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research note

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Radiation Hydrodynamics

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Subject: Introduction to Radiative Shocks

1 Review of Hydrodynamics

In this section we state the equations that govern the dynamics of fluid motion.

1.1 Navier-Stokes Equations

Assume the following:

- $|\vec{v}|/c \ll 1$ (non-relativistic flow)
- $K_n \ll 1$, where K_n is the Knudsen number, defined as

$$K_n = \frac{\text{molecular mean-free-path}}{\text{gradient length scale } (L)}.$$

- There are no external body forces or energy sources (such as radiation); these easily can be added later.

The fluid motion of a single material is then governed by the *compressible Navier-Stokes* equations:

$$\partial_t \rho + \partial_i(\rho v_i) = 0, \tag{1a}$$

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v_j) + \partial_i p = \partial_j \tau_{ij} \tag{1b}$$

$$\partial_t(\rho E) + \partial_j\left[v_j(\rho E + p)\right] = \partial_j(v_i \tau_{ij} - q_j), \tag{1c}$$

where $E = e + \frac{1}{2}|\vec{v}|^2$ (specific total energy). Note that the left-hand side of eq. (1) is the Euler equations, which you are already familiar with.

Assuming a *Newtonian Fluid* and *Stokes' Hypothesis*, the stress tensor τ_{ij} may be approximated by

$$\tau_{ij} = \mu\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}\delta_{ij}\partial_k v_k\right). \tag{2}$$

where μ is the fluid viscosity. Assuming that the heat flux follows Fourier law, then

$$q_i = -\kappa \partial_i T, \tag{3}$$

where κ is the fluid heat conductivity. Note that τ_{ij} and q_i result in diffusion terms, which smooth the solution.

In one space dimension (1-D), the N-S equations may be written very compactly as

$$\partial_t U + \partial_x F(U) = 0, \tag{4}$$

where

$$U = \begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \rho v \\ \rho v^2 + p - \tau \\ v(\rho E + p - \tau) + q \end{pmatrix}, \tag{5}$$

where

$$\tau = \frac{4}{3}\mu \partial_x v, \qquad q = -\kappa \partial_x T. \tag{6}$$

The details of the N-S equations will not be covered here. But there are some important parallels between radiation transport and hydrodynamics:

- $K_n \ll 1$ is an analogous assumption to the equilibrium-diffusion limit in transport. If this assumption does not hold, a full Boltzmann equation (or a higher-moment closure thereof) must be solved, just as in transport.
- In the equilibrium-diffusion limit of radiation hydrodynamics, we will see that one effect of the radiation is to effectively increase κ in eq. (3).

1.2 Equation of State / Constitutive Relations

Looking at eqs. (4,5), the unknowns are $(\rho, v, e, p, T, \mu, \kappa)$. But we only have three equations. In order to close the system, we will need what are referred to as *constitutive relations*, which relate the atomic behavior of the material to our macroscopic model. For the N-S equations, it is assumed the constitutive relations are algebraic relations. Keep in mind that this is a huge assumption; some materials, or regimes, require additional evolution equations that are beyond the scope of these notes.

Typically, μ and κ are given as functions of T (and maybe ρ). The relationship between the variables (ρ, e, p, T), and any other thermodynamic quantity, is a constitutive relation referred to as the *equation of state* (EOS). A complete EOS has the following very important property:

Any thermodynamic quantity may computed by knowing two other thermodynamic quantities.

Equipped with an EOS, our unknowns may now be chosen as, for example, the variables (ρ, v, T). Then the EOS supplies the functions $e(\rho, T)$ and $p(\rho, T)$. We now have the same number of unknowns as equations. An equally valid choice of unknowns is (ρ, v, e), in which case the EOS must supply $T(\rho, e)$ and $p(\rho, e)$. The form of the EOS (or its computational cost) often dictates the choice of unknowns.

