Radiative heat transfer solver with fluid motion

Jan I.C. Vermaak[1,2], Jim E. Morel[1,2]

[1]Center for Large Scale Scientific Simulations, Texas A&M Engineering Experiment Station, College Station, Texas, USA.
[2]Nuclear Engineering Department, Texas A&M University, College Station, Texas, USA.

Abstract:
Work is work for some, but for some it is play.

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Contents

1 Definitions

1.1 Independent variables

We refer to the following independent variables:

- Position in the cartesian space $\{x, y, z\}$ is denoted with $\mathbf{x}$ and each component having units $[cm]$.
- Direction, $\{\varphi, \theta\}$, is denoted with $\boldsymbol{\Omega}$ which takes on the form

$$\boldsymbol{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \text{ and/or } \boldsymbol{\Omega} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix},$$

  where φ is the azimuthal-angle and θ is the polar-angle, both in spherical coordinates. Commonly, $\cos\theta$, is denoted with μ. The general dimension of angular phase space is $[steridian]$.
- Photon frequency, ν in $[Hertz]$ or $[s^{-1}]$.
- Time, t in $[s]$.

1.2 Dependent variables

We use the following basic dependent variables:

- The foundation of the dependent unknowns is the **radiation angular intensity**, $I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)$ with units $[Joule/cm^2{-}s{-}steradian{-}Hz]$. We often use the corresponding angle-integral of this quantity, $\phi(\mathbf{x}, \nu, t)$, and define it as

$$\phi(\mathbf{x}, \nu, t) = \mathcal{E}c = \int_{4\pi} I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)\ d\boldsymbol{\Omega} \tag{1.1}$$

  with units $[Joule/cm^2{-}s{-}Hz]$. Where c is the speed of light.
- The **radiation energy density**, $\mathcal{E}$, is

$$\mathcal{E}(\mathbf{x}, \nu, t) = \frac{\phi}{c} = \frac{1}{c}\int_{4\pi} I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)\ d\boldsymbol{\Omega} \tag{1.2}$$

  with units $[Joule/cm^3{-}Hz]$.
- The **radiation energy flux**, $\boldsymbol{\mathcal{F}}$, is

$$\boldsymbol{\mathcal{F}}(\mathbf{x}, \nu, t) = \int_{4\pi} \boldsymbol{\Omega}\ I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) d\boldsymbol{\Omega} \tag{1.3}$$

- **Radiation pressure**, $\boldsymbol{\mathcal{P}}$, is

$$\boldsymbol{\mathcal{P}}(\mathbf{x}, \nu, t) = \frac{1}{c}\int_{4\pi} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) d\boldsymbol{\Omega} \tag{1.4}$$

  and is a tensor.

1.3 Blackbody radiation

A blackbody radiation source, $B(\nu, T)$, is properly described by **Planck's law**,

$$B(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{1.5}$$

with units $[Joule/cm^2{-}s{-}steridian - Hz]$ where h is Planck's constant and k_B is the Boltzmann constant.

If we integrate the blackbody source over all angle-space and frequencies then we get the mean radiation intensity from a blackbody at temperature T as

$$\begin{aligned} \int_0^\infty \int_{4\pi} B(\nu, T)\; d\mathbf{\Omega} d\nu &= \int_0^\infty \int_{4\pi} \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}\; d\mathbf{\Omega} d\nu \\ &= 4\pi \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}\; d\nu \\ &= acT^4, \end{aligned} \tag{1.6}$$

with units $[Joule/cm^2{-}s{-}steridian]$ and where a is the **blackbody radiation constant** given by

$$a = \frac{8\pi^5 k_B^4}{15h^3c^3}. \tag{1.7}$$

In both cases this unfortunately is only the intensity. Following Kirchoff's law, which states that the emission and absorption of radiation must be equal in equilibrium, we can determine the **blackbody emission rate**, S_{bb}, from the absorption rate as

$$S_{bb}(\nu, T) = \rho\kappa(\nu)B(\nu, T), \tag{1.8}$$

with units $[Joule/cm^3{-}s{-}steridian{-}Hz]$ where ρ is the material density $[g/cm^3]$ and κ is the opacity $[cm^2/g]$. The combination $\rho\kappa$ is also equal to the macroscopic absorption cross section σ_a, therefore $\rho\kappa(\nu) = \sigma_a$. Data for the opacity of a material is normally available in the form of either the **Rosseland opacity**, κ_{Rs}, or the **Planck opacity**, κ_{Pl}.

1.4 An introduction to Special Relativity

In this subsection we make a simplified derivation of special relativity, specifically how time-dilation relates to laboratory-frame velocity relative to the speed of light, c. We will refer to the laboratory-frame as S (to be consistent with Mihalas [1]), and the co-moving frame as S'. Consider the schematic in Figure 1 below. The schematic depicts a cart, having a width s, moving at velocity v. On the cart is a co-moving observer (O'), and, stationary relative to the cart, is a lab-frame observer (O). A photon is propelled from one end of the cart to the other and both observers O' and O have a clock with which they measure t_C and t_L respectively.

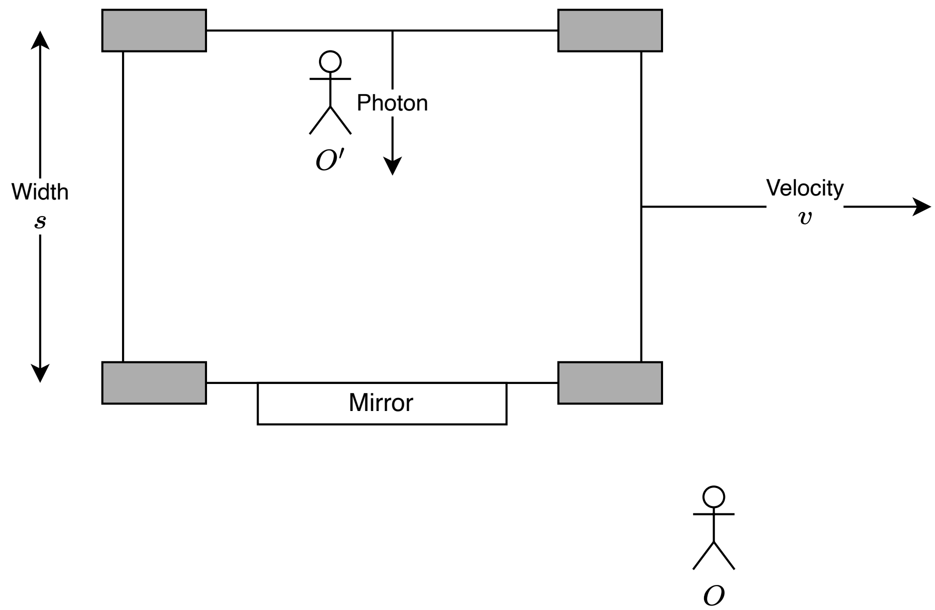

Figure 1: Schematic of a special relativity scenario. The cart, having width s, is moving at velocity v. On the cart is a co-moving observer (O'), and, stationary relative to the cart, is a lab-frame observer (O). A photon is propelled from one end of the cart to the other.

The two observers measure the time it takes for the photon to bounce of the mirror and return to its source. The path the photon takes is shown in Figure 2.

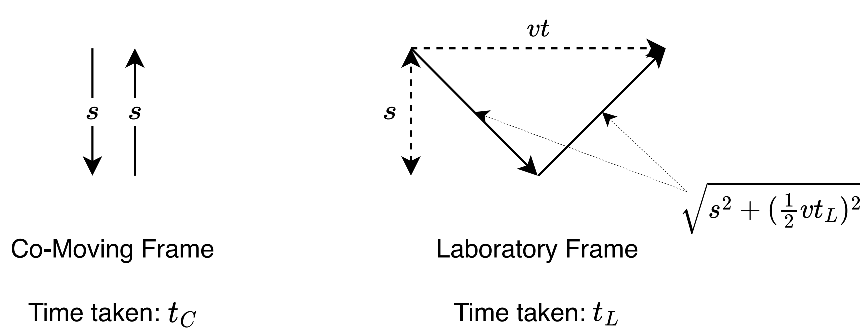

Figure 2: Schematic of the distances traveled from different frames of reference.

For O', the time is measured to be t_C, however, since the speed of light, c is a constant in any reference frame we can write

$$c = \frac{2s}{t_C}. \tag{1.9}$$

For O, the time is measured to be t_L but the distance traveled by the photon is much greater. Still, c is constant and therefore we write

$$c = \frac{2\sqrt{s^2 + (\frac{1}{2}vt_L)^2}}{t_L}. \tag{1.10}$$

From the first equation we obtain an expression for t_C, i.e.,

$$t_C = \frac{2s}{c}, \tag{1.11}$$

and from the second equation we obtain an expression for t_L, i.e.,

$$\begin{aligned} ct_L &= 2\sqrt{s^2 + (\frac{1}{2}vt_L)^2} \\ \frac{1}{4}c^2t_L^2 &= s^2 + \frac{1}{4}v^2t_L^2 \\ \frac{1}{4}t_L^2(c^2 - v^2) &= s^2 \\ t_L^2 &= 4s^2\frac{1}{c^2 - v^2} \\ \therefore t_L &= 2s\frac{1}{\sqrt{c^2 - v^2}} \end{aligned} \tag{1.12}$$

1.4.1 A useful relation from the clock-readings, e.g. time dilation

We can deduce several things from these expressions but firstly we define two important items. We define t' as the time passed in the co-moving frame relative to the lab-frame. We also define t_0 as the time passed in the lab-frame relative to the co-moving frame. With these defined we can develop expressions for these quantities as follows.

Firstly, time passed in the co-moving frame relative to the lab-frame, t', is simply the ratio of the clock readings t_C to t_L, multiplied with an arbitrary time passed in the lab-frame, t, i.e.,

$$t' = \frac{t_C}{t_L}t = \frac{\frac{2s}{c}}{2s\frac{1}{\sqrt{c^2 - v^2}}}t, \tag{1.13}$$

arriving at

$$t' = \left(\sqrt{1 - \frac{v^2}{c^2}}\right)t. \tag{1.14}$$

Secondly, time passed in the lab-frame relative to the co-moving frame, t_0, is simply the opposite ratio t_L to t_C, again multiplied with an arbitrary time passed in the lab-frame, t, i.e.,

$$t_0 = \frac{t_L}{t_C}t = \frac{2s\frac{1}{\sqrt{c^2 - v^2}}}{\frac{2s}{c}}t \tag{1.15}$$

arriving at

$$t_0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}t. \tag{1.16}$$

These two expressions, i.e., eqs. (1.14) and (1.16), can tell us some interesting things once we plug in some examples for v, e.g., for $v = \frac{6}{10}c$, we get $t' = \frac{8}{10}t$ and $t_0 = \frac{5}{4}t$. It is easy to see that the clock placed in the co-moving frame runs slow as observed from the lab-frame (i.e. $t' < 1$) and the clock placed in the lab-frame runs fast as observed from the co-moving frame (i.e. $t_0 > 1$). This effect is generally known as **time dilation**.

Time dilation forms the basis of many other aspects that follow and to that end we define the following,

$$\beta = \frac{v}{c} \tag{1.17}$$

and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{1.18}$$

With these definitions our time expresions in eqs. (1.14) and (1.16) become

$$t' = \frac{1}{\gamma}t \tag{1.19}$$

and

$$t_0 = \gamma t. \tag{1.20}$$

2 Conservation equations

2.1 Conservation equation - Radiative transfer

The basic statement of conservation, is

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},\nu,t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla I(\mathbf{x},\mathbf{\Omega},\nu,t) - \sigma_t(\mathbf{x},\nu)I(\mathbf{x},\mathbf{\Omega},\nu,t) \\ &+ \int_0^\infty \int_{4\pi} \frac{\nu}{\nu'}\sigma_s(\mathbf{x},\nu'{\to}\nu,\mathbf{\Omega}'{\cdot}\mathbf{\Omega})I(\mathbf{x},\mathbf{\Omega}',\nu',t)d\nu' d\mathbf{\Omega}' \\ &+ \sigma_a(\mathbf{x},\nu)B(\nu,T(\mathbf{x},t)) + S\end{aligned} \tag{2.1}$$

where S is any other sources/sinks of radiation intensity.

2.2 Radiative transfer assuming isotropic Thompson scattering

Assuming Thomson-scattering[1] is the only form of scattering, gives

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},\nu,t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla I(\mathbf{x},\mathbf{\Omega},\nu,t) - \sigma_t(\mathbf{x},\nu)I(\mathbf{x},\mathbf{\Omega},\nu,t) \\ &+ \frac{\sigma_s(\mathbf{x},\nu)}{4\pi}c\mathcal{E}(\mathbf{x},\nu) + \sigma_a(\mathbf{x},\nu)B(\nu,T(\mathbf{x},t)) + S\end{aligned} \tag{2.2}$$

where S is any other sources/sinks of radiation intensity.

Using energy instead of frequency, $\nu \to E$:

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},E,t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla I(\mathbf{x},\mathbf{\Omega},E,t) - \sigma_t(\mathbf{x},E)I(\mathbf{x},\mathbf{\Omega},E,t) \\ &+ \frac{\sigma_s(\mathbf{x},E)}{4\pi}c\mathcal{E}(\mathbf{x},E) + \sigma_a(\mathbf{x},E)B(E,T(\mathbf{x},t)) + S\end{aligned} \tag{2.3}$$

where S is any other sources/sinks of radiation intensity.

2.3 Radiative transfer with material motion corrections

Applying relativistic corrections for a material in motion, we can derive (e.g., see NUEN 627 lecture 4) the laboratory-frame transport equation

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},E,t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla I(\mathbf{x},\mathbf{\Omega},E,t) - \left(\frac{E_0}{E}\right)\sigma_t(\mathbf{x},E_0)I(\mathbf{x},\mathbf{\Omega},E,t) \\ &+ \left(\frac{E}{E_0}\right)^2\frac{\sigma_s(\mathbf{x},E)}{4\pi}\int_{4\pi}\left(\frac{E_0}{E'}\right)I(\mathbf{x},\mathbf{\Omega}',E',t)d\mathbf{\Omega}' + \left(\frac{E}{E_0}\right)^2\sigma_a(\mathbf{x},E_0)B(E_0,T(\mathbf{x},t)) + S,\end{aligned} \tag{2.4}$$

where

$$E_0 = E\gamma\left(1 - \mathbf{\Omega}\cdot\frac{\mathbf{u}}{c}\right) \tag{2.5}$$

$$\gamma = \left[1 - \left(\frac{||\mathbf{u}||}{c}\right)^2\right]^{-\frac{1}{2}} \tag{2.6}$$

$$\frac{E_0}{E'} = \gamma\left(1 - \mathbf{\Omega}'\cdot\frac{\mathbf{u}}{c}\right) \tag{2.7}$$

$$E' = E\frac{1 - \mathbf{\Omega}\cdot\frac{\mathbf{u}}{c}}{1 - \mathbf{\Omega}'\cdot\frac{\mathbf{u}}{c}} \tag{2.8}$$

[1] Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle. The particle's kinetic energy- as well as the photon's frequency, does not change in such a scattering. The scattering is also isotropic.

