Radiative heat transfer solver with fluid motion

Jan I.C. Vermaak[1,2], Jim E. Morel[1,2]

[1]Center for Large Scale Scientific Simulations, Texas A&M Engineering Experiment Station, College Station, Texas, USA.
[2]Nuclear Engineering Department, Texas A&M University, College Station, Texas, USA.

Abstract:
Work is work for some, but for some it is play.

Keywords: hydrodynamics

Contents

1 Definitions

1.1 Independent variables

We refer to the following independent variables:

- Position in the cartesian space $\{x, y, z\}$ is denoted with $\mathbf{x}$ and each component having units $[cm]$.
- Direction, $\{\varphi, \theta\}$, is denoted with $\mathbf{\Omega}$ which takes on the form

$$\mathbf{\Omega} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \text{ and/or } \mathbf{\Omega} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix},$$

  where φ is the azimuthal-angle and θ is the polar-angle, both in spherical coordinates. Commonly, $\cos\theta$, is denoted with μ. The general dimension of angular phase space is $[steridian]$.
- Photon frequency, ν in $[Hertz]$ or $[s^{-1}]$.
- Time, t in $[s]$.

1.2 Dependent variables

We use the following basic dependent variables:

- The foundation of the dependent unknowns is the **radiation angular intensity**, $I(\mathbf{x}, \mathbf{\Omega}, \nu, t)$ with units $[Joule/cm^2{-}s{-}steradian{-}Hz]$. We often use the corresponding angle-integral of this quantity, $\phi(\mathbf{x}, \nu, t)$, and define it as

$$\phi(\mathbf{x}, \nu, t) = \mathcal{E}c = \int_{4\pi} I(\mathbf{x}, \mathbf{\Omega}, \nu, t)\ d\mathbf{\Omega} \tag{1.1}$$

  with units $[Joule/cm^2{-}s{-}Hz]$. Where c is the speed of light.
- The **radiation energy density**, $\mathcal{E}$, is

$$\mathcal{E}(\mathbf{x}, \nu, t) = \frac{\phi}{c} = \frac{1}{c}\int_{4\pi} I(\mathbf{x}, \mathbf{\Omega}, \nu, t)\ d\mathbf{\Omega} \tag{1.2}$$

  with units $[Joule/cm^3{-}Hz]$.
- The **radiation energy flux**, $\boldsymbol{\mathcal{F}}$, is

$$\boldsymbol{\mathcal{F}}(\mathbf{x}, \nu, t) = \int_{4\pi} \mathbf{\Omega}\ I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \tag{1.3}$$

- **Radiation pressure**, $\boldsymbol{\mathcal{P}}$, is

$$\boldsymbol{\mathcal{P}}(\mathbf{x}, \nu, t) = \frac{1}{c}\int_{4\pi} \mathbf{\Omega} \otimes \mathbf{\Omega} I(\mathbf{x}, \mathbf{\Omega}, \nu, t) d\mathbf{\Omega} \tag{1.4}$$

  and is a tensor.

1.3 Blackbody radiation

A blackbody radiation source, $B(\nu, T)$, is properly described by **Planck's law**,

$$B(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} \tag{1.5}$$

with units $[Joule/cm^2{-}s{-}steridian - Hz]$ where h is Planck's constant and k_B is the Boltzmann constant.

If we integrate the blackbody source over all angle-space and frequencies then we get the mean radiation intensity from a blackbody at temperature T as

$$\begin{aligned}
\int_0^\infty \int_{4\pi} B(\nu, T)\ d\mathbf{\Omega} d\nu &= \int_0^\infty \int_{4\pi} \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}\ d\mathbf{\Omega} d\nu \\
&= 4\pi \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}\ d\nu \\
&= acT^4,
\end{aligned} \tag{1.6}$$

with units $[Joule/cm^2{-}s{-}steridian]$ and where a is the **blackbody radiation constant** given by

$$a = \frac{8\pi^5 k_B^4}{15h^3c^3}. \tag{1.7}$$

In both cases this unfortunately is only the intensity. Following Kirchoff's law, which states that the emission and absorption of radiation must be equal in equilibrium, we can determine the **blackbody emission rate**, S_{bb}, from the absorption rate as

$$S_{bb}(\nu, T) = \rho\kappa(\nu) B(\nu, T), \tag{1.8}$$

with units $[Joule/cm^3{-}s{-}steridian{-}Hz]$ where ρ is the material density $[g/cm^3]$ and κ is the opacity $[cm^2/g]$. The combination $\rho\kappa$ is also equal to the macroscopic absorption cross section σ_a, therefore $\rho\kappa(\nu) = \sigma_a$. Data for the opacity of a material is normally available in the form of either the **Rosseland opacity**, κ_{Rs}, or the **Planck opacity**, κ_{Pl}.

2 Conservation equations

2.1 Conservation equation - Radiative transfer

The basic statement of conservation, is

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\boldsymbol{\Omega},\nu,t)}{\partial t} &= -\boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I(\mathbf{x},\boldsymbol{\Omega},\nu,t) - \sigma_t(\mathbf{x},\nu)I(\mathbf{x},\boldsymbol{\Omega},\nu,t)\\ &+ \int_0^\infty \int_{4\pi} \frac{\nu}{\nu'}\sigma_s(\mathbf{x},\nu'{\rightarrow}\nu,\boldsymbol{\Omega}'{\cdot}\boldsymbol{\Omega})I(\mathbf{x},\boldsymbol{\Omega}',\nu',t)d\nu' d\boldsymbol{\Omega}'\\ &+ \sigma_a(\mathbf{x},\nu)B(\nu,T(\mathbf{x},t)) + S\end{aligned} \tag{2.1}$$

where S is any other sources/sinks of radiation intensity.

2.2 Radiative transfer assuming isotropic Thompson scattering

Assuming Thomson-scattering[1] is the only form of scattering, gives

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\boldsymbol{\Omega},\nu,t)}{\partial t} &= -\boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I(\mathbf{x},\boldsymbol{\Omega},\nu,t) - \sigma_t(\mathbf{x},\nu)I(\mathbf{x},\boldsymbol{\Omega},\nu,t)\\ &+ \frac{\sigma_s(\mathbf{x},\nu)}{4\pi}c\mathcal{E}(\mathbf{x},\nu) + \sigma_a(\mathbf{x},\nu)B(\nu,T(\mathbf{x},t)) + S\end{aligned} \tag{2.2}$$

where S is any other sources/sinks of radiation intensity.

Using energy instead of frequency, $\nu \rightarrow E$:

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\boldsymbol{\Omega},E,t)}{\partial t} &= -\boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I(\mathbf{x},\boldsymbol{\Omega},E,t) - \sigma_t(\mathbf{x},E)I(\mathbf{x},\boldsymbol{\Omega},E,t)\\ &+ \frac{\sigma_s(\mathbf{x},E)}{4\pi}c\mathcal{E}(\mathbf{x},E) + \sigma_a(\mathbf{x},E)B(E,T(\mathbf{x},t)) + S\end{aligned} \tag{2.3}$$

where S is any other sources/sinks of radiation intensity.

2.3 Radiative transfer with material motion corrections

Applying relativistic corrections for a material in motion, we can derive (e.g., see NUEN 627 lecture 4) the laboratory-frame transport equation

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{x},\boldsymbol{\Omega},E,t)}{\partial t} &= -\boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I(\mathbf{x},\boldsymbol{\Omega},E,t) - \left(\frac{E_0}{E}\right)\sigma_t(\mathbf{x},E_0)I(\mathbf{x},\boldsymbol{\Omega},E,t)\\ &+ \left(\frac{E}{E_0}\right)^2\frac{\sigma_s(\mathbf{x},E)}{4\pi}\int_{4\pi}\left(\frac{E_0}{E'}\right)I(\mathbf{x},\boldsymbol{\Omega}',E',t)d\boldsymbol{\Omega}' + \left(\frac{E}{E_0}\right)^2\sigma_a(\mathbf{x},E_0)B(E_0,T(\mathbf{x},t)) + S,\end{aligned} \tag{2.4}$$

where

$$E_0 = E\gamma\left(1 - \boldsymbol{\Omega}\cdot\frac{\mathbf{u}}{c}\right) \tag{2.5}$$

$$\gamma = \left[1 - \left(\frac{||\mathbf{u}||}{c}\right)^2\right]^{-\frac{1}{2}} \tag{2.6}$$

$$\frac{E_0}{E'} = \gamma\left(1 - \boldsymbol{\Omega}'\cdot\frac{\mathbf{u}}{c}\right) \tag{2.7}$$

$$E' = E\frac{1 - \boldsymbol{\Omega}\cdot\frac{\mathbf{u}}{c}}{1 - \boldsymbol{\Omega}'\cdot\frac{\mathbf{u}}{c}} \tag{2.8}$$

[1] Thomson scattering is the elastic scattering of electromagnetic radiation by a free charged particle. The particle's kinetic energy- as well as the photon's frequency, does not change in such a scattering. The scattering is also isotropic.

2.4 Radiative transfer with material velocity dependencies expanded to $\mathcal{O}(v/c)$

Very ugly derivations in NUEN 627 lecture 5 to get to,

$$\begin{aligned}&\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},E,t)}{\partial t}+\mathbf{\Omega}\cdot\mathbf{\nabla}I(\mathbf{x},\mathbf{\Omega},E,t)+\sigma_t(\mathbf{x},E)I(\mathbf{x},\mathbf{\Omega},E,t)\\=&\frac{\sigma_s(\mathbf{x},E)}{4\pi}\phi(E)+\sigma_a(\mathbf{x},E)B(E,T(\mathbf{x},t))\\&+\left[\left(\sigma_t+E\frac{\partial\sigma_a}{\partial E}\right)I+\frac{\sigma_s}{4\pi}\left(2\phi-E\frac{\partial\phi}{\partial E}\right)+2\sigma_aB(E,T)-B(E,T)E\frac{\partial\sigma_a}{\partial E}-\sigma_aE\frac{\partial B(E,T)}{\partial E}\right]\mathbf{\Omega}\cdot\frac{\mathbf{u}}{c}\\&-\frac{\sigma_s}{4\pi}\left(\boldsymbol{\mathcal{F}}-E\frac{\partial\boldsymbol{\mathcal{F}}}{\partial E}\right)\cdot\frac{\mathbf{u}}{c}\end{aligned}\tag{2.9}$$

Voodoo magic Grey Radiation Transport equation:
Somehow, determined by integrating over energy

$$\frac{1}{c}\frac{\partial I}{\partial t}+\mathbf{\Omega}\cdot\mathbf{\nabla}I+\sigma_t(\mathbf{x})I=\frac{\sigma_s}{4\pi}\phi+\frac{\sigma_a}{4\pi}acT^4-\frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c}+\frac{\sigma_t}{\pi}\mathcal{E}\;\mathbf{\Omega}\cdot\mathbf{u}\tag{2.10}$$

Radiation energy equation:
Obtained by integrating the transport equation over energy and angle

$$\begin{aligned}\frac{\partial\mathcal{E}(\mathbf{x},t)}{\partial t}+\mathbf{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t)=&\int_0^\infty\sigma_a(\mathbf{x},E)\big(4\pi B(E,T)-\phi(\mathbf{x},E,t)\big)dE\\&+\int_0^\infty\left(\sigma_a+E\frac{\partial\sigma_a}{\partial E}-\sigma_s(E)\right)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}dE\end{aligned}\tag{2.11}$$

Radiation momentum equation:
Obtained by first multiplying by $\frac{1}{c}\mathbf{\Omega}$, then integrating over all directions and energies,

$$\begin{aligned}\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t}+\mathbf{\nabla}\cdot\boldsymbol{\mathcal{P}}=&-\int_0^\infty\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}dE\\&+\int_0^\infty\big(\sigma_s\phi+\sigma_a4\pi B(E,T)\big)\frac{\mathbf{u}}{c^2}dE\\&+\int_0^\infty\left(\sigma_a+E\frac{\partial\sigma_a}{\partial E}+\sigma_s\right)\boldsymbol{\mathcal{P}}\cdot\frac{\mathbf{u}}{c}dE\end{aligned}\tag{2.12}$$

2.5 Grey Radiative Transfer

$$\begin{aligned}&\frac{1}{c}\frac{\partial I(\mathbf{x},\mathbf{\Omega},t)}{\partial t}+\mathbf{\Omega}\cdot\mathbf{\nabla}I(\mathbf{x},\mathbf{\Omega},t)+\sigma_t(\mathbf{x})I(\mathbf{x},\mathbf{\Omega},t)\\=&\frac{\sigma_s}{4\pi}\phi+\frac{\sigma_a}{4\pi}acT^4\\&+\left[\sigma_tI+\frac{\sigma_s}{4\pi}2\phi+2\sigma_a\frac{1}{4\pi}acT^4-\sigma_aE\frac{\partial B(E,T)}{\partial E}\right]\mathbf{\Omega}\cdot\frac{\mathbf{u}}{c}\\&-\frac{\sigma_s}{4\pi}\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}\end{aligned}\tag{2.13}$$

Radiation energy equation:
Obtained by integrating Eq. (2.13) over energy and angle

$$\begin{aligned}\frac{\partial\mathcal{E}(\mathbf{x},t)}{\partial t}+\mathbf{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t)&=\sigma_ac\big(aT^4-\mathcal{E}\big)+\big(\sigma_a-\sigma_s\big)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}\\&=\sigma_ac\big(aT^4-\mathcal{E}_0\big)-\sigma_t\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}\end{aligned}\tag{2.14}$$

Radiation momentum equation:
Obtained by first multiplying Eq. (2.13) by $\frac{1}{c}\boldsymbol{\Omega}$, then integrating over all directions and energies,

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} &= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \big(\sigma_s c\mathcal{E} + \sigma_a acT^4\big)\frac{\mathbf{u}}{c^2} + \sigma_t\boldsymbol{\mathcal{P}}\cdot\frac{\mathbf{u}}{c} \\
&= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \big((\sigma_a + \sigma_s - \sigma_a)\mathcal{E} + \sigma_a aT^4\big)\frac{\mathbf{u}}{c} + \sigma_t\boldsymbol{\mathcal{P}}\cdot\frac{\mathbf{u}}{c} \\
&= \frac{1}{c}\Big[-\sigma_t\boldsymbol{\mathcal{F}} + \big((\sigma_t - \sigma_a)\mathcal{E} + \sigma_a aT^4\big)\mathbf{u} + \sigma_t\boldsymbol{\mathcal{P}}\cdot\mathbf{u}\Big] \\
&= -\frac{1}{c}\Big[\sigma_t\boldsymbol{\mathcal{F}} - \big((\sigma_t - \sigma_a)\mathcal{E} + \sigma_a aT^4\big)\mathbf{u} - \sigma_t\boldsymbol{\mathcal{P}}\cdot\mathbf{u}\Big] \\
\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} &= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 + \sigma_a c\big(aT^4 - \mathcal{E}\big)\frac{\mathbf{u}}{c^2}
\end{aligned} \tag{2.15}$$

