

Lecture 8

Lagrangian Solution of the Radiation-Hydrodynamics Equations with Grey Radiation Diffusion

We refer to the radiation-hydrodynamics equations with grey radiation diffusion as the RHGD equations. The purpose of this lecture is to

- To put the RHGD equations in Lagrangian form.
- To define a Lagrangian scheme for numerically solving these equations.

1 The Lagrangian RHGD Equations

We begin with the RHGD equations.

$$\frac{\partial}{\partial t}\rho + \vec{\nabla} \cdot (\rho \vec{u}) = 0. \quad (1)$$

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u}) + \vec{\nabla} p = -\frac{1}{3}\vec{\nabla} \mathcal{E}, \quad (2)$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\rho u^2 + \rho e \right) + \vec{\nabla} \cdot \left[\left(\frac{1}{2}\rho u^2 + \rho e + p \right) \vec{u} \right] = \sigma_a c (\mathcal{E} - aT^4) - \frac{1}{3}\vec{\nabla} \mathcal{E} \cdot \vec{u}, \quad (3)$$

$$\frac{\partial \mathcal{E}}{\partial t} - \vec{\nabla} \cdot \frac{c}{3\langle \sigma_t \rangle} \vec{\nabla} \mathcal{E} + \frac{4}{3}\vec{\nabla} \cdot \mathcal{E} \vec{u} = \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3}\vec{\nabla} \mathcal{E} \cdot \vec{u}, \quad (4)$$

Our numerical method is based upon the use of an internal energy equation rather than a total energy equation, so we substitute the following equation for Eq. (3):

$$\frac{\partial}{\partial t}(\rho e) + \vec{\nabla} \cdot (\rho e \vec{u}) + p \vec{\nabla} \cdot \vec{u} = \sigma_a c (\mathcal{E} - aT^4) , \quad (5)$$

where we note that

$$\sigma_a c (\mathcal{E} - aT^4) = -S_{re} + \vec{S}_{rp} \cdot \vec{u} . \quad (6)$$

In a previous lecture, we derived the hydrodynamics equations in Lagrangian form. Remembering that the conservative Lagrangian derivative for a general function f is given by

$$\frac{\partial f}{\partial t} + \vec{\nabla} \cdot (f \vec{u}) \equiv \frac{1}{\mathcal{V}} \frac{D(f\mathcal{V})}{Dt} \equiv \rho \frac{D(f/\rho)}{Dt} , \quad (7)$$

we easily obtain the mass, material momentum, and material internal energy equations in Lagrangian form:

$$\frac{1}{\mathcal{V}} \frac{D(\rho\mathcal{V})}{Dt} = 0 . \quad (8)$$

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} p - \frac{1}{3} \vec{\nabla} \mathcal{E} , \quad (9)$$

$$\rho \frac{De}{Dt} = -p \vec{\nabla} \cdot \vec{u} + \sigma_a c (\mathcal{E} - aT^4) , \quad (10)$$

Two simple steps produces the radiation diffusion equation in Lagrangian form:

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot (\mathcal{E} \vec{u}) - \vec{\nabla} \cdot \frac{c}{3\langle \sigma_t \rangle} \vec{\nabla} \mathcal{E} = \\ \sigma_a c (aT^4 - \mathcal{E}) + \frac{1}{3} \vec{\nabla} \mathcal{E} \cdot \vec{u} - \frac{1}{3} \vec{\nabla} \cdot (\mathcal{E} \vec{u}) , \end{aligned}$$

$$\rho \frac{D\mathcal{E}/\rho}{Dt} - \vec{\nabla} \cdot \frac{c}{3\langle\sigma_t\rangle} \vec{\nabla} \mathcal{E} = \sigma_a c (aT^4 - \mathcal{E}) - \frac{1}{3} \mathcal{E} \vec{\nabla} \cdot \vec{u}. \quad (11)$$

