EE4150 : Control Systems Laboratory

INDIAN INSTITUTE
OF TECHNOLOGY
PALAKKAD

Post Lab Report

Experiment 5

Modelling and controller design for a rotory inverted pendulum system

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Objective

- **To perform the state space modeling, state feedback controller, and LQR controller design for an inverted pendulum system to obtain stabilization and tracking control objectives.**

Problem

1. **Using the parameters given, obtain the model of the given system. Find out the equilibrium points of the system and analyze the stability.**
2. **Analyze the controllability of the system linearized around the unstable equilibrium point and check whether the state feedback controller can be designed or not.**
3. **Design a state feedback controller for the stabilization of pendulum angle θ at the upright position. Assume initial conditions as: $x_1(0) = \pi/8$, $x_2(0) = 0$.**
4. **Design a state feedback controller for the trajectory tracking of pendulum angle θ such that the tracking error is within 5%. (Example: sinusoidal trajectory of amplitude 0.3 rad and frequency $5\pi/100$ rad/s). Assume zero initial conditions for the system states.**
5. **Repeat Problems 3 and 4 using an LQR (Linear Quadratic Regulator). An example is provided for reference. Try to automate the calculations as much as possible.**

Theory

The Inverted Pendulum is a method used to study control theories and model rotational mechanical systems.The main purpose is to create a control system for tracking and stabilizing objectives. This experiment focuses on the self-erecting control problem, which is encountered in applications like missile launching systems. The inverted pendulum is unstable and requires a stabilizing controller to maintain an upright position when disrupted.

Modelling the Inverted Pendulum System

The control system takes an input signal u which is control torque τ and the aim is to control the output, that is the pendulum angle θ. The controller tries to adjust the DC motor voltage to generate the required torque τ in order to achieve the required control objective.

The dynamical equation of motion of the inverted pendulum is as follows:

$$\ddot{\theta} = \frac{g}{L}\sin(\theta) + \frac{\tau}{mL^2}$$

Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. With this choice of system states and $u = \tau$, the equation can be written in the form of state space equations as given below:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{g}{L}\sin(x_1) + \frac{1}{mL^2}u$$

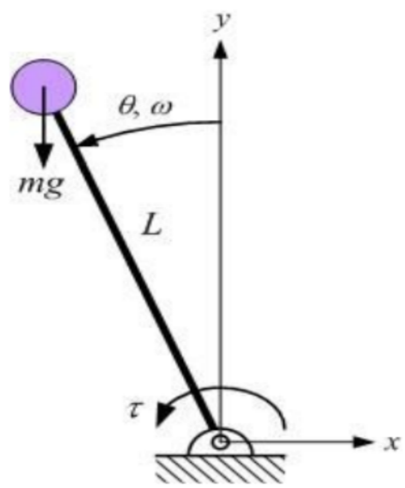

Figure 1: Inverted Pendulum system

The above nonlinear model can be expressed as:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ \frac{g}{L}\sin(x_1)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{mL^2}\end{bmatrix} u$$

Parameter	Description	Value
m	Mass	0.3 kg
L	Length of pendulum	1 m
g	Gravity Acceleration	9.81 m/s^2

Problem 1

Obtaining the equilibrium points of the system

To find the equilibrium points $\dot{x}_1$ and $\dot{x}_2$ must be zero, Hence while equating it we'll get

$$\begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}x_2\\ \frac{g}{L}\sin(x_1)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{mL^2}\end{bmatrix} u$$

$$\begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}x_2\\ \frac{g}{L}\sin(x_1) + \frac{1}{mL^2}\cdot u\end{bmatrix}$$

Hence we'll get $x_1 = 0$ and $x_2 = 0$ or π

At these points we can use taylor approximation for small disturbances for **Linearising** the system.

For case 1 (x1, x2) = (0, 0)

$$\sin x_1 \approx x_1 \text{ for } x_1 = 0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} \cdot u$$

$$A = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 9.81 & 0 \end{bmatrix}$$

Eigenvalues corresponding to A obtained through Matlab are $\lambda_1 = 3.1321$ and $\lambda_2 = -3.1321$.

For case 2 (x1, x2) = (π, 0)

$$\sin x_1 \approx -x_1 \text{ for } x_1 = \pi$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} \cdot u$$

$$A = \begin{bmatrix} 0 & 1 \\ -g/L & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9.81 & 0 \end{bmatrix}$$

Eigenvalues corresponding to A obtained through Matlab are $\lambda_1 = 3.1321i$ and $\lambda_2 = -3.1321i$.

Inference

The eigenvalues corresponding to case 1 $(x_1, x_2) = (0, 0)$ are **3.1321** and **−3.1321**. As eigenvalues are the poles of the system, the positive eigenvalue corresponds to the pole in the **right half-plane**. Hence, the system is **unstable** for $(x_1, x_2) = (0, 0)$.

The eigenvalues corresponding to case 2 $(x_1, x_2) = (\pi, 0)$ are **3.1321i** and **−3.1321i**. As eigenvalues are the poles of the system, the eigenvalues correspond to the pole on the **imaginary axis**. Hence, the system is **marginally stable**, and the system will have an **oscillatory behavior** for $(x_1, x_2) = (\pi, 0)$.

Problem 2

Controllability of the system linearized around an unstable equilibrium point

The system linearized around an equilibrium point can be written as shown below. The nonlinear model can be expressed as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9.81 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3.34 \end{bmatrix} \cdot u$$

Result

Controllability matrix obtained through Matlab is:

$$\begin{bmatrix} 0 & 3.3333 \\ 3.3333 & 0 \end{bmatrix}$$

Rank of the controllability matrix = 2.

