NUEN 627

Lecture 10

Simplified Transport Material-Motion Corrections

<u>The Grey Case</u>

The grey energy and momentum source terms correct to $O(u/c)$ are

$$S_{re} = \sigma_a c(aT^4 - \mathcal{E}) + (\sigma_a - \sigma_s)\vec{F}\cdot\vec{u}/c, \qquad (1)$$

$$\vec{S}_{rp} = -\frac{1}{c}\sigma_t \vec{F} + (\sigma_s c\mathcal{E} + \sigma_a a c T^4)\vec{u}/c + \sigma_t \mathbb{P}\cdot\vec{u}/c. \qquad (2)$$

Using

$$\mathcal{E}_0 = \mathcal{E} - \frac{2}{c^2}\vec{F}\cdot\vec{u} \qquad (3)$$

we can rewrite (1) as

$$S_{re} = \sigma_a c(aT^4 - \mathcal{E}_0) - \sigma_t \vec{F}\cdot\frac{\vec{u}}{c}. \qquad (4)$$

Using (3) and

$$\vec{F}_0 = \vec{F} - (\mathcal{E} + \overrightarrow{\overrightarrow{\mathbb{P}}})\vec{u}, \qquad (5)$$

we can re-write (2) as follows:

$$\vec{S}_{rp} = -\frac{\sigma_t \vec{F}_0}{c} + \sigma_a c \,(aT^4 - \mathcal{E})\,\frac{\vec{u}}{c^2}. \qquad (6)$$

Using (4), the grey radiation energy equation is

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla}\cdot\vec{F} = \sigma_a c\left(aT^4 - \mathcal{E} + \frac{2}{c^2}\vec{F}\cdot\vec{u}\right) - \frac{\sigma_t}{c}\vec{F}_0\cdot\vec{u}, \qquad (7)$$

and the grey radiation momentum equation is

$$\frac{1}{c^2}\frac{\partial F}{\partial t} + \vec{\nabla}\cdot\vec{\vec{P}} = -\frac{\sigma_t}{c}\left[\vec{F} - (\mathcal{E} + \vec{\vec{P}})\vec{u}\right]$$

$$+ \sigma_a c\,(aT^4 - \mathcal{E})\,\frac{\vec{u}}{c^2}. \qquad (8)$$

We know that in equilibrium,

$$\mathcal{E}_0 = aT^4, \qquad (9a)$$
$$\mathcal{E} = aT^4 + O(u^2/c^2), \qquad (9b)$$

$$\vec{F}_0 = \vec{0}, \qquad (10a)$$
$$\vec{F} = \frac{4}{3}aT^4\vec{u} + O(u^2/c^2), \qquad (10b)$$

$$\vec{\vec{P}}_0 = \frac{1}{3}aT^4, \qquad (11a)$$
$$\vec{\vec{P}} = \frac{1}{3}aT^4 + O(u^2/c^2). \qquad (11b)$$

If we substitute from (9b) and (10b) into

(7) and assume that all derivatives are zero, in equilibrium, we do not get $\vec{S}_{re} = 0$. We can fix this while retaining accuracy to $O(u/c)$ by substituting $\vec{F}_0$ for $\vec{F}$ in the right side of (7):

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla}\cdot\vec{F} = \sigma_a c\left(aT^4 - \mathcal{E} + \frac{2}{c^2}\left[\vec{F} - (\mathcal{E} + \overleftrightarrow{P})\vec{u}\right]\cdot\vec{u}\right)$$
$$- \frac{\sigma_t}{c}\left[\vec{F} - (\mathcal{E} + \overleftrightarrow{P})\vec{u}\right]\cdot\vec{u} \qquad (12)$$

On the other hand, if we substitute from (9b), (10b) and (11b) into (8) and assume that all derivatives are zero, we get $\vec{S}_{rp} = \vec{0}$.

Thus we find that the final expressions for S_{re} and $\vec{S}_{rp}$ correct to $O(u/c)$ with ad hoc equilibrium corrections are

$$S_{re} = \sigma_a c\left\{aT^4 - \mathcal{E} + \frac{2}{c^2}\left[\vec{F} - (\mathcal{E} + \overleftrightarrow{P})\vec{u}\right]\right\}\cdot\vec{u}$$
$$- \frac{\sigma_t}{c}\left[\vec{F} - (\mathcal{E} + \overleftrightarrow{P})\vec{u}\right]\cdot\vec{u} \qquad (13)$$

and

$$\vec{S}_{rp} = -\frac{\sigma_t}{c}\left[\vec{F} - (\mathcal{E} + \overleftrightarrow{P})\vec{u}\right] + \sigma_a c\,(aT^4 - \mathcal{E})\frac{\vec{u}}{c^2}. \qquad (14)$$

