EE4150 : Control Systems Laboratory

INDIAN INSTITUTE
OF TECHNOLOGY
PALAKKAD

Post Lab Report

Experiment 1 :

Time domain and frequency domain analysis of a second order system

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Problem 1

Consider the following spring-mass-damper system with parameters m = 1kg, Spring constant, k=1N/m. The damping coefficient b is a variable parameter. F is the external applied force and x the displacement.

1. **Derive the transfer function model of the given mechanical system.**

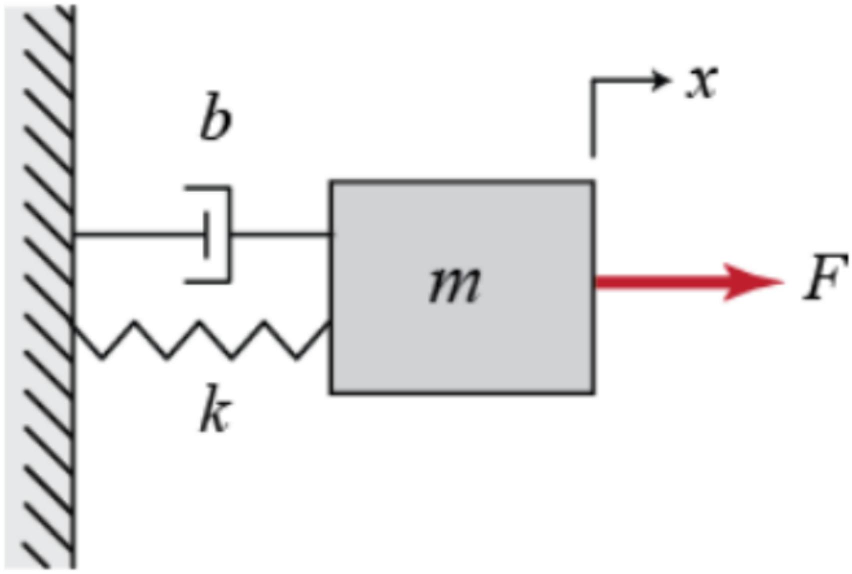


Figure 1: Free Body Diagram

In the free body diagram, the difference between the external forces and the net sum of internal forces gives the net force experienced by the body. The external force is $F(t)$, and the internal opposing forces are the spring force $F_{\text{Spring}} = kx$ and the damping force $F_{\text{Damping}} = bv$. Therefore, the body experiences a net force given by:

$$F(t) - (F_{\text{Spring}} + F_{\text{Damping}}) = F_{\text{net}}(t)$$

$$F(t) - (kx + bv) = ma$$

$$F(t) = ma + bv + kx$$

Here, the output is $x(t)$ and the input is $F(t)$. Hence, writing the equation in terms of the output and input, we get:

$$F(t) = m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + +kx(t)$$

Doing Laplace transform we'll get

$$F(s) = ms^2X(s) + bsX(s) + kX(s)$$

$$F(s) = (ms^2 + bs + k)X(s)$$

Hence the Transfer Function is,

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

Hence for the given values m = 1kg and k=1N/m, the transfer function becomes

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + bs + 1}$$

2. **Find the value of damping coefficient b such that the system is under-damped. Plot the pole locations on Matlab editor and comment on the stability. What is the value of the damping ratio with the choice of b?**

The transfer function of a typical second-order system is given by:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

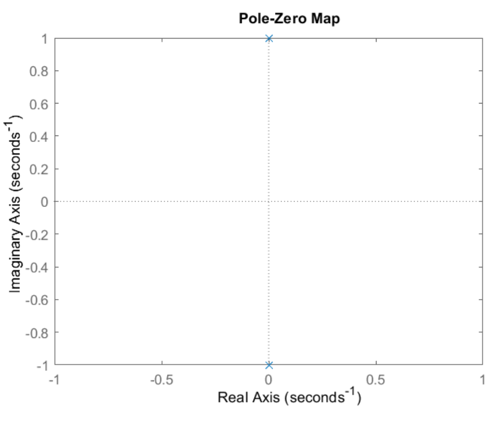
On comparing coefficients, we'll get $\omega_n = 1$ and $\zeta = \frac{b}{2}$