A γ-law gas is a very simple EOS, which may be written as

$$p = (\gamma - 1)\rho e. \tag{7}$$

For those familiar with radiation transport, this equation should look familiar, particularly if we take $\gamma = 4/3$ (which is the effective γ for photons). We obtain

$$p = \frac{1}{3}\rho e. \tag{8}$$

Since ρe is the internal energy per volume, one recognizes this is analogous to the pressure tensor for an isotropic radiation intensity, or, the P_1 closure for transport.

1.3 Euler Equations

Next, we make the following assumptions:

- The Reynolds number (R_e) is large, where

$$R_e = \frac{\rho_0 v_0 L}{\mu_0}, \tag{9}$$

  and the subscript$_0$ indicates evaluated at some reference state.

- The product $P_r R_e$ is large, where P_r is the Prandtl number.

$$P_r = \frac{C_{p_0} \mu_0}{k_0} \tag{10}$$

For many materials, P_r is order unity. Without going into more detail, with these assumptions, we're effectively assuming that the viscous and heat conduction terms in the N-S equations may be dropped, leaving the Euler equations. In 1-D, we still have eq. (4), but $F(U)$ simplifies to

$$F(U) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(\rho E + p) \end{pmatrix}. \tag{11}$$

Because the $F(U)$ here no longer uses the variables (T, μ, k), in order to close the system, the EOS only needs to supply the function $p(\rho, e)$. This is an *incomplete EOS* and you'll often see this in the literature as simply the "equation of state." However, a complete EOS is typically required in order add other physics, for more complex materials, or may be required to enforce an entropy condition.

1.4 Wave Structure of the Euler Equations

For $s$trong solutions of the Euler equations, the equations may be written in nonconservative form as

$$\partial_t V + M \partial_x V = 0, \tag{12}$$

where

$$V = \begin{pmatrix} \rho \\ v \\ e \end{pmatrix}, \qquad M = \begin{bmatrix} v & \rho & 0 \\ \frac{1}{\rho}\partial_\rho p & v & \frac{1}{\rho}\partial_e p \\ 0 & \frac{p}{\rho} & v \end{bmatrix}. \tag{13}$$

Here our EOS is of the form $p = p(\rho, e)$. Let's seek *traveling wave* solutions of eq. (12); namely, solutions of the form $V = V(x - \lambda t)$. Substituting this expression for V into eq. (12), we obtain

$$(M - \lambda I)V' = 0. \tag{14}$$

So, λ is an eigenvalue of M with corresponding eigenvalue V'. One finds the eigenvalues are

$$\lambda = v, v \pm a, \tag{15}$$

where a is the soundpseed of the material, given by

$$a = \sqrt{\partial_\rho p + \frac{p}{\rho^2}\partial_e p}. \tag{16}$$

More discussion of the eigenvalues will be given momentarily. Note that the soundspeed is another property of the EOS.

An equivalent and more common expression for a is to use an EOS of the form $p(\rho, \mathcal{S})$, where $\mathcal{S}$ is the specific entropy and is defined by the differential relation

$$T d\mathcal{S} = de - \frac{p}{\rho^2} d\rho. \tag{17}$$

Then

$$a = \sqrt{\partial_\rho p(\rho, \mathcal{S})}. \tag{18}$$

Finally, for a γ-law EOS,

$$a = \sqrt{\gamma R T}, \tag{19}$$

where R is the specific gas constant.

1.5 Shocks

Weak solutions of eq. (4) satisfy the following integral over a space-time control volume C:

$$\oint_C (F, U) \cdot \vec{n} dA = 0, \tag{20}$$

where in this section, $\vec{n} \equiv (n_x, n_t)$ is a space-time vector, normal to the boundary of C. For a discontinuity moving at speed s through C, we showed in class this relation may be used to derive:

$$(F_R - F_L) = s(U_R - U_L). \tag{21}$$

where $(\cdot)_L$, $(\cdot)_R$ are the states to the left and right of the discontinuity. In the steady frame ($s = 0$), we have the familiar jump relation

$$F_L = F_R. \tag{22}$$

Note that we may derive eq. (21) from (22) with the substitution $v \to v - s$ (Galilean transformation). Also, these relations also hold for both shocks *and* material (contact) discontinuities.