In general, red and blue shifts due to nonrelativistic material motion are relatively very small at all photon energies. For instance,

$$E_0 = E\,(1 - \vec{\Omega}\cdot\vec{u}/c) + O(u^2/c^2) \qquad (15)$$

In the worst case, $\vec{\Omega}\cdot\vec{u} = \pm u$, so

$$E_0 = E\,(1 \pm u/c) \qquad (16)$$

If we assume that $u/c \ll 1$, this means the the percent change in photon energy at all energies is no more than $100\,u/c$. This means that resolving the red and blue shifts will require an unacceptably large number of groups. This suggest that we develop approximate expressions for S_E and $\vec{S}_P$ that may not be correct to $O(u/c)$, but are nonetheless accurate in an integral sense. We have developed such expressions with the following properties:

1) Total (radiation plus material) energy conservation.

2) Total momentum conservation.

3) Correct equilibrium solutions to $O(u/c)$ for $\mathcal{E}$, $\vec{F}$, and $\overleftrightarrow{P}$.

4) Preservation of the equilibrium-diffusion limit to $O(u/c)$.

Grey expressions that meet these properties are

$$S_{re} = \sigma_a c(aT^4 - \mathcal{E}) - \frac{\sigma_t}{c}[\vec{F} - \tfrac{4}{3}\mathcal{E}\vec{u}]\cdot\vec{u} \qquad (17)$$

and

$$\vec{S}_{rp} = -\frac{\sigma_t}{c}\left[\vec{F} - \tfrac{4}{3}\mathcal{E}\vec{u}\right] \qquad (18)$$

One advantage of these simplified equations is that they avoid certain non-physical solutions associated with (13) and (14). The origin of these solutions is the second term on the right of (14) which is purely relativistic in nature. For instance, consider an infinite spatially-constant system with $\vec{u}=0$, $\mathcal{E}=\mathcal{E}_{in}$, $P=\frac{1}{3}\mathcal{E}_{in}$, $\vec{F}=\vec{0}$, and $T=T_{in}$, but $\mathcal{E}_{in} \neq aT_{in}$. Then the system relaxes to equilibrium, but remains spatially constant with $\vec{u}=0$. However, if we take the same initial conditions but set $\vec{u}$ to a constant but non-zero value, we would not expect to see the system accelerate, but if we substitute $\mathcal{E}=\mathcal{E}_{in}$, $P=\frac{1}{3}\mathcal{E}_{in}$, $\vec{F}=\frac{4}{3}\mathcal{E}_{in}$, and $T=T_{in}$ into Eq. (14), we get

$$\vec{S}_{rp} = \sigma_a c\left(aT_{in}^4 - \mathcal{E}_{in}\right)\frac{\vec{u}}{c} \neq \vec{0}\,! \tag{19}$$

This is non-physical and apparently to the inherent conflict between the relativistic nature of the photons and the non-relativistic nature of the material. Thus dropping the second term on the right side of Eq. (14) is probably a good idea anyway.

The Multigroup Case

For the multigroup case, Eqs. (17) and (18) respectively become

$$S_{rc} = \sum_g \sigma_{a,g} c\left(4\pi B_g - \mathcal{E}_g\right) - \frac{\sigma_{t,g}}{c}\left[\vec{F}_g - \frac{4}{3}\mathcal{E}_g\vec{u}\right]\cdot\vec{u}, \tag{20}$$

and

$$\vec{S}_{rp} = -\sum_g \frac{\sigma_{t,g}}{c}\left[\vec{F}_g - \frac{4}{3}\mathcal{E}_g\vec{u}\right]. \tag{21}$$

Corrections to the Transport Equation

Here we consider the correction terms that must be added to the multigroup transport equation to achieve consistency between S_{re} and the radiation energy equation, and $\vec{S}_{rp}$ and the radiation momentum equation, respectively. In particular, we write the transport equation as follows:

$$\frac{1}{c}\frac{\partial I_g}{\partial t} + \vec{\Omega}\cdot\vec{\nabla} I_g + \sigma_{t,g} I_g = \frac{1}{4\pi}\sigma_{s,g}\phi_g + \sigma_{a,g} B_g +$$

$$\frac{1}{4\pi} C_{0,g} + \frac{3}{4\pi}\vec{C}_{1,g}\cdot\vec{\Omega} , \qquad (22)$$

where the correction terms have a P-1 form and are defined as follows:

$$C_{0,g} = -\frac{\sigma_{t,g}}{c}\left(\vec{F}_g - \frac{4}{3}\mathcal{E}_g\vec{u}\right)\cdot\vec{u} , \qquad (23)$$

$$\vec{C}_{1,g} = \sigma_{t,g}\frac{4}{3}\mathcal{E}_g\vec{u} \qquad (24)$$