2.6 Grey Diffusion Approximation

Approximating the angular dependence of $I(\boldsymbol{\Omega})$ with a P_1 spherical harmonic expansion, such that the entries of $\boldsymbol{\mathcal{P}}$ are given by

$$(\boldsymbol{\mathcal{P}})_{i,j} = \frac{1}{3}\mathcal{E}\delta_{i,j}, \tag{2.16}$$

the radiation energy equation is unaffected but the radiation momentum equation changes. We repeat the radiation energy equation below, and the altered radiation moment equations:

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \big(\sigma_a - \sigma_s\big)\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}, \tag{2.17}$$

$$\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \big(\sigma_s c\mathcal{E} + \sigma_a acT^4\big)\frac{\mathbf{u}}{c^2} + \sigma_t\frac{1}{3}\mathcal{E}\frac{\mathbf{u}}{c}. \tag{2.18}$$

Useful transformations:

$$\mathcal{E}_0 = \mathcal{E} - \frac{2}{c^2}\boldsymbol{\mathcal{F}}\cdot\mathbf{u} \tag{2.19a}$$

$$\mathcal{E} = \mathcal{E}_0 + \frac{2}{c^2}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{2.19b}$$

$$\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{F}} - \big(\mathcal{E}\mathbf{u} + \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\big) \tag{2.19c}$$

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_0 + \big(\mathcal{E}_0\mathbf{u} + \boldsymbol{\mathcal{P}}_0\cdot\mathbf{u}\big) \tag{2.19d}$$

$$\boldsymbol{\mathcal{P}}_0 = \boldsymbol{\mathcal{P}} - \frac{2}{c^2}\mathbf{u}\otimes\boldsymbol{\mathcal{F}} \tag{2.19e}$$

$$\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_0 + \frac{2}{c^2}\mathbf{u}\otimes\boldsymbol{\mathcal{F}}_0 \tag{2.19f}$$

With the P_1 approximation

$$\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{F}} - \frac{4}{3}\mathcal{E}\mathbf{u} \tag{2.19g}$$

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_0 + \frac{4}{3}\mathcal{E}\mathbf{u} \tag{2.19h}$$

Applying these transformations the radiation energy- and moment equation can be expressed as

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}(\mathbf{x},t) = \sigma_a c\big(aT^4 - \mathcal{E}_0\big) - \sigma_t\boldsymbol{\mathcal{F}}\cdot\frac{\mathbf{u}}{c}, \tag{2.20}$$

$$\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 + \sigma_a c\big(aT^4 - \mathcal{E}\big)\frac{\mathbf{u}}{c^2}. \tag{2.21}$$

Several simplifications to these equations are made. Firstly arriving at the expression for the radiation energy equation,

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}(\mathbf{x}, t) = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t \boldsymbol{\mathcal{F}}_0 \cdot \frac{\mathbf{u}}{c}, \tag{2.22}$$

then the radiation momentum equation,

$$\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \tag{2.23}$$

from which we can get expression for $\boldsymbol{\mathcal{F}}_0$ and $\boldsymbol{\mathcal{F}}$ in terms of $\mathcal{E}$ as

$$\boldsymbol{\mathcal{F}}_0 = -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} \tag{2.24}$$

and

$$\begin{aligned} \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} &= -\frac{\sigma_t}{c}\left(\boldsymbol{\mathcal{F}} - \frac{4}{3}\mathcal{E}\mathbf{u}\right) \\ \therefore \boldsymbol{\mathcal{F}} &= -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} + \frac{4}{3}\mathcal{E}\mathbf{u}. \end{aligned} \tag{2.25}$$

These expressions for $\boldsymbol{\mathcal{F}}_0$ and $\boldsymbol{\mathcal{F}}$ are both then inserted into the radiation energy equation as follows

$$\begin{aligned} &\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}(\mathbf{x}, t) = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t \boldsymbol{\mathcal{F}}_0 \cdot \frac{\mathbf{u}}{c} \\ \rightarrow\ &\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} + \frac{4}{3}\mathcal{E}\mathbf{u}\right) = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \sigma_t\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\right)\cdot\frac{\mathbf{u}}{c} \\ \rightarrow\ &\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\right) + \frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}. \end{aligned} \tag{2.26}$$

Arriving at a **diffusion form** of the **radiation energy equation**,

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\left(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\right) + \frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}. \tag{2.27}$$

2.7 Conservation equation for fluid flow

The governing equations we consider here are the Euler equations defined as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) = 0 \tag{2.28}$$

$$\frac{\partial (\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \{\rho\mathbf{u} \otimes \mathbf{u}\} + \boldsymbol{\nabla} p = \mathbf{f}, \tag{2.29}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u}] = q \tag{2.30}$$

where ρ is the fluid density, $\mathbf{u} = [u_x, u_y, u_z] = [u, v, w]$ is the fluid velocity in cartesian coordinates, p is the fluid pressure, E is the material energy-density comprising kinetic energy-density, $\frac{1}{2}\rho||\mathbf{u}||^2$, and internal energy-density, ρe, such that $E = \frac{1}{2}\rho||\mathbf{u}||^2 + \rho e$, where e is the specific internal energy. The values q and $\mathbf{f}$ are abstractly used here as energy- and moment- sources/sinks, respectively.

The ideal gas law provides the closure relation

$$p = (\gamma - 1)\rho e \tag{2.31}$$

where γ is the ratio of the constant-pressure specific heat, c_p, to the constant-volume specific heat, c_v, i.e., $\gamma = \frac{c_p}{c_v}$, and is a material property.

Coupling terms:

$$\mathbf{f} = \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \\ = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \tag{2.32}$$

and

$$q = -\left(\sigma_a c(aT^4 - \mathcal{E}) - \sigma_t \boldsymbol{\mathcal{F}}_0 \cdot \frac{\mathbf{u}}{c}\right) \\ = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u} \tag{2.33}$$

3 Solver A - Radiation Hydrodynamics Grey Diffusion

The set of Radiation Hydrodynamics Grey Diffusion Equations are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{3.1a}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \boldsymbol{\nabla} p = -\frac{1}{3} \boldsymbol{\nabla} \mathcal{E}, \tag{3.1b}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3} \boldsymbol{\nabla} \mathcal{E} \cdot \mathbf{u} \tag{3.1c}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \Big(- \frac{c}{3\sigma_t} \boldsymbol{\nabla} \mathcal{E} \Big) + \frac{4}{3} \boldsymbol{\nabla} \big(\mathcal{E} \mathbf{u}\big) = \sigma_a c \big(aT^4 - \mathcal{E}\big) + \frac{1}{3} \boldsymbol{\nabla} \mathcal{E} \cdot \mathbf{u}. \tag{3.1d}$$

where

$$E = \frac{1}{2} \rho ||\mathbf{u}||^2 + \rho e, \tag{3.1e}$$

$$p = (\gamma - 1)\rho e, \tag{3.1f}$$

$$T = \frac{1}{C_v} e \tag{3.1g}$$

$$\sigma_t(T) = \sigma_s(T) + \sigma_a(T) \tag{3.1h}$$

$$\sigma_s(T) = \rho \kappa_s(T) \tag{3.1i}$$

$$\sigma_a(T) = \rho \kappa_a(T) \tag{3.1j}$$

3.1 Definitions

First we define the following terms

- The radiation emission and absorption, the radiation momentum source, and the radiation energy source

$$S_{ea} = \sigma_a c \big(aT^4 - \mathcal{E}\big) \tag{3.2a}$$

$$\mathbf{S}_{rp} = \frac{1}{3} \boldsymbol{\nabla} \mathcal{E} \tag{3.2b}$$

$$S_{re} = S_{ea} + \frac{1}{3} \boldsymbol{\nabla} \mathcal{E} \cdot \mathbf{u} \tag{3.2c}$$

- The conserved hydrodynamic variables, $\mathbf{U}$, and associated hydrodynamic flux, $\mathcal{F}^H$,

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho \mathbf{u} \\ E \end{bmatrix} \qquad \mathcal{F}^H = \begin{bmatrix} \rho u \\ \rho u u + p \\ \rho u v \\ \rho u w \\ (E+p) u \end{bmatrix} \tag{3.2d}$$

- The stationary reference frame radiation energy flux

$$\boldsymbol{\mathcal{F}}_0 = -\frac{c}{3\sigma_t} \boldsymbol{\nabla} \mathcal{E} \tag{3.2e}$$

Next, we use these terms to define a more condensed version of the RHGD equations.

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{3.3}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0 + \frac{4}{3} \boldsymbol{\nabla} \cdot \big(\mathcal{E} \mathbf{u}\big) = S_{ea} + \frac{1}{3} \boldsymbol{\nabla} \mathcal{E} \cdot \mathbf{u}. \tag{3.4}$$

3.2 Finite Volume Spatial Discretization

To apply a finite volume spatial discretization we integrate our time-discretized equations over the volume, V_c, of cell c, and afterwards divide by V_c. This leaves all the terms containing τ unchanged. In this process we develop the following terms:

3.2.1 Hydrodynamic and Radiation-energy advection

$$\frac{1}{V_c}\int_{V_c} \boldsymbol{\nabla}\cdot\mathcal{F}^H(\mathbf{U})dV = \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot\mathcal{F}^H(\mathbf{U}_f) \tag{3.5}$$

$$\frac{1}{V_c}\int_{V_c} \left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right)dV = \frac{1}{V_c}\sum_f \frac{4}{3}\mathbf{A}_f\cdot(\mathcal{E}\mathbf{u})_f \tag{3.6}$$

The face values are reconstructed from gradients in both the predictor and corrector phases. In the corrector-phase the hydrodynamic flux, $\mathcal{F}^H$, is used in its earlier defined form, whilst in the corrector-phase the flux is determined by an approximate Riemann-solver, i.e., the HLLC Riemann solver.

Predictor phases:
For the predictor phase we have the following:

$$\boldsymbol{\nabla}\cdot\mathcal{F}^H(\mathbf{U}) \mapsto \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot\mathcal{F}^H(\mathbf{U}_f) \tag{3.7}$$

$$\left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right) \mapsto \frac{1}{V_c}\sum_f \frac{4}{3}\mathbf{A}_f\cdot(\mathcal{E}\mathbf{u})_f \tag{3.8}$$

$$\mathbf{U}_f = \mathbf{U}_c + (\mathbf{x}_f - \mathbf{x}_c)\cdot\{\boldsymbol{\nabla}\mathbf{U}\}_c \tag{3.9}$$

$$\mathcal{E}_f = \mathcal{E}_c + (\mathbf{x}_f - \mathbf{x}_c)\cdot\{\boldsymbol{\nabla}\mathcal{E}\}_c \tag{3.10}$$

Corrector phases:
For the corrector phase we have the following:

$$\boldsymbol{\nabla}\cdot\mathcal{F}^H(\mathbf{U}) \mapsto \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot\mathbf{F}^{*hllc}(\mathbf{U}_f) \tag{3.11}$$

$$\left(\frac{4}{3}\boldsymbol{\nabla}\cdot(\mathcal{E}\mathbf{u})\right) \mapsto \frac{1}{V_c}\sum_f \frac{4}{3}\mathbf{A}_f\cdot(\mathcal{E}\mathbf{u})_{upw} \tag{3.12}$$

where

$$\mathbf{U}_f = \mathbf{U}_c + (\mathbf{x}_f - \mathbf{x}_c)\cdot\{\boldsymbol{\nabla}\mathbf{U}\}_c \tag{3.13}$$

$$(\mathcal{E}\mathbf{u})_{upw} = \begin{cases} (\mathcal{E}\mathbf{u})_{c,f}, & \text{if } \mathbf{u}_{c,f}\cdot\mathbf{n}_f > 0 \text{ and } \mathbf{u}_{cn,f}\cdot\mathbf{n}_f > 0 \quad \rightarrow|\rightarrow \\ (\mathcal{E}\mathbf{u})_{cn,f}, & \text{if } \mathbf{u}_{c,f}\cdot\mathbf{n}_f < 0 \text{ and } \mathbf{u}_{cn,f}\cdot\mathbf{n}_f < 0 \quad \leftarrow|\leftarrow \\ (\mathcal{E}\mathbf{u})_{cn,f} + (\mathcal{E}\mathbf{u})_{c,f}, & \text{if } \mathbf{u}_{c,f}\cdot\mathbf{n}_f > 0 \text{ and } \mathbf{u}_{cn,f}\cdot\mathbf{n}_f < 0 \quad \rightarrow|\leftarrow \\ 0, & \text{if } \mathbf{u}_{c,f}\cdot\mathbf{n}_f < 0 \text{ and } \mathbf{u}_{cn,f}\cdot\mathbf{n}_f > 0 \quad \leftarrow|\rightarrow \end{cases} \tag{3.14}$$

$$\mathcal{E}_{c,f} = \mathcal{E}_c + (\mathbf{x}_f - \mathbf{x}_c)\cdot\{\boldsymbol{\nabla}\mathcal{E}\}_c \tag{3.15}$$

3.2.2 Density and momentum updates

We apply the same process as before:

$$-\frac{1}{V_c}\int_{V_c} \mathbf{S}_{rp} dV = -\frac{1}{V_c}\sum_f \frac{1}{3}\mathbf{A}_f \mathcal{E}_f, \tag{3.16}$$

however, here we want $\mathcal{E}_f$ to satisfy the following relationship

$$\frac{D_c}{||\mathbf{x}_{cf}||}(\mathcal{E}_f - \mathcal{E}_c) = \frac{D_{cn}}{||\mathbf{x}_{fcn}||}(\mathcal{E}_{cn} - \mathcal{E}_f) \tag{3.17}$$

where

$$D_c = -\frac{c}{3\sigma_{t,c}} \tag{3.18}$$

and where $\mathbf{x}_{cf}$ is the vector from cell c's centroid to the face centroid, $\mathbf{x}_{fcn}$ is the vector from the face centroid to cell cn's centroid (where cell cn is the neighbor to c at face f). The norm $||\cdot||$ refers to the L_2 norm.

