TECHNICAL REPORT:

THERMOFLOW* - System level Thermal-Hydraulics in *ChiTech

August, 2016

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Rev 1.2

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1 Conservation equations

The overall objective is that we want to solve the following four field variables:

- Pressure, P
- Internal energy, u
- Velocity, v
- Density, ρ

In order to do this we apply the conservation equations to the volume shown in Figure 1 below:

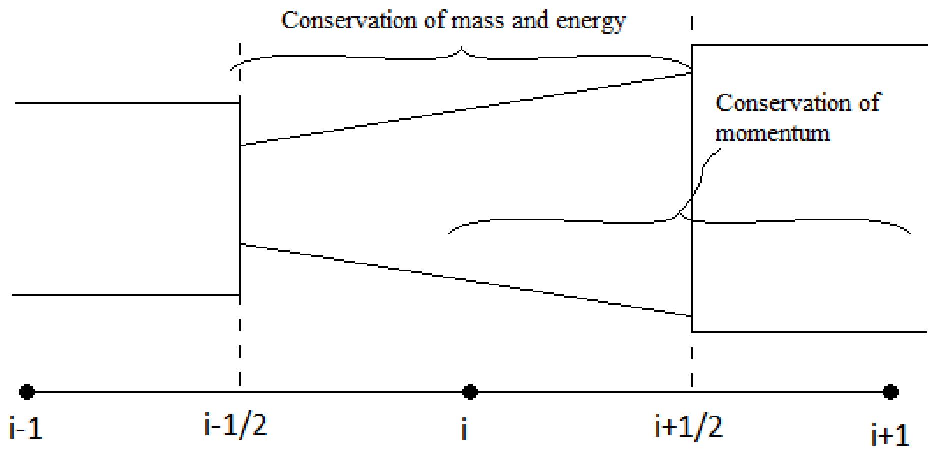

Figure 1: Simple layout of control volumes.

1.1 Conservation of Mass

Even though we will be dealing with incompressible liquids, the liquids can exist at different temperatures and therefore different densities, ρ. The conservation of mass requires as a function of time, t:

$$\frac{d\rho}{dt} = \frac{1}{V}\int_S \rho.(\vec{v}\cdot\hat{n}).dA$$

And by applying it to the control volume:

$$\frac{d\rho_i}{dt} = \frac{1}{V_i}\left[(\rho.A.v)_{i-\frac{1}{2}} - (\rho.A.v)_{i+\frac{1}{2}}\right] \tag{1}$$

Where:
v = velocity.
V = Volume.
A = Area.
$\hat{n}$ = Surface normal.

1.2 Conservation of Momentum

The change in momentum, $m_i.v_i$, in a control volume must balance the momentum of all the in and out flows as well as any forces applied:

$$\frac{d(mv)_i}{dt} = \int_S \rho.|v|.((\vec{v})\cdot\hat{n}).dA + \int_S P.\hat{n}.dA + mg - F_{friction}$$

For a control volume as shown in Figure 1 we have:

$$\begin{aligned}\frac{d(\rho v)_i}{dt} = &\frac{1}{V_i}\left[(\rho A v^2 d)_{i-\frac{1}{2}} + (\rho A v^2 d)_{i+\frac{1}{2}}\right]\\ &+\frac{1}{V_i}\left[(PA)_{i-\frac{1}{2}} - (PA)_{i+\frac{1}{2}}\right] + \rho_i g_i - \frac{1}{V_i}\Delta P_{\tau,i}A_i\end{aligned}$$

Where d is the direction value (either $+1$ or -1), F_τ is the wall friction force and g_i is the gravitational component (i.e. $g_i = g.sin\theta_i$, $g = -9.81 m.s^{-2}$). In later derivations of the numerical representation it is more convenient to define the conservation of momentum equation about a boundary by:

$$\frac{d(\rho v)}{dt} = -\rho\frac{d(v^2)}{dx} - \frac{dP}{dx} + \rho g - \frac{\tau}{L}$$

Applied to the boundary junction, $i+\frac{1}{2}$, we can write:

$$
\begin{aligned}
\frac{d(\rho v)_{i+\frac{1}{2}}}{dt} =& \frac{\rho_{i+\frac{1}{2}}}{2} \cdot \frac{d_i.(v_i)^2 - d_{i+1}.(v_{i+1})^2}{\Delta x_{i+\frac{1}{2}}} \\
&- \frac{P_{i+1}-P_i}{\Delta x_{i+\frac{1}{2}}} + \rho_{i+\frac{1}{2}}.g_{i+\frac{1}{2}} - \frac{\Delta P_{\tau,i} A_i}{2V_i} - \frac{\Delta P_{\tau,i+1} A_{i+1}}{2V_{i+1}} \\
&- \frac{\Delta P_{loss,i+\frac{1}{2}} A_{i+\frac{1}{2}}}{V_{i+1}}
\end{aligned}
\tag{2}
$$

Here $\Delta x_{i+\frac{1}{2}}$ is the distance between control volumes i and $i+1$, $g_{i+\frac{1}{2}}$ is the gravitational force component (function of inclination angle between control volume centroids). The wall friction force can be calculated from the Darcy Friction Factor, f:

$$\Delta P_\tau = \frac{1}{2} f \rho \frac{L}{D_H} v^2$$

Therefore:

$$
\begin{aligned}
\Delta P_{\tau,i} &= \frac{1}{4} f_i \rho_i \frac{L_i}{D_{H,i}} v_i^2 \\
\Delta P_{\tau,i+1} &= \frac{1}{4} f_{i+1} \rho_{i+1} \frac{L_{i+1}}{D_{H,i+1}} v_{i+1}^2
\end{aligned}
$$

In these equations L_i and $D_{H,i}$ are the length and hydraulic diameter of the i-th control volume. The Darcy friction factor is evaluated as shown below. The turbulent regime friction factor (i.e. Re>3000) is calculated from the Zigrang and Sylvester correlation.

$$
f = \begin{cases}
g(Re) = \frac{64}{Re} & , Re \leq 2200 \\
h(Re) = \frac{1}{\sqrt{f}} = -2\log\left(\frac{\epsilon}{3.7D_H} + \frac{2.51}{Re\sqrt{f}}\right) & , Re > 3000 \\
g(2200) + \frac{h(3000)-g(2200)}{800}.(Re-2200) & , 2200 < Re \leq 3000
\end{cases}
$$

The junction pressure loss $\Delta P_{loss,i+\frac{1}{2}}$ is calculated from a loss coefficient K as follows:

$$\Delta P_{loss,i+\frac{1}{2}} = \frac{1}{2} K_{i+\frac{1}{2}} \rho_{i+\frac{1}{2}} d_{i+\frac{1}{2}} v_{i+\frac{1}{2}}^2$$

1.3 Conservation of Energy

The total energy of the system, E, must balance that of the in and out flow including the work performed and the heat transfer into the system:

$$\frac{dE}{dt} = \int_S \rho.(\vec{v} \cdot \hat{n}).e.dA + Q - W + E_{loss}$$

Where:
e = Specific energy of the system.
Q = Heat transfer into the system.
W = Work leaving the system.
E_{loss} = Dissipative energy losses (includes irreversible conversion of kinetic energy to internal energy).

