EE4150 : Control Systems Laboratory

Post Lab Report

Experiment 2 :

Design of lead compensator using root locus and frequency response methods

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Problem 1

Design a lead compensator using the root locus method for the following system with open loop transfer function:

$$G(s) = \frac{K}{s(s+4)(s+6)}$$

such that the closed loop unity negative feedback system with the compensator will satisfy the following specifications:

- **Settling time (for 2% tolerance band),** $t_s \leq 2s$ **for unit step input**
- **Percentage peak overshoot for unit step input** $\leq 30\%$

Procedure

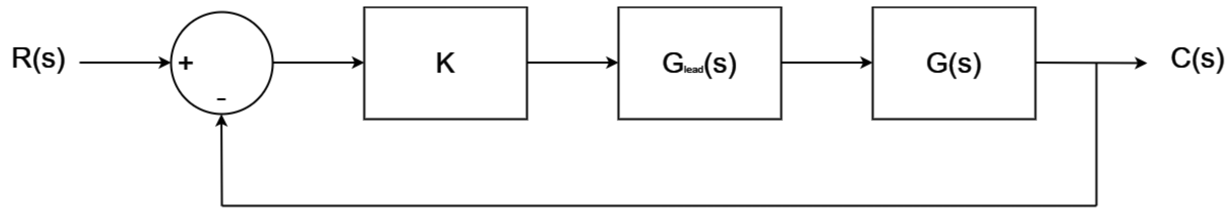

Figure 1: Lead Compensator with Negative Feedback

Dominant Poles of the Uncompensated System

The settling time for a second-order system with a tolerance band of 2% is given by:

$$t_s = \frac{-\ln\left(0.02.\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

Given the requirement $t_s \leq 2$ seconds:

$$\frac{4}{\zeta\omega_n} \leq 2$$

which simplifies to:

$$\zeta\omega_n \geq 2$$

The percentage peak overshoot M_p for a unit step input for a second-order system is given by:

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \cdot 100$$

Given the requirement $M_p\% \leq 30\%$:

$$\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)\cdot 100 \leq 30$$

which simplifies to:

$$\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \leq \ln(0.3)$$

On squaring both sides:

$$\frac{\zeta^2}{1-\zeta^2} \geq \left(\frac{\ln(0.3)}{\pi}\right)^2 = 0.1469$$

Solving for ζ:

$$\zeta \geq \sqrt{\frac{0.1469}{1.1469}} = \sqrt{0.128} = 0.3579$$

Combining this with the requirement $\zeta\omega_n \geq 2$:

$$\omega_n \geq \frac{2}{0.3579} = 5.589$$

The dominant poles of the second-order system can be theoretically found by the quadratic equation:

$$s^2 + \zeta\omega_n s + \omega_n^2 = 0$$

The roots of the above equation, which are the dominant poles of the system, are given by:

$$P_{1,2} = \frac{-\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}}{2}$$

On simplifying:

$$P_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

For specific values:

$$P_{1,2} = -2 \pm 5.216i$$

On writing the poles in Euler form, the angle θ can be obtained as:

$$\tan(\theta) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\cos(\theta) = \zeta$$

With $\zeta \geq 0.3579$:

$$\theta \geq 69.028°$$

Root Locus of Uncompensated system

The open-loop transfer function of the uncompensated system is given by:

