NUEN 627

Lecture 13

Hyperbolic Conservation Systems

We next consider hyperbolic conservation systems, which can be written as

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial}{\partial x}\vec{F}(\vec{u}) = 0 \quad , \qquad (1)$$

where $\vec{u}$ is a vector of unknowns:

$$\vec{u} = (u_1, u_2, \ldots u_n)^T, \qquad (2)$$

and $\vec{F}(\vec{u})$ is a vector flux function,

$$\vec{F} = (F_1(\vec{u}), F_2(\vec{u}), \ldots F_n(\vec{u})). \qquad (3)$$

For the case of three spatial dimensions, Eq.(1) becomes

$$\frac{\partial \vec{u}}{\partial t} + \vec{\nabla}\cdot \vec{F}(\vec{u}) = 0 \quad ,$$

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial}{\partial x}\vec{F}_x(\vec{u}) + \frac{\partial}{\partial y}\vec{F}_y(\vec{u}) + \frac{\partial}{\partial z}\vec{F}_z(\vec{u}) = 0. \qquad (4)$$

For simplicity, we will continue assuming a 1-D spatial dependence. Assuming a smooth solution, Eq (1) becomes

$$\frac{\partial \vec{u}}{\partial t} + A \frac{\partial \vec{u}}{\partial x} = 0, \qquad (5)$$

where A is a matrix:

$$a_{ij} = \frac{\partial F_i}{\partial u_j}, \quad i = 1, N, \; j = 1, N. \qquad (6)$$

The system described by Eq (1) is hyperbolic if A is diagonalizable with real eigenvalues. In this case, Eq (5) can be expressed as

$$\frac{\partial \vec{u}}{\partial t} + R \Lambda R^{-1} \frac{\partial \vec{u}}{\partial x}, \qquad (7)$$

where

$$\Lambda = \mathrm{diag}(\lambda_1(\vec{u}), \lambda_2(\vec{u}), \ldots \lambda_N(\vec{u})), \qquad (8)$$

$\lambda_i(\vec{u})$ is the ith eigenvalue of A, and R is a matrix whose ith column is the eigenvector corresponding to λ_i. More specifically,

$$A \vec{R}_j = \lambda_j \vec{R}_j , \tag{9}$$

where $\vec{R}_j$ denotes the j'th column of A. If Eq.(7) is linear, i.e., if A is constant, we can multiply Eq.(7) by R^{-1} from the left to obtain

$$\frac{\partial \vec{w}}{\partial t} + \Lambda \frac{\partial \vec{w}}{\partial x} = 0 , \tag{10}$$

where

$$\vec{w} = R^{-1} \vec{u}. \tag{11}$$

This corresponds to a similarity transformation to the characteristic variables, $(w_1, w_2 .. w_N)^T$. Note from Eq.(10) that each characteristic unknown simply advects with a constant velocity equal to its associated eigenvalue.

In the nonlinear case, we must be a little more careful. Multiplying Eq(7) by R^{-1} from the left, we obtain:

$$R^{-1} \frac{\partial \vec{u}}{\partial t} + \Lambda R^{-1} \frac{\partial \vec{u}}{\partial x} = 0 \tag{12}$$

We cannot move R^{-1} through the space and time derivatives, but we don't really need to. Rather we define $\vec{w}$ as follows

$$\frac{\partial w_i}{\partial u_j} = (R^{-1})_{ij}, \quad i=1,N, \; j=1,N. \qquad (13a)$$

Then, for $z = t$ or x,

$$\sum_{j=1}^{N} (R^{-1})_{ij} \frac{\partial u_j}{\partial z} = \sum_{j=1}^{N} \frac{\partial w_i}{\partial u_j} \frac{\partial u_j}{\partial z} = \frac{\partial w_i}{\partial z}, \quad i=1,N. \qquad (13b)$$

So we can write Eq. (12) as

$$\frac{\partial \vec{w}}{\partial t} + \Lambda \frac{\partial \vec{w}}{\partial x}, \qquad (14)$$

or equivalently

$$\frac{\partial w_i}{\partial t} + \lambda_i(\vec{w}) \frac{\partial w_i}{\partial t}, \quad i=1,N. \qquad (15)$$

System Characteristics

We can define a characteristic trajectory for the ith characteristic unknown as follows:

$$\frac{d X_i(t)}{\partial t} = \lambda_i\{\vec{w}[X_i(t), t]\}, \quad X_i(0) = X_0. \qquad (16)$$

Evaluating Eq. (15) at $\vec{\omega}(x,t)\Big|_{x=X(t)}$, we (17) get

$$\frac{\partial \omega_1}{\partial t} + \lambda_1\{\vec{\omega}[X_1(t),t]\}\frac{\partial \omega_1}{\partial x} = 0, \quad (18)$$

or equivalently,

$$\frac{D\omega_1}{Dt} = 0. \quad (19)$$

Equation (19) implies that ω_1 is constant along $X_1(t)$, but the other characteristic variables are not necessarily constant along $X_1(t)$. This implies that λ_1 is generally not constant along $X_1(t)$, so the characteristic trajectory is generally not a straight line.