Solving the above relationship for $\mathcal{E}_f$ we first set

$$k_c = \frac{D_c}{||\mathbf{x}_{cf}||}, \qquad k_{cn} = \frac{D_{cn}}{||\mathbf{x}_{fcn}||}$$

then get

$$\begin{aligned} k_c\mathcal{E}_f - k_c\mathcal{E}_c &= k_{cn}\mathcal{E}_{cn} - k_{cn}\mathcal{E}_f \\ \rightarrow \quad (k_c + k_{cn})\mathcal{E}_f &= k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c \\ \therefore \mathcal{E}_f &= \frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}}. \end{aligned} \tag{3.19}$$

Predictor and corrector phases:
We do the same for both,

$$-\mathbf{S}_{rp} \mapsto -\frac{1}{V_c}\sum_f \frac{1}{3}\mathbf{A}_f \mathcal{E}_f \tag{3.20}$$

3.2.3 Energy equations

Only two terms require special consideration here. They are: the divergence of the co-moving frame radiation energy flux, and the kinetic energy terms source terms,

$$\begin{aligned} \frac{1}{V_c}\int_{V_c} \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0 \; dV &= \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot(\boldsymbol{\mathcal{F}}_0)_f \\ \frac{1}{V_c}\int_{V_c} \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u} \; dV &= \frac{1}{V_c}\sum_f \frac{1}{3}\mathbf{A}_f\cdot(\mathcal{E}\mathbf{u})_f. \end{aligned} \tag{3.21}$$

3.2.3.1 The diffusion term

Considering the $\boldsymbol{\mathcal{F}}_0$-term first, we apply Gauss' divergence theorem to get

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0 \mapsto \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot(\boldsymbol{\mathcal{F}}_0)_f. \tag{3.22}$$

For $(\boldsymbol{\mathcal{F}}_0)_f$ we have

$$(\boldsymbol{\mathcal{F}}_0)_f = -\frac{c}{3\sigma_{tf}}\big(\boldsymbol{\nabla}\mathcal{E}\big)_f. \tag{3.23}$$

Now define

$$D_f = -\frac{c}{3\sigma_{tf}}. \tag{3.24}$$

To find D_f we seek the equivalence:

$$D_f \frac{\mathcal{E}_{cn} - \mathcal{E}_c}{||\mathbf{x}_{cn} - \mathbf{x}_c||} = D_c \frac{\mathcal{E}_f - \mathcal{E}_c}{||\mathbf{x}_f - \mathbf{x}_c||} = D_{cn} \frac{\mathcal{E}_{cn} - \mathcal{E}_f}{||\mathbf{x}_{cn} - \mathbf{x}_f||} \tag{3.25}$$

Now let us define

$$\begin{aligned} k_c &= \frac{D_c}{||\mathbf{x}_f - \mathbf{x}_c||} \\ k_{cn} &= \frac{D_{cn}}{||\mathbf{x}_{cn} - \mathbf{x}_f||} \end{aligned} \tag{3.26}$$

Now

$$\begin{aligned} k_c(\mathcal{E}_f - \mathcal{E}_c) &= k_{cn}(\mathcal{E}_{cn} - \mathcal{E}_f) \\ (k_c + k_{cn})\mathcal{E}_f &= k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c \\ \therefore \mathcal{E}_f &= \frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}} \end{aligned} \tag{3.27}$$

Now we choose any of the right-two terms in the three way equality and plug the expression for $\mathcal{E}_f$,

$$\begin{aligned} &k_c(\mathcal{E}_f - \mathcal{E}_c) \\ &= k_c\left(\frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c}{k_c + k_{cn}} - \mathcal{E}_c\right) \\ &= k_c\left(\frac{k_{cn}\mathcal{E}_{cn} + k_c\mathcal{E}_c - k_c\mathcal{E}_c - k_{cn}\mathcal{E}_c}{k_c + k_{cn}}\right) \\ \therefore D_f \frac{\mathcal{E}_{cn} - \mathcal{E}_c}{||\mathbf{x}_{cn} - \mathbf{x}_c||} &= \frac{k_c k_{cn}}{k_c + k_{cn}}\big(\mathcal{E}_{cn} - \mathcal{E}_c\big) \\ \therefore D_f &= \frac{k_c k_{cn}}{k_c + k_{cn}}||\mathbf{x}_{cn} - \mathbf{x}_c|| \end{aligned} \tag{3.28}$$

From the earlier expression for $(\boldsymbol{\mathcal{F}}_0)_f$, we can write

$$(\boldsymbol{\mathcal{F}}_0)_f = D_f\big(\mathcal{E}_{cn} - \mathcal{E}_c\big)\frac{\mathbf{x}_{cn} - \mathbf{x}_c}{||\mathbf{x}_{cn} - \mathbf{x}_c||^2} \tag{3.29}$$

for which we can define

$$\mathbf{k}_f = D_f \frac{\mathbf{x}_{cn} - \mathbf{x}_c}{||\mathbf{x}_{cn} - \mathbf{x}_c||^2} \tag{3.30}$$

such that we finally arrive at

$$(\boldsymbol{\mathcal{F}}_0)_f = \mathbf{k}_f\big(\mathcal{E}_{cn} - \mathcal{E}_c\big). \tag{3.31}$$

3.2.3.2 The kinetic energy term

For the kinetic energy source terms, we similarly have

$$\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^n \mapsto \frac{1}{V_c}\sum_f \frac{1}{3}\mathbf{A}_f \cdot (\mathcal{E}_f^n \mathbf{u}_f^n) \tag{3.32}$$

where we use the reconstructed values as in the Hydrodynamic and radiation-energy advection portion.

3.3 Temporal scheme - Implicit Euler Predictor, Crank-Nicolson Corrector

$$\frac{\partial \mathbf{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{3.33a}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0 + \frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big) = S_{re}. \tag{3.33b}$$

3.3.1 Predictor phase

$\tau = \dfrac{1}{\frac{1}{2}\Delta t}$

$$\tau(\mathbf{U}^{n*} - \mathbf{U}^n) + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}^n) = \mathbf{0} \tag{3.34a}$$

$$\tau\left(\begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+\frac{1}{2}} - \begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n*}\right) = \begin{bmatrix} 0 \\ -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \end{bmatrix}^n \tag{3.34b}$$

$$\tau(\mathcal{E}^{n*} - \mathcal{E}^n) + \left(\frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big)\right)^n = 0 \tag{3.34c}$$

$$\tau(E^{n+\frac{1}{2}} - E^{n*}) = -\theta_1 S_{ea}^{n+\frac{1}{2}} - \theta_2 S_{ea}^n - \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^n \tag{3.34d}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^{n*}) + \theta_1 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0^{n+\frac{1}{2}} + \theta_2 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0^n = \theta_1 S_{ea}^{n+\frac{1}{2}} + \theta_2 S_{ea}^n + \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^n \tag{3.34e}$$

For S_{ea} and $\boldsymbol{\mathcal{F}}_0$ both at $n + \frac{1}{2}$:

$$\sigma^{n+\frac{1}{2}} = \rho^{n+\frac{1}{2}}\kappa(T^n) \tag{3.34f}$$

$$T^{4,n+\frac{1}{2}} = T^{4,n*} + \frac{4T^{3,n*}}{C_v}(e^{n+\frac{1}{2}} - e^{n*}) \tag{3.34g}$$

3.3.2 Corrector phase

$\tau = \dfrac{1}{\Delta t}$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^n) + \boldsymbol{\nabla} \cdot \mathcal{F}^H(\mathbf{U}^{n+\frac{1}{2}}) = \mathbf{0} \tag{3.35a}$$

$$\tau\left(\begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+1} - \begin{bmatrix} \rho \\ \rho\mathbf{u} \end{bmatrix}^{n+\frac{1}{2}*}\right) = \begin{bmatrix} 0 \\ -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \end{bmatrix}^{n+\frac{1}{2}} \tag{3.35b}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}*} - \mathcal{E}^n) + \left(\frac{4}{3}\boldsymbol{\nabla}\cdot\big(\mathcal{E}\mathbf{u}\big)\right)^{n+\frac{1}{2}} = 0 \tag{3.35c}$$

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 S_{ea}^{n+1} - \theta_2 S_{ea}^n - \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.35d}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0^n = \theta_1 S_{ea}^{n+1} + \theta_2 S_{re}^n + \left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.35e}$$

For S_{ea} and $\boldsymbol{\mathcal{F}}_0$ both at $n + 1$:

$$\sigma^{n+1} = \rho^{n+1}\kappa(T^{n+\frac{1}{2}}) \tag{3.35f}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{3.35g}$$

3.3.3 General energy equations, Predictor and Corrector phase, with θ factors

Time integration scheme A uses **implicit Euler** for the predictor phase and **Crank-Nicolson** in the corrector phase. Both these schemes can be represented wit a general θ-scheme where we define:

$$\begin{aligned} \theta_1 &\in [0,1] \\ \theta_2 &= 1-\theta_1. \end{aligned} \tag{3.36}$$

For implicit Euler, $\theta_1 = 1,\ \theta_2 = 0$, and for Crank-Nicolson, $\theta_1 = \theta_2 = \frac{1}{2}$. With these factors defined we can repeat the energy equations and apply a series of manipulations. First we attempt to segregate known terms from all unknown terms. Thereafter we eliminate the internal energy, e, from the two sets of equations to get a single formulation for the radiation energy, $\mathcal{E}$. The latter formulation forms a diffusion system that needs to be assembled and solved for $\mathcal{E}$.

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1\sigma_a^{n+1}c\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) - \theta_2 S_{ea}^n - \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.37a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\mathcal{F}}_0^n\big) = \theta_1\sigma_a^{n+1}c\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) + \theta_2 S_{re}^n + \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.37b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{3.37c}$$

Define:

$$\begin{aligned} k_1 &= \theta_1\sigma_a^{n+1}c \\ k_2 &= \frac{4T^{3,n+\frac{1}{2}*}}{C_v} \end{aligned} \tag{3.38}$$

---

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) - \theta_2 S_{ea}^n - \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.39a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\mathcal{F}}_0^n\big) = k_1\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) + \theta_2 S_{re}^n + \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.39b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{3.39c}$$

---

ungroup right-hand side elements by multiplying out terms within parentheses,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 aT^{4,n+1} + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.40a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \boldsymbol{\nabla}\cdot\big(\theta_1\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\mathcal{F}}_0^n\big) = k_1 aT^{4,n+1} - k_1\mathcal{E}^{n+1} + \theta_2 S_{re}^n + \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.40b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{3.40c}$$

---

now plug in the temperature equation into both the energy equations,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a\big(T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*})\big) + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n - \Big(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{3.41a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n+\frac{1}{2}*})+\boldsymbol{\nabla}\cdot\big(\theta_1\boldsymbol{\mathcal{F}}_0^{n+1}+\theta_2\boldsymbol{\mathcal{F}}_0^{n}\big)=k_1a\big(T^{4,n+\frac{1}{2}*}+k_2(e^{n+1}-e^{n+\frac{1}{2}*})\big)-k_1\mathcal{E}^{n+1}+\theta_2S_{re}^n+\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.41b}$$

ungroup elements on the both the right-hand sides,

$$\tau(E^{n+1}-E^{n+\frac{1}{2}*})=-k_1aT^{4,n+\frac{1}{2}*}-k_1ak_2e^{n+1}+k_1ak_2e^{n+\frac{1}{2}*}+k_1\mathcal{E}^{n+1}-\theta_2S_{ea}^n-\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.42a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n+\frac{1}{2}*})+\boldsymbol{\nabla}\cdot\big(\theta_1\boldsymbol{\mathcal{F}}_0^{n+1}+\theta_2\boldsymbol{\mathcal{F}}_0^{n}\big)=k_1aT^{4,n+\frac{1}{2}*}+k_1ak_2e^{n+1}-k_1ak_2e^{n+\frac{1}{2}*}-k_1\mathcal{E}^{n+1}+\theta_2S_{re}^n+\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{3.42b}$$

Define:

$$\begin{aligned}k_3&=-k_1aT^{4,n+\frac{1}{2}*}+k_1ak_2e^{n+\frac{1}{2}*}-\theta_2S_{ea}^n-\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}\\ k_4&=-k_1ak_2\end{aligned} \tag{3.43}$$

and plug them into the equations above,

$$\tau(E^{n+1}-E^{n+\frac{1}{2}*})=k_4e^{n+1}+k_1\mathcal{E}^{n+1}+k_3 \tag{3.44a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n+\frac{1}{2}*})+\boldsymbol{\nabla}\cdot\big(\theta_1\boldsymbol{\mathcal{F}}_0^{n+1}+\theta_2\boldsymbol{\mathcal{F}}_0^{n}\big)=-k_4e^{n+1}-k_1\mathcal{E}^{n+1}-k_3 \tag{3.44b}$$

Note:

$$E^{n+1}=(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1}+\rho^{n+1}e^{n+1} \tag{3.45}$$

which gives,

$$\tau\big((\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1}+\rho^{n+1}e^{n+1}-E^{n+\frac{1}{2}*}\big)=k_4e^{n+1}+k_1\mathcal{E}^{n+1}+k_3 \tag{3.46a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n+\frac{1}{2}*})+\boldsymbol{\nabla}\cdot\big(\theta_1\boldsymbol{\mathcal{F}}_0^{n+1}+\theta_2\boldsymbol{\mathcal{F}}_0^{n}\big)=-k_4e^{n+1}-k_1\mathcal{E}^{n+1}-k_3 \tag{3.46b}$$

ungroup the material energy in the first equation,

$$\tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1}+\tau\rho^{n+1}e^{n+1}-\tau E^{n+\frac{1}{2}*}=k_4e^{n+1}+k_1\mathcal{E}^{n+1}+k_3 \tag{3.47a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n+\frac{1}{2}*})+\theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1}+\theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n}=-k_4e^{n+1}-k_1\mathcal{E}^{n+1}-k_3 \tag{3.47b}$$

and isolate the internal energy in the first equation,

$$(\tau\rho^{n+1}-k_4)e^{n+1}=k_1\mathcal{E}^{n+1}+k_3-\tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1}+\tau E^{n+\frac{1}{2}*} \tag{3.48a}$$

$$\tau(\mathcal{E}^{n+1}-\mathcal{E}^{n+\frac{1}{2}*})+\theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1}+\theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n}=-k_4e^{n+1}-k_1\mathcal{E}^{n+1}-k_3 \tag{3.48b}$$