The relations (21,22) hold for the Navier-Stokes $F(U)$ (eq. (5)), the Euler $F(U)$ (eq. (11)), or for that matter, any conservation law in the form of eq. (4). Indeed, we will use these jump relations again once we have added radiation effects.

1.5.1 Infinitesimal Jumps

Consider the case where the shock jump is very small, $dU \equiv U_R - U_L$. Then eq. (21) may be written as

$$A dU = s dU, \tag{23}$$

where the matrix A is the flux Jacobian, given by

$$A = \frac{\partial F}{\partial U}. \tag{24}$$

Recognize that eq. (23) implies s is an eigenvalue of A with corresponding eigenvector dU. The eigenvalues of A must also be the same as those for M in eq. (12), because $A = QMQ^{-1}$ (a similarity transformation), where $Q = \partial V / \partial U$. We can think of eq. (21) as a generalization of the traveling wave solutions, found in §1.4, to discontinuities.

The eigenvalues correspond to the following physics:

- $s = v$: This is known as a contact wave. For finite jumps, it corresponds to a contact discontinuity. Across the discontinuity, ρ is discontinuous, but p and v are continuous. This is known as a *linear wave*. Note that $s = v$ whether the jump is infinitesimal or discontinuous.
- $s = v \pm a$: These are known as acoustic waves. These are *nonlinear waves* that for discontinuities correspond to shocks waves. Note that infinitesimal shocks are acoustic waves.

The character of these waves may be further investigated by looking at their corresponding eigenvectors, but that exercise is beyond the scope of these notes.

1.5.2 Shocks for Navier-Stokes

What do shocks look like for Navier-Stokes? To answer this question, consider the jump relation (22) for a location $x = x_L = x_R$. Because of the diffusion terms in the variables v and T, we know that these variables *must* be continuous:

$$v_L = v_R, \tag{25}$$
$$T_L = T_R. \tag{26}$$

From the mass flux in eq. (22),

$$(\rho v)_L = (\rho v)_R. \tag{27}$$

But since $v_L = v_R$, this relation implies $\rho_L = \rho_R$. Because both ρ and T are continuous, from the EOS, we know that all thermodynamic quantities are continuous.[1] So, the diffusion terms result in a continuous solution; they *regularlize* shocks.

Note, however, that Navier-Stokes is often a poor approximation if one is interested in computing the shock transition layer accurately. This is because within the transition layer, typically K_n is no longer small. The gas-kinetic Boltzmann equation is needed, or a higher-moment closure thereof.

But the details of the transition layer are often not needed. Consider the case where x_L and x_R are far from the shock, such that at their locations the viscous and heat conduction effects may be ignored. The jump relations still hold and become *identical* to those for the Euler equations. As a result, we can view the Euler equations as ignoring viscous effects, *except at shocks*. At a shock, the viscous effects are isolated to a single point (or line in 2-D, plane in 3-D), where they result in an increase in entropy across the shock. Remarkably, the values of μ and κ need not be known to compute their overall effect.

For the remainder of these notes, we deal only with the Euler equations. But keep in mind that μ and κ also affect the shock structure in an analogous manner as radiation.

1.6 Entropy Condition

An issue with the Euler equations is that unless we enforce a so-called entropy condition, weak solutions are not unique. In other words, there are unphysical shocks that satisfy that the jump relation (21). By unphysical, we mean the entropy of the material *decreases* as it passes through the shock. From the second law of thermodynamics, the entropy must increase.

These notes will not go into the details of the entropy condition. Instead, we will simply point out three important properties of entropy-satisfying shocks:

- As the flow passes through the shock, its thermodynamic entropy increases.
- Shocks are the limiting solution of the Navier-Stokes equations as $\mu \downarrow 0$ and $\kappa \downarrow 0$.
- In the rest frame of the shock (transform to $s = 0$), the upstream Mach number $M \geq 1$ (supersonic) and downstream, $M \leq 1$ (subsonic).

Here the Mach number M is defined as

$$M = |\vec{v}|/a, \tag{28}$$

where a is the soundspeed.