2 Discretization and Solution of the RHGD Equations

Our scheme for solving the RHGD equations is a mixture of implicit and explicit time discretization in the form of a second-order predictor-corrector scheme. We consider 1-D slab, cylindrical, and spherical geometries. Some unknowns lie at vertices while others lie at cell centers. We use the standard transport notation that uses half integral indicies at vertices and integral indices at cell centers. The first vertex has an index of $\frac{1}{2}$ and the first cell center has an index of 1. Spatial cell i is defined by $r \in [r_{i-\frac{1}{2}}, r_{i+\frac{1}{2}}]$, and time cell k is defined by $t \in [t^{k-\frac{1}{2}}, t^{k+\frac{1}{2}}]$. We also define median mesh cells. In particular, median mesh cell $i + \frac{1}{2}$ is defined by $r \in [r_i, r_{i+1}]$. Velocities and radiation fluxes are located at cell edges, while mass densities, internal material energy densities, radiation energy densities, temperatures, and pressures are at cell centers. We initially neglect boundary conditions and consider an infinite mesh. To efficiently deal with multiple geometries, we define vertex areas and cell-centered volumes. Specifically,

$$\begin{aligned} A_{i+\frac{1}{2}} &= 1 && \text{for slab geometry,} \\ &= 2\pi r_{i+\frac{1}{2}} && \text{for cylindrical geometry,} \\ &= 4\pi r_{i+\frac{1}{2}}^2 && \text{for spherical geometry,} \end{aligned} \quad (12)$$

and

$$\begin{aligned}
V_i &= r_{i+\frac{1}{2}} - r_{i-\frac{1}{2}} && \text{for slab geometry,} \\
&= \pi \left(r_{i+\frac{1}{2}}^2 - r_{i-\frac{1}{2}}^2 \right) && \text{for cylindrical geometry,} \\
&= \frac{4}{3}\pi \left(r_{i+\frac{1}{2}}^3 - r_{i-\frac{1}{2}}^3 \right) && \text{for spherical geometry,}
\end{aligned} \tag{13}$$

The predictor step is defined as follows. Unless otherwise indicated all quantities at t^k are obtained by averaging corresponding quantities at $t^{k-\frac{1}{2}}$ and $t^{k+\frac{1}{2}}$, respectively.

1. We first solve for velocities at $t^{k+\frac{1}{2}}$. The discretization for Eq. (9) is

$$\frac{m_{i+\frac{1}{2}}}{\Delta t^k} \left(u_{i+\frac{1}{2}}^{k+\frac{1}{2}} - u_{i+\frac{1}{2}}^{k-\frac{1}{2}} \right) = -A_{i+\frac{1}{2}}^{k-\frac{1}{2}} \left(P_{i+1}^{k-\frac{1}{2}} + \frac{1}{3}\mathcal{E}_{i+1}^{k-\frac{1}{2}} - P_i^{k-\frac{1}{2}} - \frac{1}{3}\mathcal{E}_i^{k-\frac{1}{2}} \right), \tag{14}$$

where $m_{i+\frac{1}{2}}$ is the mass in median-mesh cell $i + \frac{1}{2}$, and $\Delta t^k = t^{k+\frac{1}{2}} - t^{k-\frac{1}{2}}$. This discretization can be obtained by rigorously integrating Eq. (9) over time step k and median-mesh cell $i + \frac{1}{2}$, and substituting the areas and pressures at $t^{k-\frac{1}{2}}$ for the areas and pressures at t^k . This is an explicit approximation. Since all quantities at $t^{k-\frac{1}{2}}$ are known, the solution of Eq. (14) is trivial.

2. Next we solve for the new spatial coordinates at $t^{k+\frac{1}{2}}$.

$$r_{i+\frac{1}{2}}^{k+\frac{1}{2}} = r_{i+\frac{1}{2}}^{k-\frac{1}{2}} + u_{i+\frac{1}{2}}^k \Delta t^k. \tag{15}$$

Note that this is an implicit Crank-Nicholson-like approximation.

3. Next we solve for the new mass densities $t^{k+\frac{1}{2}}$.

$$\rho_i^{k+\frac{1}{2}} = m_i / V_i^{k+\frac{1}{2}}, \tag{16}$$

Note that this is an implicit approximation that is essentially exact given exact cell volumes.

