Chapter 10

Integration methods

10.1 Introduction

The finite element analysis techniques are always based on an integral formulation. At the very minimum it will always be necessary to integrate at least an element square matrix. This means that every coefficient function in the matrix must be integrated. In the following sections various methods will be considered for evaluating the typical integrals that arise. Most simple finite element matrices for two-dimensional problems are based on the use of linear triangular or quadrilateral elements. Since a quadrilateral can be divided into two or more triangles, only exact integrals over arbitrary triangles will be considered here. Integrals over triangular elements commonly involve integrands of the form

$$I = \int_A x^m y^n \, dx \, dy \tag{10.1}$$

where A is the area of a typical triangle. When $0 \leq (m + n) \leq 2$, the above integral can easily be expressed in closed form in terms of the spatial coordinates of the three corner points. For a right-handed coordinate system, the corners must be numbered in counter-clockwise order. In this case, the above integrals are given in Table 10.1. These integrals should be recognized as the area, and first and second moments of the area. If one had a volume of revolution that had a triangular cross-section in the $\rho - z$ plane, then one has

$$I = \int_V \rho f(\rho, z) \, d\rho \, dz \, d\phi = 2\pi \int_A \rho f(\rho, z) \, d\rho \, dz$$

so that similar expressions could be used to evaluate the volume integrals.

10.2 Unit coordinate integration

The utilization of global coordinate interpolation is becoming increasingly rare. However, as we have seen, the use of non-dimensional local coordinates is common. Thus we often see local coordinate polynomials integrated over the physical domain of an element. Sec. 4.3 presented some typical unit coordinate integrals in 1-D, written in exact closed form. These concepts can be extended to two- and three-dimensional

Table 10.1 *Exact integrals for a triangle*

m	n	$I = \int_A x^m y^n \, dx \, dy$
0	0	$\int dA = A = [\, x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \,]/2$
0	1	$\int y \, dA = A\bar{y} = A(y_1 + y_2 + y_3)/3$
1	0	$\int x \, dA = A\bar{x} = A(x_1 + x_2 + x_3)/3$
0	2	$\int y^2 \, dA = A(y_1^2 + y_2^2 + y_3^2 + 9\bar{y}^2)/12$
1	1	$\int xy \, dA = A(x_1\, y_1 + x_2\, y_2 + x_3\, y_3 + 9\overline{x\, y})/12$
2	0	$\int x^2 \, dA = A(x_1^2 + x_2^2 + x_3^2 + 9\bar{x}^2)/12$

elements. For example, consider an integration over a triangular element. It is known that for an element with a constant Jacobian

$$I = \int_A r^m \, s^n \, da = \frac{2A\,\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(3+m+n)} \tag{10.2}$$

where Γ denote the Gamma function. Restricting consideration to positive integer values of the exponents, m and n, yields

$$I = 2\,A^e \, \frac{m!\, n!}{(2+m+n)\,!} = \frac{A^e}{K_{mn}} \,, \tag{10.3}$$

where ! denotes the factorial and K_{mn} is an integer constant given in Table 10.2 for common values of m and n. Similarly for the tetrahedral element

$$I^e = \int_{V^e} r^m \, s^n \, t^p \, dv = 6V^e \, \frac{m!\, n!\, p!}{(3+m+n+p)\,!} \,. \tag{10.4}$$

Thus, one notes that common integrals of this type can be evaluated by simply multiplying the element characteristic (i.e., global length, area, or volume) by known constants which could be stored in a data statement. To illustrate the application of these equations in evaluating element matrices, we consider the following example for the three node triangle in unit coordinates:

$$I = \int_{A^e} \mathbf{H}^T \, da = \int_{A^e} \begin{Bmatrix} (1 - r - s) \\ r \\ s \end{Bmatrix} da = \begin{Bmatrix} A^e - A^e/3 - A^e/3 \\ A^e/3 \\ A^e/3 \end{Bmatrix} = \frac{A^e}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} .$$

$$\mathbf{I}_V = 2\pi \int_{A^e} \mathbf{H}^T \rho \, da = 2\pi \left(\int_{A^e} \mathbf{H}^T \mathbf{H} \, da \right) \rho^{\mathrm{e}} = \frac{2\pi \mathrm{A}^{\mathrm{e}}}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rho^{\mathrm{e}}.$$

10.3 Simplex coordinate integration

A simplex region is one where the minimum number of vertices is one more than the dimension of the space. These were illustrated in Fig. 3.2. Some analysts like to define a set of *simplex coordinates* or *barycentric coordinates*. If there are N vertices

Table 10.2 *Denominator, K, for unit triangle* $\boldsymbol{I = \int_A r^m s^n \, da = A/K}$

M	N: 0	1	2	3	4	5	6	7	8
0	1	3	6	10	15	21	28	36	45
1	3	12	30	60	105	168	252	360	495
2	6	30	90	210	420	756	1260	1980	2970
3	10	60	210	560	1260	2520	4620	7920	12870
4	15	105	420	1260	3150	6930	13860	25740	45045
5	21	168	756	2520	6930	16632	36036	72072	135135
6	28	252	1260	4620	13860	36036	84084	180180	360360
7	36	360	1980	7920	25740	72072	180180	411840	875160
8	45	495	2970	12870	45045	135135	360360	875160	1969110

