboratory-frame photon directions are related by

$$\overrightarrow{\Omega}_0 = (E/E_0)\left\{\overrightarrow{\Omega} - \gamma(\overrightarrow{u}/c)\left[1 - \left(\gamma\overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right)/\left(\gamma+1\right)\right]\right\}\,, \tag{5a}$$

and

$$\overrightarrow{\Omega} = (E_0/E)\left\{\overrightarrow{\Omega}_0 + \gamma(\overrightarrow{u}/c)\left[1 + \left(\gamma\overrightarrow{\Omega}_0 \cdot \overrightarrow{u}/c\right)/\left(\gamma+1\right)\right]\right\}\,. \tag{5b}$$

Note that the relationship between photon directions is actually a function of direction only. Equations (4a) through (5b) can be used to derive the following results.

$$dE_0 = (E_0/E)\, dE\,, \tag{6}$$

$$d\xi_0 = (E/E_0)^2 d\xi \,, \tag{7}$$

where $\xi = \cos(\theta)$ is the cosine of the standard polar angle,

$$d\Phi_0 = d\Phi \,, \tag{8}$$

where Φ is the standard azimuthal angle. Combining Eqs. (7) and (8), we find that

$$d\Omega_0 = (E/E_0)^2 \, d\Omega \,. \tag{9}$$

Combining Eqs. (6) and (9), we obtain

$$d\Omega_0 \, dE_0 = (E/E_0) \, d\Omega \, dE \,. \tag{10}$$

3 Transformations for Transport Quantities

We first derive a transformation for the angular intensity. Consider a differential area, dS, fixed in space and time in the laboratory frame and oriented such that it is normal to the material velocity vector. Then the number of photons having directions about $\overrightarrow{\Omega}$ and energies about E that pass through this surface over a differential time dt is

$$N = I\left(\overrightarrow{\Omega}, E\right) \frac{1}{E} \, d\Omega \, dE \, |\overrightarrow{\Omega} \cdot \overrightarrow{n}| \, dS \, dt \,, \tag{11}$$

where

$$\overrightarrow{n} = \overrightarrow{u}/u \,. \tag{12}$$

To an observer in the comoving frame, dS appears to have a velocity equal to $-\overrightarrow{u}$. The number of photons that pass through dS as observed from the comoving frame is

$$N_0 = I_0\left(\overrightarrow{\Omega}_0, E_0\right)\frac{1}{E_0}\,d\Omega_0\,dE_0\,|\,\overrightarrow{\Omega}_0\cdot\overrightarrow{n} + u/c\,|\,dS\,dt_0\,. \tag{13}$$

The justification for Eq. (13) is as follows. The term proportional to $|\overrightarrow{\Omega}_0\cdot\overrightarrow{n}|$ yields the number of photons that would pass through dS if it were stationary, while the term proportional to u/c represents the number of photons within the volume traced out by dS over time dt_0. The total number of photons passing through dS must be the sum of these two terms.

Observers in both frames must observe the same number of photons to pass through dS, so it follows that

$$\begin{aligned} &I(\overrightarrow{\Omega}, E)\frac{1}{E}\,|\,\overrightarrow{\Omega}\cdot\overrightarrow{n}\,|\,d\Omega\,dE\,dS\,dt = \\ &I_0(\overrightarrow{\Omega}_0, E_0)\frac{1}{E_0}\,|\,\overrightarrow{\Omega}_0\cdot\overrightarrow{n} + u/c\,|\,d\Omega_0\,dE_0\,dS\,dt_0\,, \end{aligned} \tag{14}$$

Substituting from Eqs. (1) and (10) into Eq. (14), we obtain

$$\begin{aligned} &I(\overrightarrow{\Omega}, E)\frac{1}{E}\,|\,\overrightarrow{\Omega}\cdot\overrightarrow{n}\,|\,d\Omega\,dE\,dS\,dt = \\ &I_0(\overrightarrow{\Omega}_0, E_0)\frac{1}{E_0}\,|\,\overrightarrow{\Omega}_0\cdot\overrightarrow{n} + u/c\,|\,(E/E_0)\gamma\,d\Omega\,dE\,dS\,dt\,. \end{aligned} \tag{15}$$