$$G(s) = \frac{K}{s(s+4)(s+6)}$$

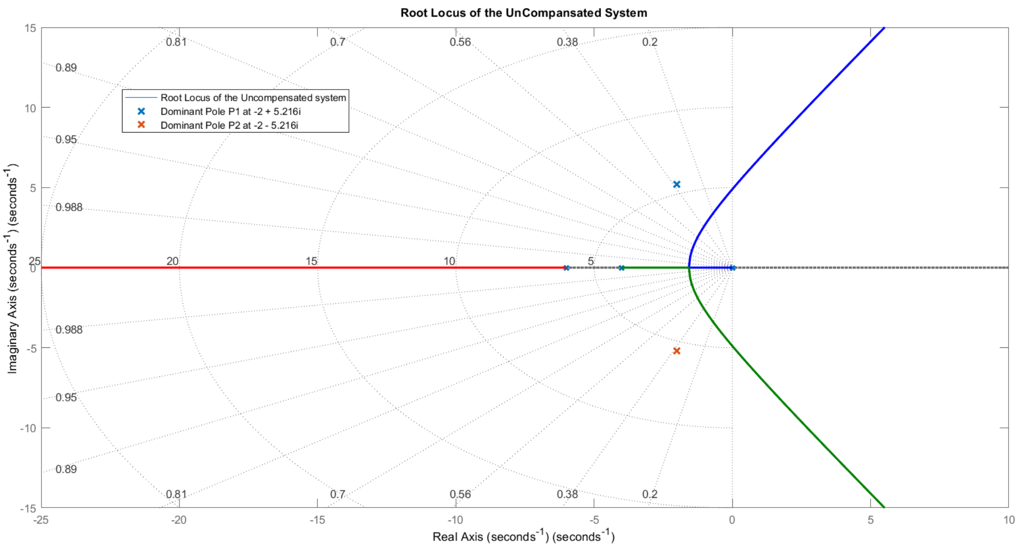

Figure 2: Root Locus of the Uncompensated System

Inference : The root locus plot gives the trajectory of poles of the closed-loop transfer function of a system as the gain varies. From the root locus plot, it is observed that the dominant poles of the uncompensated system will not pass through the root locus for any value of K. Hence, gain adjustment alone cannot give the required closed-loop poles.

Dominant Pole Approximation

We can consider the third order system to be a second order system as from the below figure it is seen that the real part of two of the poles is seven times that of the other pole . Hence we can ignore that pole and consider only the two dominant poles out of the three poles of the system. Thus we approximate the third order system to a scond order system.

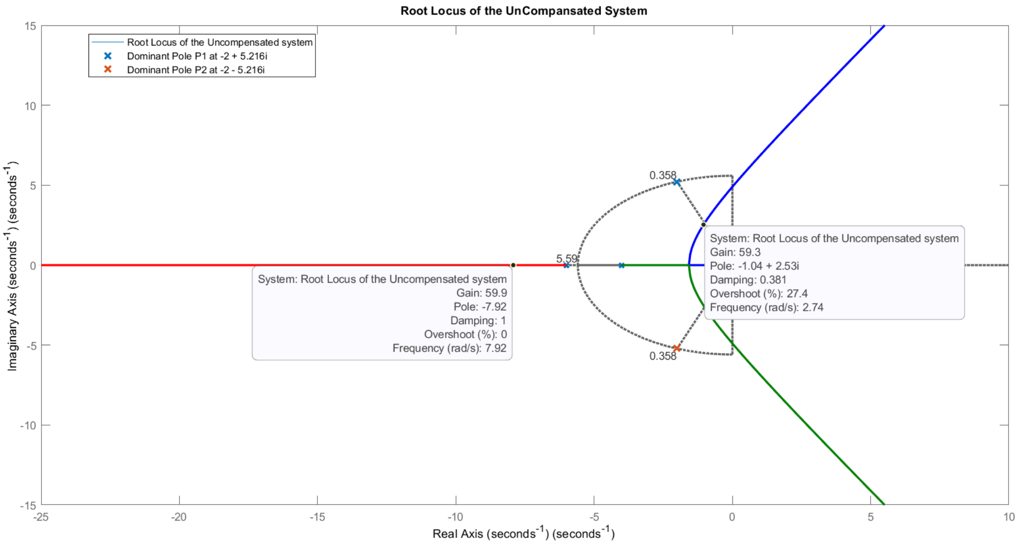

Figure 3: Validity of Dominant Pole Approximation

Phase Angle Calculation

The phase angle contribution by the poles of $G(s)$ with respect to the dominant poles is given by:

$$\theta = \sum (\text{angles made by each zero to dominant pole}) - \sum (\text{angles made by each pole to dominant pole})$$

The zero of the compensator is placed just below the dominant poles, hence the zero is placed at -2.

Poles/Zeros Angle Made to Dominant Pole

Poles/Zeros	Angle made to Dominant Pole
Zero of $G(s) = 0$	$110.9687°$
Pole $= -4$	$69.0313°$
Pole $= -6$	$52.5308°$
Zero of $G_{\text{lead}}(s) = -2$	$90°$

$$\theta = 0° - (110.9687° + 69.0313° + 52.5308°) = -232.5308°$$

Combined Open loop Transfer function of the Compensated system is

$$G_{\text{lead}}(s).G(s)$$

Using the angle criterion, for the point to lie in the root locus:

$$\angle G_{\text{lead}}(s)G(s) = \angle G_{\text{lead}}(s) + \angle G(s) = 180^\circ$$

The phase angle contribution by the pole of the compensator is:

$$\angle \text{Pole}_{\text{compensator}} = \phi$$

The phase angle ϕ is given by:

$$\phi = -180^\circ + 232.5308^\circ - 90^\circ = 37.4692^\circ$$

Zero and Pole of the Compensator

The zero (Z_c) of the compensator is placed at the real part of the dominant pole:

$$Z_c = -2$$

The pole of the lead compensator is determined as follows:

$$\tan\phi = \frac{\text{Imaginary part of dominant pole}}{\text{Real part of compensator pole}}$$

Substituting the values:

$$\tan 37.4692^\circ = \frac{5.2187}{\text{Real part of compensator pole}}$$

$$\text{Real part of compensator pole} = \frac{5.2187}{\tan 37.4692^\circ} = 6.8087$$

$$\text{Real part of compensator pole } (P_c) = -2 - 6.807 = -8.8087$$

Gain of the Compensator Using Magnitude Criterion

The distance between the poles of the compensator and the dominant poles, zeros of the compensator and the dominant poles, and the distance between the poles of the open-loop system with dominant poles are found using MATLAB. The Magnitude Criterion is then applied to find the gain K.