The components of energy are internal energy, U, kinetic energy, $\frac{1}{2}mv^2$, and potential energy, mgz:

$$E = m.e = m.(u + \frac{1}{2}v^2 + gz)$$

For most control volume fluid flows where the control volume flow rate is small compared to its area it is often acceptable to neglect the kinetic energy term. Also, since it undergoes no change in elevation we can neglect the elevation change (but only for the control volume, not the junctions).

The heat transfer into the system is normally associated with some heat flux, $\dot{q}$, and the total heat transfer surface, A_s, therefore:

$$Q_i = \dot{q}_i.A_{s,i}$$

The components of work include shaft work, W_{shaft}, and pressure work, $W_{pressure}$. For this case we will consider only pressure work:

$$W_{pressure} = (P.A.v)_{i+\frac{1}{2}} - (P.A.v)_{i-\frac{1}{2}}$$

From here the energy conservation equation becomes:

$$\begin{aligned}\frac{d}{dt}\bigg(m.u\bigg) &= (\rho A v)_{i-\frac{1}{2}}.(u+gz)_{i-\frac{1}{2}} - (\rho A v)_{i+\frac{1}{2}}.(u+gz)_{i+\frac{1}{2}} \\ &+ \dot{q}_i.A_{s,i} + \bigg[(P.A.v)_{i-\frac{1}{2}} - (P.A.v)_{i+\frac{1}{2}}\bigg] + E_{loss}\end{aligned}$$

Dividing by the volume we get:

$$\begin{aligned}\frac{d(\rho.u)_i}{dt} &= \frac{1}{V_i}\left[(\rho Av)_{i-\frac{1}{2}}.(u+gz)_{i-\frac{1}{2}} - (\rho Av)_{i+\frac{1}{2}}.(u+gz)_{i+\frac{1}{2}}\right] \\ &+ \frac{A_{s,i}}{V_i}\dot{q}_i + \frac{1}{V_i}\left[(P.A.v)_{i-\frac{1}{2}} - (P.A.v)_{i+\frac{1}{2}}\right] + \frac{E_{loss}}{V_i}\end{aligned} \tag{3}$$

1.4 Equation of state

In addition to the conservation equations, there is the equation of state. ChiTech has the ability to use the IAPWS-95 steam tables which is captured within a function with the main entry specified as pressure [bar]. Therefore the property Y, which can be any thermal property (i.e. temperature, density, internal energy, enthalpy, entropy, heat capacity, dynamic viscosity, Prandtl number), can be determined from the equation below given X (a known quantity in the subset listed for Y):

$$Y_i = f(P_i, X_i) \tag{4}$$

2 Numerical solution

The conservation of momentum equations conveniently couples the pressure field and velocities and therefore is used implicitly to determine the pressures at time $n+1$.

2.1 Conservation of Mass - finite difference formulation

The finite difference formulation is as follows:

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{1}{V_i}\left[(\rho.A)^n_{i-\frac{1}{2}} v^{n+1}_{i-\frac{1}{2}} - (\rho.A)^n_{i+\frac{1}{2}} v^{n+1}_{i+\frac{1}{2}}\right] \tag{5}$$

2.2 Conservation of Momentum - finite difference formulation

Because no other future time equation has pressure as a variable we opt to include pressure at time $n+1$ as an implicit variable. We manipulate the original conservation of mass equation for junction $i+\frac{1}{2}$:

$$\begin{aligned}\frac{d(\rho v)_{i+\frac{1}{2}}}{dt} = &- \frac{\rho_{i+\frac{1}{2}}}{2} \cdot \frac{d_i.(v_i)^2 + d_{i+1}.(v_{i+1})^2}{\Delta x_{i+\frac{1}{2}}} \\ &- \frac{P_{i+1} - P_i}{\Delta x_{i+\frac{1}{2}}} + \rho_{i+\frac{1}{2}}.g_{i+\frac{1}{2}} - \frac{\Delta P_{\tau,i} A_i}{2V_i} - \frac{\Delta P_{\tau,i+1} A_{i+1}}{2V_{i+1}} \\ &- \frac{\Delta P_{loss,i+\frac{1}{2}} A_{i+\frac{1}{2}}}{V_{i+1}}\end{aligned}$$

We use the following time notation:

$$\begin{aligned}\frac{\rho^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}} - \rho^n_{i+\frac{1}{2}}.v^n_{i+\frac{1}{2}}}{\Delta t} = &\rho^n_{i+\frac{1}{2}} \cdot \frac{d_i.(v_i^n)^2 - d_{i+1}.(v^n_{i+1})^2}{\Delta x_{i+\frac{1}{2}}} \\ &- \frac{P^{n+1}_{i+1} - P^{n+1}_i}{\Delta x_{i+\frac{1}{2}}} + \rho^n_{i+\frac{1}{2}}.g_{i+\frac{1}{2}} \\ &- \frac{\Delta P^n_{\tau,i}}{2\Delta x_{i+\frac{1}{2}}} - \frac{\Delta P^n_{\tau,i+1}}{2\Delta x_{i+\frac{1}{2}}} - \frac{\Delta P^n_{loss,i+\frac{1}{2}}}{\Delta x_{i+\frac{1}{2}}}\end{aligned} \tag{6}$$

In the equations above we included implicit time instances to pressure in order to semi-implicitly couple the system.

We can find the same equation for junction $i-\frac{1}{2}$:

$$\begin{aligned}\frac{\rho^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} - \rho^n_{i-\frac{1}{2}}.v^n_{i-\frac{1}{2}}}{\Delta t} &= \rho^n_{i-\frac{1}{2}} \cdot \frac{d_{i-1}.(v^n_{i-1})^2 + d_i.(v^n_i)^2}{\Delta x_{i-\frac{1}{2}}} \\ &- \frac{P^{n+1}_i - P^{n+1}_{i-1}}{\Delta x_{i-\frac{1}{2}}} + \rho^n_{i-\frac{1}{2}}.g_{i-\frac{1}{2}} \\ &- \frac{\Delta P^n_{\tau,i-1}}{2\Delta x_{i-\frac{1}{2}}} - \frac{\Delta P^n_{\tau,i}}{2\Delta x_{i-\frac{1}{2}}} - \frac{\Delta P^n_{loss,i-\frac{1}{2}}}{\Delta x_{i-\frac{1}{2}}}\end{aligned} \tag{7}$$

2.3 Conservation of Energy - finite difference formulation

We start with the original conservation of energy equation:

$$\begin{aligned}\frac{d(\rho.u)_i}{dt} = \frac{1}{V_i}\Big[(\rho A v)_{i-\frac{1}{2}}.(u+gz)_{i-\frac{1}{2}} - (\rho A v)_{i+\frac{1}{2}}.(u+gz)_{i+\frac{1}{2}}\Big] \\ + \frac{A_{s,i}}{V_i}\dot{q}_i + \frac{1}{V_i}\Big[(P.A.v)_{i-\frac{1}{2}} - (P.A.v)_{i+\frac{1}{2}}\Big] - \frac{E_{loss,i}}{V_i}\end{aligned}$$

We now discretize the left side and set the right side to correspond to time n.

$$\begin{aligned}\frac{(\rho^{n+1}_i.u^{n+1}_i - \rho^n_i.u^n_i)}{\Delta t} &= \frac{1}{V_i}\Big[(\rho A u)^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} - (\rho A u)^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}}\Big] \\ &+ \frac{1}{V_i}\Big[(\rho A g z)^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} - (\rho A g z)^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}}\Big] \\ &+ \frac{A_{s,i}}{V_i}\dot{q}^n_i + \frac{1}{V_i}\Big[(P.A)^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} - (P.A)^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}}\Big] \\ &+ \frac{E^n_{loss,i}}{V_i}\end{aligned} \tag{8}$$