For a system to be under-damped it's damping ratio (ζ) should be between 0 and 1

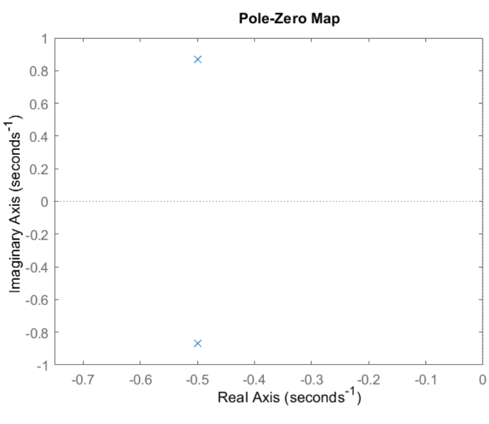
$$\zeta = \frac{b}{2} \in (0, 1)$$

$$b \in (0, 2)$$

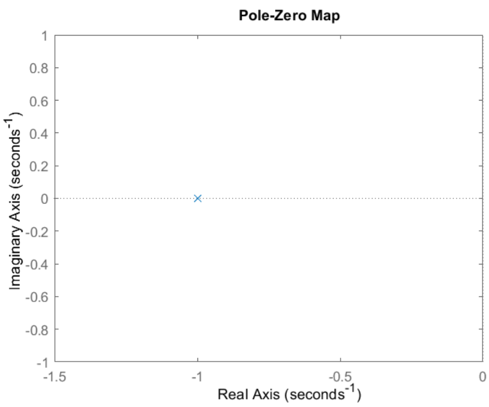
From the pole zero map, if both the poles of the system lie on the left-hand plane, the system will be stable. Hence the system will be stable for all values of b greater than 0, $b \in (0, \infty)$.



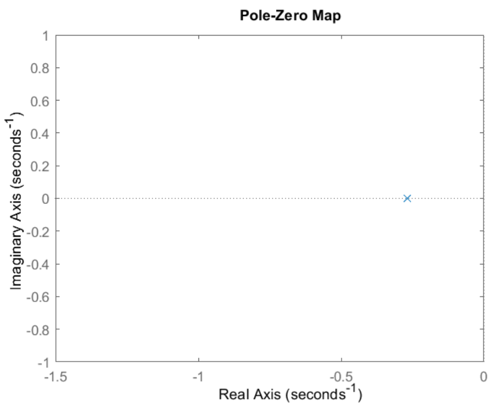
(a) Undamped System with $b = 0$



(b) Under-Damped System with $b = 1$



(a) Critically-Damped System with $b = 2$



(b) Over-Damped System with $b = 4$

Problem 2

Choose damping coefficient, b=4 Ns/m.

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + 4s + 1}$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + 4s + 1}$$

1. **Determine the type of response exhibited by the system in MATLAB under unit step, unit ramp and unit impulse excitation. Plot the system responses.**