Inference

- The system is **controllable** as the rank of the controllability matrix is 2, which is a full-rank matrix. Hence, a state feedback controller can be designed at the unstable position $(x_1, x_2) = (0, 0)$ to make the system stable.
- We can see that the system is both **controllable** and **observable** as the rank of the controllability and observability matrices is 2, which is the same as their dimension. Therefore, the system is both controllable and observable.

Problem 3

The linearized equation for the inverted pendulum system without any damping is as follows:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g}{L}x_1 + \frac{1}{mL^2}u$$

This can be rewritten in state-space representation as:

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ g/L & 0\end{bmatrix} \cdot \begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \frac{1}{mL^2}\end{bmatrix} \cdot u$$

where x_1 is θ, the angular displacement in radians, and x_2 is the angular velocity ω.

Since we want to stabilize the system upright, we have to choose the optimal values of K, the gain matrix. Then the control input will have the form:

$$u = -Kx$$

For applying the pole-placement method, we will have to choose the desired pole first. In order to get a stable system, the pole should be in the negative half-plane.

$$p_1 = -5 \text{ and } p_2 = -9$$

where p_1 and p_2 are the desired poles.

$$|sI - A_{CL}| = (s - p_1)(s - p_2)$$
$$A_{CL} = A - BK$$
$$(s - p_1)(s - p_2) = s^2 + 14s + 45$$
$$A - BK = \begin{bmatrix}0 & 1 \\ \frac{g}{L} & 0\end{bmatrix} - \begin{bmatrix}0 \\ \frac{1}{mL^2}\end{bmatrix} \begin{bmatrix}K_1 & K_2\end{bmatrix}$$
$$A_{CL} = \begin{bmatrix}0 & 1 \\ \frac{g}{L} - \frac{K_1}{mL^2} & -\frac{K_2}{mL^2}\end{bmatrix}$$
$$sI - A_{CL} = \begin{bmatrix}s & -1 \\ \frac{K_1}{0.3} - 9.81 & \frac{K_2}{0.3} + s\end{bmatrix}$$
$$|sI - A_{CL}| = s^2 + 3.33K_2 \cdot s + 3.33K_1 - 9.81$$

Comparing equations, we can relate:

$$3.33K_2 = 14$$

$$3.33K_1 - 9.81 = 45$$

Solving the above two equations, we get the Gain Matrix as:

$$K = \begin{bmatrix} 16.4430 & 4.2000 \end{bmatrix}$$

Using this K matrix, we can find the output signal x_1 and x_2.

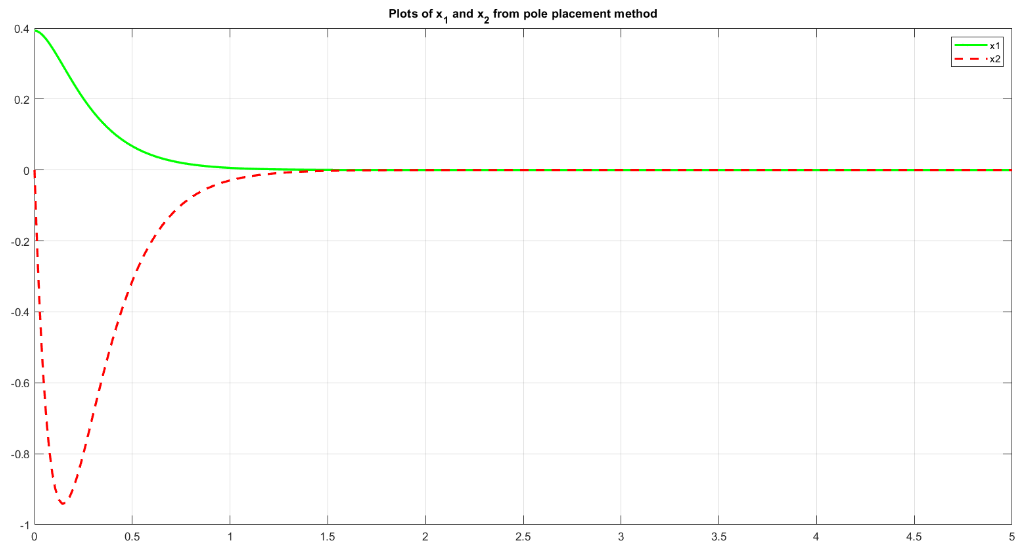

Figure 2: State feedback control designed with the help of pole placement method. The plot shows that the signals x_1 and x_2 stabilizing at zero

Problem 4

We need the pendulum angle θ to follow a sinusoidal trajectory,

$$\theta(t) = 0.3\sin(0.05\pi t)$$

and

$$\dot{\theta}(t) = 0.015\pi\cos(0.05\pi t)$$

The control input for trajectory tracking will have the form:

$$u = -K(x - x_{\text{desired}})$$

where K is the gain matrix and

$$x_{\text{desired}} = \begin{bmatrix} 0.3\sin(0.05\pi t) \\ 0.015\pi\cos(0.05\pi t) \end{bmatrix}$$

For this problem, we can only find the K matrix values using the trial-and-error method.