Pole/Zero	**Distance to Dominant Pole**
Zero of $G(s)$	5.5888
Pole of $G(s)$ = -4	5.5888
Pole of $G(s)$ = -6	6.5753
Zero of $G_{\text{lead}}(s)$ = -2	5.2187
Pole of $G_{\text{lead}}(s)$ = -8.8087	8.5787

Using the Magnitude Criterion:

$$K \cdot \frac{\pi_z}{\pi_u \cdot \pi_p} = 1$$

where

- d_z = Product of Distances from zero of the compensator
- π_u = Product of Distances from poles of the uncompensated system
- π_p = Product of Distances from poles of the compensator

$$K = \frac{5.5888 \times 6.5753 \times 5.5888 \times 8.5787}{5.2187} = 337.6101$$

Root locus of the compensated system

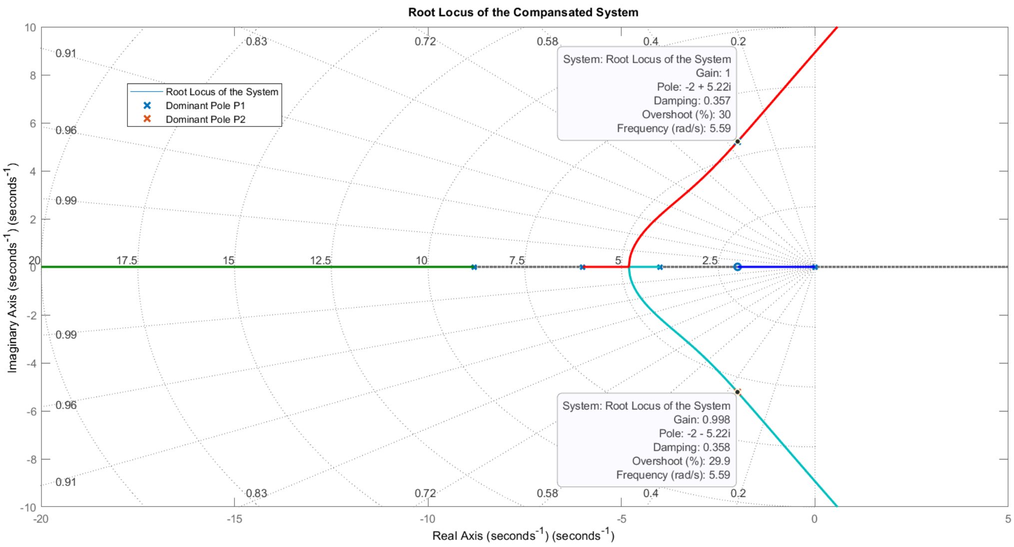

Figure 4: Root Locus of the Compensated System

Compensated System

The compensated system is given by:

$$G_{\text{Compensated System}}(s) = KG_{\text{lead}}(s)G(s)$$

$$G_{\text{Compensated System}}(s) = 337.6101 \frac{(s+2)}{s(s+4)(s+6)(s+8.809)}$$

Results

Compensator Transfer Function

The compensator transfer function is $\dfrac{s+2}{s+8.809}$

Compensated System Transfer Function

The compensated system transfer function is $\dfrac{337.61(s+2)}{s(s+4)(s+6)(s+8.809)}$