Simplifying Eq. (15), we get

$$I(\overrightarrow{\Omega}, E)\frac{1}{E}\,|\,\overrightarrow{\Omega}\cdot\overrightarrow{n}\,| = I_0(\overrightarrow{\Omega}_0, E_0)\frac{1}{E_0}\,(E/E_0)\gamma|\,\overrightarrow{\Omega}_0\cdot\overrightarrow{n} + u/c\,|\,. \tag{16}$$

Substituting from Eqs. (5b) and (12) into Eq. (16), we obtain

$$I(\overrightarrow{\Omega}, E)\frac{1}{E}(E_0/E)\left|\overrightarrow{\Omega}_0 \cdot \overrightarrow{n} + \gamma\frac{u}{c}\left[1 + \frac{\gamma}{\gamma+1}\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}\frac{u}{c}\right]\right| =$$

$$I_0(\overrightarrow{\Omega}_0, E_0)\frac{1}{E_0}(E/E_0)\gamma|\overrightarrow{\Omega}_0 \cdot \overrightarrow{n} + u/c|\,. \tag{17}$$

Collecting terms in Eq. (17), we get

$$I(\overrightarrow{\Omega}, E)\frac{1}{E}(E_0/E)\left|\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}\left[1 + \frac{\gamma^2}{\gamma+1}\frac{u^2}{c^2}\right] + \gamma\frac{u}{c}\right| =$$

$$I_0(\overrightarrow{\Omega}_0, E_0)\frac{1}{E_0}(E/E_0)\gamma|\overrightarrow{\Omega}_0 \cdot \overrightarrow{n} + u/c|\,. \tag{18}$$

It is easily shown that

$$\frac{\gamma^2}{\gamma+1} = \frac{\gamma^2(\gamma-1)}{\gamma^2-1} = \frac{c^2}{u^2}(\gamma-1)\,. \tag{19}$$

Substituting from Eq. (19) into Eq. (18), we get

$$I(\overrightarrow{\Omega}, E)\frac{1}{E}(E_0/E)\gamma|\overrightarrow{\Omega}_0 \cdot \overrightarrow{n} + u/c| =$$

$$I_0(\overrightarrow{\Omega}_0, E_0)\frac{1}{E_0}(E/E_0)\gamma|\overrightarrow{\Omega}_0 \cdot \overrightarrow{n} + u/c|\,. \tag{20}$$

Solving Eq. (20) for I, we obtain

$$I(\overrightarrow{\Omega}, E) = (E/E_0)^3 I_0(\overrightarrow{\Omega}_0, E_0)\,. \tag{21}$$

The above transformation enables the comoving-frame and laboratory-frame intensities to be expressed in terms of one another. Note that the ratio E_0/E can be expressed either in terms of laboratory-frame quantities (see Eq. (4a)) or comoving-frame quantities (see Eq. (4b)). Thus the laboratory-frame intensity can be expressed in terms of the comoving-frame intensity and comoving quantities, and vice versa. We next consider radiation sources. Observers in each frame must observe the same number of photons being emitted into a phase-space volume over a differential time. Thus we obtain the following equation:

$$Q_0(\overrightarrow{\Omega}_0, E_0)\frac{1}{E_0}\, d\Omega_0\, dE_0\, dV_0\, dt_0 = Q(\overrightarrow{\Omega}, E)\frac{1}{E}\, d\Omega\, dE\, dV\, dt\,, \tag{22}$$

where Q denotes any volumetric source ($energy/volume - time - steradian - energy$). Using Eqs. (3a), and (10), we easily obtain

$$Q(\overrightarrow{\Omega}, E) = (E/E_0)^2\, Q_0(\overrightarrow{\Omega}_0, E_0)\,. \tag{23}$$

Finally, we consider macroscopic cross sections. To relate cross sections, we simply equate the number of photons in each frame observed to undergo an interaction within a differential phase-space volume over a differential time:

$$\sigma_0(E_0)\, I_0(\overrightarrow{\Omega}_0, E_0)\frac{1}{E_0}\, d\Omega_0\, dE_0\, dV_0\, dt_0 = \sigma(\overrightarrow{\Omega}, E)\, I(\overrightarrow{\Omega}, E)\frac{1}{E}\, d\Omega\, dE\, dV\, dt\,. \tag{24}$$

Using Eqs. (3a), 10), and (21), we easily get

$$\sigma(\overrightarrow{\Omega}, E) = (E_0/E)\sigma_0(E_0)\ . \tag{25}$$

Note that the comoving-frame cross section is standard and thus independent of direction, but the laboratory-frame cross section is directionally dependent.