2.4 Equation of state numerical implementation

In order to implement a linear approximation to the equation of state we linearize the Equation of State by determine the derivative at the old time values. Specifically this applies to the relation of internal energy, u, to the density, ρ:

$$\frac{d(\rho_i^n u_i^n)}{d\rho} = \frac{\rho_{EOS}(P_i^n, T_i^n + \Delta T) u_{EOS}(P_i^n, T_i^n + \Delta T) - \rho_{EOS}(P_i^n, T_i^n - \Delta T) u_{EOS}(P_i^n, T_i^n - \Delta T)}{\rho_{EOS}(P_i^n, T_i^n + \Delta T) - \rho_{EOS}(P_i^n, T_i^n - \Delta T)}$$

Where ρ_{EOS} and u_{EOS} are determined from interpolating the steam tables. Also, ΔT is an arbitrary but small difference in temperature sufficient to locally linearize the equation of state. From this derivative we find the linearized equation of state:

$$\rho_i^{n+1} u_i^{n+1} - \rho_i^n u_i^n = \frac{d(\rho_i^n u_i^n)}{d\rho} \cdot (\rho_i^{n+1} - \rho_i^n) \tag{9}$$

2.5 Semi-implicit approach

From the set of equations 5 to 9 we can develop a system of unknowns. Before we depict how this is done, let us repeat these equations all together:

$$
\begin{aligned}
eq.5,\quad & \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} && = \frac{1}{V_i}\left[(\rho.A)_{i-\frac{1}{2}}^n v_{i-\frac{1}{2}}^{n+1} - (\rho.A)_{i+\frac{1}{2}}^n v_{i+\frac{1}{2}}^{n+1} \right] \\
& && \ldots \\
eq.6,\quad & \frac{\rho_{i+\frac{1}{2}}^n . v_{i+\frac{1}{2}}^{n+1} - \rho_{i+\frac{1}{2}}^n . v_{i+\frac{1}{2}}^n}{\Delta t} && = \rho_{i+\frac{1}{2}}^n \cdot \frac{d_i.(v_i^n)^2 - d_{i+1}.(v_{i+1}^n)^2}{\Delta x_{i+\frac{1}{2}}} \\
& && - \frac{P_{i+1}^{n+1} - P_i^{n+1}}{\Delta x_{i+\frac{1}{2}}} + \rho_{i+\frac{1}{2}}^n . g_{i+\frac{1}{2}} \\
& && - \frac{\Delta P_{\tau,i}^n}{2\Delta x_{i+\frac{1}{2}}} - \frac{\Delta P_{\tau,i+1}^n}{2\Delta x_{i+\frac{1}{2}}} - \frac{\Delta P_{loss,i+\frac{1}{2}}^n}{\Delta x_{i+\frac{1}{2}}} \\
& && \ldots \\
eq.7,\quad & \frac{\rho_{i-\frac{1}{2}}^n . v_{i-\frac{1}{2}}^{n+1} - \rho_{i-\frac{1}{2}}^n . v_{i-\frac{1}{2}}^n}{\Delta t} && = \rho_{i-\frac{1}{2}}^n \cdot \frac{d_{i-1}.(v_{i-1}^n)^2 + d_i.(v_i^n)^2}{\Delta x_{i-\frac{1}{2}}} \\
& && - \frac{P_i^{n+1} - P_{i-1}^{n+1}}{\Delta x_{i-\frac{1}{2}}} + \rho_{i-\frac{1}{2}}^n . g_{i-\frac{1}{2}} \\
& && - \frac{\Delta P_{\tau,i-1}^n}{2\Delta x_{i-\frac{1}{2}}} - \frac{\Delta P_{\tau,i}^n}{2\Delta x_{i-\frac{1}{2}}} - \frac{\Delta P_{loss,i-\frac{1}{2}}^n}{\Delta x_{i-\frac{1}{2}}} \\
& && \ldots \\
eq.8,\quad & \frac{(\rho_i^{n+1} . u_i^{n+1} - \rho_i^n . u_i^n)}{\Delta t} && = \frac{1}{V_i}\left[(\rho A u)_{i-\frac{1}{2}}^n . v_{i-\frac{1}{2}}^{n+1} - (\rho A u)_{i+\frac{1}{2}}^n . v_{i+\frac{1}{2}}^{n+1} \right] \\
& && + \frac{1}{V_i}\left[(\rho A g z)_{i-\frac{1}{2}}^n . v_{i-\frac{1}{2}}^{n+1} - (\rho A g z)_{i+\frac{1}{2}}^n . v_{i+\frac{1}{2}}^{n+1} \right] \\
& && + \frac{A_{s,i}}{V_i} \dot{q}_i^n + \frac{1}{V_i}\left[(P.A)_{i-\frac{1}{2}}^n . v_{i-\frac{1}{2}}^{n+1} - (P.A)_{i+\frac{1}{2}}^n . v_{i+\frac{1}{2}}^{n+1} \right] \\
& && - \frac{E_{loss,i}^n}{V_i} \\
& && \ldots \\
eq.9,\quad & \rho_i^{n+1} u_i^{n+1} - \rho_i^n u_i^n && = \frac{d(\rho_i^n u_i^n)}{d\rho} \cdot (\rho_i^{n+1} - \rho_i^n)
\end{aligned}
$$

The "new" time values that require solving are found in the following terms:

$$\rho_i^{n+1}, v_{i+\frac{1}{2}}^{n+1}, v_{i-\frac{1}{2}}^{n+1},\ from\ equation\ 5$$
$$v_{i+\frac{1}{2}}^{n+1}, P_{i+1}^{n+1}, P_i^{n+1},\ from\ equation\ 6$$
$$v_{i-\frac{1}{2}}^{n+1}, P_{i-1}^{n+1}, P_i^{n+1},\ from\ equation\ 7$$
$$\rho_i^{n+1}, u_i^{n+1}, v_{i-\frac{1}{2}}^{n+1}, v_{i+\frac{1}{2}}^{n+1},\ from\ equation\ 8$$
$$\rho_i^{n+1}, u_i^{n+1},\ from\ equation\ 9$$

There are also the unknown junction quantities $\rho_{i+\frac{1}{2}}^n$ and $u_{i+\frac{1}{2}}^n$, however, these can be determined by a simple upwind scheme as follows:

$$\frac{d\rho_{i+\frac{1}{2}}}{dt} = -v_{i+\frac{1}{2}} \cdot \frac{d\rho}{dx} \qquad and \qquad \frac{du_{i+\frac{1}{2}}}{dt} = -v_{i+\frac{1}{2}} \cdot \frac{du}{dx}$$
$$\therefore \rho_{i+\frac{1}{2}}^n = \rho_{i+\frac{1}{2}}^{n-1} - v_{i+\frac{1}{2}}^n \cdot \frac{\Delta t}{\Delta x_{i+\frac{1}{2}}} \cdot (\rho_{i+1}^n - \rho_i^n)$$
$$u_{i+\frac{1}{2}}^n = u_{i+\frac{1}{2}}^{n-1} - v_{i+\frac{1}{2}}^n \cdot \frac{\Delta t}{\Delta x_{i+\frac{1}{2}}} \cdot (u_{i+1}^n - u_i^n)$$

By using simple mathematical manipulation we can reduce the set of equations (i.e. eq. 5 to 9) to a single equation in terms of P_{i-1}^{n+1}, P_i^{n+1} and P_{i+1}^{n+1}, a process indicated in the solution algorithm. We can then implicitly solve for the control volume pressure by constructing the system $Ax = b$ and solving it with a suitable sparse solver. After this has been done the junction velocities, $v_{i+\frac{1}{2}}^{n+1}$ can be determined from the advancement of the momentum equations (i.e. equation 6) and from those values one can determine ρ_i^{n+1} from the conservation of mass equation and finally u_i^{n+1} from the equation of state interpolation.

