NUEN 627
Lecture 15

Steady-State Hydrodynamic Shock Solutions

We start by assuming a steady-state shock for a hyperbolic system:

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial}{\partial x}\vec{F}(\vec{u}) = 0 \quad (1)$$

If the shock is steady, then the solution satisfies

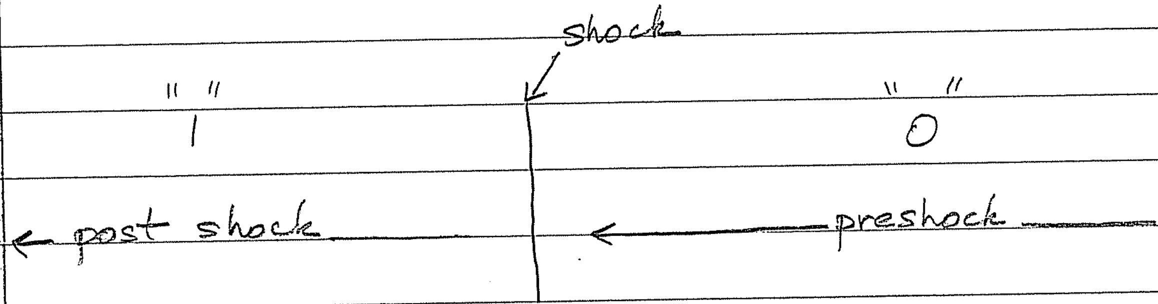
$$\frac{\partial}{\partial x}\vec{F}(\vec{u}) = 0 \quad (2)$$

Integrating Eq.(2) across the shock, we find that

$$\vec{F}(\vec{u}_L) = \vec{F}(\vec{u}_R), \quad (3)$$

i.e., the fluxes must be continuous. Let "0" denote a preshock quantity, and let "1" denote a postshock quantity. Then it can be shown that the shock solution (at least the physical one) has the following properties, where c_s denotes the speed of sound:

$$c_s = \sqrt{e\gamma(\gamma-1)} = \sqrt{\gamma p/\rho} \quad (4)$$



$|V_1| < C_{s1}$ $\qquad$ $|V_0| > C_{s0}$

$\rho_1 > \rho_0$ $\qquad$ ρ_0 arbitrary

$T_1 > T_0$ $\qquad$ T_0 arbitrary

The flux continuity equations corresponding to (3) are

$$\rho_0 u_0 = \rho_1 u_1 , \tag{5}$$

$$\rho_0 u_0^2 + p_0 = \rho_1 u_1^2 + p_1 , \tag{6}$$

$$\frac{1}{2}\rho_0 u_0^3 + \rho_0 e_0 u_0 + p_0 u_0 = \frac{1}{2}\rho_1 u_1^3 + \rho_1 e_1 u_1 + p_1 u_1 . \tag{7}$$

The pressure is given by

$$p = \rho e(\gamma - 1) \tag{8}$$

Substituting from (8) into (6) and (7), we get

$$\rho_0 u_0^2 + \rho_0 e_0 (\gamma - 1) = \rho_1 u_1^2 + \rho_1 e_1 (\gamma - 1), \tag{9}$$

$$\tfrac{1}{2}\rho_0 u_0^3 + \rho_0 e_0 u_0 + \rho_0 e_0 (\gamma-1) u_0 = \tfrac{1}{2}\rho_1 u_1^3 + \rho_1 e_1 u_1 + \rho_1 e_1 (\gamma-1) u_1,$$

$$\tfrac{1}{2}\rho_0 u_0^3 + \rho_0 e_0 \gamma u_0 = \tfrac{1}{2}\rho_1 u_1^3 + \rho_1 e_1 \gamma u_1. \tag{10}$$