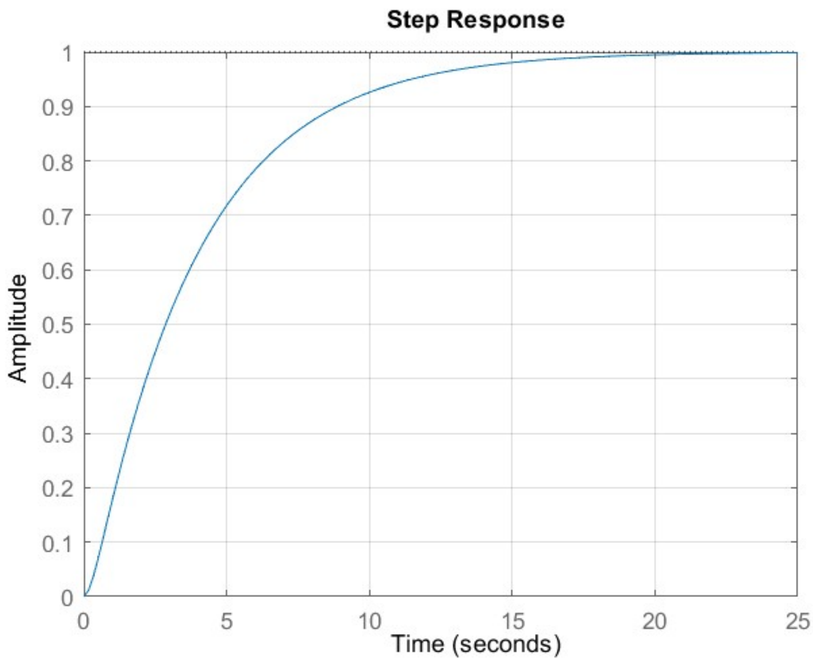


Figure 4: Step Response of the System

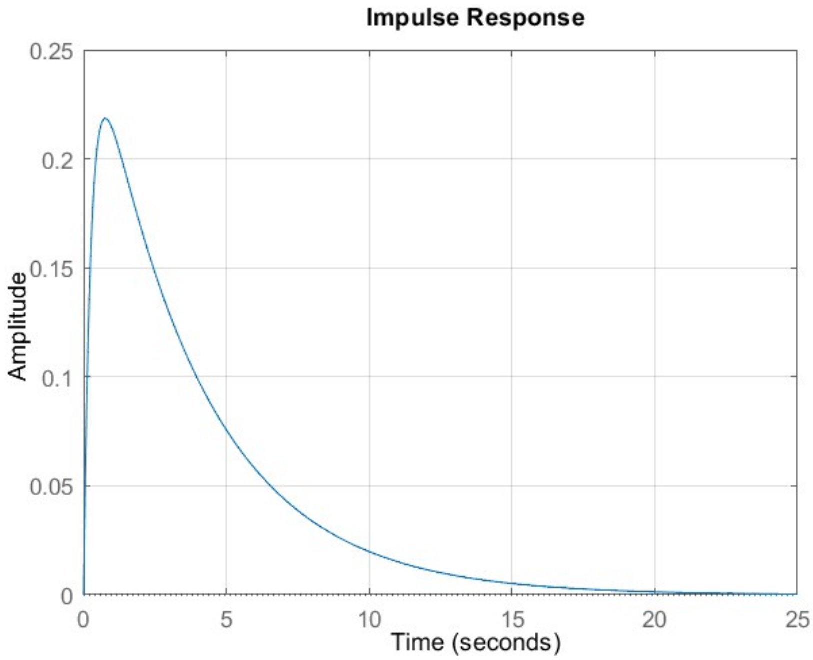


Figure 5: Impulse Response of the System

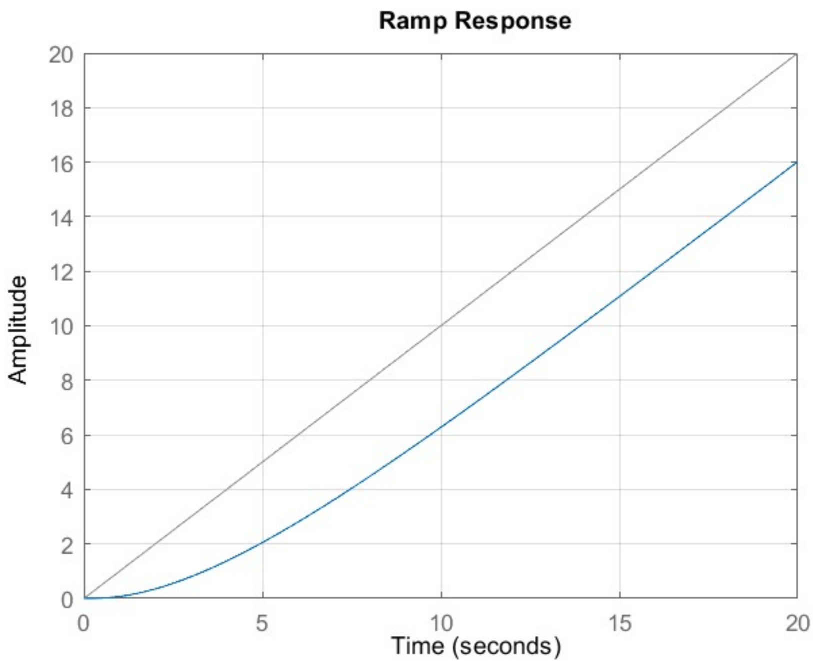


Figure 6: Ramp Response of the System

2. **Determine the steady-state error of the system subjected to these inputs using simulation and tabulate the results**

   For this system the error is given by

$$E(t) = F(t) - X(t)$$

Appling Laplace transform

$$E(s) = F(s) - X(s)$$

$$E(s) = F(s) - F(s).G(s)$$

$$E(s) = F(s).(1 - G(s))$$

Using final value theorem

$$\lim_{t\to\infty} E(t) = \lim_{s\to 0} sE(s)$$

$$E_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sF(s).(1 - G(s))$$

For unit step excitation

$$E_{ss} = \lim_{s\to 0} sR(s)[1 - G(s)] = \lim_{s\to 0} s \cdot \frac{1}{s}\left[1 - \frac{1}{s^2 + 4s + 1}\right]$$

$$E_{ss} = \lim_{s\to 0}\left[\frac{s^2 + 4s}{s^2 + 4s + 1}\right] = 0$$

For unit impulse excitation

$$E_{ss} = \lim_{s\to 0} sR(s)[1 - G(s)] = \lim_{s\to 0} s \cdot 1\left[1 - \frac{1}{s^2 + 4s + 1}\right]$$

$$E_{ss} = \lim_{s\to 0}\left[\frac{s^3 + 4s^2}{s^2 + 4s + 1}\right] = 0$$

For unit ramp excitation

$$E_{ss} = \lim_{s\to 0} sR(s)[1 - G(s)] = \lim_{s\to 0} s \cdot \frac{1}{s^2}\left[1 - \frac{1}{s^2 + 4s + 1}\right]$$

$$E_{ss} = \lim_{s\to 0} \frac{1}{s}\left[\frac{s^2 + 4s}{s^2 + 4s + 1}\right] = 4$$

Steady state error for different excitation

Type of Excitation	**Theoretical E_{ss}**	**Experimental E_{ss}**