4. Next we simultaneously solve the following two equations for the material internal specific energy densities and the radiation energy densities at $t^{k+\frac{1}{2}}$.

$$\frac{m_i}{\Delta t^k} \left(e_i^{k+\frac{1}{2}} - e_i^{k-\frac{1}{2}} \right) = -P_i^{k-\frac{1}{2}} \left(A_{i+\frac{1}{2}}^{k-\frac{1}{2}} u_{i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^{k-\frac{1}{2}} u_{i-\frac{1}{2}}^k \right) + m_i \kappa_{a,i}^{k-\frac{1}{2}} c \left[\mathcal{E}_i^k - a T_i^{4,k-\frac{1}{2}} - 4a T_i^{3,k-\frac{1}{2}} \left(e_i^{k+\frac{1}{2}} - e_i^{k-\frac{1}{2}} \right) / (2C_v) \right], \quad (17)$$

$$\begin{aligned} \frac{m_i}{\Delta t^k} \left(\frac{\mathcal{E}_i^{k+\frac{1}{2}}}{\rho_i^{k+\frac{1}{2}}} - \frac{\mathcal{E}_i^{k-\frac{1}{2}}}{\rho_i^{k-\frac{1}{2}}} \right) + A_{i+\frac{1}{2}}^k \mathcal{F}_{0,i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^k \mathcal{F}_{0,i-\frac{1}{2}}^k = \\ -\frac{1}{3} \mathcal{E}_i^{k-\frac{1}{2}} \left(A_{i+\frac{1}{2}}^{k-\frac{1}{2}} u_{i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^{k-\frac{1}{2}} u_{i-\frac{1}{2}}^k \right) + \\ m_i \kappa_{a,i}^{k-\frac{1}{2}} c \left[a T_i^{4,k-\frac{1}{2}} + 4a T_i^{3,k-\frac{1}{2}} \left(e_i^{k+\frac{1}{2}} - e_i^{k-\frac{1}{2}} \right) / (2C_v) - \mathcal{E}_i^k \right], \end{aligned} \quad (18)$$

where

$$\mathcal{F}_{0,i+\frac{1}{2}}^k = \frac{-2c (\mathcal{E}_{i+1}^k - \mathcal{E}_i^k)}{3 \left(\rho_i^k \Delta r_i^k \kappa_{t,i,i+\frac{1}{2}}^{k-\frac{1}{2}} + \rho_{i+1}^k \Delta r_{i+1}^k \kappa_{t,i+1,i+\frac{1}{2}}^{k-\frac{1}{2}} \right)}, \quad (18a)$$

$$\Delta r_i = r_{i+\frac{1}{2}} - r_{i-\frac{1}{2}}, \quad (18b)$$

$$T_i^{n,k-\frac{1}{2}} = \left[T_i^{k-\frac{1}{2}} \right]^n, \quad (18c)$$

and $\kappa_{t,i,i+\frac{1}{2}}$ denotes the opacity in cell i evaluated at the following vertex temperature:

$$T_{i+\frac{1}{2}} = \left[(T_i^4 + T_{i+1}^4) / 2 \right]^{\frac{1}{4}}. \quad (19)$$

Note that $T_{i+\frac{1}{2}}$ carries the time indexing of κ_t . The system defined by Eqs. (17) and (18) has terms that are treated both implicitly and explicitly in time. The explicit approximation is based upon the Crank-Nicholson time discretization. The specific material internal energy densities at time level k can be eliminated from Eq. (18) via Eq. (16):