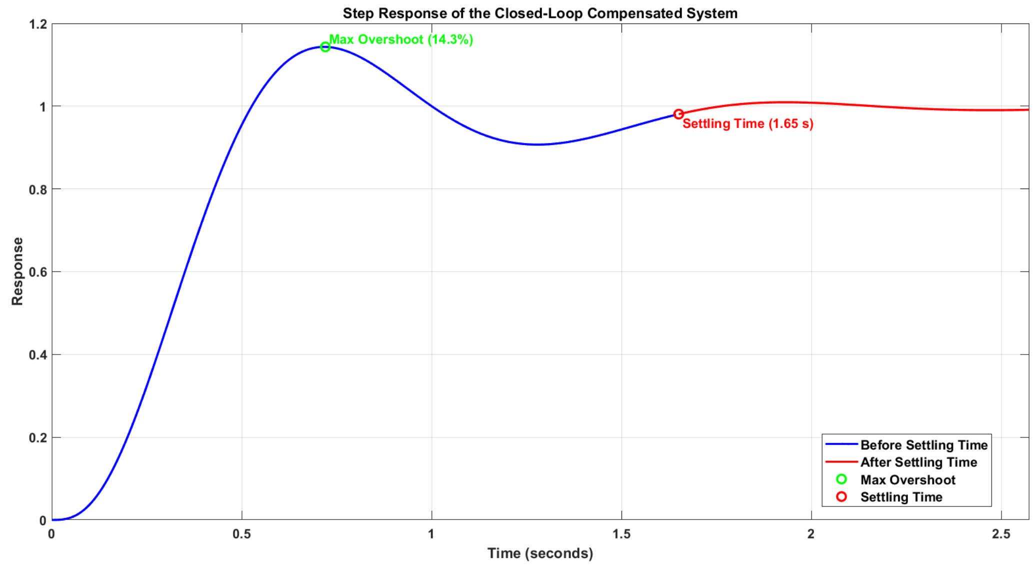

Figure 5: Step Response of the Compensated system

For the compensated system, the settling time is observed to be **1.65 seconds** and the maximum overshoot is **14.3%**.

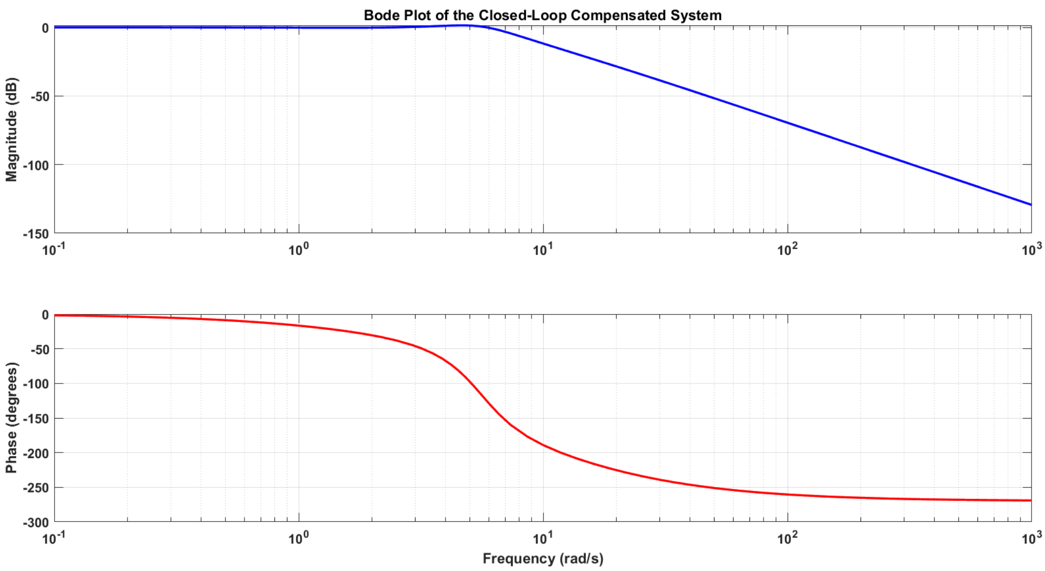

Figure 6: Bode plot of the Compensated system

Inference

From the Root Locus plot of the uncompensated system, it is evident that adjusting the gain alone is insufficient to position the dominant poles on the Root Locus. Therefore, a Phase

Lead Compensator is employed. The compensator's zero is positioned near the real part of the dominant pole, while the poles are determined based on the Angle Criterion. The system gain is adjusted according to the Magnitude Criterion. As a result, the dominant poles are more accurately aligned with the Root Locus.

The compensator's pole is located further to the left compared to its zero. Consequently, the lead compensator causes the asymptotes and Root Locus plot to shift towards the left side of the S-plane. This adjustment improves the system's stability, which is evidenced by the reduced settling time in the steady-state response.

Problem 2

Design a lead compensator using the Bode plot for the following system with open-loop transfer function

$$G(s) = \frac{1}{s(s+1)}$$

The required specifications are:

- Phase margin $\leq 45°$
- Steady state error subjected to unit ramp input $\leq \frac{1}{15}$ units
- Gain crossover frequency ≤ 7.5 rad/s

Assuming the phase lead compensator has the following form:

$$G_c(s) = K\left(\frac{s+z}{s+p}\right)$$

$$G_c(s) = K\left(\frac{z}{p}\right)\left(\frac{s+z+1}{s+p+1}\right)$$

$$K_1 = K\left(\frac{z}{p}\right)$$

The value of K_1 can be found from the steady state error specification. The steady state error should be less than or equal to $\frac{1}{15}$ units.

