NUEN 627

Lecture 12

Scalar Conservation Equations

Scalar conservation equations have the following general form:

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0, \qquad (1)$$

where $F(u)$ is called the flux and is generally a nonlinear convex function of u: $F''(u) > 0$. Eq. (1) is conservative because the total integral of $u(x,t)$ over $(-\infty, +\infty)$ is constant in time assuming $F(\infty,t) = F(-\infty,t) = 0$, we find that

$$\int_{-\infty}^{+\infty} \frac{\partial u}{\partial t} dx = -\left(F(\infty,t) - F(-\infty,t)\right) = 0,$$

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{+\infty} u \, dx\right) = 0 \qquad (2)$$

From (2) it follows from (2) that

$$\int_{-\infty}^{+\infty} u \, dx = \text{constant}, \qquad (3)$$

If the solution is smooth, we can manipulate Eq.(1) as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} = 0 \quad . \qquad (4)$$

Note that $\frac{\partial F}{\partial u}$ has units of velocity, and it can be interpreted as as such.

Characteristics

Given that $\frac{\partial F}{\partial u}(u(x,t))$ is a velocity, consider the following ODE for a trajectory or "characteristic", $X(t)$:

$$\frac{dX}{dt} = \frac{\partial F}{\partial u}\left\{u[X(t),t]\right\} \quad , \quad X(0) = X_0 . \qquad (5)$$

Note that $X(t)$ represents the trajectory that a point starting at $(x = x_0, t=0)$ will follow with the velocity defined by the right side of Eq.(5). We can re-express Eq.(4) as

$$\frac{\partial u}{\partial t} + \frac{dX}{dt}\frac{\partial u}{\partial x} = 0 \quad . \qquad (6)$$

Recognizing that Eq. (6) is just the Lagrangian derivative with the velocity defined by $X(t)$, we see that this equation can be written as

$$\frac{Du}{Dt} = 0 , \tag{7}$$

which implies that the solution is <u>constant</u> along the characteristic. Further note that if u is constant, $\frac{\partial F}{\partial u}(u)$ is constant! Thus all characteristics are straight lines:

$$X(t) = X_0 + F'[u_0(X_0)]\,t \tag{8}$$

where

$$F' \equiv \frac{\partial F}{\partial u} , \tag{8a}$$

and

$$u_0(X) = u(X_0, 0) . \tag{8b}$$

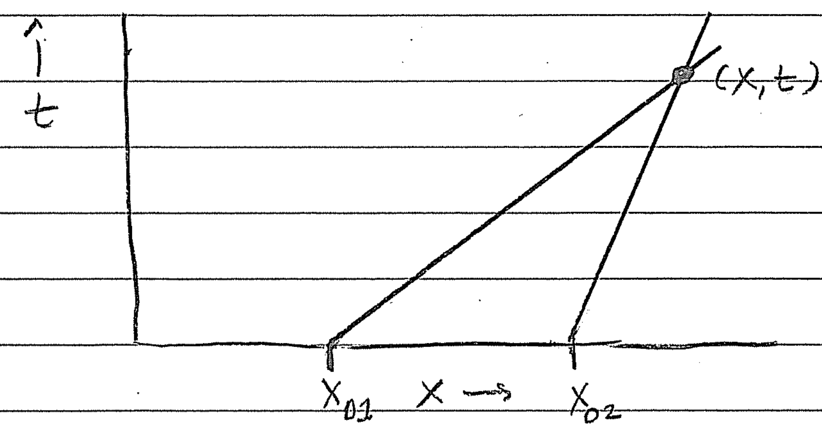
It follows from Eq (8) that if

$$X = X_0 + F'[u_0(X_0)]\,t, \tag{9}$$

then

$$u(X,t) = u(X_0) = u(X - F'[u_0(X_0)]\,t), \tag{10}$$

It is important to realize that the solution given by Eq(10) is only good as long as there is only one x_0 for which Eq.(9) is satisfied, i.e, as long as two or more characteristics do not cross at the point (x,t):



Crossing characteristics imply a discontinuity in the solution. Our equations are not well-defined with discontinuities. Further analysis is required to understand what happens with discontinuities.