4 The Laboratory-Frame Transport Equation

Using the transformations derived in the previous section, we can now derive the correct laboratory-frame transport equation. Of course, only the interaction terms need be derived since all other terms remain valid. We express all quantities in terms of the comoving-frame cross sections, i.e., the standard cross sections. Hence we will not carry a subscript "0" on material properties. Using Eq. (25), we immediately obtain the laboratory removal term:

$$Q_r(\overrightarrow{\Omega}, E) \equiv (E_0/E)\sigma_t(E_0)I(\overrightarrow{\Omega}, E) \,. \tag{26}$$

The comoving-frame emission source is

$$Q_{e,0} = \sigma_a(E_0)B(E_0) \,. \tag{27}$$

Using Eq. (23), we immediately obtain the laboratory-frame emission source:

$$Q_e(\overrightarrow{\Omega}, E) \equiv (E/E_0)^2\, \sigma_a(E_0)B(E_0) \,. \tag{28}$$

The inscatter source is a bit tricky to obtain. The comoving-frame inscatter source is:

$$Q_{i,0} \equiv \frac{\sigma_s}{4\pi} \int_{4\pi} I_0(\overrightarrow{\Omega}'_0, E_0)\, d\Omega'_0 \,. \tag{29}$$

Since this term represents a source, we first use Eq. (23) to obtain the laboratory-frame inscatter source:

$$Q_i(\overrightarrow{\Omega}, E) \equiv \, (E/E_0)^2\, \frac{\sigma_s}{4\pi} \int_{4\pi} I_0(\overrightarrow{\Omega}'_0, E_0)\, d\Omega'_0 \,. \tag{30}$$

However, this expression cannot be used directly because we do not know what the comoving-frame intensity is. Before proceeding, it is useful to re-express Eq. (30) in the following form with integration over both energy and direction:

$$Q_i(\overrightarrow{\Omega}, E) \equiv (E/E_0)^2 \frac{\sigma_s}{4\pi} \int_{4\pi} \int_0^\infty I_0(\overrightarrow{\Omega}'_0, E'_0) \delta\left(E'_0 - E_0\right) \, dE'_0 \, d\Omega'_0 \,. \tag{31}$$

Substituting from Eqs. (10) and (21) into Eq. (31), we obtain the inscatter source in terms of the laboratory-frame intensity with laboratory-frame integration variables.

$$Q_i(\overrightarrow{\Omega}, E) \equiv (E/E_0)^2 \frac{\sigma_s}{4\pi} \int_{4\pi} \int_0^\infty (E'_0/E')^2 I(\overrightarrow{\Omega}', E') \delta\left(E'_0 - E_0\right) \, dE' \, d\Omega' \,. \tag{32}$$

It is important to recognize that E_0 is defined in Eq. (32) by $\overrightarrow{\Omega}$ and E via Eq. (4a):

$$E_0 = E \, \gamma \left(1 - \overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right) \,, \tag{33}$$

and that E'_0 is similarly defined in Eq. (32) by $\overrightarrow{\Omega}'$ and E' via Eq. (4a)

$$E'_0 = E' \, \gamma \left(1 - \overrightarrow{\Omega}' \cdot \overrightarrow{u}/c\right) \,, \tag{34}$$

The integral over E' that appears in Eq. (32) requires special treatment because the integration variable appearing within the delta-function is not E', but rather E'_0. Although one can formally evaluate the integral over E', the simplest way to deal with the delta-function is to change the integration variable back to E'_0. Because $\overrightarrow{\Omega}'$ is held constant during the integration over E', it follows from Eq. (34) that

$$dE' = \frac{dE'_0}{\gamma\left(1 - \overrightarrow{\Omega}' \cdot \overrightarrow{u}/c\right)} = (E'/E'_0) \, dE'_0 \,. \tag{35}$$