Unit Step	0	0.03
Unit Impulse	0	0.17
Unit Ramp	4	3.98

The theoretical steady-state error is calculated assuming an infinite time horizon, while simulations are typically conducted over a finite time interval. This can lead to discrepancies between theoretical and experimental results.

Generally, the following relationship holds for steady-state error (SSD):

$$SSD_{Ramp} \gg SSD_{Step}, SSD_{Impulse}$$

Due to the continuous increase in error for ramp inputs and the eventual stabilization of step and impulse responses, the steady-state errors for the latter two are often negligible.

3. **Comment on the steady state error of the system subject to unit ramp input, if a second damper element is added in series/parallel to the existing damper.**

   Steady state error for a unit ramp excitation under an arbitrarily chosen damping coefficient:

$$E_{ss} = \lim_{s \to 0} sR(s)[1 - G(s)]$$

$$E_{ss} = \lim_{s \to 0} s \cdot \frac{1}{s^2} \left[1 - \frac{1}{s^2 + bs + 1}\right]$$

$$E_{ss} = \lim_{s \to 0} \frac{1}{s} \left[\frac{s^2 + bs}{s^2 + bs + 1}\right]$$

$$E_{ss} = b$$

   For parallel connection of dampers,

$$b_{eff} = b_1 + b_2$$

   For series connection of dampers,

$$b_{eff} = \frac{b_1 b_2}{b_1 + b_2}$$

   Hence, the steady state error will be higher for the parallel connection of dampers as the effect of b is higher compared to the steady state error of the series connection of dampers.