2.4 Radiative transfer with material velocity dependencies expanded to $\mathcal{O}(v/c)$

Very ugly derivations in NUEN 627 lecture 5 to get to,

$$\begin{aligned}
&\frac{1}{c}\frac{\partial I(\mathbf{x},\boldsymbol{\Omega},E,t)}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I(\mathbf{x},\boldsymbol{\Omega},E,t) + \sigma_t(\mathbf{x},E)I(\mathbf{x},\boldsymbol{\Omega},E,t) \\
=&\frac{\sigma_s(\mathbf{x},E)}{4\pi}\phi(E) + \sigma_a(\mathbf{x},E)B(E,T(\mathbf{x},t)) \\
&+\left[\left(\sigma_t + E\frac{\partial\sigma_a}{\partial E}\right)I + \frac{\sigma_s}{4\pi}\left(2\phi - E\frac{\partial\phi}{\partial E}\right) + 2\sigma_a B(E,T) - B(E,T)E\frac{\partial\sigma_a}{\partial E} - \sigma_a E\frac{\partial B(E,T)}{\partial E}\right]\boldsymbol{\Omega}\cdot\frac{\mathbf{u}}{c} \\
&-\frac{\sigma_s}{4\pi}\left(\boldsymbol{\mathcal{F}} - E\frac{\partial\boldsymbol{\mathcal{F}}}{\partial E}\right)\cdot\frac{\mathbf{u}}{c}
\end{aligned} \tag{2.9}$$

Voodoo magic Grey Radiation Transport equation:
Somehow, determined by integrating over energy

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \boldsymbol{\Omega}\cdot\mathbf{u} \tag{2.10}$$

Radiation energy equation:
Obtained by integrating the transport equation over energy and angle

$$\begin{aligned}
\frac{\partial\mathcal{E}(\mathbf{x},t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) =& \int_0^\infty \sigma_a(\mathbf{x},E)\big(4\pi B(E,T) - \phi(\mathbf{x},E,t)\big)dE \\
&+ \int_0^\infty \left(\sigma_a + E\frac{\partial\sigma_a}{\partial E} - \sigma_s(E)\right)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}dE
\end{aligned} \tag{2.11}$$

Radiation momentum equation:
Obtained by first multiplying by $\frac{1}{c}\boldsymbol{\Omega}$, then integrating over all directions and energies,

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} =& -\int_0^\infty \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}dE \\
&+ \int_0^\infty \big(\sigma_s\phi + \sigma_a 4\pi B(E,T)\big)\frac{\mathbf{u}}{c^2}dE \\
&+ \int_0^\infty \left(\sigma_a + E\frac{\partial\sigma_a}{\partial E} + \sigma_s\right)\boldsymbol{\mathcal{P}}\cdot\frac{\mathbf{u}}{c}dE
\end{aligned} \tag{2.12}$$

2.5 Grey Radiative Transfer

$$\begin{aligned}
&\frac{1}{c}\frac{\partial I(\mathbf{x},\boldsymbol{\Omega},t)}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I(\mathbf{x},\boldsymbol{\Omega},t) + \sigma_t(\mathbf{x})I(\mathbf{x},\boldsymbol{\Omega},t) \\
=&\frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 \\
&+\left[\sigma_t I + \frac{\sigma_s}{4\pi}2\phi + 2\sigma_a\frac{1}{4\pi}acT^4 - \sigma_a E\frac{\partial B(E,T)}{\partial E}\right]\boldsymbol{\Omega}\cdot\frac{\mathbf{u}}{c} \\
&-\frac{\sigma_s}{4\pi}\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}
\end{aligned} \tag{2.13}$$

Radiation energy equation:
Obtained by integrating Eq. (2.13) over energy and angle

$$\begin{aligned}
\frac{\partial\mathcal{E}(\mathbf{x},t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) &= \sigma_a c\big(aT^4 - \mathcal{E}\big) + \big(\sigma_a - \sigma_s\big)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c} \\
&= \sigma_a c\big(aT^4 - \mathcal{E}_0\big) - \sigma_t\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}
\end{aligned} \tag{2.14}$$

Radiation momentum equation:
Obtained by first multiplying Eq. (2.13) by $\frac{1}{c}\boldsymbol{\Omega}$, then integrating over all directions and energies,

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} &= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \big(\sigma_s c\mathcal{E} + \sigma_a a c T^4\big)\frac{\mathbf{u}}{c^2} + \sigma_t\boldsymbol{\mathcal{P}}\cdot\frac{\mathbf{u}}{c} \\
&= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \big((\sigma_a + \sigma_s - \sigma_a)\mathcal{E} + \sigma_a a T^4\big)\frac{\mathbf{u}}{c} + \sigma_t\boldsymbol{\mathcal{P}}\cdot\frac{\mathbf{u}}{c} \\
&= \frac{1}{c}\Big[-\sigma_t\boldsymbol{\mathcal{F}} + \big((\sigma_t - \sigma_a)\mathcal{E} + \sigma_a a T^4\big)\mathbf{u} + \sigma_t\boldsymbol{\mathcal{P}}\cdot\mathbf{u}\Big] \\
&= -\frac{1}{c}\Big[\sigma_t\boldsymbol{\mathcal{F}} - \big((\sigma_t - \sigma_a)\mathcal{E} + \sigma_a a T^4\big)\mathbf{u} - \sigma_t\boldsymbol{\mathcal{P}}\cdot\mathbf{u}\Big] \\
\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} &= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 + \sigma_a c\big(aT^4 - \mathcal{E}\big)\frac{\mathbf{u}}{c^2}
\end{aligned}
\tag{2.15}
$$

2.6 Grey Diffusion Approximation

Approximating the angular dependence of $I(\boldsymbol{\Omega})$ with a P_1 spherical harmonic expansion, such that the entries of $\boldsymbol{\mathcal{P}}$ are given by

$$
(\boldsymbol{\mathcal{P}})_{i,j} = \frac{1}{3}\mathcal{E}\delta_{i,j}, \tag{2.16}
$$

the radiation energy equation is unaffected but the radiation momentum equation changes. We repeat the radiation energy equation below, and the altered radiation moment equations:

$$
\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \big(\sigma_a - \sigma_s\big)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}, \tag{2.17}
$$

$$
\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \big(\sigma_s c\mathcal{E} + \sigma_a a c T^4\big)\frac{\mathbf{u}}{c^2} + \sigma_t\frac{1}{3}\mathcal{E}\frac{\mathbf{u}}{c}. \tag{2.18}
$$

Useful transformations:

$$
\mathcal{E}_0 = \mathcal{E} - \frac{2}{c^2}\boldsymbol{\mathcal{F}}\cdot\mathbf{u} \tag{2.19a}
$$

$$
\mathcal{E} = \mathcal{E}_0 + \frac{2}{c^2}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{2.19b}
$$

$$
\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{F}} - \big(\mathcal{E}\mathbf{u} + \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\big) \tag{2.19c}
$$

$$
\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_0 + \big(\mathcal{E}_0\mathbf{u} + \boldsymbol{\mathcal{P}}_0\cdot\mathbf{u}\big) \tag{2.19d}
$$

$$
\boldsymbol{\mathcal{P}}_0 = \boldsymbol{\mathcal{P}} - \frac{2}{c^2}\mathbf{u}\otimes\boldsymbol{\mathcal{F}} \tag{2.19e}
$$

$$
\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_0 + \frac{2}{c^2}\mathbf{u}\otimes\boldsymbol{\mathcal{F}}_0 \tag{2.19f}
$$

With the P_1 approximation

$$
\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{F}} - \frac{4}{3}\mathcal{E}\mathbf{u} \tag{2.19g}
$$

$$
\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_0 + \frac{4}{3}\mathcal{E}\mathbf{u} \tag{2.19h}
$$

Applying these transformations the radiation energy- and moment equation can be expressed as

$$
\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) = \sigma_a c\big(aT^4 - \mathcal{E}_0\big) - \sigma_t\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}, \tag{2.20}
$$

$$
\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 + \sigma_a c\big(aT^4 - \mathcal{E}\big)\frac{\mathbf{u}}{c^2}. \tag{2.21}
$$

Several simplifications to these equations are made. Firstly arriving at the expression for the radiation energy equation,

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}(\mathbf{x}, t) = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t \boldsymbol{\mathcal{F}}_0 \cdot \frac{\mathbf{u}}{c}, \tag{2.22}$$

then the radiation momentum equation,

$$\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \tag{2.23}$$

from which we can get expression for $\boldsymbol{\mathcal{F}}_0$ and $\boldsymbol{\mathcal{F}}$ in terms of $\mathcal{E}$ as

$$\boldsymbol{\mathcal{F}}_0 = -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} \tag{2.24}$$

and

$$\begin{aligned} \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} &= -\frac{\sigma_t}{c}\left(\boldsymbol{\mathcal{F}} - \frac{4}{3}\mathcal{E}\mathbf{u}\right) \\ \therefore \boldsymbol{\mathcal{F}} &= -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} + \frac{4}{3}\mathcal{E}\mathbf{u}. \end{aligned} \tag{2.25}$$

These expressions for $\boldsymbol{\mathcal{F}}_0$ and $\boldsymbol{\mathcal{F}}$ are both then inserted into the radiation energy equation as follows

$$\begin{aligned} &\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}(\mathbf{x}, t) = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t \boldsymbol{\mathcal{F}}_0 \cdot \frac{\mathbf{u}}{c} \\ \to &\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} + \frac{4}{3}\mathcal{E}\mathbf{u}\right) = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\right) \cdot \frac{\mathbf{u}}{c} \\ \to &\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\right) + \frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}. \end{aligned} \tag{2.26}$$

Arriving at a **diffusion form** of the **radiation energy equation**,

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\right) + \frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}. \tag{2.27}$$

2.7 Conservation equation for fluid flow

The governing equations we consider here are the Euler equations defined as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) = 0 \tag{2.28}$$

$$\frac{\partial (\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \{\rho\mathbf{u} \otimes \mathbf{u}\} + \boldsymbol{\nabla} p = \mathbf{f}, \tag{2.29}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = q \tag{2.30}$$

where ρ is the fluid density, $\mathbf{u} = [u_x, u_y, u_z] = [u, v, w]$ is the fluid velocity in cartesian coordinates, p is the fluid pressure, E is the material energy-density comprising kinetic energy-density, $\frac{1}{2}\rho||\mathbf{u}||^2$, and internal energy-density, ρe, such that $E = \frac{1}{2}\rho||\mathbf{u}||^2 + \rho e$, where e is the specific internal energy. The values q and $\mathbf{f}$ are abstractly used here as energy- and moment- sources/sinks, respectively.

The ideal gas law provides the closure relation

$$p = (\gamma - 1)\rho e \tag{2.31}$$

where γ is the ratio of the constant-pressure specific heat, c_p, to the constant-volume specific heat, c_v, i.e., $\gamma = \frac{c_p}{c_v}$, and is a material property.

Coupling terms:

$$\begin{aligned}\mathbf{f} &= \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \\ &= -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}q &= -\left(\sigma_a c(aT^4 - \mathcal{E}) - \sigma_t \boldsymbol{\mathcal{F}}_0 \cdot \frac{\mathbf{u}}{c}\right) \\ &= \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\end{aligned} \tag{2.33}$$

3 Solver A - Radiation Hydrodynamics Grey Diffusion

The set of Radiation Hydrodynamics Grey Diffusion Equations are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{3.1a}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \boldsymbol{\nabla} p = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}, \tag{3.1b}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u} \tag{3.1c}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\right) + \frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}\mathbf{u}\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}. \tag{3.1d}$$

where

$$E = \frac{1}{2}\rho||\mathbf{u}||^2 + \rho e, \tag{3.1e}$$

$$p = (\gamma - 1)\rho e, \tag{3.1f}$$

$$T = \frac{1}{C_v} e \tag{3.1g}$$

$$\sigma_t(T) = \sigma_s(T) + \sigma_a(T) \tag{3.1h}$$

$$\sigma_s(T) = \rho\kappa_s(T) \tag{3.1i}$$

$$\sigma_a(T) = \rho\kappa_a(T) \tag{3.1j}$$

3.1 Definitions

First we define the following terms

- The radiation emission and absorption, the radiation momentum source, and the radiation energy source

$$S_{ea} = \sigma_a c\big(aT^4 - \mathcal{E}\big) \tag{3.2a}$$

$$\mathbf{S}_{rp} = \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \tag{3.2b}$$

$$S_{re} = S_{ea} + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u} \tag{3.2c}$$

- The conserved hydrodynamic variables, $\mathbf{U}$, and associated hydrodynamic flux, $\mathcal{F}^H$,

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ E \end{bmatrix} \qquad \mathcal{F}^H = \begin{bmatrix} \rho u \\ \rho u u + p \\ \rho u v \\ \rho u w \\ (E+p)u \end{bmatrix} \tag{3.2d}$$

- The stationary reference frame radiation energy flux

$$\boldsymbol{\mathcal{F}}_0 = -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} \tag{3.2e}$$

Next, we use these terms to define a more condensed version of the RHGD equations.

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{3.3}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0 + \frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big) = S_{ea} + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}. \tag{3.4}$$

3.2 Finite Volume Spatial Discretization

To apply a finite volume spatial discretization we integrate our time-discretized equations over the volume, V_c, of cell c, and afterwards divide by V_c. This leaves all the terms containing τ unchanged. In this process we develop the following terms:

3.2.1 Hydrodynamic and Radiation-energy advection

$$\frac{1}{V_c}\int_{V_c} \boldsymbol{\nabla}\cdot\mathcal{F}^H(\mathbf{U})dV = \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot\mathcal{F}^H(\mathbf{U}_f) \tag{3.5}$$

$$\frac{1}{V_c}\int_{V_c}\left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right)dV = \frac{1}{V_c}\sum_f \frac{4}{3}\mathbf{A}_f\cdot(\mathcal{E}\mathbf{u})_f \tag{3.6}$$

The face values are reconstructed from gradients in both the predictor and corrector phases. In the corrector-phase the hydrodynamic flux, $\mathcal{F}^H$, is used in its earlier defined form, whilst in the corrector-phase the flux is determined by an approximate Riemann-solver, i.e., the HLLC Riemann solver.