$$\begin{aligned} \frac{m_i}{\Delta t^k} \left(\frac{\mathcal{E}_i^{k+\frac{1}{2}}}{\rho_i^{k+\frac{1}{2}}} - \frac{\mathcal{E}_i^{k-\frac{1}{2}}}{\rho_i^{k-\frac{1}{2}}} \right) + A_{i+\frac{1}{2}}^k \mathcal{F}_{0,i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^k \mathcal{F}_{0,i-\frac{1}{2}}^k = \\ m_i \kappa_{a,i}^{k-\frac{1}{2}} c \left(1 - \nu_i^{k-\frac{1}{2}} \right) \left(a T_i^{4,k-\frac{1}{2}} - \mathcal{E}_i^k \right) + \\ \nu_i^{k-\frac{1}{2}} \xi_i^{k-\frac{1}{2}} - \frac{1}{3} \mathcal{E}_i^{k-\frac{1}{2}} \left(A_{i+\frac{1}{2}}^{k-\frac{1}{2}} u_{i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^{k-\frac{1}{2}} u_{i-\frac{1}{2}}^k \right), \end{aligned} \quad (20)$$

where

$$\nu_i^{k-\frac{1}{2}} = \frac{\Delta t^k \kappa_{a,i}^{k-\frac{1}{2}} c 2a T_i^{3,k-\frac{1}{2}}}{C_{v,i} + \Delta t^k \kappa_{a,i}^{k-\frac{1}{2}} c 2a T_i^{3,k-\frac{1}{2}}}, \quad (20a)$$

and

$$\xi_i^{k-\frac{1}{2}} = -P_i^{k-\frac{1}{2}} \left(A_{i+\frac{1}{2}}^{k-\frac{1}{2}} u_{i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^{k-\frac{1}{2}} u_{i-\frac{1}{2}}^k \right). \quad (20b)$$

The solution of Eq. (20) requires the inversion of a tridiagonal matrix. A direct technique is the most efficient. Once Eq. (20) has been solved, the material internal specific energy densities can be calculated via the following expression:

$$\begin{aligned} e_i^{k+\frac{1}{2}} = e_i^{k-\frac{1}{2}} + \\ \left\{ \frac{\Delta t^k C_{v,i} \left[m_i \kappa_{a,i}^{k-\frac{1}{2}} c \left(\mathcal{E}_i^k - a T_i^{4,k-\frac{1}{2}} \right) + \xi_i^{k-\frac{1}{2}} \right]}{m_i C_{v,i} + \Delta t^k m_i \kappa_{a,i}^{k-\frac{1}{2}} c 2a T_i^{3,k-\frac{1}{2}}} \right\}. \end{aligned} \quad (21)$$

5. Calculate material temperatures and pressures at $t^{k+\frac{1}{2}}$:

$$T_i^{k+\frac{1}{2}} = \frac{e_i^{k+\frac{1}{2}}}{C_{v,i}} , \quad (22)$$

$$P_i^{k+\frac{1}{2}} = (\gamma - 1) \rho_i^{k+\frac{1}{2}} e_i^{k+\frac{1}{2}} . \quad (23)$$

The corrector step can be described as follows. A superscript "p" indicates a quantity computed in the predictor step.

1. We first solve for the velocities.

$$\frac{m_{i+\frac{1}{2}}}{\Delta t^k} \left(u_{i+\frac{1}{2}}^{k+\frac{1}{2}} - u_{i+\frac{1}{2}}^{k-\frac{1}{2}} \right) = -A_{i+\frac{1}{2}}^{p,k} \left(P_{i+1}^{p,k} + \frac{1}{3} \mathcal{E}_{i+1}^{p,k} - P_i^{p,k} - \frac{1}{3} \mathcal{E}_i^{p,k} \right) . \quad (24)$$

This is a semi-explicit expression. The only difference between Eq. (14) and Eq. (24) is the use of the predicted areas and pressures.

