Radiative heat transfer solver with fluid motion

Jan I.C. Vermaak[1,2], Jim E. Morel[1,2]

[1]Center for Large Scale Scientific Simulations, Texas A&M Engineering Experiment Station, College Station, Texas, USA.
[2]Nuclear Engineering Department, Texas A&M University, College Station, Texas, USA.

Abstract:
Work is work for some, but for some it is play.

Keywords: hydrodynamics

Contents

1 Definitions

1.1 Independent variables

We refer to the following independent variables:

- Position in the cartesian space $\{x, y, z\}$ is denoted with $\mathbf{x}$ and each component having units $[cm]$.
- Direction, $\{\varphi, \theta\}$, is denoted with $\mathbf{\Omega}$ which takes on the form

$$\mathbf{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \text{ and/or } \mathbf{\Omega} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix},$$

  where φ is the azimuthal-angle and θ is the polar-angle, both in spherical coordinates. Commonly, $\cos\theta$, is denoted with μ. The general dimension of angular phase space is $[steridian]$.

- Photon frequency, ν in $[Hertz]$ or $[s^{-1}]$.
- Time, t in $[s]$.

1.2 Dependent variables

We use the following basic dependent variables:

- The foundation of the dependent unknowns is the **radiation angular intensity**, $I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)$ with units $[Joule/cm^2{-}s{-}steradian{-}Hz]$. We often use the corresponding angle-integral of this quantity, $\phi(\mathbf{x}, \nu, t)$, and define it as
$$\phi(\mathbf{x}, \nu, t) = \mathcal{E}c = \int_{4\pi} I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)\ d\boldsymbol{\Omega} \tag{1.1}$$
with units $[Joule/cm^2{-}s{-}Hz]$. Where c is the speed of light.

- The **radiation energy density**, $\mathcal{E}$, is
$$\mathcal{E}(\mathbf{x}, \nu, t) = \frac{\phi}{c} = \frac{1}{c}\int_{4\pi} I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t)\ d\boldsymbol{\Omega} \tag{1.2}$$
with units $[Joule/cm^3{-}Hz]$.

- The **radiation energy flux**, $\boldsymbol{\mathcal{F}}$, is
$$\boldsymbol{\mathcal{F}}(\mathbf{x}, \nu, t) = \int_{4\pi} \boldsymbol{\Omega}\ I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) d\boldsymbol{\Omega} \tag{1.3}$$

- **Radiation pressure**, $\boldsymbol{\mathcal{P}}$, is
$$\boldsymbol{\mathcal{P}}(\mathbf{x}, \nu, t) = \frac{1}{c}\int_{4\pi} \boldsymbol{\Omega} \otimes \boldsymbol{\Omega} I(\mathbf{x}, \boldsymbol{\Omega}, \nu, t) d\boldsymbol{\Omega} \tag{1.4}$$
and is a tensor.

1.3 Blackbody radiation

A blackbody radiation source, $B(\nu, T)$, is properly described by **Planck's law**,
$$B(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{1.5}$$
with units $[Joule/cm^2{-}s{-}steridian - Hz]$ where h is Planck's constant and k_B is the Boltzmann constant.

If we integrate the blackbody source over all angle-space and frequencies then we get the mean radiation intensity from a blackbody at temperature T as
$$\begin{aligned}\int_0^\infty \int_{4\pi} B(\nu, T)\ d\boldsymbol{\Omega} d\nu &= \int_0^\infty \int_{4\pi} \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1}\ d\boldsymbol{\Omega} d\nu \\ &= 4\pi \int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1}\ d\nu \\ &= acT^4,\end{aligned} \tag{1.6}$$
with units $[Joule/cm^2{-}s{-}steridian]$ and where a is the **blackbody radiation constant** given by
$$a = \frac{8\pi^5 k_B^4}{15h^3c^3}. \tag{1.7}$$