Predictor phases:
For the predictor phase we have the following:

$$\boldsymbol{\nabla}\cdot\mathcal{F}^H(\mathbf{U}) \mapsto \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot\mathcal{F}^H(\mathbf{U}_f) \tag{3.7}$$

$$\left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right) \mapsto \frac{1}{V_c}\sum_f \frac{4}{3}\mathbf{A}_f\cdot(\mathcal{E}\mathbf{u})_f \tag{3.8}$$

$$\mathbf{U}_f = \mathbf{U}_c + (\mathbf{x}_f - \mathbf{x}_c)\cdot\{\boldsymbol{\nabla}\mathbf{U}\}_c \tag{3.9}$$

$$\mathcal{E}_f = \mathcal{E}_c + (\mathbf{x}_f - \mathbf{x}_c)\cdot\{\boldsymbol{\nabla}\mathcal{E}\}_c \tag{3.10}$$

Corrector phases:
For the corrector phase we have the following:

$$\boldsymbol{\nabla}\cdot\mathcal{F}^H(\mathbf{U}) \mapsto \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot\mathbf{F}^{*hllc}(\mathbf{U}_f) \tag{3.11}$$

$$\left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right) \mapsto \frac{1}{V_c}\sum_f \frac{4}{3}\mathbf{A}_f\cdot(\mathcal{E}\mathbf{u})_{upw} \tag{3.12}$$

where

$$\mathbf{U}_f = \mathbf{U}_c + (\mathbf{x}_f - \mathbf{x}_c)\cdot\{\boldsymbol{\nabla}\mathbf{U}\}_c \tag{3.13}$$

$$(\mathcal{E}\mathbf{u})_{upw} = \begin{cases} (\mathcal{E}\mathbf{u})_{c,f}, & \text{if } \mathbf{u}_{c,f}\cdot\mathbf{n}_f > 0 \text{ and } \mathbf{u}_{cn,f}\cdot\mathbf{n}_f > 0 \quad \rightarrow|\rightarrow \\ (\mathcal{E}\mathbf{u})_{cn,f}, & \text{if } \mathbf{u}_{c,f}\cdot\mathbf{n}_f < 0 \text{ and } \mathbf{u}_{cn,f}\cdot\mathbf{n}_f < 0 \quad \leftarrow|\leftarrow \\ (\mathcal{E}\mathbf{u})_{cn,f} + (\mathcal{E}\mathbf{u})_{c,f}, & \text{if } \mathbf{u}_{c,f}\cdot\mathbf{n}_f > 0 \text{ and } \mathbf{u}_{cn,f}\cdot\mathbf{n}_f < 0 \quad \rightarrow|\leftarrow \\ 0, & \text{if } \mathbf{u}_{c,f}\cdot\mathbf{n}_f < 0 \text{ and } \mathbf{u}_{cn,f}\cdot\mathbf{n}_f > 0 \quad \leftarrow|\rightarrow \end{cases} \tag{3.14}$$

$$\mathcal{E}_{c,f} = \mathcal{E}_c + (\mathbf{x}_f - \mathbf{x}_c)\cdot\{\boldsymbol{\nabla}\mathcal{E}\}_c \tag{3.15}$$

3.2.2 Density and momentum updates

We apply the same process as before:

$$-\frac{1}{V_c}\int_{V_c}\mathbf{S}_{rp}dV = -\frac{1}{V_c}\sum_f \frac{1}{3}\mathbf{A}_f\mathcal{E}_f, \tag{3.16}$$

however, here we want $\mathcal{E}_f$ to satisfy the following relationship

$$\frac{D_c}{||\mathbf{x}_{cf}||}(\mathcal{E}_f - \mathcal{E}_c) = \frac{D_{cn}}{||\mathbf{x}_{fcn}||}(\mathcal{E}_{cn} - \mathcal{E}_f) \tag{3.17}$$

where

$$D_c = -\frac{c}{3\sigma_{t,c}} \tag{3.18}$$

and where $\mathbf{x}_{cf}$ is the vector from cell c's centroid to the face centroid, $\mathbf{x}_{fcn}$ is the vector from the face centroid to cell cn's centroid (where cell cn is the neighbor to c at face f). The norm $||\cdot||$ refers to the L_2 norm.

Solving the above relationship for $\mathcal{E}_f$ we first set

$$k_c = \frac{D_c}{||\mathbf{x}_{cf}||}, \qquad k_{cn} = \frac{D_{cn}}{||\mathbf{x}_{fcn}||}$$

then get

$$\begin{aligned} k_c\mathcal{E}_f - k_c\mathcal{E}_c &= k_{cn}\mathcal{E}_{cn} - k_{cn}\mathcal{E}_f \\ \rightarrow \quad (k_c + k_{cn})\mathcal{E}_f &= k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c \\ \therefore \mathcal{E}_f &= \frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}}. \end{aligned} \tag{3.19}$$

Predictor and corrector phases:
We do the same for both,

$$-\mathbf{S}_{rp} \mapsto -\frac{1}{V_c}\sum_f \frac{1}{3}\mathbf{A}_f\mathcal{E}_f \tag{3.20}$$

3.2.3 Energy equations

Only two terms require special consideration here. They are: the divergence of the co-moving frame radiation energy flux, and the kinetic energy terms source terms,

$$\begin{aligned} \frac{1}{V_c}\int_{V_c}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0 \ dV &= \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot(\boldsymbol{\mathcal{F}}_0)_f \\ \frac{1}{V_c}\int_{V_c}\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u} \ dV &= \frac{1}{V_c}\sum_f \frac{1}{3}\mathbf{A}_f\cdot(\mathcal{E}\mathbf{u})_f. \end{aligned} \tag{3.21}$$

3.2.3.1 The diffusion term

Considering the $\boldsymbol{\mathcal{F}}_0$-term first, we apply Gauss' divergence theorem to get

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0 \mapsto \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot(\boldsymbol{\mathcal{F}}_0)_f. \tag{3.22}$$

For $(\boldsymbol{\mathcal{F}}_0)_f$ we have

$$(\boldsymbol{\mathcal{F}}_0)_f = -\frac{c}{3\sigma_{tf}}\big(\boldsymbol{\nabla}\mathcal{E}\big)_f. \tag{3.23}$$

Now define

$$D_f = -\frac{c}{3\sigma_{tf}}. \tag{3.24}$$

To find D_f we seek the equivalence:

$$D_f \frac{\mathcal{E}_{cn} - \mathcal{E}_c}{||\mathbf{x}_{cn} - \mathbf{x}_c||} = D_c \frac{\mathcal{E}_f - \mathcal{E}_c}{||\mathbf{x}_f - \mathbf{x}_c||} = D_{cn} \frac{\mathcal{E}_{cn} - \mathcal{E}_f}{||\mathbf{x}_{cn} - \mathbf{x}_f||} \tag{3.25}$$

Now let us define

$$\begin{aligned} k_c &= \frac{D_c}{||\mathbf{x}_f - \mathbf{x}_c||} \\ k_{cn} &= \frac{D_{cn}}{||\mathbf{x}_{cn} - \mathbf{x}_f||} \end{aligned} \tag{3.26}$$

Now

$$\begin{aligned} k_c(\mathcal{E}_f - \mathcal{E}_c) &= k_{cn}(\mathcal{E}_{cn} - \mathcal{E}_f) \\ (k_c + k_{cn})\mathcal{E}_f &= k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c \\ \therefore \mathcal{E}_f &= \frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}} \end{aligned} \tag{3.27}$$

Now we choose any of the right-two terms in the three way equality and plug the expression for $\mathcal{E}_f$,

$$\begin{aligned} & k_c(\mathcal{E}_f - \mathcal{E}_c) \\ &= k_c\left(\frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}} - \mathcal{E}_c\right) \\ &= k_c\left(\frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c - k_c\mathcal{E}_c - k_{cn}\mathcal{E}_c}{k_c + k_{cn}}\right) \\ \therefore D_f \frac{\mathcal{E}_{cn} - \mathcal{E}_c}{||\mathbf{x}_{cn} - \mathbf{x}_c||} &= \frac{k_c k_{cn}}{k_c + k_{cn}}\big(\mathcal{E}_{cn} - \mathcal{E}_c\big) \\ \therefore D_f &= \frac{k_c k_{cn}}{k_c + k_{cn}}||\mathbf{x}_{cn} - \mathbf{x}_c|| \end{aligned} \tag{3.28}$$

From the earlier expression for $(\boldsymbol{\mathcal{F}}_0)_f$, we can write

$$(\boldsymbol{\mathcal{F}}_0)_f = D_f\big(\mathcal{E}_{cn} - \mathcal{E}_c\big)\frac{\mathbf{x}_{cn} - \mathbf{x}_c}{||\mathbf{x}_{cn} - \mathbf{x}_c||^2} \tag{3.29}$$

for which we can define

$$\mathbf{k}_f = D_f \frac{\mathbf{x}_{cn} - \mathbf{x}_c}{||\mathbf{x}_{cn} - \mathbf{x}_c||^2} \tag{3.30}$$

such that we finally arrive at

$$(\boldsymbol{\mathcal{F}}_0)_f = \mathbf{k}_f\big(\mathcal{E}_{cn} - \mathcal{E}_c\big). \tag{3.31}$$

3.2.3.2 The kinetic energy term

For the kinetic energy source terms, we similarly have

$$\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^n \mapsto \frac{1}{V_c}\sum_f \frac{1}{3}\mathbf{A}_f \cdot (\mathcal{E}_f^n \mathbf{u}_f^n) \tag{3.32}$$

where we use the reconstructed values as in the Hydrodynamic and radiation-energy advection portion.

3.3 Temporal scheme - Implicit Euler Predictor, Crank-Nicolson Corrector

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{3.33a}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0 + \frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u}) = S_{re}. \tag{3.33b}$$

3.3.1 Predictor phase

$\tau = \dfrac{1}{\frac{1}{2}\Delta t}$

$$\tau(\mathbf{U}^{n*} - \mathbf{U}^n) + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}^n) = \mathbf{0} \tag{3.34a}$$

$$\tau\left(\begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+\frac{1}{2}} - \begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n*}\right) = \begin{bmatrix} 0 \\ -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \end{bmatrix}^n \tag{3.34b}$$

$$\tau(\mathcal{E}^{n*} - \mathcal{E}^n) + \left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right)^n = 0 \tag{3.34c}$$

$$\tau(E^{n+\frac{1}{2}} - E^{n*}) = -\theta_1 S_{ea}^{n+\frac{1}{2}} - \theta_2 S_{ea}^n - \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^n \tag{3.34d}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^{n*}) + \theta_1 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0^{n+\frac{1}{2}} + \theta_2 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0^n = \theta_1 S_{ea}^{n+\frac{1}{2}} + \theta_2 S_{ea}^n + \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^n \tag{3.34e}$$

For S_{ea} and $\boldsymbol{\mathcal{F}}_0$ both at $n + \frac{1}{2}$:

$$\sigma^{n+\frac{1}{2}} = \rho^{n+\frac{1}{2}}\kappa(T^n) \tag{3.34f}$$

$$T^{4,n+\frac{1}{2}} = T^{4,n*} + \frac{4T^{3,n*}}{C_v}(e^{n+\frac{1}{2}} - e^{n*}) \tag{3.34g}$$

3.3.2 Corrector phase

$\tau = \dfrac{1}{\Delta t}$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^n) + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}^{n+\frac{1}{2}}) = \mathbf{0} \tag{3.35a}$$

$$\tau\left(\begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+1} - \begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+\frac{1}{2}*}\right) = \begin{bmatrix} 0 \\ -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \end{bmatrix}^{n+\frac{1}{2}} \tag{3.35b}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}*} - \mathcal{E}^n) + \left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right)^{n+\frac{1}{2}} = 0 \tag{3.35c}$$

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 S_{ea}^{n+1} - \theta_2 S_{ea}^n - \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.35d}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0^n = \theta_1 S_{ea}^{n+1} + \theta_2 S_{re}^n + \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.35e}$$

For S_{ea} and $\boldsymbol{\mathcal{F}}_0$ both at $n + 1$:

$$\sigma^{n+1} = \rho^{n+1}\kappa(T^{n+\frac{1}{2}}) \tag{3.35f}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{3.35g}$$

3.3.3 General energy equations, Predictor and Corrector phase, with θ factors

Time integration scheme A uses **implicit Euler** for the predictor phase and **Crank-Nicolson** in the corrector phase. Both these schemes can be represented wit a general θ-scheme where we define:

$$\begin{aligned} &\theta_1 \in [0,1] \\ &\theta_2 = 1-\theta_1. \end{aligned} \tag{3.36}$$

For implicit Euler, $\theta_1 = 1,\ \theta_2 = 0$, and for Crank-Nicolson, $\theta_1 = \theta_2 = \frac{1}{2}$. With these factors defined we can repeat the energy equations and apply a series of manipulations. First we attempt to segregate known terms from all unknown terms. Thereafter we eliminate the internal energy, e, from the two sets of equations to get a single formulation for the radiation energy, $\mathcal{E}$. The latter formulation forms a diffusion system that needs to be assembled and solved for $\mathcal{E}$.

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 \sigma_a^{n+1} c\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) - \theta_2 S_{ea}^n - \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.37a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1 \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}_0^{n}\big) = \theta_1 \sigma_a^{n+1} c\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) + \theta_2 S_{re}^n + \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.37b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{3.37c}$$

Define:

$$\begin{aligned} k_1 &= \theta_1 \sigma_a^{n+1} c \\ k_2 &= \frac{4T^{3,n+\frac{1}{2}*}}{C_v} \end{aligned} \tag{3.38}$$

---

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) - \theta_2 S_{ea}^n - \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.39a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1 \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}_0^{n}\big) = k_1\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) + \theta_2 S_{re}^n + \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.39b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{3.39c}$$

---

ungroup right-hand side elements by multiplying out terms within parentheses,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 aT^{4,n+1} + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.40a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1 \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}_0^{n}\big) = k_1 aT^{4,n+1} - k_1\mathcal{E}^{n+1} + \theta_2 S_{re}^n + \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.40b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{3.40c}$$

---

now plug in the temperature equation into both the energy equations,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a\big(T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*})\big) + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.41a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1 \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}_0^{n}\big) = k_1 a\big(T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*})\big) - k_1\mathcal{E}^{n+1} + \theta_2 S_{re}^n + \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.41b}$$

ungroup elements on the both the right-hand sides,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a T^{4,n+\frac{1}{2}*} - k_1 a k_2 e^{n+1} + k_1 a k_2 e^{n+\frac{1}{2}*} + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.42a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1 \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}_0^{n}\big) = k_1 a T^{4,n+\frac{1}{2}*} + k_1 a k_2 e^{n+1} - k_1 a k_2 e^{n+\frac{1}{2}*} - k_1\mathcal{E}^{n+1} + \theta_2 S_{re}^n + \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.42b}$$

Define:

$$\begin{aligned} k_3 &= -k_1 a T^{4,n+\frac{1}{2}*} + k_1 a k_2 e^{n+\frac{1}{2}*} - \theta_2 S_{ea}^n - \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \\ k_4 &= -k_1 a k_2 \end{aligned} \tag{3.43}$$

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = k_4 e^{n+1} + k_1\mathcal{E}^{n+1} + k_3 \tag{3.44a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1 \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}_0^{n}\big) = -k_4 e^{n+1} - k_1\mathcal{E}^{n+1} - k_3 \tag{3.44b}$$