Substituting from Eq. (35) into Eq. (32), we get

$$Q_i(\overrightarrow{\Omega}, E) \equiv (E/E_0)^2 \frac{\sigma_s}{4\pi} \int_{4\pi} \int_0^{\infty} (E'_0/E') I(\overrightarrow{\Omega}', E') \delta\left(E'_0 - E_0\right) \, dE'_0 \, d\Omega' \, . \tag{36}$$

Evaluating the integral over E'_0 in Eq. (36), we obtain the final expression for the inscatter source:

$$Q_i(\overrightarrow{\Omega}, E) \equiv (E/E_0)^2 \frac{\sigma_s}{4\pi} \int_{4\pi} (E_0/E') I(\overrightarrow{\Omega}', E') \, d\Omega' \, . \tag{37}$$

where E' takes on the value that forces $E'_0 = E_0$:

$$E'\gamma \left(1 - \overrightarrow{\Omega}' \cdot \overrightarrow{u}/c\right) = E_0 = E\gamma \left(1 - \overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right) \, . \tag{38}$$

Solving Eq. (38) for E', we get

$$E' = E \, \frac{1 - \overrightarrow{\Omega} \cdot \overrightarrow{u}/c}{1 - \overrightarrow{\Omega}' \cdot \overrightarrow{u}/c} \, , \tag{39}$$

and solving Eq. (38) for E_0/E', we get

$$E_0/E' = \gamma \left(1 - \overrightarrow{\Omega}' \cdot \overrightarrow{u}/c\right) \, . \tag{40}$$

We now have sufficient information to construct the thermal radiation transport equation in the laboratory frame with material motion. Recognizing that the time-derivative and gradient terms in the transport equation without material motion remain valid, and using Eqs. (26), (28), and (37), we obtain

$$\frac{1}{c}\frac{\partial I}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I + (E_0/E)\sigma_t(E_0) I(\overrightarrow{\Omega}, E) =$$

$$(E/E_0)^2 \frac{\sigma_s}{4\pi} \int_{4\pi} (E_0/E')\, I(\overrightarrow{\Omega}', E')\, d\Omega' + (E/E_0)^2\, \sigma_a(E_0) B(E_0)\,, \tag{41}$$

where E_0 is defined by Eq. (4a), E_0/E' is defined by Eq. (42a), and E' is defined by Eq. (39).

5 The Comoving-Frame Transport Equation

Although we will not work further with the comoving-frame transport equation, it is nonetheless useful to show how it is derived. We will consider only 1-D slab-geometry equation. There are a few more relationships between laboratory and comoving-frame quantities needed for the derivation. In 1-D slab geometry, Eqs. (4a) and (4b) respectively become

$$E_0 = E\gamma(1 - \beta\mu)\,, \tag{42a}$$

and

$$E = E_0\, \gamma\, (1 + \beta\mu_0)\ . \tag{42b}$$

We also need to relate the laboratory-frame and comoving-frame polar cosines:

$$\mu_0 = \frac{\mu - \beta}{1 - \beta\mu}\,, \tag{43a}$$

and

$$\mu = \frac{\mu_0 + \beta}{1 + \beta\mu_0}\,. \tag{43b}$$

We begin with the laboratory-frame equation in 1-D slab geometry:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial z} = Q\,, \tag{44}$$

where Q denotes all interaction terms (the total source). Next we use Eq. (23) to express the laboratory-frame total source in terms of the comoving-frame total source:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial z} = (E/E_0)^2 Q_0\,, \tag{45}$$

where

$$Q_0 = \frac{1}{4\pi}\sigma_s\varphi_0 + \sigma_a B - \sigma_t I_0\,. \tag{46}$$

Next we use Eq. (21) to express the laboratory-frame intensity, $I(t, z, \mu, E)$ in terms of the comoving-frame intensity, $I(t, z, \mu_0, E_0)$:

$$\frac{1}{c}\frac{\partial (E/E_0)^3 I_0}{\partial t} + \mu\frac{\partial (E/E_0)^3 I_0}{\partial z} = (E/E_0)^2 Q_0\,. \tag{47}$$

