Lecture 7

The Grey-Diffusion Approximation

The purpose of this lecture is to

- To derive the $\mathrm{O}(u/c)$ grey diffusion approximation for radiation-hydrodynamics.
- To modify the basic equations via $\mathrm{O}(u^2/c^2)$ terms to achieve certain desired properties while retaining full accuracy to $\mathrm{O}(u/c)$
- To further modify the equations to make them computationally simpler while retaining full $\mathrm{O}(u/c)$ accuracy only in the equilibrium-diffusion limit.

1 Derivation of the Grey Diffusion Approximation

We begin with the $\mathrm{O}(u/c)$ radiation-hydrodynamic equations, replacing the transport equation with the radiation energy and momentum equations:

$$\frac{\partial}{\partial t}\rho + \overrightarrow{\nabla}\cdot(\rho \overrightarrow{u}) = 0 \,. \tag{1}$$

$$\frac{\partial}{\partial t}(\rho \overrightarrow{u}) + \overrightarrow{\nabla}\cdot\left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right) + \overrightarrow{\nabla} p = -\overrightarrow{S}_{rp} \,, \tag{2}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \overrightarrow{\nabla}\cdot\left[(\frac{1}{2}\rho u^2 + \rho e + p)\overrightarrow{u}\right] = -S_{re} \,, \tag{3}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \overrightarrow{\nabla}\cdot\overrightarrow{\mathcal{F}} = S_{re} \,, \tag{4}$$

$$\frac{1}{c^2}\frac{\partial \overrightarrow{F}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\overrightarrow{\mathcal{P}}} = \overrightarrow{S}_{rp} \,, \tag{5}$$

where

$$S_{re} = \int_0^\infty \sigma_a \left(4\pi B - \varphi\right) \, dE \, + \int_0^\infty \left(\sigma_a + E\frac{\partial \sigma_a}{\partial E} - \sigma_s\right) \overrightarrow{F} \cdot \overrightarrow{u}/c \, dE \,, \tag{6}$$

$$\overrightarrow{S}_{rp} = -\int_0^\infty \frac{1}{c}\sigma_t \overrightarrow{F} \, dE +$$

$$\int_0^\infty \left(\sigma_s \varphi + \sigma_a 4\pi B\right) \overrightarrow{u}/c^2 \;\; dE + \int_0^\infty \left(\sigma_a + E\frac{\partial \sigma_a}{\partial E} + \sigma_s\right) \overrightarrow{\overrightarrow{P}} \cdot \overrightarrow{u}/c \;\; dE \,. \tag{7}$$

If we assume that the cross sections do not depend upon energy, the interaction source terms become

$$S_{re} = \sigma_a c \left(aT^4 - \mathcal{E}\right) \, + \left(\sigma_a - \sigma_s\right) \overrightarrow{\mathcal{F}} \cdot \overrightarrow{u}/c \,, \tag{8}$$

and

$$\overrightarrow{S}_{rp} = -\frac{1}{c}\sigma_t \overrightarrow{\mathcal{F}} +$$

$$\left(\sigma_s c\mathcal{E} + \sigma_a a c T^4\right) \overrightarrow{u}/c^2 \, + \sigma_t \overrightarrow{\overrightarrow{\mathcal{P}}} \cdot \overrightarrow{u}/c \,. \tag{9}$$

To obtain a diffusion approximation, we simply set the time-derivative of the radiation flux in Eq. (5) to zero and assume that

$$\mathcal{P}_{i,j} = \frac{1}{3}\mathcal{E}\delta_{i,j} \,. \tag{10}$$

Applying these two steps, Eqs. (5) and (9) respectively become,

$$\frac{1}{3}\overrightarrow{\nabla}\mathcal{E} = \overrightarrow{S}_{rp} \,, \tag{11}$$

and

$$\overrightarrow{S}_{rp} = -\frac{1}{c}\sigma_t \overrightarrow{\mathcal{F}} +$$

$$\left(\sigma_s c\mathcal{E} + \sigma_a acT^4\right)\overrightarrow{u}/c^2 + \sigma_t \frac{1}{3}\mathcal{E}\overrightarrow{u}/c\,. \tag{12}$$

The other equations are unaffected.