Define:

$$k_5 = \frac{k_1}{\tau\rho^{n+1} - k_4}$$
$$k_6 = \frac{k_3 - \tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau E^{n+\frac{1}{2}*}}{\tau\rho^{n+1} - k_4} \tag{3.49}$$

---

and plug these constants into the first equation above,

$$e^{n+1} = k_5\mathcal{E}^{n+1} + k_6 \tag{3.50a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n} = -k_1\mathcal{E}^{n+1} - k_3 - k_4 e^{n+1} \tag{3.50b}$$

---

now plug the first equation into the second,

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^{n+\frac{1}{2}*}) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n} = -k_1\mathcal{E}^{n+1} - k_3 - k_4k_5\mathcal{E}^{n+1} - k_4k_6 \tag{3.51a}$$

---

now collect all the $\mathcal{E}^{n+1}$ terms on the left-hand side,

$$\big(\tau + k_1 + k_4k_5\big)\mathcal{E}^{n+1} + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1} = -k_3 - k_4k_6 + \tau\mathcal{E}^{n+\frac{1}{2}*} - \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n} \tag{3.52a}$$

Recall:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0 \mapsto \frac{1}{V_c}\sum_f \mathbf{A}_f\cdot(\boldsymbol{\mathcal{F}}_0)_f \tag{3.53}$$

and

$$(\boldsymbol{\mathcal{F}}_0)_f = \mathbf{k}_f(\mathcal{E}_{cn} - \mathcal{E}_c) \tag{3.54}$$

---

which gives the system,

$$\big(\tau + k_1 + k_4k_5\big)\mathcal{E}^{n+1} + \frac{\theta_1}{V_c}\sum_f \mathbf{A}_f\cdot\mathbf{k}_f^{n+1}(\mathcal{E}_{cn}^{n+1} - \mathcal{E}_c^{n+1}) = -k_3 - k_4k_6 + \tau\mathcal{E}^{n+\frac{1}{2}*} - \frac{\theta_2}{V_c}\sum_f \mathbf{A}_f\cdot\mathbf{k}_f^{n}(\mathcal{E}_{cn}^{n} - \mathcal{E}_c^{n}) \tag{3.55a}$$

This system is SPD and in one dimension forms a tridiagonal system.

3.3.4 Using the energy related algebra for both the predictor and the corrector

To perform the energy related algebra for the corrector step we need the following inputs:

κ_a^n	For σ_a^n in S_{ea}^n
κ_t^n	For σ_t^n in $\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^n$
$\kappa_a^{n+\frac{1}{2}}$	For σ_a^{n+1} in S_{ea}^{n+1}
$\kappa_t^{n+\frac{1}{2}}$	For σ_t^{n+1} in $\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_0^{n+1}$
C_v	For the linearization of $T^{4,n+1}$
τ	For the time constant
θ_1, θ_2	For the time scheme
$\mathbf{U}^n$	For T, ρ in S_{ea}^n
$\mathbf{U}^{n+\frac{1}{2}}$	For $\mathbf{u}$ in $\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathbf{U}^{n+\frac{1}{2}*}$	For $E^{n+\frac{1}{2}*}$ and $e^{n+\frac{1}{2}*}$
$\mathbf{U}_{0,1}^{n+1} = \begin{bmatrix}\rho \\ \rho\mathbf{u}\end{bmatrix}_{n+1}$	For the kinetic energy in E^{n+1}, and $\rho^{n+1} \to \sigma_a^{n+1}, \sigma_t^{n+1}$
$\boldsymbol{\nabla}\mathbf{U}^{n+\frac{1}{2}}$	For the reconstructions in $\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathcal{E}^n$	For S_{ea}^n
$\mathcal{E}^{n+\frac{1}{2}}$	For $\mathcal{E}$ in $\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}$
$\mathcal{E}^{n+\frac{1}{2}*}$	For itself
$\boldsymbol{\nabla}\mathcal{E}^{n+\frac{1}{2}}$	For the reconstructions in $\left(\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}\cdot\mathbf{u}\right)^{n+\frac{1}{2}}$

To following remapping(s) then applies to the predictor:

$$
\begin{aligned}
\kappa_a^n &\to \kappa_a^n \\
\kappa_t^n &\to \kappa_t^n \\
\kappa_a^n &\to \kappa_a^{n+\frac{1}{2}} \\
\kappa_t^n &\to \kappa_t^{n+\frac{1}{2}} \\
\mathbf{U}^n &\to \mathbf{U}^n \\
\mathbf{U}^n &\to \mathbf{U}^{n+\frac{1}{2}} \\
\mathbf{U}^{n*} &\to \mathbf{U}^{n+\frac{1}{2}*} \\
\mathbf{U}^{n+\frac{1}{2}} &\to \mathbf{U}^{n+1} \\
\boldsymbol{\nabla}\mathbf{U}^n &\to \boldsymbol{\nabla}\mathbf{U}^{n+\frac{1}{2}}
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
\mathcal{E}^n &\to \mathcal{E}^n \\
\mathcal{E}^n &\to \mathcal{E}^{n+\frac{1}{2}} \\
\mathcal{E}^{n*} &\to \mathcal{E}^{n+\frac{1}{2}*} \\
\boldsymbol{\nabla}\mathcal{E}^n &\to \boldsymbol{\nabla}\mathcal{E}^{n+\frac{1}{2}}
\end{aligned}
$$

4 Solver B - Radiation Hydrodynamics Grey Diffusion - Mixed finite element

We now derive a general mixed finite element approach for

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0(\mathbf{x}) = 1, \quad \mathbf{x} \in \mathcal{D} \tag{4.1a}$$

$$\boldsymbol{\mathcal{F}}_0(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\mathcal{D} \tag{4.1b}$$

where

$$\boldsymbol{\mathcal{F}}_0(\mathbf{x}) = D(\mathbf{x})\boldsymbol{\nabla}\mathcal{E}(\mathbf{x}). \tag{4.2}$$

4.1 Auxiliary notation and variables for $\boldsymbol{\mathcal{F}}_0$

First we discretize $\boldsymbol{\mathcal{F}}_0$ on N_n number of nodes per cell c, using continuous basis functions $b_j(\mathbf{x})$ such that

$$\boldsymbol{\mathcal{F}}_0(\mathbf{x}) \approx \sum_{j=1}^{N_n} (\boldsymbol{\mathcal{F}}_0)_j b_j(\mathbf{x}), \tag{4.3}$$

whilst keeping the cell-centered representation for $\mathcal{E}$. Next we discretize eq. (4.2) by applying a weight function $b_i(\mathbf{x})$ and integrating over the volume of the cell c,

$$\begin{aligned} \int_{V_c} b_i \boldsymbol{\mathcal{F}}_0 dV &= \int_{V_c} b_i D \boldsymbol{\nabla}\mathcal{E} dV \\ \sum_j \biggl[\int_{V_c} b_i b_j dV \biggr] (\boldsymbol{\mathcal{F}}_0)_j &= \int_{V_c} b_i D \boldsymbol{\nabla}\mathcal{E} dV. \end{aligned} \tag{4.4}$$

The integral coefficients on the left-hand side are generally known as the ij coefficients in the standard finite element mass-matrix, which we shall use in a moment to define a general scheme. The right hand side of the equation requires some treatment. We introduce the values $\mathcal{E}_{c,j}$ at the cell surface to remedy the discontinuities in the cell-centered $\mathcal{E}$ and to define auxiliary unknowns for developing the $\boldsymbol{\mathcal{F}}_0$ in finite element form. With these new variables declared, we next apply integration by parts to the right-hand side,

$$\int_{V_c} b_i D \boldsymbol{\nabla}\mathcal{E} dV = \int_{V_c} D \boldsymbol{\nabla}(b_i \mathcal{E}) dV - \int_{V_c} D\mathcal{E} \boldsymbol{\nabla} b_i dV. \tag{4.5}$$

Next we apply Gauss's divergence theorem on the first term on the right-hand side,

$$\int_{V_c} b_i D \boldsymbol{\nabla}\mathcal{E} dV = \sum_f \int_{S_f} D\mathbf{n}_f b_i \mathcal{E} dA - \int_{V_c} D\mathcal{E} \boldsymbol{\nabla} b_i dV, \tag{4.6}$$

after which we insert $\mathcal{E}_{c,j}$ in the first term on the right, since they are designated unknowns on the surface of the cell, and $\mathcal{E}_c$ into the right most term. Since $\mathcal{E}_c$ is cell-constant within the cell-domain, the D coefficient is also dependent only on $\mathcal{E}_c$ and therefore constant within cell c, hence denoted as D_c,

$$\int_V b_i D \boldsymbol{\nabla}\mathcal{E} dV = \sum_f \sum_j \biggl[D_c \mathbf{n}_f \int_{S_f} b_i b_j dA \biggr] \mathcal{E}_{c,j} - \biggl[D_c \int_V \boldsymbol{\nabla} b_i dV \biggr] \mathcal{E}_c. \tag{4.7}$$

Putting the developed right- and left-hand sides back together we then get,

$$\sum_j \biggl[\int_V b_i b_j dV \biggr] (\boldsymbol{\mathcal{F}}_0)_j = \sum_f \sum_j \biggl[D_c \mathbf{n}_f \int_{S_f} b_i b_j dA \biggr] \mathcal{E}_{c,j} - \biggl[D_c \int_V \boldsymbol{\nabla} b_i dV \biggr] \mathcal{E}_c. \tag{4.8}$$

This equation can be written in more succinct form as

$$\bar{M}_c \bar{\mathbf{F}}_c = D_c \bar{C}_c \boldsymbol{\mathcal{E}}_c \tag{4.9}$$

where the structure still needs to be defined (which follows). $\bar{M}_c$ is a square block-matrix with block-dimension $N_n{\times}N_n$, $\bar{\mathbf{F}}_c$ is a block-vector with block-dimension $N_n{\times}1$, $\bar{C}_c$ is a rectangular block-matrix with block-dimension $N_n{\times}(N_f{+}1)$. The vector $\boldsymbol{\mathcal{E}}_c$ is simply the cell-centered and surface unknowns for cell c, i.e., $\boldsymbol{\mathcal{E}}_c = [\mathcal{E}_c, \mathcal{E}_{n=0}, \dots, \mathcal{E}_{n=N_n-1}]^T$.

The dimension of the inner blocks of $\bar{M}_c$, $\bar{\mathbf{F}}_c$ and $\bar{C}_c$, depend on the number of dimensions, N_d, in the problem. For further reference we shall denote dimensions with d but it generally refers to $d \in [0,1,2] \mapsto [x,y,z]$ and vice-versa.

The block entries of $\bar{M}$ are small diagonal matrices,

$$(\bar{M})_{ij} = \text{diag}(M_{ij}, \dots, M_{ij})^{N_d \times N_d} \tag{4.10}$$

where M_{ij} are the elements of the standard finite element mass-matrix for cell c, i.e.,

$$M_{ij} = \int_V b_i b_j dV. \tag{4.11}$$

The block entries of $\bar{\mathbf{F}}$ are

$$(\bar{\mathbf{F}})_i = \begin{bmatrix} (\boldsymbol{\mathcal{F}}_0)_{i,x} \\ (\boldsymbol{\mathcal{F}}_0)_{i,y} \\ (\boldsymbol{\mathcal{F}}_0)_{i,z} \end{bmatrix}^{N_d \times 1} \tag{4.12}$$

obviously only up to y for 2D and only up to x for 1D. The entries of $\bar{C}_c$ are formed as follows. First the structure of $\bar{C}_c$ is such that

$$\text{block-row } i \text{ of } \bar{C}_c = \text{columns} \begin{pmatrix} \mathbf{C}_i^c & \mathbf{C}_{i,j=0}^s & \dots & \mathbf{C}_{i,j=N_n-1}^s \end{pmatrix}^{N_d \times (N_n+1)}. \tag{4.13}$$

We then define the vectors

$$\begin{aligned} \mathbf{G}_i &= \int_V \boldsymbol{\nabla} b_i dV \\ M_{ij}^f &= \int_{S_f} b_i b_j dA \end{aligned} \tag{4.14}$$

Then,

$$\mathbf{C}_i^c = \begin{bmatrix} -(\mathbf{G}_i)_x \\ -(\mathbf{G}_i)_y \\ -(\mathbf{G}_i)_z \end{bmatrix}^{N_d \times 1} \tag{4.15}$$

and

$$\mathbf{C}_{ij}^s = \begin{bmatrix} \sum_f n_{f,x} M_{ij}^f \\ \sum_f n_{f,y} M_{ij}^f \\ \sum_f n_{f,z} M_{ij}^f \end{bmatrix}^{N_d \times 1} \tag{4.16}$$

With these definitions in-hand we can see that the true dimensions of $\bar{M}$ is $N_d N_n {\times} N_d N_n$, that of $\bar{\mathbf{F}}$ is $N_d N_n {\times} 1$, and the true dimensions of $\bar{C}$ is $N_d N_n {\times} (N_n + 1)$.

Finally, we have the vector $\boldsymbol{\mathcal{E}}$ as

$$\boldsymbol{\mathcal{E}}_c = \begin{bmatrix} \mathcal{E}_c \\ \mathcal{E}_{n=0} \\ \vdots \\ \mathcal{E}_{n=N_n-1} \end{bmatrix}^{(N_n+1)\times 1}. \tag{4.17}$$

To get an expression for all of the nodal $\boldsymbol{\mathcal{F}}_0$'s we take the system form of the equation and we invert $\bar{M}$ to get, in coefficient form, expressions for nodal $\boldsymbol{\mathcal{F}}_0$'s,

$$\mathbf{F}_c = \begin{bmatrix} (\boldsymbol{\mathcal{F}}_0)_0 \\ \vdots \\ (\boldsymbol{\mathcal{F}}_0)_{N_n-1} \end{bmatrix} = \bar{M}_c^{-1} \bar{C}_c \boldsymbol{\mathcal{E}} = C_c^* \boldsymbol{\mathcal{E}}_c, \tag{4.18}$$

where $C_c^* = \bar{M}_c^{-1} \bar{C}_c$.