In both cases this unfortunately is only the intensity. Following Kirchoff's law, which states that the emission and absorption of radiation must be equal in equilibrium, we can determine the **blackbody emission rate**, S_{bb}, from the absorption rate as

$$S_{bb}(\nu, T) = \rho\kappa(\nu)B(\nu, T), \tag{1.8}$$

with units $[Joule/cm^3{-}s{-}steridian{-}Hz]$ where ρ is the material density $[g/cm^3]$ and κ is the opacity $[cm^2/g]$. The combination $\rho\kappa$ is also equal to the macroscopic absorption cross section σ_a, therefore $\rho\kappa(\nu) = \sigma_a$. Data for the opacity of a material is normally available in the form of either the **Rosseland opacity**, κ_{Rs}, or the **Planck opacity**, κ_{Pl}.

2 Conservation equations

2.1 Conservation equation - Radiative transfer

The basic statement of conservation, is

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},\nu,t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla I(\mathbf{x},\mathbf{\Omega},\nu,t) - \sigma_t(\mathbf{x},\nu)I(\mathbf{x},\mathbf{\Omega},\nu,t) \\ &+ \int_0^\infty \int_{4\pi} \frac{\nu}{\nu'}\sigma_s(\mathbf{x},\nu'\to\nu,\mathbf{\Omega}'\cdot\mathbf{\Omega})I(\mathbf{x},\mathbf{\Omega}',\nu',t)d\nu' d\mathbf{\Omega}' \\ &+ \sigma_a(\mathbf{x},\nu)B(\nu,T(\mathbf{x},t)) + S\end{aligned} \tag{2.1}$$

where S is any other sources/sinks of radiation intensity.

2.2 Radiative transfer assuming isotropic Thompson scattering

Assuming Thomson-scattering[1] is the only form of scattering, gives

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},\nu,t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla I(\mathbf{x},\mathbf{\Omega},\nu,t) - \sigma_t(\mathbf{x},\nu)I(\mathbf{x},\mathbf{\Omega},\nu,t) \\ &+ \frac{\sigma_s(\mathbf{x},\nu)}{4\pi}c\mathcal{E}(\mathbf{x},\nu) + \sigma_a(\mathbf{x},\nu)B(\nu,T(\mathbf{x},t)) + S\end{aligned} \tag{2.2}$$

where S is any other sources/sinks of radiation intensity.

Using energy instead of frequency, $\nu \to E$:

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},E,t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla I(\mathbf{x},\mathbf{\Omega},E,t) - \sigma_t(\mathbf{x},E)I(\mathbf{x},\mathbf{\Omega},E,t) \\ &+ \frac{\sigma_s(\mathbf{x},E)}{4\pi}c\mathcal{E}(\mathbf{x},E) + \sigma_a(\mathbf{x},E)B(E,T(\mathbf{x},t)) + S\end{aligned} \tag{2.3}$$

where S is any other sources/sinks of radiation intensity.

2.3 Radiative transfer with material motion corrections

Applying relativistic corrections for a material in motion, we can derive

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},E,t)}{\partial t} &= -\mathbf{\Omega}\cdot\nabla I(\mathbf{x},\mathbf{\Omega},E,t) - \left(\frac{E_0}{E}\right)\sigma_t(\mathbf{x},E_0)I(\mathbf{x},\mathbf{\Omega},E,t) \\ &+ \left(\frac{E}{E_0}\right)^2 \frac{\sigma_s(\mathbf{x},E)}{4\pi}\int_{4\pi}\left(\frac{E_0}{E'}\right)I(\mathbf{x},\mathbf{\Omega}',E',t)d\mathbf{\Omega}' + \left(\frac{E}{E_0}\right)^2\sigma_a(\mathbf{x},E_0)B(E_0,T(\mathbf{x},t)) + S,\end{aligned} \tag{2.4}$$

