Lecture 2

Basic Concepts of Hydrodynamics

1 Introduction

The basic unknowns of fluid dynamics are the material density ($mass/volume$), the material velocity, $\overrightarrow{u}$ ($length/time$), the material specific internal energy e ($energy/mass$), and the the material pressure p ($force/area$). Since we are primarily interested in shock waves, we consider the fluid dynamics equations known as the Euler equations. These equations are simpler than the Navier-Stokes equations and admit true shock waves. They do not account for viscous effects or heat conduction. The fluid dynamics equations can be obtained via an asymptotic expansion from the nonlinear Boltzmann equation for self-interacting particles. This limit is associated with a particle-particle mean-free-path that is very small relative to the spatial scalelength of the phase-space density function. The Euler equations are obtained to leading order in this limit and the Navier-Stokes equations are obtained through first order. The fluid flow limit corresponds to particle phase-space density function that is isotropic in direction with a superposed drift velocity and a Maxwellian energy dependence.

2 The Euler Equations

The first Euler equation expresses the conservation of mass:

$$\frac{\partial}{\partial t}\rho + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{u}) = 0\,. \tag{1}$$

The second equation expresses the conservation of material momentum:

$$\frac{\partial}{\partial t}(\rho \overrightarrow{u}) + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right) + \overrightarrow{\nabla} p = 0\,, \tag{2}$$

where $\overrightarrow{u} \otimes \overrightarrow{u}$ denotes the dyadic tensor formed by the outer product of $\overrightarrow{u}$ with itself:

$$\left[\overrightarrow{u} \otimes \overrightarrow{u}\right]_{i,j} \equiv u_i u_j\,, \tag{2a}$$

The divergence of such a tensor is vector, and the i'th component of that vector is just the divergence of the vector formed by the i'th row of the tensor:

$$\left[\overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right)\right]_i = \overrightarrow{\nabla} \cdot \left(\rho u_i \overrightarrow{u}\right)\,. \tag{2b}$$

The third equation expresses the conservation of total (internal plus kinetic) material energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \overrightarrow{\nabla} \cdot \left[(\frac{1}{2}\rho u^2 + \rho e + p)\overrightarrow{u}\right] = 0\,. \tag{3}$$

The fourth equation required to close the system is the equation of state, which relates the pressure to the density and specific internal energy,

$$p = p(\rho, e)\,. \tag{4}$$

For an ideal gas,

$$p = (\gamma - 1)\rho e\,, \tag{5}$$

where γ is a constant that is equal to 5/3 for a monotonic gas and approaches 1 as the number of thermodynamic degrees of freedom approaches infinity. The equation of state can also be used to similarly express the temperature, T (*temperature*):

$$T = T(\rho, e)\,. \tag{6}$$

For an ideal gas,

$$T = \frac{\rho e}{C_v}\,, \tag{7}$$

where C_v ($energy/volume - temperature$) is the material heat capacity. The temperature need not be calculated to solve the hydrodynamics equations. As we shall later see, it is required to solve the radiation-hydrodynamics equations because the thermal radiation transport equation explicitly contains the material temperature. Note that we have assumed no external sources of momentum, or energy.

2.1 Einstein Notation

Einstein notation is a simple and compact means of expressing equations containing vectors and tensors. All such quantities are expressed in terms of their components. For instance, the i'th component of the vector $\overrightarrow{v}$ is denoted by v_i. The partial derivative with respect to the i'th coordinate is denoted by ∂_i. A repeated index in an expression implies a sum over all the values of that index, so the dot product of the vectors $\overrightarrow{f}$ and $\overrightarrow{g}$ is just denoted by $f_i g_i$.

