TECHNICAL REPORT:

$THERMOFLOW$ - System level Thermal-Hydraulics in $ChiTech$

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Rev 1.01

Simulation Engine

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1 Conservation equations

The overall objective is that we want to solve the following four field variables:

- Pressure, P
- Internal energy, u
- Velocity, v
- Density, ρ

In order to do this we apply the conservation equations to the volume shown in Figure 1 below:

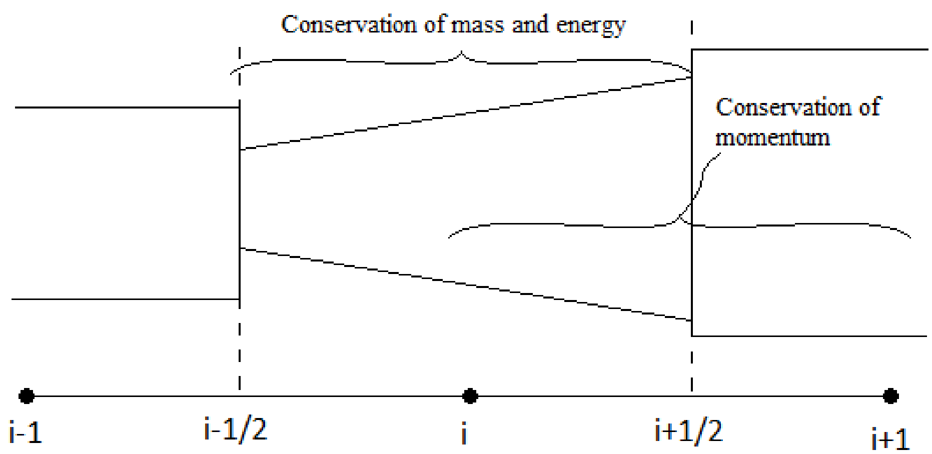

Figure 1: Simple layout of control volumes.

1.1 Conservation of Mass

Even though we will be dealing with incompressible liquids, the liquids can exist at different temperatures and therefore different densities, ρ. The conservation of mass requires as a function of time, t:

$$\frac{d\rho}{dt} = -\frac{1}{V}\int_S \rho.(\vec{v}\cdot\hat{n}).dA$$

And by applying it to the control volume:

$$\frac{d\rho_i}{dt} = \frac{1}{V_i}\left[(\rho.A.v)_{i-\frac{1}{2}} - (\rho.A.v)_{i+\frac{1}{2}}\right] \tag{1}$$

Where:
v = velocity.
V = Volume.
A = Area.
$\hat{n}$ = Surface normal.

1.2 Conservation of Momentum

The change in momentum, $\dot{m}_i.v_i$, in a control volume must balance the momentum of all the in and out flows as well as any forces applied:

$$\frac{d}{dt}\left(\int_{CV}\rho.v.dV\right) + \int_S \rho.v.(\vec{v}\cdot\hat{n}).dA = \sum F_{cv}$$

For a control volume as shown in Figure 1 we have:

$$\begin{aligned}\frac{d(\rho_{i+\frac{1}{2}}.v_{i+\frac{1}{2}})}{dt} + \frac{1}{V_{i+\frac{1}{2}}}\left[(\rho A v^2)_{i+1} - (\rho A v^2)_i\right] = -\frac{1}{V_{i+\frac{1}{2}}}\left[(PA)_{i+1} - (PA)_i\right] + \rho_{i+\frac{1}{2}}.g^* \\ -\frac{1}{2}\frac{1}{V_i}F_{\tau,i} - \frac{1}{2}\frac{1}{V_{i+1}}F_{\tau,i+1}\end{aligned} \tag{2}$$

Where $V_{i+\frac{1}{2}} = \frac{1}{2}V_i + \frac{1}{2}V_{i+1}$ is the distance between control volumes, g^* is the gravitational force component (function of inclination angle between control volume centroids) and F_τ is the wall frictional force that can be calculated from the Darcy Friction Factor, f:

$$F_\tau = \frac{1}{2}f\rho\frac{L}{D}Av^2$$

Therefore:

$$F_{\tau,i} = \frac{1}{4} f_i \rho_i \frac{L_i}{D_i} A_i v_i^2$$
$$F_{\tau,i+1} = \frac{1}{4} f_{i+1} \rho_{i+1} \frac{L_{i+1}}{D_{i+1}} A_{i+1} v_{i+1}^2$$