Steady state error:

$$E_{ss} = \lim_{s\to 0} \frac{sR(s)}{1 + G_c(s)G(s)}$$

$$E_{ss} = \lim_{s\to 0} \frac{s \times \frac{A}{s^2}}{1 + K_1\left(\frac{s+z+1}{s+p+1}\right)\left(\frac{1}{s(s+1)}\right)}$$

$$E_{ss} = \frac{A}{K_1} = \frac{A}{15}$$

$$K_1 = 15$$

Using the value of K_1, the open-loop transfer function of the uncompensated system is

$$K_1 G(s) = \frac{K_1}{s(s+1)}$$

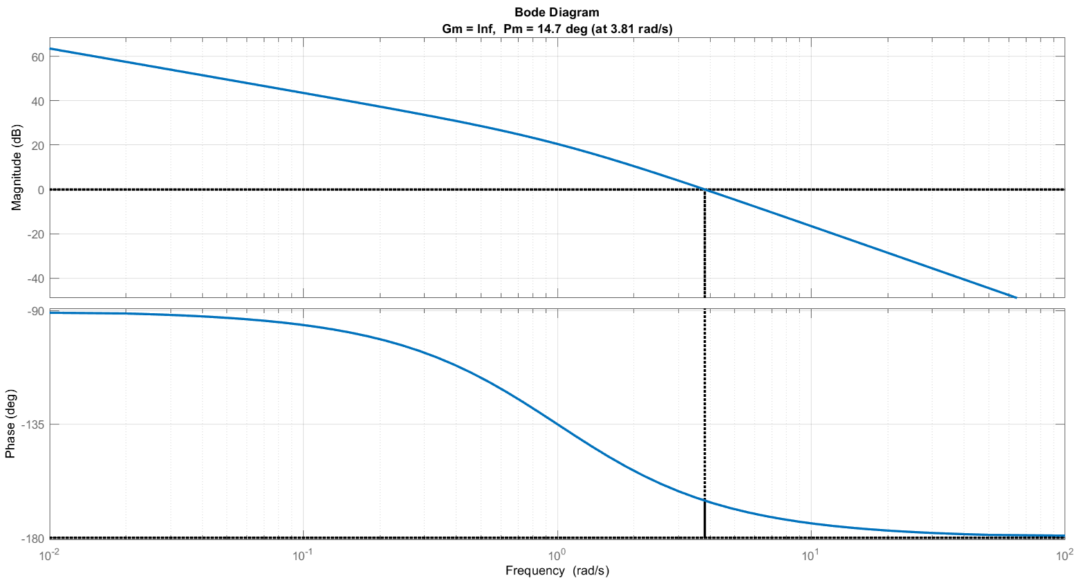

Figure 7: Bode plot of uncompensated system with Gain K1

Phase Lead Compensator Design

For the uncompensated system, we have:

- Phase Margin $= 14.7°$
- $\omega_{gc} = 3.81\,\text{rad/s}$
- Additional Phase required $= 45° - 14.7° = 30.3°$
- Increasing the phase by 10% to account for phase lags, Additional Phase, $\Phi_m = 33.3°$

  Using this additional phase, the value of $\alpha = \frac{z}{p}$ can be determined as follows:

$$\sin(\Phi_m) = \frac{\alpha - 1}{\alpha + 1}$$

$$\alpha = \frac{1 + \sin(\Phi_m)}{1 - \sin(\Phi_m)}$$
$$\alpha = 3.4375$$

This value of α provides a gain of $10\log(\alpha)$ to the system at ω_m. To avoid this, we can place ω_m at the frequency where the magnitude of the uncompensated system $K_1 G(j\omega)$ is equal to $-10\log(\alpha)$. This frequency will be the new ω_m as well as the new 0-dB crossover frequency:

$$-40 = \frac{0 + 10\log(\alpha)}{\log(\omega_{gc}) - \log(\omega_m)}$$

$$40\log(\omega_m) = 10\log(\alpha) + 40\log(\omega_{gc})$$

$$\log(\omega_m) = \frac{1}{4}\log(\alpha) + \log(\omega_{gc})$$

$$\log(\omega_m) = \log(\alpha^{1/4}\omega_{gc})$$

$$\omega_m = \alpha^{1/4}\omega_{gc}$$

$$\omega_m = 5.1864\,\mathrm{rad/s}$$

The poles and zero of the phase lead compensator are related to ω_m as follows:

$$p = \omega_m\sqrt{\alpha} = 9.6157$$

$$z = \frac{p}{\alpha} = 2.7973$$

And the gain K is:

$$K = K_1\alpha = 51.5620$$

The open-loop transfer function of the phase lead compensator is:

$$G_c(s) = K \cdot (\frac{s+z}{s+p})$$

$$G_c(s) = 51.5620 \cdot (\frac{s+2.7973}{s+9.6157})$$

Compensated System with Gain K

The compensated system open-loop transfer function is:

$$G_c(s)G(s) = 51.5620 \cdot (\frac{s+2.7973}{s+9.6157}) \cdot \frac{1}{s(s+1)}$$