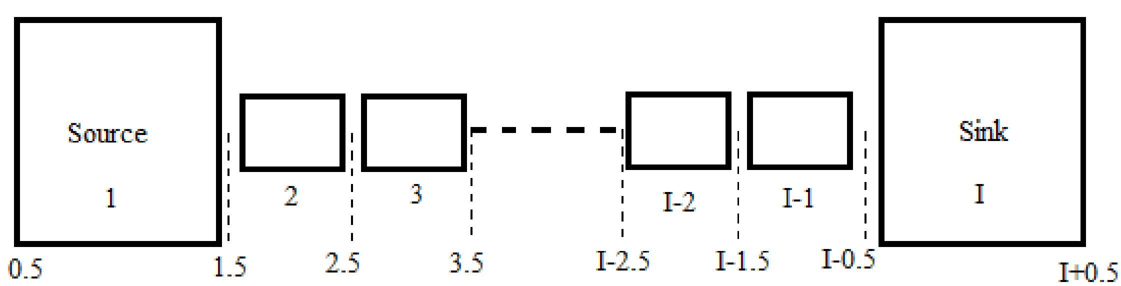

Figure 2: Simple layout of multiple control volumes.

2.6 Solution algorithm

Step 1 - Preliminaries

Load all control volumes. Input parameters:

- Area, A
- Length, L
- Pressure, P
- Temperature, T
- Elevation change, Δz
- Wall roughness, ϵ
- Hydraulic diameter, D_H
- Surface area, A_s

Load all junctions. Input parameters:

- Junction velocity, v
- Area, A
- Loss factor, K

Step 2

Determine control volume connectivity to junctions. This step merely requires that control volumes know what they are connected to.

Step 3

Initialize control volume pressure, P_i^n, and junction velocities, $v_{i+\frac{1}{2}}^n$.

Determine density, ρ_i, from:

$$\rho = \rho_{EOS}(P, T)$$

Internal energy, u_i, from:

$$u = u_{EOS}(P, T)$$

Dynamic viscosity, μ_i, from:

$$\mu = \mu_{EOS}(P,T)$$

Initialize temperature and calculate control volume velocity, v_i, from:

$$v_i = \begin{cases} v_{i+\frac{1}{2}} \frac{A_{i+\frac{1}{2}}}{A_i} & , left\ junction\ missing \\ v_{i-\frac{1}{2}} \frac{A_{i-\frac{1}{2}}}{A_i} & , right\ junction\ missing \\ v_{i-\frac{1}{2}} \frac{A_{i-\frac{1}{2}}}{2A_i} + v_{i+\frac{1}{2}} \frac{A_{i+\frac{1}{2}}}{2A_i} & , both\ junctions\ exist \end{cases}$$

Step 4

Determine all junction and control volume z values using control volume Δz values. Also initialize the mass error and cumulative mass error.

************************ **End of Initialization phase** ************************

******************* **Repeat the steps below for each time step** *************************

Step 5

Determine $\rho^n_{i+\frac{1}{2}}$ as follows:

$$\rho^n_{i+\frac{1}{2}} = \rho^{n-1}_{i+\frac{1}{2}} - v^n_{i+\frac{1}{2}} \cdot \frac{\Delta t}{\Delta x_{i+\frac{1}{2}}} \cdot (\rho^n_{i+1} - \rho^n_i)$$

And similarly for $u_{i+\frac{1}{2}}$ from:

$$u^n_{i+\frac{1}{2}} = u^{n-1}_{i+\frac{1}{2}} - v^n_{i+\frac{1}{2}} \cdot \frac{\Delta t}{\Delta x_{i+\frac{1}{2}}} \cdot (u^n_{i+1} - u^n_i)$$

Step 6

Calculate control volume quantities, v_i, from: Velocity, v_i, from:

$$v_i = \begin{cases} v_{i+\frac{1}{2}} \frac{A_{i+\frac{1}{2}}}{A_i} & , left\ junction\ missing \\ v_{i-\frac{1}{2}} \frac{A_{i-\frac{1}{2}}}{A_i} & , right\ junction\ missing \\ v_{i-\frac{1}{2}} \frac{A_{i-\frac{1}{2}}}{2A_i} + v_{i+\frac{1}{2}} \frac{A_{i+\frac{1}{2}}}{2A_i} & , both\ junctions\ exist \end{cases}$$

Dynamic viscosity, μ_i, from:

$$\mu = \mu_{EOS}(P, T)$$

Reynolds number, Re_i, from:

$$Re_i = \frac{\rho^n_i v^n_n D_{H,i}}{\mu^n_i}$$

Friction pressure drop, ΔP_τ, from:

$$\Delta P_{\tau,i} = \frac{1}{2} f_i \rho_i \frac{L_i}{D_{H,i}} v_i^2$$

With:

$$f = \begin{cases} 6.5 \times 10^6 & , Re = 0.0 \\ g(Re) = \frac{64}{Re} & , Re \leq 2200 \\ h(Re) = \frac{1}{\sqrt{f}} = -2\log\left(\frac{\epsilon}{3.7D_H} + \frac{2.51}{Re\sqrt{f}}\right) & , Re > 3000 \\ g(2200) + \frac{h(3000)-g(2200)}{800}.(Re - 2200) & , 2200 < Re \leq 3000 \end{cases}$$

Junction pressure drop losses, E_{loss}, from:

$$E_{loss,i+\frac{1}{2}} = (\Delta P.A.v)_{i+\frac{1}{2}} = \frac{1}{2}K_{i+\frac{1}{2}}(\rho A v^3)_{i+\frac{1}{2}}$$

Step 7

Determine the elements of equation 5:

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{1}{V_i}\left[(\rho.A)^n_{i-\frac{1}{2}} v^{n+1}_{i-\frac{1}{2}} - (\rho.A)^n_{i+\frac{1}{2}} v^{n+1}_{i+\frac{1}{2}}\right]$$

Where the elements are:

$$B_5 = \frac{\Delta t(\rho.A)^n_{i-\frac{1}{2}}}{V_i}$$
$$C_5 = \frac{\Delta t(\rho.A)^n_{i+\frac{1}{2}}}{V_i}$$

To get:

$$\rho_i^{n+1} = \rho_i^n + B_5.v^{n+1}_{i-\frac{1}{2}} - C_5.v^{n+1}_{i+\frac{1}{2}} \tag{10}$$