In these equations L_i and D_i is the length and diameter of the i-th control volume

1.3 Conservation of Energy

The total energy of the system, E, must balance that of the in and out flow including the work performed and the heat transfer into the system:

$$\frac{dE}{dt} = \int_S \rho.(\vec{v} \cdot \hat{n}).E.dA + Q - W - E_{loss}$$

Where:
Q = Heat transfer into the system.
W = Work leaving the system.
E_{loss} = Dissipative energy losses.

The components of energy are internal energy, U, kinetic energy, $\frac{1}{2}mv^2$, and potential energy, mgz:

$$E = m.e = m.(u + \frac{1}{2}v^2 + gz)$$

The heat transfer into the system is normally associated with some heat flux, $\dot{q}$, and the total heat transfer surface, A_s, therefore:

$$Q_i = \dot{q}_i.A_{s,i}$$

The components of work include shaft work, W_{shaft}, and pressure work, $W_{pressure}$. For this case we will consider only pressure work:

$$W_{pressure} = (P.A.v)_{i-\frac{1}{2}} - (P.A.v)_{i+\frac{1}{2}}$$

From here the energy conservation equation becomes:

$$\begin{aligned}\frac{d}{dt}\bigg(m.(u + \frac{1}{2}v^2 + gz)\bigg) &= (\rho A v)_{i-\frac{1}{2}}.(u + \frac{1}{2}v^2 + gz)_{i-\frac{1}{2}} - (\rho A v)_{i+\frac{1}{2}}.(u + \frac{1}{2}v^2 + gz)_{i+\frac{1}{2}} \\ &+ \dot{q}_i.A_{s,i} - \bigg[(P.A.v)_{i-\frac{1}{2}} - (P.A.v)_{i+\frac{1}{2}}\bigg] - E_{loss}\end{aligned}$$

Dividing by the volume we get:

$$\begin{aligned}\frac{d}{dt}\left(\rho.(u+\frac{1}{2}v^2)\right) = \frac{1}{V_i}\Big[(\rho Av)_{i-\frac{1}{2}}.(u+\frac{1}{2}v^2+gz)_{i-\frac{1}{2}} - (\rho Av)_{i+\frac{1}{2}}.(u+\frac{1}{2}v^2+gz)_{i+\frac{1}{2}}\Big] \\ + \frac{A_{s,i}}{V_i}\dot{q}_i - \frac{1}{V_i}\Big[(P.A.v)_{i-\frac{1}{2}} - (P.A.v)_{i+\frac{1}{2}}\Big] - \frac{1}{V_i}E_{loss}\end{aligned} \tag{3}$$

The energy loss term includes both dynamic losses at the junctions and those within the control volume, however, for simplicity we will only consider the loss associated with junction losses for which the pressure loss is given by:

$$\Delta P = \frac{1}{2}K\rho v^2$$

Where K is a dimensionless parameter dependent on the geometry. The energy loss associated with this factor is:

$$E_{loss,i-\frac{1}{2}} = \begin{cases} (\Delta P.A.v)_{i-\frac{1}{2}} = \frac{1}{2}K_{i-\frac{1}{2}}(\rho Av^3)_{i-\frac{1}{2}} & , v_{i-\frac{1}{2}} > 0 \\ 0 & , v_{i-\frac{1}{2}} < 0 \end{cases}$$

Similarly:

$$E_{loss,i+\frac{1}{2}} = \begin{cases} (\Delta P.A.v)_{i+\frac{1}{2}} = \frac{1}{2}K_{i+\frac{1}{2}}(\rho Av^3)_{i+\frac{1}{2}} & , v_{i+\frac{1}{2}} < 0 \\ 0 & , v_{i+\frac{1}{2}} > 0 \end{cases}$$

1.4 Equation of state

In addition to the conservation equations, there is the equation of state. For the incompressible liquids of this simulation we can approximate the equation of state as:

$$\begin{aligned} u &= -0.002053148\ \rho^3 + 5.927524805\ \rho^2 - 5710.176493\ \rho + 1835863.516 \\ \rho &= 7.88656E-06\ T^3 - 0.004477273\ T^2 - 0.059652292\ T + 1001.25303 \end{aligned} \tag{4}$$

2 Numerical solution

The conservation of momentum equations conveniently couples the pressure field and velocities and therefore is used implicitly to determine the pressures at time $n+1$.