Linear Advection

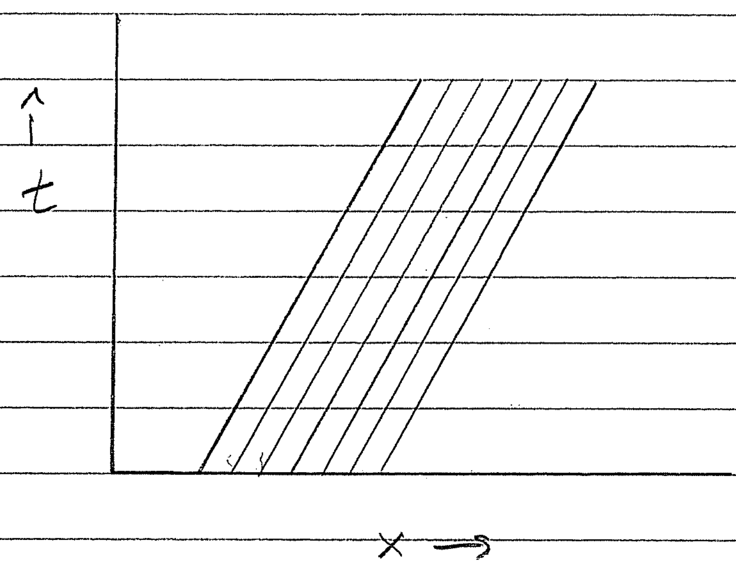
When $F(u) = au$, where "a" is a constant, we obtain the linear advection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0. \qquad (11)$$

Note that this equation yields the same

slope for every characteristic:

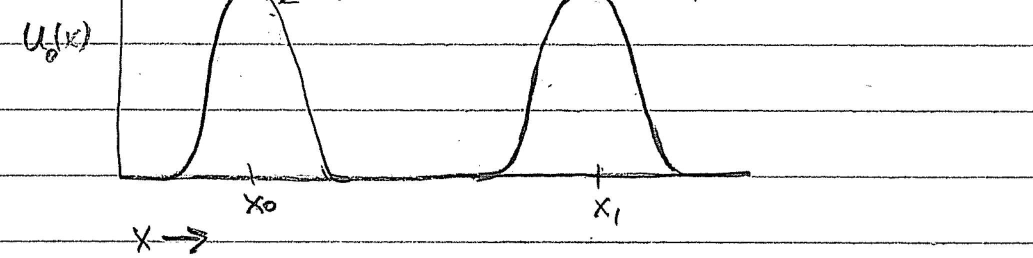
$$\frac{dx}{dt} = a. \qquad (12)$$



The above picture corresponds to $a>0$. Note that the characteristics never cross, so the solution given by Eq. (10) is always valid.

$$u(x,t) = u_0(x-at). \qquad (13)$$

This corresponds to simple advection. The function $u_0(x)$ just moves down the x-axis with velocity "a":



Burgers' Equation

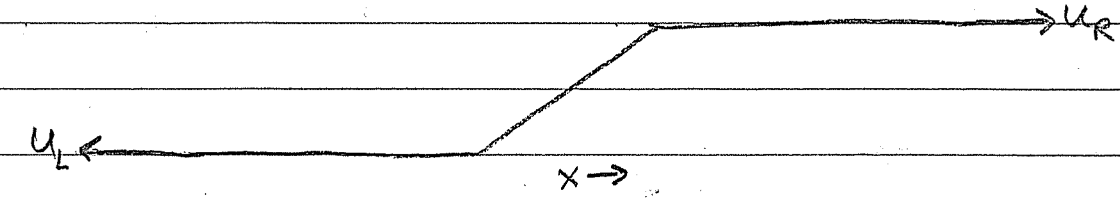
This is a simple equation that yields all the complex behavior of non-linear conservation equations: $F(u) = \frac{1}{2} u^2$, so

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \tag{14}$$

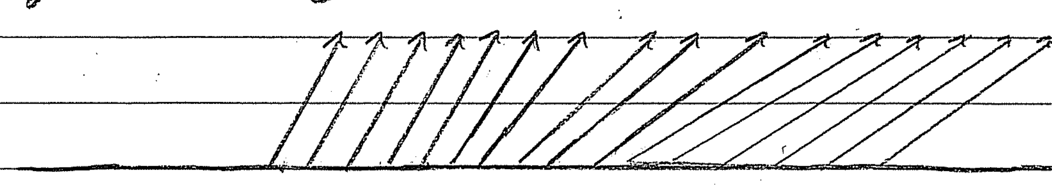
The velocity is the unknown itself. The characteristics are defined by

$$X(t) = x_0 + u_0(x_0) t. \tag{15}$$

Consider the following distribution for $u_0(x)$:

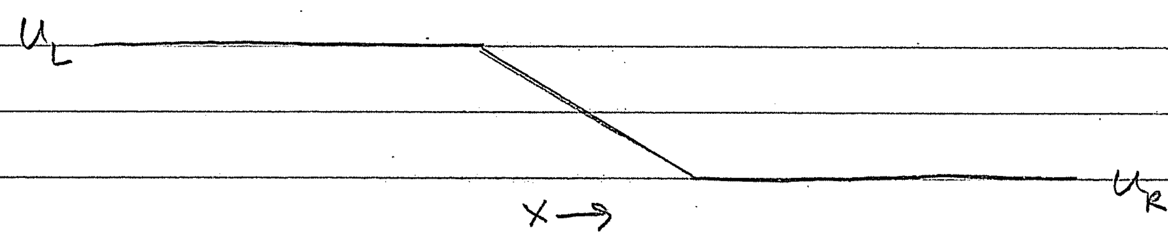


where $u_R > u_L > 0$. The corresponding characteristics are qualitatively given as follows:

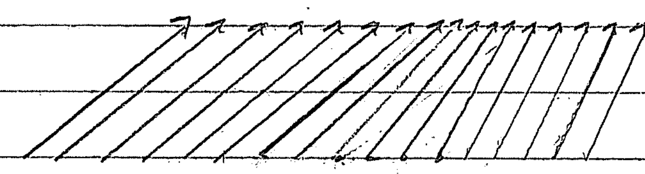


Note that the characteristics never cross, and the initial distribution gets more and more "spread out" with time because points to the right move faster than points to the left. This correspond to a "rarefaction" wave.

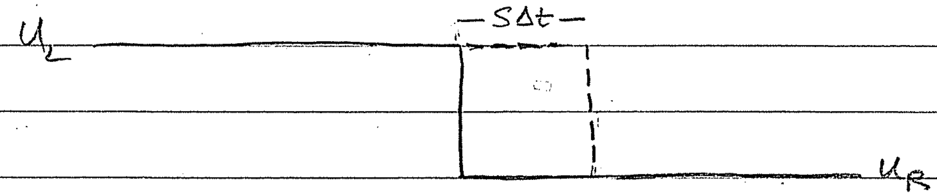
Now let us consider the following distribution:



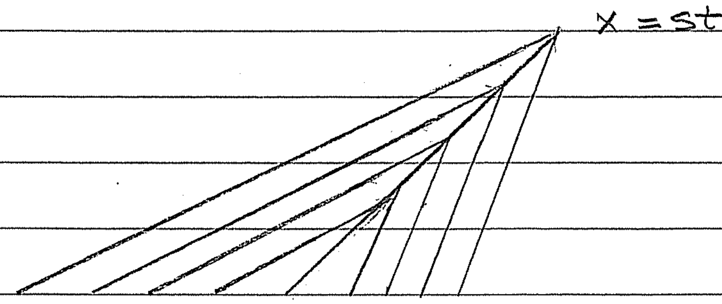
where $u_L > u_R > 0$. The corresponding characteristics are qualitatively given as follows:



In this case, the initial distribution "steepens" in time because the points left are moving faster than the points moving to the right. This eventually results in the formation of a shock when the characteristics begin to cross.



The shock (or discontinuity) propagates from left to right with speed $s = \frac{1}{2}(u_L + u_R)$. All characteristics move into the shock.



Weak Solutions

To deal with discontinuous solutions we must introduce the concept of weak solutions. We first consider the space of two-dimensional functions that are continuously differentiable and have compact support. Compact support means that the functions are zero outside some bounded domain in $R \times R^+$, where $R \equiv (-\infty, +\infty)$ and $R^+ \equiv (0, \infty)$. We denote an arbitrary element of $C_0^1(R \times R^+)$ by ϕ. Next we multiply Eq. (1) by ϕ and integrate over $R \times R^+$:

$$\int_0^{\infty}\int_{-\infty}^{+\infty}\left[\phi\frac{\partial u}{\partial t}+\phi\frac{\partial}{\partial x}[F(u)]\right]dx\,dt=0 \qquad (16)$$

Next we integrate by parts:

$$\int_0^{\infty}\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial t}(\phi u)-u\frac{\partial\phi}{\partial t}+\frac{\partial}{\partial x}[\phi F(u)]-F(u)\frac{\partial\phi}{\partial x}\right]dx\,dt=0,$$

$$-\int_{-\infty}^{+\infty}\phi(x,0)\,u(x,0)\,dx-\int_0^{\infty}\int_{-\infty}^{+\infty}\left[u\frac{\partial\phi}{\partial t}+F(u)\frac{\partial\phi}{\partial x}\right]dx\,dt=0,$$

$$\int_0^{\infty}\int_{-\infty}^{+\infty}\left[u\frac{\partial\phi}{\partial t}+F(u)\frac{\partial\phi}{\partial x}\right]dx\,dt=-\int_{-\infty}^{+\infty}\phi(x,0)\,u(x,0). \qquad (17)$$