Step 8

Determine the elements of equation 6:

$$\frac{\rho^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}} - \rho^n_{i+\frac{1}{2}}.v^n_{i+\frac{1}{2}}}{\Delta t} = \rho^n_{i+\frac{1}{2}} \cdot \frac{d_i.(v^n_i)^2 - d_{i+1}.(v^n_{i+1})^2}{\Delta x_{i+\frac{1}{2}}} - \frac{P^{n+1}_{i+1} - P^{n+1}_i}{\Delta x_{i+\frac{1}{2}}} + \rho^n_{i+\frac{1}{2}}.g_{i+\frac{1}{2}} - \frac{\Delta P^n_{\tau,i}}{2\Delta x_{i+\frac{1}{2}}} - \frac{\Delta P^n_{\tau,i+1}}{2\Delta x_{i+\frac{1}{2}}} - \frac{\Delta P^n_{loss,i+\frac{1}{2}}}{\Delta x_{i+\frac{1}{2}}}$$

Where the elements are:

$$B_6 = \rho^n_{i+\frac{1}{2}}.v^n_{i+\frac{1}{2}}$$

$$C_6 = \Delta t \rho^n_{i+\frac{1}{2}} \cdot \frac{d_i.(v^n_i)^2 - d_{i+1}.(v^n_{i+1})^2}{\Delta x_{i+\frac{1}{2}}}$$

$$D_6 = \frac{\Delta t}{\Delta x_{i+\frac{1}{2}}}, \quad E_6 = \Delta t \rho^n_{i+\frac{1}{2}}.g_{i+\frac{1}{2}}$$

$$F_6 = \frac{\Delta t \Delta P^n_{\tau,i}}{2\Delta x_{i+\frac{1}{2}}}, \quad G_6 = \frac{\Delta t \Delta P^n_{\tau,i+1}}{2\Delta x_{i+\frac{1}{2}}} + \frac{\Delta t \Delta P^n_{loss,i+\frac{1}{2}}}{\Delta x_{i+\frac{1}{2}}}$$

$$H_6 = \frac{B_6 + C_6 + E_6 - F_6 - G_6}{\rho^n_{i+\frac{1}{2}}}, \quad I_6 = \frac{D6}{\rho^n_{i+\frac{1}{2}}}$$

To get:

$$\rho^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}} = B_6 + C_6 - D_6.P^{n+1}_{i+1} + D_6.P^{n+1}_i + E_6 - F_6 - G_6$$

And:

$$v^{n+1}_{i+\frac{1}{2}} = H_6 - I_6.P^{n+1}_{i+1} + I_6.P^{n+1}_i \tag{11}$$

Step 9

Determine the elements of equation 7:

$$\begin{aligned}\frac{\rho^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} - \rho^n_{i-\frac{1}{2}}.v^n_{i-\frac{1}{2}}}{\Delta t} =& \rho^n_{i-\frac{1}{2}} \cdot \frac{d_{i-1}.(v^n_{i-1})^2 - d_i.(v^n_i)^2}{\Delta x_{i-\frac{1}{2}}} \\ &- \frac{P^{n+1}_i - P^{n+1}_{i-1}}{\Delta x_{i-\frac{1}{2}}} + \rho^n_{i-\frac{1}{2}}.g_{i-\frac{1}{2}} \\ &- \frac{\Delta P^n_{\tau,i-1}}{2\Delta x_{i-\frac{1}{2}}} - \frac{\Delta P^n_{\tau,i}}{2\Delta x_{i-\frac{1}{2}}} - \frac{\Delta P^n_{loss,i-\frac{1}{2}}}{\Delta x_{i-\frac{1}{2}}}\end{aligned}$$

Where the elements are:

$$\begin{aligned}B_7 &= \rho^n_{i-\frac{1}{2}}.v^n_{i-\frac{1}{2}} \\ C_7 &= \Delta t \rho^n_{i-\frac{1}{2}} \cdot \frac{d_{i-1}.(v^n_{i-1})^2 - d_i.(v^n_i)^2}{\Delta x_{i-\frac{1}{2}}} \\ D_7 &= \frac{\Delta t}{\Delta x_{i-\frac{1}{2}}}, \quad E_7 = \Delta t \rho^n_{i-\frac{1}{2}}.g_{i-\frac{1}{2}} \\ F_7 &= \frac{\Delta t \Delta P^n_{\tau,i-1}}{2\Delta x_{i-\frac{1}{2}}}, \quad G_7 = \frac{\Delta t \Delta P^n_{\tau,i}}{2\Delta x_{i-\frac{1}{2}}} + \frac{\Delta t \Delta P^n_{loss,i-\frac{1}{2}}}{\Delta x_{i-\frac{1}{2}}} \\ H_7 &= \frac{B_7 + C_7 + E_7 - F_7 - G_7}{\rho^n_{i-\frac{1}{2}}}, \quad I_7 = \frac{D7}{\rho^n_{i-\frac{1}{2}}}\end{aligned}$$

To get:

$$\rho^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} = B_7 + C_7 - D_7.P^{n+1}_i + D_7.P^{n+1}_{i-1} + E_7 - F_7 - G_7$$

And:

$$v^{n+1}_{i-\frac{1}{2}} = H_7 - I_7.P^{n+1}_i + I_7.P^{n+1}_{i-1} \tag{12}$$

Step 10

Determine the elements of equation 8:

$$
\begin{aligned}
\frac{(\rho_i^{n+1}.u_i^{n+1} - \rho_i^n.u_i^n)}{\Delta t} &= \frac{1}{V_i}\left[(\rho A u)^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} - (\rho A u)^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}}\right] \\
&+ \frac{1}{V_i}\left[(\rho A g z)^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} - (\rho A g z)^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}}\right] \\
&+ \frac{A_{s,i}}{V_i}\dot{q}_i^n + \frac{1}{V_i}\left[(P.A)^n_{i-\frac{1}{2}}.v^{n+1}_{i-\frac{1}{2}} - (P.A)^n_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}}\right] \\
&- \frac{E^n_{loss,i}}{V_i}
\end{aligned}
$$

Where the elements are:

$$
\begin{aligned}
C_8 &= \frac{\Delta t(\rho A u)^n_{i-\frac{1}{2}}}{V_i}, \quad D_8 = \frac{\Delta t(\rho A u)^n_{i+\frac{1}{2}}}{V_i} \\
E_8 &= \frac{\Delta t(\rho A g z)^n_{i-\frac{1}{2}}}{V_i}, \quad F_8 = \frac{\Delta t(\rho A g z)^n_{i+\frac{1}{2}}}{V_i} \\
G_8 &= \frac{\Delta t A_{s,i}}{V_i}\dot{q}_i^n \\
H_8 &= \frac{\Delta t(P.A)^n_{i-\frac{1}{2}}}{V_i}, \quad I_8 = \frac{\Delta t(P.A)^n_{i+\frac{1}{2}}}{V_i} \\
J_8 &= \frac{\Delta t E^n_{loss,i}}{V_i} \\
K_8 &= G_8 - J_8 \\
L_8 &= C_8 + E_8 + H_8 \\
M_8 &= D_8 + F_8 + I_8
\end{aligned}
$$

To get:

$$
\rho_i^{n+1}.u_i^{n+1} - \rho_i^n.u_i^n = C_8.v^{n+1}_{i-\frac{1}{2}} - D_8.v^{n+1}_{i+\frac{1}{2}} + E_8.v^{n+1}_{i-\frac{1}{2}} - F_8.v^{n+1}_{i+\frac{1}{2}} + G_8 + H_8.v^{n+1}_{i-\frac{1}{2}} - I_8.v^{n+1}_{i+\frac{1}{2}} - J_8
$$