2.1 Conservation of Mass - finite difference formulation

The finite difference formulation is as follows:

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}=\frac{1}{V_i}\left[(\rho.A.v)_{i-\frac{1}{2}}^n-(\rho.A.v)_{i+\frac{1}{2}}^n\right]$$

Therefore we get:

$$\rho_i^{n+1}=\rho_i^n+\frac{\Delta t}{V_i}\left[(\rho.A.v)_{i-\frac{1}{2}}^n-(\rho.A.v)_{i+\frac{1}{2}}^n\right] \tag{5}$$

2.2 Conservation of Momentum - finite difference formulation

Because no other future time equation has pressure as a variable we opt to include pressure at time $n+1$ as an implicit variable. We manipulate the original conservation of mass equation:

$$\begin{aligned}\frac{d(\rho_{i+\frac{1}{2}}.v_{i+\frac{1}{2}})}{dt}+\frac{1}{V_{i+\frac{1}{2}}}\left[(\rho^*Av^2)_{i+1}-(\rho^*Av^2)_i\right]=-\frac{1}{V_{i+\frac{1}{2}}}\left[(PA)_{i+1}-(PA)_i\right]+\rho_{i+\frac{1}{2}}.g^*\\ -\frac{1}{2}\frac{1}{V_i}F_{\tau,i}-\frac{1}{2}\frac{1}{V_{i+1}}F_{\tau,i+1}\end{aligned}$$

We use the following time notation:

$$\begin{aligned}\rho_{i+\frac{1}{2}}^{n+1}.v_{i+\frac{1}{2}}^{n+1}-\rho_{i+\frac{1}{2}}^{n}.v_{i+\frac{1}{2}}^{n}+\frac{\Delta t}{V_{i+\frac{1}{2}}}\left[(\rho Av^2)_{i+1}^n-(\rho Av^2)_i^n\right]=-\frac{\Delta t}{V_{i+\frac{1}{2}}}\left[(PA)_{i+1}^{n+1}-(PA)_i^{n+1}\right]+\rho_{i+\frac{1}{2}}^n.g^*\Delta t\\ -\frac{1}{2}\frac{\Delta t}{V_i}F_{\tau,i}^n-\frac{1}{2}\frac{\Delta t}{V_{i+1}}F_{\tau,i+1}^n\end{aligned}$$

And rearrange the terms:

$$\begin{aligned}\rho_{i+\frac{1}{2}}^{n+1}.v_{i+\frac{1}{2}}^{n+1}+\frac{\Delta t}{V_{i+\frac{1}{2}}}\left[(PA)_{i+1}^{n+1}-(PA)_i^{n+1}\right]=\rho_{i+\frac{1}{2}}^{n}.v_{i+\frac{1}{2}}^{n}-\frac{\Delta t}{V_{i+\frac{1}{2}}}\left[(\rho Av^2)_{i+1}^n-(\rho Av^2)_i^n\right]\\ +\rho_{i+\frac{1}{2}}^n.g^*\Delta t-\frac{1}{2}\frac{\Delta t}{V_i}F_{\tau,i}^n-\frac{1}{2}\frac{\Delta t}{V_{i+1}}F_{\tau,i+1}^n\end{aligned} \tag{6}$$

In the equations above we included implicit time instances to pressure in order to semi-implicitly couple the system.