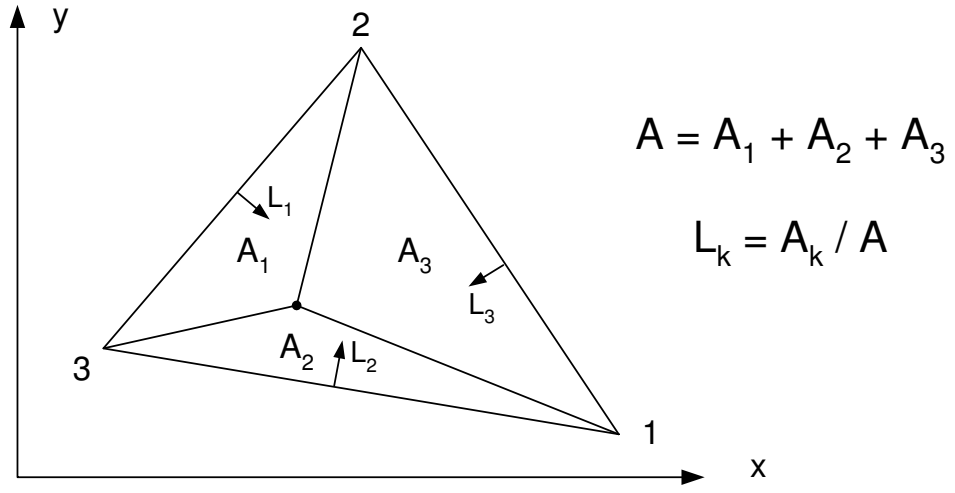

Figure 10.1 *Area coordinates*

then N non-dimensional coordinates, L_i, $1 \le i \le N$, are defined and constrained so that

$$1 \;=\; \sum_{i=1}^{N} \; L_i$$

at any point in space. Thus, they are not independent. However, they can be used to simplify certain recursion relations. In physical spaces these coordinates are sometimes called *line coordinates*, *area coordinates*, and *volume coordinates*. At a given point in the region we can define the simplex coordinate for node j, L_j, in a generalized manner. It is the ratio of the generalized volume from the point to all other vertices (other than j) and the total generalized volume of the simplex. This is illustrated in Fig. 10.1. If the simplex has a constant Jacobian (e.g., straight sides and flat faces), then the exact form of the integrals of the simplex coordinates are simple. They are

$$\begin{aligned}
\int_L L_1^a \; L_2^b \; dL &= \frac{a!b!}{(a+b+1)!} \; (L) \\
\int_A L_1^a \; L_2^b \; L_3^c \; da &= \frac{a!b!c!}{(a+b+c+2)!} \; (2A) \\
\int_V L_1^a \; L_2^b \; L_3^c \; L_4^d \; dv &= \frac{a!b!c!d!}{(a+b+c+d+3)!} \; (6V) \; .
\end{aligned} \tag{10.5}$$

The independent coordinates are those we have generally referred to as the unit coordinates of an element. Since a lot of references make use of barycentric coordinates it is useful to learn how to manipulate them correctly. The barycentric coordinates, say L_j, essentially measure the percent of total volume contained in the region from the face (lower dimensional simplex) opposite to node j to any point in the simplex. Therefore, $L_j \equiv 0$ when the point lies on the opposite face and $L_j \equiv 1$ when the point is located at node j. Clearly, the sum of all these volumes is the total volume of the simplex.

We have referred to the independent coordinates in the set as the unit coordinates. For simplex elements, the use of barycentric coordinates simplifies the algebra needed to define the interpolation functions; however, it complicates the calculation of their derivatives. Barycentric coordinates are often used to tabulate numerical integration rules for simplex domains.

For example, consider the three-dimensional case where $L_1 = r$, $L_2 = s$, $L_3 = t$, and $L_1 + L_2 + L_3 + L_4 = 1$. The interpolation functions for the linear tetrahedral (P4) are simply $G_j = L_j$. The expressions for the Lagrangian quadratic tetrahedral (P10) vertices are

$$\begin{aligned}
G_1 &= L_1 \, (\, 2 \, L_1 - 1 \,) & \qquad G_2 &= L_2 \, (\, 2 \, L_2 - 1 \,) \\
G_3 &= L_3 \, (\, 2 \, L_3 - 1 \,) & \qquad G_4 &= L_4 \, (\, 2 \, L_4 - 1 \,)
\end{aligned}$$

and the six mid-edge values are

$$\begin{aligned}
G_5 &= 4 \; L_1 \; L_2 & \qquad G_6 &= 4 \; L_1 \; L_3 \\
G_7 &= 4 \; L_1 \; L_4 & \qquad G_8 &= 4 \; L_2 \; L_3 \\
G_9 &= 4 \; L_3 \; L_4 & \qquad G_{10} &= 4 \; L_2 \; L_4 \; .
\end{aligned}$$

