Lecture 5

The Radiation-Hydrodynamics Equations

The purpose of this lecture is to

- To introduce the nonrelativistic hydrodynamics equations coupled to the relativistic laboratory-frame radiation transport equation.
- To derive the radiation-hydrodynamics equations to O(u/c).

1 The Radiation-Hydrodynamics Equations

Recalling our result from the previous lecture, we write the laboratory-frame thermal radiation transport equation as follows:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \overrightarrow{\Omega} \cdot \overrightarrow{\nabla} I = Q \,, \tag{1}$$

where

$$Q \equiv -(E_0/E)\sigma_t(E_0) I(\overrightarrow{\Omega}, E) +$$

$$(E/E_0)^2 \frac{\sigma_s}{4\pi} \int_{4\pi} (E_0/E')\, I(\overrightarrow{\Omega}', E')\, d\Omega' + (E/E_0)^2\, \sigma_a(E_0) B(E_0) \,, \tag{2}$$

$$E_0 = E\, \gamma \left(1 - \overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right) \,, \tag{3}$$

$$E' = E\, \frac{1 - \overrightarrow{\Omega} \cdot \overrightarrow{u}/c}{1 - \overrightarrow{\Omega}' \cdot \overrightarrow{u}/c} \,. \tag{4}$$

It is fairly straightforward to couple this equation to the radiation-hydrodynamics equations. We begin by defining the following sources:

$$S_{re} = \int_0^\infty \int_{4\pi} Q \, d\Omega \, dE \,, \tag{5}$$

and

$$\overrightarrow{S}_{rp} = \int_0^\infty \int_{4\pi} \frac{1}{c} \overrightarrow{\Omega} \, Q \, d\Omega \, dE \,. \tag{6}$$

Note that $S_{re}(t, \overrightarrow{r})$ and $\overrightarrow{S}_{rp}(t, \overrightarrow{r})$ represent the net source rates for radiation energy and radiation momentum, respectively, due to radiation-material interactions. Since total material and radiation energy must be preserved in the interaction process, it follows that $-S_{re}(t, \overrightarrow{r})$ and $-\overrightarrow{S}_{rp}(t, \overrightarrow{r})$ represent the net source rates for total (internal plus kinetic) material energy and material momentum, respectively, due to radiation-material interactions. Thus it follows that the radiation-hydrodynamic equations consist of Eq. (1) together with the following equations:

$$\frac{\partial}{\partial t}\rho + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{u}) = 0 \,. \tag{7}$$

$$\frac{\partial}{\partial t}(\rho \overrightarrow{u}) + \overrightarrow{\nabla} \cdot \left(\rho \overrightarrow{u} \otimes \overrightarrow{u}\right) + \overrightarrow{\nabla} p = -\overrightarrow{S}_{rp} \,, \tag{8}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \overrightarrow{\nabla} \cdot \left[(\frac{1}{2}\rho u^2 + \rho e + p)\overrightarrow{u}\right] = -S_{re} \,. \tag{9}$$

For illustrative purposes, it is also use useful to add the radiation energy conservation equation and the radiation momentum conservation equation, respectively:

$$\frac{\partial \mathcal{E}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{F}} = S_{re} \,, \tag{10}$$

$$\frac{1}{c^2}\frac{\partial \overrightarrow{\mathcal{F}}}{\partial t} + \overrightarrow{\nabla} \cdot \overrightarrow{\overrightarrow{\mathcal{P}}} = \overrightarrow{S}_{rp} \,. \tag{11}$$

It is also useful to construct a source for the material internal energy equation since it is often substituted for the total energy equation. Toward this end we note that since the radiation momentum source for the material is $-\overrightarrow{S}_{rp}$, it follows that the corresponding material kinetic energy source is $-\overrightarrow{S}_{rp} \cdot \overrightarrow{u}$. Since the total material energy source is the sum of the internal and kinetic energy sources, it follows that the the material internal energy source due to radiation-material interactions is

$$-S_{ri} \equiv -S_{re} + \overrightarrow{S}_{rp} \cdot \overrightarrow{u} \,. \tag{12}$$