2.3 Conservation of Energy - finite difference formulation

We start with the original conservation of energy equation:

$$\frac{d}{dt}\left(\rho.(u+\frac{1}{2}v^2)\right) = \frac{1}{V_i}\left[(\rho Av)_{i-\frac{1}{2}}.(u+\frac{1}{2}v^2+gz)_{i-\frac{1}{2}} - (\rho Av)_{i+\frac{1}{2}}.(u+\frac{1}{2}v^2+gz)_{i+\frac{1}{2}}\right] + \frac{A_{s,i}}{V_i}\dot{q}_i - \frac{1}{V_i}\left[(P.A.v)_{i-\frac{1}{2}} - (P.A.v)_{i+\frac{1}{2}}\right] - \frac{1}{V_i}E_{loss}$$

We now discretize the left side and set the right side to correspond to time n.

$$\begin{aligned}\frac{(\rho_i^{n+1}.u_i^{n+1} - \rho_i^n.u_i^n) + (\frac{1}{2}.\rho_i^{n+1}.(v_i^{n+1})^2 - \frac{1}{2}.\rho_i^n.(v_i^n)^2)}{\Delta t} &= \frac{1}{V_i}\Big[(\rho Av)^n_{i-\frac{1}{2}}.(u+\frac{1}{2}v^2+gz)^n_{i-\frac{1}{2}} \\ &- (\rho Av)^n_{i+\frac{1}{2}}.(u+\frac{1}{2}v^2+gz)^n_{i+\frac{1}{2}}\Big] \\ &+ \frac{A_{s,i}}{V_i}\dot{q}_i^n - \frac{1}{V_i}\left[(P.A.v)^n_{i-\frac{1}{2}} - (P.A.v)^n_{i+\frac{1}{2}}\right] \\ &- \frac{1}{V_i}E^n_{loss}\end{aligned}$$

And finally:

$$\begin{aligned}\rho_i^{n+1}.u_i^{n+1} + \frac{1}{2}.\rho_i^{n+1}.(v_i^{n+1})^2 &= \rho_i^n.u_i^n + \frac{1}{2}.\rho_i^n.(v_i^n)^2 + \frac{1}{V_i}\Big[(\rho Av)^n_{i-\frac{1}{2}}.(u+\frac{1}{2}v^2+gz)^n_{i-\frac{1}{2}} \\ &- (\rho Av)^n_{i+\frac{1}{2}}.(u+\frac{1}{2}v^2+gz)^n_{i+\frac{1}{2}}\Big] \\ &+ \frac{A_{s,i}}{V_i}\dot{q}_i^n - \frac{1}{V_i}\left[(P.A.v)^n_{i-\frac{1}{2}} - (P.A.v)^n_{i+\frac{1}{2}}\right] \\ &- \frac{1}{V_i}E^n_{loss}\end{aligned} \tag{7}$$

2.4 Unknowns in the available equations

The "new" time values that require solving are found in the following left hand terms:

$$\begin{aligned}&\rho_i^{n+1} \\ &\rho^{n+1}_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}} + \frac{\Delta t}{V_{i+\frac{1}{2}}}\left[(PA)^{n+1}_{i+1} - (PA)^{n+1}_i\right] \\ &\rho_i^{n+1}.u_i^{n+1} + \frac{1}{2}.\rho_i^{n+1}.(v_i^{n+1})^2\end{aligned}$$

We change $\rho^{n+1}_{i+\frac{1}{2}}$ according to:

$$\rho_{i+\frac{1}{2}}^{n+1} = \begin{cases} \rho_i^{n+1} & , v_{i+\frac{1}{2}} > 0 \\ \rho_{i+1}^{n+1} & , v_{i+\frac{1}{2}} < 0 \end{cases}$$

We then change v_i^{n+1} according to $v_{i+\frac{1}{2}}^{n+1}\frac{A_{i+\frac{1}{2}}}{A_i}$. A "thing" that will be tested later but will allow us to determine $v_{i+\frac{1}{2}}$ from equation 7.

2.5 Semi-implicit approach

From equation 5 we can solve for ρ_i^{n+1} and from equation 4 we can solve for u_i^{n+1}. From equation 7 we can solve for $v_{i+\frac{1}{2}}^{n+1}$. Now the only variables left unsolved is the pressures P_{i+1}^{n+1} and P_i^{n+1}.