where

$$E_0 = E\gamma\left(1 - \mathbf{\Omega}\cdot\frac{\mathbf{u}}{c}\right) \tag{2.5}$$

$$\gamma = \left[1 - \left(\frac{||\mathbf{u}||}{c}\right)^2\right]^{-\frac{1}{2}} \tag{2.6}$$

$$\frac{E_0}{E'} = \gamma\left(1 - \mathbf{\Omega}'\cdot\frac{\mathbf{u}}{c}\right) \tag{2.7}$$

$$E' = E\frac{1 - \mathbf{\Omega}\cdot\frac{\mathbf{u}}{c}}{1 - \mathbf{\Omega}'\cdot\frac{\mathbf{u}}{c}} \tag{2.8}$$

[1]Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle. The particle's kinetic energy- as well as the photon's frequency, does not change in such a scattering. The scattering is also isotropic.

2.4 Radiative transfer with material velocity dependencies expanded to $\mathcal{O}(v/c)$

$$
\begin{aligned}
&\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},E,t)}{\partial t}+\mathbf{\Omega}\cdot\boldsymbol{\nabla} I(\mathbf{x},\mathbf{\Omega},E,t)+\sigma_t(\mathbf{x},E)I(\mathbf{x},\mathbf{\Omega},E,t)\\
=&\frac{\sigma_s(\mathbf{x},E)}{4\pi}\phi(E)+\sigma_a(\mathbf{x},E)B(E,T(\mathbf{x},t))\\
&+\left[\left(\sigma_t+E\frac{\partial\sigma_a}{\partial E}\right)I+\frac{\sigma_s}{4\pi}\left(2\phi-E\frac{\partial\phi}{\partial E}\right)+2\sigma_a B(E,T)-B(E,T)\frac{\partial\sigma_a}{\partial E}-\sigma_a E\frac{\partial B(E,T)}{\partial E}\right]\mathbf{\Omega}\cdot\frac{\mathbf{u}}{c}\\
&-\frac{\sigma_s}{4\pi}\left(\boldsymbol{\mathcal{F}}-E\frac{\partial\boldsymbol{\mathcal{F}}}{\partial E}\right)\cdot\frac{\mathbf{u}}{c}
\end{aligned}
\tag{2.9}
$$

Radiation energy equation:
Obtained by integrating the transport equation over energy and angle

$$
\begin{aligned}
\frac{\partial\mathcal{E}(\mathbf{x},t)}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t)=&\int_0^\infty\sigma_a(\mathbf{x},E)\big(4\pi B(E,T)-\phi(\mathbf{x},E,t)\big)dE\\
&+\int_0^\infty\left(\sigma_a+E\frac{\partial\sigma_a}{\partial E}-\sigma_s(E)\right)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}dE
\end{aligned}
\tag{2.10}
$$

Radiation momentum equation:
Obtained by first multiplying by $\frac{1}{c}\mathbf{\Omega}$, then integrating over all directions and energies,

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}}=&-\int_0^\infty\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}dE\\
&+\int_0^\infty\big(\sigma_s\phi+\sigma_a 4\pi B(E,T)\big)\frac{\mathbf{u}}{c^2}dE\\
&+\int_0^\infty\left(\sigma_a+E\frac{\partial\sigma_a}{\partial E}+\sigma_s\right)\boldsymbol{\mathcal{P}}\cdot\frac{\mathbf{u}}{c}dE
\end{aligned}
\tag{2.11}
$$

2.5 Grey Radiative Transfer

$$
\begin{aligned}
&\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},t)}{\partial t}+\mathbf{\Omega}\cdot\boldsymbol{\nabla} I(\mathbf{x},\mathbf{\Omega},t)+\sigma_t(\mathbf{x})I(\mathbf{x},\mathbf{\Omega},t)\\
=&\frac{\sigma_s}{4\pi}\phi+\frac{\sigma_a}{4\pi}acT^4\\
&+\left[\sigma_t I+\frac{\sigma_s}{4\pi}2\phi+2\sigma_a\frac{1}{4\pi}acT^4-\sigma_a E\frac{\partial B(E,T)}{\partial E}\right]\mathbf{\Omega}\cdot\frac{\mathbf{u}}{c}\\
&-\frac{\sigma_s}{4\pi}\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}
\end{aligned}
\tag{2.12}
$$