Note that the compact support property of ϕ ensures that all the boundary terms except those related to the initial condition are zero. We define $u(x,t)$ to be a weak solution of Eq.(1) if Eq.(17) holds for all $\phi \in C_0^1(R\times R^+)$. Note that there are no derivatives of u or F appearing in Eq.(17), so discontinuities are not a problem. Given an appropriate initial condition, Eq.(1) has a unique strong solution, but there may be any number of weak solutions. However, only <u>one</u> weak solution will be physically correct. There are two approaches to identifying the unique physically-correct weak solution. The first is to add dissipation to the equation. For instance, if we replace Burgers' equation with

$$\frac{\partial u}{\partial t} + \frac{\partial(\frac{1}{2}u^2)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}, \tag{18}$$

we will find that the physically correct weak solution to Eq. (17) will be obtained from Eq. (18) in the limit as $\epsilon \to 0$. The other approach is to use entropy conditions.

Entropy Pairs

Assume some entropy function, $\eta(u)$, and an associated entropy flux, $\psi(u)$, satisfy the following equation:

$$\frac{\partial \eta(u)}{\partial t} + \frac{\partial}{\partial x}\psi(u) = 0. \tag{19}$$

Assuming smoothness, we can rewrite Eq (19) as

$$\frac{\partial \eta}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x} = 0. \tag{20}$$

If we multiply Eq (1) by $\frac{\partial \eta}{\partial u}$, we get

$$\frac{\partial \eta}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial u}\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} = 0 \quad . \qquad (21)$$

Comparing Eqs. (20) and (21), we find that that they will be equal if

$$\frac{\partial \psi}{\partial \mu} = \frac{\partial \eta}{\partial \mu}\frac{\partial F}{\partial \mu} . \qquad (22)$$

Thus given any $\eta(u)$, we can use Eq. (22) to solve for the corresponding entropy flux, thereby ensuring that Eq. (20) is satisfied by the entropy pair, $\{\eta(u), \psi(u)\}$, under the assumption that u is smooth.

If u is not smooth, then under the assumption that η is convex, i.e.,

$$\frac{\partial^2 \eta}{\partial u^2} > 0 , \qquad (23)$$

the following theorem applies:

The physically correct weak solution to Eq. (1) is $u(x,t)$ if for all convex entropy functions and corresponding entropy fluxes, the inequality

$$\frac{\partial \eta}{\partial t} + \frac{\partial \psi}{\partial x} \le 0 \qquad (24)$$

is satisfied in a weak sense. To demonstrate this we first consider the following viscous equation:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} . \qquad (25)$$

As previously discussed, the solution to Eq.(25) converges to the physically correct weak solution to Eq.(1) in the limit as $\epsilon \to 0$. Multiplying Eq.(25) by $\frac{\partial \eta}{\partial u}$, we obtain

$$\frac{\partial \eta}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial u}\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} = \epsilon \frac{\partial \eta}{\partial u}\frac{\partial^2 u}{\partial x^2} ,$$

which is equivalent to

$$\frac{\partial \eta}{\partial t} + \frac{\partial \psi}{\partial x} = \epsilon \left\{ \frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x}\right) - \frac{\partial^2 \eta}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 \right\}. \qquad (26)$$

Integrating Eq.(26) over the arbitrary domain, $[x_1, x_2] \times [t_1, t_2]$ gives

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left(\frac{\partial \eta}{\partial t} + \frac{\partial \psi}{\partial x}\right) dx\,dt = \int_{t_1}^{t_2} \epsilon \left\{ \left(\frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x}\right)\Bigg|_{x=x_2} - \left(\frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x}\right)\Bigg|_{x=x_1} \right\} dt$$

$$- \epsilon \int_{t_1}^{t_2}\int_{x_1}^{x_2} \frac{\partial^2 \eta}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 dx\,dt. \qquad (27)$$

It can be shown that the first term on the right side of Eq. (27) must vanish as $\epsilon \to 0$ even if u becomes discontinuous at x_1 or x_2. However, if the limiting weak solution is discontinuous along a curve in the domain of integration, the second term will not vanish. Furthermore the convexity of η implies that this term will be non-positive. Thus we can conclude that the vanishing viscosity weak solution satisfies