2. Next we solve for the radii at $t^{k+\frac{1}{2}}$.

$$r_{i+\frac{1}{2}}^{k+\frac{1}{2}} = r_{i+\frac{1}{2}}^{k-\frac{1}{2}} + u_{i+\frac{1}{2}}^k \Delta t^k . \quad (25)$$

3. Next we solve for the mass densities at $t^{k+\frac{1}{2}}$.

$$\rho_i^{k+\frac{1}{2}} = m_i / V_i^{k+\frac{1}{2}} . \quad (26)$$

4. Next we simultaneously solve the following two equations for the material internal specific energy densities and the radiation energy densities at $t^{k+\frac{1}{2}}$.

$$\frac{m_i}{\Delta t^k} \left(e_i^{k+\frac{1}{2}} - e_i^{k-\frac{1}{2}} \right) = -P_i^{p,k} \left(A_{i+\frac{1}{2}}^{p,k} u_{i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^{p,k} u_{i-\frac{1}{2}}^k \right) +$$

$$m_i \kappa_{a,i}^{p,k} c \left[\mathcal{E}_i^k - aT_i^{p,4,k} - 4aT_i^{p,3,k+\frac{1}{2}} \left(e_i^{k+\frac{1}{2}} - e_i^{p,k+\frac{1}{2}} \right) / (2C_{v,i}) \right], \quad (27)$$

$$\begin{aligned} \frac{m_i}{\Delta t^k} \left(\frac{\mathcal{E}_i^{k+\frac{1}{2}}}{\rho_i^{k+\frac{1}{2}}} - \frac{\mathcal{E}_i^{k-\frac{1}{2}}}{\rho_i^{k-\frac{1}{2}}} \right) + A_{i+\frac{1}{2}}^k \mathcal{F}_{0,i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^k \mathcal{F}_{0,i-\frac{1}{2}}^k = \\ -\frac{1}{3} \mathcal{E}_i^{p,k} \left(A_{i+\frac{1}{2}}^{p,k} u_{i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^{p,k} u_{i-\frac{1}{2}}^k \right) + \\ m_i \kappa_{a,i}^{p,k} c \left[aT_i^{p,4,k} + 4aT_i^{p,3,k+\frac{1}{2}} \left(e_i^{k+\frac{1}{2}} - e_i^{p,k+\frac{1}{2}} \right) / (2C_{v,i}) - \mathcal{E}_i^k \right], \end{aligned} \quad (28)$$

where

$$\mathcal{F}_{0,i+\frac{1}{2}}^k = \frac{-2c (\mathcal{E}_{i+1}^k - \mathcal{E}_i^k)}{3 \left(\rho_i^k \Delta x_i^k \kappa_{t,i+\frac{1}{2}}^{p,k} + \rho_{i+1}^k \Delta x_{i+1}^k \kappa_{t,i+1,i+\frac{1}{2}}^{p,k} \right)}. \quad (28a)$$

$$T_i^{p,n,k+\frac{1}{2}} = \left[T_i^{p,k+\frac{1}{2}} \right]^n, \quad (28b)$$

and

$$T_i^{p,n,k} = \left[\left(T_i^{k-\frac{1}{2}} \right)^n + \left(T_i^{p,k+\frac{1}{2}} \right)^n \right] / 2. \quad (28c)$$

The specific material internal energy densities at time level $k + \frac{1}{2}$ can be eliminated from Eq. (28) via Eq. (27):

$$\begin{aligned} \frac{m_i}{\Delta t^k} \left(\frac{\mathcal{E}_i^{k+\frac{1}{2}}}{\rho_i^{k+\frac{1}{2}}} - \frac{\mathcal{E}_i^{k-\frac{1}{2}}}{\rho_i^{k-\frac{1}{2}}} \right) + A_{i+\frac{1}{2}}^k \mathcal{F}_{0,i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^k \mathcal{F}_{0,i-\frac{1}{2}}^k = \\ m_i \kappa_{a,i}^{p,k} c (1 - \nu_i^k) \left(aT_i^{p,4,k} - \mathcal{E}_i^k \right) + \\ \nu_i^k \xi_i^k - \frac{1}{3} \mathcal{E}_i^{p,k} \left(A_{i+\frac{1}{2}}^{p,k} u_{i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^{p,k} u_{i-\frac{1}{2}}^k \right), \end{aligned} \quad (29)$$

where

$$\nu_i^k = \frac{\Delta t^k \kappa_{a,i}^{p,k} c 2aT_i^{p,3,k+\frac{1}{2}}}{C_{v,i} + \Delta t^k \kappa_{a,i}^{p,k} c 2aT_i^{p,3,k+\frac{1}{2}}}, \quad (29a)$$

and

$$\xi_i^k = -\frac{m_i}{\Delta t^k} \left(e_i^{p,k+\frac{1}{2}} - e_i^{k-\frac{1}{2}} \right) - P_i^{p,k} \left(A_{i+\frac{1}{2}}^{p,k} u_{i+\frac{1}{2}}^k - A_{i-\frac{1}{2}}^{p,k} u_{i-\frac{1}{2}}^k \right) . \quad (29b)$$