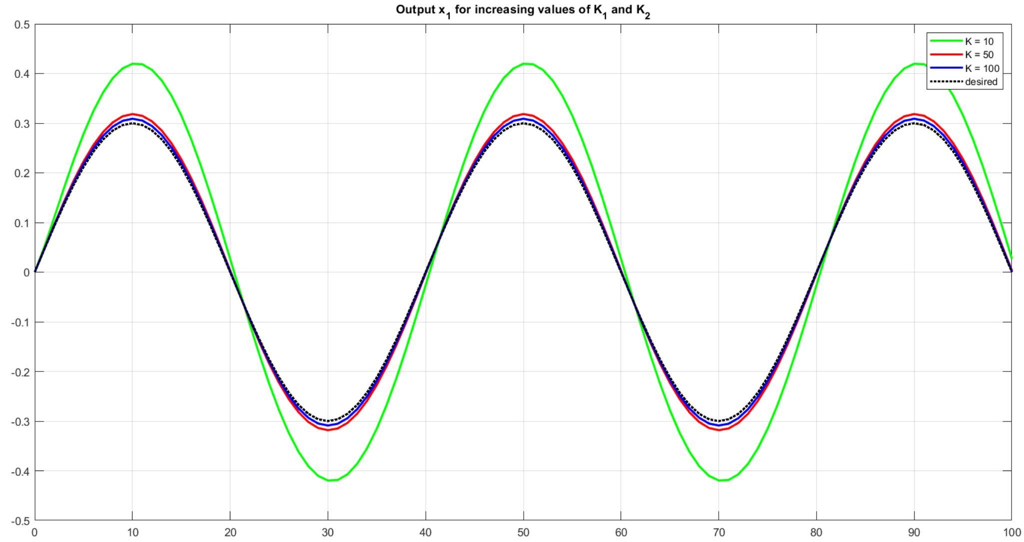

Figure 3: The effect of increasing K1 and K2 on x_1

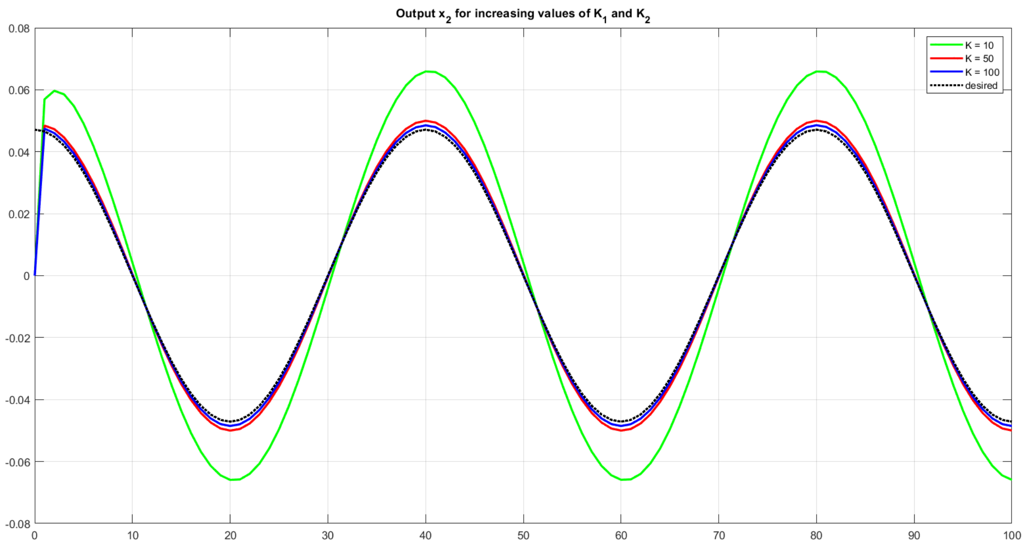

Figure 4: The effect of increasing K1 and K2 on x_2

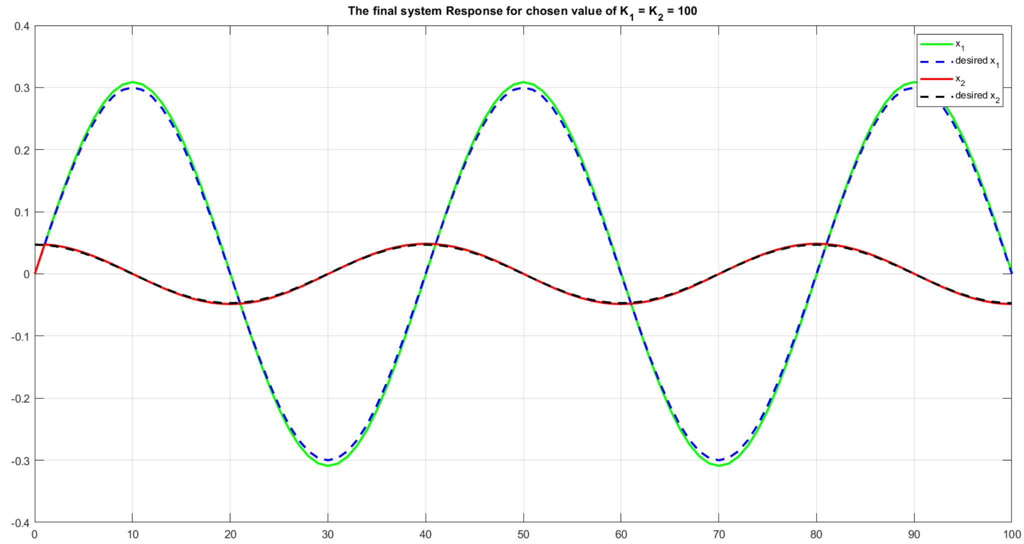

Figure 5: The system response for K1 = K2 = 100

Problem 5

The linearized equation for the inverted pendulum system without any damping is as follows:

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = \frac{g}{L}x_1 + \frac{1}{mL^2}u \tag{2}$$

This can be rewritten in state-space representation as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u$$

where x_1 is θ, the angular displacement in radians, and x_2 is the angular velocity ω.