All the tetrahedra have the condition that

$$L_4 \;=\; 1 - L_1 - L_2 - L_3 \;=\; 1 - r - s - t$$

so that we can write the unit coordinate partial derivatives as

$$\frac{\partial L_j}{\partial r} \;=\; 1,\, 0,\, 0,\, -1\,, \qquad \frac{\partial L_j}{\partial s} \;=\; 0,\, 1,\, 0,\, -1\,, \qquad \frac{\partial L_j}{\partial t} \;=\; 0,\, 0,\, 1,\, -1$$

for $j = 1,\ 2,\ 3,\ 4$, respectively. The Jacobian calculation requires the derivatives of the geometric interpolation functions, $\mathbf{G}$. Here we have

$$\begin{aligned} \frac{\partial \mathbf{G}}{\partial r} \;&=\; \frac{\partial \mathbf{G}}{\partial L_1}\,\frac{\partial L_1}{\partial r} + \frac{\partial \mathbf{G}}{\partial L_2}\,\frac{\partial L_2}{\partial r} + \frac{\partial \mathbf{G}}{\partial L_3}\,\frac{\partial L_3}{\partial r} + \frac{\partial \mathbf{G}}{\partial L_4}\,\frac{\partial L_4}{\partial r} \\ &=\; \frac{\partial \mathbf{G}}{\partial L_1} - \frac{\partial \mathbf{G}}{\partial L_4}\,. \end{aligned}$$

Likewise,

$$\frac{\partial \mathbf{G}}{\partial s} \;=\; \frac{\partial \mathbf{G}}{\partial L_2} - \frac{\partial \mathbf{G}}{\partial L_4}\,, \qquad \frac{\partial \mathbf{G}}{\partial t} \;=\; \frac{\partial \mathbf{G}}{\partial L_3} - \frac{\partial \mathbf{G}}{\partial L_4}\,.$$

For a general simplex, we have

$$\partial_l\, \mathbf{G} \;=\; \partial_L\, \mathbf{G} \;-\; \mathbf{I}\,\frac{\partial \mathbf{G}}{\partial L}\,.$$

To illustrate these rules for derivatives, consider the linear triangle (T3) in barycentric coordinates ($n_n = 3$). The geometric interpolation array is

$$\mathbf{G} \;=\; \lfloor\; L_3 \qquad L_1 \qquad L_2 \;\rfloor$$

and the two independent local space derivatives are

$$\Delta \;=\; \partial_l\, \mathbf{G} \;=\; \left\{ \begin{array}{c} \dfrac{\partial}{\partial r} \\[2ex] \dfrac{\partial}{\partial s} \end{array} \right\} \mathbf{G} \;=\; \left\{ \begin{array}{c} \dfrac{\partial}{\partial L_1} - \dfrac{\partial}{\partial L_3} \\[2ex] \dfrac{\partial}{\partial L_2} - \dfrac{\partial}{\partial L_3} \end{array} \right\} \mathbf{G}$$

$$\Delta \;=\; \left[\begin{array}{ccc} (0-1) & (1-0) & (0-0) \\ (0-1) & (0-0) & (1-0) \end{array} \right] \;=\; \left[\begin{array}{ccc} -1 & 1 & 0 \\ -1 & 0 & 1 \end{array} \right],$$

which is the same as the previous result in Sec. 10.2.

If one is willing to restrict the elements to having a constant Jacobian (straight edges and flat faces), then the inverse global to barycentric mapping is simple to develop. Then the global derivatives that we desire are easy to write

$$\frac{\partial}{\partial x} \;=\; \sum_{j=1}^{n+1} \frac{\partial}{\partial L_j}\,\frac{\partial L_j}{\partial x}\,,$$

where $\partial L_j / \partial x$ is a known value, say V_j. For example, in 1-D we have

$$\left\{ \begin{array}{c} L_1 \\ L_2 \end{array} \right\} \;=\; \frac{1}{L^e} \left[\begin{array}{cc} x_2 & -1 \\ -x_1 & 1 \end{array} \right]^e \left\{ \begin{array}{c} 1 \\ x \end{array} \right\},$$

and in 2-D

$$\left\{ \begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array} \right\} = \frac{1}{2A^e} \left[\begin{array}{ccc} 2A_{23} & (y_2 - y_3) & (x_3 - x_2) \\ 2A_{13} & (y_3 - y_1) & (x_1 - x_3) \\ 2A_{12} & (y_1 - y_2) & (x_2 - x_1) \end{array} \right] \left\{ \begin{array}{c} 1 \\ x \\ y \end{array} \right\}$$

where A_{ij} is the triangular area enclosed by the origin $(0, 0)$ and nodes i and j.

10.4 Numerical integration

In many cases it is impossible or impractical to integrate the expression in closed form and numerical integration must therefore be utilized. If one is using sophisticated elements, it is almost always necessary to use numerical integration. Similarly, if the application is complicated, e.g., the solution of a nonlinear ordinary differential equation, then even simple one-dimensional elements can require numerical integration. Many analysts have found that the use of numerical integration simplifies the programming of the element matrices. This results from the fact that lengthy algebraic expressions are avoided and thus the chance of algebraic and/or programming errors is reduced. There are many numerical integration methods available. Only those methods commonly used in finite element applications will be considered here.