Once Eq. (29) has been solved, the material internal specific energy densities can be calculated via the following expression:

$$e_i^{k+\frac{1}{2}} = e_i^{p,k+\frac{1}{2}} + \left\{ \frac{\Delta t^k C_{v,i} \left[m_i \kappa_{a,i}^{p,k} c \left(\mathcal{E}_i^k - a T_i^{p,4,k} \right) + \xi_i^k \right]}{m_i C_{v,i} + \Delta t^k m_i \kappa_{a,i}^{p,k} c 2a T_i^{p,3,k+\frac{1}{2}}} \right\} . \quad (30)$$

5. Calculate material temperatures and pressures at $t^{k+\frac{1}{2}}$:

$$T_i^{k+\frac{1}{2}} = \frac{e_i^{k+\frac{1}{2}}}{C_v} , \quad (31)$$

$$P_i^{k+\frac{1}{2}} = (\gamma - 1) \rho_i^{k+\frac{1}{2}} e_i^{k+\frac{1}{2}} . \quad (32)$$

2.1 Boundary Conditions

Only two of the equations in the RHGD set require boundary conditions: the velocity equation and the radiation diffusion equation. Boundary conditions for the cell-centered diffusion discretization that we have used are standard and completely analogous to those used for neutronics. They arise from the application of a continuity-of-flux principle that is also applied at cell interfaces on the mesh interior to obtain the expressions for the flux

given in Eqs. (18a) and (28a). See Lecture 12 of the NUEN-625 notes for a complete description. The radiation energy flux at the left boundary is respectively given by

$$\mathcal{F}_{0,\frac{1}{2}} = -\frac{2c}{3\rho_1\Delta r_1\kappa_{t,1,\frac{1}{2}} + 4} (\mathcal{E}_1 - \mathcal{E}_L^b) , \quad (33)$$

where \mathcal{E}_L^b is the extrapolated value of the radiation density intensity at the left boundary and $\kappa_{t,1,\frac{1}{2}}$ is evaluated at the following edge temperature:

$$T_{\frac{1}{2}} = \left[\left(\frac{1}{a} \mathcal{E}_L^b + T_1^4 \right) / 2 \right]^{\frac{1}{4}} . \quad (34)$$

Analogous expressions for the right boundary are

$$\mathcal{F}_{0,N+\frac{1}{2}} = -\frac{2c}{3\rho_N\Delta r_N\kappa_{t,N,N+\frac{1}{2}} + 4} (\mathcal{E}_R^b - \mathcal{E}_N) , \quad (35)$$

and

$$T_{N+\frac{1}{2}} = \left[\left(\frac{1}{a} \mathcal{E}_R^b + T_N^4 \right) / 2 \right]^{\frac{1}{4}} , \quad (36)$$

where N is the total number of spatial cells. Note that we have suppressed all time indexing in these equations because different indexings occur in the predictor and corrector steps. The interior and extrapolated densities must carry the same time index as the boundary values. A vacuum condition simply corresponds to a zero extrapolated density value, and a reflective condition corresponds to equal cell center and extrapolated densities, or equivalently, a zero boundary flux. The extrapolated density is generally defined in terms of an effective black-body density, i.e., $\mathcal{E}^b \equiv aT^4$.