With this expression-form of the individual nodal $\boldsymbol{\mathcal{F}}_0$'s we need to modify the primary equation, eq. (4.1). Additionally, since we introduced additional variables in the form of the face-baced $\mathcal{E}_f$'s, we need to define additional equations to close the system. For the primary equations we will simply plug in the expressions for $\boldsymbol{\mathcal{F}}_0$, which is detailed in the next subsection. For additional equations we will use the interface between cells to enforce continuity of $\boldsymbol{\mathcal{F}}_0$ at the face, for each cell of the face.

4.2 Using the auxiliary notation in the primary equation

Using this coefficient-form in the primary equations is done by first integrating eq. (4.1) over the volume of cell c, assuming the coefficient matrix $\bar{M}^{-1}\bar{C}$ has been developed for cell c, after which we apply Gauss's divergence theorem,

$$\begin{aligned}\int_{V_c} \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0 dV &= V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0)\\ \int_{S_c} \mathbf{n} \cdot \boldsymbol{\mathcal{F}}_0 dA &= V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0)\\ \sum_f \biggl[\mathbf{n}_f \cdot \int_{S_f} \boldsymbol{\mathcal{F}}_0 dA\biggr] &= V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0).\end{aligned} \tag{4.19}$$

We now expand $\boldsymbol{\mathcal{F}}_0$,

$$\sum_j \sum_f \biggl[\mathbf{n}_f \cdot (\boldsymbol{\mathcal{F}}_0)_j \int_{S_f} b_j dA\biggr] = V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0), \tag{4.20}$$

define

$$S_{i,f} = \int_{S_f} b_i dA \tag{4.21}$$

$$\sum_j \sum_f \biggl[n_{f,x} S_{j,f} (\boldsymbol{\mathcal{F}}_0)_{j,x} + n_{f,y} S_{j,f} (\boldsymbol{\mathcal{F}}_0)_{j,y} + n_{f,z} S_{j,f} (\boldsymbol{\mathcal{F}}_0)_{j,z}\biggr] = V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0), \tag{4.22}$$

or

$$\sum_j \sum_f \sum_d \biggl[n_{f,d} S_{j,f} (\boldsymbol{\mathcal{F}}_0)_{j,d}\biggr] = V_c(\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0), \tag{4.23}$$

where d denotes dimension such that $d \in [0,1,2] \mapsto [x,y,z]$, the indices (j,d) of $(\boldsymbol{\mathcal{F}}_0)_{j,d}$ maps to a row in C^*, i.e.,

$$(j,d) \mapsto k \ : \ k = N_d j + d, \tag{4.24}$$

from which we get

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}_0 = \frac{1}{V_c} \sum_j \sum_f \sum_d \biggl[n_{f,d} S_{j,f} C^*_{(j,d)\mapsto \text{ row } k} \cdot \boldsymbol{\mathcal{E}}_c\biggr], \qquad \forall c, \tag{4.25}$$

If the indices of $\mathcal{E}_c$ are then mapped to global system indexes for the corresponding $\mathcal{E}_c$ and collection of $\mathcal{E}_f$'s then the system can be constructed.

4.3 Auxiliary equations

For each face-node we now require continuity of flux. This can generally be expressed as

$$\sum_c \sum_f \int_{S_f} \mathbf{n}_f \cdot (\boldsymbol{\mathcal{F}}_0)_j dA = 0 \tag{4.26}$$

from which we get

$$\sum_c \sum_f \sum_d \biggl[n_{f,d} S_{j,f} C^*_{(j,d)\mapsto \text{ row } k} \cdot \boldsymbol{\mathcal{E}}_c\biggr] = 0, \qquad \forall j. \tag{4.27}$$

5 Solver C - Radiation Hydrodynamics Grey Radiation with the Variable Eddington Factor (VEF) method

We first repeat eqs. (2.14) and (2.15),

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0 \cdot \frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \boldsymbol{\Omega}\cdot\mathbf{u} \tag{5.1a}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \tag{5.1b}$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\{\rho\mathbf{u}\otimes\mathbf{u}\} + \boldsymbol{\nabla}p = \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0, \tag{5.1c}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) + \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{5.1d}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}. \tag{5.1e}$$

$$\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\big) \tag{5.1f}$$

where the radiation moment equation has been obtained by dropping the energy exchange terms,

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} &= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 + \cancelto{0}{\sigma_a c\big(aT^4 - \mathcal{E}\big)\frac{\mathbf{u}}{c^2}} \\
&= -\frac{\sigma_t}{c}\Big[\boldsymbol{\mathcal{F}} - \mathcal{E}\mathbf{u} - \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\Big] \\
&= -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \boldsymbol{\mathcal{P}}\cdot\mathbf{u}\big).
\end{aligned}$$

Now, recall the definition of the radiation pressure tensor, $\boldsymbol{\mathcal{P}}$,

$$\boldsymbol{\mathcal{P}}(\mathbf{x},\nu,t) = \frac{1}{c}\int_{4\pi}\boldsymbol{\Omega}\otimes\boldsymbol{\Omega}\ I(\mathbf{x},\boldsymbol{\Omega},\nu,t)d\boldsymbol{\Omega}. \tag{5.2}$$

If we expand the tensor-product we get

$$\boldsymbol{\mathcal{P}} = \frac{1}{c}\int_{4\pi}\begin{bmatrix}\boldsymbol{\Omega}_x\boldsymbol{\Omega}_x & \boldsymbol{\Omega}_x\boldsymbol{\Omega}_y & \boldsymbol{\Omega}_x\boldsymbol{\Omega}_z \\ \boldsymbol{\Omega}_y\boldsymbol{\Omega}_x & \boldsymbol{\Omega}_y\boldsymbol{\Omega}_y & \boldsymbol{\Omega}_y\boldsymbol{\Omega}_z \\ \boldsymbol{\Omega}_z\boldsymbol{\Omega}_x & \boldsymbol{\Omega}_z\boldsymbol{\Omega}_y & \boldsymbol{\Omega}_z\boldsymbol{\Omega}_z\end{bmatrix} I(\boldsymbol{\Omega})\ d\boldsymbol{\Omega}. \tag{5.3}$$

The VEF-method involves the approximation

$$\begin{aligned}
\boldsymbol{\mathcal{P}} &\approx \{f\}\frac{1}{c}\int_{4\pi} I(\boldsymbol{\Omega})d\boldsymbol{\Omega} \\
\therefore \boldsymbol{\mathcal{P}} &= \{f\}\ \mathcal{E}
\end{aligned} \tag{5.4}$$

where $\{f\}$ is the variable Eddington factor computed as an angular-intensity weighted-average such that the entries of the tensor are given by

$$\{f\}\ :\ f_{ij} = \frac{\dfrac{1}{c}\displaystyle\int_{4\pi}\boldsymbol{\Omega}_i\boldsymbol{\Omega}_j I(\boldsymbol{\Omega})d\boldsymbol{\Omega}}{\dfrac{1}{c}\displaystyle\int_{4\pi} I(\boldsymbol{\Omega})d\boldsymbol{\Omega}} \qquad i,j\in[x,y,z]. \tag{5.5}$$

Now, rewriting our set of equations we get

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \boldsymbol{\Omega}\cdot\mathbf{u} \tag{5.6a}$$

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \tag{5.6b}$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\{\rho\mathbf{u}\otimes\mathbf{u}\} + \boldsymbol{\nabla}p = \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0, \tag{5.6c}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot[(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) + \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{5.6d}$$

$$\frac{\partial\mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}. \tag{5.6e}$$

$$\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot(\{f\}\mathcal{E}) = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big) \tag{5.6f}$$

with

$$\boldsymbol{\mathcal{F}}_0 = \boldsymbol{\mathcal{F}} - (\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}) \tag{5.6g}$$

5.1 Definitions

We can cast the above equations into the following form

$$\frac{1}{c}\frac{\partial I}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \boldsymbol{\Omega}\cdot\mathbf{u} \tag{5.7a}$$

$$\frac{\partial\mathbf{U}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{5.7b}$$

$$\frac{\partial\mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}. \tag{5.7c}$$

$$\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot(\{f\}\mathcal{E}) = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big) \tag{5.7d}$$

where

$$\mathbf{S}_{rp} = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \tag{5.7e}$$

$$S_{ea} = \sigma_a c(aT^4 - \mathcal{E}) \tag{5.7f}$$

$$S_{re} = S_{ea} - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u} \tag{5.7g}$$

5.2 Temporal scheme - Implicit Euler Predictor, Crank-Nicolson Corrector

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{\Omega}\cdot\nabla I + \sigma_t(\mathbf{x})I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t}{4\pi}\boldsymbol{\mathcal{F}}_0\cdot\frac{\mathbf{u}}{c} + \frac{\sigma_t}{\pi}\mathcal{E}\ \mathbf{\Omega}\cdot\mathbf{u} \tag{5.8a}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla\cdot\mathcal{F}^H(\mathbf{U}) = \begin{bmatrix} 0 \\ -\mathbf{S}_{rp} \\ -S_{re} \end{bmatrix} \tag{5.8b}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot\boldsymbol{\mathcal{F}} = \sigma_a c\big(aT^4 - \mathcal{E}\big) - \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}. \tag{5.8c}$$

5.2.1 Transport prephase

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mathbf{\Omega}\cdot\nabla I + \sigma_t(\mathbf{x})I = \frac{\sigma_s^n}{4\pi}c\mathcal{E}^n + \frac{\sigma_a^n}{4\pi}ac(T^4)^n - \frac{\sigma_t^n}{4\pi}\boldsymbol{\mathcal{F}}_0^n\cdot\frac{\mathbf{u}^n}{c} + \frac{\sigma_t^n}{\pi}\mathcal{E}^n\ \mathbf{\Omega}\cdot\mathbf{u}^n \tag{5.9a}$$

Develop $\{f\}^n$.

5.2.2 Predictor phase

$\tau = \frac{1}{\frac{1}{2}\Delta t}$

$$\tau(\mathbf{U}^{n*} - \mathbf{U}^n) + \nabla\cdot\mathcal{F}^H(\mathbf{U}^n) = \mathbf{0} \tag{5.10a}$$

$$\tau\left(\begin{bmatrix}\rho\\ \rho\mathbf{u}\end{bmatrix}^{n+\frac{1}{2}} - \begin{bmatrix}\rho\\ \rho\mathbf{u}\end{bmatrix}^{n*}\right) = \begin{bmatrix}0\\ \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\end{bmatrix}^n \tag{5.10b}$$

$$\tau(E^{n+\frac{1}{2}} - E^{n*}) = -\theta_1 S_{ea}^{n+\frac{1}{2}} - \theta_2 S_{ea}^n + \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^n \tag{5.10c}$$

$$\tau(\mathcal{E}^{n+\frac{1}{2}} - \mathcal{E}^n) + \theta_1\nabla\cdot\boldsymbol{\mathcal{F}}^{n+\frac{1}{2}} + \theta_2\nabla\cdot\boldsymbol{\mathcal{F}}^n = \theta_1 S_{ea}^{n+\frac{1}{2}} + \theta_2 S_{ea}^n - \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^n \tag{5.10d}$$

For S_{ea} and $\boldsymbol{\mathcal{F}}$ both at $n+\frac{1}{2}$:

$$\sigma^{n+\frac{1}{2}} = \rho^{n+\frac{1}{2}}\kappa(T^n) \tag{5.10e}$$

$$T^{4,n+\frac{1}{2}} = T^{4,n*} + \frac{4T^{3,n*}}{C_v}(e^{n+\frac{1}{2}} - e^{n*}) \tag{5.10f}$$

5.2.3 Corrector phase

$\tau = \frac{1}{\Delta t}$

$$\tau(\mathbf{U}^{n+\frac{1}{2}*} - \mathbf{U}^n) + \nabla\cdot\mathcal{F}^H(\mathbf{U}^{n+\frac{1}{2}}) = \mathbf{0} \tag{5.11a}$$

$$\tau\left(\begin{bmatrix}\rho\\ \rho\mathbf{u}\end{bmatrix}^{n+1} - \begin{bmatrix}\rho\\ \rho\mathbf{u}\end{bmatrix}^{n+\frac{1}{2}*}\right) = \begin{bmatrix}0\\ \frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\end{bmatrix}^{n+\frac{1}{2}} \tag{5.11b}$$

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1 S_{ea}^{n+1} - \theta_2 S_{ea}^n + \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.11c}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \theta_1\nabla\cdot\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\nabla\cdot\boldsymbol{\mathcal{F}}^n = \theta_1 S_{ea}^{n+1} + \theta_2 S_{ea}^n - \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.11d}$$

For S_{ea} and $\boldsymbol{\mathcal{F}}$ both at $n+1$:

$$\sigma^{n+1} = \rho^{n+1}\kappa(T^{n+\frac{1}{2}}) \tag{5.11e}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{5.11f}$$

5.2.4 General energy equations, Predictor and Corrector phase, with θ factors

Time integration scheme uses **implicit Euler** for the predictor phase and **Crank-Nicolson** in the corrector phase. Both these schemes can be represented wit a general θ-scheme where we define:

$$\begin{aligned}\theta_1 &\in [0,1]\\ \theta_2 &= 1-\theta_1.\end{aligned} \tag{5.12}$$

For implicit Euler, $\theta_1 = 1,\ \theta_2 = 0$, and for Crank-Nicolson, $\theta_1 = \theta_2 = \frac{1}{2}$. With these factors defined we can repeat the energy equations and apply a series of manipulations. First we attempt to segregate known terms from all unknown terms. Thereafter we eliminate the internal energy, e, from the two sets of equations to get a single formulation for the radiation energy, $\mathcal{E}$. The latter formulation forms a diffusion system that needs to be assembled and solved for $\mathcal{E}$.