Note:

$$E^{n+1} = (\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \rho^{n+1}e^{n+1} \tag{3.45}$$

which gives,

$$\tau\big((\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \rho^{n+1}e^{n+1} - E^{n+\frac{1}{2}*}\big) = k_4 e^{n+1} + k_1\mathcal{E}^{n+1} + k_3 \tag{3.46a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1 \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}_0^{n}\big) = -k_4 e^{n+1} - k_1\mathcal{E}^{n+1} - k_3 \tag{3.46b}$$

ungroup the material energy in the first equation,

$$\tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau\rho^{n+1}e^{n+1} - \tau E^{n+\frac{1}{2}*} = k_4 e^{n+1} + k_1\mathcal{E}^{n+1} + k_3 \tag{3.47a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n} = -k_4 e^{n+1} - k_1\mathcal{E}^{n+1} - k_3 \tag{3.47b}$$

and isolate the internal energy in the first equation,

$$(\tau\rho^{n+1} - k_4)e^{n+1} = k_1\mathcal{E}^{n+1} + k_3 - \tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau E^{n+\frac{1}{2}*} \tag{3.48a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n} = -k_4 e^{n+1} - k_1\mathcal{E}^{n+1} - k_3 \tag{3.48b}$$

Define:

$$k_5 = \frac{k_1}{\tau\rho^{n+1} - k_4}$$
$$k_6 = \frac{k_3 - \tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau E^{n+\frac{1}{2}*}}{\tau\rho^{n+1} - k_4} \tag{3.49}$$

and plug these constants into the first equation above,

$$e^{n+1} = k_5\mathcal{E}^{n+1} + k_6 \tag{3.50a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n} = -k_1\mathcal{E}^{n+1} - k_3 - k_4e^{n+1} \tag{3.50b}$$

now plug the first equation into the second,

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n} = -k_1\mathcal{E}^{n+1} - k_3 - k_4k_5\mathcal{E}^{n+1} - k_4k_6 \tag{3.51a}$$

now collect all the $\mathcal{E}^{n+1}$ terms on the left-hand side,

$$\big(\tau + k_1 + k_4k_5\big)\mathcal{E}^{n+1} + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1} = -k_3 - k_4k_6 + \tau\mathcal{E}^{n+\frac{1}{2}*} - \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n} \tag{3.52a}$$

Recall:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0 \mapsto \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot(\boldsymbol{\mathcal{F}}_0)_f \tag{3.53}$$

and

$$(\boldsymbol{\mathcal{F}}_0)_f = \mathbf{k}_f(\mathcal{E}_{cn} - \mathcal{E}_c) \tag{3.54}$$

which gives the system,

$$\big(\tau + k_1 + k_4k_5\big)\mathcal{E}^{n+1} + \frac{\theta_1}{V_c}\sum_f \mathbf{A}_f\cdot\mathbf{k}_f^{n+1}(\mathcal{E}_{cn}^{n+1} - \mathcal{E}_c^{n+1}) = -k_3 - k_4k_6 + \tau\mathcal{E}^{n+\frac{1}{2}*} - \frac{\theta_2}{V_c}\sum_f \mathbf{A}_f\cdot\mathbf{k}_f^{n}(\mathcal{E}_{cn}^{n} - \mathcal{E}_c^{n}) \tag{3.55a}$$

This system is SPD and in one dimension forms a tridiagonal system.

3.3.4 Using the energy related algebra for both the predictor and the corrector

To perform the energy related algebra for the corrector step we need the following inputs:

κ_a^n	For σ_a^n in S_{ea}^n
κ_t^n	For σ_t^n in $\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^n$
$\kappa_a^{n+\frac{1}{2}}$	For σ_a^{n+1} in S_{ea}^{n+1}
$\kappa_t^{n+\frac{1}{2}}$	For σ_t^{n+1} in $\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1}$
C_v	For the linearization of $T^{4,n+1}$
τ	For the time constant
θ_1,θ_2	For the time scheme
$\mathbf{U}^n$	For T,ρ in S_{ea}^n
$\mathbf{U}^{n+\frac{1}{2}}$	For $\mathbf{u}$ in $\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathbf{U}^{n+\frac{1}{2}*}$	For $E^{n+\frac{1}{2}*}$ and $e^{n+\frac{1}{2}*}$
$\mathbf{U}_{0,1}^{n+1}=\begin{bmatrix}\rho\\ \rho\mathbf{u}\end{bmatrix}_{n+1}$	For the kinetic energy in E^{n+1}, and $\rho^{n+1}\to\sigma_a^{n+1},\sigma_t^{n+1}$
$\boldsymbol{\nabla}\mathbf{U}^{n+\frac{1}{2}}$	For the reconstructions in $\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathcal{E}^n$	For S_{ea}^n
$\mathcal{E}^{n+\frac{1}{2}}$	For $\mathcal{E}$ in $\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathcal{E}^{n+\frac{1}{2}*}$	For itself
$\boldsymbol{\nabla}\mathcal{E}^{n+\frac{1}{2}}$	For the reconstructions in $\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}$

To following remapping(s) then applies to the predictor:

$$
\begin{aligned}
\kappa_a^n &\to \kappa_a^n\\
\kappa_t^n &\to \kappa_t^n\\
\kappa_a^n &\to \kappa_a^{n+\frac{1}{2}}\\
\kappa_t^n &\to \kappa_t^{n+\frac{1}{2}}\\
\mathbf{U}^n &\to \mathbf{U}^n\\
\mathbf{U}^n &\to \mathbf{U}^{n+\frac{1}{2}}\\
\mathbf{U}^{n*} &\to \mathbf{U}^{n+\frac{1}{2}*}\\
\mathbf{U}^{n+\frac{1}{2}} &\to \mathbf{U}^{n+1}\\
\boldsymbol{\nabla}\mathbf{U}^n &\to \boldsymbol{\nabla}\mathbf{U}^{n+\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}^n &\to \mathcal{E}^n\\
\mathcal{E}^n &\to \mathcal{E}^{n+\frac{1}{2}}\\
\mathcal{E}^{n*} &\to \mathcal{E}^{n+\frac{1}{2}*}\\
\boldsymbol{\nabla}\mathcal{E}^n &\to \boldsymbol{\nabla}\mathcal{E}^{n+\frac{1}{2}}
\end{aligned}
$$

4 Solver B - Radiation Hydrodynamics Grey Diffusion - Mixed finite element

We now derive a general mixed finite element approach for

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0(\mathbf{x}) = 1, \quad \mathbf{x} \in \mathcal{D} \tag{4.1a}$$

$$\boldsymbol{\mathcal{F}}_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\mathcal{D} \tag{4.1b}$$

where

$$\boldsymbol{\mathcal{F}}_0(\mathbf{x}) = D(\mathbf{x})\boldsymbol{\nabla}\mathcal{E}(\mathbf{x}). \tag{4.2}$$

4.1 Auxiliary notation and variables for $\boldsymbol{\mathcal{F}}_0$

First we discretize $\boldsymbol{\mathcal{F}}_0$ on N_n number of nodes per cell c, using continuous basis functions $b_j(\mathbf{x})$ such that

$$\boldsymbol{\mathcal{F}}_0(\mathbf{x}) \approx \sum_{j=1}^{N_n} (\boldsymbol{\mathcal{F}}_0)_j b_j(\mathbf{x}), \tag{4.3}$$

whilst keeping the cell-centered representation for $\mathcal{E}$. Next we discretize eq. (4.2) by applying a weight function $b_i(\mathbf{x})$ and integrating over the volume of the cell c,

$$\begin{aligned} \int_{V_c} b_i \boldsymbol{\mathcal{F}}_0 dV &= \int_{V_c} b_i D \boldsymbol{\nabla}\mathcal{E} dV \\ \sum_j \biggl[\int_{V_c} b_i b_j dV \biggr] (\boldsymbol{\mathcal{F}}_0)_j &= \int_{V_c} b_i D \boldsymbol{\nabla}\mathcal{E} dV. \end{aligned} \tag{4.4}$$

The integral coefficients on the left-hand side are generally known as the ij coefficients in the standard finite element mass-matrix, which we shall use in a moment to define a general scheme. The right hand side of the equation requires some treatment. We introduce the values $\mathcal{E}_{c,j}$ at the cell surface to remedy the discontinuities in the cell-centered $\mathcal{E}$ and to define auxiliary unknowns for developing the $\boldsymbol{\mathcal{F}}_0$ in finite element form. With these new variables declared, we next apply integration by parts to the right-hand side,

$$\int_{V_c} b_i D \boldsymbol{\nabla}\mathcal{E} dV = \int_{V_c} D \boldsymbol{\nabla}(b_i \mathcal{E}) dV - \int_{V_c} D \mathcal{E} \boldsymbol{\nabla} b_i dV. \tag{4.5}$$

Next we apply Gauss's divergence theorem on the first term on the right-hand side,

$$\int_{V_c} b_i D \boldsymbol{\nabla}\mathcal{E} dV = \sum_f \int_{S_f} D \mathbf{n}_f b_i \mathcal{E} dA - \int_{V_c} D \mathcal{E} \boldsymbol{\nabla} b_i dV, \tag{4.6}$$

after which we insert $\mathcal{E}_{c,j}$ in the first term on the right, since they are designated unknowns on the surface of the cell, and $\mathcal{E}_c$ into the right most term. Since $\mathcal{E}_c$ is cell-constant within the cell-domain, the D coefficient is also dependent only on $\mathcal{E}_c$ and therefore constant within cell c, hence denoted as D_c,

$$\int_V b_i D \boldsymbol{\nabla}\mathcal{E} dV = \sum_f \sum_j \biggl[D_c \mathbf{n}_f \int_{S_f} b_i b_j dA \biggr] \mathcal{E}_{c,j} - \biggl[D_c \int_V \boldsymbol{\nabla} b_i dV \biggr] \mathcal{E}_c. \tag{4.7}$$

Putting the developed right- and left-hand sides back together we then get,

$$\sum_j \biggl[\int_V b_i b_j dV \biggr] (\boldsymbol{\mathcal{F}}_0)_j = \sum_f \sum_j \biggl[D_c \mathbf{n}_f \int_{S_f} b_i b_j dA \biggr] \mathcal{E}_{c,j} - \biggl[D_c \int_V \boldsymbol{\nabla} b_i dV \biggr] \mathcal{E}_c. \tag{4.8}$$

This equation can be written in more succinct form as

$$\bar{M}_c \bar{\mathbf{F}}_c = D_c \bar{C}_c \boldsymbol{\mathcal{E}}_c \tag{4.9}$$

where the structure still needs to be defined (which follows). $\bar{M}_c$ is a square block-matrix with block-dimension $N_n \times N_n$, $\bar{\mathbf{F}}_c$ is a block-vector with block-dimension $N_n \times 1$, $\bar{C}_c$ is a rectangular block-matrix with block-dimension $N_n \times (N_f + 1)$. The vector $\boldsymbol{\mathcal{E}}_c$ is simply the cell-centered and surface unknowns for cell c, i.e., $\boldsymbol{\mathcal{E}}_c = [\mathcal{E}_c, \mathcal{E}_{n=0}, \dots, \mathcal{E}_{n=N_n-1}]^T$.

The dimension of the inner blocks of $\bar{M}_c$, $\bar{\mathbf{F}}_c$ and $\bar{C}_c$, depend on the number of dimensions, N_d, in the problem. For further reference we shall denote dimensions with d but it generally refers to $d \in [0,1,2] \mapsto [x,y,z]$ and vice-versa.

The block entries of $\bar{M}$ are small diagonal matrices,

$$(\bar{M})_{ij} = \text{diag}(M_{ij}, \dots, M_{ij})^{N_d \times N_d} \tag{4.10}$$

where M_{ij} are the elements of the standard finite element mass-matrix for cell c, i.e.,

$$M_{ij} = \int_V b_i b_j dV. \tag{4.11}$$

The block entries of $\bar{\mathbf{F}}$ are

$$(\bar{\mathbf{F}})_i = \begin{bmatrix} (\boldsymbol{\mathcal{F}}_0)_{i,x} \\ (\boldsymbol{\mathcal{F}}_0)_{i,y} \\ (\boldsymbol{\mathcal{F}}_0)_{i,z} \end{bmatrix}^{N_d \times 1} \tag{4.12}$$

obviously only up to y for 2D and only up to x for 1D. The entries of $\bar{C}_c$ are formed as follows. First the structure of $\bar{C}_c$ is such that

$$\text{block-row } i \text{ of } \bar{C}_c = \text{columns} \begin{pmatrix} \mathbf{C}_i^c & \mathbf{C}_{i,j=0}^s & \dots & \mathbf{C}_{i,j=N_n-1}^s \end{pmatrix}^{N_d \times (N_n+1)}. \tag{4.13}$$

We then define the vectors

$$\begin{aligned} \mathbf{G}_i &= \int_V \boldsymbol{\nabla} b_i dV \\ M_{ij}^f &= \int_{S_f} b_i b_j dA \end{aligned} \tag{4.14}$$

Then,

$$\mathbf{C}_i^c = \begin{bmatrix} -(\mathbf{G}_i)_x \\ -(\mathbf{G}_i)_y \\ -(\mathbf{G}_i)_z \end{bmatrix}^{N_d \times 1} \tag{4.15}$$

and

$$\mathbf{C}_{ij}^s = \begin{bmatrix} \sum_f n_{f,x} M_{ij}^f \\ \sum_f n_{f,y} M_{ij}^f \\ \sum_f n_{f,z} M_{ij}^f \end{bmatrix}^{N_d \times 1} \tag{4.16}$$

With these definitions in-hand we can see that the true dimensions of $\bar{M}$ is $N_d N_n \times N_d N_n$, that of $\bar{\mathbf{F}}$ is $N_d N_n \times 1$, and the true dimensions of $\bar{C}$ is $N_d N_n \times (N_n + 1)$.

Finally, we have the vector $\boldsymbol{\mathcal{E}}$ as

$$\boldsymbol{\mathcal{E}}_c = \begin{bmatrix} \mathcal{E}_c \\ \mathcal{E}_{n=0} \\ \vdots \\ \mathcal{E}_{n=N_n-1} \end{bmatrix}^{(N_n+1) \times 1}. \tag{4.17}$$

To get an expression for all of the nodal $\boldsymbol{\mathcal{F}}_0$'s we take the system form of the equation and we invert $\bar{M}$ to get, in coefficient form, expressions for nodal $\boldsymbol{\mathcal{F}}_0$'s,

$$\mathbf{F}_c = \begin{bmatrix} (\boldsymbol{\mathcal{F}}_0)_0 \\ \vdots \\ (\boldsymbol{\mathcal{F}}_0)_{N_n-1} \end{bmatrix} = \bar{M}_c^{-1} \bar{C}_c \boldsymbol{\mathcal{E}} = C_c^* \boldsymbol{\mathcal{E}}_c, \tag{4.18}$$

where $C_c^* = \bar{M}_c^{-1} \bar{C}_c$.

With this expression-form of the individual nodal $\boldsymbol{\mathcal{F}}_0$'s we need to modify the primary equation, eq. (4.1). Additionally, since we introduced additional variables in the form of the face-baced $\mathcal{E}_f$'s, we need to define additional equations to close the system. For the primary equations we will simply plug in the expressions for $\boldsymbol{\mathcal{F}}_0$, which is detailed in the next subsection. For additional equations we will use the interface between cells to enforce continuity of $\boldsymbol{\mathcal{F}}_0$ at the face, for each cell of the face.

