TECHNICAL REPORT:
Nuclear Fission Reactor Point Kinetics in JIC^{Lib2}

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Rev 1

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1 Derivation of the Point Kinetics Equations

A very simple approximation of the kinetics of a reactor can be made by integrating the neutron transport equation over the entire core volume. This results in the reduced transport equation as shown below:

$$\begin{aligned} \frac{1}{v}\frac{d\phi}{dt} + \nabla\vec{J} + \Sigma_a\phi(t) &= (1-\beta_{\text{eff}})\bar{\nu}\Sigma_f\phi(t) + \sum_{i=1}^{6}\lambda_i C_i(t) \\ \frac{dC_i}{dt} &= \beta_{i,\text{eff}}\bar{\nu}\Sigma_f\phi(t) - \lambda_i C_i(t) \quad i = 1,\ldots,6 \end{aligned} \tag{1}$$

Where:

ν	, Neutron velocity [m.s^{-1}]
ϕ	, Average neutron flux [cm^{-2}.s^{-1}]
$\vec{J}$	, Neutron current [cm^{-2}.s^{-1}]
Σ_a	, Effective macroscopic absorption cross-section [cm^{-1}]
β_{eff}	, Effective delayed neutron fraction
$\bar{\nu}$	, Average amount of neutrons released per fission
Σ_f	, Effective macroscopic fission cross-section [cm^{-1}]
λ_i	, Decay constant for delayed neutron precursor i [s^{-1}]
C_i	, Average concentration of delayed neutron precursor i [cm^{-3}]
$\beta_{i,\text{eff}}$	, Effective delayed neutron fraction of precursor i

In order to simplify equation 1 we need to define three additional parameters, the multiplication factor, k_{eff}, the prompt neutron lifetime, l_p, and reactivity, ρ.

1.1 The effective multiplication factor, k_{eff}

To get some context on where we will use this parameter we first lump all the components of criticality together as follows:

$$\begin{aligned} \frac{1}{v}\frac{d\phi}{dt} &= (1-\beta_{\text{eff}})\bar{\nu}\Sigma_f\phi(t) - \nabla\vec{J} - \Sigma_a\phi(t) + \sum_{i=1}^{6}\lambda_i C_i(t) \\ &= \bar{\nu}\Sigma_f\phi(t)\left((1-\beta_{\text{eff}}) - \left(\frac{\nabla\vec{J} + \Sigma_a\phi(t)}{\bar{\nu}\Sigma_f\phi(t)}\right)\right) + \sum_{i=1}^{6}\lambda_i C_i(t) \end{aligned} \tag{2}$$

We then define the effective multiplication factor as the ratio of sources to losses (i.e. $\frac{\text{born}}{\text{lost}}$):

$$k_{\text{eff}} = \frac{(1-\beta_{\text{eff}})\bar{\nu}\Sigma_f\phi(t) + \sum_{i=1}^{6}\lambda_i C_i(t)}{\nabla\vec{J} + \Sigma_a\phi(t)} \tag{3}$$

The next parameter, the prompt neutron lifetime, l_p, is a parameter that is both defined and measured at steady state. Therefore, at steady state, all the $\frac{dC_i}{dt}$ terms in equation 1 reduce to zero and we can deduce:

$$\lambda_i C_i(t) = \beta_{i,\text{eff}} \bar{\nu} \Sigma_f \phi(t)$$

Therefore:

$$\begin{aligned}
\sum_{i=1}^{6} \lambda_i C_i(t) &= \sum_{i=1}^{6} \beta_{i,\text{eff}} \bar{\nu} \Sigma_f \phi(t) \\
&= \bar{\nu} \Sigma_f \phi(t) \left(\sum_{i=1}^{6} \beta_{i,\text{eff}} \right) \\
\sum_{i=1}^{6} \lambda_i C_i(t) &= \beta_{\text{eff}} \bar{\nu} \Sigma_f \phi(t)
\end{aligned}$$

Equation 3 then becomes:

$$\begin{aligned}
k_{\text{eff}} &= \frac{(1 - \beta_{\text{eff}}) \bar{\nu} \Sigma_f \phi(t) + \beta_{\text{eff}} \bar{\nu} \Sigma_f \phi(t)}{\nabla \vec{J} + \Sigma_a \phi(t)} \\
\therefore k_{\text{eff}} &= \frac{\bar{\nu} \Sigma_f \phi(t)}{\nabla \vec{J} + \Sigma_a \phi(t)}
\end{aligned}$$

Substituting this value of k_{eff} into equation 2, one obtains:

$$\frac{1}{v} \frac{d\phi}{dt} = \bar{\nu} \Sigma_f \phi(t) \left((1 - \beta_{\text{eff}}) - \frac{1}{k_{\text{eff}}} \right) + \sum_{i=1}^{6} \lambda_i C_i(t) \tag{4}$$

Next we will remove the $\bar{\nu} \Sigma_f$ term by defining the prompt neutron lifetime, l_p.