Thus the material internal energy equation is

$$\frac{\partial}{\partial t}(\rho e) + \overrightarrow{\nabla} \cdot \left(\rho e \overrightarrow{u}\right) + p \overrightarrow{\nabla} \cdot \overrightarrow{v} = -S_{re} + \overrightarrow{S}_{rp} \cdot \overrightarrow{u} \,. \tag{13}$$

2 The First-Order Transport Equation

It clearly makes sense to simplify the transport equation (and the radiation-hydrodynamics equations) in the non-relativistic limit by expanding material velocity dependencies to $O(v/c)$. We first perform some expansions for fundamental expressions.

$$E_0/E = 1 - \overrightarrow{\Omega} \cdot \overrightarrow{u}/c \,, \tag{14}$$

$$E/E_0 = 1 + \overrightarrow{\Omega} \cdot \overrightarrow{u}/c \,, \tag{15}$$

$$\begin{aligned} \sigma(E_0) &= \sigma(E) + \frac{\partial \sigma}{\partial E}(E_0 - E)\,, \\ &= \sigma(E) - \frac{\partial \sigma}{\partial E} E\, \overrightarrow{\Omega} \cdot \overrightarrow{u}/c\,, \end{aligned} \tag{16}$$

$$E' = E\left[1 + \left(\overrightarrow{\Omega}' - \overrightarrow{\Omega}\right) \cdot \overrightarrow{u}/c\right]\,, \tag{17}$$

We next expand each term in the transport source. First, using Eqs. (14) and (16), we expand the removal source:

$$\begin{aligned} (E_0/E)\sigma_t(E_0) I(\overrightarrow{\Omega}, E) &\approx \left(1 - \overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right)\left(\sigma_t - E\frac{\partial \sigma_a}{\partial E}\overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right) I(\overrightarrow{\Omega}, E)\,, \\ &\approx \left[\sigma_t - \left(\sigma_t + E\frac{\partial \sigma_a}{\partial E}\right)\overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right] I(\overrightarrow{\Omega}, E)\,. \end{aligned} \tag{18}$$

Next, using Eqs. (14) through (17), we expand the Thompson inscattering source:

$$(E/E_0)^2\, \frac{\sigma_s}{4\pi} \int_{4\pi} (E_0/E')\, I(\overrightarrow{\Omega}', E')\, d\Omega'$$

$$\begin{aligned} &\approx \left(1 + 2\overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right) \frac{\sigma_s}{4\pi} \int_{4\pi} \left(1 - \overrightarrow{\Omega}' \cdot \overrightarrow{u}/c\right) \left[I(\overrightarrow{\Omega}', E) + \frac{\partial I}{\partial E} E \left(\overrightarrow{\Omega}' - \overrightarrow{\Omega}\right) \cdot \overrightarrow{u}/c\right] d\Omega'\,, \\ &\approx \left(1 + 2\overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right) \frac{\sigma_s}{4\pi} \int_{4\pi} \left[I(\overrightarrow{\Omega}', E)\left(1 - \overrightarrow{\Omega}' \cdot \overrightarrow{u}/c\right) + \frac{\partial I}{\partial E} E \left(\overrightarrow{\Omega}' - \overrightarrow{\Omega}\right) \cdot \overrightarrow{u}/c\right] d\Omega'\,, \\ &\approx \left(1 + 2\overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right) \frac{\sigma_s}{4\pi} \left[\varphi - \overrightarrow{F} \cdot \overrightarrow{u}/c + E\frac{\partial \overrightarrow{F}}{\partial E} \cdot \overrightarrow{u}/c - E\frac{\partial \varphi}{\partial E}\overrightarrow{\Omega} \cdot \overrightarrow{u}/c\right]\,, \\ &\approx \frac{\sigma_s}{4\pi}\left\{\varphi + \left[\left(2\varphi - E\frac{\partial \varphi}{\partial E}\right)\overrightarrow{\Omega} - \left(\overrightarrow{F} - E\frac{\partial \overrightarrow{F}}{\partial E}\right)\right] \cdot \overrightarrow{u}/c\right\}\,, \end{aligned} \tag{19}$$

where $\overrightarrow{F}$ denotes the energy-dependent flux:

$$\overrightarrow{F} = \int_{4\pi} \overrightarrow{\Omega}\, I(\overrightarrow{\Omega}, E)\, d\Omega\,, \tag{20}$$