Hence we are left with the conservation of momentum equation with the unknowns P_i and P_{i+1}. Therefore, in a single time step where we have I amount of control volumes as shown in Figure 2 below, we will have I unknowns and I amount of momentum equations.

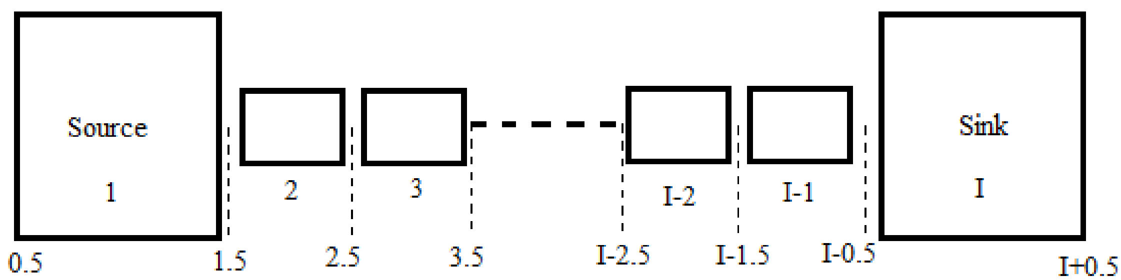

Figure 2: Simple layout of multiple control volumes.

2.6 Solution algorithm

Step 1

Load all control volumes. Input parameters:

- Area, A
- Length, L
- Pressure, P
- Temperature, T
- Elevation change, Δz
- Wall roughness, ϵ

Determine density, ρ_i, from:

$$\rho = 7.88656E - 06\ T^3 - 0.004477273\ T^2 - 0.059652292\ T + 1001.25303$$

And internal energy from:

$$u = -0.002053148\ \rho^3 + 5.927524805\ \rho^2 - 5710.176493\ \rho + 1835863.516$$

Step 2

Load all junctions. Input parameters:

- Junction velocity, v
- Area, A
- Loss factor, K

Determine all junction z values using control volume Δz values.

Step 3

Determine $\rho^n_{i-\frac{1}{2}}$ and $\rho^n_{i+\frac{1}{2}}$ as follows:

$$\rho^n_{i-\frac{1}{2}} = \begin{cases} \rho^n_{i-1} & , v_{i-\frac{1}{2}} > 0 \\ \rho^n_i & , v_{i-\frac{1}{2}} < 0 \end{cases}$$

$$\rho^n_{i+\frac{1}{2}} = \begin{cases} \rho^n_{i} & , v_{i+\frac{1}{2}} > 0 \\ \rho^n_{i+1} & , v_{i+\frac{1}{2}} < 0 \end{cases}$$

And similarly for $u^n_{i-\frac{1}{2}}$ and $u_{i+\frac{1}{2}}$ from:

$$u^n_{i-\frac{1}{2}} = \begin{cases} u^n_{i-1} & , v_{i-\frac{1}{2}} > 0 \\ u^n_i & , v_{i-\frac{1}{2}} < 0 \end{cases}$$

$$u^n_{i+\frac{1}{2}} = \begin{cases} u^n_{i} & , v_{i+\frac{1}{2}} > 0 \\ u^n_{i+1} & , v_{i+\frac{1}{2}} < 0 \end{cases}$$

Step 4

Run through each control volume and apply equation 5:

$$\rho_i^{n+1} = \rho_i^n + \frac{\Delta t}{V_i}\Big[(\rho.A.v)^n_{i-\frac{1}{2}} - (\rho.A.v)^n_{i+\frac{1}{2}}\Big]$$

Step 5

Determine internal energy, u^{n+1}, using density, ρ^{n+1} from equation 4:

$$u = -0.002053148\ \rho^3 + 5.927524805\ \rho^2 - 5710.176493\ \rho + 1835863.516$$