$$G_c(s)G(s) = \frac{51.5620(s+2.7973)}{s^3 + 10.6157s^2 + 9.6157s}$$

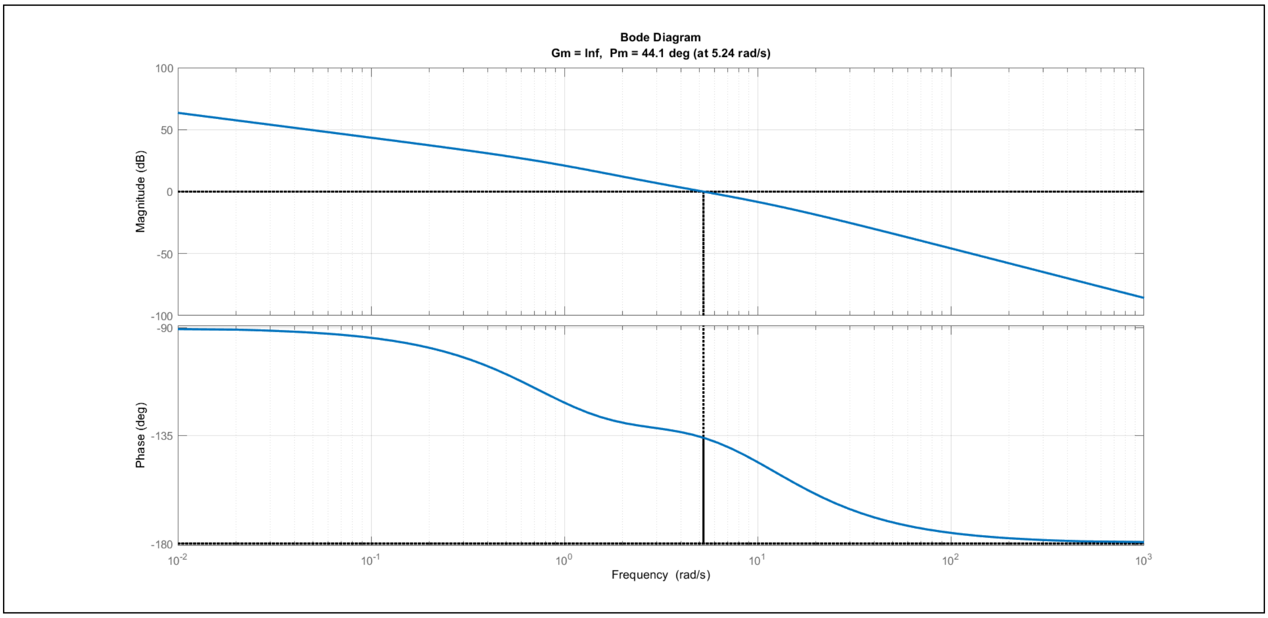

Figure 8: Bode Plot of Compensated System with 10% Additional Phase, Φ_m

- Phase Margin, PM $= 44.1°$
- Gain cross-over frequency, $\omega_{gc} = 5.24\,\text{rad/s}$

The Phase margin of the compensated system is still not equal to 45°. To increase the Phase margin, repeat the above process by changing the percentage increase of the additional phase, Φ_m, from 10% to 20%.

$$\Phi_m = 36.3474° \quad (49)$$

$$\alpha = 3.9101 \quad (50)$$

$$\omega_m = 5.3561\,\text{rad/s} \quad (51)$$

$$p = 10.5913\,\text{rad/s} \quad (52)$$

$$z = 2.7087\,\text{rad/s} \quad (53)$$

$$K = 58.6522 \quad (54)$$

The open-loop transfer function of the phase lead compensator is:

$$G_c(s) = K\left(\frac{s+z}{s+p}\right)$$

$$G_c(s) = 58.6522\left(\frac{s+2.7087}{s+10.5913}\right) \quad (55)$$

The compensated system open-loop transfer function is:

$$G_c(s)G(s) = 58.6522\left(\frac{s+2.7087}{s+10.5913}\right)\frac{1}{s(s+1)}$$

$$G_c(s)G(s) = \frac{58.6522s + 158.9}{s^3 + 11.59s^2 + 10.59s}$$

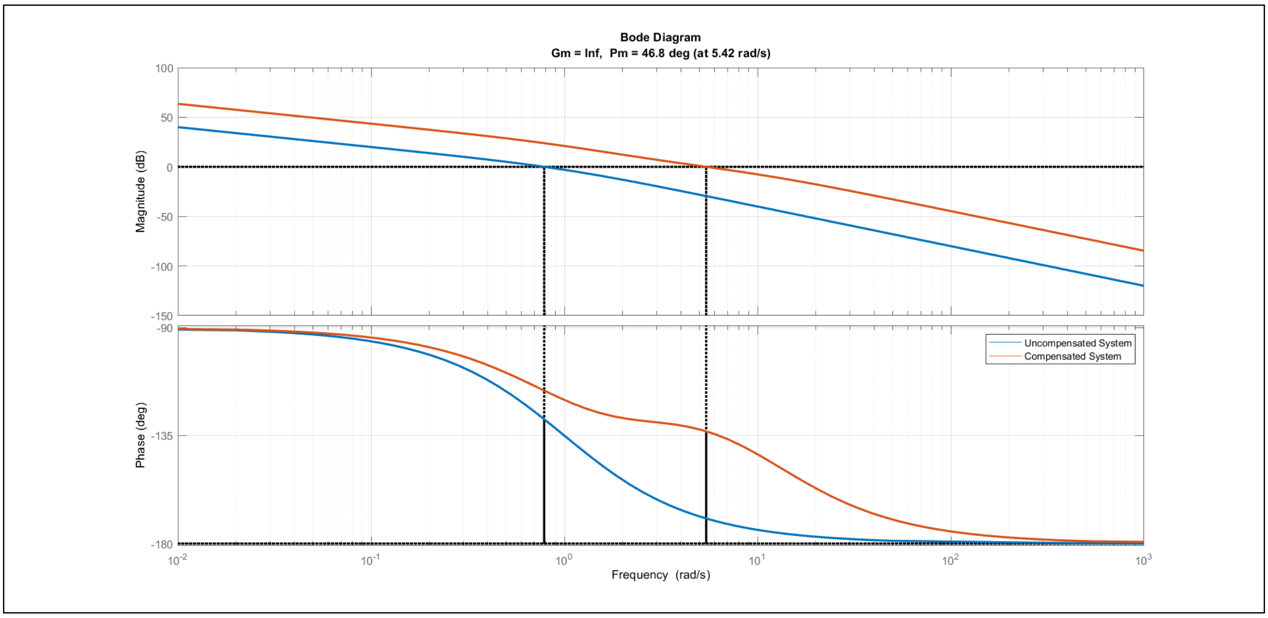

Figure 9: Bode Plot of Compensated System with 20% Additional Phase, Φ_m

- Phase Margin, PM $= 46.8°$
- Gain cross-over frequency, $\omega_{gc} = 5.42\,\text{rad/s}$

By increasing the additional phase, ϕ_m, there is an increase in the Phase margin of the compensated system $G_c(s)G(s)$. The gain cross-over frequency is also less than the specified value of $7.5\,\text{rad/s}$.

Steady-State Error of the Final Compensated System

The steady-state error for a unit ramp input is calculated as follows:

$$E_{\text{ss}} = \lim_{s\to 0} \frac{sR(s)}{1 + G_c(s)G(s)}$$

$$E_{\text{ss}} = \lim_{s\to 0} \frac{s \times \frac{1}{s^2}}{1 + \frac{58.6522s+158.9}{s^3+11.59s^2+10.59s}}$$

$$E_{\text{ss}} = \lim_{s\to 0} \frac{s^3 + 11.59s^2 + 10.59s}{s \times (s^3 + 11.59s^2 + 10.59s + 58.6522s + 158.9)}$$

$$E_{\text{ss}} = \frac{10.59}{158.9} \times 100 = 6.66\%$$

From MATLAB simulation, the steady-state error $E_{\text{ss}} = 6.6667\%$. The steady-state error for the compensated system must be less than or equal to $\frac{1}{15} = 0.0666$ units. The steady-state error of the compensated system meets the required specification.

