Lecture 6

The Equilibrium-Diffusion Limit

The purpose of this lecture is to

- To give the physical scaling associated with the asymptotic equilibrium-diffusion limit.
- To give the scaled equations associated with the asymptotic equilibrium-diffusion limit.
- To give the leading-order radiation-hydrodynamics equations associated with the asymptotic equilibrium-diffusion limit.

1 The Radiation-Hydrodynamics Equations to O(u/c)

It is convenient to begin the derivation of the equilibrium-diffusion limit with the radiation-hydrodyamics equations to O(u/c):

$$\frac{\partial}{\partial t}\rho + \overrightarrow{\nabla}\cdot(\rho\overrightarrow{u}) = 0\,, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho\overrightarrow{u}) + \overrightarrow{\nabla}\cdot\left(\rho\overrightarrow{u}\otimes\overrightarrow{u}\right) + \overrightarrow{\nabla}p = -\overrightarrow{S}_{rp}\,, \tag{2}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \overrightarrow{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + \rho e + p\right)\overrightarrow{u}\right] = -S_{re}\,, \tag{3}$$

$$\frac{1}{c}\frac{\partial I}{\partial t} + \overrightarrow{\Omega}\cdot\overrightarrow{\nabla}I + \sigma_t I = \frac{\sigma_s}{4\pi}\varphi + \sigma_a B +$$

$$\left[\left(\sigma_t + E\frac{\partial \sigma_a}{\partial E}\right) I + \frac{\sigma_s}{4\pi}\left(2\varphi - E\frac{\partial \varphi}{\partial E}\right) + 2\sigma_a B - B\,E\frac{\partial \sigma_a}{\partial E} - \sigma_a\,E\frac{\partial B}{\partial E}\right] \overrightarrow{\Omega} \cdot \overrightarrow{u}/c -$$
$$\frac{\sigma_s}{4\pi}\left(\overrightarrow{F} - E\frac{\partial \overrightarrow{F}}{\partial E}\right) \cdot \overrightarrow{u}/c\,, \tag{4}$$

where

$$S_{re} = \int_0^\infty \sigma_a\left(4\pi B - \varphi\right)\,dE + \left[\int_0^\infty \left(\sigma_a + E\frac{\partial \sigma_a}{\partial E} - \sigma_s\right)\overrightarrow{F}\,dE\right] \cdot \overrightarrow{u}/c\,, \tag{5}$$

and

$$\overrightarrow{S}_{rp} = -\int_0^\infty \frac{1}{c}\sigma_t \overrightarrow{F}\,dE +$$
$$\left[\int_0^\infty \left(\sigma_s\varphi + \sigma_a 4\pi B\right)\,dE\right]\overrightarrow{u}/c^2 + \left[\int_0^\infty \left(\sigma_a + E\frac{\partial \sigma_a}{\partial E} + \sigma_s\right)\overrightarrow{\overrightarrow{\mathcal{P}}}\,dE\right] \cdot \overrightarrow{u}/c\,. \tag{6}$$

2 Non-Dimensionalization

To non-dimensionalize the radiation-hydrodyamics equations, we first define the following non-dimensional variables. It is important to remember that steradians are already non-dimensional, so dimensional quantities that have steradian units will still have steradian units in non-dimensional form.

$$\begin{aligned}
&\overrightarrow{r} = \overrightarrow{r}'\ell_\infty\,, && t = t'\ell_\infty/u_\infty\,, && \rho = \rho'\rho_\infty\,, && \overrightarrow{u} = \overrightarrow{u}'u_\infty\,, \\
&p = p'\rho_\infty u_\infty^2\,, && T = T'T_\infty\,, && E = E'kT_\infty\,, && I = I'acT_\infty^3/k\,, \\
&B = B'acT_\infty^3/k\,, && \sigma_t = \sigma_t'/\lambda_{t,\infty}\,, && \sigma_s = \sigma_s'/\lambda_{s,\infty}\,, && e = e'u_\infty^2\,.
\end{aligned} \tag{7}$$

where non-dimensional quantities carry a superscript "$'$", and reference quantities carry a subscript "∞". Note that both the radiation and material variables assume the same space and time scales, that the temperature is only used to scale transport quantities. Further note that the intensity is scaled so that

$$\mathcal{E} = \frac{1}{c}\int_0^\infty \int_{4\pi} I' acT_\infty^4/(kT_\infty)\, d\Omega\, dE = \int_0^\infty \int_{4\pi} I' aT_\infty^4\, d\Omega\, dE' = \mathcal{E}' aT_\infty^4\,, \tag{8a}$$

$$\overrightarrow{\mathcal{F}} = \overrightarrow{\mathcal{F}}' acT_\infty^4\,, \tag{8b}$$

$$\overrightarrow{\overrightarrow{\mathcal{P}}} = \overrightarrow{\overrightarrow{\mathcal{P}}}' aT_\infty^4\,. \tag{8c}$$