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left(\frac{\partial \eta}{\partial t} + \frac{\partial \psi}{\partial x}\right) dx\,dt \le 0 \,. \qquad (28)$$

Since the domain of integration is arbitrary, Eq (28) implies that the vanishing viscosity weak solution satisfies Eq. (24) in a weak sense. Setting $x_1 = -\infty$ and $x_2 = +\infty$ in Eq. (28), we find that

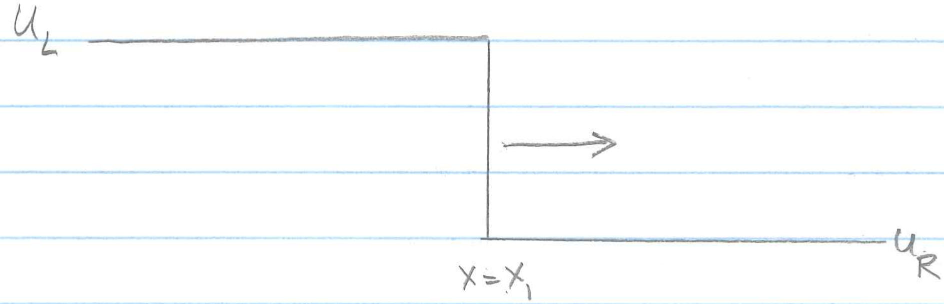
$$\int_0^\infty \eta(x,t_2)\,dx \le \int_0^\infty \eta(x,t_1)\,dx - \int_{t_1}^{t_2} \cancelto{0}{\left\{ \psi(+\infty,t) - \psi(-\infty,t)\right\}} dt,$$

$$\le \int_0^\infty \eta(x,t_1)\,dx \,.$$

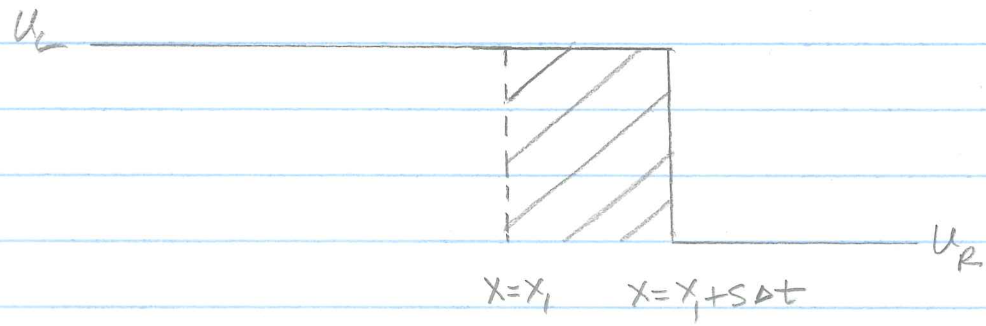
Thus the total entropy is either conserved (Smooth u) <u>decreases</u> with time. Remember that the physical entropy <u>increases</u> because $\frac{\partial^2 \eta}{\partial u^2} < 0$.

Shock Speed

A shock is a propagating discontinuity. The speed of a shock is easily derived using nothing more than the principle of conservation. For instance, assume the following shock solution to Eq.(1) at some arbitrary time, t_0,



where the shock is moving to the right at speed s. At time $t_0+\Delta t$, the solution is



If we integrate Eq(1) over $x \in [x_1, x_1+s\Delta t]$, and $t \in [t_0, t_0+\Delta t]$, we get the following:

$$\int_{t_0}^{t_0+\Delta t}\int_{X_1}^{X_1+S\Delta t}\left(\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}F(u)\right)dx\,dt=0,$$

$$\int_{X_1}^{X_1+S\Delta t}\left[u(X,t+\Delta t)-u(x,t)\right]dX+\int_{t_1}^{t_1+\Delta t}\left\{F[u(x_1+s\Delta t,t)]-F[u(x_1,t)]\right\}dt$$

$$=0,$$

$$(U_L-U_R)s\Delta t+[F(U_R)-F(U_L)]\Delta t=0 \qquad (29)$$

Solving Eq(21) for the shock speed, we get

$$S=\frac{F(U_R)-F(U_L)}{U_R-U_L}. \qquad (30)$$

There is a weak solution of Eq.(1) that has the shock traveling from right to left rather than left to right. However this is not the entropy solution. For convex $F(u)$, one can show that a shock always propagates from the higher value of u to the lower value of u. As we shall later see, one can fairly easily identify entropy solutions for the Euler equations.