Radiation energy equation:
Obtained by integrating Eq. (2.12) over energy and angle

$$
\frac{\partial\mathcal{E}(\mathbf{x},t)}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t)=\sigma_a c\big(aT^4-\mathcal{E}\big)+\big(\sigma_a-\sigma_s\big)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}
\tag{2.13}
$$

Radiation momentum equation:
Obtained by first multiplying Eq. (2.12) by $\frac{1}{c}\mathbf{\Omega}$, then integrating over all directions and energies,

$$
\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}}=-\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}+\big(\sigma_s c\mathcal{E}+\sigma_a acT^4\big)\frac{\mathbf{u}}{c^2}+\sigma_t\boldsymbol{\mathcal{P}}\cdot\frac{\mathbf{u}}{c}
\tag{2.14}
$$

2.6 Grey Diffusion Approximation

Approximating the angular dependence of $I(\boldsymbol{\Omega})$ with a P_1 spherical harmonic expansion, such that the entries of $\boldsymbol{\mathcal{P}}$ are given by

$$(\boldsymbol{\mathcal{P}})_{i,j} = \frac{1}{3}\mathcal{E}\delta_{i,j}, \tag{2.15}$$

the radiation energy equation is unaffected but the radiation momentum equation changes. We repeat the radiation energy equation below, and the altered radiation moment equations:

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \big(\sigma_a - \sigma_s\big)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}, \tag{2.16}$$

$$\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \big(\sigma_s c\mathcal{E} + \sigma_a acT^4\big)\frac{\mathbf{u}}{c^2} + \sigma_t\frac{1}{3}\mathcal{E}\frac{\mathbf{u}}{c}. \tag{2.17}$$

Useful transformations:

$$\mathcal{E}_0 = \mathcal{E} - \frac{2}{c^2}\boldsymbol{\mathcal{F}}\cdot\mathbf{u} \tag{2.18a}$$

$$\mathcal{E} = \mathcal{E}_0 + \frac{2}{c^2}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{2.18b}$$

$$\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{F}} - \big(\mathcal{E}\mathbf{u} + \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\big) \tag{2.18c}$$

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_0 + \big(\mathcal{E}_0\mathbf{u} + \boldsymbol{\mathcal{P}}_0\cdot\mathbf{u}\big) \tag{2.18d}$$

$$\boldsymbol{\mathcal{P}}_0 = \boldsymbol{\mathcal{P}} - \frac{2}{c^2}\mathbf{u}\otimes\boldsymbol{\mathcal{F}} \tag{2.18e}$$

$$\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_0 + \frac{2}{c^2}\mathbf{u}\otimes\boldsymbol{\mathcal{F}}_0 \tag{2.18f}$$

With the P_1 approximation

$$\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{F}} - \frac{4}{3}\mathcal{E}\mathbf{u} \tag{2.18g}$$

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_0 + \frac{4}{3}\mathcal{E}\mathbf{u} \tag{2.18h}$$

Applying these transformations the radiation energy- and moment equation can be expressed as

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) = \sigma_a c\big(aT^4 - \mathcal{E}_0\big) - \sigma_t\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}, \tag{2.19}$$

$$\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 + \sigma_a c\big(aT^4 - \mathcal{E}\big)\frac{\mathbf{u}}{c^2}. \tag{2.20}$$

Several simplifications to these equations are made. Firstly arriving at the expression for the radiation energy equation,

