NEN 627

Lecture 11

Diffusion Limit Asymptotics of Grey Transport

We begin with the grey transport equation with simplified material-motion corrections,

$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{\Omega}\cdot\vec{\nabla} I + \sigma_t I = \frac{1}{4\pi}\sigma_s \phi + \frac{1}{4\pi}\sigma_a acT^4$$

$$-\frac{1}{4\pi}\frac{\sigma_t}{c}\left(\vec{F} - \frac{4}{3}\phi\frac{\vec{u}}{c}\right)\cdot\vec{u} + \frac{3}{4\pi}\sigma_t\frac{4}{3}\phi\frac{\vec{u}}{c}\cdot\vec{\Omega}, \qquad (1)$$

and the hydro equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0, \qquad (2a)$$

$$\frac{\partial(\rho\vec{u})}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}\otimes\vec{u}) + \vec{\nabla}p = \frac{\sigma_t}{c}\left(\vec{F} - \frac{4}{3}\phi\frac{\vec{u}}{c}\right) \qquad (2b)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + \rho e + p\right)\vec{u}\right] =$$

$$\sigma_a\left(\phi - acT^4\right) + \frac{\sigma_t}{c}\left(\vec{F} - \frac{4}{3}\phi\frac{\vec{u}}{c}\right)\cdot\vec{u} \qquad (2c)$$

We next proceed with non-dimensionalization of the equations. The non-dimensional variables are defined as follows, where a superscript "prime" denotes a dimensionless quantity and a subscript "infinity" denotes a reference quantity.

$$
\begin{aligned}
\vec{r} &= l_\infty \vec{r}\,', \\
t &= (l_\infty / u_\infty) t', \\
\rho &= \rho_\infty \rho', \\
\vec{u} &= u_\infty \vec{u}\,', \\
p &= \rho_\infty u_\infty^2 p', \\
T &= T_\infty T', \\
I &= a c T_\infty^4 I', \\
\sigma_t &= \sigma_{t,\infty} \sigma_t', \\
\sigma_s &= \sigma_{s,\infty} \sigma_s', \\
e &= u_\infty^2 e'.
\end{aligned}
\tag{3}
$$

As we will eventually see, the following dimensionless parameters appear in the non-dimensional equations, and the are scaled as indicated.

$$\mathcal{U} \equiv \frac{u_\infty}{c} \Longrightarrow \epsilon, \tag{4a}$$

$$\mathcal{R} \equiv \rho_\infty u_\infty^2 / a T_\infty^4 \Longrightarrow 1, \tag{4b}$$

$$\mathcal{L} \equiv 1 / (\sigma_{t,\infty} l_\infty) \Longrightarrow \epsilon, \tag{4c}$$

$$\mathcal{L}_s \equiv \sigma_{s,\infty} / \sigma_{t,\infty} \Longrightarrow \epsilon. \tag{4d}$$

Proceeding with the non-dimensionalization of the transport equation we first note that

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \frac{u_\infty}{l_\infty}\frac{\partial}{\partial t'}, \qquad (5a)$$

$$\vec{\nabla} = \frac{\partial \vec{r}'}{\partial \vec{r}} \cdot \vec{\nabla} = \frac{1}{l_\infty}\vec{\nabla}'. \qquad (5b)$$

Now,

$$\frac{1}{c}\frac{u_\infty}{l_\infty}\frac{\partial}{\partial t'} acT_\infty^4 I' + \frac{1}{l_\infty}\vec{\Omega}\cdot\vec{\nabla}' acT_\infty^4 I' + \sigma_{t,\infty}\sigma_t' acT_\infty^4 I' =$$

$$\frac{1}{4\pi}\sigma_{s,\infty}\sigma_s' acT_\infty^4 \phi' + \frac{1}{4\pi}(\sigma_{t,\infty}\sigma_t' - \sigma_{s,\infty}\sigma_s') acT_\infty^4 (T')^4$$

$$-\frac{1}{4\pi}\sigma_{t,\infty}\sigma_t' acT_\infty^4 (\vec{F}' - \frac{4}{3}\phi' \frac{u_\infty}{c}\vec{u}')\cdot\frac{u_\infty}{c}\vec{u}' +$$

$$\frac{3}{4\pi}\sigma_{t\infty}\sigma_t' \frac{4}{3} acT_\infty^4 \phi' \frac{u_\infty}{c}\vec{u}'\cdot\vec{\Omega}. \qquad (6)$$