Finally, using Eqs. (15) and (16)we expand the emission source:

$$
\begin{aligned}
(E/E_0)^2\,\sigma_a(E_0)B(E_0) &\approx \left(1+2\overrightarrow{\Omega}\cdot\overrightarrow{u}/c\right)\left(\sigma_a - E\frac{\partial\sigma_a}{\partial E}\overrightarrow{\Omega}\cdot\overrightarrow{u}/c\right)\left(B - E\frac{\partial B}{\partial E}\overrightarrow{\Omega}\cdot\overrightarrow{u}/c\right), \\
&\approx \left(1+2\overrightarrow{\Omega}\cdot\overrightarrow{u}/c\right)\left[\sigma_a B - \left(B\,E\frac{\partial\sigma_a}{\partial E} + \sigma_a\,E\frac{\partial B}{\partial E}\right)\overrightarrow{\Omega}\cdot\overrightarrow{u}/c\right], \\
&\approx \sigma_a B + \left(2\sigma_a B - B\,E\frac{\partial\sigma_a}{\partial E} - \sigma_a\,E\frac{\partial B}{\partial E}\right)\overrightarrow{\Omega}\cdot\overrightarrow{u}/c\,. \qquad (21)
\end{aligned}
$$

Using Eqs. (1), (18), (19), and (21), we construct the transport equation to $\mathrm{O}(u/c)$:

$$
\begin{gathered}
\frac{1}{c}\frac{\partial I}{\partial t} + \overrightarrow{\Omega}\cdot\overrightarrow{\nabla} I + \sigma_t I = \frac{\sigma_s}{4\pi}\varphi + \sigma_a B + \\
\left[\left(\sigma_t + E\frac{\partial\sigma_a}{\partial E}\right)I + \frac{\sigma_s}{4\pi}\left(2\varphi - E\frac{\partial\varphi}{\partial E}\right) + 2\sigma_a B - B\,E\frac{\partial\sigma_a}{\partial E} - \sigma_a\,E\frac{\partial B}{\partial E}\right]\overrightarrow{\Omega}\cdot\overrightarrow{u}/c - \\
\frac{\sigma_s}{4\pi}\left(\overrightarrow{F} - E\frac{\partial\overrightarrow{F}}{\partial E}\right)\cdot\overrightarrow{u}/c\,. \qquad (22)
\end{gathered}
$$

To obtain the radiation energy equation to $\mathrm{O}(u/c)$, we integrate Eq. (22) over all directions and energies:

$$
\frac{\partial\mathcal{E}}{\partial t} + \overrightarrow{\nabla}\cdot\overrightarrow{\mathcal{F}} = \int_0^\infty \sigma_a\left(4\pi B - \varphi\right)\,dE + \int_0^\infty\left(\sigma_a + E\frac{\partial\sigma_a}{\partial E} - \sigma_s\right)\overrightarrow{F}\cdot\overrightarrow{u}/c\,dE\,. \qquad (23)
$$

To obtain the radiation momentum equation to $\mathrm{O}(u/c)$, we first multiply Eq. (22) by $\overrightarrow{\Omega}/c$ and then integrate it over all directions and energies:

$$
\begin{gathered}
\frac{1}{c^2}\frac{\partial\overrightarrow{\mathcal{F}}}{\partial t} + \overrightarrow{\nabla}\cdot\overrightarrow{\overrightarrow{\mathcal{P}}} = -\int_0^\infty\frac{1}{c}\sigma_t\overrightarrow{F}\,dE + \\
\int_0^\infty\left(\sigma_s\varphi + \sigma_a 4\pi B\right)\overrightarrow{u}/c^2\;dE + \int_0^\infty\left(\sigma_a + E\frac{\partial\sigma_a}{\partial E} + \sigma_s\right)\overrightarrow{\overrightarrow{P}}\cdot\overrightarrow{u}/c\;dE\,, \qquad (24)
\end{gathered}
$$

where $\overrightarrow{\overrightarrow{P}}$ is the energy-dependent radiation pressure:

$$P_{i,j} = \frac{1}{c} \int_{4\pi} \Omega_i \Omega_j \, I(\overrightarrow{\Omega}', E) \, d\Omega' \, . \tag{25}$$