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c}, \tag{2.21}$$

then the radiation momentum equation,

$$\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \tag{2.22}$$

from which we can get expression for $\boldsymbol{\mathcal{F}}_0$ and $\boldsymbol{\mathcal{F}}$ in terms of $\mathcal{E}$ as

$$\boldsymbol{\mathcal{F}}_0 = -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} \tag{2.23}$$

and

$$\begin{aligned} \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} &= -\frac{\sigma_t}{c}\left(\boldsymbol{\mathcal{F}} - \frac{4}{3}\mathcal{E}\mathbf{u}\right) \\ \therefore \boldsymbol{\mathcal{F}} &= -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} + \frac{4}{3}\mathcal{E}\mathbf{u}. \end{aligned} \tag{2.24}$$

These expressions for $\boldsymbol{\mathcal{F}}_0$ and $\boldsymbol{\mathcal{F}}$ are both then inserted into the radiation energy equation as follows

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) &= \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t \boldsymbol{\mathcal{F}}_0 \cdot \frac{\mathbf{u}}{c} \\ \rightarrow \frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\bigg(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} + \frac{4}{3}\mathcal{E}\mathbf{u}\bigg) &= \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t\bigg(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\bigg)\cdot\frac{\mathbf{u}}{c} \\ \rightarrow \frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\bigg(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\bigg) + \frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u}) &= \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}.\end{aligned} \tag{2.25}$$

Arriving at a **diffusion form** of the **radiation energy equation**,

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\bigg(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\bigg) + \frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u}) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}. \tag{2.26}$$

2.7 Conservation equation for fluid flow

The governing equations we consider here are the Euler equations defined as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \tag{2.27}$$

$$\frac{\partial (\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\{\rho\mathbf{u}\otimes\mathbf{u}\} + \boldsymbol{\nabla}p = \mathbf{f}, \tag{2.28}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = q \tag{2.29}$$

where ρ is the fluid density, $\mathbf{u} = [u_x, u_y, u_z] = [u, v, w]$ is the fluid velocity in cartesian coordinates, p is the fluid pressure, E is the material energy-density comprising kinetic energy-density, $\frac{1}{2}\rho||\mathbf{u}||^2$, and internal energy-density, ρe, such that $E = \frac{1}{2}\rho||\mathbf{u}||^2 + \rho e$, where e is the specific internal energy. The values q and $\mathbf{f}$ are abstractly used here as energy- and moment- sources/sinks, respectively.

The ideal gas law provides the closure relation

$$p = (\gamma - 1)\rho e \tag{2.30}$$

where γ is the ratio of the constant-pressure specific heat, c_p, to the constant-volume specific heat, c_v, i.e., $\gamma = \frac{c_p}{c_v}$, and is a material property.

Coupling terms:

$$\begin{aligned}\mathbf{f} &= \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \\ &= -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}q &= -\bigg(\sigma_a c(aT^4 - \mathcal{E}) - \sigma_t\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c}\bigg) \\ &= \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\end{aligned} \tag{2.32}$$

2.8 The set of Radiation Hydrodynamics Grey Diffusion Equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{2.33a}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \boldsymbol{\nabla} p = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}, \tag{2.33b}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u} \tag{2.33c}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\Big(- \frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\Big) + \frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}\mathbf{u}\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}. \tag{2.33d}$$

where

$$E = \frac{1}{2}\rho||\mathbf{u}||^2 + \rho e, \tag{2.33e}$$

$$p = (\gamma - 1)\rho e, \tag{2.33f}$$

$$T = \frac{1}{C_v} e \tag{2.33g}$$

3 Notations

First we define the following terms

- The radiation momentum source

$$\mathbf{S}_{rp} = \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \tag{3.1a}$$

- The radiation energy source

$$S_{re} = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u} \tag{3.1b}$$

- The conserved hydrodynamic variables

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho \mathbf{u} \\ E \end{bmatrix} \tag{3.1c}$$

- The hydrodynamic flux

$$\mathcal{F}^H = \begin{bmatrix} \rho u \\ \rho u u + p \\ \rho u v \\ \rho u w \\ (E+p)u \end{bmatrix} \tag{3.1d}$$

- The radiation energy current

$$\mathbf{J} = -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} \tag{3.1e}$$

Next, we use these terms to define a more condensed version of the RHGD equations.