Dividing (6) by $\sigma_{t,\infty} acT_\infty^4$, we get

$$\left(\frac{u_\infty}{c}\right)\left(\frac{1}{\sigma_{t,\infty} l_\infty}\right)\frac{\partial}{\partial t'} I' + \left(\frac{1}{\sigma_{t,\infty} l_\infty}\right)\vec{\Omega}\cdot\vec{\nabla}' I' + \sigma_t' I' =$$

$$\frac{1}{4\pi}\left(\frac{\sigma_{s,\infty}}{\sigma_{t,\infty}}\right)\sigma_s'\phi' + \frac{1}{4\pi}\left(\sigma_t' - \frac{\sigma_{s,\infty}}{\sigma_{t,\infty}}\sigma_s'\right)(T')^4$$

$$-\frac{1}{4\pi}\sigma_t'(\vec{F}' - \frac{4}{3}\phi' \frac{u_\infty}{c}\vec{u}')\cdot\frac{u_\infty}{c}\vec{u}' +$$

$$\frac{3}{4\pi}\sigma_t' \frac{4}{3}\phi' \frac{u_\infty}{c}\vec{u}'\cdot\vec{\Omega} \qquad (7)$$

Next we substitute from Eqs. (4) into (7) to obtain the scaling for each term and convert back to dimensional form:

$$\frac{\epsilon^2}{c}\frac{\partial I}{\partial t} + \epsilon\, \vec{\Omega}\cdot\vec{\nabla} I + \sigma_t I = \frac{\epsilon}{4\pi}\sigma_s \phi + \frac{1}{4\pi}(\sigma_t - \epsilon\sigma_s)\, a c T^4$$

$$- \frac{\epsilon}{4\pi}\frac{\sigma_t}{c}\left(\vec{F} - \epsilon\frac{4}{3}\phi\frac{\vec{u}}{c}\right)\cdot\vec{u} + \frac{3\epsilon}{4\pi}\sigma_t\frac{4}{3}\phi\frac{\vec{u}}{c}\cdot\vec{\Omega} \qquad (8)$$

Next we proceed to the mass conservation equation, (2a):

$$\frac{u_\infty}{l_\infty}\frac{\partial}{\partial t'}\rho_\infty\rho' + \frac{1}{l_\infty}\vec{\nabla}'(\rho_\infty\rho' u_\infty\vec{u}') = 0 \qquad (9)$$

Dividing (9) by $\left(\frac{\rho_\infty u_\infty}{l_\infty}\right)$, we obtain no dimensionless parameters, so there is no scaling of the mass conservation equation. One simply obtains Eq. (2a).

The momentum conservation equation is next.

$$\frac{u_\infty}{l_\infty}\frac{\partial}{\partial t'}\rho_\infty\rho' u_\infty\vec{u}' + \frac{1}{l_\infty}\vec{\nabla}'\cdot(\rho_\infty\rho' u_\infty\vec{u}'\otimes u_\infty\vec{u}') + \frac{1}{l_\infty}\vec{\nabla}'(\rho_\infty u_\infty^2 p') =$$

$$\sigma_{t,\infty}\frac{\sigma_t'}{c}\left(acT_\infty^4\vec{F}' - \frac{4}{3}acT_\infty^4\phi'\frac{u_\infty}{c}\vec{u}'\right) \qquad (10)$$