Using Einstein notation, the Eqs.(1), (2), and (3) take the following respective forms Cartesian geometry:

$$\partial_t \rho + \partial_j \rho u_j = 0\,, \tag{8}$$

$$\partial_t(\rho u_i) + \partial_j\left(\rho u_i u_j\right) + \partial_i p = 0\,, \tag{9}$$

$$\partial_t\left(\frac{1}{2}\rho v^2 + \rho e\right) + \partial_j\left[\left(\frac{1}{2}\rho u^2 + \rho e + p\right) u_j\right] = 0\,. \tag{10}$$

2.2 Conservation

Equations. (1) through (3) follow from the universal conservation principle that the time rate of change for any conserved quantity within a volume is equal to the source rates for that quantity minus the sink rates. For instance, if we integrate Eq. (1) over an arbitrary volume, we obtain

$$\frac{\partial}{\partial t}\left[\int_V \rho\, dV\right] = -\oint \overrightarrow{F}_\rho \cdot \overrightarrow{n}\, dA\,, \tag{11}$$

where $\overrightarrow{F}_\rho \equiv \rho\overrightarrow{u}$ $(mass/area - time)$ is the mass flux (or current in neutron terminology) and $\overrightarrow{n}$ is the outward-directed surface normal. The term on the left side of Eq. (11) represents the time-rate of change of the mass within the volume V, and the term on the right side represents the *net* rate at which mass flows into the volume (inflow minus outflow).

Note from Eq. (2) that $\rho \overrightarrow{u}$ is both the mass flux ($mass/area{-}time$) and the momentum density vector ($momentum/volume$). If we integrate Eq. (2) over an arbitrary volume, we get

$$\partial_t \left[\int_V \rho \overrightarrow{u} \, dV \right] = - \oint \overrightarrow{F}_{\rho u_i} \cdot \overrightarrow{n} \, dA - \oint p \overrightarrow{n} \, dA \,, \tag{12}$$

where $\overrightarrow{F}_{\rho u_i} \equiv \rho u_i \overrightarrow{u}$ is the flux for momentum component i. The term on the left side of Eq. (12) represents the time-rate of change of the momentum within the volume V, and the first term on the right side of Eq. (12) represents the net rate at which the i'th component of momentum flows into the volume. The second term second term on the right side of the equation requires a bit of additional information to interpret. The gradient of the pressure has units of ($force/cm^3$), so $-\overrightarrow{\nabla} p \, dV$ represents the force applied to the differential fluid mass contained within the differential volume dV. From classical physics it follows that the force applied to a rigid body is equal to the time-derivative of the momentum of that body. Thus $-\overrightarrow{\nabla} p \, dV$ represents the time rate of change of the momentum of the fluid within dV due to the force arising from the pressure gradient. Given this interpretation, it is clear that the second term on the right side of Eq. (12) represents the total time rate of change of the momentum within the volume V due to the forces arising from pressure gradients within that volume.

If we integrate Eq. (3) over an arbitrary volume, we get

$$\frac{\partial}{\partial t}\left[\int_V \left(\frac{1}{2}\rho u^2 + \rho e\right)\, dV\right] = -\oint \left(\overrightarrow{F}_{\frac{1}{2}\rho u^2} + \overrightarrow{F}_{\rho e}\right)\cdot \overrightarrow{n}\, dA - \oint p\,\overrightarrow{u}\cdot\overrightarrow{n}\, dA\,, \tag{13}$$

where $\overrightarrow{F}_{\frac{1}{2}\rho u^2} \equiv \frac{1}{2}\rho v^2\,\overrightarrow{u}$ is the kinetic energy flux, and $\overrightarrow{F}_{\rho e} \equiv \rho e\,\overrightarrow{v}$ is the internal energy flux. The first term on the right side of Eq. (13) represents the rate at which the kinetic energy and the internal energy flows into the volume minus the rate at which it flows out of the volume. The second term on the right side of Eq. (13) requires a bit of additional information to interpret. It is not diffficult to show that this term represents the time rate of change of the total energy within the volume due to both material acceleration via internal pressure gradients and work done via compression and expansion of the material. To see this, we first note that

$$-\oint p\,\overrightarrow{u}\cdot\overrightarrow{n}\, dA = -\int_V \overrightarrow{\nabla}\cdot\left(p\,\overrightarrow{u}\right)\, dV\,. \tag{14}$$

Decomposing the volumetric integrand in Eq. (14) into a sum, we get

$$-\overrightarrow{\nabla}\cdot\left(p\,\overrightarrow{u}\right) = -\overrightarrow{u}\cdot\overrightarrow{\nabla}p - p\,\overrightarrow{\nabla}\cdot\overrightarrow{u}\,. \tag{15}$$