The vector of state-feedback control gains K can be determined using the Linear Quadratic Regulation (LQR) method. The MATLAB function `lqr()` allows us to balance the relative importance of the control effort (u) and the error (deviation from 0) in the cost function that we are trying to optimize by selecting two parameters, R and Q. Increasing the values of the Q elements will improve the response, but will require a greater control force u. This is a trade-off between performance and the control effort, as more control effort generally corresponds to greater cost (more energy, larger actuator, etc.).

Finding Q and R Matrices by Manual Tuning

The initial condition of the inverted pendulum is $(x_1, x_2) = (\pi, 0)$. We need to stabilize the system in an upright position, that is, $(x_1, x_2) = (0, 0)$. Since the system is of second order, we will fix R as a constant and Q as a diagonal matrix of the form:

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \tag{44}$$

The values of R and Q have to be found by manual tuning. The Q matrix specifies the relative importance of each state variable in the cost function. Higher values in the Q matrix indicate that the controller places greater emphasis on minimizing the deviations of specific state variables from their desired values. The R matrix specifies the relative importance of the control effort in the cost function. Higher values in the R matrix indicate that the controller places greater emphasis on minimizing the control effort (the magnitudes of the control inputs).

Varying R and Keeping Q Fixed

For finding the optimal value of R, we will vary R from 10 to 0.001 while keeping the Q matrix fixed as:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

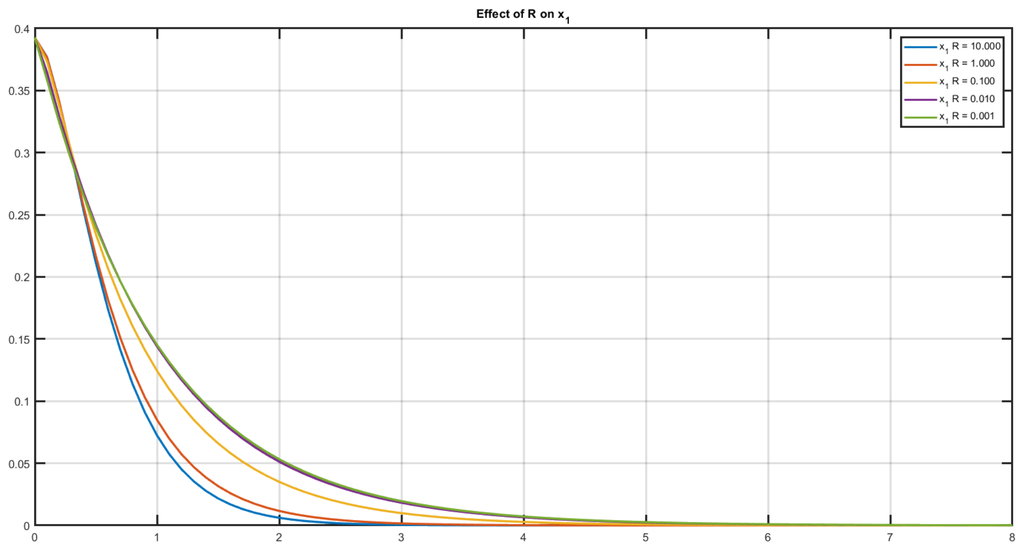

Figure 6: The effect of decreasing R on x1

As we decrease the value of R from 10 to 0.001, x_1 and x_2 decay slowly to zero. For larger values of R, x_2 has a more negative amplitude, although it quickly decays to zero. We can choose $R = 0.1$ as the optimal value.

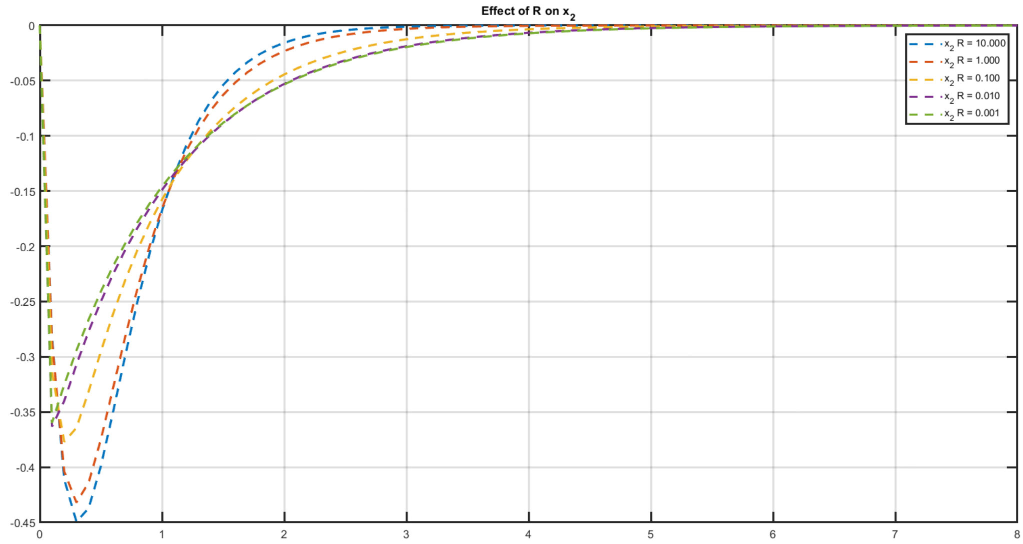

Figure 7: The effect of decreasing R on x2

Varying Q and keeping R fixed

To find the optimal weights for the cost function, we fix R = 0.1, while changing qii of the Q-matrix to 1, 5 and 10.