10.4.1 Unit coordinate quadrature

Numerical quadrature in one-dimension was introduced in Sec. 5.4. There we saw that an integral is replaced with a summation of functions evaluated at tabulated points and then multiplied by tabulated weights. The same procedure applies to all numerical integration rules. The main difficulty is to obtain the tabulated data. For triangular unit coordinate regions the weights, W_i, and abscissae (r_i, s_i) are less well known. Typical points for rules on the unit triangle are shown in Fig. 10.2. It presents rules that yield points that are symmetric with respect to all corners of the triangle. These low order data are placed in subroutine *SYMRUL*. As before, one approximates an integral of $f(x, y) = F(r, s)$ over a triangle by

$$I = \int f(x, y)\, dx\, dy = \sum_{i=1}^{n} W_i\, F(r_i, s_i)\, |J_i|.$$

As a simple example of integration over a triangle, let $f = y$ and consider the integral over a triangle with its three vertices at $(0, 0)$, $(3, 0)$, and $(0, 6)$, respectively, in (x, y) coordinates. Then the area $A = 9$ and the Jacobian is a constant $|J| = 18$. For a three point quadrature rule the integral is thus given by

$$I = \sum_{i=1}^{3} W_i\, y_i\, |J_i|.$$

Since our interpolation defines $y(r, s) = y_1 + (y_2 - y_1)r + (y_3 - y_1)s = 0 + 0 + 6s$, the transformed integrand is $F(r, s) = 6s$. Thus, at integration point, i, $F(r_i, s_i) = 6s_i$. Substituting a three-point quadrature rule and factoring out the constant Jacobian gives $I = 18\,[\,(\,6(1/6)\,)\,(1/6) + (\,6(1/6)(1/6) + (6(2/3)\,)\,(1/6)\,] = 18$ which is the exact solution.

Table 10.3 *Symmetric quadrature for the unit triangle:*

$$\int_0^1 \int_0^{1-r} f(r,s)\,dr\,ds = \sum_{i=1}^{n} f(r_i, s_i)\,W_i$$

n	$p^\dagger$	i	r_i	s_i	W_i
1	1	1	1/3	1/3	1/2
3	2	1	1/6	1/6	1/6
		2	2/3	1/6	1/6
		3	1/6	2/3	1/6
4	3	1	1/3	1/3	–9/32
		2	3/5	1/5	25/96
		3	1/5	3/5	25/96
		4	1/5	1/5	25/96
7	4	1	0	0	1/40
		2	1/2	0	1/15
		3	1	0	1/40
		4	1/2	1/2	1/15
		5	0	1	1/40
		6	0	1/2	1/15
		7	1/3	1/3	9/40

P = Degree of Polynomial for exact integration.
See subroutine *DUNAVANT_UNIT_TRIANGLE_RULE*

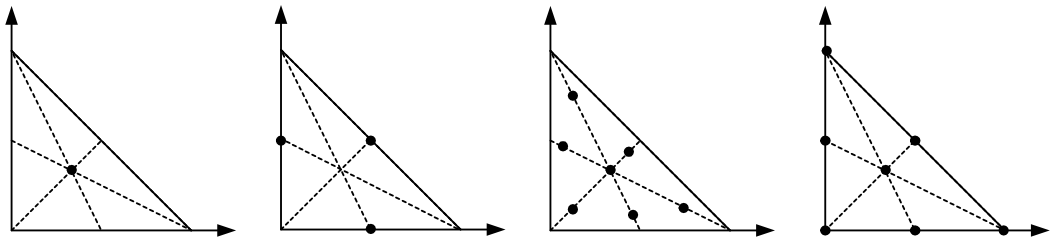

Figure 10.2 *Symmetric quadrature locations for unit triangle*

Table 10.3 gives a tabulation of symmetric quadrature rules over the unit triangle. Decimal versions are given in subroutine *SYMRUL* of values of n_q up to 13. A similar set of rules for extension to the three-dimensional tetrahedra in unit coordinates are given in Table 10.4 for polynomials up to degree four [7]. Quadrature rules for high degree polynomials on triangles have been published by Dunavant [5]. They are suitable for use

Table 10.4 *Quadrature for unit tetrahedra*

Number of points N	Degree of precision	Unit coordinates r_i	s_i	t_i	Weights W_i
1	1	1/4	1/4	1/4	1/6
4	2	a	b	b	1/24
		b	a	b	1/24
		b	b	a	1/24
		b	b	b	1/24
$a = (5+3\sqrt{5})/20 = 0.5854101966249685$ $b = (5-\sqrt{5})/20 = 0.1381966011250105$					
5	3	1/4	1/4	1/4	−4/30
		1/2	1/6	1/6	9/120
		1/6	1/2	1/6	9/120
		1/6	1/6	1/2	9/120
		1/6	1/6	1/6	9/120
11	4	1/4	1/4	1/4	−74/5625
		11/14	1/14	1/14	343/45000
		1/14	11/14	1/14	343/45000
		1/14	1/14	11/14	343/45000
		1/14	1/14	1/14	343/45000
		a	a	b	56/2250
		a	b	a	56/2250
		a	b	b	56/2250
		b	a	a	56/2250
		b	a	b	56/2250
		b	b	a	56/2250
$a = (1+\sqrt{(5/14)})/4 = 0.3994035761667992$ $b = (1-\sqrt{(5/14)})/4 = 0.1005964238332008$					
See subroutine KEAST_UNIT_TET_RULE					

with hierarchical elements. Those rules are given in Table 10.5 in area coordinates, since that form requires the smallest table size. Most of the lines are used multiple times by cycling through the area coordinates. The number N in the table indicates if the line is for the centroid, three symmetric points, or six symmetric locations. These data are expanded to their full form (up to 61 points for a polynomial of degree 17) in subroutine *DUNAVANT_UNIT_TRIANGLE_RULE*. The corresponding unit triangle coordinate data are also given in subroutine *D_Q_RULE*.