4.2 Using the auxiliary notation in the primary equation

Using this coefficient-form in the primary equations is done by first integrating eq. (4.1) over the volume of cell c, assuming the coefficient matrix $\bar{M}^{-1}\bar{C}$ has been developed for cell c, after which we apply Gauss's divergence theorem,

$$\begin{aligned}\int_{V_c} \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0 dV &= V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0)\\ \int_{S_c} \mathbf{n} \cdot \boldsymbol{\mathcal{F}}_0 dA &= V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0)\\ \sum_f \left[\mathbf{n}_f \cdot \int_{S_f} \boldsymbol{\mathcal{F}}_0 dA \right] &= V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0).\end{aligned} \tag{4.19}$$

We now expand $\boldsymbol{\mathcal{F}}_0$,

$$\sum_j \sum_f \left[\mathbf{n}_f \cdot (\boldsymbol{\mathcal{F}}_0)_j \int_{S_f} b_j dA \right] = V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0), \tag{4.20}$$

define

$$S_{i,f} = \int_{S_f} b_i dA \tag{4.21}$$

$$\sum_j \sum_f \left[n_{f,x} S_{j,f} (\boldsymbol{\mathcal{F}}_0)_{j,x} + n_{f,y} S_{j,f} (\boldsymbol{\mathcal{F}}_0)_{j,y} + n_{f,z} S_{j,f} (\boldsymbol{\mathcal{F}}_0)_{j,z} \right] = V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0), \tag{4.22}$$

or

$$\sum_j \sum_f \sum_d \left[n_{f,d} S_{j,f} (\boldsymbol{\mathcal{F}}_0)_{j,d} \right] = V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0), \tag{4.23}$$

where d denotes dimension such that $d \in [0,1,2] \mapsto [x,y,z]$, the indices (j,d) of $(\boldsymbol{\mathcal{F}}_0)_{j,d}$ maps to a row in C^*, i.e.,

$$(j,d) \mapsto k \ : \ k = N_d j + d, \tag{4.24}$$

from which we get

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0 = \frac{1}{V_c} \sum_j \sum_f \sum_d \left[n_{f,d} S_{j,f} C^*_{(j,d)\mapsto \text{ row } k} \cdot \boldsymbol{\mathcal{E}}_c \right], \qquad \forall c, \tag{4.25}$$

If the indices of $\mathcal{E}_c$ are then mapped to global system indexes for the corresponding $\mathcal{E}_c$ and collection of $\mathcal{E}_f$'s then the system can be constructed.

4.3 Auxiliary equations

For each face-node we now require continuity of flux. This can generally be expressed as

$$\sum_c \sum_f \int_{S_f} \mathbf{n}_f \cdot (\boldsymbol{\mathcal{F}}_0)_j dA = 0 \tag{4.26}$$

from which we get

$$\sum_c \sum_f \sum_d \left[n_{f,d} S_{j,f} C^*_{(j,d)\mapsto \text{ row } k} \cdot \boldsymbol{\mathcal{E}}_c \right] = 0, \qquad \forall j. \tag{4.27}$$

5 Solver C - Radiation Hydrodynamics Grey Radiation with the Variable Eddington Factor (VEF) method

We first repeat eqs. (2.14) and (2.15),

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \boldsymbol{\Omega}\cdot\mathbf{u} \tag{5.1a}$$

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \tag{5.1b}$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\{\rho\mathbf{u}\otimes\mathbf{u}\} + \boldsymbol{\nabla}p = \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0, \tag{5.1c}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) + \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{5.1d}$$

$$\frac{\partial\mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}. \tag{5.1e}$$

$$\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\big) \tag{5.1f}$$

where the radiation moment equation has been obtained by dropping the energy exchange terms,

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} &= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 + \cancelto{0}{\sigma_a c\big(aT^4 - \mathcal{E}\big)\frac{\mathbf{u}}{c^2}} \\
&= -\frac{\sigma_t}{c}\Big[\boldsymbol{\mathcal{F}} - \mathcal{E}\mathbf{u} - \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\Big] \\
&= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\big).
\end{aligned}$$

Now, recall the definition of the radiation pressure tensor, $\boldsymbol{\mathcal{P}}$,

$$\boldsymbol{\mathcal{P}}(\mathbf{x},\nu,t) = \frac{1}{c}\int_{4\pi}\boldsymbol{\Omega}\otimes\boldsymbol{\Omega}\ I(\mathbf{x},\boldsymbol{\Omega},\nu,t)d\boldsymbol{\Omega}. \tag{5.2}$$

If we expand the tensor-product we get

$$\boldsymbol{\mathcal{P}} = \frac{1}{c}\int_{4\pi}\begin{bmatrix}\boldsymbol{\Omega}_x\boldsymbol{\Omega}_x & \boldsymbol{\Omega}_x\boldsymbol{\Omega}_y & \boldsymbol{\Omega}_x\boldsymbol{\Omega}_z \\ \boldsymbol{\Omega}_y\boldsymbol{\Omega}_x & \boldsymbol{\Omega}_y\boldsymbol{\Omega}_y & \boldsymbol{\Omega}_y\boldsymbol{\Omega}_z \\ \boldsymbol{\Omega}_z\boldsymbol{\Omega}_x & \boldsymbol{\Omega}_z\boldsymbol{\Omega}_y & \boldsymbol{\Omega}_z\boldsymbol{\Omega}_z\end{bmatrix} I(\boldsymbol{\Omega})\ d\boldsymbol{\Omega}. \tag{5.3}$$

The VEF-method involves the approximation

$$\begin{aligned}
\boldsymbol{\mathcal{P}} &\approx \{f\}\frac{1}{c}\int_{4\pi} I(\boldsymbol{\Omega})d\boldsymbol{\Omega} \\
\therefore \boldsymbol{\mathcal{P}} &= \{f\}\ \mathcal{E}
\end{aligned} \tag{5.4}$$

where $\{f\}$ is the variable Eddington factor computed as an angular-intensity weighted-average such that the entries of the tensor are given by

$$\{f\}\ :\ f_{ij} = \frac{\dfrac{1}{c}\displaystyle\int_{4\pi}\boldsymbol{\Omega}_i\boldsymbol{\Omega}_j I(\boldsymbol{\Omega})d\boldsymbol{\Omega}}{\dfrac{1}{c}\displaystyle\int_{4\pi} I(\boldsymbol{\Omega})d\boldsymbol{\Omega}} \qquad i,j\in[x,y,z]. \tag{5.5}$$

Now, rewriting our set of equations we get

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \boldsymbol{\Omega}\cdot\mathbf{u} \tag{5.6a}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \tag{5.6b}$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\{\rho\mathbf{u}\otimes\mathbf{u}\} + \boldsymbol{\nabla}p = \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0, \tag{5.6c}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) + \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{5.6d}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}. \tag{5.6e}$$

$$\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot(\{f\}\mathcal{E}) = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big) \tag{5.6f}$$

with

$$\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{F}} - (\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}) \tag{5.6g}$$

5.1 Definitions

We can cast the above equations into the following form

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \boldsymbol{\Omega}\cdot\mathbf{u} \tag{5.7a}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{5.7b}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}. \tag{5.7c}$$

$$\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot(\{f\}\mathcal{E}) = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big) \tag{5.7d}$$

where

$$\mathbf{S}_{rp} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \tag{5.7e}$$

$$S_{ea} = \sigma_a c(aT^4 - \mathcal{E}) \tag{5.7f}$$

$$S_{re} = S_{ea} - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{5.7g}$$

5.2 Temporal scheme - Implicit Euler Predictor, Crank-Nicolson Corrector

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{\Omega}\cdot\nabla I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \mathbf{\Omega}\cdot\mathbf{u} \tag{5.8a}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla\cdot\mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{5.8b}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot\boldsymbol{\mathcal{F}} = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}. \tag{5.8c}$$

5.2.1 Transport prephase

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{\Omega}\cdot\nabla I + \sigma_t(\mathbf{x})I = \frac{\sigma_s^n}{4\pi}c\mathcal{E}^n + \frac{\sigma_a^n}{4\pi}ac(T^4)^n - \frac{\sigma_t^n}{4\pi}\boldsymbol{\mathcal{F}}_0^n\cdot\frac{\mathbf{u}^n}{c} + \frac{\sigma_t^n}{\pi}\mathcal{E}^n\ \mathbf{\Omega}\cdot\mathbf{u}^n \tag{5.9a}$$

Develop $\{f\}^n$.

5.2.2 Predictor phase

$\tau = \frac{1}{\frac{1}{2}\Delta t}$

$$\tau(\mathbf{U}^{n*} - \mathbf{U}^n) + \nabla\cdot\mathcal{F}^H(\mathbf{U}^n) = \mathbf{0} \tag{5.10a}$$

$$\tau\left(\begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+\frac{1}{2}} - \begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n*}\right) = \begin{bmatrix} 0 \\ \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \end{bmatrix}^n \tag{5.10b}$$

$$\tau(E^{n+\frac{1}{2}} - E^{n*}) = -\theta_1 S_{ea}^{n+\frac{1}{2}} - \theta_2 S_{ea}^n + \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^n \tag{5.10c}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^n) + \theta_1\nabla\cdot\boldsymbol{\mathcal{F}}^{n+\frac{1}{2}} + \theta_2\nabla\cdot\boldsymbol{\mathcal{F}}^n = \theta_1 S_{ea}^{n+\frac{1}{2}} + \theta_2 S_{ea}^n - \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^n \tag{5.10d}$$

For S_{ea} and $\boldsymbol{\mathcal{F}}$ both at $n+\frac{1}{2}$:

$$\sigma^{n+\frac{1}{2}} = \rho^{n+\frac{1}{2}}\kappa(T^n) \tag{5.10e}$$

$$T^{4,n+\frac{1}{2}} = T^{4,n*} + \frac{4T^{3,n*}}{C_v}(e^{n+\frac{1}{2}} - e^{n*}) \tag{5.10f}$$

5.2.3 Corrector phase

$\tau = \frac{1}{\Delta t}$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^n) + \nabla\cdot\mathcal{F}^H(\mathbf{U}^{n+\frac{1}{2}}) = \mathbf{0} \tag{5.11a}$$

$$\tau\left(\begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+1} - \begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+\frac{1}{2}*}\right) = \begin{bmatrix} 0 \\ \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \end{bmatrix}^{n+\frac{1}{2}} \tag{5.11b}$$

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 S_{ea}^{n+1} - \theta_2 S_{ea}^n + \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.11c}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \theta_1\nabla\cdot\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\nabla\cdot\boldsymbol{\mathcal{F}}^n = \theta_1 S_{ea}^{n+1} + \theta_2 S_{ea}^n - \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.11d}$$

For S_{ea} and $\boldsymbol{\mathcal{F}}$ both at $n+1$:

$$\sigma^{n+1} = \rho^{n+1}\kappa(T^{n+\frac{1}{2}}) \tag{5.11e}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{5.11f}$$

5.2.4 General energy equations, Predictor and Corrector phase, with θ factors

Time integration scheme uses **implicit Euler** for the predictor phase and **Crank-Nicolson** in the corrector phase. Both these schemes can be represented wit a general θ-scheme where we define:

$$\begin{aligned} \theta_1 &\in [0,1] \\ \theta_2 &= 1-\theta_1. \end{aligned} \tag{5.12}$$

For implicit Euler, $\theta_1 = 1,\ \theta_2 = 0$, and for Crank-Nicolson, $\theta_1 = \theta_2 = \frac{1}{2}$. With these factors defined we can repeat the energy equations and apply a series of manipulations. First we attempt to segregate known terms from all unknown terms. Thereafter we eliminate the internal energy, e, from the two sets of equations to get a single formulation for the radiation energy, $\mathcal{E}$. The latter formulation forms a diffusion system that needs to be assembled and solved for $\mathcal{E}$.

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1\sigma_a^{n+1}c\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) - \theta_2 S_{ea}^n + \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.13a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \boldsymbol{\nabla\cdot}\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\mathcal{F}}^n\big) = \theta_1\sigma_a^{n+1}c\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) + \theta_2 S_{re}^n - \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.13b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{5.13c}$$

Define:

$$\begin{aligned} k_1 &= \theta_1\sigma_a^{n+1}c \\ k_2 &= \frac{4T^{3,n+\frac{1}{2}*}}{C_v} \end{aligned} \tag{5.14}$$

---

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) - \theta_2 S_{ea}^n + \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.15a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \boldsymbol{\nabla\cdot}\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\mathcal{F}}^n\big) = k_1\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) + \theta_2 S_{re}^n - \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.15b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{5.15c}$$

---

ungroup right-hand side elements by multiplying out terms within parentheses,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 aT^{4,n+1} + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n + \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.16a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \boldsymbol{\nabla\cdot}\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\mathcal{F}}^n\big) = k_1 aT^{4,n+1} - k_1\mathcal{E}^{n+1} + \theta_2 S_{re}^n - \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.16b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{5.16c}$$

---

now plug in the temperature equation into both the energy equations,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a\big(T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*})\big) + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n + \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.17a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n})+\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1}+\theta_2\boldsymbol{\mathcal{F}}^{n}\big)=k_1a\big(T^{4,n+\frac{1}{2}*}+k_2(e^{n+1}-e^{n+\frac{1}{2}*})\big)-k_1\mathcal{E}^{n+1}+\theta_2S^n_{re}-\left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.17b}$$

---

ungroup elements on the both the right-hand sides,

$$\tau(E^{n+1}-E^{n+\frac{1}{2}*})=-k_1aT^{4,n+\frac{1}{2}*}-k_1ak_2e^{n+1}+k_1ak_2e^{n+\frac{1}{2}*}+k_1\mathcal{E}^{n+1}-\theta_2S^n_{ea}+\left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.18a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n})+\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1}+\theta_2\boldsymbol{\mathcal{F}}^{n}\big)=k_1aT^{4,n+\frac{1}{2}*}+k_1ak_2e^{n+1}-k_1ak_2e^{n+\frac{1}{2}*}-k_1\mathcal{E}^{n+1}+\theta_2S^n_{re}-\left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.18b}$$

Define:

$$\begin{aligned}k_3&=-k_1aT^{4,n+\frac{1}{2}*}+k_1ak_2e^{n+\frac{1}{2}*}-\theta_2S^n_{ea}+\left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^{n+\frac{1}{2}}\\ k_4&=-k_1ak_2\end{aligned} \tag{5.19}$$

---

and plug them into the equations above,

$$\tau(E^{n+1}-E^{n+\frac{1}{2}*})=k_4e^{n+1}+k_1\mathcal{E}^{n+1}+k_3 \tag{5.20a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n})+\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1}+\theta_2\boldsymbol{\mathcal{F}}^{n}\big)=-k_4e^{n+1}-k_1\mathcal{E}^{n+1}-k_3 \tag{5.20b}$$

Note:

$$E^{n+1}=(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1}+\rho^{n+1}e^{n+1} \tag{5.21}$$