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{3.2}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big) = S_{re}. \tag{3.3}$$

4 Overview of temporal numerical scheme

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{4.1a}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big) = S_{re}. \tag{4.1b}$$

4.1 Predictor phase

$\tau = \dfrac{1}{\frac{1}{2}\Delta t}$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^n) + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}^n) = \mathbf{0} \tag{4.2a}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}*} - \mathcal{E}^n) + \left(\frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big)\right)^n = 0 \tag{4.2b}$$

$$\tau(\mathbf{U}^{n+\frac{1}{2}} - \mathbf{U}^{n+\frac{1}{2}*})_{0,1} = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \end{bmatrix}^n \tag{4.2c}$$

$$\tau(\mathbf{U}^{n+\frac{1}{2}} - \mathbf{U}^{n+\frac{1}{2}*})_0 = \frac{1}{2}\sigma_a^{n+\frac{1}{2}} c\Big(\mathcal{E}^{n+\frac{1}{2}} + \mathcal{E}^n - a\big(T^{4,n+\frac{1}{2}} + T^{4,n}\big)\Big) - \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^n \tag{4.2d}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^{n+\frac{1}{2}*}) + \frac{1}{2}\boldsymbol{\nabla}\cdot\big(\mathbf{J}^{n+\frac{1}{2}} + \mathbf{J}^n\big) = \frac{1}{2}\sigma_a^{n+\frac{1}{2}} c\Big(a\big(T^{4,n+\frac{1}{2}} + T^{4,n}\big) - \mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^n\Big) + \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^n \tag{4.2e}$$

$$T^{4,n+\frac{1}{2}} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+\frac{1}{2}} - e^{n+\frac{1}{2}*}) \tag{4.2f}$$

4.2 Corrector phase

$\tau = \dfrac{1}{\Delta t}$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^{n}) + \boldsymbol{\nabla}\cdot\mathcal{F}^{H}(\mathbf{U}^{n+\frac{1}{2}}) = \mathbf{0} \tag{4.3a}$$

$$\tau(\mathbf{U}^{n+1} - \mathbf{U}^{n+\frac{1}{2}*})_{0,1} = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \end{bmatrix}^{n+\frac{1}{2}} \tag{4.3b}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}*} - \mathcal{E}^{n}) + \left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right)^{n+\frac{1}{2}} = 0 \tag{4.3c}$$

$$\tau(\mathbf{U}^{n+1} - \mathbf{U}^{n+\frac{1}{2}*})_{0} = \frac{1}{2}\sigma_a^{n+1}c\Big(\mathcal{E}^{n+1} + \mathcal{E}^{n} - a\big(T^{4,n+1} + T^{4,n}\big)\Big) - \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{4.3d}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \frac{1}{2}\boldsymbol{\nabla}\cdot\big(\mathbf{J}^{n+1} + \mathbf{J}^{n}\big) = \frac{1}{2}\sigma_a^{n+1}c\Big(a\big(T^{4,n+1} + T^{4,n}\big) - \mathcal{E}^{n+1} - \mathcal{E}^{n}\Big) + \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{4.3e}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{4.3f}$$