4. **If the damping coefficient, b of the dashpot is zero, what will be the nature of the system when subjected to unit step input? Plot the response.**

   When the damping coefficient b becomes equal to zero, the output of the system will oscillate at its natural frequency. The system is underdamped and marginally stable.

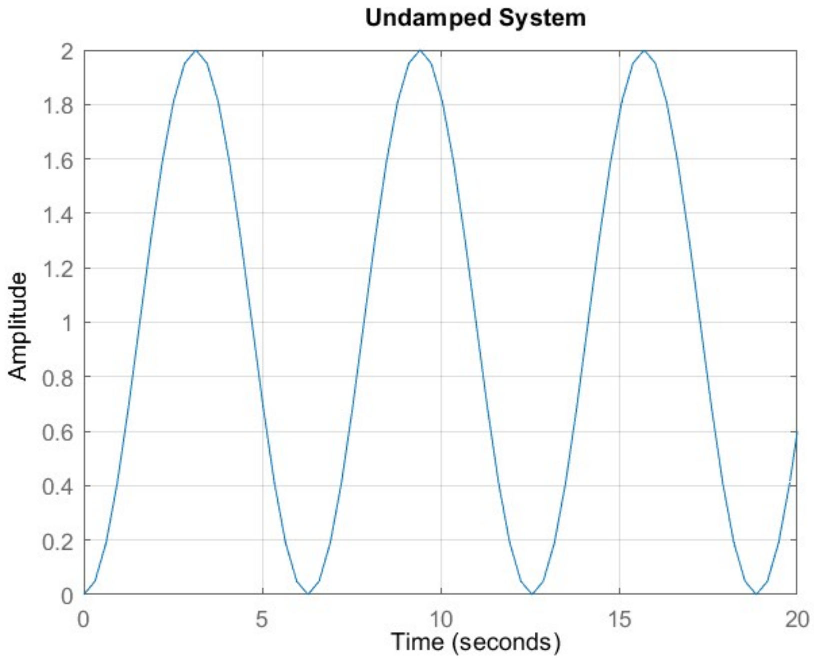


Figure 7: System Response for b = 0

Problem 3

1. **Obtain the unit step response of the mechanical system by choosing damping coefficient b such that the system is underdamped. Compute the time domain parameters manually and in the MATLAB editor. Tabulate the results**

Damping Ratio (ζ)

It is a measure of how rapidly the oscillations will decay. Based on the value of ζ, we can classify the system as underdamped ($\zeta < 1$), overdamped ($\zeta > 1$), and critically damped ($\zeta = 1$). From the general form of a second-order system, we can find ζ as:

$$\zeta = \frac{b}{2\sqrt{mk}}$$

For underdamped condition, ($\zeta < 1$), the value of b should be: $b < 2\sqrt{mk}$. For $b = 1$, the system is underdamped and damping ratio, $\zeta = 0.5$.

Settling Time (t_s)

It is the time required for the output to reach and remain within a given tolerance band following an input stimulus. For 5% tolerance band (0.95 to 1.05):

$$t_s = -\frac{\ln(0.05\sqrt{1-\zeta^2})}{\zeta\omega_n} = -\frac{\ln(0.05\sqrt{1-0.25})}{0.5 \times 1} = 6.27s$$

Rise Time (t_r)

It is the time taken for the output to reach within the final value for the first time.

$$t_r = \frac{\pi - \cos^{-1}\zeta}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi - \cos^{-1} 0.5}{\sqrt{1-0.25}} = 2.41s$$

Delay Time (t_d)

It is the time required for the response to reach half its final value from zero instant.

$$t_d = \frac{1+0.7\zeta}{\omega_n} = 1.35s$$

Time Constant (τ)