The velocities may be explicitly specified at the boundaries if desired, or an equation for the boundary velocity can be defined that is analogous to those defined on the mesh interior. More specifically, the velocity equations at the left and right boundaries are respectively given by

$$\frac{m_{\frac{1}{2}}}{\Delta t^k} \left(u_{\frac{1}{2}}^{k+\frac{1}{2}} - u_{\frac{1}{2}}^{k-\frac{1}{2}} \right) = -A_{\frac{1}{2}} \left(P_1 + \frac{1}{3}\mathcal{E}_1 - P_{\frac{1}{2}} - \frac{1}{3}\mathcal{E}_{\frac{1}{2}} \right), \quad (37)$$

and

$$\frac{m_{N+\frac{1}{2}}}{\Delta t^k} \left(u_{N+\frac{1}{2}}^{k+\frac{1}{2}} - u_{N+\frac{1}{2}}^{k-\frac{1}{2}} \right) = -A_{N+\frac{1}{2}} \left(P_{N+\frac{1}{2}} + \frac{1}{3}\mathcal{E}_{N+\frac{1}{2}} - P_N - \frac{1}{3}\mathcal{E}_N \right), \quad (38)$$

where $m_{\frac{1}{2}} = \frac{1}{2}m_1$ and $m_{N+\frac{1}{2}} = \frac{1}{2}m_N$, and the surface intensities at the left and right boundaries are respectively given by:

$$\mathcal{E}_{\frac{1}{2}} = \frac{3\rho_1\Delta r_1\kappa_{t,1}\mathcal{E}_L^b + 4\mathcal{E}_1}{3\rho_1\Delta r_1\kappa_{t,1} + 4}, \quad (39)$$

and

$$\mathcal{E}_{N+\frac{1}{2}} = \frac{3\rho_N\Delta r_N\kappa_{t,N}\mathcal{E}_R^b + 4\mathcal{E}_N}{3\rho_N\Delta r_N\kappa_{t,N} + 4}. \quad (40)$$

Note that these equations require the definition of a boundary material pressure. A boundary pressure can be explicitly specified by the physical boundary condition, but if the boundary velocity is explicitly specified, these can still be used with the boundary pressure computed to yield the desired boundary velocity. The latter procedure may seem more complicated than necessary, but it is useful if one is to compute a global energy balance expression (see the section on global energy conservation). Note that the quantities on

the right sides of Eqs. (37) through (40) carry different time indexes (which have been suppressed) dependent upon whether they appear in the predictor or corrector steps.

3 Global Energy Conservation

One can derive a total energy conservation statement for radiation-hydrodynamics system that is evaluated at the end of each time step. This statement can be a very useful for code verification. To derive this expression for the corrector step:

1. Multiply the material velocity equation for each node by the time-centered velocity at that node, to obtain a balance equation for the kinetic energy at each node.
2. Sum the kinetic energy equation over all nodes and the internal energy and radiation energy equations over all cells to obtain an equation for the time-derivative of the total energy in the system.
3. Integrate this total energy equation from $t^{\frac{1}{2}}$ to $t^{k+\frac{1}{2}}$.

The conservation statement that results from this process can be expressed as follows:

$$\sum_{i=0}^N \frac{1}{2} m_{i+\frac{1}{2}} (u_{i+\frac{1}{2}}^{k+\frac{1}{2}})^2 + \sum_{i=1}^N m_i \left(e_i^{k+\frac{1}{2}} + \mathcal{E}_i^{k+\frac{1}{2}} / \rho_i^{k+\frac{1}{2}} \right) -$$

$$\sum_{i=0}^N \frac{1}{2} m_{i+\frac{1}{2}} (u_{i+\frac{1}{2}}^{\frac{1}{2}})^2 + \sum_{i=1}^N m_i \left(e_i^{\frac{1}{2}} + \mathcal{E}_i^{\frac{1}{2}} / \rho_i^{\frac{1}{2}} \right) =$$