Dividing (10) by $\sigma_{t,\infty} a T_\infty^4$, we get

$$\left(\frac{\rho_\infty u_\infty^2}{a T_\infty^4}\right)\left(\frac{1}{\sigma_{t,\infty} l_\infty}\right)\left\{\frac{\partial}{\partial t'}(\rho'\vec{u}') + \vec{\nabla}'\cdot[\rho'(\vec{u}'\otimes\vec{u}')] + \vec{\nabla}'p'\right\} =$$

$$\sigma_t'\left(\vec{F}' - \frac{u_\infty}{c}\frac{4}{3}\phi'\vec{u}'\right) \qquad (11)$$

Next we substitute from Eqs. (4) into (11) to obtain the scaling for each term and convert back to dimensional form:

$$\epsilon\frac{\partial(\rho\vec{u})}{\partial t} + \epsilon\vec{\nabla}\cdot[\rho(\vec{u}\otimes\vec{u})] + \epsilon\vec{\nabla}p =$$

$$\frac{\sigma_t}{c}\left(\vec{F} - \epsilon\frac{4}{3}\phi\frac{\vec{u}}{c}\right). \qquad (12)$$

Finally, we proceed with the non-dimensionalization of the material energy equation:

$$\frac{u_\infty}{l_\infty}\frac{\partial}{\partial t'}\left[\frac{1}{2}\rho_\infty\rho' u_\infty^2(u')^2 + \rho_\infty\rho' u_\infty^2 e'\right] +$$

$$\frac{1}{l_\infty}\vec{\nabla}'\cdot\left[\left(\frac{1}{2}\rho_\infty\rho' u_\infty^2(u')^2 + \rho_\infty\rho' u_\infty^2 e' + \rho_\infty u_\infty^2 p'\right)u_\infty\vec{u}'\right] =$$

$$(\sigma_{t,\infty}\sigma_t' - \sigma_{s,\infty}\sigma_s')\left[a c T_\infty^4\phi' - a c T_\infty^4 (T')^4\right] +$$

$$\sigma_{t,\infty}\sigma_t'\left(a c T_\infty^4\vec{F}' - \frac{4}{3}a c T_\infty^4\phi'\frac{u_\infty}{c}\vec{u}'\right)\cdot\frac{u_\infty}{c}\vec{u}' \qquad (13)$$

Dividing Eq. (13) by $\sigma_{t,\infty} a c T_\infty^4$, we get

$$\left(\frac{\rho_\infty u_\infty^2}{a T_\infty^4}\right)\left(\frac{u_\infty}{c}\right)\left(\frac{1}{\sigma_{t,\infty} l_\infty}\right)\left\{\frac{\partial}{\partial t'}\left(\frac{1}{2}\rho'(u')^2+\rho' e'\right)+\right.$$

$$\left.\vec{\nabla}'\cdot\left[\left(\frac{1}{2}\rho'(u')^2+\rho' e'+p'\right)\vec{u}'\right]\right\} =$$

$$\left(\sigma_t' - \frac{\sigma_{s,\infty}}{\sigma_{t,\infty}}\sigma_s'\right)\left[\varphi' - (T')^4\right] +$$

$$\frac{u_\infty}{c}\sigma_t'\left(\vec{F}' - \frac{u_\infty}{c}\frac{4}{3}\varphi'\vec{u}'\right)\cdot\vec{u}' , \qquad (14)$$

Substituting from Eqs. (4) into (14), we obtain the scaling for each term and return the equation to dimensional form:

$$\epsilon^2\frac{\partial}{\partial t}\left[\frac{1}{2}\rho u^2+\rho e\right]+\epsilon^2\vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2+\rho e+p\right)\vec{u}\right]+\epsilon^2\vec{\nabla}p =$$

$$(\sigma_t - \epsilon\sigma_s)\left(\varphi - acT^4\right) + \epsilon\frac{\sigma_t}{c}\left(\vec{F} - \epsilon\frac{4}{3}\varphi\frac{\vec{u}}{c}\right)\cdot\vec{u} . \qquad (15)$$

We are now ready to proceed with the derivation of the asymptotic equations. In particular, we assume a power-series expansion in ϵ for each unknown, e.g.,

$$I = \sum_{n=0}^{\infty} I^{(n)}\epsilon^n ,$$

$$\rho = \sum_{n=0}^{\infty} \rho^{(n)}\epsilon^n ,$$

$$\vdots$$

etc., substitute the expansions into the scaled equations, and then generate a hierarchy of equations by forcing each scaled equation to hold for each power of ϵ. We begin with the transport equation.