10.4.2 Natural coordinate quadrature

Here we assume that the coordinates are in the range of −1 to +1. In this space it is common to employ Gaussian quadratures. The one-dimensional rules were discussed in Sec. 5.4. For a higher number of space dimensions one obtains a multiple summation (tensor product) for evaluating the integral. For example, a typical integration in two dimensions

$$I = \int_{-1}^{1}\int_{-1}^{1} f(r, s)\, dr\, ds \approx \sum_{j=1}^{n}\sum_{k=1}^{n} f(r_j, s_k)\, W_j\, W_k$$

for n integration points in each dimension. This can be written as a single summation as

$$I \approx \sum_{i=1}^{m} f(r_i, s_i)\, W_i$$

where $m = n^2$, $i = j + (k-1)n$, and where $r_i = \alpha_j$, $s_i = \alpha_k$, and $W_i = W_j W_k$. Here α_j and W_j denote the tabulated one-dimensional abscissae and weights given in Sec. 5.4. A similar rule can be given for a three-dimensional region. The result of the above summation is given in Table 10.6. The extension of the 1-D data to the quadrilateral and hexahedra are done by subroutines *GAUSS_2D* and *GAUSS_3D* (see Fig. 10.3).

10.5 Typical source distribution integrals *

Previously we introduced the contributions of distributed source terms. For the C° continuity line elements we had

$$\mathbf{C}_Q^e = \int_{L^e} \mathbf{H}^{e^T} Q^e\, dx\ .$$

Similar forms occur in two-dimensional problems. Then typically one has

$$\mathbf{C}_Q^e = \int_{A^e} \mathbf{H}^{e^T} Q^e\, da\ .$$

If the typical source or forcing term, Q^e, varies with position we usually use the interpolation functions to define it in terms of the nodal values, Q^e, as

$$Q^e = \mathbf{H}^{e^T} \mathbf{Q}^e\ . \tag{10.6}$$

Thus, a common element integral for the consistent nodal sources is

$$\mathbf{C}_Q^e = \int_{\Omega^e} \mathbf{H}^{e^T}\, \mathbf{H}^e\, d\Omega\, \mathbf{Q}^e\ . \tag{10.7}$$

The previous sections present analytic and numerical methods for evaluating these integrals. Figure 10.4 shows the typical analytic results for the two and three node line integrals. For linear or constant source distributions the normalized nodal resultants are summarized in Fig. 10.5. Once one goes beyond the linear (two-node) element the consistent results usually differ from physical intuition estimates. Thus, you must rely on the mathematics or the summaries in the above figures. Many programs will numerically integrate the source distributions for any element shape. If the source acts on an area shaped like the parent element (constant Jacobian) then we can again easily evaluate the integrals analytically. For a uniform source over an area the consistent nodal contributions for triangles and quadrilaterals are shown in Figs. 10.6 and 10.7, respectively. Note that the Serendipity families can actually develop negative contributions. Triangular and Lagrangian elements do not have that behavior for uniform sources. Of course, a general loading can be treated by numerical integration.

```
SUBROUTINE GAUSS_3D (M_QP, N_IP, PT, WT)                          ! 1
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * *      ! 2
!     USE 1-D GAUSSIAN DATA TO GENERATE                           ! 3
!       QUADRATURE DATA FOR A CUBE                                ! 4
! * * * * * * * * * * * * * * * * * * * * * * * * * * * * *      ! 5
Use Precision_Module                                              ! 6
 IMPLICIT  NONE                                                   ! 7
 INTEGER,  INTENT(IN)  :: M_QP, N_IP                              ! 8
 REAL(DP), INTENT(OUT) :: PT (3, N_IP), WT (N_IP)                 ! 9
 REAL(DP) :: GPT (M_QP), GWT (M_QP) ! Automatic Arrays            !10
 INTEGER  :: I, J, K, L, N_GP       ! Loops                       !11
                                                                  !12
! M_QP = NUMBER OF TABULATED 1-D POINTS                           !13
! N_IP = M_QP**3 = NUMBER OF 3-D POINTS                           !14
! GPT  = TABULATED 1-D QUADRATURE POINTS                          !15
! GWT  = TABULATED 1-D QUADRATURE WEIGHTS                         !16
! PT   = CALCULATED COORDS  IN A CUBE                             !17
! WT   = CALCULATED WEIGHTS IN A CUBE                             !18
                                                                  !19
  N_GP = M_QP                                                     !20
  CALL GAUSS_COEFF (N_GP, GPT, GWT) !  GET 1-D DATA               !21
                                                                  !22
!  LOOP OVER GENERATED POINTS                                     !23
  K = 0                                                           !24
  DO L = 1, N_GP                                                  !25
    DO I = 1, N_GP                                                !26
      DO J = 1, N_GP                                              !27
        K = K + 1                                                 !28
        WT (K) = GWT (I) * GWT (J) * GWT (L)                      !29
        PT (1, K) = GPT (J)                                       !30
        PT (2, K) = GPT (I)                                       !31
        PT (3, K) = GPT (L)                                       !32
      END DO                                                      !33
    END DO                                                        !34
  END DO                                                          !35
END SUBROUTINE GAUSS_3D                                           !36
```