Several of the following transformations are useful for physically interpreting the grey diffusion interaction sources. We stress that these transformations are only correct for the indicated angle-energy-integrated quantities, and are only correct to O(u/c):

$$\mathcal{E}_0 = \mathcal{E} - \frac{2}{c^2}\overrightarrow{\mathcal{F}}\cdot\overrightarrow{u}\,, \tag{13a}$$

$$\mathcal{E} = \mathcal{E}_0 + \frac{2}{c^2}\overrightarrow{\mathcal{F}}_0\cdot\overrightarrow{u}\,, \tag{13b}$$

$$\overrightarrow{\mathcal{F}}_0 = \overrightarrow{\mathcal{F}} - \left(\mathcal{E}\overrightarrow{u} + \overrightarrow{\overrightarrow{\mathcal{P}}}\cdot\overrightarrow{u}\right)\,, \tag{14a}$$

$$\overrightarrow{\mathcal{F}} = \overrightarrow{\mathcal{F}}_0 + \left(\mathcal{E}_0\overrightarrow{u} + \overrightarrow{\overrightarrow{\mathcal{P}}}_0\cdot\overrightarrow{u}\right)\,, \tag{14b}$$

$$\overrightarrow{\overrightarrow{\mathcal{P}}}_0 = \overrightarrow{\overrightarrow{\mathcal{P}}} - \frac{2}{c^2}\overrightarrow{u}\otimes\overrightarrow{\mathcal{F}}\,, \tag{15a}$$

$$\overrightarrow{\overrightarrow{\mathcal{P}}} = \overrightarrow{\overrightarrow{\mathcal{P}}}_0 + \frac{2}{c^2}\overrightarrow{u}\otimes\overrightarrow{\mathcal{F}}_0\,. \tag{15b}$$

Motivated by Eq. (13a) we first manipulate and then respectively rewrite Eq. (8) as follows:

$$\begin{aligned} S_{re} &= \sigma_a c\left[aT^4 - \left(\mathcal{E} - \frac{2}{c^2}\overrightarrow{\mathcal{F}}\right)\cdot\overrightarrow{u}\right] - \sigma_t\overrightarrow{\mathcal{F}}\cdot\overrightarrow{u}/c\,, \\ &= \sigma_a c\left(aT^4 - \mathcal{E}_0\right) - \sigma_t\overrightarrow{\mathcal{F}}\cdot\overrightarrow{u}/c\,, \end{aligned} \tag{16}$$

Motivated by Eq. (14a) and remembering Eq. (10), we first manipulate and then respectively rewrite Eq. (11) as follows:

$$\begin{aligned}\overrightarrow{S}_{rp} &= -\frac{1}{c}\sigma_t\left(\overrightarrow{\mathcal{F}} - \frac{4}{3}\mathcal{E}\overrightarrow{u}\right) + \sigma_a c\left(aT^4 - \mathcal{E}\right)\overrightarrow{u}/c^2\,, \\ &= -\frac{1}{c}\sigma_t\overrightarrow{\mathcal{F}}_0 + \sigma_a c\left(aT^4 - \mathcal{E}\right)\overrightarrow{u}/c^2\,. \qquad (17)\end{aligned}$$

Because we have truncated at $\mathrm{O}(u/c)$, there are some subtle inconsistencies in the grey interaction terms. For instance, $\sigma_a c\,(aT^4 - \mathcal{E}_0)$ is readily identified as the comoving-frame radiation-material internal energy source. Since internal energy must be the same in both frames, this must also be the laboratory-frame radiation-material internal energy source. Furthermore, since the total radiation-material source must be the sum of the internal and kinetic sources, it follows that we can identify $-\sigma_t\overrightarrow{\mathcal{F}}\cdot\overrightarrow{u}/c$ as the kinetic energy source. Setting $\overrightarrow{S}_{rp}\cdot\overrightarrow{u} = -\sigma_t\overrightarrow{\mathcal{F}}\cdot\overrightarrow{u}/c$ and solving for $\overrightarrow{S}_{rp}$ we get

$$\overrightarrow{S}_{rp} = -\frac{1}{c}\sigma_t\overrightarrow{\mathcal{F}}\,, \qquad (18)$$

which does not agree with Eq. (17). Of course, as one would expect, Eqs. (17) and (18) do agree to $\mathrm{O}(u/c)$, i.e., the difference between the two expressions is $\mathrm{O}(u^2/c^2)$. These subtle inconsistencies generate spurious equilibrium solutions. The equilibrium solutions are spatially and temporally constant. Such solutions must yield zero interaction sources. We know from basic physics consideratons that in equilibrium,

$$\mathcal{E}_0 = aT^4\,, \qquad (19a)$$

$$\overrightarrow{\mathcal{F}}_0 = \overrightarrow{0}\,, \tag{19b}$$

$$\mathcal{P}_{0,i,j} = \frac{1}{3}\mathcal{E}_0\delta_{i,j} = \frac{1}{3}aT^4\delta_{i,j}\,. \tag{19c}$$