It is the time taken for the step response to rise up to 63% of its final value.

$$\tau = \frac{1}{\zeta\omega_n} = \frac{1}{0.5} = 2s$$

Natural Frequency (ω_n)

It is the frequency at which the system tends to oscillate in the absence of any driving force.

$$\omega_n = \sqrt{\frac{k}{m}} = 1$$

Percentage Overshoot(M_p)

It is the maximum amount by which the output exceeds the steady-state value. For step input maximum overshoot percentage (M_p) can be calculated as:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = 16.30\%$$

Table 1: Calculation Result

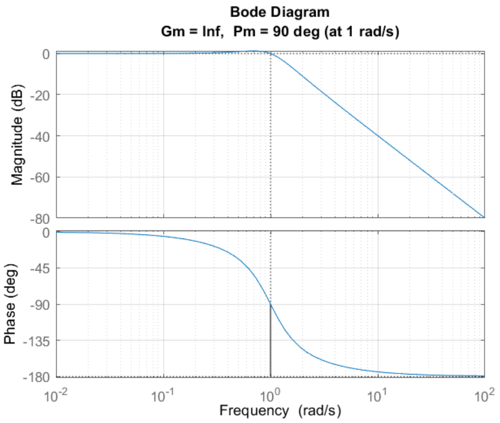
$b(Ns/m)$	Type of Response	ζ	t_s	t_r	t_d	τ	ω_n	M_p
1	Underdamped	0.5	6.27	2.41	1.35	2	1	16.3

Table 2: Simulation Result

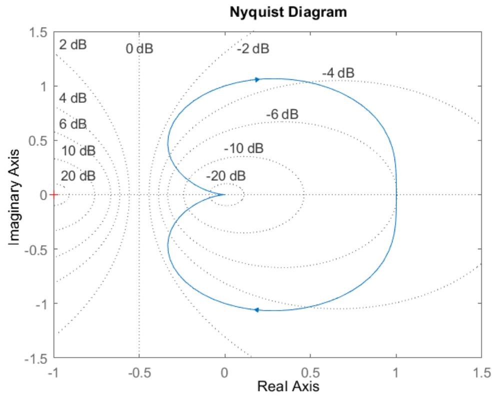
$b(Ns/m)$	Type of Response	ζ	t_s	t_r	t_d	τ	ω_n	M_p
1	Underdamped	0.5	8.0759	1.63	1.29	2	1	16.29

2. **Choose the value of damping coefficient, b such that the system exhibits underdamped, critically damped and overdamped responses.**

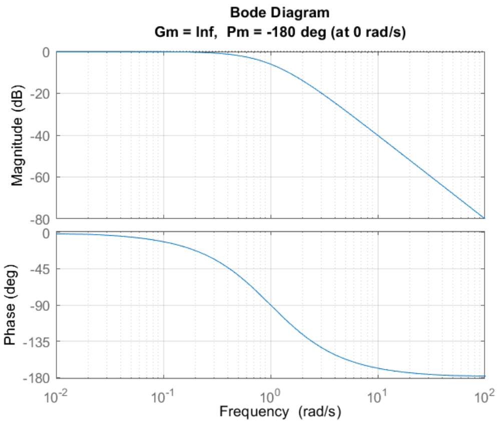
Plot the Bode and Nyquist diagrams.