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -\theta_1\sigma_a^{n+1}c\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) - \theta_2 S_{ea}^n + \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.13a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \nabla\cdot\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\mathcal{F}}^n\big) = \theta_1\sigma_a^{n+1}c\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) + \theta_2 S_{re}^n - \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.13b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + \frac{4T^{3,n+\frac{1}{2}*}}{C_v}(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{5.13c}$$

Define:

$$\begin{aligned}k_1 &= \theta_1\sigma_a^{n+1}c\\ k_2 &= \frac{4T^{3,n+\frac{1}{2}*}}{C_v}\end{aligned} \tag{5.14}$$

---

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) - \theta_2 S_{ea}^n + \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.15a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \nabla\cdot\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\mathcal{F}}^n\big) = k_1\Big(aT^{4,n+1} - \mathcal{E}^{n+1}\Big) + \theta_2 S_{re}^n - \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.15b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{5.15c}$$

---

ungroup right-hand side elements by multiplying out terms within parentheses,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 aT^{4,n+1} + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n + \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.16a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \nabla\cdot\big(\theta_1\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\mathcal{F}}^n\big) = k_1 aT^{4,n+1} - k_1\mathcal{E}^{n+1} + \theta_2 S_{re}^n - \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.16b}$$

$$T^{4,n+1} = T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*}) \tag{5.16c}$$

---

now plug in the temperature equation into both the energy equations,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a\big(T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*})\big) + k_1\mathcal{E}^{n+1} - \theta_2 S_{ea}^n + \Big(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0\cdot\mathbf{u}\Big)^{n+\frac{1}{2}} \tag{5.17a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \boldsymbol{\nabla}\boldsymbol{\cdot}\big(\theta_1 \boldsymbol{\mathcal{F}}^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}^n\big) = k_1 a\big(T^{4,n+\frac{1}{2}*} + k_2(e^{n+1} - e^{n+\frac{1}{2}*})\big) - k_1 \mathcal{E}^{n+1} + \theta_2 S^n_{re} - \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \cdot \mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.17b}$$

---

ungroup elements on the both the right-hand sides,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = -k_1 a T^{4,n+\frac{1}{2}*} - k_1 a k_2 e^{n+1} + k_1 a k_2 e^{n+\frac{1}{2}*} + k_1 \mathcal{E}^{n+1} - \theta_2 S^n_{ea} + \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \cdot \mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.18a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \boldsymbol{\nabla}\boldsymbol{\cdot}\big(\theta_1 \boldsymbol{\mathcal{F}}^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}^n\big) = k_1 a T^{4,n+\frac{1}{2}*} + k_1 a k_2 e^{n+1} - k_1 a k_2 e^{n+\frac{1}{2}*} - k_1 \mathcal{E}^{n+1} + \theta_2 S^n_{re} - \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \cdot \mathbf{u}\right)^{n+\frac{1}{2}} \tag{5.18b}$$

Define:

$$\begin{aligned} k_3 &= -k_1 a T^{4,n+\frac{1}{2}*} + k_1 a k_2 e^{n+\frac{1}{2}*} - \theta_2 S^n_{ea} + \left(\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}}_0 \cdot \mathbf{u}\right)^{n+\frac{1}{2}} \\ k_4 &= -k_1 a k_2 \end{aligned} \tag{5.19}$$

---

and plug them into the equations above,

$$\tau(E^{n+1} - E^{n+\frac{1}{2}*}) = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \tag{5.20a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \boldsymbol{\nabla}\boldsymbol{\cdot}\big(\theta_1 \boldsymbol{\mathcal{F}}^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}^n\big) = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \tag{5.20b}$$

Note:

$$E^{n+1} = (\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \rho^{n+1} e^{n+1} \tag{5.21}$$

---

which gives,

$$\tau\big((\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \rho^{n+1} e^{n+1} - E^{n+\frac{1}{2}*}\big) = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \tag{5.22a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \boldsymbol{\nabla}\boldsymbol{\cdot}\big(\theta_1 \boldsymbol{\mathcal{F}}^{n+1} + \theta_2 \boldsymbol{\mathcal{F}}^n\big) = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \tag{5.22b}$$

---

ungroup the material energy in the first equation,

$$\tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau\rho^{n+1} e^{n+1} - \tau E^{n+\frac{1}{2}*} = k_4 e^{n+1} + k_1 \mathcal{E}^{n+1} + k_3 \tag{5.23a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \theta_1 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}^{n+1} + \theta_2 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}^n = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \tag{5.23b}$$

---

and isolate the internal energy in the first equation,

$$(\tau\rho^{n+1} - k_4) e^{n+1} = k_1 \mathcal{E}^{n+1} + k_3 - \tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau E^{n+\frac{1}{2}*} \tag{5.24a}$$

$$\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \theta_1 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}^{n+1} + \theta_2 \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}}^n = -k_4 e^{n+1} - k_1 \mathcal{E}^{n+1} - k_3 \tag{5.24b}$$

Define:

$$
\begin{aligned}
k_5 &= \frac{k_1}{\tau\rho^{n+1} - k_4} \\
k_6 &= \frac{k_3 - \tau(\frac{1}{2}\rho||\mathbf{u}||^2)^{n+1} + \tau E^{n+\frac{1}{2}*}}{\tau\rho^{n+1} - k_4}
\end{aligned}
\tag{5.25}
$$

and plug these constants into the first equation above,

$$
e^{n+1} = k_5\mathcal{E}^{n+1} + k_6 \tag{5.26a}
$$

$$
\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^n = -k_1\mathcal{E}^{n+1} - k_3 - k_4 e^{n+1} \tag{5.26b}
$$

now plug the first equation into the second,

$$
\tau(\mathcal{E}^{n+1} - \mathcal{E}^n) + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n+1} + \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^n = -k_1\mathcal{E}^{n+1} - k_3 - k_4k_5\mathcal{E}^{n+1} - k_4k_6 \tag{5.27a}
$$

now collect all the $\mathcal{E}^{n+1}$ terms on the left-hand side,

$$
\big(\tau + k_1 + k_4k_5\big)\mathcal{E}^{n+1} + \theta_1\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^{n+1} = -k_3 - k_4k_6 + \tau\mathcal{E}^n - \theta_2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}^n \tag{5.28a}
$$

5.3 Mixed Finite Element Method

We now turn our attention to just the radiation momentum equation,

$$
\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{F}}}{\partial t} + \boldsymbol{\nabla}\cdot(\{f\}\mathcal{E}) = -\frac{\sigma_t}{c}\boldsymbol{\mathcal{F}} + \frac{\sigma_t}{c}\big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big). \tag{5.29}
$$

We assume that all the unknowns in this equation have a linear FE representation on a cell, except $\mathcal{E}$ and $\mathbf{u}$.

5.3.1 The radiation-flux at $n = 0$

One of the first items we will need in any temporal discretization is the old $\boldsymbol{\mathcal{F}}^n$. In order to get value of $\boldsymbol{\mathcal{F}}^n$, when starting the iterations, we simply use the radiation-momentum equation with no time derivative to get

$$
\boldsymbol{\mathcal{F}}^n = -\frac{c}{\sigma_t^n}\boldsymbol{\nabla}\cdot(\{f\}^n\mathcal{E}^n) + (\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u})^n, \qquad \text{if } n = 0. \tag{5.30}
$$

This equation, however, still requires a suitable spatial discretization. Applying a linear FEM first requires multiplying by a trial function, then integrating over volume

$$
\int_V b_i\Big[\boldsymbol{\mathcal{F}}^n = -\frac{c}{\sigma_t^n}\boldsymbol{\nabla}\cdot(\{f\}^n\mathcal{E}^n) + (\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u})^n\Big]dV \tag{5.31}
$$

Let us now consider the terms one-by-one. First are the $\boldsymbol{\mathcal{F}}$ terms. Since $\boldsymbol{\mathcal{F}} \approx \sum_j \boldsymbol{\mathcal{F}}_j b_j(\mathbf{x})$ we get

$$
\int_V b_i\boldsymbol{\mathcal{F}}dV = \sum_j \boldsymbol{\mathcal{F}}_j \int_V b_i b_j dV \tag{5.32}
$$

Second are the divergence terms. First we rewrite

$$
\begin{aligned}
\int_V b_i\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})dV &= \int_S \mathbf{n}\cdot(b_i\{f\}\mathcal{E})dA - \int_V \{f\}\mathcal{E}\cdot\boldsymbol{\nabla}b_i dV \\
&= \sum_f \int_{S_f} \mathbf{n}_f\cdot(b_i\{f\}\mathcal{E})dA - \int_V \{f\}\mathcal{E}\cdot\boldsymbol{\nabla}b_i dV \\
&= \sum_j\sum_f \mathbf{n}_f\cdot(\{f\}\mathcal{E})_j \int_{S_f} b_i b_j dA - \sum_j(\{f\}\mathcal{E})_j\cdot\int_V b_j\boldsymbol{\nabla}b_i dV
\end{aligned}
\tag{5.33}
$$

Last are the advection terms. If the velocity and radiation-energy are only considered to be cell-constant then we have

$$\int_V b_i \big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big) dV = \big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big)_c \int_V b_i dV \tag{5.34}$$

Putting all this together, we get

$$\sum_j \boldsymbol{\mathcal{F}}_j^n \int_V b_i b_j dV = -\frac{c}{\sigma_t^n}\left[\sum_j \sum_f \mathbf{n}_f \cdot (\{f\}\mathcal{E})_j^n \int_{S_f} b_i dA - (\{f\}\mathcal{E})_c^n \cdot \int_V \boldsymbol{\nabla} b_i dV\right] + \big(\mathcal{E}\mathbf{u} + \{f\}\mathcal{E}\cdot\mathbf{u}\big)_c^n \int_V b_i dV \tag{5.35}$$

and for 1D $\{f\}$

$$\sum_j \boldsymbol{\mathcal{F}}_j^n \int_V b_i b_j dV = -\frac{c}{\sigma_t^n}\left[\sum_j \sum_f \mathbf{n}_f \cdot (f\mathcal{E})_j^n \int_{S_f} b_i dA - (f\mathcal{E})_c^n \int_V \boldsymbol{\nabla} b_i dV\right] + \big(1 + f_c^n\big)(\mathcal{E}\mathbf{u})_c^n \int_V b_i dV \tag{5.36}$$

which can be written as

$$\bar{M}_c \bar{\mathbf{F}}_c^n = -\frac{c}{\sigma_t}\bar{C}_c(\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + (1 + f_c^n)\mathcal{E}_c^n C_{vol}\mathbf{u}_c^n \tag{5.37}$$

where $\bar{M}_c$ is the dimension-extended mass-matrix as defined for Solver B and the matrix $\bar{C}_c$ is also the same as defined for Solver B. The nodal vector $(\mathbf{f}\boldsymbol{\mathcal{E}})$ is a stack of firstly the cell-centered $f\mathcal{E}$, then the list of nodal values,

$$(\mathbf{f}\boldsymbol{\mathcal{E}}) = \begin{bmatrix} f_c\mathcal{E}_c \\ f_{c,0}\mathcal{E}_{c,0} \\ \vdots \\ f_{c,N_n-1}\mathcal{E}_{c,N_n-1} \end{bmatrix} \tag{5.38}$$

the matrix C_{vol} is a square block-matrix with block dimension $N_n \times 1$ with block-structure

$$C_{vol} = \begin{bmatrix} V_{c,0} I \\ \vdots \\ V_{c,N_n-1} I \end{bmatrix} \tag{5.39}$$

where the identity matrices, I, all have dimension $N_d \times N_d$ and the coefficients $V_{c,i}$ are given by

$$V_{c,i} = \int_V b_i dV. \tag{5.40}$$

The resulting true dimensions of C_{vol} is therefore $N_d N_n \times N_d$.

In order to obtain an expression for the nodal radiation-fluxes we then take the inverse of $\bar{M}_c$ to get

$$\bar{\mathbf{F}}_c^n = -\frac{c}{\sigma_t}\bar{M}_c^{-1}\bar{C}_c(\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + (1 + f_c^n)\mathcal{E}_c^n \bar{M}_c^{-1} C_{vol}\mathbf{u}_c^n \tag{5.41}$$

which we can express as

$$\bar{\mathbf{F}}_c^n = -\frac{c}{\sigma_t}\bar{C}_c^*(\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + (1 + f_c^n)\mathcal{E}_c^n C_{vol}^*\mathbf{u}_c^n \tag{5.42}$$

where $\bar{C}_c^* = \bar{M}_c^{-1}\bar{C}_c$ and $C_{vol}^* = \bar{M}_c^{-1} C_{vol}$. However, using the identity

Additionally we require an expression for $\boldsymbol{\mathcal{F}}_0$. If we use the same notation for building $\bar{\mathbf{F}}$ from $\boldsymbol{\mathcal{F}}$, for $\bar{\mathbf{F}}_0$ from $\boldsymbol{\mathcal{F}}_0$, we get

$$\bar{\mathbf{F}}_{0,c}^n = \bar{\mathbf{F}}_c^n - (1 + f_c^n)\mathcal{E}_c^n C_{vol}^*\mathbf{u}_c^n \tag{5.43}$$

5.3.2 The radiation-flux at $n+1$

We now seek a similar expression for the radiation-flux at timestep $n+1$. Our first discretization is a temporal theta-scheme discretization where we lag the advection terms,

$$\begin{aligned}&\frac{\tau}{c^2}(\boldsymbol{\mathcal{F}}^{n+1}-\boldsymbol{\mathcal{F}}^n)+\theta_1\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}+\theta_2\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^n\\&=-\frac{\theta_1}{c}\sigma_t^{n+1}\boldsymbol{\mathcal{F}}^{n+1}-\frac{\theta_2}{c}\sigma_t^n\boldsymbol{\mathcal{F}}^n+\frac{\sigma_t^{n+\frac{1}{2}}}{c}\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}\end{aligned}\tag{5.44}$$

where $\tau=\dfrac{1}{\Delta t}$. We then multiply by $\dfrac{c^2}{\tau}$

$$\begin{aligned}&\boldsymbol{\mathcal{F}}^{n+1}-\boldsymbol{\mathcal{F}}^n+\theta_1\frac{c^2}{\tau}\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}+\theta_2\frac{c^2}{\tau}\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^n\\&=-\frac{\theta_1c^2}{c\tau}\sigma_t^{n+1}\boldsymbol{\mathcal{F}}^{n+1}-\frac{\theta_2c^2}{c\tau}\sigma_t^n\boldsymbol{\mathcal{F}}^n+\frac{\sigma_t^{n+\frac{1}{2}}c^2}{c\tau}\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}.\end{aligned}\tag{5.45}$$

Next we define,

$$\begin{aligned}a_1&=\theta_1\frac{c^2}{\tau}\\a_2&=\theta_2\frac{c^2}{\tau}\\a_3&=\sigma_t^{n+\frac{1}{2}}\frac{c}{\tau}\end{aligned}\tag{5.46}$$

to get,

$$\begin{aligned}&\boldsymbol{\mathcal{F}}^{n+1}-\boldsymbol{\mathcal{F}}^n+a_1\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}+a_2\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^n\\&=-\frac{a_1}{c}\sigma_t^{n+1}\boldsymbol{\mathcal{F}}^{n+1}-\frac{a_2}{c}\sigma_t^n\boldsymbol{\mathcal{F}}^n+a_3\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}.\end{aligned}\tag{5.47}$$

which we can rearrange as

$$\big(1+\frac{a_1}{c}\sigma_t^{n+1}\big)\boldsymbol{\mathcal{F}}^{n+1}+a_1\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}=\big(1-\frac{a_2}{c}\sigma_t^n\big)\boldsymbol{\mathcal{F}}^n-a_2\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^n+a_3\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}.\tag{5.48}$$