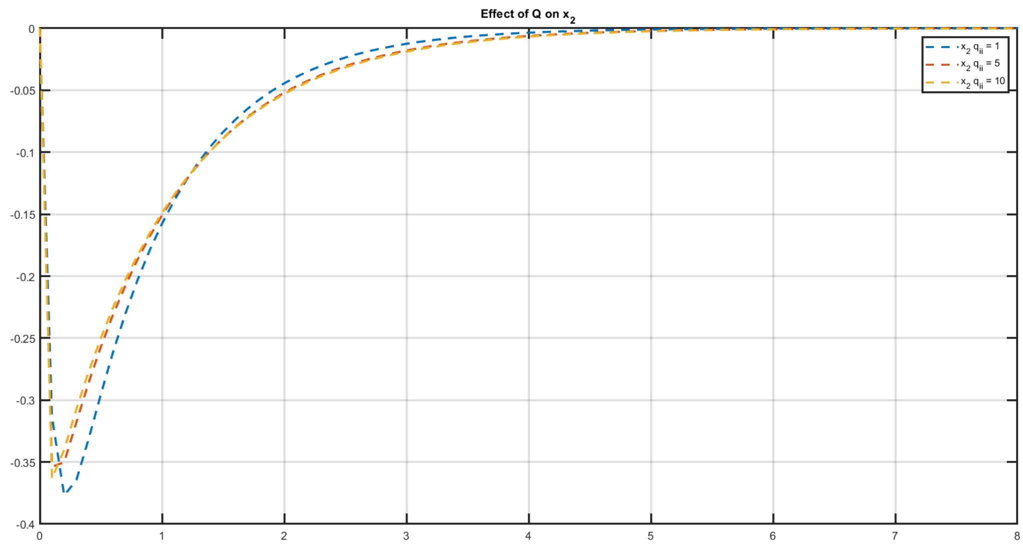

Figure 8: The effect of decreasing Q on x1

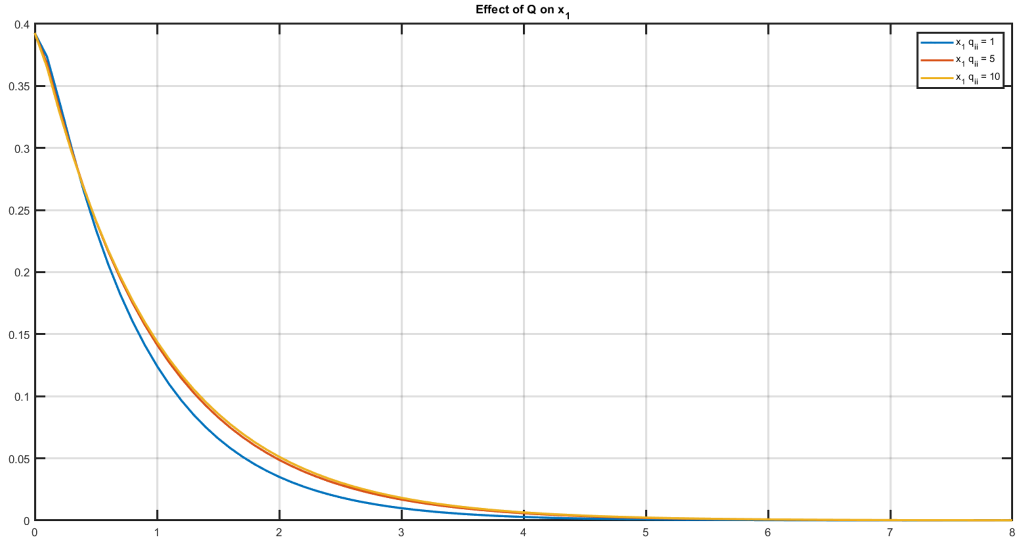

Figure 9: The effect of decreasing Q on x2

Final System

From our earlier observation, we can set the Q and R matrices as:

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$R = 0.1$$

The Gain matrix for the given Q and R matrices is found using the `lqr()` function available in MATLAB. The control input for the system is of the form $u = -Kx$:

$$K = \begin{bmatrix} 13.3671 & 10.3933 \end{bmatrix}$$

The closed-loop system for the system is:

$$A_{CL} = A - BK$$

$$A_{CL} = \begin{bmatrix} 0 & 1.0000 \\ -34.7469 & -34.6443 \end{bmatrix}$$

The closed-loop poles are:

$$p = \begin{bmatrix} -33.6105 \\ -1.0338 \end{bmatrix}$$

Since both poles lie in the left half-plane, the system is closed-loop stable.