$$\begin{aligned}
& \sum_{k'=1}^k \left[A_{\frac{1}{2}}^{k'} \overrightarrow{\mathcal{F}}_{0, \frac{1}{2}}^{k'} - A_{N+\frac{1}{2}}^{k'} \overrightarrow{\mathcal{F}}_{0, N+\frac{1}{2}}^{k'} \right] \Delta t^{k'} + \\
& \sum_{k'=1}^k \left[A_{\frac{1}{2}}^{p, k'} \left(\frac{1}{3} \mathcal{E}_{\frac{1}{2}}^{p, k'} + P_{\frac{1}{2}}^{p, k'} \right) \overrightarrow{u}_{\frac{1}{2}}^{k'} - A_{N+\frac{1}{2}}^{p, k'} \left(\frac{1}{3} \mathcal{E}_{N+\frac{1}{2}}^{p, k'} + P_{N+\frac{1}{2}}^{p, k'} \right) \overrightarrow{u}_{N+\frac{1}{2}}^{k'} \right] \Delta t^{k'}. \quad (41)
\end{aligned}$$

4 Artificial Viscosity

Although it may not be obvious, the discrete hydrodynamics and radiation-hydrodynamics equations that we have described require the addition of numerical diffusion to avoid excessive oscillation. This is achieved through what is called an artificial viscosity term. It is analogous to physical viscosity, but is only activated in cells that are compressing. The artificial viscosity is simply added to the material pressure, i.e., replace P in the equations with $P + Q$, where the artificial viscosity Q is defined as follows:

$$Q_i = \begin{cases} (1 - \Gamma) \bar{\rho} |\Delta u_i| \left(c_Q |\Delta u_i| + \sqrt{c_Q^2 \Delta u_i^2 + \bar{c}_s^2} \right) & \text{if } u_{i+\frac{1}{2}} < u_{i-\frac{1}{2}}, \\ 0, & \text{otherwise,} \end{cases} \quad (42)$$

where

$$\bar{\rho} = 2 \frac{\rho^- \rho^+}{\rho^- + \rho^+}, \quad (43a)$$

$$\rho^- = \frac{\rho_{i-1} \Delta r_{i-1} + \rho_i \Delta r_i}{\Delta r_{i-1} + \Delta r_i}, \quad (43b)$$

$$\rho^+ = \frac{\rho_i \Delta r_i + \rho_{i+1} \Delta r_{i+1}}{\Delta r_i + \Delta r_{i+1}}, \quad (43c)$$

$$c_Q = (\gamma + 1) / 4, \quad (43d)$$

$$\Delta u_i = u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}, \quad (43e)$$

$$\bar{c}_s = \min(c_s^-, c_s^+), \quad (43f)$$

$$c_s^- = \frac{\sqrt{\gamma p_{i-1}/\rho_{i-1}} \Delta r_{i-1} + \sqrt{\gamma p_i/\rho_i} \Delta r_i}{\Delta r_{i-1} + \Delta r_i}, \quad (43g)$$

$$c_s^+ = \frac{\sqrt{\gamma p_i/\rho_i} \Delta r_i + \sqrt{\gamma p_{i+1}/\rho_{i+1}} \Delta r_{i+1}}{\Delta r_i + \Delta r_{i+1}}, \quad (43h)$$

and

$$\Gamma = \max \left\{ 0, \min \left[1, 2R^-, 2R^+, \frac{1}{2} (R^- + R^+) \right] \right\}, \quad (43i)$$

where

$$R^- = \frac{\Delta u_{i-1} \Delta r_i}{\Delta r_{i-1} \Delta u_i}, \quad (43j)$$

$$R^+ = \frac{\Delta u_{i+1} \Delta r_i}{\Delta r_{i+1} \Delta u_i}. \quad (43k)$$

For boundary cells, set the non-existent value of ρ^\pm to the mass density in the boundary cell, set the non-existent value of c_s^\pm to the sound speed in the boundary cell, and set the non-existent value of R^\pm to one. Equation (42) is known as the Kuropatenko artificial viscosity. It is just one of very many artificial viscosities that have been defined. Note that the artificial viscosity is only evaluated at the beginning of each time step. Thus the same values are used for both the predictor and corrector steps. Also note that the above definition is used for *all* geometries - not just slab geometry. Finally, we note that there are no boundary values of the artificial viscosity even when boundary material and radiation pressures are specified.