It is important to recognize that μ_0 and E_0 in Eq. (47) are explicit functions of the laboratory-frame variables. Applying the time and space derivatives first to $(E/E_0)^3$ and then to I_0, we get

$$(E/E_0)^3\left[\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial z}\right] - 3(E^3/E_0^4)\left[\frac{1}{c}\frac{\partial E_0}{\partial t} + \mu\frac{\partial E_0}{\partial z}\right] I_0 = (E/E_0)^2 Q_0\,. \tag{48}$$

Multiplying Eq. (48) by $(E_0/E)^2$, we obtain

$$(E/E_0)\left[\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial z}\right] - 3(E/E_0^2)\left[\frac{1}{c}\frac{\partial E_0}{\partial t} + \mu\frac{\partial E_0}{\partial z}\right] I_0 = Q_0\,. \tag{49}$$

At this point we have a comoving-frame equation with the comoving-frame source on the right side, but we must apply the time and space derivatives to the comoving-frame intensity via the chain rule so that we can treat μ_0 and E_0 as independent variables and eliminate μ and E from the equation:

$$\frac{\partial}{\partial t} I_0\left[\mu_0(t,z), E_0(t,z)\right] = \frac{\partial I_0}{\partial t} + \frac{\partial \mu_0}{\partial t}\frac{\partial I_0}{\partial \mu_0} + \frac{\partial E_0}{\partial t}\frac{\partial I_0}{\partial E_0}\,, \tag{50}$$

and

$$\frac{\partial}{\partial x} I_0\left[\mu_0(t,z), E_0(t,z)\right] = \frac{\partial I_0}{\partial z} + \frac{\partial \mu_0}{\partial z}\frac{\partial I_0}{\partial \mu_0} + \frac{\partial E_0}{\partial z}\frac{\partial I_0}{\partial E_0}\,, \tag{51}$$

where the partial derivative of I_0 with respect to t on the right side of Eq. (50) and the partial derivative of I_0 with respect to z on the right side of Eq. (51) are taken with μ_0 and E_0 held constant. Substituting from Eqs. (50) and (51) into (49), we obtain

$$\begin{aligned}(E/E_0)\left\{\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial z} + \left[\frac{1}{c}\frac{\partial \mu_0}{\partial t} + \mu\frac{\partial \mu_0}{\partial z}\right]\frac{\partial I_0}{\partial \mu_0} + \left[\frac{1}{c}\frac{\partial E_0}{\partial t} + \mu\frac{\partial E_0}{\partial z}\right]\frac{\partial I_0}{\partial E_0}\right\} - \\ 3(E/E_0^2)\left[\frac{1}{c}\frac{\partial E_0}{\partial t} + \mu\frac{\partial E_0}{\partial z}\right] I_0 = Q_0\,.\end{aligned} \tag{52}$$

Substituting from Eq. (42b) and Eq. (43b) into Eq. (52), we eliminate μ and E to obtain

$$\begin{aligned}\gamma(1+\beta\mu_0)\left\{\frac{1}{c}\frac{\partial I_0}{\partial t} + \frac{\mu_0+\beta}{1+\beta\mu_0}\frac{\partial I_0}{\partial z} + \left[\frac{1}{c}\frac{\partial \mu_0}{\partial t} + \frac{\mu_0+\beta}{1+\beta\mu_0}\frac{\partial \mu_0}{\partial z}\right]\frac{\partial I_0}{\partial \mu_0} + \right. \\ \left.\left[\frac{1}{c}\frac{\partial E_0}{\partial t} + \frac{\mu_0+\beta}{1+\beta\mu_0}\frac{\partial E_0}{\partial z}\right]\frac{\partial I_0}{\partial E_0}\right\} - \\ \frac{3}{E_0}\gamma(1+\beta\mu_0)\left[\frac{1}{c}\frac{\partial E_0}{\partial t} + \frac{\mu_0+\beta}{1+\beta\mu_0}\frac{\partial E_0}{\partial z}\right] I_0 = Q_0\,.\end{aligned} \tag{53}$$

Our next task is to compute the space and time derivatives of μ_0 and E_0. First we differentiate Eq. (42a) with respect to t to obtain

$$\frac{\partial E_0}{\partial t} = E\left[\gamma^3\beta\frac{\partial\beta}{\partial t}(1-\beta\mu) - \gamma\frac{\partial\beta}{\partial t}\mu\right] . \tag{54}$$