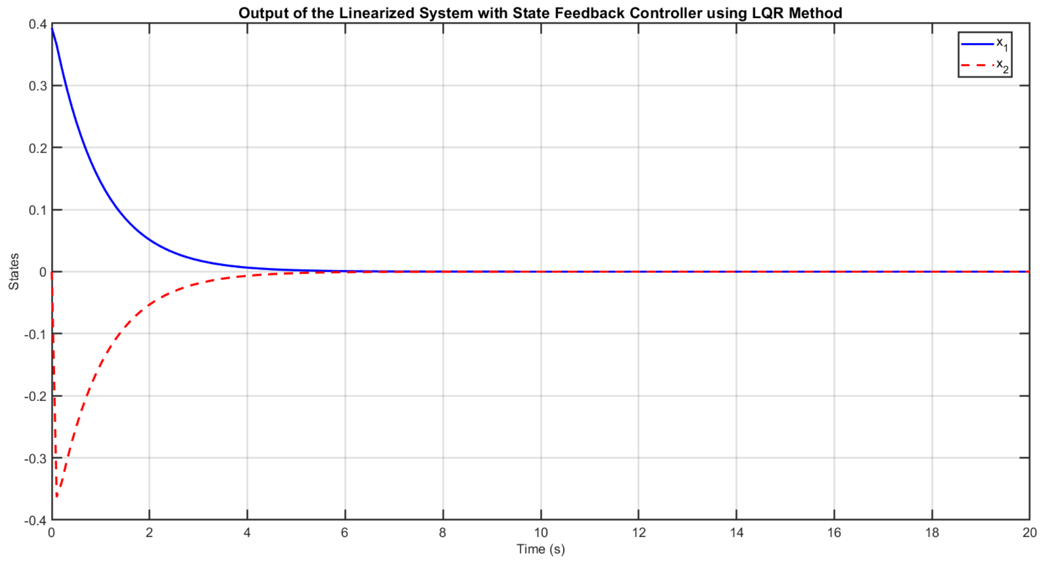

Figure 10: The output x_1 and x_2 of the final system

We need the pendulum angle θ to follow a sinusoidal trajectory:

$$\theta(t) = 0.3 \sin\left(\frac{5\pi}{100}t\right)$$

and

$$\dot{\theta}(t) = 0.015\pi \cos\left(\frac{5\pi}{100}t\right)$$

The control input for trajectory tracking will have the form:

$$u = -K(x - x_{\text{desired}})$$

where K is the gain matrix and

$$x_{\text{desired}} = \begin{bmatrix} 0.3 \sin\left(\frac{5\pi}{100}t\right) \\ 0.015\pi \cos\left(\frac{5\pi}{100}t\right) \end{bmatrix}$$

Finding Q and R matrices by manual tuning

Since the system is of second order, we will fix R as a constant and Q as a diagonal matrix of the form:

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$$

Similar to what we have done in Problem 3, we first modify Q and find the desired weights, and then vary R to find their optimal values. We have fixed $R = 0.1$, the same value used in Problem 3. Then, the value of q_{ii} is varied from 1 to 20 by a step of 5.