Equations for $\epsilon^{(0)}$

$$\sigma_t^{(0)} I^{(0)} = \frac{1}{4\pi} \sigma_t^{(0)} a c T^{4,(0)} \tag{16}$$

Solving (16) for $I^{(0)}$, we get

$$I^{(0)} = \frac{1}{4\pi} a c T^{4,(0)}. \tag{17}$$

The above result implies that

$$\vec{F}^{(0)} = 0, \tag{18a}$$

$$\varphi^{(0)} = a c T^{4,(0)}, \tag{18b}$$

$$P_{ij}^{(0)} = \delta_{ij} \frac{1}{3c} \varphi^{(0)}. \tag{18c}$$

Note that the temperature coefficients for T and T^4 are directly related, but for simplicity, we do not explicitly express this relationship, e.g., $T^{4,(0)} = (T^{(0)})^4$, and $T^{4,(1)} = 4(T^{(0)})^3 T^{(1)}$.

Equations for $\epsilon^{(1)}$

$$\vec{\Omega}\cdot\vec{\nabla} I^{(0)} + \sigma_t^{(0)} I^{(1)} + \sigma_t^{(1)} I^{(0)} = \frac{1}{4\pi}\sigma_s^{(0)}\phi^{(0)}$$

$$+\frac{1}{4\pi}\sigma_t^{(0)} acT^{4,(1)} + \frac{1}{4\pi}\sigma_t^{(1)} acT^{4,(0)} - \frac{1}{4\pi}\sigma_s^{(0)} acT^{4,(1)}$$

$$-\frac{1}{4\pi}\frac{\sigma_t^{(0)}}{c}\vec{F}^{(0)}\cdot\vec{u}^{(0)} + \frac{3}{4\pi}\sigma_t^{(0)}\frac{4}{3}\phi^{(0)}\frac{\vec{u}^{(0)}}{c}\cdot\vec{\Omega}. \qquad (18)$$

Including previous results, (18) becomes

$$\vec{\Omega}\cdot\vec{\nabla}\frac{1}{4\pi}acT^{4,(0)} + \sigma_t^{(0)} I^{(1)} = \sigma_t^{(0)}\frac{1}{4\pi}acT^{4,(1)}$$

$$+\frac{3}{4\pi}\sigma_t^{(0)}\frac{4}{3}\phi^{(0)}\frac{\vec{u}^{(0)}}{c}\cdot\vec{\Omega} \qquad (19)$$

Equation (19) implies that

$$\vec{F}^{(1)} = -\frac{1}{3\sigma_t^{(0)}}\vec{\nabla} acT^{4,(0)} + \frac{4}{3}aT^{4,(0)}\vec{u}^{(0)}, \qquad (20a)$$

$$\phi^{(1)} = acT^{4,(1)}, \qquad (20b)$$

$$P_{ij}^{(1)} = \delta_{ij}\frac{1}{3}acT^{4,(1)}. \qquad (20c)$$

Equations for $\epsilon^{(2)}$

$$\frac{1}{c}\frac{\partial I^{(0)}}{\partial t} + \vec{\Omega}\cdot\vec{\nabla} I^{(1)} + \sigma_t^{(0)} I^{(2)} + \sigma_t^{(1)} I^{(1)} + \sigma_t^{(2)} I^{(0)} = \frac{1}{4\pi}\sigma_s^{(0)}\phi^{(1)}$$