---

which gives,

$$\tau\big((\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1}+\rho^{n+1}e^{n+1}-E^{n+\frac{1}{2}*}\big)=k_4e^{n+1}+k_1\mathcal{E}^{n+1}+k_3 \tag{5.22a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n})+\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1}+\theta_2\boldsymbol{\mathcal{F}}^{n}\big)=-k_4e^{n+1}-k_1\mathcal{E}^{n+1}-k_3 \tag{5.22b}$$

---

ungroup the material energy in the first equation,

$$\tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1}+\tau\rho^{n+1}e^{n+1}-\tau E^{n+\frac{1}{2}*}=k_4e^{n+1}+k_1\mathcal{E}^{n+1}+k_3 \tag{5.23a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n})+\theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n+1}+\theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n}=-k_4e^{n+1}-k_1\mathcal{E}^{n+1}-k_3 \tag{5.23b}$$

---

and isolate the internal energy in the first equation,

$$(\tau\rho^{n+1}-k_4)e^{n+1}=k_1\mathcal{E}^{n+1}+k_3-\tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1}+\tau E^{n+\frac{1}{2}*} \tag{5.24a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n})+\theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n+1}+\theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n}=-k_4e^{n+1}-k_1\mathcal{E}^{n+1}-k_3 \tag{5.24b}$$

Define:

$$k_5 = \frac{k_1}{\tau\rho^{n+1} - k_4}$$
$$k_6 = \frac{k_3 - \tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau E^{n+\frac{1}{2}*}}{\tau\rho^{n+1} - k_4} \tag{5.25}$$

and plug these constants into the first equation above,

$$e^{n+1} = k_5\mathcal{E}^{n+1} + k_6 \tag{5.26a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^n = -k_1\mathcal{E}^{n+1} - k_3 - k_4e^{n+1} \tag{5.26b}$$

now plug the first equation into the second,

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^n = -k_1\mathcal{E}^{n+1} - k_3 - k_4k_5\mathcal{E}^{n+1} - k_4k_6 \tag{5.27a}$$

now collect all the $\mathcal{E}^{n+1}$ terms on the left-hand side,

$$\big(\tau + k_1 + k_4k_5\big)\mathcal{E}^{n+1} + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n+1} = -k_3 - k_4k_6 + \tau\mathcal{E}^n - \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^n \tag{5.28a}$$

5.3 Mixed Finite Element Method

We now turn our attention to just the radiation momentum equation,

$$\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot(\{f\}\mathcal{E}) = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big). \tag{5.29}$$

We assume that all the unknowns in this equation have a linear FE representation on a cell, except $\mathcal{E}$ and $\mathbf{u}$.

5.3.1 The radiation-flux at $n = 0$

One of the first items we will need in any temporal discretization is the old $\boldsymbol{\mathcal{F}}^n$. In order to get value of $\boldsymbol{\mathcal{F}}^n$, when starting the iterations, we simply use the radiation-momentum equation with no time derivative to get

$$\boldsymbol{\mathcal{F}}^n = -\frac{c}{\sigma_t^n}\boldsymbol{\nabla}\cdot(\{f\}^n\mathcal{E}^n) + (\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u})^n, \qquad \text{if } n = 0. \tag{5.30}$$

This equation, however, still requires a suitable spatial discretization. Applying a linear FEM first requires multiplying by a trial function, then integrating over volume

$$\int_V b_i\Big[\boldsymbol{\mathcal{F}}^n = -\frac{c}{\sigma_t^n}\boldsymbol{\nabla}\cdot(\{f\}^n\mathcal{E}^n) + (\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u})^n\Big]dV \tag{5.31}$$

Let us now consider the terms one-by-one. First are the $\boldsymbol{\mathcal{F}}$ terms. Since $\boldsymbol{\mathcal{F}} \approx \sum_j \boldsymbol{\mathcal{F}}_j b_j(\mathbf{x})$ we get

$$\int_V b_i\boldsymbol{\mathcal{F}}dV = \sum_j \boldsymbol{\mathcal{F}}_j\int_V b_ib_jdV \tag{5.32}$$

Second are the divergence terms. First we rewrite

$$\begin{aligned}\int_V b_i\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})dV &= \int_S \mathbf{n}\cdot(b_i\{f\}\mathcal{E})dA - \int_V \{f\}\mathcal{E}\cdot\boldsymbol{\nabla}b_idV \\ &= \sum_f\int_{S_f}\mathbf{n}_f\cdot(b_i\{f\}\mathcal{E})dA - \int_V\{f\}\mathcal{E}\cdot\boldsymbol{\nabla}b_idV \\ &= \sum_j\sum_f\mathbf{n}_f\cdot(\{f\}\mathcal{E})_j\int_{S_f}b_ib_jdA - \sum_j(\{f\}\mathcal{E})_j\cdot\int_V b_j\boldsymbol{\nabla}b_idV\end{aligned} \tag{5.33}$$

Last are the advection terms. If the velocity and radiation-energy are only considered to be cell-constant then we have

$$\int_V b_i \big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big) dV = \big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big)_c \int_V b_i dV \tag{5.34}$$

Putting all this together, we get

$$\sum_j \boldsymbol{\mathcal{F}}_j^n \int_V b_i b_j dV = -\frac{c}{\sigma_t^n}\Bigg[\sum_j\sum_f \mathbf{n}_f\cdot(\{f\}\mathcal{E})_j^n \int_{S_f} b_i dA - (\{f\}\mathcal{E})_c^n\cdot\int_V \boldsymbol{\nabla} b_i dV\Bigg] + \big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big)_c^n \int_V b_i dV \tag{5.35}$$

and for 1D $\{f\}$

$$\sum_j \boldsymbol{\mathcal{F}}_j^n \int_V b_i b_j dV = -\frac{c}{\sigma_t^n}\Bigg[\sum_j\sum_f \mathbf{n}_f\cdot(f\mathcal{E})_j^n \int_{S_f} b_i dA - (f\mathcal{E})_c^n\int_V \boldsymbol{\nabla} b_i dV\Bigg] + \big(1 + f_c^n\big)(\mathcal{E}\mathbf{u})_c^n \int_V b_i dV \tag{5.36}$$

which can be written as

$$\bar{M}_c \bar{\mathbf{F}}_c^n = -\frac{c}{\sigma_t}\bar{C}_c(\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + (1+f_c^n)\mathcal{E}_c^n C_{vol}\mathbf{u}_c^n \tag{5.37}$$

where $\bar{M}_c$ is the dimension-extended mass-matrix as defined for Solver B and the matrix $\bar{C}_c$ is also the same as defined for Solver B. The nodal vector $(\mathbf{f}\boldsymbol{\mathcal{E}})$ is a stack of firstly the cell-centered $f\mathcal{E}$, then the list of nodal values,

$$(\mathbf{f}\boldsymbol{\mathcal{E}}) = \begin{bmatrix} f_c\mathcal{E}_c \\ f_{c,0}\mathcal{E}_{c,0} \\ \vdots \\ f_{c,N_n-1}\mathcal{E}_{c,N_n-1} \end{bmatrix} \tag{5.38}$$

the matrix C_{vol} is a square block-matrix with block dimension $N_n{\times}1$ with block-structure

$$C_{vol} = \begin{bmatrix} V_{c,0}I \\ \vdots \\ V_{c,N_n-1}I \end{bmatrix} \tag{5.39}$$

where the identity matrices, I, all have dimension $N_d{\times}N_d$ and the coefficients $V_{c,i}$ are given by

$$V_{c,i} = \int_V b_i dV. \tag{5.40}$$

The resulting true dimensions of C_{vol} is therefore $N_dN_n{\times}N_d$.

In order to obtain an expression for the nodal radiation-fluxes we then take the inverse of $\bar{M}_c$ to get

$$\bar{\mathbf{F}}_c^n = -\frac{c}{\sigma_t}\bar{M}_c^{-1}\bar{C}_c(\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + (1+f_c^n)\mathcal{E}_c^n \bar{M}_c^{-1} C_{vol}\mathbf{u}_c^n \tag{5.41}$$

which we can express as

$$\bar{\mathbf{F}}_c^n = -\frac{c}{\sigma_t}\bar{C}_c^*(\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + (1+f_c^n)\mathcal{E}_c^n C_{vol}^*\mathbf{u}_c^n \tag{5.42}$$

where $\bar{C}_c^* = \bar{M}_c^{-1}\bar{C}_c$ and $C_{vol}^* = \bar{M}_c^{-1}C_{vol}$. However, using the identity

Additionally we require an expression for $\boldsymbol{\mathcal{F}}_0$. If we use the same notation for building $\bar{\mathbf{F}}$ from $\boldsymbol{\mathcal{F}}$, for $\bar{\mathbf{F}}_0$ from $\boldsymbol{\mathcal{F}}_0$, we get

$$\bar{\mathbf{F}}_{0,c}^n = \bar{\mathbf{F}}_c^n - (1+f_c^n)\mathcal{E}_c^n C_{vol}^*\mathbf{u}_c^n \tag{5.43}$$

5.3.2 The radiation-flux at $n+1$

We now seek a similar expression for the radiation-flux at timestep $n+1$. Our first discretization is a temporal theta-scheme discretization where we lag the advection terms,

$$\begin{aligned}&\frac{\tau}{c^2}(\boldsymbol{\mathcal{F}}^{n+1}-\boldsymbol{\mathcal{F}}^{n})+\theta_1\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}+\theta_2\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n}\\&=-\frac{\theta_1}{c}\sigma_t^{n+1}\boldsymbol{\mathcal{F}}^{n+1}-\frac{\theta_2}{c}\sigma_t^{n}\boldsymbol{\mathcal{F}}^{n}+\frac{\sigma_t^{n+\frac{1}{2}}}{c}\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}\end{aligned}\tag{5.44}$$

where $\tau=\dfrac{1}{\Delta t}$. We then multiply by $\dfrac{c^2}{\tau}$

$$\begin{aligned}&\boldsymbol{\mathcal{F}}^{n+1}-\boldsymbol{\mathcal{F}}^{n}+\theta_1\frac{c^2}{\tau}\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}+\theta_2\frac{c^2}{\tau}\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n}\\&=-\frac{\theta_1c^2}{c\tau}\sigma_t^{n+1}\boldsymbol{\mathcal{F}}^{n+1}-\frac{\theta_2c^2}{c\tau}\sigma_t^{n}\boldsymbol{\mathcal{F}}^{n}+\frac{\sigma_t^{n+\frac{1}{2}}c^2}{c\tau}\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}.\end{aligned}\tag{5.45}$$

Next we define,

$$\begin{aligned}a_1&=\theta_1\frac{c^2}{\tau}\\a_2&=\theta_2\frac{c^2}{\tau}\\a_3&=\sigma_t^{n+\frac{1}{2}}\frac{c}{\tau}\end{aligned}\tag{5.46}$$

to get,

$$\begin{aligned}&\boldsymbol{\mathcal{F}}^{n+1}-\boldsymbol{\mathcal{F}}^{n}+a_1\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}+a_2\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n}\\&=-\frac{a_1}{c}\sigma_t^{n+1}\boldsymbol{\mathcal{F}}^{n+1}-\frac{a_2}{c}\sigma_t^{n}\boldsymbol{\mathcal{F}}^{n}+a_3\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}.\end{aligned}\tag{5.47}$$

which we can rearrange as

$$\big(1+\frac{a_1}{c}\sigma_t^{n+1}\big)\boldsymbol{\mathcal{F}}^{n+1}+a_1\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}=\big(1-\frac{a_2}{c}\sigma_t^{n}\big)\boldsymbol{\mathcal{F}}^{n}-a_2\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n}+a_3\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}.\tag{5.48}$$

Next we define

$$\begin{aligned}a_4&=\frac{a_1}{1+\frac{a_1}{c}\sigma_t^{n+1}}\\a_5&=\frac{1-\frac{a_2}{c}\sigma_t^{n}}{1+\frac{a_1}{c}\sigma_t^{n+1}}\\a_6&=\frac{a_2}{1+\frac{a_1}{c}\sigma_t^{n+1}}\\a_7&=\frac{a_3}{1+\frac{a_1}{c}\sigma_t^{n+1}}\end{aligned}\tag{5.49}$$

to get

$$\boldsymbol{\mathcal{F}}^{n+1}+a_4\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}=a_5\boldsymbol{\mathcal{F}}^{n}-a_6\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n}+a_7\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}.\tag{5.50}$$

Now we apply our spatial discretization scheme by multiplying by trial functions, defined as the basis functions on each cell, then integrating over volume

$$\int_V b_i(\mathbf{x})\Big[\boldsymbol{\mathcal{F}}^{n+1}+a_4\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}=a_5\boldsymbol{\mathcal{F}}^{n}-a_6\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n}+a_7\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}\Big]dV.\tag{5.51}$$

For 1D $\{f\}$

$$\int_V b_i(\mathbf{x})\Big[\boldsymbol{\mathcal{F}}^{n+1}+a_4\boldsymbol{\nabla}\cdot(f\mathcal{E})^{n+1}=a_5\boldsymbol{\mathcal{F}}^{n}-a_6\boldsymbol{\nabla}(f\mathcal{E})^{n}+a_7\big(1+f_c^{n+\frac{1}{2}}\big)\mathcal{E}_c^{n+\frac{1}{2}}\mathbf{u}_c^{n+\frac{1}{2}}\Big]dV.\tag{5.52}$$

Using the expressions we developed for the $n=0$ case, we can similarly write

$$\bar{M}_c \bar{\mathbf{F}}_c^{n+1} + a_4 \bar{C}_c (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n+1} = a_5 \bar{M}_c \bar{\mathbf{F}}_c^{n} - a_6 \bar{C}_c (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n} + a_7 (1 + f_c^{n+\frac{1}{2}}) \mathcal{E}_c^{n+\frac{1}{2}} C_{vol} \mathbf{u}_c^{n+\frac{1}{2}}. \tag{5.53}$$

An expression for the nodal fluxes is then obtained by multiplying with the inverse of $\bar{M}_c$ to get

$$\bar{\mathbf{F}}_c^{n+1} = -a_4 \bar{M}_c^{-1} \bar{C}_c (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n+1} + a_5 \bar{\mathbf{F}}_c^{n} - a_6 \bar{M}_c^{-1} \bar{C}_c (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n} + a_7 (1 + f_c^{n+\frac{1}{2}}) \mathcal{E}_c^{n+\frac{1}{2}} \bar{M}_c^{-1} C_{vol} \mathbf{u}_c^{n+\frac{1}{2}}. \tag{5.54}$$

which we can express as

$$\bar{\mathbf{F}}_c^{n+1} = -a_4 \bar{C}_c^* (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n+1} + a_5 \bar{\mathbf{F}}_c^{n} - a_6 \bar{C}_c^* (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n} + a_7 (1 + f_c^{n+\frac{1}{2}}) \mathcal{E}_c^{n+\frac{1}{2}} C_{vol}^* \mathbf{u}_c^{n+\frac{1}{2}}. \tag{5.55}$$

using the definitions of $\bar{C}_c^*$ and C_{vol}^* as developed for the $n=0$ case.