Step 6

Determine the elements of equation 7:

$$\begin{aligned} \rho_i^{n+1}.u_i^{n+1} + \frac{1}{2}.\rho_i^{n+1}.(v^{n+1}_{i+\frac{1}{2}}\frac{A_{i+\frac{1}{2}}}{A_i})^2 = \rho_i^n.u_i^n + \frac{1}{2}.\rho_i^n.(v^n_{i+\frac{1}{2}}\frac{A_{i+\frac{1}{2}}}{A_i})^2 &+ \frac{1}{V_i}\Big[(\rho A v)^n_{i-\frac{1}{2}}.(u + \frac{1}{2}v^2 + gz)^n_{i-\frac{1}{2}} \\ &- (\rho A v)^n_{i+\frac{1}{2}}.(u + \frac{1}{2}v^2 + gz)^n_{i+\frac{1}{2}}\Big] \\ &+ \frac{A_{s,i}}{V_i}\dot{q}_i^n - \frac{1}{V_i}\Big[(P.A.v)^n_{i-\frac{1}{2}} - (P.A.v)^n_{i+\frac{1}{2}}\Big] \\ &- \frac{1}{V_i}E^n_{loss} \end{aligned}$$

In elemental form we want to determine $v^{n+1}_{i+\frac{1}{2}}$ from:

$$J + M(v^{n+1}_{i+\frac{1}{2}}N)^2 = B + \frac{1}{V_i}\Big[C - D\Big] + E - \frac{1}{V_i}\Big[F - G\Big] - H$$

Where:

$$B = \rho_i^n.u_i^n + \frac{1}{2}.\rho_i^n.(v_{i+\frac{1}{2}}^n \frac{A_{i+\frac{1}{2}}}{A_i})^2$$
$$C = (\rho A v)_{i-\frac{1}{2}}^n.(u + \frac{1}{2}v^2 + gz)_{i-\frac{1}{2}}^n$$
$$D = (\rho A v)_{i+\frac{1}{2}}^n.(u + \frac{1}{2}v^2 + gz)_{i+\frac{1}{2}}^n$$
$$E = \frac{A_{s,i}}{V_i}\dot{q}_i^n \quad F = P_i^n(A.v)_{i-\frac{1}{2}}^n \quad G = P_i^n(A.v)_{i+\frac{1}{2}}^n$$
$$H = \frac{1}{V_i}E_{loss}^n \quad J = \rho_i^{n+1}.u_i^{n+1} \quad M = \frac{1}{2}.\rho_i^{n+1} \quad N = \frac{A_{i+\frac{1}{2}}}{A_i}$$

The $P_{i-\frac{1}{2}}$ terms in these elements were exchanged with P_i terms. This is a valid approximation since $P_i - P_{i-\frac{1}{2}} \approx P_{i+\frac{1}{2}} - P_i$. The E_{loss}^n term needs to be calculated as follows:

$$E_{loss}^n = E_{loss,i+\frac{1}{2}}^n + E_{loss,i-\frac{1}{2}}^n$$

Where:

$$E_{loss,i-\frac{1}{2}} = \begin{cases} (\Delta P.A.v)_{i-\frac{1}{2}} = \frac{1}{2}K_{i-\frac{1}{2}}(\rho A v^3)_{i-\frac{1}{2}}^n & , v_{i-\frac{1}{2}} > 0 \\ 0 & , v_{i-\frac{1}{2}} < 0 \end{cases}$$

And:

$$E_{loss,i+\frac{1}{2}} = \begin{cases} (\Delta P.A.v)_{i+\frac{1}{2}} = \frac{1}{2}K_{i+\frac{1}{2}}(\rho A v^3)_{i+\frac{1}{2}}^n & , v_{i+\frac{1}{2}} < 0 \\ 0 & , v_{i+\frac{1}{2}} > 0 \end{cases}$$

Then calculate:

$$B_1 = B + \frac{1}{V_i}\Big[C - D\Big] + E - \frac{1}{V_i}\Big[F - G\Big] - H - J$$
$$B_2 = \sqrt{\frac{B_1}{M}}$$

And finally:

$$v_{i+\frac{1}{2}}^{n+1} = \frac{B_2}{N}$$

Step 6

Preparing for step 7 we set:

$$\rho_{i+\frac{1}{2}}^{n+1} = \begin{cases} \rho_i^{n+1} & , v_{i+\frac{1}{2}}^{n+1} > 0 \\ \rho_{i+1}^{n+1} & , v_{i+\frac{1}{2}}^{n+1} < 0 \end{cases}$$