$$+\frac{1}{4\pi}\sigma_s^{(1)}\phi^{(0)} + \frac{1}{4\pi}\sigma_t^{(0)} acT^{4,(2)} + \frac{1}{4\pi}\sigma_t^{(1)} acT^{4,(1)} + \frac{1}{4\pi}\sigma_t^{(2)} acT^{4,(0)}$$

$$-\frac{1}{4\pi}\sigma_s^{(0)} acT^{4,(1)} - \frac{1}{4\pi}\sigma_s^{(1)} acT^{4,(0)} - \frac{1}{4\pi}\frac{\sigma_t^{(0)}}{c}\left(\vec{F}^{(1)} - \frac{4}{3}\phi^{(0)}\frac{\vec{u}^{(0)}}{c}\right)\cdot\vec{u}^{(0)}$$

$$-\frac{1}{4\pi}\frac{\sigma_t^{(1)}}{c}\vec{F}^{(0)}\cdot\vec{u}^{(0)} - \frac{1}{4\pi}\frac{\sigma_t^{(0)}}{c}\vec{F}^{(0)}\cdot\vec{u}^{(1)} + \frac{3}{4\pi}\sigma_t^{(0)}\frac{4}{3}\varphi^{(1)}\frac{\vec{u}^{(0)}}{c}\cdot\vec{\Omega}$$

$$+\frac{3}{4\pi}\sigma_t^{(0)}\frac{4}{3}\varphi^{(0)}\frac{\vec{u}^{(1)}}{c}\cdot\vec{\Omega} + \frac{3}{4\pi}\sigma_t^{(1)}\frac{4}{3}\varphi^{(0)}\frac{\vec{u}^{(0)}}{c}\cdot\vec{\Omega}. \qquad (21)$$

Including previous results and integrating (21) over all angles, we get

$$\frac{\partial}{\partial t}aT^{4,(0)} - \vec{\nabla}\cdot\frac{1}{3\sigma_t^{(0)}}\vec{\nabla}acT^{4,(0)} + \vec{\nabla}\cdot\left(\frac{4}{3}aT^{4,(0)}\right)\cdot\vec{u}^{(0)}$$

$$\sigma_t^{(0)}\left(acT^{4,(2)} - \varphi^{(2)}\right) + \vec{\nabla}\left(\frac{1}{3}aT^{4(0)}\right)\cdot\vec{u}^{(0)}. \qquad (22)$$

Next we generate the asymptotic equations for the material energy equation.

Equations for $\epsilon^{(0)}$

$$\sigma_t^{(0)}\left(\varphi^{(0)} - acT^{4,(0)}\right) = 0, \qquad (23)$$

which is satisfied given previous results.

Equations for $\epsilon^{(1)}$

$$\sigma_t^{(0)}\left(\varphi^{(1)} - acT^{4,(1)}\right) + \sigma_t^{(1)}\left(\varphi^{(0)} - acT^{4,(0)}\right) - \sigma_s^{(0)}\left(\varphi^{(0)} - acT^{4,(0)}\right)$$

$$+ \frac{\sigma_t^{(0)}}{c}\vec{F}^{(0)}\cdot\vec{u}^{(0)}, \qquad (24)$$

which is satisfied given previous results.

Equations for $\epsilon^{(2)}$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho^{(0)}u^{2,(0)}+\rho^{(0)}e^{(0)}\right]+\vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho^{(0)}u^{2,(0)}+\rho^{(0)}e^{(0)}+p^{(0)}\right)\cdot\vec{u}^{(0)}\right]=$$

$$\sigma_t^{(0)}\left(\phi^{(2)}-acT^{4,(2)}\right)+\sigma_t^{(1)}\left(\phi^{(1)}-acT^{4,(1)}\right)+\sigma_t^{(2)}\left(\phi^{(0)}-acT^{4,(0)}\right)$$

$$-\sigma_s^{(0)}\left(\phi^{(1)}-acT^{4,(1)}\right)-\sigma_s^{(1)}\left(\phi^{(0)}-acT^{4,(0)}\right)+$$