Figure 10.3 *Gaussian rules for a cube*

Table 10.5 *Dunavant quadrature for area coordinate triangle*

P	N	Wt	L_1	L_2	L_3
1	1	1.000000000000000	0.333333333333333	0.333333333333333	0.333333333333333
2	3	0.333333333333333	0.666666666666667	0.166666666666667	0.166666666666667
3	1	−0.562500000000000	0.333333333333333	0.333333333333333	0.333333333333333
	3	0.520833333333333	0.600000000000000	0.200000000000000	0.200000000000000
4	3	0.223381589678011	0.108103018168070	0.445948490915965	0.445948490915965
	3	0.109951743655322	0.816847572980459	0.091576213509771	0.091576213509771
5	1	0.225000000000000	0.333333333333333	0.333333333333333	0.333333333333333
	3	0.132394152788506	0.059715871789770	0.470142064105115	0.470142064105115
	3	0.125939180544827	0.797426985353087	0.101286507323456	0.101286507323456
6	3	0.116786275726379	0.501426509658179	0.249286745170910	0.249286745170910
	3	0.050844906370207	0.873821971016996	0.063089014491502	0.063089014491502
	6	0.082851075618374	0.053145049844817	0.310352451033784	0.636502499121399
7	1	−0.149570044467682	0.333333333333333	0.333333333333333	0.333333333333333
	3	0.175615257433208	0.479308067841920	0.260345966079040	0.260345966079040
	3	0.053347235608838	0.869739794195568	0.065130102902216	0.065130102902216
	6	0.077113760890257	0.048690315425316	0.312865496004874	0.638444188569810
8	1	0.144315607677787	0.333333333333333	0.333333333333333	0.333333333333333
	3	0.095091634267285	0.081414823414554	0.459292588292723	0.459292588292723
	3	0.103217370534718	0.658861384496480	0.170569307751760	0.170569307751760
	3	0.032458497623198	0.898905543365938	0.050547228317031	0.050547228317031
	6	0.027230314174435	0.008394777409958	0.263112829634638	0.728492392955404
9	1	0.097135796282799	0.333333333333333	0.333333333333333	0.333333333333333
	3	0.031334700227139	0.020634961602525	0.489682519198738	0.489682519198738
	3	0.077827541004774	0.125820817014127	0.437089591492937	0.437089591492937
	3	0.079647738927210	0.623592928761935	0.188203535619033	0.188203535619033
	3	0.025577675658698	0.910540973211095	0.044729513394453	0.044729513394453
	6	0.043283539377289	0.036838412054736	0.221962989160766	0.741198598784498
10	1	0.090817990382754	0.333333333333333	0.333333333333333	0.333333333333333
	3	0.036725957756467	0.028844733232685	0.485577633383657	0.485577633383657
	3	0.045321059435528	0.781036849029926	0.109481575485037	0.109481575485037
	6	0.072757916845420	0.141707219414880	0.307939838764121	0.550352941820999
	6	0.028327242531057	0.025003534762686	0.246672560639903	0.728323904597411
	6	0.009421666963733	0.009540815400299	0.066803251012200	0.923655933587500

P = Degree of complete polynomial exactly integrated, N = Number of cyclic uses
Wt = Weight at point, L_j = Area coordinates at the point
(See subroutine D_Q_RULE for $P \leq 17$)

Table 10.6 *Gaussian quadrature on a quadrilateral*

$$\int_{-1}^{1}\int_{-1}^{1} f(r,s)\,dr\,ds = \sum_{i=1}^{n} f(r_i,\ s_i)\,W_i$$

n	i	r_i	s_i	W_i
1	1	0	0	4
4	1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1
	2	$+1/\sqrt{3}$	$-1/\sqrt{3}$	1
	3	$-1/\sqrt{3}$	$+1/\sqrt{3}$	1
	4	$+1/\sqrt{3}$	$+1/\sqrt{3}$	1
9	1	$-\sqrt{3/5}$	$-\sqrt{3/5}$	25/81
	2	0	$-\sqrt{3/5}$	40/81
	3	$+\sqrt{3/5}$	$-\sqrt{3/5}$	25/81
	4	$-\sqrt{3/5}$	0	40/81
	5	0	0	64/81
	6	$+\sqrt{3/5}$	0	40/81
	7	$-\sqrt{3/5}$	$+\sqrt{3/5}$	25/81
	8	0	$+\sqrt{3/5}$	40/81
	9	$+\sqrt{3/5}$	$+\sqrt{3/5}$	25/81

10.6 Minimal, optimal, reduced and selected integration *

Since the numerical integration of the element square matrix can represent a large part of the total cost it is desirable to use low order integration rules. Care must be taken when selecting the *minimal order* of integration. Usually the integrand will contain global derivatives so that in the limit, as the element size h approaches zero, the integrand can be assumed to be constant, and then only the integral $I = \int dv = \int |J| dr\,ds\,dt$ remains to be integrated exactly. Such a rule could be considered the minimal order. However, the order is often too low to be practical since it may lead to a rank deficient element (and system) square matrix, if the rule does not exactly integrate the equations. Typical integrands involve terms such as the strain energy density per unit volume: $\mathbf{B}^T\mathbf{D}\mathbf{B}/2$.

