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Spline Interpolation Whitepaper

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1 Preliminaries

We first proceed with the derivation of the basic equations. For an interval t_{i+1} to t_i we specify the second derivative of the piecewise linearly defined spline as a function of the unknown second derivatives at the knots $S_i^{''}(t_i) = z_i$:

$$\begin{aligned}
\frac{S_i^{''}(x) - z_i}{x - t_i} &= \frac{z_{i+1} - z_i}{t_{i+1} - t_i} \\
S_i^{''}(x) &= z_i + (x - t_i) \cdot \frac{z_{i+1} - z_i}{t_{i+1} - t_i} \\
S_i^{''}(x) &= \frac{z_i(t_{i+1} - t_i) + (x - t_i)(z_{i+1} - z_i)}{t_{i+1} - t_i} \\
S_i^{''}(x) &= \frac{z_i t_{i+1} - z_i t_i + (x - t_i) z_{i+1} - z_i x + z_i t_i}{t_{i+1} - t_i}
\end{aligned}$$

By letting $h_i = t_{i+1} - t_i$ we get:

$$S_i^{''}(x) = \frac{z_i}{h_i}(t_{i+1} - x) + \frac{z_{i+1}}{h_i}(x - t_i) \tag{1}$$

Integrating equation 1 once, we get the first derivative of the spline (plus an integration constant B):

$$S_i^{'}(x) = -\frac{z_i}{2h_i}(t_{i+1} - x)^2 - \frac{z_{i+1}}{2h_i}(x - t_i)^2 + A \tag{2}$$

And after another integration, we get the spline itself(plus integration constants A and B):

$$S_i(x) = \frac{z_i}{6h_i}(t_{i+1} - x)^3 - \frac{z_{i+1}}{2h_i}(x - t_i)^3 + Ax + B$$

We do a bit of juggling ($A = C - D$ and $B = Dt_{i+1} - Ct_i$) to get:

$$S_i(x) = \frac{z_i}{6h_i}(t_{i+1} - x)^3 - \frac{z_{i+1}}{2h_i}(x - t_i)^3 + C(x - t_i) + D(t_{i+1} - x) \tag{3}$$

We then proceed to determining the values of the unknown integration constants C and D.

1.1 Interior intervals $(1 < i < (n-1)))$

For interior intervals we can impose the spline values at the knots equal the knot values, $S_i(t_i) = y_i$ and $S_i(t_{i+1}) = y_{i+1}$ to find C and D:

$$\begin{aligned}
C &= \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right) \\
D &= \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right)
\end{aligned}$$

Resulting in the equations:

$$\begin{aligned}
S_i(x) =& \frac{z_i}{6h_i}(t_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - t_i)^3 \\
&+ \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right)(x - t_i) + \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right)(t_{i+1} - x)
\end{aligned} \tag{4}$$

And:

$$S_i^{'}(x) = -\frac{z_i}{2h_i}(t_{i+1}-x)^2 + \frac{z_{i+1}}{2h_i}(x-t_i)^2 \\ + (\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}) - (\frac{y_i}{h_i} - \frac{z_i h_i}{6}) \tag{5}$$

From this equation we can then determine $S_i^{'}(x)$ and use the continuity of first derivatives $(S_{i-1}^{'}(t_i) = S_i^{'}(t_i))$ to get:

$$h_{i-1}z_{i-1} + 2(h_i + h_{i-1})z_i + h_i z_{i+1} = \frac{6}{h_i}(y_{i+1}-y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1}) \tag{6}$$

1.2 Left most interval $(i = 0)$

Free end

For a free end there is no value to the second derivative:

$$z_0 = 0 \tag{7}$$

Fixed end (derivative specified)

For the special case where a derivative is specified on the left, equation 6 will not hold. Therefore we write from equation 5:

$$\begin{aligned} S_i^{'}(t_i) = y_i^{'} =& -\frac{z_i}{2h_i}(t_{i+1}-t_i)^2 - \frac{z_{i+1}}{2h_i}(t_i-t_i)^2 \\ &+ (\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}) - (\frac{y_i}{h_i} - \frac{z_i h_i}{6}) \\ y_i^{'} =& -\frac{h_i}{2}z_i + \frac{1}{h_i}(y_{i+1}-y_i) - \frac{h_i}{6}z_{i+1} + \frac{h_i}{6}z_i \\ \frac{h_i}{3}z_i + \frac{h_i}{6}z_{i+1} =& \frac{1}{h_i}(y_{i+1}-y_i) - y_i^{'} \end{aligned}$$

Therefore at $i = 0$:

$$2h_0 z_0 + h_0 z_1 = \frac{6}{h_0}(y_1 - y_0) - 6y_0^{'} \tag{8}$$

1.3 Right most interval $(i = (n-1))$

For this interval we have two boundary conditions; on the left the first derivative needs to be continuous, but at the right we can have either a free end condition or a fixed end condition. For the left condition equation 6 holds. Another equation however is needed.