$$\frac{\sigma_t^{(0)}}{c}\left(\vec{F}^{(1)}-\frac{4}{3}\phi^{(0)}\frac{\vec{u}^{(0)}}{c}\right)\cdot\vec{u}^{(0)}+\frac{\sigma_t^{(0)}}{c}\vec{F}^{(0)}\cdot\vec{u}^{(1)}+\frac{\sigma_t^{(1)}}{c}\vec{F}^{(0)}\cdot\vec{u}^{(0)}. \quad (25)$$

Including previous results, (25) reduces to

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho^{(0)}u^{2,(0)}+\rho^{(0)}e^{(0)}\right]+\vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho^{(0)}u^{2,(0)}+\rho^{(0)}e^{(0)}+p^{(0)}\right)\cdot\vec{u}^{(0)}\right]=$$

$$\sigma_t^{(0)}\left(\phi^{(2)}-acT^{4,(2)}\right)-\vec{\nabla}\left(\frac{1}{3}aT^{4,(0)}\right)\cdot\vec{u}^{(0)}. \quad (26)$$

Adding (22) and (26), we obtain the equilibrium-diffusion limit total energy equation:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho^{(0)}u^{2,(0)}+\rho^{(0)}e^{(0)}+aT^{4,(0)}\right]+$$

$$\vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho^{(0)}u^{2,(0)}+\rho^{(0)}e^{(0)}+aT^{4,(0)}+p^{(0)}+\frac{1}{3}aT^{4,(0)}\right)\vec{u}^{(0)}\right]$$

$$=\vec{\nabla}\cdot\frac{1}{3\sigma_t^{(0)}}\vec{\nabla}acT^{4,(0)} \quad (27)$$

Next we generate the asymptotic equations for the material momentum equation:

Equations for $\epsilon^{(0)}$

$$\frac{\sigma_t^{(0)}}{c}\vec{F}^{(0)} = 0 . \tag{28}$$

This equation is satisfied by previous results.

Equations for $\epsilon^{(1)}$

$$\frac{\partial}{\partial t}(\rho^{(0)}\vec{u}^{(0)}) + \vec{\nabla}\cdot\left[\rho^{(0)}\vec{u}^{(0)}\otimes\vec{u}^{(0)}\right] + \vec{\nabla}p^{(0)} =$$

$$\frac{\sigma_t^{(0)}}{c}\left(\vec{F}^{(1)} - \frac{4}{3}\mathcal{E}^{(0)}\frac{\vec{u}^{(0)}}{c}\right) + \frac{\sigma_t^{(1)}}{c}\vec{F}^{(0)} . \tag{29}$$

Using previous results, (29) reduces to the equilibrium diffusion-limit material momentum equation:

$$\frac{\partial}{\partial t}(\rho^{(0)}\vec{u}^{(0)}) + \vec{\nabla}\cdot\left[\rho^{(0)}\vec{u}^{(0)}\otimes\vec{u}^{(0)}\right] +$$

$$\vec{\nabla}p^{(0)} + \vec{\nabla}\left(\frac{1}{3}aT^{4,(0)}\right) = 0 \tag{30}$$

Finally we generate the asymptotic equations for the mass conservation equation. Since this equation is not scaled, we simply obtain the following equation for $\epsilon^{(0)}$:

$$\frac{\partial}{\partial t}\rho^{(0)} + \vec{\nabla}\cdot\rho^{(0)}u^{(0)} = 0 \qquad (31)$$

Equations (31), (30), and (27) constitute the leading-order equilibrium-diffusion limit equations which are closed via the equations of state:

$$p^{(0)} = p^{(0)}(\rho^{(0)}, \epsilon^{(0)}) , \qquad (32a)$$

$$T^{(0)} = T^{(0)}(\rho^{(0)}, \epsilon^{(0)}) . \qquad (32b)$$