There are many other properties unique to entropy-satisfying shocks that can be useful, but these are beyond the scope of these notes.

[1] This assumes no phase changes. However, assuming an EOS exists is invalid in the region of a phase change.

1.7 Jump Relations for the Euler Equations

For the Euler equations, we can solve the steady jump relation (22) to obtain

$$\frac{\rho_R}{\rho_L} = \frac{(\gamma+1)M_L^2}{2+(\gamma-1)M_L^2}, \tag{29a}$$

$$\frac{v_R}{v_L} = \frac{\rho_L}{\rho_R}, \tag{29b}$$

$$\frac{T_R}{T_L} = \frac{(1-\gamma+2\gamma M_L^2)(2+(\gamma-1)M_L^2)}{(\gamma+1)^2 M_L^2}, \tag{29c}$$

where M_L is the pre-shock Mach number at state-$_L$:

$$M_L = \frac{v_L}{\sqrt{\gamma R T_L}}. \tag{29d}$$

If $\mathcal{S}$ is the entropy, one can show that $\mathcal{S}_R \geq \mathcal{S}_L$ whenever $M_L \geq 1$ (in which case $M_R \leq 1$).

Note that the jump relations (29) remain satisfied if we change the sign on v_L. However, this would mean that the entropy of the flow *decreases* as it passes through the shock, violating the entropy condition. If such a flow configuration was an initial condition, what actually happens is an unsteady expansion wave is formed.

2 Radiative Shock Overview

You nearly have enough background information to understand radiative shocks, and in particular, the study by Lowrie and Rauenzahn [1]. This section will fill in the remaining gaps.

For coupled radiation and an Euler-hydrodynamics model, the evolution of *total* mass, momentum, and energy follow eq. (4), but with

$$U = \begin{pmatrix} \rho \\ \rho v + \mathcal{F}/c^2 \\ \rho E + \mathcal{E} \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \rho v \\ \rho v^2 + p + \mathcal{P} \\ v(\rho E + p) + \mathcal{F} \end{pmatrix}, \tag{30}$$

where $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$ are the radiative energy, flux, and pressure. This system holds for any radiation model, but obviously is not closed. We still have to specify $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$. Note that we could instead couple with Navier-Stokes by including its τ and q terms; here, we assume these terms are small compared with the radiation effects.

Consider a shock problem, where far from the shock we assume constant states and that the radiation comes into equilibrium with the material. Specifically, to $O(v^2/c^2)$, we have

$$\mathcal{E}_L = a_R T_L^4, \quad \mathcal{F}_L = \frac{4}{3} v_L a_R T_L^4, \quad \mathcal{P}_L = \frac{1}{3} a_R T_L^4, \tag{31}$$

and

$$\mathcal{E}_R = a_R T_R^4, \quad \mathcal{F}_R = \frac{4}{3} v_R a_R T_R^4, \quad \mathcal{P}_R = \frac{1}{3} a_R T_R^4. \tag{32}$$

where a_R is the radiation constant. Now the jump relation (21) can be solved, *without specifying the radiation model*. This is directly analogous to Navier-Stokes, where the Euler equations give the jump relations, without needing to know the details of the shock structure. In the radiative shock case, the radiation model provides the details of the shock structure, but the overall jump relations are independent of the radiation model.

Finally, in Ref. [1], the jump relations are given with the radiation momentum term dropped from U, so that

$$U = \begin{pmatrix} \rho \\ \rho v \\ \rho E + \mathcal{E} \end{pmatrix}. \tag{33}$$

Under equilibrium conditions, the radiation momentum term is $O(v^2/c^2)$, so dropping this term is within the approximation of the physics. Moreover, if this term is included, the system is no longer Galilean invariant. For non-Galilean invariant systems, solutions to the jump relation (21) depend on your frame of reference.

References

[1] R. B. Lowrie and R. M. Rauenzahn, "Shock wave solutions for equilibrium-diffusion radiation hydrodynamics," Tech. Rep. LA-UR-06-3853, Los Alamos National Laboratory, 2006.