5.3.3 Using the expression for $\boldsymbol{\mathcal{F}}^{n+1}$ in the primary equation

We start with the form that $\boldsymbol{\mathcal{F}}$ appears in the radiation-energy equation,

$$\begin{aligned}
\nabla \cdot \boldsymbol{\mathcal{F}} &= \frac{1}{V_c} \int_V \nabla \cdot \boldsymbol{\mathcal{F}} dV \\
&= \frac{1}{V_c} \int_S \mathbf{n} \cdot \boldsymbol{\mathcal{F}} dA \\
&= \frac{1}{V_c} \sum_f \mathbf{n}_f \cdot \int_{S_f} \boldsymbol{\mathcal{F}} dA \\
&= \frac{1}{V_c} \sum_j \sum_f \mathbf{n}_f \cdot \boldsymbol{\mathcal{F}}_j \int_{S_f} b_j dA \\
\nabla \cdot \boldsymbol{\mathcal{F}} &= \frac{1}{V_c} \sum_f \sum_j \sum_d n_{f,d} (\boldsymbol{\mathcal{F}}_j)_d \int_{S_f} b_j dA.
\end{aligned} \tag{5.56}$$

This now serves as a template for each of the terms in the expression. The term containing $\boldsymbol{\mathcal{F}}^n$ is already covered by the template itself, therefore, by substituting $\int_{S_f} b_j dA = S_{j,f}$, we get

$$\nabla \cdot \bar{\mathbf{F}}_c^n \mapsto \frac{1}{V_c} \sum_f \sum_j \sum_d n_{f,d} S_{j,f} (\boldsymbol{\mathcal{F}}_j)_d. \tag{5.57}$$

The second set of terms we need to address have the general form,

$$-a \bar{C}_c^* (\mathbf{f}\boldsymbol{\mathcal{E}})_c$$

where a is either a_4 or a_6 and thus,

$$-a \bar{C}_c^* (\mathbf{f}\boldsymbol{\mathcal{E}})_c \mapsto \frac{-a}{V_c} \sum_f \sum_j \sum_d \Big[n_{f,d} S_{j,f} C^*_{(j,d) \mapsto \text{ row } k} \cdot (\mathbf{f}\boldsymbol{\mathcal{E}}) \Big]. \tag{5.58}$$

The last term to consider is the advection term,

$$a_7 (1 + f_c^{n+\frac{1}{2}}) \mathcal{E}_c^{n+\frac{1}{2}} C_{vol}^* \mathbf{u}_c^{n+\frac{1}{2}} \mapsto \frac{a_7}{V_c} \sum_f \sum_j \sum_d \Big[n_{f,d} S_{j,f} * C^*_{vol,(j,d) \mapsto \text{ row } k} \cdot \mathbf{u}_c^{n+\frac{1}{2}} \Big] \tag{5.59}$$

5.3.4 Using the expression for $\boldsymbol{\mathcal{F}}^{n+1}$ in the auxiliary equations

For each face-node we now require continuity of flux. This can generally be expressed as

$$\sum_c \sum_f \int_{S_f} \mathbf{n}_f \cdot \boldsymbol{\mathcal{F}}_j^{n+1} dA = 0 \tag{5.60}$$

from which we again have to develop the three types of term, i.e., the term with $\boldsymbol{\mathcal{F}}$, the term with $\{f\}\mathcal{E}$ and the term with $\mathcal{E}\mathbf{u}$. Fortunately, the previously determined forms are easily extended to here, i.e., with only the relevant face and node indices changing.

$$\begin{aligned}
&\sum_c \sum_f \sum_d n_{f,d} S_{j,f} \left[-a_4 C^*_{(j,d)\mapsto \text{ row } k} \cdot (\mathbf{f}\boldsymbol{\mathcal{E}})^{n+1} \right] \\
&= \sum_c \sum_f \sum_d n_{f,d} S_{j,f} \left[-a_5 (\boldsymbol{\mathcal{F}}_j)^n_d + a_6 C^*_{(j,d)\mapsto \text{ row } k} \cdot (\mathbf{f}\boldsymbol{\mathcal{E}})^n - a_7 n_{f,d} S_{j,f} * C^*_{vol,(j,d)\mapsto \text{ row } k} \cdot \mathbf{u}_c^{n+\frac{1}{2}} \right]
\end{aligned} \tag{5.61}$$

6 Multifrequency

6.1 Definitions

From fundamental transformations we have, for differential quantities,

$$\begin{aligned} dE_0 &= \frac{E_0}{E} dE \\ d\mathbf{\Omega}_0 &= \left(\frac{E}{E_0}\right)^2 d\mathbf{\Omega} \\ d\mathbf{\Omega}_0 dE_0 &= \frac{E}{E_0} d\mathbf{\Omega} dE. \end{aligned} \tag{6.1}$$

The ratio of energies $\frac{E_0}{E}$ is given by

$$\frac{E_0}{E} = \gamma(1 - \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c}) \tag{6.2}$$

for which we can apply a first-order Taylor-series expansion about $\mathbf{u} = 0$ to get

$$\frac{E_0}{E} \approx (1 - \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c}). \tag{6.3}$$

Also,

$$\frac{E}{E_0} \approx (1 + \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c}). \tag{6.4}$$

Both these derivations are shown in the appendices. Co-moving cross sections are approximated by

$$\begin{aligned} \sigma(E_0) &= \sigma(E) + \frac{\partial \sigma}{\partial E}(E_0 - E) \\ &= \sigma(E) + \frac{\partial \sigma}{\partial E} E\left(\frac{E_0}{E} - 1\right) \\ &= \sigma(E) + \frac{\partial \sigma}{\partial E} E\left((1 - \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c}) - 1\right) \\ &= \sigma(E) - \frac{\partial \sigma}{\partial E} E \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c}. \end{aligned} \tag{6.5}$$

Note that this cross section seems to drop the directional dependence of σ_0.

$$I_0(\mathbf{\Omega}_0, E_0) = \left(\frac{E_0}{E}\right)^3 I(\mathbf{\Omega}, E) \tag{6.6}$$

6.2 Equilibrium quantities

$$\begin{aligned} &\frac{1}{c}\frac{\partial I(\mathbf{x}, \mathbf{\Omega}, E, t)}{\partial t} + \mathbf{\Omega} \cdot \nabla I(\mathbf{x}, \mathbf{\Omega}, E, t) + \sigma_t(\mathbf{x}, E) I(\mathbf{x}, \mathbf{\Omega}, E, t) \\ =&\frac{\sigma_s(\mathbf{x}, E)}{4\pi}\phi(E) + \sigma_a(\mathbf{x}, E) B(E, T(\mathbf{x}, t)) \\ &+\left[\left(\sigma_t + E\frac{\partial \sigma_a}{\partial E}\right) I + \frac{\sigma_s}{4\pi}\left(2\phi - E\frac{\partial \phi}{\partial E}\right) + 2\sigma_a B(E, T) - B(E, T) E \frac{\partial \sigma_a}{\partial E} - \sigma_a E \frac{\partial B(E, T)}{\partial E}\right] \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c} \\ &- \frac{\sigma_s}{4\pi}\left(\boldsymbol{\mathcal{F}} - E\frac{\partial \boldsymbol{\mathcal{F}}}{\partial E}\right) \cdot \frac{\mathbf{u}}{c} \end{aligned} \tag{6.7}$$

6.2.1 Equilibrium Angular intensity, I_{eq}

From eq. (6.6) we get

$$I(\mathbf{\Omega}, E) = \left(\frac{E}{E_0}\right)^3 I_0(\mathbf{\Omega}_0, E_0) \tag{6.8}$$

but in equilibrium I_0 becomes the Planck function, and therefore

$$I_{eq}(\mathbf{\Omega}, E) = \left(\frac{E}{E_0}\right)^3 B(E_0), \tag{6.9}$$

into which we can plug the first-order expansion for $\frac{E_0}{E}$,

$$= \left(1 + \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c}\right)^3 B(E_0) \tag{6.10}$$

Next we expand the first and second portion separately (I think, but wtfk)

$$= \left(1 + 3\mathbf{\Omega} \cdot \frac{\mathbf{u}}{c}\right)\left(B(E_0)|_{\beta=0} + O(\beta^2)\frac{\partial B}{\partial E_0}\frac{\partial E_0}{\partial \beta}\bigg|_{\beta=0}\right)\mathbf{\Omega} \cdot \frac{\mathbf{u}}{c} \tag{6.11}$$

No fucking clue what happened here, only that $E = E_0$ if $\beta = 0$,

$$= B(E) + (3B(E) - E\frac{\partial B}{\partial E})\mathbf{\Omega} \cdot \frac{\mathbf{u}}{c} + O(\beta^2) \tag{6.12}$$

6.3 Radiation-energy equation and radiation-momentum equation

The radiation energy and momentum equation to $O(v/c)$ is

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = \int_0^\infty \sigma_a(4\pi B - \phi)dE + \int_0^\infty (\sigma_a + E\frac{\partial \sigma_a}{\partial E} - \sigma_s)\overrightarrow{F} dE \cdot \frac{\mathbf{u}}{c} \tag{6.13}$$

$$\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{P}} = -\int_0^\infty \frac{1}{c}\sigma_t \overrightarrow{F} dE + \int_0^\infty dE(\sigma_a + E\frac{\partial \sigma_a}{\partial E} + \sigma_s)\boldsymbol{\mathcal{P}} \cdot \frac{\mathbf{u}}{c} + \int_0^\infty dE(\sigma_s\phi + 4\pi\sigma_a B)\frac{\mathbf{u}}{c^2} \tag{6.14}$$

The latter we write in the form

$$\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{P}} = -\int_0^\infty \frac{1}{c}\sigma_t \overrightarrow{F} dE + \int_0^\infty \overrightarrow{S}^{(1)}_{rp} dE \tag{6.15}$$

where

$$\overrightarrow{S}^{(1)}_{rp} = (\sigma_a + E\frac{\partial \sigma_a}{\partial E} + \sigma_s)\boldsymbol{\mathcal{P}} \cdot \frac{\mathbf{u}}{c} + (\sigma_s\phi + 4\pi\sigma_a B)\frac{\mathbf{u}}{c^2} \tag{6.16}$$

A Angular integration identities

Identity A-1

$$\int_{4\pi} d\mathbf{\Omega} = 4\pi.$$

Identity A-2

$$\int_{4\pi} \mathbf{\Omega}\ d\mathbf{\Omega} = 0.$$

Identity A-3 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \cdot \mathbf{v}\ d\mathbf{\Omega} = 0.$$

Identity A-4 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \cdot \boldsymbol{\nabla}(\mathbf{\Omega} \cdot \mathbf{v})\ d\mathbf{\Omega} = \frac{4\pi}{3} \boldsymbol{\nabla} \cdot \mathbf{v}.$$

Identity A-5 Given the scalar, a,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \boldsymbol{\nabla} a \Big)\ d\mathbf{\Omega} = \frac{4\pi}{3} \boldsymbol{\nabla} a.$$

Identity A-6 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \mathbf{v} \Big)\ d\mathbf{\Omega} = \frac{4\pi}{3} \mathbf{v}.$$

Identity A-7 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \boldsymbol{\nabla}(\mathbf{\Omega} \cdot \mathbf{v}) \Big)\ d\mathbf{\Omega} = 0.$$

B Boundary and initial conditions for radiation hydrodynamic problems

In a one dimensional simulation we can simulate steady-state shocks by setting the appropriate pre- and post-shock conditions. Pre-shock conditions will be denoted with a subscript L whereas post-shock conditions will be denoted with a subscript R.

B.1 Hydrodynamics only

With no radiation energy present we wish to have $\mathcal{F}_L^H = \mathcal{F}_R^H$, therefore

$$\begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}_R . \tag{B.1}$$

Here we have three equations but 4 unknowns, i.e., ρ, u, p and e. Fortunately, we can express both e and p in terms of temperature since

$$p = (\gamma - 1)\rho e$$

and

$$e = C_v T.$$

Therefore,

$$p = (\gamma - 1)\rho C_v T$$

and

$$\begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \rho C_v T u + pu \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \rho C_v T u + pu \end{bmatrix}_R . \tag{B.2}$$

When the left state is known then we can frame these equations as seeking the non-linear solution of

$$\mathbf{F}\begin{pmatrix} \rho_R \\ T_R \\ u_R \end{pmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \gamma \rho C_v T u \end{bmatrix}_R - \mathcal{F}_L^H = \mathbf{0} \tag{B.3}$$

or simply

$$\mathbf{F}(\mathbf{x}) = \mathcal{F}^H(\mathbf{x}_R) - \mathcal{F}_L^H = \mathbf{0}. \tag{B.4}$$

to which Newton-iteration can be applied in the form

$$\mathbf{x}_R^{\ell+1} = \mathbf{x}_R^{\ell} - J^{-1}(\mathbf{x}_R^{\ell})\mathbf{F}(\mathbf{x}_R^{\ell})$$

where the Jacobian matrix, J, is given by

$$J = \begin{bmatrix} u & 0 & \rho \\ u^2 + (\gamma-1)C_v T & (\gamma-1)\rho C_v & 2\rho u \\ \frac{1}{2}u^3 + \gamma C_v T u & \gamma \rho C_v u & \frac{3}{2}\rho u^2 + \gamma \rho C_v T \end{bmatrix} \tag{B.5}$$

Note: The initial guess, $\mathbf{x}^0$ cannot be the same as $\mathbf{x}_L$ since the iteration will terminate immediately. Generally the values need to be perturbed sufficiently such that $\rho_R > \rho_L$, $T_R > T_L$ and $u_R < a_R$ where a is the sound-speed.