Compensator Bode Plot

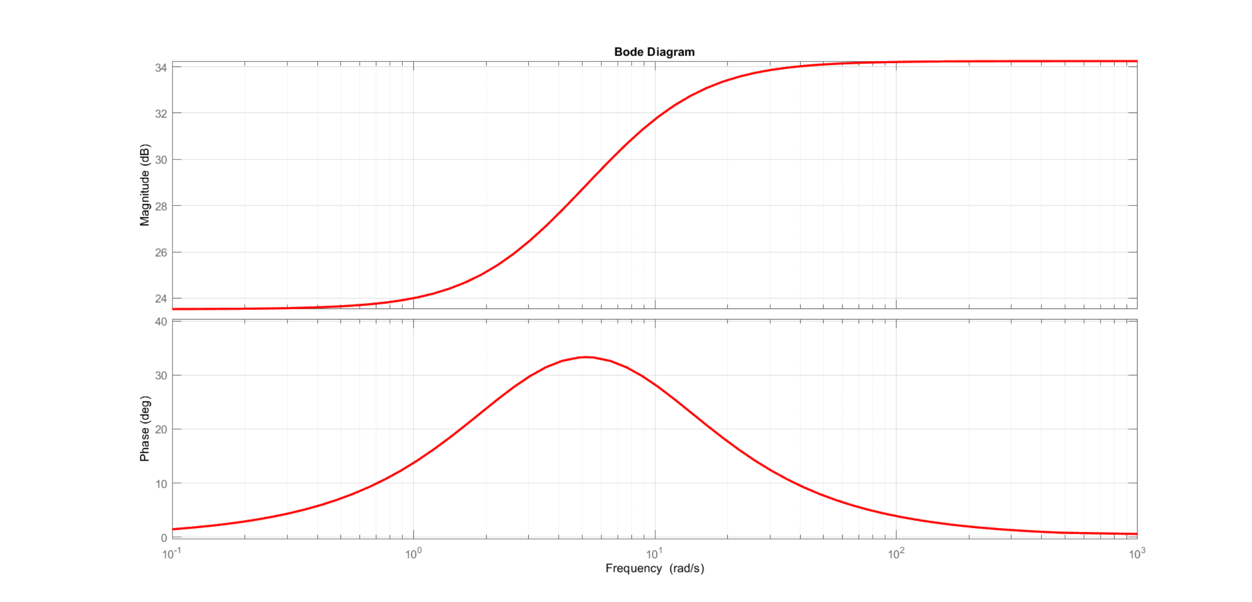

Figure 10: Unit ramp response of the final compensated system

From the above plot , we observe that the lead compensator system is a High pass filter.

Response to Ramp Input

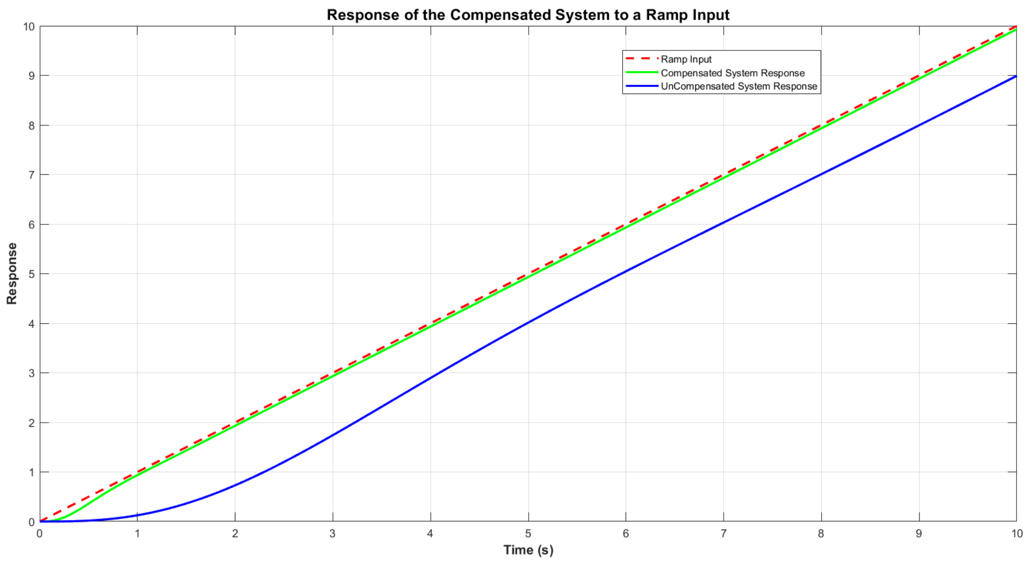

Figure 11: Unit ramp response of the final compensated system

Velocity Error Constant, K_v

$$K_v = \lim_{s\to 0} sG_c(s)G(s)$$

$$K_v = \lim_{s\to 0} \frac{s \times (58.6522s + 158.9)}{s^3 + 11.59s^2 + 10.59s}$$

$$K_v = \lim_{s \to 0} \frac{58.6522s + 158.9}{s^2 + 11.59s + 10.59}$$
$$K_v = \frac{158.9}{10.59} = 15$$
$$K_v = \frac{1}{E_{\text{ss}}} = 15$$

Results

1. The compensator transfer function: $G_c(s) = 58.6522\left(\frac{s+2.7087}{s+10.5913}\right)$
2. The compensated system transfer function: $G_c(s)G(s) = \frac{58.6522s+158.9}{s^3+11.59s^2+10.59s}$
3. For the compensated system:
   - Phase margin, $PM = 46.8°$
   - Velocity error constant, $K_v = 15$
   - Steady-state error, $E_{\text{ss}} = 6.6667\%$
   - Gain crossover frequency, $\omega_{gc} = 5.42$ rad/s

Inferences

A phase lead compensator can be designed using both the Bode plot and root locus methods. These approaches provide a deep understanding of system dynamics, aiding in precise control design. Bode plot analysis helps identify initial system deficiencies and determine the required phase margin and gain crossover frequency to meet performance specifications. The root locus method allows for the visualization of compensator parameters' impact on pole locations, facilitating iterative adjustments for stability and transient response. This combined approach ensures that the compensator enhances system stability, satisfies specified requirements, minimizes steady-state errors, and optimizes control system performance.