When a force $\overrightarrow{F}$ is applied to a rigid body, classical physics states that the change in the kinetic energy of that body over a pathlength $\overrightarrow{ds}$ is $\overrightarrow{F}\cdot\overrightarrow{ds}$. Since $-\overrightarrow{\nabla}p\,dV$ is the force applied to the fluid mass in dV, it follows that the time rate of change in the kinetic energy of the fluid mass in dV is $-\overrightarrow{v}\cdot\overrightarrow{\nabla}p\,dV$. Although it is far from obvious, $\overrightarrow{\nabla}\cdot\overrightarrow{u}\,dV$, is the time rate of change of dV assuming it moves with the fluid. We will later derive this property in a discussion of the Lagrangian derivative. It follows from thermodynamics that the internal energy of a fluid is increased when the fluid is compressed and decreased when it is expanded. More specifically, the time rate of change of the internal energy is equal to the product of the pressure and the time rate of change of the volume. Thus it follows that $-p\overrightarrow{\nabla}\cdot\overrightarrow{u}\,dV$ represents the time rate of change of the internal energy of the mass within dV due to the compression or expansion of that mass. Given these interpretations for the two terms in Eq. (15), it follows that the second term on the right side of Eq. (13) represents the time rate of change of the total energy within the volume V due to accelerations arising from pressure gradients within that volume and due to compression or expansion of the fluid within that volume.

One can derive the following equation for the kinetic energy density from the mass density equation and the momentum density equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right)+\overrightarrow{\nabla}\cdot\left(\frac{1}{2}\rho u^2\overrightarrow{u}\right)+\overrightarrow{u}\cdot\overrightarrow{\nabla}p=0\,. \tag{16}$$

One can derive the following equation for the internal energy density from the kinetic energy density equation and the and the total energy density equation:

$$\frac{\partial}{\partial t}(\rho e)+\overrightarrow{\nabla}\cdot\left(\rho e\overrightarrow{u}\right)+p\overrightarrow{\nabla}\cdot\overrightarrow{u}=0\,. \tag{17}$$

The internal energy equation can be substituted for the total energy equation if desired, but one generally discretizes the equations to ensure that mass, momentum, and total energy are conserved.

3 The Lagrangian Viewpoint

The Lagrangian viewpoint is one in which one casts the hydrodynamics equations in terms of total time derivative taken as one moves with the fluid. It is useful from the viewpoint of physical insight and it is useful from the viewpoint of numerics. Consider a function of time and position, $f(t, \overrightarrow{r})$, defined over a hydrodynamic domain. Since the velocity of the fluid is defined at each time and point in space, we can use the velocity to compute the total time derivative of f as we move with the fluid. In particular we want to calculate

$$\frac{Df}{Dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t, \overrightarrow{r} + \Delta \overrightarrow{r}) - f(t, \overrightarrow{r})}{\Delta t}\,, \tag{18}$$

where

$$\frac{D\overrightarrow{r}}{Dt} = \overrightarrow{u}\,. \tag{19}$$

Using the chain rule of basic calculus, we get

$$\begin{aligned}\frac{Df}{Dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{Dx}{Dt} + \frac{\partial f}{\partial y}\frac{Dy}{Dt} + \frac{\partial f}{\partial x}\frac{Dz}{Dt}\,. \\ &= \frac{\partial f}{\partial t} + \overrightarrow{u} \cdot \overrightarrow{\nabla} f\,. \end{aligned} \tag{20}$$

The derivative denoted by $\frac{Df}{Dt}$ is known as the Lagrangian derivative.