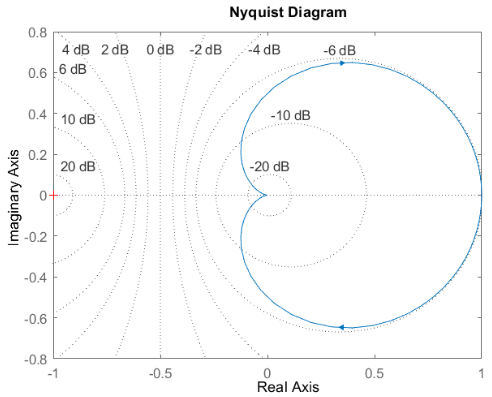
(a) UnderDamped System's Bode Plot



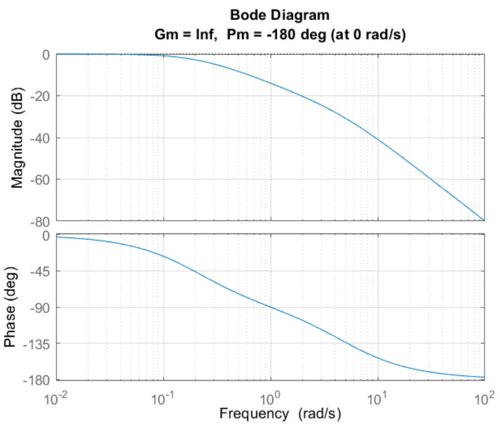
(b) UnderDamped System's Nyquist Plot



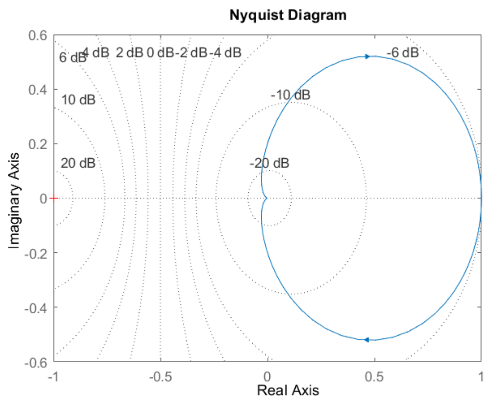
(a) Critically-Damped System's Bode Plot



(b) Critically-Damped System's Nyquist Plot



(a) Over-Damped System's Bode Plot



(b) Over-Damped System's Nyquist Plot

Analyze the stability of the system using Bode and Nyquist plot and tabulate the results

From Bode plot we can check the open loop stability of the system. If both the Gain Margin (GM) and Phase Margin (PM) are positive, then the system is open loop stable.

If both GM and PM are equal to zero, then the system is marginally stable. If either GM or PM is negative, then the system is unstable.

From Nyquist Diagram we can check the closed loop stability of the system. From Nyquist criterion, we have,

$$Z = P - N$$

where:

- Z = No. of Zeros of the characteristic equation $1 + G(s)H(s) = 0$
- P = No. of poles of the system on the Right-hand side of the complex plane
- N = No. of encirclements of -1 in the Nyquist plot

For the system to be stable, Z should be equal to zero.

b (Ns/m)	Type of Response	Comments on Stability (Bode plot)	Comments on Stability (Nyquist Diagram)
1	Underdamped	Open loop stable. Poles are present only in LHS.	Closed loop stable. No -1 encirclements. $P = 0$, $N = 0$, $Z = 0$
2	Critically damped	Open loop stable. Poles are present only in LHS.	Closed loop stable. No -1 encirclements. $P = 0$, $N = 0$, $Z = 0$
4	Overdamped	Open loop stable. Poles are present only in LHS.	Closed loop stable. No -1 encirclements. $P = 0$, $N = 0$, $Z = 0$

Show the frequency response parameters (Gain margin, phase margin, Gain crossover frequency, phase crossover frequency) on the Bode plot. Compute resonance peak, bandwidth, and resonance frequency manually

Gain Margin (GM):
It is the amount of gain that needs to be increased at the frequency where the phase is $-180°$ (ω_{pc}), to make the system unstable. The greater the GM, the greater the stability of the system. From the Bode plot, we can find GM as the vertical difference between the magnitude curve and 0dB at the frequency where the Bode phase plot is at $-180°$.

Phase Margin (PM):
It is the amount of phase that needs to be increased at the frequency where the gain is 0dB (ω_{gc}), to make the system unstable. The greater the PM, the greater the stability. In the Bode plot, PM is the vertical difference between the phase curve and $-180°$ at the frequency where the gain is 0dB.