From (5) and (10), we get

$$\tfrac{1}{2} u_0^2 + e_0 \gamma = \tfrac{1}{2} u_1^2 + e_1 \gamma. \tag{11}$$

Substituting from (5) into (9), we get

$$\frac{u_1}{u_0}\left\{u_0^2 + e_0(\gamma-1)\right\} = u_1^2 + e_1(\gamma - 1),$$

$$u_1 u_0^2 + u_1 e_0 (\gamma - 1) = u_0 u_1^2 + u_0 e_1 (\gamma - 1). \tag{12}$$

Solving (12) for e_1, we get

$$e_1 = \frac{u_1 u_0^2 - u_0 u_1^2 + u_1 e_0 (\gamma - 1)}{u_0(\gamma - 1)}. \tag{13}$$

Substituting from (13) into (11), we get

$$\tfrac{1}{2} u_0^2 + e_0 \gamma = \tfrac{1}{2} u_1^2 + \frac{\gamma u_1 u_0^2 - \gamma u_0 u_1^2 + \gamma u_1 e_0 (\gamma - 1)}{u_0 (\gamma - 1)},$$

$$u_0^3(\gamma-1)+2e_0\gamma(\gamma-1)u_0 = u_1^2 u_0(\gamma-1) +$$

$$2\gamma u_1 u_0^2 - 2\gamma u_0 u_1^2 + 2\gamma u_1 e_0(\gamma-1),$$

$$u_1^2[u_0(\gamma-1)-2\gamma u_0] + u_1[2\gamma u_0^2 + 2\gamma e_0(\gamma-1) =$$

$$u_0^3(\gamma-1)+2e_0\gamma(\gamma-1)u_0,$$

$$u_1^2 u_0(\gamma-1) - u_1\left[2\gamma u_0^2 + 2\gamma e_0(\gamma-1)\right] + u_0^3(\gamma-1)+2e_0\gamma(\gamma-1)u_0 = 0. \quad (14)$$

Remembering that

$$C_{s0} = \sqrt{e_0\gamma(\gamma-1)}, \quad (15)$$

we substitute from (15) into (14):

$$u_1^2 u_0(\gamma+1) - u_1[2\gamma u_0^2 + 2C_{s0}^2] + u_0[u_0^2(\gamma-1)+2C_{s0}^2] = 0. \quad (16)$$

Equation (6) can be written

$$a u_1^2 + b u_1 + c = 0. \quad (17)$$

Thus the solution to (17) is

$$u_1 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad (18)$$

where

$$b^2 = 4\gamma^2 u_0^4 + 8\gamma u_0^2 c_{s0}^2 + 4c_{s0}^4 \qquad (19)$$

$$4ac = 4u_0(\gamma+1)u_0\left[u_0^2(\gamma-1) + 2c_{s0}^2\right]$$

$$= 4u_0^4(\gamma^2-1) + 8u_0^2(\gamma+1)c_{s0}^2$$

$$b^2 - 4ac = 4\gamma^2 u_0^4 - 4u_0^4\gamma^2 + 4u_0^4 + 8\gamma u_0^2 c_{s0}^2 - 8u_0^2\gamma c_{s0}^2 - 8u_0^2 c_{s0}^2 + 4c_{s0}^4$$

$$= 4u_0^4 - 8u_0^2 c_{s0}^2 + 4c_{s0}^4$$

$$= 4\left(u_0^2 - c_{s0}^2\right)^2 \qquad (20)$$

Substituting from (16), (17), (19), and (20), into (18), we get

$$u_1 = \frac{2(\gamma u_0^2 + c_{s0}^2) \pm \sqrt{4(u_0^2 - c_{s0}^2)^2}}{2u_0(\gamma+1)},$$

$$= \frac{\gamma u_0^2 + c_{s0}^2 \pm (u_0^2 - c_{s0}^2)}{u_0(\gamma+1)},$$

$$= u_0$$

or

$$u_1 = \frac{u_0^2(\gamma-1) + 2c_{s0}^2}{u_0(\gamma+1)} . \qquad (21)$$

$u_0 = u_1$ is not a physical solution. Only Eq. (21) is physical.