And:

$$
\rho_i^{n+1}.u_i^{n+1} - \rho_i^n.u_i^n = K_8 + L_8.v^{n+1}_{i-\frac{1}{2}} - M_8.v^{n+1}_{i+\frac{1}{2}} \tag{13}
$$

Step 11

Determine the elements of equation 9:

$$\rho_i^{n+1} u_i^{n+1} - \rho_i^n u_i^n = \frac{d(\rho_i^n u_i^n)}{d\rho} \cdot (\rho_i^{n+1} - \rho_i^n)$$

Where the elements are:

$$B_9 = \frac{d(\rho_i^n u_i^n)}{d\rho} = \frac{\rho_{EOS}(P_i^n, T_i^n + \Delta T) u_{EOS}(P_i^n, T_i^n + \Delta T) - \rho_{EOS}(P_i^n, T_i^n - \Delta T) u_{EOS}(P_i^n, T_i^n - \Delta T)}{\rho_{EOS}(P_i^n, T_i^n + \Delta T) - \rho_{EOS}(P_i^n, T_i^n - \Delta T)}$$

$$C_9 = B_9 . \rho_i^n$$

To get:

$$\rho_i^{n+1} u_i^{n+1} - \rho_i^n u_i^n = B_9 . \rho_i^{n+1} - C_9 \tag{14}$$

Step 12

Reduce the given equations to a single equation. This is possible because we have the following set of equations:

$$\begin{aligned}
&eq\ 10, \quad \rho_i^{n+1} = \rho_i^n + B_5.v_{i-\frac{1}{2}}^{n+1} - C_5.v_{i+\frac{1}{2}}^{n+1} \\
&eq\ 11, \quad v_{i+\frac{1}{2}}^{n+1} = H_6 - I_6.P_{i+1}^{n+1} + I_6.P_i^{n+1} \\
&eq\ 12, \quad v_{i-\frac{1}{2}}^{n+1} = H_7 - I_7.P_i^{n+1} + I_7.P_{i-1}^{n+1} \\
&eq\ 13, \quad \rho_i^{n+1}u_i^{n+1} - \rho_i^n.u_i^n = K_8 + L_8.v_{i-\frac{1}{2}}^{n+1} - M_8.v_{i+\frac{1}{2}}^{n+1} \\
&eq\ 14, \quad \rho_i^{n+1}u_i^{n+1} - \rho_i^n u_i^n = B_9.\rho_i^{n+1} - C_9
\end{aligned}$$

Now the reduction goal is find the unknowns in equation 13 by locally solving the set of equations until only the pressures remain, i.e. P_{i-1}^{n+1}, P_i^{n+1} and P_{i+1}^{n+1}.

We start by inserting equation 11 and 12 into 10:

$$\begin{aligned}
\rho_i^{n+1} &= \rho_i^n + B_5.\Big(H_7 - I_7.P_i^{n+1} + I_7.P_{i-1}^{n+1}\Big) - C_5.\Big(H_6 - I_6.P_{i+1}^{n+1} + I_6.P_i^{n+1}\Big) \\
&= \rho_i^n + B_5H_7 - B_5I_7.P_i^{n+1} + B_5I_7.P_{i-1}^{n+1} - C_5H_6 + C_5I_6.P_{i+1}^{n+1} - C_5I_6.P_i^{n+1} \\
&= \rho_i^n + B_5H_7 - C_5H_6 + B_5I_7.P_{i-1}^{n+1} - B_5I_7.P_i^{n+1} - C_5I_6.P_i^{n+1} + C_5I_6.P_{i+1}^{n+1} \\
&= \Big(\rho_i^n + B_5H_7 - C_5H_6\Big) + \Big(B_5I_7\Big)P_{i-1}^{n+1} - \Big(B_5I_7 + C_5I_6\Big)P_i^{n+1} + \Big(C_5I_6\Big)P_{i+1}^{n+1}
\end{aligned}$$

More simplistically we define:

$$\begin{aligned}
B_{10} &= \rho_i^n + B_5H_7 - C_5H_6 \\
C_{10} &= B_5I_7 \\
D_{10} &= B_5I_7 + C_5I_6 \\
E_{10} &= C_5I_6
\end{aligned}$$

To arrive at:

$$\rho_i^{n+1} = B_{10} + C_{10}.P_{i-1}^{n+1} - D_{10}.P_i^{n+1} + E_{10}.P_{i+1}^{n+1} \tag{15}$$

Next we aim to eliminate the internal energy by inserting equation 14 into equation 13:

$$B_9.\rho_i^{n+1} - C_9 = K_8 + L_8.v_{i-\frac{1}{2}}^{n+1} - M_8.v_{i+\frac{1}{2}}^{n+1}$$

For simplicity we define:

$$B_{14} = \frac{K_8 + C_9}{B_9} \quad C_{14} = \frac{L_8}{B_9} \quad D_{14} = \frac{M_8}{B_9}$$

To get:

$$\rho_i^{n+1} = B_{14} + C_{14}.v_{i-\frac{1}{2}}^{n+1} - D_{14}.v_{i+\frac{1}{2}}^{n+1}$$

We need to also insert equations 11 and 12:

$$\begin{aligned}
\rho_i^{n+1} &= B_{14} + C_{14}.\Big(H_7 - I_7.P_i^{n+1} + I_7.P_{i-1}^{n+1}\Big) - D_{14}.\Big(H_6 - I_6.P_{i+1}^{n+1} + I_6.P_i^{n+1}\Big) \\
&= B_{14} + C_{14}H_7 - C_{14}I_7.P_i^{n+1} + C_{14}I_7.P_{i-1}^{n+1} - D_{14}H_6 + D_{14}I_6.P_{i+1}^{n+1} - D_{14}I_6.P_i^{n+1} \\
&= B_{14} + C_{14}H_7 - D_{14}H_6 + C_{14}I_7.P_{i-1}^{n+1} - C_{14}I_7.P_i^{n+1} - D_{14}I_6.P_i^{n+1} + D_{14}I_6.P_{i+1}^{n+1} \\
&= \Big(B_{14} + C_{14}H_7 - D_{14}H_6\Big) + \Big(C_{14}I_7\Big).P_{i-1}^{n+1} - \Big(C_{14}I_7 + D_{14}I_6\Big).P_i^{n+1} + \Big(D_{14}I_6\Big).P_{i+1}^{n+1}
\end{aligned}$$

Again we define:

$$\begin{aligned}
E_{14} &= B_{14} + C_{14}H_7 - D_{14}H_6 \\
F_{14} &= C_{14}I_7 \\
G_{14} &= C_{14}I_7 + D_{14}I_6 \\
H_{14} &= D_{14}I_6
\end{aligned}$$

To get:

$$\rho_i^{n+1} = E_{14} + F_{14}.P_{i-1}^{n+1} - G_{14}.P_i^{n+1} + H_{14}.P_{i+1}^{n+1} \tag{16}$$

Now all thats left is to equate equation 15 and 16:

$$
\begin{aligned}
B_{10}+C_{10}.P_{i-1}^{n+1}-D_{10}.P_i^{n+1}+E_{10}.P_{i+1}^{n+1} &= E_{14}+F_{14}.P_{i-1}^{n+1}-G_{14}.P_i^{n+1}+H_{14}.P_{i+1}^{n+1} \\
C_{10}.P_{i-1}^{n+1}-F_{14}.P_{i-1}^{n+1}+G_{14}.P_i^{n+1}-D_{10}.P_i^{n+1}+E_{10}.P_{i+1}^{n+1}-H_{14}.P_{i+1}^{n+1} &= E_{14}-B_{10}
\end{aligned}
$$