Let n_q denote the number of element integration points while m_I represents the number of independent relations at each integration point; then the rank of the element is $n_q \times m_I$. Generally, m_I corresponds to the number of rows in $\mathbf{B}$ in the usual symbolic integrand $\mathbf{B}^T\mathbf{D}\mathbf{B}$. For a typical element, we want $n_q \times (m_i - m_c) \geq n_i$, where m_c represents the number of element constraints, if any. For a non-singular system matrix a similar expression is $n_e \times (n_q \times m_i - m_c) \geq n_d - m_r$, where $m_r \geq 1$ denotes the number of nodal parameter restraints. These relations can be used as guides in selecting a minimal value of n_q. Consider a problem involving a governing integral statement with m-th order derivatives. If the interpolation (trial) functions are complete polynomials of

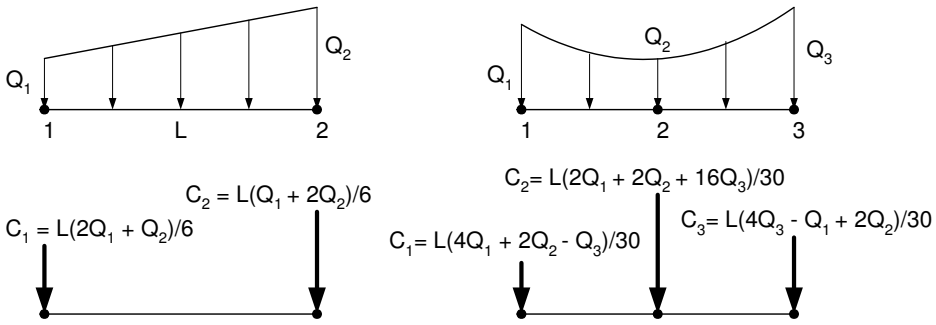

Figure 10.4 *General consistent line sources*

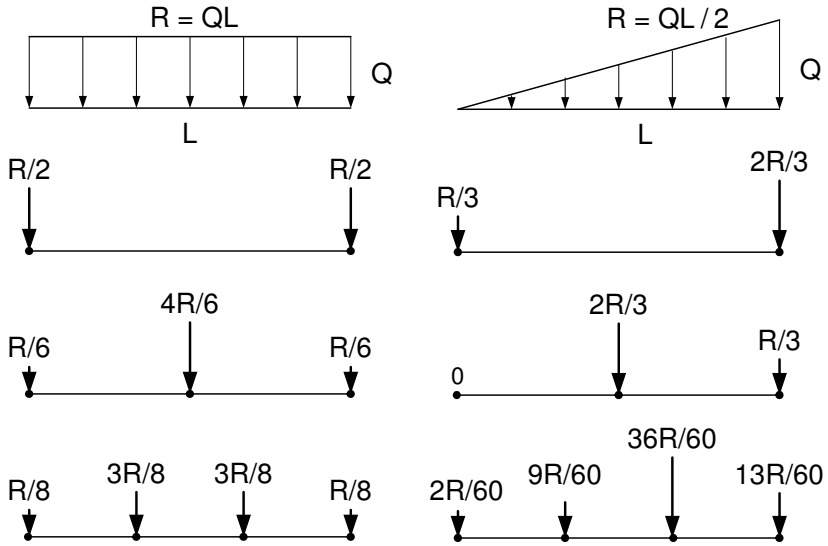

Figure 10.5 *Consistent resultants for a unit source*

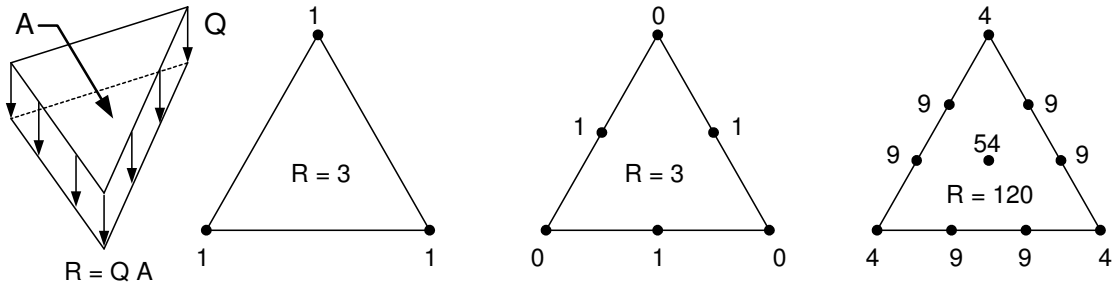

Figure 10.6 *Resultants for a constant source on a triangle*

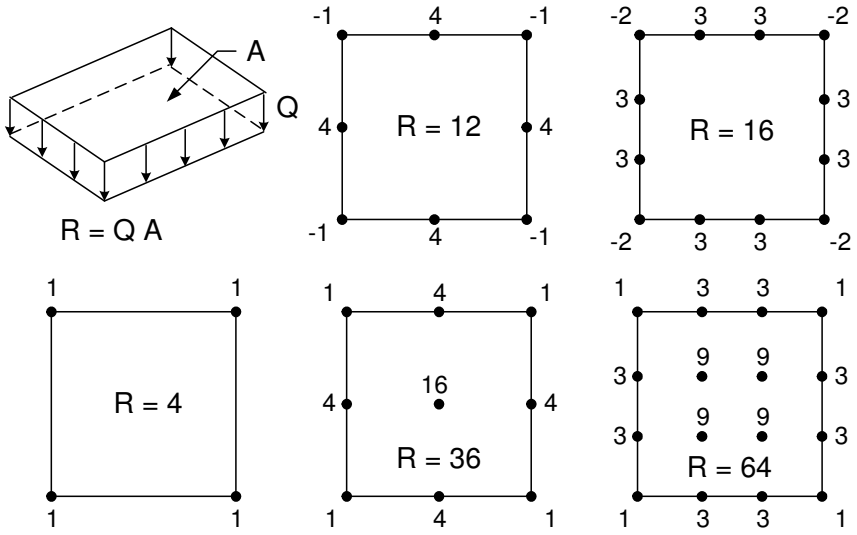

Figure 10.7 *Resultants for a constant source rectangle*

order p then to maintain the theoretical convergence rate n_q should be selected [14] to give accuracy of order $0(h^{2(p-m)+1})$. That is, to integrate polynomial terms of order $(2p - m)$ exactly.