Note that as $u_0 \to \infty$ $\quad u_1 = u_0\left(\frac{\gamma-1}{\gamma+1}\right)$

Solving Eq. (5) for ρ_1, we get

$$\rho_1 = \rho_0 \frac{u_0}{u_1} = \frac{\rho_0 u_0^2(\gamma+1)}{u_0^2(\gamma-1)+2c_{s0}^2} \qquad (22)$$

From (11) and (21), we find that

$$e_1 = \frac{1}{2\gamma}(u_0^2 - u_1^2) + e_0$$

$$= e_0 + \frac{1}{2\gamma}\left\{u_0^2 - \left[\frac{u_0^2(\gamma-1)+2c_{s0}^2}{u_0(\gamma+1)}\right]^2\right\} \qquad (23)$$

Other checks that should be done are to make sure that (5), (6), and (7) hold.

We can use Galilean invariance to transform to a frame where the pre-shock velocity is zero. Then the shock speed will be $-u_0$:

$$S = -u_0 , \tag{24a}$$

$$u_0^* = u_0 + S = 0 , \tag{24b}$$

$$u_1^* = u_1 + S = u_1 - u_0 . \tag{24c}$$

To check the validity of this simple transformation, one should be able to show that the Hugoniot-Rankine conditions hold:

$$\frac{\Delta F_i}{\Delta U_i} = S \quad , \; i = 1,3 . \tag{25}$$

In particular, that

$$\frac{\rho_0 u_0^* - \rho_1 u_1^*}{\rho_0 - \rho_1} = S , \tag{26a}$$

$$\frac{\rho_0 (u_0^*)^2 + p_0 - \rho_1 (u_1^*)^2 - p_1}{\rho_0 u_0^* - \rho_1 u_1^*} = S , \tag{26b}$$

and

$$\frac{\frac{1}{2}\rho_0 (u_0^*)^3 + \rho_0 e_0 u_0^* + p_0 u_0^* - \frac{1}{2}\rho_1 (u_1^*)^3 - \rho_1 e_1 u_1^* - p_1 u_1^*}{\frac{1}{2}\rho_0 (u_0^*)^2 + \rho_0 e_0 - \frac{1}{2}\rho_1 (u_1^*)^2 - \rho_1 e_1} = S. \quad (26c)$$

For example, substituting from (24b) and (24c) into (26a), we get

$$\frac{\rho_1 (u_1 - u_0)}{\rho_1 - \rho_0} = S,$$

$$\frac{u_1 - u_0}{1 - \rho_0/\rho_1} = S. \quad (27)$$

Now, from (22), we get

$$1 - \rho_0/\rho_1 = 1 - \frac{u_0^2(\gamma-1) + 2c_{s0}^2}{u_0^2(\gamma+1)},$$

$$= \frac{u_0^2(\gamma+1) - u_0^2(\gamma-1) - 2c_{s0}^2}{u_0^2(\gamma+1)},$$

$$= \frac{2(u_0^2 - c_{s0}^2)}{u_0^2(\gamma+1)}. \quad (28)$$

From (21), we get

$$u_1 - u_0 = \frac{u_0^2(\gamma-1) + 2c_{s0}^2}{u_0(\gamma+1)} - u_0,$$

$$u_1 - u_0 = \frac{u_0^2(\gamma-1) + 2C_{s0}^2 - u_0^2(\gamma+1)}{u_0(\gamma+1)}$$

$$= \frac{2(C_{s0}^2 - u_0^2)}{u_0(\gamma+1)} \qquad (29)$$

Substituting from (28) and (29) into the left side of (27), we get

$$\frac{u_1 - u_0}{1 - \rho_0/\rho_1} = \frac{u_0^2(\gamma+1)}{2(u_0^2 - C_{s0}^2)} \; \frac{2(C_{s0}^2 - u_0^2)}{u_0(\gamma+1)}$$

$$= -u_0 . \qquad (30)$$

Thus we have shown that Eq. (26a) does hold for the proposed Galilean transformation to the "natural" frame in which the pre-shock region is at rest and the shock moves into it