1.2 The prompt neutron lifetime, l_p

The average time from the birth of a prompt neutron to the time it is absorbed or lost is termed the prompt neutron lifetime. In order to derive a more intuitive definition, let us first define the rate of neutron loss (either through absorption or leaking):

$$\text{rate of loss} = \text{leakage rate} + \text{absorption rate}$$

This, however, is familiar:

$$\text{rate of loss} = \nabla \vec{J} + \Sigma_a \phi(t)$$

It features in the equation for k_{eff} above and therefore we can write:

$$\text{rate of loss} = \frac{\bar{\nu} \Sigma_f \phi(t)}{\nabla \vec{J} + \Sigma_a \phi(t)} \cdot \left(\frac{1}{\bar{\nu} \Sigma_f \phi(t)} \right) = k_{\text{eff}} \cdot \left(\frac{1}{\bar{\nu} \Sigma_f \phi(t)} \right)$$

The prompt neutron lifetime is then defined as the time it would take to lose a certain neutron population $n(t)$. And this is defined as the total amount lost ($n(t)$) multiplied by the rate of loss:

$$\begin{aligned} l_p &= n(t) \times \text{rate of loss} \\ &= n(t) \times k_{\text{eff}} \cdot \left(\frac{1}{\bar{\nu}\Sigma_f \phi(t)} \right) \end{aligned}$$

Including the definition of flux, $\phi(t) = n(t).v$, we obtain the final form of the prompt neutron lifetime:

$$\begin{aligned} l_p &= n(t) \times k_{\text{eff}} \cdot \left(\frac{1}{\bar{\nu}\Sigma_f n(t) v} \right) \\ &= k_{\text{eff}} \cdot \left(\frac{1}{v.\bar{\nu}\Sigma_f} \right) \\ \therefore l_p &= \frac{k_{\text{eff}}}{v.\bar{\nu}\Sigma_f} \end{aligned}$$

Now extraction from this equation an expression for $\bar{\nu}\Sigma_f$ we get:

$$\bar{\nu}\Sigma_f = \frac{k_{\text{eff}}}{v.l_p}$$

We can then substitute this into equation 4 to obtain:

$$\begin{aligned} \frac{1}{v}\frac{d\phi}{dt} &= \frac{k_{\text{eff}}}{v.l_p}\phi(t)\left((1 - \beta_{\text{eff}}) - \frac{1}{k_{\text{eff}}} \right) + \sum_{i=1}^{6} \lambda_i C_i(t) \\ &= \frac{\phi(t)}{v.l_p}\left(k_{\text{eff}}(1 - \beta_{\text{eff}}) - 1 \right) + \sum_{i=1}^{6} \lambda_i C_i(t) \end{aligned} \tag{5}$$

1.3 Reactivity ρ

In practical reactor facilities the value of k_{eff} is never determined in the classical sense because it involves numbers that are hard to comprehend (i.e. 1.001 is controllable whereas 1.01 is a massive excursion). In order to center the concept of criticality around the value of zero (i.e. positive means power increases and negative means power decreases). For this, the term "reactivity", ρ, was defined as:

$$\rho = \frac{k_{\text{eff}} - 1}{k_{\text{eff}}}$$

However, the nature of this number was still unsatisfactory since it depended on the kinetic parameter β_{eff}. When normalizing by this factor, reactivity, ρ', with units of dollars (\$) becomes analogous between reactor shapes and sizes and is defined by:

$$\rho' = \frac{1}{\beta_{\text{eff}}} \cdot \frac{k_{\text{eff}} - 1}{k_{\text{eff}}} \tag{6}$$

Solving for k_{eff} from this equation we get:

$$\begin{aligned}
\rho' &= \frac{1}{\beta_{\text{eff}}} \cdot (1 - 1/k_{\text{eff}}) \\
\rho'.\beta_{\text{eff}} &= 1 - 1/k_{\text{eff}} \\
1/k_{\text{eff}} &= 1 - \rho'.\beta_{\text{eff}} \\
k_{\text{eff}} &= \frac{1}{1 - \rho'.\beta_{\text{eff}}}
\end{aligned}$$

Substituting this into equation 5 we get:

$$\begin{aligned}
\frac{1}{v}\frac{d\phi}{dt} &= \frac{\phi(t)}{v.l_p}\left(\frac{1}{1-\rho'.\beta_{\text{eff}}}(1-\beta_{\text{eff}}) - 1\right) + \sum_{i=1}^{6} \lambda_i C_i(t) \\
&= \frac{\beta_{\text{eff}}(\rho' - 1)}{(1 - \rho'\beta_{\text{eff}})} \cdot \frac{\phi(t)}{v.l_p} + \sum_{i=1}^{6} \lambda_i C_i(t)
\end{aligned}$$

