EE4150 : Control Systems Laboratory

INDIAN INSTITUTE OF TECHNOLOGY PALAKKAD

Post Lab Report

Experiment 3 :

Design of lag compensator using root locus and frequency response methods

Nakul C	**122101024**
Rittika Ghosh	**122101047**

Problem 1

Design a lag compensator using root locus method for the following system with open loop transfer function:

$$G(s) = \frac{K}{s(s+2)(s+8)}$$

such that the closed loop unity negative feedback system with the compensator will satisfy the following specifications:

- **Damping ratio is grater than or equal to 0.6.** $\zeta \geq 0.6$
- **Settling time (for 2% tolerance band),** $t_s \leq 5s$ **for unit step input**
- **Velocity error constant** $K_v \geq 8$ $\mathbf{s}^{-1}$

Procedure

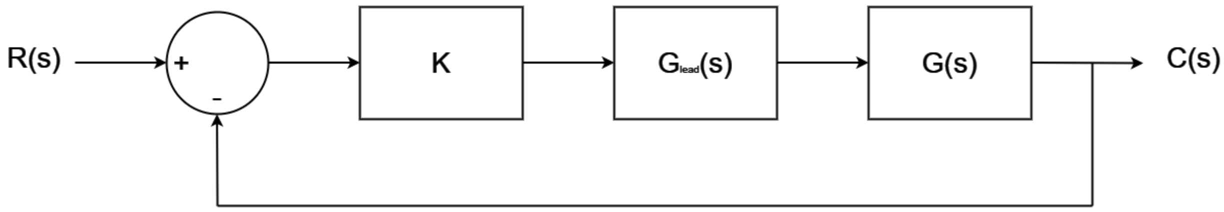

Figure 1: Lag Compensator with Negative Feedback

Dominant Poles of the Uncompensated System

The settling time for a second-order system with a tolerance band of 2% is given by:

$$t_s = \frac{-\ln\left(0.02.\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

Given the requirement $t_s \leq 5$ seconds:

$$\frac{4}{\zeta\omega_n} \leq 5$$

which simplifies to:

$$\zeta\omega_n \geq 0.8$$

According to the requirement

$$\zeta \geq 0.6$$

Due to which the value of the Natural frequency (ω_n) from these two equations would be:

$$\omega_n \leq \frac{0.8}{0.6}$$

$$\omega_n \leq \frac{4}{3}$$

$$\omega_n \approx 1.333$$

The dominant poles of the second-order system can be theoretically found by the quadratic equation:

$$s^2 + \zeta\omega_n s + \omega_n^2 = 0$$

The roots of the above equation, which are the dominant poles of the system, are given by:

$$P_{1,2} = \frac{-\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}}{2}$$

On simplifying:

$$P_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

For specific values:

$$P_{1,2} = -0.8 \pm 1.0666i$$

Root Locus of Uncompensated system

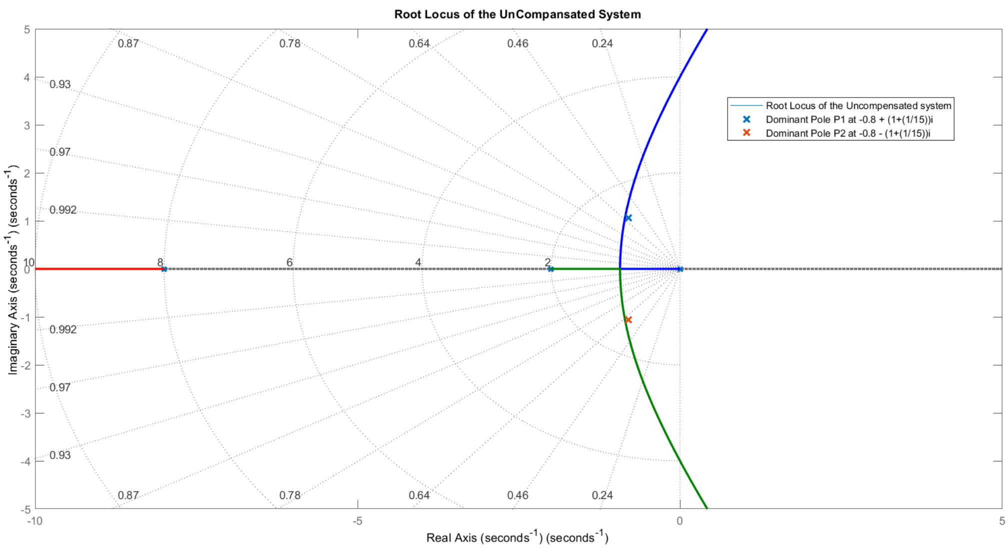

Figure 2: Root Locus of the Uncompensated System

The open-loop transfer function of the uncompensated system is given by:

$$G(s) = \frac{K}{s(s+2)(s+8)}$$

Inference : Root locus plot gives the trajectory of poles of the system's closed loop Transfer function Gain varies. The root locus plot shows that the dominant poles of the uncompensated system passes through the Root locus which says that the natural behaviour of the system which is uncompensated system itself meets the transient response that meets the design specification .