Next we define

$$\begin{aligned}a_4&=\frac{a_1}{1+\frac{a_1}{c}\sigma_t^{n+1}}\\a_5&=\frac{1-\frac{a_2}{c}\sigma_t^n}{1+\frac{a_1}{c}\sigma_t^{n+1}}\\a_6&=\frac{a_2}{1+\frac{a_1}{c}\sigma_t^{n+1}}\\a_7&=\frac{a_3}{1+\frac{a_1}{c}\sigma_t^{n+1}}\end{aligned}\tag{5.49}$$

to get

$$\boldsymbol{\mathcal{F}}^{n+1}+a_4\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}=a_5\boldsymbol{\mathcal{F}}^n-a_6\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^n+a_7\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}.\tag{5.50}$$

Now we apply our spatial discretization scheme by multiplying by trial functions, defined as the basis functions on each cell, then integrating over volume

$$\int_V b_i(\mathbf{x})\Big[\boldsymbol{\mathcal{F}}^{n+1}+a_4\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^{n+1}=a_5\boldsymbol{\mathcal{F}}^n-a_6\boldsymbol{\nabla}\cdot(\{f\}\mathcal{E})^n+a_7\big(\mathcal{E}\mathbf{u}+\{f\}\mathcal{E}\cdot\mathbf{u}\big)^{n+\frac{1}{2}}\Big]dV.\tag{5.51}$$

For 1D $\{f\}$

$$\int_V b_i(\mathbf{x})\Big[\boldsymbol{\mathcal{F}}^{n+1}+a_4\boldsymbol{\nabla}\cdot(f\mathcal{E})^{n+1}=a_5\boldsymbol{\mathcal{F}}^n-a_6\boldsymbol{\nabla}(f\mathcal{E})^n+a_7\big(1+f_c^{n+\frac{1}{2}}\big)\mathcal{E}_c^{n+\frac{1}{2}}\mathbf{u}_c^{n+\frac{1}{2}}\Big]dV.\tag{5.52}$$

Using the expressions we developed for the $n = 0$ case, we can similarly write

$$\bar{M}_c \bar{\mathbf{F}}_c^{n+1} + a_4 \bar{C}_c (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n+1} = a_5 \bar{M}_c \bar{\mathbf{F}}_c^n - a_6 \bar{C}_c (\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + a_7 (1 + f_c^{n+\frac{1}{2}}) \mathcal{E}_c^{n+\frac{1}{2}} C_{vol} \mathbf{u}_c^{n+\frac{1}{2}}. \tag{5.53}$$

An expression for the nodal fluxes is then obtained by multiplying with the inverse of $\bar{M}_c$ to get

$$\bar{\mathbf{F}}_c^{n+1} = -a_4 \bar{M}_c^{-1} \bar{C}_c (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n+1} + a_5 \bar{\mathbf{F}}_c^n - a_6 \bar{M}_c^{-1} \bar{C}_c (\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + a_7 (1 + f_c^{n+\frac{1}{2}}) \mathcal{E}_c^{n+\frac{1}{2}} \bar{M}_c^{-1} C_{vol} \mathbf{u}_c^{n+\frac{1}{2}}. \tag{5.54}$$

which we can express as

$$\bar{\mathbf{F}}_c^{n+1} = -a_4 \bar{C}_c^* (\mathbf{f}\boldsymbol{\mathcal{E}})_c^{n+1} + a_5 \bar{\mathbf{F}}_c^n - a_6 \bar{C}_c^* (\mathbf{f}\boldsymbol{\mathcal{E}})_c^n + a_7 (1 + f_c^{n+\frac{1}{2}}) \mathcal{E}_c^{n+\frac{1}{2}} C_{vol}^* \mathbf{u}_c^{n+\frac{1}{2}}. \tag{5.55}$$

using the definitions of $\bar{C}_c^*$ and C_{vol}^* as developed for the $n = 0$ case.

5.3.3 Using the expression for $\boldsymbol{\mathcal{F}}^{n+1}$ in the primary equation

We start with the form that $\boldsymbol{\mathcal{F}}$ appears in the radiation-energy equation,

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}} &= \frac{1}{V_c} \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}} dV \\
&= \frac{1}{V_c} \int_S \mathbf{n} \cdot \boldsymbol{\mathcal{F}} dA \\
&= \frac{1}{V_c} \sum_f \mathbf{n}_f \cdot \int_{S_f} \boldsymbol{\mathcal{F}} dA \\
&= \frac{1}{V_c} \sum_j \sum_f \mathbf{n}_f \cdot \boldsymbol{\mathcal{F}}_j \int_{S_f} b_j dA \\
\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}} &= \frac{1}{V_c} \sum_f \sum_j \sum_d n_{f,d} (\boldsymbol{\mathcal{F}}_j)_d \int_{S_f} b_j dA.
\end{aligned} \tag{5.56}$$

This now serves as a template for each of the terms in the expression. The term containing $\boldsymbol{\mathcal{F}}^n$ is already covered by the template itself, therefore, by substituting $\int_{S_f} b_j dA = S_{j,f}$, we get

$$\boldsymbol{\nabla} \cdot \bar{\mathbf{F}}_c^n \mapsto \frac{1}{V_c} \sum_f \sum_j \sum_d n_{f,d} S_{j,f} (\boldsymbol{\mathcal{F}}_j)_d. \tag{5.57}$$

The second set of terms we need to address have the general form,

$$-a \bar{C}_c^* (\mathbf{f}\boldsymbol{\mathcal{E}})_c$$

where a is either a_4 or a_6 and thus,

$$-a \bar{C}_c^* (\mathbf{f}\boldsymbol{\mathcal{E}})_c \mapsto \frac{-a}{V_c} \sum_f \sum_j \sum_d \Big[n_{f,d} S_{j,f} C^*_{(j,d) \mapsto \text{ row } k} \cdot (\mathbf{f}\boldsymbol{\mathcal{E}}) \Big]. \tag{5.58}$$

The last term to consider is the advection term,

$$a_7 (1 + f_c^{n+\frac{1}{2}}) \mathcal{E}_c^{n+\frac{1}{2}} C_{vol}^* \mathbf{u}_c^{n+\frac{1}{2}} \mapsto \frac{a_7}{V_c} \sum_f \sum_j \sum_d \Big[n_{f,d} S_{j,f} * C^*_{vol,(j,d) \mapsto \text{ row } k} \cdot \mathbf{u}_c^{n+\frac{1}{2}} \Big] \tag{5.59}$$

5.3.4 Using the expression for $\boldsymbol{\mathcal{F}}^{n+1}$ in the auxiliary equations

For each face-node we now require continuity of flux. This can generally be expressed as

$$\sum_c \sum_f \int_{S_f} \mathbf{n}_f \cdot \boldsymbol{\mathcal{F}}_j^{n+1} dA = 0 \tag{5.60}$$

from which we again have to develop the three types of term, i.e., the term with $\boldsymbol{\mathcal{F}}$, the term with $\{f\}\mathcal{E}$ and the term with $\mathcal{E}\mathbf{u}$. Fortunately, the previously determined forms are easily extended to here, i.e., with only the relevant face and node indices changing.

$$
\begin{aligned}
&\sum_c \sum_f \sum_d n_{f,d} S_{j,f} \Big[- a_4 C^*_{(j,d)\mapsto \text{ row } k} \cdot (\mathbf{f}\boldsymbol{\mathcal{E}})^{n+1} \Big] \\
&= \sum_c \sum_f \sum_d n_{f,d} S_{j,f} \Big[- a_5 (\boldsymbol{\mathcal{F}}_j)^n_d + a_6 C^*_{(j,d)\mapsto \text{ row } k} \cdot (\mathbf{f}\boldsymbol{\mathcal{E}})^n - a_7 n_{f,d} S_{j,f} * C^*_{vol,(j,d)\mapsto \text{ row } k} \cdot \mathbf{u}_c^{n+\frac{1}{2}} \Big]
\end{aligned}
\tag{5.61}
$$

A Angular integration identities

Identity A-1

$$\int_{4\pi} d\mathbf{\Omega} = 4\pi.$$

Identity A-2

$$\int_{4\pi} \mathbf{\Omega}\ d\mathbf{\Omega} = 0.$$

Identity A-3 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \cdot \mathbf{v}\ d\mathbf{\Omega} = 0.$$

Identity A-4 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \cdot \boldsymbol{\nabla}(\mathbf{\Omega} \cdot \mathbf{v})\ d\mathbf{\Omega} = \frac{4\pi}{3} \boldsymbol{\nabla} \cdot \mathbf{v}.$$

Identity A-5 Given the scalar, a,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \boldsymbol{\nabla} a \Big)\ d\mathbf{\Omega} = \frac{4\pi}{3} \boldsymbol{\nabla} a.$$

Identity A-6 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \mathbf{v} \Big)\ d\mathbf{\Omega} = \frac{4\pi}{3} \mathbf{v}.$$

Identity A-7 Given the known three component vector, $\mathbf{v}$,

$$\int_{4\pi} \mathbf{\Omega} \Big(\mathbf{\Omega} \cdot \boldsymbol{\nabla}(\mathbf{\Omega} \cdot \mathbf{v}) \Big)\ d\mathbf{\Omega} = 0.$$

B Boundary and initial conditions for radiation hydrodynamic problems

In a one dimensional simulation we can simulate steady-state shocks by setting the appropriate pre- and post-shock conditions. Pre-shock conditions will be denoted with a subscript L whereas post-shock conditions will be denoted with a subscript R.

B.1 Hydrodynamics only

With no radiation energy present we wish to have $\mathcal{F}_L^H = \mathcal{F}_R^H$, therefore

$$\begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}_R . \tag{B.1}$$

Here we have three equations but 4 unknowns, i.e., ρ, u, p and e. Fortunately, we can express both e and p in terms of temperature since

$$p = (\gamma - 1)\rho e$$

and

$$e = C_v T.$$

Therefore,

$$p = (\gamma - 1)\rho C_v T$$

and

$$\begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \rho C_v T u + pu \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \rho C_v T u + pu \end{bmatrix}_R . \tag{B.2}$$

When the left state is known then we can frame these equations as seeking the non-linear solution of

$$\mathbf{F}\begin{pmatrix} \rho_R \\ T_R \\ u_R \end{pmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T \\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u \end{bmatrix}_R - \mathcal{F}_L^H = \mathbf{0} \tag{B.3}$$

or simply

$$\mathbf{F}(\mathbf{x}) = \mathcal{F}^H(\mathbf{x}_R) - \mathcal{F}_L^H = \mathbf{0}. \tag{B.4}$$

to which Newton-iteration can be applied in the form

$$\mathbf{x}_R^{\ell+1} = \mathbf{x}_R^{\ell} - J^{-1}(\mathbf{x}_R^{\ell})\mathbf{F}(\mathbf{x}_R^{\ell})$$

where the Jacobian matrix, J, is given by

$$J = \begin{bmatrix} u & 0 & \rho \\ u^2 + (\gamma-1)C_v T & (\gamma-1)\rho C_v & 2\rho u \\ \frac{1}{2}u^3 + \gamma C_v T u & \gamma\rho C_v u & \frac{3}{2}\rho u^2 + \gamma\rho C_v T \end{bmatrix} \tag{B.5}$$

Note: The initial guess, $\mathbf{x}^0$ cannot be the same as $\mathbf{x}_L$ since the iteration will terminate immediately. Generally the values need to be perturbed sufficiently such that $\rho_R > \rho_L$, $T_R > T_L$ and $u_R < a_R$ where a is the sound-speed.

B.2 Hydrodynamics with radiation energy

With radiation energy present we are concerned with the following set of equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{B.6a}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \{\rho \mathbf{u} \otimes \mathbf{u}\} + \boldsymbol{\nabla} p = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}, \tag{B.6b}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u} \tag{B.6c}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\Big(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\Big) + \frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}\mathbf{u}\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot \mathbf{u}. \tag{B.6d}$$

which for a steady-steady, one dimensional simulation becomes

$$\boldsymbol{\nabla} \cdot (\rho u) = 0 \tag{B.7a}$$

$$\boldsymbol{\nabla} \cdot (\rho u^2) + \boldsymbol{\nabla} p = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}, \tag{B.7b}$$

$$\boldsymbol{\nabla} \cdot [(E+p)u] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot u \tag{B.7c}$$

$$\boldsymbol{\nabla}\cdot\Big(-\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E}\Big) + \frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}u\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}u. \tag{B.7d}$$

Additionally, far away from the interface the co-moving frame radiation flux, $\boldsymbol{\mathcal{F}}_0$, is zero, therefore

$$\boldsymbol{\mathcal{F}}_0 = -\frac{c}{3\sigma_t}\boldsymbol{\nabla}\mathcal{E} = 0$$

and the equation set becomes

$$\boldsymbol{\nabla} \cdot (\rho u) = 0 \tag{B.8a}$$

$$\boldsymbol{\nabla} \cdot (\rho u^2) + \boldsymbol{\nabla} p = -\frac{1}{3}\boldsymbol{\nabla}\mathcal{E}, \tag{B.8b}$$

$$\boldsymbol{\nabla} \cdot [(E+p)u] = \sigma_a c(\mathcal{E} - aT^4) - \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} \cdot u \tag{B.8c}$$

$$\frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}u\big) = \sigma_a c\big(aT^4 - \mathcal{E}\big) + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E}u. \tag{B.8d}$$

Now, adding the last equation to the third ,we get

$$\boldsymbol{\nabla} \cdot (\rho u) = 0 \tag{B.9a}$$

$$\boldsymbol{\nabla} \cdot (\rho u^2) + \boldsymbol{\nabla} p + \frac{1}{3}\boldsymbol{\nabla}\mathcal{E} = 0, \tag{B.9b}$$

$$\boldsymbol{\nabla} \cdot [(E+p)u] + \frac{4}{3}\boldsymbol{\nabla}\big(\mathcal{E}u\big) = 0. \tag{B.9c}$$