We now have a single equation for each control volume:

$$
\Bigg(C_{10}-F_{14}\Bigg)P_{i-1}^{n+1}+\Bigg(G_{14}-D_{10}\Bigg)P_i^{n+1}+\Bigg(E_{10}-H_{14}\Bigg)P_{i+1}^{n+1}=E_{14}-B_{10} \tag{17}
$$

Step 13
Using equation 17, in the form $\quad a_i.P_{i-1}^{n+1}+b_i.P_i^{n+1}+c_i.P_{i+1}^{n+1}=d_i \quad$ we can construct a linear system $Ax=b$ as follows:

$$
A=\begin{bmatrix}
b_1 & c_1 & \cdots & \cdots & \cdots & 0 \\
a_2 & b_2 & c_2 & \cdots & \cdots & \vdots \\
\vdots & & \ddots & & & \vdots \\
\vdots & & & \ddots & & \vdots \\
\vdots & & & a_{I-1} & b_{I-1} & c_{I-1} \\
0 & \cdots & \cdots & \cdots & a_I & b_I
\end{bmatrix}
x=\begin{bmatrix}
P_1^{n+1} \\ P_2^{n+1} \\ \vdots \\ \vdots \\ P_{I-1}^{n+1} \\ P_I^{n+1}
\end{bmatrix}
b=\begin{bmatrix}
d1 \\ d2 \\ \vdots \\ \vdots \\ d_{I-1} \\ d_I
\end{bmatrix}
$$

This system can be solved using a conjugate gradient numerical solver.

Step 14

Calculate junction velocities from equation 11:

$$v_{i+\frac{1}{2}}^{n+1} = H_6 - I_6.P_{i+1}^{n+1} + I_6.P_i^{n+1}$$

Step 15

Calculate control volume new densities from equation 10:

$$\rho_i^{n+1} = \rho_i^n + B_5.v_{i-\frac{1}{2}}^{n+1} - C_5.v_{i+\frac{1}{2}}^{n+1}$$

Step 16

Calculate control volume new internal energies from equation 14:

$$\begin{aligned}\rho_i^{n+1}u_i^{n+1} - \rho_i^n u_i^n &= B_9.\rho_i^{n+1} - C_9 \\ \rho_i^{n+1}u_i^{n+1} &= \rho_i^n u_i^n - C_9 + B_9.\rho_i^{n+1} \\ \therefore u_i^{n+1} &= \frac{\rho_i^n u_i^n - C_9}{\rho_i^{n+1}} + B_9\end{aligned}$$

Step 17

Advance the time values.

3 Test cases

In order to test various aspects of the thermal-hydraulic code it is necessary to construct a collection of test cases. These are:

1. Simple horizontal flow. No heat transfer, no junction form loss.
2. Simple horizontal flow with junction form loss.
3. Horizontal flow with heat transfer.
4. Vertical flow without heat transfer.
5. Vertical flow with heat transfer.

For all these cases, let us use the following geometry:

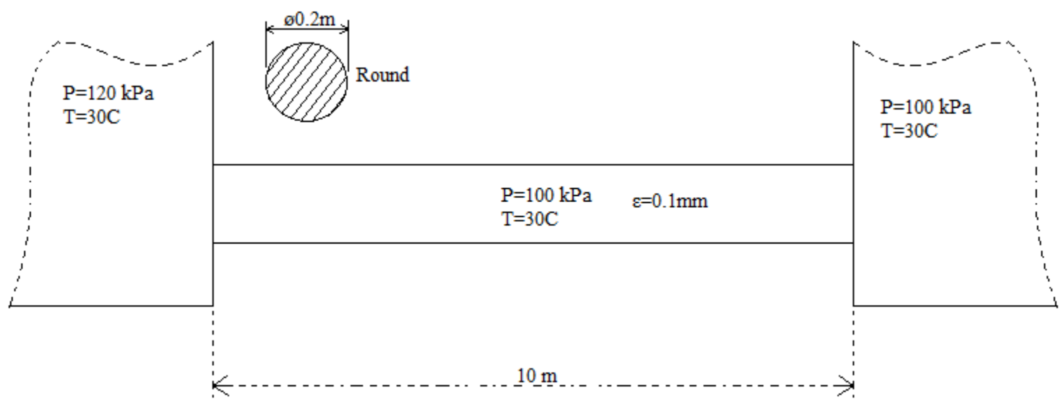

Figure 3: Test case geometry.

4 Heat block submodel

4.1 The heat conduction equation

The generalized heat conduction equation takes the following form:

$$\frac{dT}{dt} = \alpha \nabla^2 T + \frac{1}{\rho C_p} \dot{e}_{gen} \tag{18}$$

In slab geometry this equation takes the form:

$$\frac{dT}{dt} = \frac{1}{\rho C_p} \frac{d}{dx} \left(k \frac{dT}{dx} \right) + \frac{1}{\rho C_p} \dot{e}_{gen} \tag{19}$$

And in cylindrical geometry this equation takes the form:

$$\frac{dT}{dt} = \frac{1}{\rho C_p} \frac{1}{r} \frac{d}{dr} \left(k.r. \frac{dT}{dr} \right) + \frac{1}{\rho C_p} \dot{e}_{gen} \tag{20}$$

Typical boundary conditions for such second order differential equations are the symmetry or insulated boundary condition:

$$\left. \frac{dT}{dx} \right|_L = 0 \quad or \quad \left. \frac{dT}{dr} \right|_R = 0$$

The constant temperature boundary condition:

$$\left. \frac{dT}{dt} \right|_L = 0 \quad or \quad \left. \frac{dT}{dt} \right|_R = 0$$

And the surface convection boundary condition:

$$\begin{gathered} k \left. \frac{dT}{dx} \right|_L^{+,-} = h(T_L - T_{\text{inf}}) \\ or \\ k \left. \frac{dT}{dr} \right|_R^{+,-} = h(T_R - T_{\text{inf}}) \end{gathered}$$

4.2 Numerical form of the heat conduction equation

Consider the nodalization as shown in Figure 4 below.

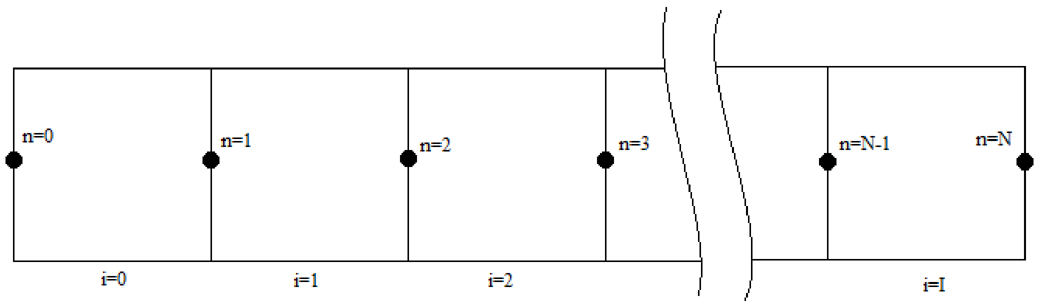

Figure 4: Heat block nodalization.