5 Finite Volume Spatial Discretization

We now integrate $\mathbf{S}_{rp}$, S_{re}, the hydrodynamic equations and the radiation energy equation over the volume of a cell with index c and volume V_c to get

$$V_c \mathbf{S}_{rp} = \frac{1}{3} \sum_{f}^{N_{f,c}-1} \left(\mathbf{A}_f \mathcal{E}_f \right) \tag{5.1}$$

$$V_c \mathbf{S}_{re} = V_c \sigma_a c (a T_c^4 - \mathcal{E}_c) + \frac{1}{3} \sum_{f}^{N_{f,c}-1} \left(\mathbf{A}_f \cdot \mathcal{E}_f \mathbf{u}_f \right) \tag{5.2}$$

$$V_c \frac{\partial \mathbf{U}}{\partial t} + \sum_{f}^{N_{f,c}-1} \left(\mathbf{A}_f \cdot \mathcal{F}_f^H \right) = \begin{bmatrix} 0 \\ -V_c \mathbf{S}_{rp} \\ -V_c S_{re} \end{bmatrix} \tag{5.3}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \sum_{f}^{N_{f,c}-1} \left(\mathbf{A}_f \cdot \mathbf{J}_f \right) + \frac{4}{3} \sum_{f}^{N_{f,c}-1} \left(\mathbf{A}_f \cdot \mathcal{E}_f \mathbf{u}_f \right) = V_c S_{re} \tag{5.4}$$

where $\mathbf{A}_f$ denotes the area-vector $A_f \mathbf{n}$ and all quantities with an f subscript as of yet unresolved.

A Angular integration identities

Identity A-1

$$\int_{4\pi} d\mathbf{\Omega} = 4\pi.$$

Identity A-2

$$\int_{4\pi} \mathbf{\Omega}\ d\mathbf{\Omega} = 0.$$

Identity A-3 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \cdot \mathbf{v}\ d\mathbf{\Omega} = 0.$$

Identity A-4 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \cdot \boldsymbol{\nabla}(\mathbf{\Omega} \cdot \mathbf{v})\ d\mathbf{\Omega} = \frac{4\pi}{3} \boldsymbol{\nabla} \cdot \mathbf{v}.$$

Identity A-5 Given the scalar, a,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \boldsymbol{\nabla} a \Big)\ d\mathbf{\Omega} = \frac{4\pi}{3} \boldsymbol{\nabla} a.$$

Identity A-6 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \mathbf{v} \Big)\ d\mathbf{\Omega} = \frac{4\pi}{3} \mathbf{v}.$$

Identity A-7 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \boldsymbol{\nabla}(\mathbf{\Omega} \cdot \mathbf{v}) \Big)\ d\mathbf{\Omega} = 0.$$

References

[1] John I. Castor, *Radiation Hydrodynamics*, UCRL-BOOK-134209-Rev-1, Lawrence Livermore National Laboratory, November 2003.

[2] Toro E.F., *Riemann Solvers and Numerical Methods for Fluid Dynamics - A Practical Introduction*, third edition, Springer, 2009.

[3] Moukalled F., Mangani L., Darwish M., *The Finite Volume Method in Computational Fluid Dynamics - An Advanced Introduction with OpenFOAM® and Matlab®*, Springer, 2016.

[4] McClarren R.G., Lowrie R.B., *The effects of slope limiting on asymptotic-preserving numerical methods for hyperbolic conservation laws*, Journal of Computational Physics, vol 227 p9711-9726, 2008.

[5] Timmers F.X., *Exact Riemann Solver*, Website: https://cococubed.com/code_pages/exact_riemann.shtml, accessed April 22, 2022.

B Roderigues's formula

Roderigues' formula for the rotation of a vector **v** about a unit vector **a** with right-hand rule

$$\mathbf{v}_{rotated} = \cos\theta\mathbf{v} + (\mathbf{a}\cdot\mathbf{v})(1-\cos\theta)\mathbf{a} + \sin\theta(\mathbf{a}\times\mathbf{v}) \tag{B.1}$$

In matrix form

$$\mathbf{v}_{rotated} = A\mathbf{v} \tag{B.2}$$

where

$$A = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{B.3}$$

and

$$R = I + \sin\theta A + (1-\cos\theta)A^2 \tag{B.4}$$