Let us define the specific volume as $\mathcal{V} \equiv rho^{-1}$ $(volume/mass)$. The differential volume associated with a differential mass, dM is clearly $dV = \mathcal{V}\, dM = \rho^{-1}\rho\, dV$. Let us now calculate the Lagrangian derivative of the specific volume divided by the specific volume.

$$\begin{aligned} \frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt} &= \rho\left[\frac{\partial\rho^{-1}}{\partial t} + \overrightarrow{u}\cdot\overrightarrow{\nabla}\rho^{-1}\right], \\ &= -\rho^{-1}\left[\frac{\partial\rho}{\partial t} + \overrightarrow{u}\cdot\overrightarrow{\nabla}\rho\right]. \end{aligned} \tag{21}$$

From Eq. (1) it follows that

$$\frac{\partial\rho}{\partial t} + \overrightarrow{u}\cdot\overrightarrow{\nabla}\rho = -\rho\overrightarrow{\nabla}\cdot\overrightarrow{u}\,. \tag{22}$$

Substituting from Eq. (22) into Eq. (21), we find that

$$\frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt} = \overrightarrow{\nabla}\cdot\overrightarrow{u}\,. \tag{23}$$

The interpretation of $\overrightarrow{\nabla}\cdot\overrightarrow{u}\,dV$ as the time rate of change of a differential volume moving with the fluid now becomes clear.

There is a "conservative" version of the Lagrangian derivative. In particular, let us assume some arbitrary density function $f(t, \overrightarrow{r})$ with arbitrary units of $(quantity/vol)$. Then

$$\begin{aligned} \frac{1}{\mathcal{V}}\frac{Df\mathcal{V}}{Dt} &= \frac{Df}{Dt} + f\frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt}, \\ &= \frac{\partial f}{\partial t} + \overrightarrow{\nabla}\cdot\left(f\overrightarrow{u}\right). \end{aligned} \tag{24}$$

The physical interpretation of $\frac{1}{\mathcal{V}}\frac{Df\mathcal{V}}{Dt}\,dV$ is the time rate of change of the quantity within dV when dV is moving with the fluid. The justification for this interpretation can be demonstrated as follows. Remembering the definitions of the Lagrangian derivative and the specific volume, it follows that

$$\frac{1}{\mathcal{V}}\frac{Df\mathcal{V}}{Dt}dV \equiv \rho\frac{Df/\rho}{Dt}dV\,. \tag{25}$$

Note that f/ρ is just f re-expressed as a specific density function ($quantity/mass$). Furthermore, $\rho\,dV$ is equal to the differential mass dM associated with dV. Thus we can write Eq. (25) as

$$\frac{1}{\mathcal{V}}\frac{Df\mathcal{V}}{Dt}dV \equiv \frac{Df/\rho}{Dt}dM\,. \tag{26}$$

Since the mass in a volume that moves with the flow does not change, dM can pass through the Lagrangian time derivative to yield

$$\frac{1}{\mathcal{V}}\frac{Df\mathcal{V}}{Dt}dV \equiv \frac{Df\,dV}{Dt}\,. \tag{27}$$

which is clearly the the time rate of change of the quantity within dV when dV is moving with the fluid.

Let us now return to the Euler equations and re-express them in terms of the conservative Lagrangian derivative. The mass conservation equation takes the following form:

$$\frac{1}{\mathcal{V}}\frac{D\rho\mathcal{V}}{Dt} = 0\,. \tag{28}$$

The physical interpretation of this equation is fairly obvious: the mass within a differential volume that moves with the fluid does not change. The momentum conservation equation takes the following form:

$$\frac{1}{\mathcal{V}} \frac{D\left(\rho \overrightarrow{u}\right)\mathcal{V}}{Dt} = -\overrightarrow{\nabla} p\,. \tag{29}$$

The physical interpretation of this equation is that the time rate of change of the momentum within a differential volume moving with the fluid is equation is equal to the force within that volume arising from the pressure gradient. The total energy conservation equation takes the following form:

$$\frac{1}{\mathcal{V}} \frac{D\left(\frac{1}{2}\rho u^2 + \rho e\right)\mathcal{V}}{Dt} = -\overrightarrow{\nabla} \cdot \left(p \overrightarrow{u}\right)\,. \tag{30}$$

The physical interpretation of this equation is that the time rate of change of the total energy within a differential volume moving with the fluid is equal to the time rate of change in kinetic energy due to the forces arising from the pressure gradient within the volume and the time rate of change of the internal energy within the volume due to rate at which the fluid within the volume is being compressed or expanded.

The numerical advantages of the Lagrangian form of the Euler equations will be discussed in a later lecture.