We can linearize the derivatives of equation 20 as follows:

$$\frac{T_n^{t+1}-T_n^t}{\Delta t}=\frac{1}{\rho_n C_{p,n}}\frac{1}{r_n \Delta r_n}\left(k_i^t.r_i.\frac{T_{n+1}^{t+1}-T_n^{t+1}}{\Delta r_i}-k_{i-1}^t.r_{i-1}\frac{T_n^{t+1}-T_{n-1}^{t+1}}{\Delta r_{i-1}}\right)+\frac{1}{\rho_n C_{p,n}}\dot{e}_{gen,n}^t \tag{21}$$

In this formulation the points $r_1, r_2, \ldots, r_N$ are defined by the user. Δr_i is defined only on intervals. The value of r_i is the halfway point between the nodes on either side of the interval. The values of ρ_n and $C_{p,n}$ are calculated from the arithmetic average of the adjoining intervals as follows:

$$\rho_n=\frac{(r_n+\frac{1}{2}\Delta r_n)^2-(r_n)^2}{r_{i=n-1}^2-r_{i=n}^2}\cdot\rho_{i=n-1}+\frac{(r_n)^2-(r_n-\frac{1}{2}\Delta r_n)^2}{r_{i=n-1}^2-r_{i=n}^2}\cdot\rho_{i=n}$$

$$C_{p,n}=\frac{(r_n+\frac{1}{2}\Delta r_n)^2-(r_n)^2}{r_{i=n-1}^2-r_{i=n}^2}\cdot C_{p,i=n-1}+\frac{(r_n)^2-(r_n-\frac{1}{2}\Delta r_n)^2}{r_{i=n-1}^2-r_{i=n}^2}\cdot C_{p,i=n}$$

$$\dot{e}_{gen,n}=\frac{(r_n+\frac{1}{2}\Delta r_n)^2-(r_n)^2}{r_{i=n-1}^2-r_{i=n}^2}\cdot\dot{e}_{gen,i=n-1}+\frac{(r_n)^2-(r_n-\frac{1}{2}\Delta r_n)^2}{r_{i=n-1}^2-r_{i=n}^2}\cdot\dot{e}_{gen,i=n}$$

4.3 Solution algorithm

Step 1
Load all heat block:

- Load mesh points, r_n, and initial temperatures, T_n^t
- Calculate Δr_n and Δr_i
- Calculate intervals, r_i
- Load thermal properties, k_i, $C_{p,i}$ and ρ_i
- Load initial temperature, T_n
- Set boundary conditions, $Convection$, $Symmetry$ or $Temperature$
- Set interval heat generation, $\dot{e}_{gen,n}$

************************** End of Initialization phase **************************

******************* Repeat the steps below for each time step **************************

Step 2
Set thermal conditions:

- Set boundary conditions, $Convection(h, T_{\inf})$, $Symmetry$ or $Temperature(T)$
- Set interval heat generation, $\dot{e}_{gen,n}$

Step 3a - Inner mesh points
Determine the elements of equation 21:

$$\frac{T_n^{t+1} - T_n^t}{\Delta t} = \frac{1}{\rho_n C_{p,n}} \frac{1}{r_n \Delta r_n} \left(k_i^t . r_i . \frac{T_{n+1}^{t+1} - T_n^{t+1}}{\Delta r_i} - k_{i-1}^t . r_{i-1} \frac{T_n^{t+1} - T_{n-1}^{t+1}}{\Delta r_{i-1}} \right) + \frac{1}{\rho_n C_{p,n}} \dot{e}_{gen,n}^t$$

Where the elements are:

$$B_{21} = \frac{\Delta t}{\rho_n C_{p,n}} \cdot \frac{k_i^t . r_i}{r_n . \Delta r_n . \Delta r_i} \quad and \quad C_{21} = \frac{\Delta t}{\rho_n C_{p,n}} \cdot \frac{k_i^t . r_i}{r_n . \Delta r_n . \Delta r_i}$$

$$D_{21} = \frac{\Delta t}{\rho_n C_{p,n}} \cdot \frac{k_{i-1}^t . r_{i-1}}{r_n . \Delta r_n . \Delta r_{i-1}} \quad and \quad E_{21} = \frac{\Delta t}{\rho_n C_{p,n}} \cdot \frac{k_{i-1}^t . r_{i-1}}{r_n . \Delta r_n . \Delta r_{i-1}}$$

$$F_{21} = \frac{\Delta t}{\rho_n C_{p,n}} \dot{e}_{gen,n}^t$$

To get:

$$\therefore T_n^{t+1} - T_n^t = B_{21}.T_{n+1}^{t+1} - C_{21}.T_n^{t+1} - D_{21}.T_n^{t+1} + E_{21}.T_{n-1}^{t+1} + F_{21}$$

With further simplification we can write this as:

$$-E_{21}.T_{n-1}^{t+1} + T_n^{t+1} + C_{21}.T_n^{t+1} + D_{21}.T_n^{t+1} - B_{21}.T_{n+1}^{t+1} = F_{21} + T_n^t$$

And finally:

$$\Big(-E_{21}\Big)T_{n-1}^{t+1} + \Big(1 + C_{21} + D_{21}\Big)T_n^{t+1} + \Big(-B_{21}\Big)T_{n+1}^{t+1} = \Big(F_{21} + T_n^t\Big) \tag{22}$$

Step 3b - Left convection boundary condition

This is the same as **step 3a** but with a change. For a convection boundary condition the following applies:

$$k_{i-1}^t.r_{i-1}\frac{T_n^{t+1} - T_{n-1}^{t+1}}{\Delta r_{i-1}} \quad \rightarrow \quad h_n^t.r_n(T_n^{t+1} - T_{\text{inf},n})$$

Therefore, E_{21} is zero and D_{21} becomes:

$$D_{21} = \frac{\Delta t}{\rho_n C_{p,n}} \cdot \frac{h_n^t}{\Delta r_n}$$

Also F_{21} now receives the known contribution from $T_{inf,n}$ and becomes:

$$F_{21} = \frac{\Delta t}{\rho_n C_{p,n}} \dot{e}_{gen,n}^t + D_{21}.T_{\text{inf},n}$$

Step 3c - Right symmetry boundary condition

This is the same as **step 3a** but with a change. For a symmetry boundary condition the following applies:

$$k_i^t.r_i.\frac{T_{n+1}^{t+1} - T_n^{t+1}}{\Delta r_i} \quad \rightarrow \quad 0$$

Therefore, B_{21} and C_{21} is zero with no changes to F_{21}.

Step 4

Using equation 22, in the form $a_i.T_{n-1}^{t+1} + b_i.T_n^{t+1} + c_i.T_{n+1}^{t+1} = d_i$ we can construct a linear system $Ax = b$ as follows:

$$A = \begin{bmatrix} b_1 & c_1 & \cdots & \cdots & \cdots & 0 \\ a_2 & b_2 & c_2 & \cdots & \cdots & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & a_{I-1} & b_{I-1} & c_{I-1} \\ 0 & \cdots & \cdots & \cdots & a_I & b_I \end{bmatrix} x = \begin{bmatrix} P_1^{n+1} \\ P_2^{n+1} \\ \vdots \\ \vdots \\ P_{I-1}^{n+1} \\ P_I^{n+1} \end{bmatrix} b = \begin{bmatrix} d1 \\ d2 \\ \vdots \\ \vdots \\ d_{I-1} \\ d_I \end{bmatrix}$$

This system can be solved using a conjugate gradient numerical solver.

References