Finally, note that σ_a must be expressed as $\sigma_t - \sigma_s$ in the equations, thereby avoiding the need for an explicit non-dimensionalization of σ_a. The next step is to express all the equations in non-dimensional form. Derivatives are transformed as follows:

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = \frac{1}{\ell_\infty}\frac{\partial}{\partial x'}\,, \tag{9}$$

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \frac{u_\infty}{\ell_\infty}\frac{\partial}{\partial t'}\,, \tag{10}$$

$$\frac{\partial}{\partial E} = \frac{\partial E'}{\partial E}\frac{\partial}{\partial E'} = \frac{1}{kT_\infty}\frac{\partial}{\partial E'}\,, \tag{11}$$

and the dimensional variables are expressed in terms of the non-dimensional variables. The equations are then manipulated to identify the non-dimensional parameters. These non-dimensional parameters take the following form:

$$\mathcal{U} \equiv u_\infty / c\,, \tag{12}$$

$$\mathcal{R} \equiv \rho_\infty u_\infty^2 / \, aT_\infty^4 \,, \tag{13}$$

$$\mathcal{L} \equiv \ell_\infty / \, \lambda_{t,\infty} \,, \tag{14}$$

$$\mathcal{L}_s \equiv \lambda_{t,\infty} / \, \lambda_{s,\infty} \,. \tag{15}$$

The parameter $\mathcal{U}$ represents the ratio of the material speed to the speed of light. It is scaled O(ϵ) in accordance with the assumption of a non-relativistic limit. The parameter $\mathcal{R}$ represents the ratio of material energy to radiation energy. It is scaled O(1). The parameter $\mathcal{L}$ represents the ratio of the characteristic spatial scalelength of the radiation-hydrodynamic solution to the radiation mean-free-path. It is scaled O(ϵ^{-1}) in accordance with the physics of diffusion. The parameter $\mathcal{L}_s$ is the ratio of the total mean-free-path to the scattering mean-free-path. It is scaled O(ϵ), which is physically plausible for radiative transfer.

Once the non-dimensional parameters are scaled within the non-dimensional equations, the equations are returned to physical form yielding:

$$\frac{\partial}{\partial t}\rho + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{u}) = 0 \,, \tag{16}$$

$$\begin{aligned} \epsilon \frac{\partial}{\partial t}(\rho \overrightarrow{u}) + \epsilon \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right) + \epsilon \overrightarrow{\nabla} p = - \int_0^\infty \frac{1}{c}\sigma_t \overrightarrow{F} \, dE - \\ \epsilon \left\{ \int_0^\infty \left[\epsilon \sigma_s \varphi + (\sigma_t - \epsilon \sigma_s) \, 4\pi B\right] \, dE \right\} \overrightarrow{u}/c^2 - \epsilon \left\{ \int_0^\infty \left(\sigma_t + E\frac{\partial \sigma_a}{\partial E}\right) \overrightarrow{\overrightarrow{\mathcal{P}}} \, dE \right\} \cdot \overrightarrow{u}/c \,, \end{aligned} \tag{17}$$

$$\epsilon^2 \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \epsilon^2 \overrightarrow{\nabla} \cdot \left[\left(\frac{1}{2}\rho u^2 + \rho e + p\right) \overrightarrow{u}\right] =$$

$$- \int_0^\infty \left(\sigma_t - \epsilon\sigma_s\right)\left(4\pi B - \varphi\right)\, dE - \epsilon \int_0^\infty \left(\sigma_t + E\frac{\partial \sigma_a}{\partial E} - \epsilon 2\sigma_s\right) \overrightarrow{F} \cdot \overrightarrow{u}/c\,, \tag{18}$$