Free end

For a free end there is no value to the second derivative:

$$z_n = 0 \tag{9}$$

Fixed end (derivative specified)

From equation 5 we can write the derivative at t_n:

$$
\begin{aligned}
S_i^{'}(t_{i+1}) &= \cancelto{0}{-\frac{z_i}{2h_i}(t_{i+1}-t_{i+1})^2} + \frac{z_{i+1}}{2h_i}(t_{i+1}-t_i)^2 \\
&\quad + (\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}) - (\frac{y_i}{h_i} - \frac{z_i h_i}{6}) \\
S_i^{'}(t_{i+1}) &= \frac{h_i}{2}z_{i+1} + \frac{1}{h_i}(y_{i+1}-y_i) + \frac{h_i}{6}z_i - \frac{h_i}{6}z_{i+1}
\end{aligned}
$$

And for the right boundary condition we set this derivative equal to the specified value $y_n^{'}$ to get:

$$
\begin{aligned}
S_i^{'}(t_{i+1}) = y_{i+1}^{'} &= \frac{h_i}{2}z_{i+1} + \frac{1}{h_i}(y_{i+1}-y_i) + \frac{h_i}{6}z_i - \frac{h_i}{6}z_{i+1} \\
-\frac{h_i}{6}z_i - \frac{h_i}{3}z_{i+1} &= \frac{1}{h_i}(y_{i+1}-y_i) - y_{i+1}^{'} \\
h_i z_i + 2h_i z_{i+1} &= 6y_{i+1}^{'} - \frac{6}{h_i}(y_{i+1}-y_i)
\end{aligned}
$$

Thus, for $i = (n-1)$ we have:

$$
h_{n-1}z_{n-1} + 2h_{n-1}z_n = 6y_n^{'} - \frac{6}{h_{n-1}}(y_n - y_{n-1}) \tag{10}
$$

1.4 Constructing the system $Ax = b$

Since we have the unknowns $z_0, z_1, \cdots, z_n$ we will need a matrix A of size $(n+1)\times(n+1)$ and a corresponding $(n+1)\times 1$ vector. The coefficients for row 0 will be obtained from equation 7 or 8 depending on the boundary condition. The coefficients for rows 1 to (n-1) will be obtained from equation 6. Finally, the coefficients for row n will be obtained from equation 9 or 10 depending on the boundary condition.

2 Calculational Results

Suppose we need to interpolate data between the given distance markers of a car (Table 1). When assuming free ends, we can find the cars position at $t = 10\ s$ as:

$$
Distance = 757.715\ ft
$$

When assuming fixed ends with the start and end speeds set at 75 and 72 feet per second respectively, we find at $t = 10\ s$:

$$
Distance = 747.956\ ft
$$

Table 1: Car distance markers

Time[s]	0	3	5	8	13
Distance[ft]	0	225	385	623	933

For the case where we want to interpolate population data (Table 2) when assuming free ends, we find that in 1965, 1975 and 1985 we have:

$$1965 = 191,834\ K$$
$$1975 = 214,775\ K$$
$$1985 = 238,121\ K$$

Table 2: Population data

Year	1930	1940	1950	1960	1970	1980	1990
Population (in K)	123203	131669	150697	179323	203212	226505	249643

We can also develop the form of a half circle by defining a smaller resolution of points, $x(t) = cost, y(t) = sint$ to get the plot shown in (Figure 1). For 11 knot points the maximum error on a 1000 interval mesh was 0.004047 or 0.4% and for 21 knot points this reduced to 0.001102 or 0.1%. Therefore we can see that spline interpolation can be fairly accurate.

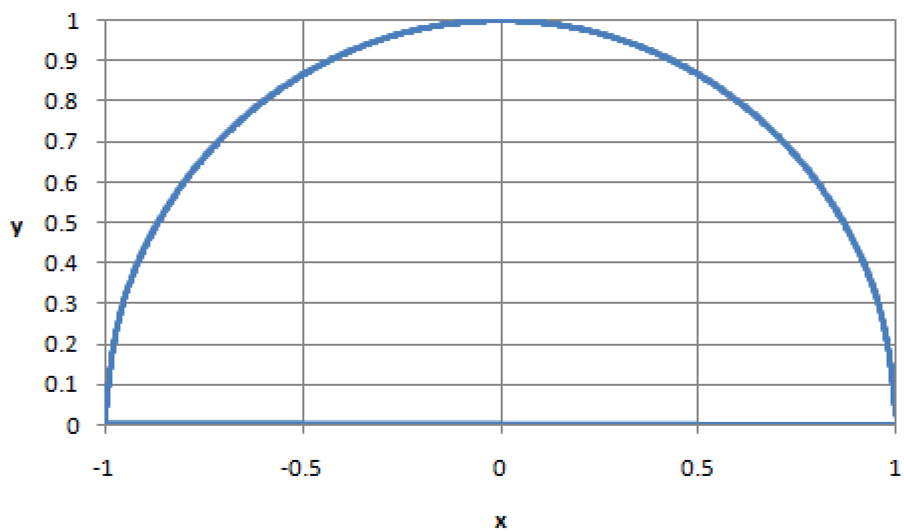

Figure 1: Half circle generated by spline interpolation of 11 knot points.

Similar to the half circle example we can create symbols with splines that look very elegant as shown in Figure 2 below.

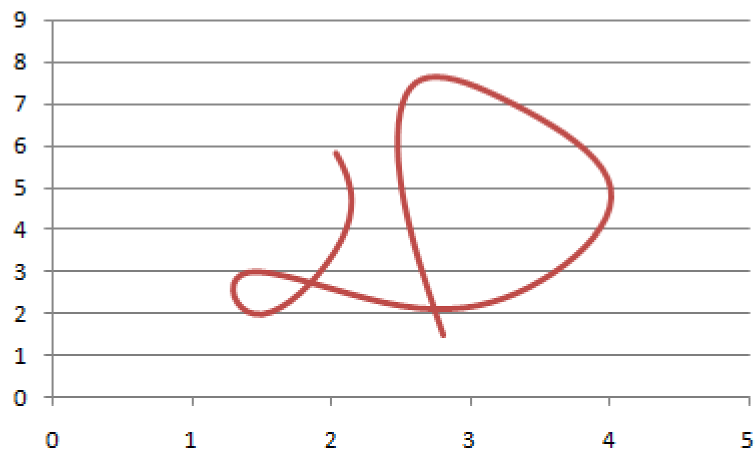

Figure 2: A symbol drawn with a spline.