It has long been known that a finite element model gives a stiffness which is too high. Using reduced integration so as to underestimate the element stiffness has been accepted as one way to improve the results. These procedures have been investigated by several authors including Zienkiewicz [14], Zienkiewicz and Hinton [14], Hughes, Cohen and Haroun [8] and Malkus and Hughes [12]. Reduced integration has been especially useful in problems with constraints, such as material incompressibility. A danger of low order integration rules is that *zero energy modes* may arise in an element. That is, the element energy is $\mathbf{D}^{eT}\,\mathbf{S}^{e}\,\mathbf{D}^{e} = 0$ for $\mathbf{D}^{e} \neq 0$. Usually these zero energy modes, $\mathbf{D}^{e}$, are incompatible with the same modes in an adjacent element. Thus, the assembly of elements may have no zero energy modes (except for the standard *rigid body* modes). Cook [3] illustrates that an eigen-analysis of the element square matrix can be used as a check since zero eigenvalues correspond to zero energy modes.

The integrand usually involves derivatives of the function of interest. Many solutions require the post-solution calculation of these derivatives for auxiliary calculations. Thus a related question is which points give the most accurate estimates for those derivatives. These points are often called *optimal points* or Barlow points. Their locations have been derived by Barlow [1, 2] and Moan [13]. The optimal points usually are the common quadrature points. For low order elements the optimal points usually correspond to the minimal integration points. This is indeed fortunate. As discussed in Chapter 1, it is possible in some cases to obtain exact derivative estimates from the optimal points. Barlow considered line elements, quadrilaterals and hexahedra while Moan considered the triangular elements. The points were found by assuming that the p-th order polynomial solution, in a small element, is approximately equal to the $(p + 1)$

order exact polynomial solution. The derivatives of the two forms were equated and the coordinates of points where the identity is satisfied were determined. For triangles the optimal rules are the symmetric rules involving 1, 4, 7, and 13 points. For machines with small word lengths the 4 and 13 point rules may require higher precision due to the negative centroid weights. Generally, all interior point quadrature rules can be used to give more accurate derivative estimates. The derivatives of the interpolation functions are least accurate at the nodes. Later we will show how patch methods can be used to generate much more accurate derivatives at the nodes.

For element formulations involving element constraints, or penalties, it is now considered best to employ selective integration rules [14]. For penalty formulations it is common to have equations of the form $(\mathbf{S}_1 + \alpha \mathbf{S}_2)\,\mathbf{D} = \mathbf{C}$ where the constant $\alpha \to \infty$ in the case where the penalty constraint is exactly satisfied. In the limit as $\alpha \to \infty$ the system degenerates to $\mathbf{S}_2\,\mathbf{D} = 0$, where the solution approaches the trivial result, $\mathbf{D} = \mathbf{0}$. To obtain a non-trivial solution in this limit it is necessary for $\mathbf{S}_2$ to be singular. Therefore, the two contributing element parts $\mathbf{S}_1^e$ and $\mathbf{S}_2^e$ are *selectively* integrated. That is, $\mathbf{S}_2^e$ is under integrated so as to be rank deficient (singular) while $\mathbf{S}_1^e$ is integrated with a rule which renders $\mathbf{S}_1$ non-singular. Typical applications of selective integration were cited above and include problems such as plate bending where the bending contributions are in $\mathbf{S}_1^e$ while the shear contributions are in $\mathbf{S}_2^e$.

10.7 Exercises

1. Explain why in Tables 10.3, 10.4, 10.5, and 10.6 and in Fig. 10.3 the sum of the weights are exactly 1/2, 1/6, 1, 4, and 8, respectively.

2. Assume a constant Jacobian and numerically evaluate the matrices:

$$a)\quad \mathbf{C}^e = \int_{\Omega^e} \mathbf{H}^T\, dx, \quad b)\quad \mathbf{M}^e = \int_{\Omega^e} \mathbf{H}^T\, \mathbf{H}\, dx\,,$$

$$c)\quad \mathbf{S}^e = \int_{\Omega^e} \frac{d\mathbf{H}^T}{dx}\,\frac{d\mathbf{H}}{dx}\, dx\,,\ d)\quad \mathbf{U}^e = \int_{\Omega^e} \mathbf{H}^T\,\frac{d\mathbf{H}}{dx}\, dx.$$

   for: a) a unit right angle triangle, b) a unit square, based on linear interpolation.

3. Confirm the nodal resultants of Fig. 10.4.

4. Confirm the nodal resultants of Fig. 10.5.

5. Confirm the nodal resultants of Fig. 10.6.

6. Confirm the nodal resultants of Fig. 10.7.

10.8 Bibliography

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