$$\epsilon^2 \frac{1}{c}\frac{\partial I}{\partial t} + \epsilon\, \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I + \sigma_t I = \epsilon \frac{\sigma_s}{4\pi}\varphi + \left(\sigma_t - \epsilon\sigma_s\right) B +$$
$$\epsilon \left\{ \left(\sigma_t + E\frac{\partial \sigma_a}{\partial E}\right) I + \epsilon \frac{\sigma_s}{4\pi}\left(2\varphi - E\frac{\partial \varphi}{\partial E}\right) + 2\left(\sigma_t - \epsilon\sigma_s\right) B - \right.$$
$$\left. B\, E\frac{\partial \sigma_a}{\partial E} - \left(\sigma_t - \epsilon\sigma_s\right)\, E\frac{\partial B}{\partial E} \right\} \overrightarrow{\Omega} \cdot \overrightarrow{u}/c \, - \epsilon^2 \frac{\sigma_s}{4\pi}\left(\overrightarrow{F} - E\frac{\partial \overrightarrow{F}}{\partial E}\right) \cdot \overrightarrow{u}/c\,. \tag{19}$$

Next, all of the radiation-hydrodynamic unknowns, the Planck function, and the material properties are expanded in a power series in ϵ, e.g.,

$$\rho = \sum_{n=0}^{\infty} \rho^{(n)} \epsilon^n\,, \tag{20a}$$

$$\overrightarrow{u} = \sum_{n=0}^{\infty} \overrightarrow{u}^{(n)} \epsilon^n\,, \tag{20b}$$

$$e = \sum_{n=0}^{\infty} e^{(n)} \epsilon^n\,, \tag{20c}$$

$$T = \sum_{n=0}^{\infty} T^{(n)} \epsilon^n\,, \tag{20d}$$

$$p = \sum_{n=0}^{\infty} p^{(n)} \epsilon^n\,, \tag{20e}$$

$$I = \sum_{n=0}^{\infty} I^{(n)} \epsilon^n\,, \tag{20f}$$

$$B = \sum_{n=0}^{\infty} B^{(n)} \epsilon^n\,, \tag{20g}$$

$$\sigma_t = \sum_{n=0}^{\infty} \sigma_t^{(n)} \epsilon^n\,. \tag{20h}$$

The dependence of the Planck function and the material properties upon ϵ arises from their dependence upon the temperature. Thus, for instance, $B^{(0)} = B(T^{(0)})$ and $B^{(1)} = \frac{\partial B}{\partial T}^{(0)} T^{(1)}$. The dependence of the material pressure upon ϵ arises from its dependence upon ρ and e through the equation of state.

Next the terms multiplying each power of ϵ are gathered to obtain a hierarchical set of equations. We next give the most important equations associated with each order of ϵ. The ϵ^0 equations yield:

$$\frac{\partial}{\partial t}\rho^{(0)} + \overrightarrow{\nabla}\cdot(\rho^{(0)}\overrightarrow{u}^{(0)}) = 0\,, \tag{21}$$

$$I^{(0)} = B^{(0)}\,. \tag{22}$$

This implies that

$$\mathcal{E}^{(0)} = aT^{(0)^4}\,, \tag{23a}$$

$$\overrightarrow{\mathcal{F}}^{(0)} = \overrightarrow{0}\,, \tag{23b}$$

$$\overrightarrow{\overrightarrow{\mathcal{P}}}^{(0)} = \frac{1}{3}aT^{(0)^4}\delta_{i,j}\,. \tag{23c}$$

The ϵ^1 equations yield

$$\frac{\partial}{\partial t}(\rho^{(0)}\overrightarrow{u}^{(0)}) + \overrightarrow{\nabla}\cdot\left(\rho^{(0)}\overrightarrow{u}^{(0)}\otimes\overrightarrow{u}^{(0)}\right) + \overrightarrow{\nabla}p^{(0)} = -\frac{1}{3}\overrightarrow{\nabla}aT^{(0)^4}\,, \tag{24}$$

$$\overrightarrow{\mathcal{F}}^{(1)} = -\frac{1}{3\langle\sigma_t^{(0)}\rangle}\overrightarrow{\nabla}acT^{(0)^4} + \frac{4}{3}aT^{(0)^4}\overrightarrow{u}^{(0)}\,, \tag{25}$$

where $\langle\sigma_t\rangle$ is the Rosselund-averaged total cross section:

$$
\begin{aligned}
\frac{1}{\langle\sigma_t\rangle} &= \int_0^\infty \frac{4\pi}{\sigma_t}\frac{\partial B}{\partial T}\,dE \Big/ \int_0^\infty 4\pi\frac{\partial B}{\partial T}\,dE \\
&= \int_0^\infty \frac{\pi}{acT^3\sigma_t}\frac{\partial B}{\partial T}\,dE\,. \qquad (26)
\end{aligned}
$$