And we calculate the arithmetic average volume velocity:

$$v_i^n = \frac{(Av)_{i-\frac{1}{2}} + (Av)_{i+\frac{1}{2}}}{2.A_i}$$

Step 7

Determine the elements of equation 6:

$$\rho^{n+1}_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}} + \frac{\Delta t}{V_{i+\frac{1}{2}}}\Big[(PA)^{n+1}_{i+1} - (PA)^{n+1}_{i}\Big] = \rho^{n}_{i+\frac{1}{2}}.v^{n}_{i+\frac{1}{2}} - \frac{\Delta t}{V_{i+\frac{1}{2}}}\Big[(\rho A v^2)^{n}_{i+1} - (\rho A v^2)^{n}_{i}\Big]$$
$$+ \rho^{n}_{i+\frac{1}{2}}.g^*\Delta t - \frac{1}{2}\frac{\Delta t}{V_i}F^{n}_{\tau,i} - \frac{1}{2}\frac{\Delta t}{V_{i+1}}F^{n}_{\tau,i+1}$$

In elemental form:

$$H + \frac{\Delta t}{V_{i+\frac{1}{2}}}\Big[(PA)^{n+1}_{i+1} - (PA)^{n+1}_{i}\Big] = B - \frac{\Delta t}{V_{i+\frac{1}{2}}}\Big[C - D\Big] + E - F - G$$

Where:

$$B = \rho^{n}_{i+\frac{1}{2}}.v^{n}_{i+\frac{1}{2}} \quad C = (\rho A v^2)^{n}_{i+1} \quad D = (\rho A v^2)^{n}_{i} \quad E = \rho^{n}_{i+\frac{1}{2}}.g^*\Delta t$$
$$F = \frac{1}{2}\frac{\Delta t}{V_i}F^{n}_{\tau,i} \quad G = \frac{1}{2}\frac{\Delta t}{V_{i+1}}F^{n}_{\tau,i+1} \quad H = \rho^{n+1}_{i+\frac{1}{2}}.v^{n+1}_{i+\frac{1}{2}}$$

In order to calculate $F^{n}_{\tau,i}$ and $F^{n}_{\tau,i+1}$ we need to apply the following:

$$F^{n}_{\tau,i} = \frac{1}{4}f^{n}_{i}\rho^{n}_{i}\frac{L^{n}_{i}}{D^{n}_{i}}A^{n}_{i}(v^{n}_{i})^2$$
$$F^{n}_{\tau,i+1} = \frac{1}{4}f^{n}_{i+1}\rho^{n}_{i+1}\frac{L^{n}_{i+1}}{D^{n}_{i+1}}A^{n}_{i+1}(v^{n}_{i+1})^2$$

Where f^{n}_{i} and f^{n}_{i+1} are calculated from the transcendental equation:

$$\frac{1}{\sqrt{f}} = -2\,\log\left(\frac{\epsilon}{3.7D_h} + \frac{2.51}{Re\sqrt{f}}\right)$$

Where Re is the dimensionless Reynold's number defined as:

$$Re = \frac{\rho^{n}_{i}v^{n}_{i}D_h}{\mu^{n}_{i}}$$

Now calculate the equation in its final form calculate:

$$A_{1,i} = \frac{\Delta t A_i}{V_{i+\frac{1}{2}}}$$
$$A_{2,i} = \frac{\Delta t A_{i+1}}{V_{i+\frac{1}{2}}}$$
$$B_i = B - \frac{\Delta t}{V_{i+\frac{1}{2}}}\Big[C - D\Big] + E - F - G - H$$

Now one can construct a linear system $Ax = b$ with elements:

$$A_{j,i} = \begin{cases} A_{1,j} & , i = j \\ A_{2,j} & , i = j+1 \\ 0 & , i < j \; and \; i > j+1 \end{cases}$$

$$x = [P_1 \; P_2 \; \dots \; P_{i-1} \; P_i]^T$$

$$b = [b_1 \; b_2 \; \dots \; b_{i-1} \; b_i]^T, \quad b_j = B_i$$

This system is a diagonal system of unknowns and can be solved by any suitable solver.

References

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