Using Eqs. (13a) through (15b) and (13a) through (15b), we obtain the following expressions for the laboratory-frame equilibrium solutions:

$$\mathcal{E} = aT^4 + O(u^2/c^2)\,, \tag{20a}$$

$$\overrightarrow{\mathcal{F}} = \frac{4}{3}aT^4\overrightarrow{u} + O(u^2/c^2)\,, \tag{20b}$$

$$\mathcal{P}_{i,j} = \frac{1}{3}aT^4 + O(u^2/c^2)\,. \tag{20c}$$

We can eliminate all inconsistencies and spurious equilibrium solutions while retaining first-order accuracy by defining the interaction sources as follows:

$$S_{re} = \sigma_a c\left(aT^4 - \mathcal{E}_0\right) + \overrightarrow{S}_{rp}\cdot\overrightarrow{u}/c\,, \tag{21}$$

$$\overrightarrow{S}_{rp} = -\frac{1}{c}\sigma_t\overrightarrow{\mathcal{F}}_0 + \sigma_a c\left(aT^4 - \mathcal{E}_0\right)\overrightarrow{u}/c^2\,, \tag{22}$$

where we replace Eq. (13a) with

$$\mathcal{E}_0 = E - \frac{2}{c^2}\overrightarrow{\mathcal{F}}_0\cdot\overrightarrow{u}\,. \tag{23}$$

The laboratory-frame equilibrium solutions that are obtained are

$$\mathcal{E} = aT^4\,, \tag{24a}$$

$$\overrightarrow{\mathcal{F}} = \frac{4}{3} a T^4 \overrightarrow{u}\,, \tag{24b}$$

$$\mathcal{P}_{i,j} = \frac{1}{3} a T^4\,. \tag{24c}$$

The diffusion approximation that we have derived is fairly complex. Considering the fact that $v/c < .01$ in the non-relativistic limit, one might ask if it is worth all this trouble when we have made a gross approximation in the first place by replacing the transport equation with a grey diffusion equation. We can make considerable simplifications and still retain the following properties:

- Total (radiation plus material) energy conservation.
- Equilibrium solutions correct to O(v/c).
- Preservation of the equilbrium-diffusion limit.

The properties given above are obtained with the following definitions for the interaction sources:

$$S_{re} = \sigma_a c \left(a T^4 - \mathcal{E} \right) - \overrightarrow{S}_{rp} \cdot \overrightarrow{u} / c\,, \tag{25}$$

$$\overrightarrow{S}_{rp} = -\frac{1}{c} \sigma_t \overrightarrow{\mathcal{F}}_0\,. \tag{26}$$

The equilibrium solutions obtained with these definitions are identical to those given in Eqs. (24a) through (24c).

It is important to remember that the diffusion approximation does not preserve radiation momentum because the time-derivative of the flux is set to zero in the radiation momentum equation. Of course, a momentum-conserving P_1 version of the our grey diffusion approximation is easily defined simply by retaining that time-derivative in the equation. When this is done, our interaction source definitions result in total momentum conservation.

Our final radiation-hydrodynamics equations with grey radiation diffusion can be expressed as follows:

$$\frac{\partial}{\partial t}\rho + \overrightarrow{\nabla}\cdot(\rho \overrightarrow{u}) = 0\,. \tag{27}$$

$$\frac{\partial}{\partial t}(\rho \overrightarrow{u}) + \overrightarrow{\nabla}\cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right) + \overrightarrow{\nabla} p = -\frac{1}{3}\overrightarrow{\nabla}\mathcal{E}\,, \tag{28}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \overrightarrow{\nabla}\cdot \left[(\frac{1}{2}\rho u^2 + \rho e + p)\overrightarrow{u}\right] = \sigma_a c\left(\mathcal{E} - aT^4\right) - \frac{1}{3}\overrightarrow{\nabla}\mathcal{E}\cdot \overrightarrow{u}\,, \tag{29}$$

$$\frac{\partial \mathcal{E}}{\partial t} - \overrightarrow{\nabla}\cdot\frac{c}{3\langle\sigma_t\rangle}\overrightarrow{\nabla}\mathcal{E} + \frac{4}{3}\overrightarrow{\nabla}\cdot\left(\mathcal{E}\overrightarrow{u}\right) = \sigma_a c\left(aT^4 - \mathcal{E}\right) + \frac{1}{3}\overrightarrow{\nabla}\mathcal{E}\cdot \overrightarrow{u}\,, \tag{30}$$

$$\overrightarrow{\mathcal{F}} = -\frac{c}{3\langle\sigma_t\rangle}\overrightarrow{\nabla}\mathcal{E} + \frac{4}{3}\mathcal{E}\overrightarrow{u}\,, \tag{31}$$

where

$$\mathcal{E}_0 = \mathcal{E}\,, \tag{32}$$

$$\overrightarrow{\mathcal{F}}_0 = -\frac{c}{3\langle\sigma_t\rangle}\overrightarrow{\nabla}\mathcal{E}\,, \tag{33}$$

$$\mathcal{P}_{0,i,j} = \frac{1}{3}\mathcal{E}\delta_{i,j}\,. \tag{34}$$