The ϵ^2 equations yield

$$
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho^{(0)}{u^{(0)}}^2 + \rho^{(0)}e^{(0)} + a{T^{(0)}}^4\right) +
$$

$$
\overrightarrow{\nabla}\cdot\left[\left(\frac{1}{2}\rho^{(0)}{u^{(0)}}^2 + \rho^{(0)}e^{(0)} + a{T^{(0)}}^4 + p^{(0)} + \frac{1}{3}a{T^{(0)}}^4\right)\overrightarrow{u}^{(0)}\right] = \overrightarrow{\nabla}\cdot\frac{1}{3\langle\sigma_t^{(0)}\rangle}\overrightarrow{\nabla}ac{T^{(0)}}^4\,. \quad (27)
$$

It would seem the transport equation must expanded to $\mathrm{O}(u^2/c^2)$ to evaluate the equilibrium diffusion limit to leading order, but this is actually not the case. A second-order interaction source term appears in two of the $\mathrm{O}(\epsilon^2)$ equations:

$$
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho^{(0)}{u^{(0)}}^2 + \rho^{(0)}e^{(0)}\right) + \overrightarrow{\nabla}\cdot\left[\left(\frac{1}{2}\rho^{(0)}{u^{(0)}}^2 + \rho^{(0)}e^{(0)} + p^{(0)}\right)\overrightarrow{u}^{(0)}\right] = -S_{re}^{(2)} \qquad (28)
$$

$$
\frac{\partial}{\partial t}a{T^{(0)}}^4 - \overrightarrow{\nabla}\cdot\frac{1}{3\langle\sigma_t^{(0)}\rangle}\overrightarrow{\nabla}ac{T^{(0)}}^4 + \overrightarrow{\nabla}\cdot\frac{4}{3}a{T^{(0)}}^4\overrightarrow{u}^{(0)} = S_{re}^{(2)}\,, \qquad (29)
$$

This second-order interaction source term need not be explicitly evaluated because it is eliminated when Eqs. (28) and (29) are summed to obtain Eq. (27).

In summary the radiation-hydrodynamic equations in the equilibrium-diffusion limit can be expressed as follows:

$$
\frac{\partial}{\partial t}\rho + \overrightarrow{\nabla}\cdot(\rho\overrightarrow{u}) = 0\,, \qquad (30)
$$

$$\frac{\partial}{\partial t}(\rho \overrightarrow{u}) + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right) + \overrightarrow{\nabla} p + \frac{1}{3} \overrightarrow{\nabla} aT^4 = \overrightarrow{0}\,, \tag{31}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e + aT^4\right) + \\ \overrightarrow{\nabla} \cdot \left[\left(\frac{1}{2}\rho u^2 + \rho e + aT^4 + p + \frac{1}{3}aT^4\right)\overrightarrow{u}\right] = \overrightarrow{\nabla} \cdot \frac{1}{3\langle\sigma_t\rangle} \overrightarrow{\nabla} acT^4\,, \tag{32}$$

where the material pressure and temperature are given as a functions of density and internal energy by the equation of state. The radiation variables are given by

$$I = B(T, E)\,, \tag{33}$$

$$\mathcal{E} = aT^4\,, \tag{34}$$

$$\overrightarrow{\mathcal{F}} = -\frac{1}{3\langle\sigma_t\rangle} \overrightarrow{\nabla} acT^4 + \frac{4}{3} aT^4 \overrightarrow{u}\,, \tag{35}$$

$$\mathcal{P}_{i,j} = -\frac{1}{3} aT^4 \delta_{i,j}\,. \tag{36}$$

Interestingly, the equilibrium-diffusion equations are Gallilean-invariant. The relativistic nature of the radiation seemingly disappears in this limit. Furthermore, there exists a strong-equilibrium limit in which the diffusion term on the right side of Eq. (32) becomes negligible. In this limit, a purely hyperbolic system system is obtained and the radiation becomes one with the material. One can define a material-radiation specific internal energy density,

$$e^* \equiv e + aT^4/\rho\,, \tag{37}$$

and a material-radiation pressure,

$$p^* = p + \frac{1}{3}aT^4 \,, \tag{38}$$

which leads to the standard hydrodynamics equations with $e*$ and p^* replacing e and p, respectively.