For the 1-D slab-geometry Euler equations,

$$\vec{U} = (u_1, u_2, u_3)^T$$

$$= (\rho, \rho u, E_m)^T, \quad (20)$$

where

$$E_m = \tfrac{1}{2}\rho u^2 + \rho e, \quad (20a)$$

and

$$\vec{F} = (f_1, f_2, f_3)^T, \tag{21}$$

$$= [\rho u, \rho u^2 + p, (E_m + p) u], \tag{21a}$$

$$= \left\{ u_2, \frac{u_2^2}{u_1} + \left(u_3 - \frac{1}{2}\frac{u_2^2}{u_1}\right)(\gamma-1), \left[u_3 + \left(u_3 - \frac{1}{2}\frac{u_2^2}{u_1}\right)(\gamma-1)\right]\frac{u_2}{u_1} \right\}. \tag{21b}$$

So,

$$A(\vec{U}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)\left(\frac{u_2}{u_1}\right)^2 & (3-\gamma)\frac{u_2}{u_1} & \gamma-1 \\ -\frac{\gamma u_2 u_3}{u_1^2} + (\gamma-1)\left(\frac{u_2}{u_1}\right)^3 & \frac{\gamma u_3}{u_1} - \frac{3}{2}(\gamma-1)\left(\frac{u_2}{u_1}\right)^2 & \gamma\left(\frac{u_2}{u_1}\right) \end{bmatrix} \tag{22}$$

where we have assumed an ideal gas EOS

$$p = \rho e(\gamma-1). \tag{23}$$

It is particularly useful to re-express the Jacobian matrix in terms of the speed of sound, a, and the material velocity:

$$A(U) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1 \\ \frac{1}{2}(\gamma-2)u^3 - \frac{a^2 u}{(\gamma-1)} & \frac{3-2\gamma}{2}u^2 + \frac{a^2}{\gamma-1} & \gamma u \end{bmatrix}, \tag{24}$$

where

$$a = \sqrt{\frac{\gamma P}{\rho}}. \tag{25}$$

The eigenvalues of A are

$$\begin{aligned} \lambda_1 &= u - a, \\ \lambda_2 &= u, \\ \lambda_3 &= u + a, \end{aligned} \tag{26}$$

The eigenvectors of A are

$$\vec{K}_1 = (1, u-a, H-ua)^T,$$

$$\vec{K}_2 = (1, u, \tfrac{1}{2}u^2)^T, \qquad (27)$$

$$\vec{K}_3 = (1, u+a, H+ua)^T,$$

where H is the total specific enthalpy

$$H = (E+p)/\rho. \qquad (28)$$

The Riemann Problem

The Riemann problem is defined for Eq.(1) as follows:

$$\vec{u}_0(x) = \vec{u}_L, \text{ for } x<0,$$

$$= \vec{u}_R, \text{ for } x>0,$$

where $\vec{u}_L$ and $\vec{u}_R$ are constant vectors. It is important to note that Eq.(1) is invariant to a constant scaling of the space and time coordinates. For instance, let

$$x' = \alpha x, \qquad (29)$$
$$t' = \alpha t,$$

where α is a constant. Then

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = \alpha\frac{\partial}{\partial x'},$$
$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \alpha\frac{\partial}{\partial t'}. \qquad (30)$$

Substituting from Eq.(30) into Eq.(1), we get

$$\alpha\frac{\partial\vec{u}}{\partial t'} + \alpha\frac{\partial\vec{F}}{\partial x'} = 0,$$
$$\frac{\partial\vec{u}}{\partial t'} + \frac{\partial\vec{F}}{\partial x'} = 0. \qquad (31)$$

This Eq.(1) is invariant under this scaling. In addition, the initial condition is also invariant. In particular, at $t'=0$,

$$\vec{u}_0(x') = \vec{u}_L, \quad x'<0,$$
$$\vec{u}_0(x') = \vec{u}_R, \quad x'>0. \qquad (32)$$

Thus the solution to the Riemann problem is invariant under this scaling. This implies the the solution must be of the following form:

$$u(x,t) = u\left(\frac{x}{t}\right). \qquad (33)$$

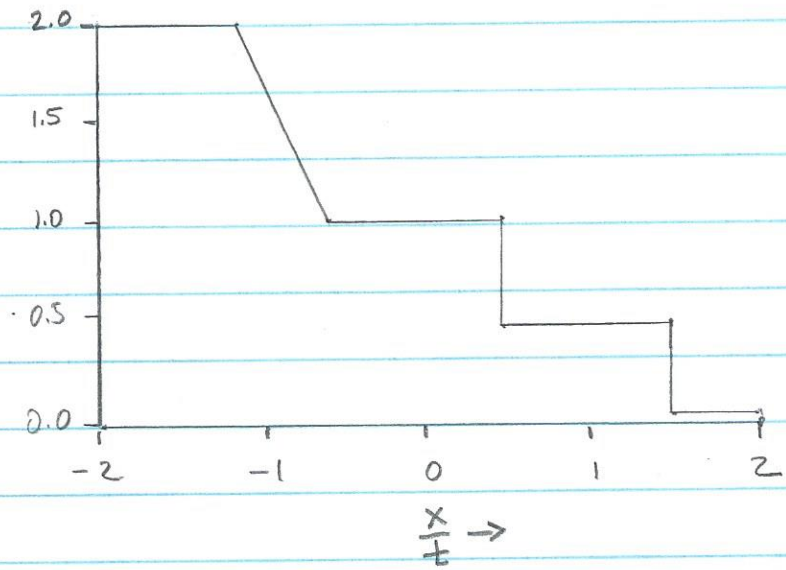
This means that waves travel at constant speed and that the solution is constant along any ray $\frac{x}{t}$ = constant.

The Shock Tube Problem

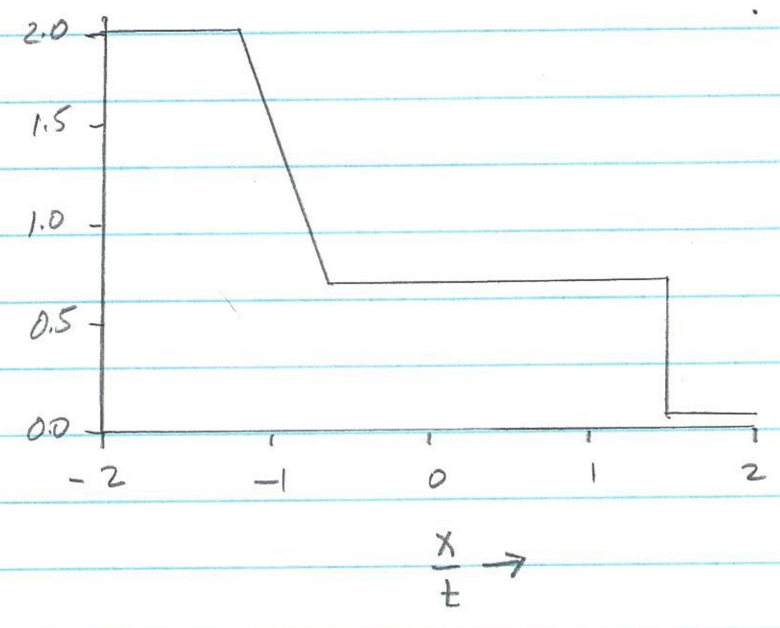
The shock-tube problem for the Euler equations is a special case of the Riemann problem characterized by a zero initial velocity everywhere. The solution to this problem is such that there are three distinct waves separating regions in which the state variables are constant. Across two of these waves there are discontinuities in some of the state variables. In particular, a shock wave propagates into the region of lower pressure, across which the density and pressure jump to higher values and all of the state variables are discontinuous. This is followed by a contact discontinuity, across which the density is discontinuous, but the velocity and pressure are constant. The third wave moves into the high pressure region (the opposite direction of the other two) and has a very different structure; all of the state variables are continuous and there is a smooth transition. This is called a rarefaction wave since the density of the gas decreases (is rarefied) as the wave passes through.

Solutions to the shock tube problem are shown on the next two pages.

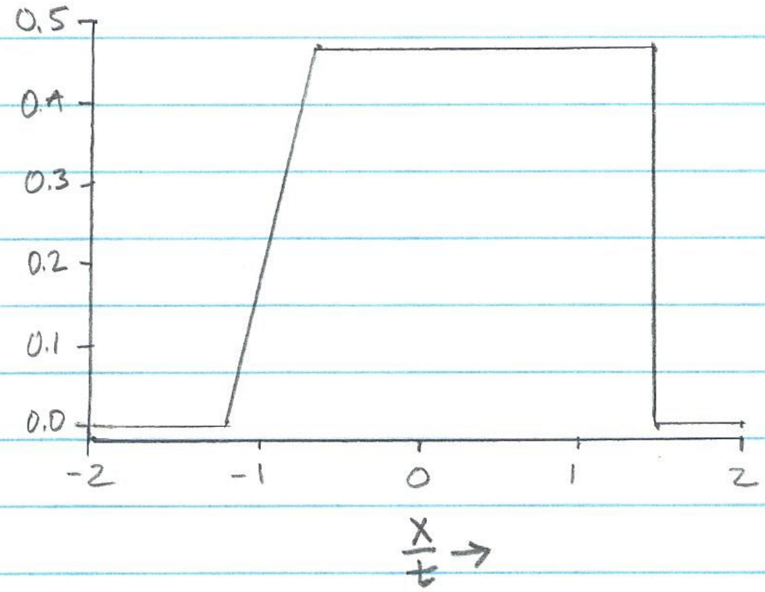
Density



Pressure



Velocity



Wave Structure in the x-t Plane