B.2 Hydrodynamics with radiation energy

With radiation energy present we are concerned with the following set of equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{B.6a}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \boldsymbol{\nabla} p = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}, \tag{B.6b}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u} \tag{B.6c}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\Big(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\Big) + \frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}\mathbf{u}\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}. \tag{B.6d}$$

which for a steady-steady, one dimensional simulation becomes

$$\boldsymbol{\nabla} \cdot (\rho u) = 0 \tag{B.7a}$$

$$\boldsymbol{\nabla} \cdot (\rho u^2) + \boldsymbol{\nabla} p = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}, \tag{B.7b}$$

$$\boldsymbol{\nabla} \cdot [(E+p)u] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot u \tag{B.7c}$$

$$\boldsymbol{\nabla}\cdot\Big(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\Big) + \frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}u\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}u. \tag{B.7d}$$

Additionally, far away from the interface the co-moving frame radiation flux, $\boldsymbol{\mathcal{F}}_0$, is zero, therefore

$$\boldsymbol{\mathcal{F}}_0 = -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} = 0$$

and the equation set becomes

$$\boldsymbol{\nabla} \cdot (\rho u) = 0 \tag{B.8a}$$

$$\boldsymbol{\nabla} \cdot (\rho u^2) + \boldsymbol{\nabla} p = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}, \tag{B.8b}$$

$$\boldsymbol{\nabla} \cdot [(E+p)u] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot u \tag{B.8c}$$

$$\frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}u\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}u. \tag{B.8d}$$

Now, adding the last equation to the third ,we get

$$\boldsymbol{\nabla} \cdot (\rho u) = 0 \tag{B.9a}$$

$$\boldsymbol{\nabla} \cdot (\rho u^2) + \boldsymbol{\nabla} p + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = 0, \tag{B.9b}$$

$$\boldsymbol{\nabla} \cdot [(E+p)u] + \frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}u\big) = 0. \tag{B.9c}$$

We now express internal energy, e, the pressure, p, and the radiation energy, $\mathcal{E}$, in terms of temperature

$$\boldsymbol{\nabla} \cdot (\rho u) = 0 \tag{B.10a}$$

$$\boldsymbol{\nabla} \cdot (\rho u^2) + \boldsymbol{\nabla}\big((\gamma-1)\rho C_v T\big) + \frac{1}{3}\boldsymbol{\nabla} aT^4 = 0, \tag{B.10b}$$

$$\boldsymbol{\nabla} \cdot [(\frac{1}{2}\rho u^2 + \rho C_v T + (\gamma-1)\rho C_v T)u] + \frac{4}{3}\boldsymbol{\nabla}\big(aT^4 u\big) = 0. \tag{B.10c}$$

Finally we integrate this equation set over the entire domain to get

$$\begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T + \frac{1}{3}aT^4 \\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u + \frac{4}{3}aT^4 u \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T + \frac{1}{3}aT^4 \\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u + \frac{4}{3}aT^4 u \end{bmatrix}_R \tag{B.11a}$$

Similar to the previous case we can now define

$$\mathbf{F}\begin{pmatrix}\rho_R\\ T_R\\ u_R\end{pmatrix} = \begin{bmatrix}\rho u\\ \rho u^2+(\gamma-1)\rho C_v T+\frac{1}{3}aT^4\\ \frac{1}{2}\rho u^3+\gamma\rho C_v T u+\frac{4}{3}aT^4 u\end{bmatrix}_R - \begin{bmatrix}\rho u\\ \rho u^2+(\gamma-1)\rho C_v T+\frac{1}{3}aT^4\\ \frac{1}{2}\rho u^3+\gamma\rho C_v T u+\frac{4}{3}aT^4 u\end{bmatrix}_L = \mathbf{0}, \tag{B.12}$$

where the subscript L quantities are all known. Applying Newton-iteration to this equation again is

$$\mathbf{x}_R^{\ell+1} = \mathbf{x}_R^{\ell} - J^{-1}(\mathbf{x}_R^{\ell})\mathbf{F}(\mathbf{x}_R^{\ell})$$

where the Jacobian matrix, J is given by

$$J = \begin{bmatrix} u & 0 & \rho\\ u^2+(\gamma-1)C_v T & (\gamma-1)\rho C_v+\frac{4}{3}aT^3 & 2\rho u\\ \frac{1}{2}u^3+\gamma C_v T u & \gamma\rho C_v u+\frac{16}{3}aT^3 u & \frac{3}{2}\rho u^2+\gamma\rho C_v T+\frac{4}{3}aT^4\end{bmatrix} \tag{B.13}$$

Example mach 3 conditions, $C_v = 0.14472799784454$ and $\gamma = \frac{5}{3}$:

```
[0]   rho0  1.00000000e+00 u0    3.80431331e-01 T0    1.00000000e-01 e0    1.44727998e-02 radE0 1.37223549e-06
[0]   rho1  3.00185103e+00 u1    1.26732249e-01 T1    3.66260705e-01 e1    5.30081785e-02 radE1 2.46939153e-04
```

Note: The initial guess, $\mathbf{x}^0$ cannot be the same as $\mathbf{x}_L$ since the iteration will terminate immediately. Generally the values need to be perturbed sufficiently such that $\rho_R > \rho_L$, $T_R > T_L$, $u_R < a_R$ where a is the sound-speed, and $\mathcal{E}_R > \mathcal{E}_L$.

C Tensor Algebra

Most of these notations are obtained from [4].

C.1 Identities

Identity C-1 Given $f(x,y,z)$ and $\mathbf{a} = [a_x, a_y, a_z]$.

$$f\boldsymbol{\nabla}\cdot\mathbf{a} = \boldsymbol{\nabla}\cdot(f\mathbf{a}) - \mathbf{a}\cdot\boldsymbol{\nabla}f$$

Proof:

$$f\boldsymbol{\nabla}\cdot\mathbf{a} = f\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) \tag{C.1}$$

$$= f\frac{\partial a_x}{\partial x} + f\frac{\partial a_y}{\partial y} + f\frac{\partial a_z}{\partial z}, \tag{C.2}$$

applying the product rule of differentiation,

$$= \frac{\partial}{\partial x}(fa_x) - a_x\frac{\partial f}{\partial x} \tag{C.3}$$

$$+ \frac{\partial}{\partial y}(fa_y) - a_y\frac{\partial f}{\partial y} \tag{C.4}$$

$$+ \frac{\partial}{\partial z}(fa_z) - a_z\frac{\partial f}{\partial z} \tag{C.5}$$

by observing the vertical alignment here we get

$$f\boldsymbol{\nabla}\cdot\mathbf{a} = \boldsymbol{\nabla}\cdot(f\mathbf{a}) - \mathbf{a}\cdot\boldsymbol{\nabla}f \tag{C.6}$$

C.2 Tensor product of two vectors $\mathbf{a}\otimes\mathbf{b}$

Also called the *dyadic product*. Given vector $\mathbf{a}$ of size $N\times 1$ and vector $\mathbf{b}$ of size $M\times 1$, then the tensor product of $\mathbf{a}$ and $\mathbf{b}$ results in a rank 2 tensor and is given by

$$\mathbf{a}\otimes\mathbf{b} = \begin{bmatrix} a_0b_0 & a_0b_1 & \dots & a_0b_{M-1} \\ a_1b_0 & a_1b_1 & \dots & a_1b_{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1}b_0 & a_{N-1}b_1 & \dots & a_{N-1}b_{M-1} \end{bmatrix} \tag{C.7}$$

with resulting dimensions $N\times M$.

C.3 Dot-product of a vector with a tensor, $\mathbf{a}\bullet\{t\}$

Under the same topic of tensor product notation we can also discuss the **dot product of scalar and a rank 2 tensor**. The dot-product of a vector $\mathbf{a}$ and a tensor $\{\mathbf{t}\}$, commonly written as $\mathbf{a}\cdot\{\mathbf{t}\}$, which results in a vector of size N, can be understood using one of two thought patterns:

- Thought pattern 1: Classical component-wise dot product

$$\begin{aligned}
\mathbf{a}\cdot\{\mathbf{t}\} &= \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} \cdot \begin{bmatrix} t_{00} & t_{01} & \dots & t_{0(N-1)} \\ t_{10} & t_{11} & \dots & t_{1(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ t_{(N-1)0} & t_{(N-1)1} & \dots & t_{(N-1)(N-1)} \end{bmatrix} \\
&= \begin{bmatrix} a_0 t_{00} & + & a_1 t_{01} & + & \dots & + & a_{N-1} t_{0(N-1)} \\ a_0 t_{10} & + & a_1 t_{11} & + & \dots & + & a_{N-1} t_{1(N-1)} \\ \vdots & + & \vdots & + & \ddots & + & \vdots \\ a_0 t_{(N-1)0} & + & a_1 t_{(N-1)1} & + & \dots & + & a_{N-1} t_{(N-1)(N-1)} \end{bmatrix} \\
&= \{t\}\cdot\mathbf{a}
\end{aligned} \tag{C.8}$$

- Thought pattern 2 (preferred): Dot product of vectors

$$\mathbf{a}\cdot\{\mathbf{t}\} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_{(N-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{a}\ \cdot\ \mathbf{t}_0 \\ \mathbf{a}\ \cdot\ \mathbf{t}_1 \\ \vdots \\ \mathbf{a}\ \cdot\ \mathbf{t}_{(N-1)} \end{bmatrix} = \{t\}\cdot\mathbf{a} \tag{C.9}$$

where the latter thought pattern requires the rank 2 tensor two be represented as a vector of vectors (or a matrix if you prefer).

C.4 Finite element discretization of the divergence of a tensor, i.e., $\boldsymbol{\nabla}\cdot\boldsymbol{\tau}$

Short-hand notation. $b_i \equiv b_i(\mathbf{x})$. For this section we seek a general way to handle

$$\int_V b_i \boldsymbol{\nabla}\cdot\boldsymbol{\tau} dV. \tag{C.10}$$

We start by writing the tensor as a block vector

$$\boldsymbol{\tau} = \begin{bmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_{N-1} \end{bmatrix}, \tag{C.11}$$

therefore,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\nabla}\cdot\mathbf{t}_0 \\ \vdots \\ \boldsymbol{\nabla}\cdot\mathbf{t}_{N-1} \end{bmatrix}. \tag{C.12}$$

If we now multiply by the trial space function, b_i, and integrate then we essentially have

$$\int_V b_i \boldsymbol{\nabla}\cdot\boldsymbol{\tau} dV = \begin{bmatrix} \int_V b_i \boldsymbol{\nabla}\cdot\mathbf{t}_0 dV \\ \vdots \\ \int_V b_i \boldsymbol{\nabla}\cdot\mathbf{t}_{N-1} dV \end{bmatrix}, \tag{C.13}$$

after we use identity C-1 to get

$$\int_V b_i \boldsymbol{\nabla}\cdot\boldsymbol{\tau} dV = \begin{bmatrix} \int_V b_i \boldsymbol{\nabla}\cdot\mathbf{t}_0 dV \\ \vdots \\ \int_V b_i \boldsymbol{\nabla}\cdot\mathbf{t}_{N-1} dV \end{bmatrix} = \begin{bmatrix} \int_V \boldsymbol{\nabla}\cdot(b_i\mathbf{t}_0) dV - \int_V \mathbf{t}_0\cdot\boldsymbol{\nabla} b_i dV \\ \vdots \\ \int_V \boldsymbol{\nabla}\cdot(b_i\mathbf{t}_{N-1}) dV - \int_V \mathbf{t}_{N-1}\cdot\boldsymbol{\nabla} b_i dV \end{bmatrix}, \tag{C.14}$$

Now we apply Gauss' divergence theorem on the first terms

$$\int_V b_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau} dV = \begin{bmatrix} \displaystyle\int_S \mathbf{n} \cdot (b_i \mathbf{t}_0) dA \\ \vdots \\ \displaystyle\int_S \mathbf{n} \cdot (b_i \mathbf{t}_{N-1}) dA \end{bmatrix} - \begin{bmatrix} \displaystyle\int_V \mathbf{t}_0 \cdot \boldsymbol{\nabla} b_i dV \\ \vdots \\ \displaystyle\int_V \mathbf{t}_{N-1} \cdot \boldsymbol{\nabla} b_i dV \end{bmatrix}, \tag{C.15}$$

which we can write as

$$\int_V b_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau} dV = \int_S \mathbf{n} \cdot (b_i \boldsymbol{\tau}) dA - \int_V \boldsymbol{\tau} \cdot \boldsymbol{\nabla} b_i dV. \tag{C.16}$$

We can now expand $\boldsymbol{\tau}$ into basis functions and segregate the surface-integrals into face-integrals to get

$$\int_V b_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau} dV = \sum_j \sum_f \left[\mathbf{n}_f \cdot \boldsymbol{\tau}_j \int_{S_f} b_i b_j dA \right] - \sum_j \left[\boldsymbol{\tau}_j \cdot \int_V b_j \boldsymbol{\nabla} b_i dV \right]. \tag{C.17}$$

What we can additionally do here is to lump the $\boldsymbol{\tau}_j$'s in the last term to cell-centered values $\boldsymbol{\tau}_c$ so that we have

$$\int_V b_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau} dV = \sum_j \sum_f \left[\mathbf{n}_f \cdot \boldsymbol{\tau}_j \int_{S_f} b_i b_j dA \right] - \boldsymbol{\tau}_c \cdot \int_V \boldsymbol{\nabla} b_i dV, \tag{C.18}$$

which enables us to develop a system form.

Define the following,

$$M_{ij}^f = \int_{S_f} b_i b_j dA$$

$$\mathbf{G}_i = \int_V \boldsymbol{\nabla} b_i dV.$$

We then start to define the matrix C_c as

$$\begin{aligned} \boldsymbol{\tau}_c \cdot \int_V \boldsymbol{\nabla} b_i dV &= \begin{bmatrix} \mathbf{G}_{i,x} & \mathbf{G}_{i,y} & \mathbf{G}_{i,z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{G}_{i,x} & \mathbf{G}_{i,y} & \mathbf{G}_{i,z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{G}_{i,x} & \mathbf{G}_{i,y} & \mathbf{G}_{i,z} \end{bmatrix} \begin{bmatrix} \mathbf{t}_{c,x} \\ \mathbf{t}_{c,y} \\ \mathbf{t}_{c,z} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{G}_i^T & 0 & 0 \\ 0 & \mathbf{G}_i^T & 0 \\ 0 & 0 & \mathbf{G}_i^T \end{bmatrix} \begin{bmatrix} \mathbf{t}_{c,x} \\ \mathbf{t}_{c,y} \\ \mathbf{t}_{c,z} \end{bmatrix} \end{aligned} \tag{C.19}$$

D Taylor expansions

D.1 $\frac{E_0}{E}$

$$\frac{E_0}{E} = \gamma(1 - \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c}) \tag{D.1}$$

Pretend velocity along z-axis

$$= \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} (1 - \mu u) = f(u) \tag{D.2}$$

$$= f(u = 0) + \left.\frac{\partial f}{\partial u}\right|_{u=0} (u) \tag{D.3}$$

using Wolfram alpha

$$= 1 - \mu u \tag{D.4}$$

Unpretending

$$\frac{E_0}{E} = 1 - \mathbf{\Omega} \cdot \mathbf{u} \tag{D.5}$$

D.2 $\frac{E}{E_0}$

$$\frac{E}{E_0} = \frac{1}{\gamma(1 - \mathbf{\Omega} \cdot \frac{\mathbf{u}}{c})} \tag{D.6}$$

Similar process to that above

$$\frac{E}{E_0} = 1 + \mathbf{\Omega} \cdot \mathbf{u} \tag{D.7}$$

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E Roderigues's formula

Roderigues' formula for the rotation of a vector $\mathbf{v}$ about a unit vector $\mathbf{a}$ with right-hand rule

$$\mathbf{v}_{rotated} = \cos\theta\mathbf{v} + (\mathbf{a}\cdot\mathbf{v})(1-\cos\theta)\mathbf{a} + \sin\theta(\mathbf{a}\times\mathbf{v}) \tag{E.1}$$

In matrix form

$$\mathbf{v}_{rotated} = A\mathbf{v} \tag{E.2}$$

where

$$A = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{E.3}$$

and

$$R = I + \sin\theta A + (1-\cos\theta)A^2 \tag{E.4}$$