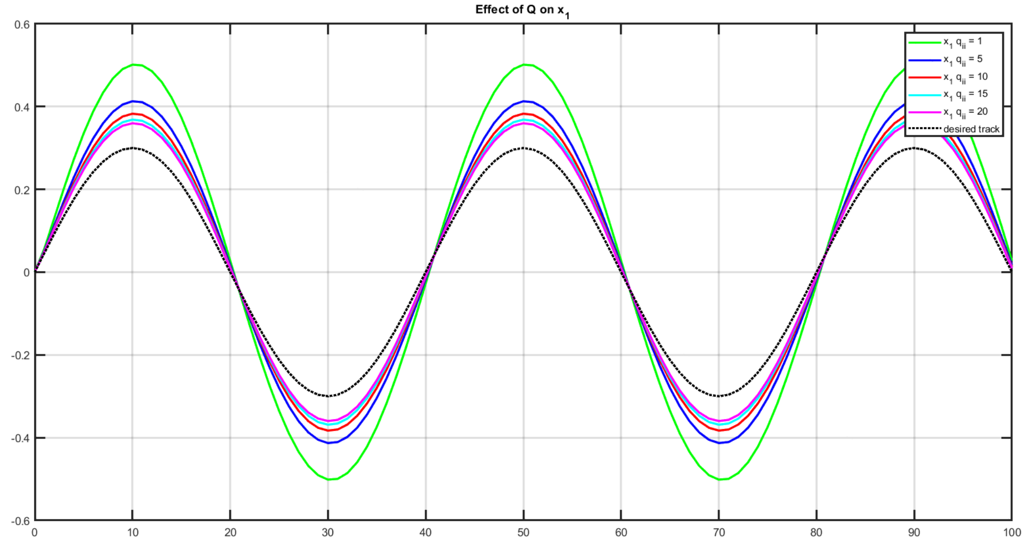

Figure 11: The effect increasing qii on X_1

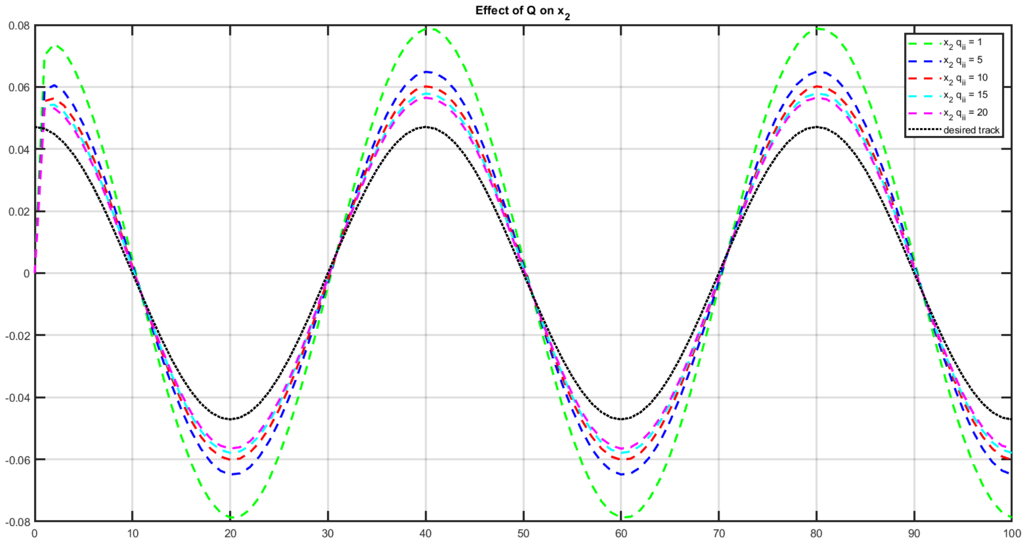

Figure 12: The effect increasing qii on X_2

Only when the weight $q_{ii} = 20$, does the system become almost closer to the desired trajectory. To further reduce the tracking error, we can increase the value of R even further. The value of R is varied from 0.1 to 0.001.

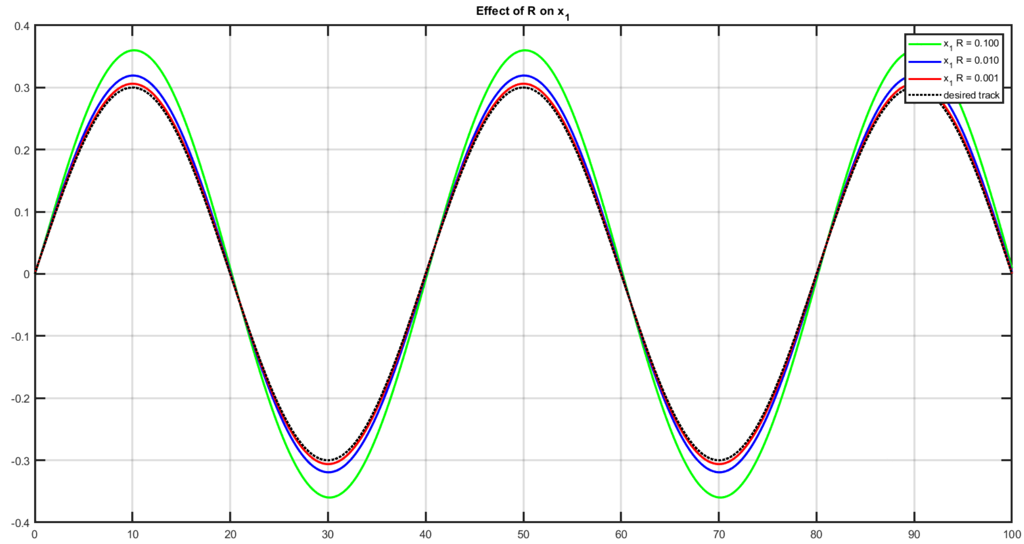

Figure 13: The effect increasing R on X_1

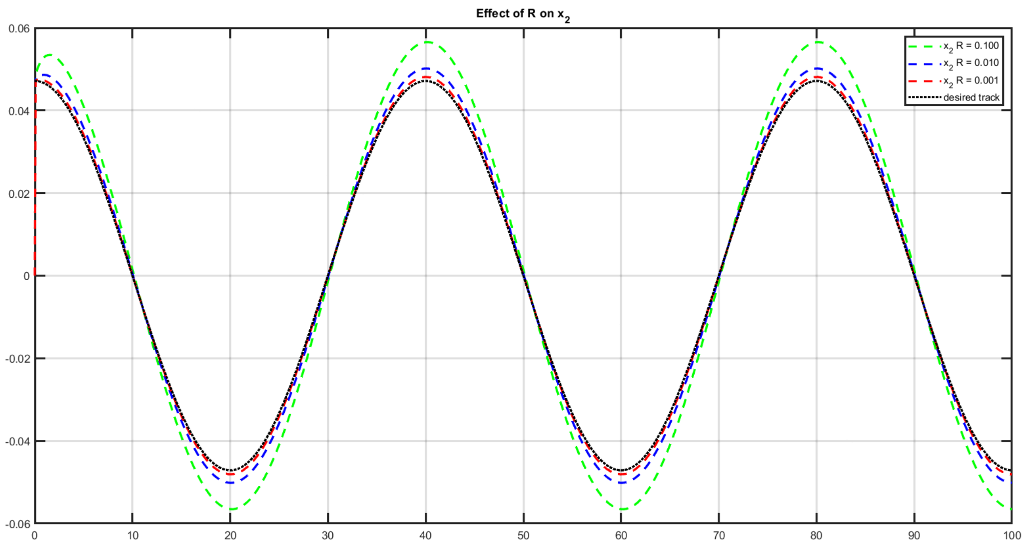

Figure 14: The effect increasing R on X_2

When the value of $R = 0.001$, the tracking error is minimum.
So the optimal values of Q and R matrices are:

$$R = 0.001$$

$$Q = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$

Final System

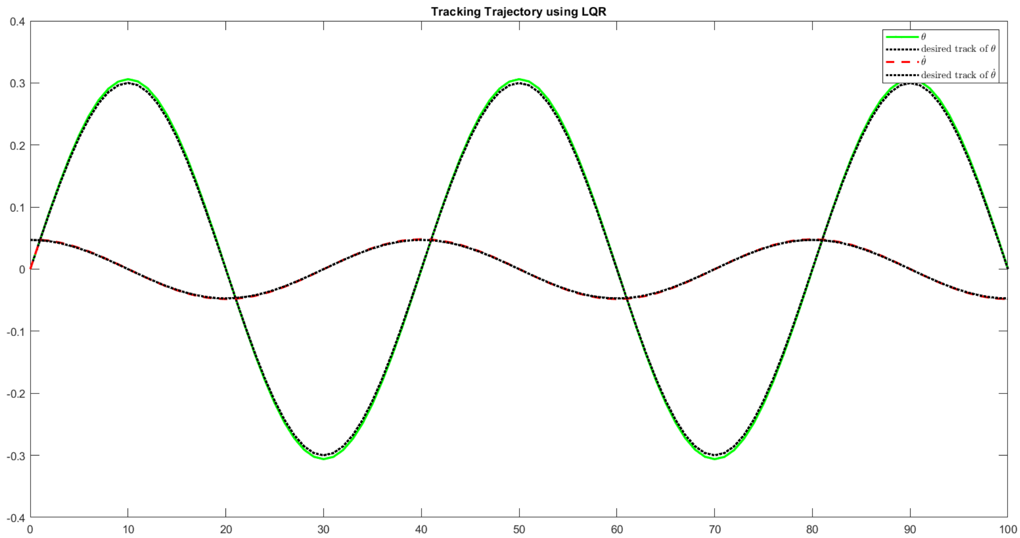

Figure 15: The output signal x1 and x2 of the final system.

Gain Matrix

$$K = \begin{bmatrix} 144.3950 & 141.7273 \end{bmatrix}$$

Tracking Error

$$\theta_{\text{error}} = 0.61\%$$

The tracking error is much lower than the required error.

The Closed-Loop System

$$A_{CL} = \begin{bmatrix} 0 & 1.0000 \\ -471.5066 & -472.4244 \end{bmatrix}$$

The Closed-Loop Poles

$$p = \begin{bmatrix} -471.4242 \\ -1.0002 \end{bmatrix}$$

Since both poles lie in the left half-plane, the system is closed-loop stable.

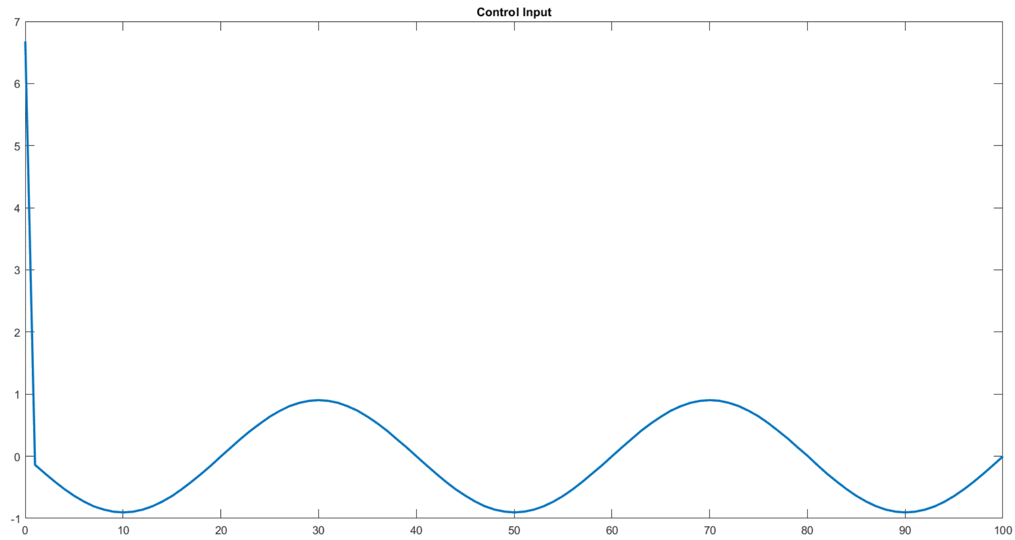

Figure 16: The control input. Since the required trajectory is a sinusoidal path, the control input is also a sinusoidal signal.

Results

- The inverted pendulum system has two equilibrium points: $(\theta, \dot{\theta}) = (0, 0)$, which represents an unstable equilibrium, and $(\theta, \dot{\theta}) = (\pi, 0)$, which is a marginally stable equilibrium. The marginal stability of $(\pi, 0)$ is due to the absence of damping in the system, meaning that a small perturbation will cause the pendulum to oscillate without returning to rest. On the other hand, $(0, 0)$ is an unstable equilibrium, as any small perturbation will prevent the system from returning to its initial upright position.
- The system was linearized to represent it in state-space form. Since the system was controllable, both the pole placement and LQR methods were applicable for controlling the system.
- A state feedback controller was designed to stabilize the inverted pendulum in the upright position (Problem 3), and a trajectory tracking controller was designed (Problem 4), both using pole placement and LQR methods.
- The Q and R matrices in the LQR controller allow for adjusting the trade-off between state tracking (controllability) and control effort (energy consumption, actuator size). By appropriately tuning these matrices, a balance can be achieved to meet the specific control objectives and constraints of the system.
- For trajectory tracking, the control input follows a sinusoidal form, reflecting the requirement for a sinusoidal trajectory.