1.4 Final simplified form

As a final modification to equation 1 we need to transform the differential equations for the delayed neutron precursors:

$$\frac{dC_i}{dt} = \beta_{i,\text{eff}} \bar{\nu} \Sigma_f \phi(t) - \lambda_i C_i(t) \quad i = 1, \ldots, 6$$

We do this by first substituting the derived expression for $\bar{\nu}\Sigma_f$:

$$\frac{dC_i}{dt} = \beta_{i,\text{eff}} \frac{k_{\text{eff}}}{v.l_p} \phi(t) - \lambda_i C_i(t) \quad i = 1, \ldots, 6$$

And then the expression for k_{eff}:

$$\frac{dC_i}{dt} = \beta_{i,\text{eff}} \frac{\frac{1}{1-\rho'.\beta_{\text{eff}}}}{v.l_p} \phi(t) - \lambda_i C_i(t) \quad i = 1, \ldots, 6$$

To finally get:

$$\frac{dC_i}{dt} = \frac{\beta_{i,\text{eff}}}{(1 - \rho'.\beta_{\text{eff}})} \cdot \frac{\phi(t)}{v.l_p} - \lambda_i C_i(t) \quad i = 1, \ldots, 6$$

Therefore the final simplified form of the point kinetics equations, after substituting the expression for flux ($\phi(t) = n(t).v$) become:

$$\begin{aligned}
\frac{dn}{dt} &= \frac{\beta_{\text{eff}}(\rho' - 1)}{(1 - \rho'\beta_{\text{eff}})} \cdot \frac{n(t)}{l_p} + \sum_{i=1}^{6} \lambda_i C_i(t) \\
\frac{dC_i}{dt} &= \frac{\beta_{i,\text{eff}}}{(1 - \rho'.\beta_{\text{eff}})} \cdot \frac{n(t)}{l_p} - \lambda_i C_i(t) \quad i = 1, \ldots, 6
\end{aligned} \tag{7}$$

2 Discretized forms of the Point Kinetics Equations

For any method of discretization, the time derivatives need to be linearized:

$$\frac{dn}{dt} = \frac{n^{j+1} - n^j}{\Delta t}$$
$$\frac{dC_i}{dt} = \frac{C_i^{j+1} - C_i^j}{\Delta t}$$

Where j and $j+1$ refers to time steps in the solution domain.

2.1 Initial first-order discretization

As an initial approach, let us consider calculating the future time step from strictly using values from the previous time step. This is known as Euler's Method for numerically solving ordinary differential equations:

$$\frac{n^{j+1} - n^j}{\Delta t} = \frac{\beta_{\text{eff}}(\rho' - 1)}{(1 - \rho'\beta_{\text{eff}})} \cdot \frac{n^j}{l_p} + \sum_{i=1}^{6} \lambda_i C_i^j$$
$$\frac{C_i^{j+1} - C_i^j}{\Delta t} = \frac{\beta_{i,\text{eff}}}{(1 - \rho'.\beta_{\text{eff}})} \cdot \frac{n^j}{l_p} - \lambda_i C_i^j \quad i = 1, \ldots, 6$$

In this form, each of the 7 equations can be calculated individually and the implementation in computer code becomes fairly easy:

$$\begin{aligned} n^{j+1} &= n^j + \Delta t\left(\frac{\beta_{\text{eff}}(\rho' - 1)}{(1 - \rho'\beta_{\text{eff}})} \cdot \frac{n^j}{l_p} + \sum_{i=1}^{6} \lambda_i C_i^j\right) \\ C_i^{j+1} &= C_i^j + \Delta t\left(\frac{\beta_{i,\text{eff}}}{(1 - \rho'.\beta_{\text{eff}})} \cdot \frac{n^j}{l_p} - \lambda_i C_i^j\right) \quad i = 1, \ldots, 6 \end{aligned} \tag{8}$$

The detriment of this method however is that it is fairly unstable and induces large errors when the time step is not extremely small.

2.2 Second-order Runge-Kutta method

As an improvement to the first-order Euler method let us consider writing equation 7 as:

$$n'(t) = f(n, C_1, C_2, C_3, C_4, C_5, C_6)$$
$$C_i' = f(n, C_i) \quad i = 1, \ldots, 6$$

For simplicity, let us write the equations in a more familiar format:

$$
\begin{aligned}
x_1' &= a_{11}x_1 + \cdots + a_{16}x_6 \\
x_2' &= a_{21}x_1 + \cdots + a_{26}x_6 \\
&\vdots \\
x_7' &= a_{71}x_1 + \cdots + a_{76}x_6
\end{aligned}
$$