Next we eliminate E and μ from Eq. (54) using Eqs. (42b) and (43b), respectively, to obtain

$$\frac{\partial E_0}{\partial t} = -\gamma^2 E_0\mu_0\frac{\partial\beta}{\partial t} . \tag{55}$$

The derivative of E_0 with respect to z is completely analogous:

$$\frac{\partial E_0}{\partial z} = -\gamma^2 E_0\mu_0\frac{\partial\beta}{\partial z} . \tag{56}$$

We next differentiate Eq. (43a) to get

$$\frac{\partial\mu_0}{\partial t} = -\frac{\partial\beta}{\partial t}\frac{1}{1-\beta\mu} + \frac{(\mu-\beta)}{(1-\beta\mu)^2}\frac{\partial\beta}{\partial t}\mu . \tag{57}$$

Using Eq. (43a) to eliminate μ from Eq. (57), we obtain

$$\frac{\partial\mu_0}{\partial t} = -\gamma^2(1-\mu_0^2)\frac{\partial\beta}{\partial t} . \tag{58}$$

The derivative of μ_0 with respect to z is completely analogous:

$$\frac{\partial\mu_0}{\partial z} = -\gamma^2(1-\mu_0^2)\frac{\partial\beta}{\partial z} . \tag{59}$$

Substituting from Eqs. (55), (56), (58), and (59) into Eq. (53), we obtain the comoving-frame transport equation:

$$\gamma(1+\beta\mu_0)\left\{\frac{1}{c}\frac{\partial I_0}{\partial t} + \frac{\mu_0+\beta}{1+\beta\mu_0}\frac{\partial I_0}{\partial z} - \left[\frac{1}{c}\gamma^2(1-\mu_0^2)\frac{\partial\beta}{\partial t} + \frac{\mu_0+\beta}{1+\beta\mu_0}\gamma^2(1-\mu_0^2)\frac{\partial\beta}{\partial z}\right]\frac{\partial I_0}{\partial\mu_0} - \right.$$

$$\left[\frac{1}{c}\gamma^2 E_0\mu_0\frac{\partial\beta}{\partial t}+\frac{\mu_0+\beta}{1+\beta\mu_0}\gamma^2 E_0\mu_0\frac{\partial\beta}{\partial z}\right]\frac{\partial I_0}{\partial E_0}\Bigg\}+$$
$$\frac{3}{E_0}\gamma(1+\beta\mu_0)\left[\frac{1}{c}\gamma^2 E_0\mu_0\frac{\partial\beta}{\partial t}+\frac{\mu_0+\beta}{1+\beta\mu_0}\gamma^2 E_0\mu_0\frac{\partial\beta}{\partial z}\right]I_0=Q_0\,. \tag{60}$$

In the literature, an integration by parts of the each term containing derivatives of I_0 with respect to μ_0 and E_0 is performed to put the equation in so-called "conservative" form. However, as previously noted energy and momentum are generally not conserved in the comoving frame. Finally, performing the integrations by parts (together with some other algebraic manipulations), we obtain the conservative form of the comoving-frame 1-D slab-geometry transport equation:

$$\gamma(1+\beta\mu_0)\frac{1}{c}\frac{\partial I_0}{\partial t}+\gamma(\mu_0+\beta)\frac{\partial I_0}{\partial z}-\frac{\partial}{\partial\mu_0}\left\{\gamma^3(1-\mu_0^2)\left[(1+\beta\mu_0)\frac{1}{c}\frac{\partial\beta}{\partial t}+(\mu_0+\beta)\frac{\partial\beta}{\partial z}\right]I_0\right\}-$$
$$\frac{\partial}{\partial E_0}\left\{\gamma^3 E_0\mu_0\left[(1+\beta\mu_0)\frac{1}{c}\frac{\partial\beta}{\partial t}+(\mu_0+\beta)\frac{\partial\beta}{\partial z}\right]I_0\right\}+$$
$$\gamma^3\left\{\left[2\mu_0+\beta(1+\mu_0^2)\right]\frac{1}{c}\frac{\partial\beta}{\partial t}+\left[1+\mu_0^2+2\mu_0\beta\right]\frac{\partial\beta}{\partial z}\right\}I_0=Q_0\,. \tag{61}$$