We now express internal energy, e, the pressure, p, and the radiation energy, $\mathcal{E}$, in terms of temperature

$$\boldsymbol{\nabla} \cdot (\rho u) = 0 \tag{B.10a}$$

$$\boldsymbol{\nabla} \cdot (\rho u^2) + \boldsymbol{\nabla}\big((\gamma-1)\rho C_v T\big) + \frac{1}{3}\boldsymbol{\nabla} aT^4 = 0, \tag{B.10b}$$

$$\boldsymbol{\nabla} \cdot [(\frac{1}{2}\rho u^2 + \rho C_v T + (\gamma-1)\rho C_v T)u] + \frac{4}{3}\boldsymbol{\nabla}\big(aT^4 u\big) = 0. \tag{B.10c}$$

Finally we integrate this equation set over the entire domain to get

$$\begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T + \frac{1}{3}aT^4 \\ \frac{1}{2}\rho u^3 + \gamma \rho C_v T u + \frac{4}{3}aT^4 u \end{bmatrix}_L = \begin{bmatrix} \rho u \\ \rho u^2 + (\gamma-1)\rho C_v T + \frac{1}{3}aT^4 \\ \frac{1}{2}\rho u^3 + \gamma \rho C_v T u + \frac{4}{3}aT^4 u \end{bmatrix}_R \tag{B.11a}$$

Similar to the previous case we can now define

$$\mathbf{F}\begin{pmatrix}\rho_R\\ T_R\\ u_R\end{pmatrix} = \begin{bmatrix}\rho u\\ \rho u^2 + (\gamma-1)\rho C_v T + \frac{1}{3}aT^4\\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u + \frac{4}{3}aT^4 u\end{bmatrix}_R - \begin{bmatrix}\rho u\\ \rho u^2 + (\gamma-1)\rho C_v T + \frac{1}{3}aT^4\\ \frac{1}{2}\rho u^3 + \gamma\rho C_v T u + \frac{4}{3}aT^4 u\end{bmatrix}_L = \mathbf{0}, \tag{B.12}$$

where the subscript L quantities are all known. Applying Newton-iteration to this equation again is

$$\mathbf{x}_R^{\ell+1} = \mathbf{x}_R^{\ell} - J^{-1}(\mathbf{x}_R^{\ell})\mathbf{F}(\mathbf{x}_R^{\ell})$$

where the Jacobian matrix, J is given by

$$J = \begin{bmatrix} u & 0 & \rho\\ u^2 + (\gamma-1)C_v T & (\gamma-1)\rho C_v + \frac{4}{3}aT^3 & 2\rho u\\ \frac{1}{2}u^3 + \gamma C_v T u & \gamma\rho C_v u + \frac{16}{3}aT^3 u & \frac{3}{2}\rho u^2 + \gamma\rho C_v T + \frac{4}{3}aT^4\end{bmatrix} \tag{B.13}$$

Example mach 3 conditions, $C_v = 0.14472799784454$ and $\gamma = \frac{5}{3}$:

```
[0]   rho0  1.00000000e+00 u0    3.80431331e-01 T0    1.00000000e-01 e0    1.44727998e-02 radE0 1.37223549e-06
[0]   rho1  3.00185103e+00 u1    1.26732249e-01 T1    3.66260705e-01 e1    5.30081785e-02 radE1 2.46939153e-04
```

Note: The initial guess, $\mathbf{x}^0$ cannot be the same as $\mathbf{x}_L$ since the iteration will terminate immediately. Generally the values need to be perturbed sufficiently such that $\rho_R > \rho_L$, $T_R > T_L$, $u_R < a_R$ where a is the sound-speed, and $\mathcal{E}_R > \mathcal{E}_L$.

C Tensor Algebra

Most of these notations are obtained from [3].

C.1 Identities

Identity C-1 Given $f(x, y, z)$ and $\mathbf{a} = [a_x, a_y, a_z]$.

$$f\boldsymbol{\nabla} \cdot \mathbf{a} = \boldsymbol{\nabla} \cdot (f\mathbf{a}) - \mathbf{a} \cdot \boldsymbol{\nabla} f$$

Proof:

$$f\boldsymbol{\nabla} \cdot \mathbf{a} = f\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) \tag{C.1}$$

$$= f\frac{\partial a_x}{\partial x} + f\frac{\partial a_y}{\partial y} + f\frac{\partial a_z}{\partial z}, \tag{C.2}$$

applying the product rule of differentiation,

$$= \frac{\partial}{\partial x}(fa_x) - a_x\frac{\partial f}{\partial x} \tag{C.3}$$

$$+ \frac{\partial}{\partial y}(fa_y) - a_y\frac{\partial f}{\partial y} \tag{C.4}$$

$$+ \frac{\partial}{\partial z}(fa_z) - a_z\frac{\partial f}{\partial z} \tag{C.5}$$

by observing the vertical alignment here we get

$$f\boldsymbol{\nabla} \cdot \mathbf{a} = \boldsymbol{\nabla} \cdot (f\mathbf{a}) - \mathbf{a} \cdot \boldsymbol{\nabla} f \tag{C.6}$$

C.2 Tensor product of two vectors $\mathbf{a} \otimes \mathbf{b}$

Also called the *dyadic product*. Given vector $\mathbf{a}$ of size $N\times1$ and vector $\mathbf{b}$ of size $M\times1$, then the tensor product of $\mathbf{a}$ and $\mathbf{b}$ results in a rank 2 tensor and is given by

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_0b_0 & a_0b_1 & \dots & a_0b_{M-1} \\ a_1b_0 & a_1b_1 & \dots & a_1b_{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1}b_0 & a_{N-1}b_1 & \dots & a_{N-1}b_{M-1} \end{bmatrix} \tag{C.7}$$

with resulting dimensions $N\times M$.

C.3 Dot-product of a vector with a tensor, $\mathbf{a} \bullet \{t\}$

Under the same topic of tensor product notation we can also discuss the **dot product of scalar and a rank 2 tensor**. The dot-product of a vector $\mathbf{a}$ and a tensor $\{\mathbf{t}\}$, commonly written as $\mathbf{a} \cdot \{\mathbf{t}\}$, which results in a vector of size N, can be understood using one of two thought patterns:

- Thought pattern 1: Classical component-wise dot product

$$\begin{aligned}
\mathbf{a}\cdot\{\mathbf{t}\} &= \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} \cdot \begin{bmatrix} t_{00} & t_{01} & \dots & t_{0(N-1)} \\ t_{10} & t_{11} & \dots & t_{1(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ t_{(N-1)0} & t_{(N-1)1} & \dots & t_{(N-1)(N-1)} \end{bmatrix} \\
&= \begin{bmatrix} a_0 t_{00} & + & a_1 t_{01} & + & \dots & + & a_{N-1} t_{0(N-1)} \\ a_0 t_{10} & + & a_1 t_{11} & + & \dots & + & a_{N-1} t_{1(N-1)} \\ \vdots & + & \vdots & + & \ddots & + & \vdots \\ a_0 t_{(N-1)0} & + & a_1 t_{(N-1)1} & + & \dots & + & a_{N-1} t_{(N-1)(N-1)} \end{bmatrix} \\
&= \{t\}\cdot\mathbf{a}
\end{aligned} \tag{C.8}$$

- Thought pattern 2 (preferred): Dot product of vectors

$$\mathbf{a}\cdot\{\mathbf{t}\} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_{(N-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{a}\ \cdot\ \mathbf{t}_0 \\ \mathbf{a}\ \cdot\ \mathbf{t}_1 \\ \vdots \\ \mathbf{a}\ \cdot\ \mathbf{t}_{(N-1)} \end{bmatrix} = \{t\}\cdot\mathbf{a} \tag{C.9}$$

where the latter thought pattern requires the rank 2 tensor two be represented as a vector of vectors (or a matrix if you prefer).

C.4 Finite element discretization of the divergence of a tensor, i.e., $\boldsymbol{\nabla}\cdot\boldsymbol{\tau}$

Short-hand notation. $b_i \equiv b_i(\mathbf{x})$. For this section we seek a general way to handle

$$\int_V b_i \boldsymbol{\nabla}\cdot\boldsymbol{\tau} dV. \tag{C.10}$$

We start by writing the tensor as a block vector

$$\boldsymbol{\tau} = \begin{bmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_{N-1} \end{bmatrix}, \tag{C.11}$$

therefore,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\nabla}\cdot\mathbf{t}_0 \\ \vdots \\ \boldsymbol{\nabla}\cdot\mathbf{t}_{N-1} \end{bmatrix}. \tag{C.12}$$

If we now multiply by the trial space function, b_i, and integrate then we essentially have

$$\int_V b_i \boldsymbol{\nabla}\cdot\boldsymbol{\tau} dV = \begin{bmatrix} \int_V b_i \boldsymbol{\nabla}\cdot\mathbf{t}_0 dV \\ \vdots \\ \int_V b_i \boldsymbol{\nabla}\cdot\mathbf{t}_{N-1} dV \end{bmatrix}, \tag{C.13}$$

after we use identity C-1 to get

$$\int_V b_i \boldsymbol{\nabla}\cdot\boldsymbol{\tau} dV = \begin{bmatrix} \int_V b_i \boldsymbol{\nabla}\cdot\mathbf{t}_0 dV \\ \vdots \\ \int_V b_i \boldsymbol{\nabla}\cdot\mathbf{t}_{N-1} dV \end{bmatrix} = \begin{bmatrix} \int_V \boldsymbol{\nabla}\cdot(b_i\mathbf{t}_0) dV - \int_V \mathbf{t}_0\cdot\boldsymbol{\nabla} b_i dV \\ \vdots \\ \int_V \boldsymbol{\nabla}\cdot(b_i\mathbf{t}_{N-1}) dV - \int_V \mathbf{t}_{N-1}\cdot\boldsymbol{\nabla} b_i dV \end{bmatrix}, \tag{C.14}$$

Now we apply Gauss' divergence theorem on the first terms

$$\int_V b_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau} dV = \begin{bmatrix} \displaystyle\int_S \mathbf{n} \cdot (b_i \mathbf{t}_0) dA \\ \vdots \\ \displaystyle\int_S \mathbf{n} \cdot (b_i \mathbf{t}_{N-1}) dA \end{bmatrix} - \begin{bmatrix} \displaystyle\int_V \mathbf{t}_0 \cdot \boldsymbol{\nabla} b_i dV \\ \vdots \\ \displaystyle\int_V \mathbf{t}_{N-1} \cdot \boldsymbol{\nabla} b_i dV \end{bmatrix}, \tag{C.15}$$

which we can write as

$$\int_V b_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau} dV = \int_S \mathbf{n} \cdot (b_i \boldsymbol{\tau}) dA - \int_V \boldsymbol{\tau} \cdot \boldsymbol{\nabla} b_i dV. \tag{C.16}$$

We can now expand $\boldsymbol{\tau}$ into basis functions and segregate the surface-integrals into face-integrals to get

$$\int_V b_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau} dV = \sum_j \sum_f \left[\mathbf{n}_f \cdot \boldsymbol{\tau}_j \int_{S_f} b_i b_j dA \right] - \sum_j \left[\boldsymbol{\tau}_j \cdot \int_V b_j \boldsymbol{\nabla} b_i dV \right]. \tag{C.17}$$

What we can additionally do here is to lump the $\boldsymbol{\tau}_j$'s in the last term to cell-centered values $\boldsymbol{\tau}_c$ so that we have

$$\int_V b_i \boldsymbol{\nabla} \cdot \boldsymbol{\tau} dV = \sum_j \sum_f \left[\mathbf{n}_f \cdot \boldsymbol{\tau}_j \int_{S_f} b_i b_j dA \right] - \boldsymbol{\tau}_c \cdot \int_V \boldsymbol{\nabla} b_i dV, \tag{C.18}$$

which enables us to develop a system form.

Define the following,

$$M_{ij}^f = \int_{S_f} b_i b_j dA$$
$$\mathbf{G}_i = \int_V \boldsymbol{\nabla} b_i dV.$$

We then start to define the matrix C_c as

$$\begin{aligned} \boldsymbol{\tau}_c \cdot \int_V \boldsymbol{\nabla} b_i dV &= \begin{bmatrix} \mathbf{G}_{i,x} & \mathbf{G}_{i,y} & \mathbf{G}_{i,z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{G}_{i,x} & \mathbf{G}_{i,y} & \mathbf{G}_{i,z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{G}_{i,x} & \mathbf{G}_{i,y} & \mathbf{G}_{i,z} \end{bmatrix} \begin{bmatrix} \mathbf{t}_{c,x} \\ \mathbf{t}_{c,y} \\ \mathbf{t}_{c,z} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{G}_i^T & 0 & 0 \\ 0 & \mathbf{G}_i^T & 0 \\ 0 & 0 & \mathbf{G}_i^T \end{bmatrix} \begin{bmatrix} \mathbf{t}_{c,x} \\ \mathbf{t}_{c,y} \\ \mathbf{t}_{c,z} \end{bmatrix} \end{aligned} \tag{C.19}$$

References

[1] John I. Castor, *Radiation Hydrodynamics*, UCRL-BOOK-134209-Rev-1, Lawrence Livermore National Laboratory, November 2003.

[2] Toro E.F., *Riemann Solvers and Numerical Methods for Fluid Dynamics - A Practical Introduction*, third edition, Springer, 2009.

[3] Moukalled F., Mangani L., Darwish M., *The Finite Volume Method in Computational Fluid Dynamics - An Advanced Introduction with OpenFOAM® and Matlab®*, Springer, 2016.

[4] McClarren R.G., Lowrie R.B., *The effects of slope limiting on asymptotic-preserving numerical methods for hyperbolic conservation laws*, Journal of Computational Physics, vol 227 p9711-9726, 2008.

[5] Timmers F.X., *Exact Riemann Solver*, Website: https://cococubed.com/code_pages/exact_riemann.shtml, accessed April 22, 2022.

D Roderigues's formula

Roderigues' formula for the rotation of a vector $\mathbf{v}$ about a unit vector $\mathbf{a}$ with right-hand rule

$$\mathbf{v}_{rotated} = \cos\theta\mathbf{v} + (\mathbf{a}\cdot\mathbf{v})(1-\cos\theta)\mathbf{a} + \sin\theta(\mathbf{a}\times\mathbf{v}) \tag{D.1}$$

In matrix form

$$\mathbf{v}_{rotated} = A\mathbf{v} \tag{D.2}$$

where

$$A = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{D.3}$$

and

$$R = I + \sin\theta A + (1-\cos\theta)A^2 \tag{D.4}$$