Gain Cross Over Frequency (ω_{gc})
It is the frequency at which the gain of the system is equal to 0dB.

Phase Cross Over Frequency (ω_{pc})
It is the frequency at which the phase of the system is equal to $-180°$.

Resonance Peak (M_r)
It is the peak value of the magnitude of the transfer function.

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

For an underdamped system ($b = 1$, $\zeta = 0.5$):

$$M_r = \frac{1}{2 \times 0.5 \times \sqrt{1-0.25}} = 1.154$$

For critically damped and overdamped systems, there are no oscillations. The maximum value of the magnitude of the transfer function is 1 (0dB).

Resonance Frequency
It is the frequency at which the magnitude of the transfer function reaches its peak value.

$$\omega_r = \omega_n\sqrt{1-\zeta^2}$$

For an underdamped system, the resonance frequency is:

$$\omega_r = \sqrt{1-0.25} = 0.866 \text{ rad/s}$$

For critically damped and overdamped systems, the magnitude of the transfer function is at its peak value at 0 rad/s.

Bandwidth (BW)
It is the frequency range where the magnitude of the transfer function decreases to 70.7% (-3dB in log scale) of its initial magnitude at 0 rad/s.

For an underdamped system ($b = 1$, $\zeta = 0.5$):

$$BW = \omega_n\sqrt{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}}$$

$$BW = 1 \times \sqrt{1-2\times 0.5^2+\sqrt{2-4\times 0.5^2+4\times 0.5^4}} = 1.27 \text{ rad/s}$$

For a critically damped system ($b = 2$, $\zeta = 1$):

$$BW = 1 \times \sqrt{1-2+\sqrt{2-4+4}} = 0.6435 \text{ rad/s}$$

For an overdamped system ($b = 4$, $\zeta = 2$):

$$BW = 1 \times \sqrt{1-(2\times 4)+\sqrt{2-16+64}} = 0.2665 \text{ rad/s}$$

b **(Ns/m)**	**Type of Response**	**GM**	**PM**	ω_{gc}	ω_{pc}	**BW (rad/s)**	M_r	ω_n **(rad/s)**
1	Under Damped	∞	$90°$	∞	1	1.26	1.154	0.866
2	Critically damped	∞	$-180°$	∞	0	0.6435	-	-
4	Over Damped	∞	$-180°$	∞	0	0.2665	-	-

Comment on the stability of a system with infinite gain margin and -180 phase margin. Justify your answer.

Infinite Gain Margin indicates that the system can withstand any increase in gain without becoming unstable.

The Phase Margin is the additional phase lag required to bring the system to the verge of instability (i.e., where the open-loop phase is $-180°$). A Phase Margin of $-180°$ means that the system is already at the point where any further phase lag will cause the system to become unstable. In fact, this implies that the system's phase has already reached $-180°$ at the gain crossover frequency (ω_{gc}) where the gain is 0 dB.

Since the Phase Margin is $-180°$, the system is at the edge of instability or is already unstable, despite having an infinite gain margin. The phase lag has reduced the margin to zero or below, indicating that any slight disturbance or change in the system could push it into instability.

Therefore, the system is Marginally Stable or Unstable due to the critical phase condition, even though it has an infinite gain margin. Hence we can't properly determine the system's stablity based on the bode plot alone for infinite gain.

Suppose that the system has a dead time. Comment on the relative stability of the system

Dead Time is the amount of time that lapses from the point at which the input signal is specified and passes through the controlled system through to the output signal being output.

- Dead time introduces additional phase lag, which increases with frequency, reducing the phase margin and pushing the system closer to instability.
- Gain crossover frequency and amplitude characteristic of the Bode plot is unaffected by a time delay.
- Instability in feedback control systems results from an imbalance between system dynamic lags and the strength of the corrective action.
- The result is poor performance and sluggish responses.
- Unbounded negative phase angle aggravates stability problems in feedback systems with DT's.
- The time delay increases the phase shift proportional to frequency, with the proportionality constant being equal to the time delay.