Gain of the System

Distance between the poles and zeroes of the open-loop system with dominant poles are found using MATLAB. The Magnitude Criterion is then applied to find the gain K.

Pole/Zero	Distance to Dominant Pole
Pole of $G(s)$ = 0	1.3333
Pole of $G(s)$ = -2	1.6055
Pole of $G(s)$ = -8	7.2786

Using the Magnitude Criterion:

$$\frac{K}{\pi_{\mathrm{u}}} = 1$$

- K = Gain of the Uncompensated System
- π_{u} = Product of Distances from poles of the uncompensated system

$$K = 1.3333 \times 1.6055 \times 7.2786 = 15.5815$$

Transient response of the Uncompensated System

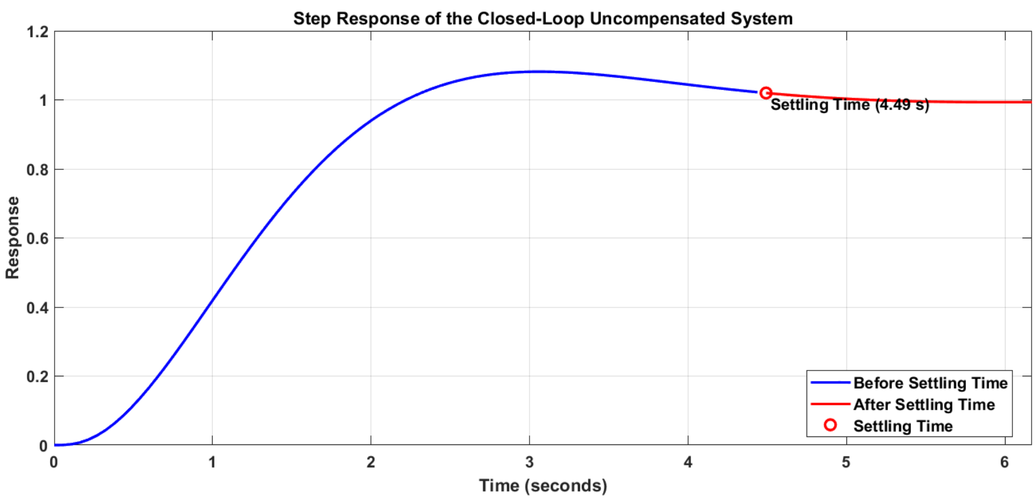

Figure 3: Transient Response of Uncompensated System

If the transient response is already satisfied then we should go for a lag compensater to improve the steady state error of the system.

K_v of the Uncompensated system

The velocity error constant of the uncompensated system is:

$$\lim_{s \to 0} (s \cdot K \cdot G(s)) = K_v$$

$$\lim_{s \to 0} \left(s \cdot \frac{15.5875 \cdot K}{s \cdot (s+2) \cdot (s+8)} \right) = K_v$$

For the uncompensated system:

$$K_{v_{uncompensated}} = \frac{K}{16} = \frac{15.5875}{16} = 0.9738$$

K_v of the Compensated system

The velocity error constant of the **Compensated system** is:

$$\lim_{s \to 0} (s \cdot K \cdot G_{lag}(s) \cdot G(s)) = K_v$$

$$\lim_{s \to 0} \left(s \cdot \frac{15.5875 \cdot K \cdot (s+z)}{s \cdot (s+2) \cdot (s+8) \cdot (s+p)} \right) = K_v$$

$$K_{v_{uncompensated}} \cdot \frac{z}{p} = K_{v_{compensated}}$$

But we know that $K_{v_{compensated}} = 8$ and $K_{v_{uncompensated}} = 0.9738$.

$$\alpha = \frac{z}{p} = 8.215$$

Finding zero and pole of the lag Compensator

Zero of the lag compensator is chosen close to origin so as to have a minimal effect on angle subtended by the uncompensated system's pole and zero on the required pole enabling after lag compensation. This also ensures that the pole of the compensator is close to zero.

Hence Zero of the compensator is choosen as 0.015

$$z = 0.015$$

$$\text{Pole of the compensator(p)} = \frac{\text{Zero of the compensator(z)}}{\alpha}$$

$$p = 0.0009626$$

Root Locus of Compensated system

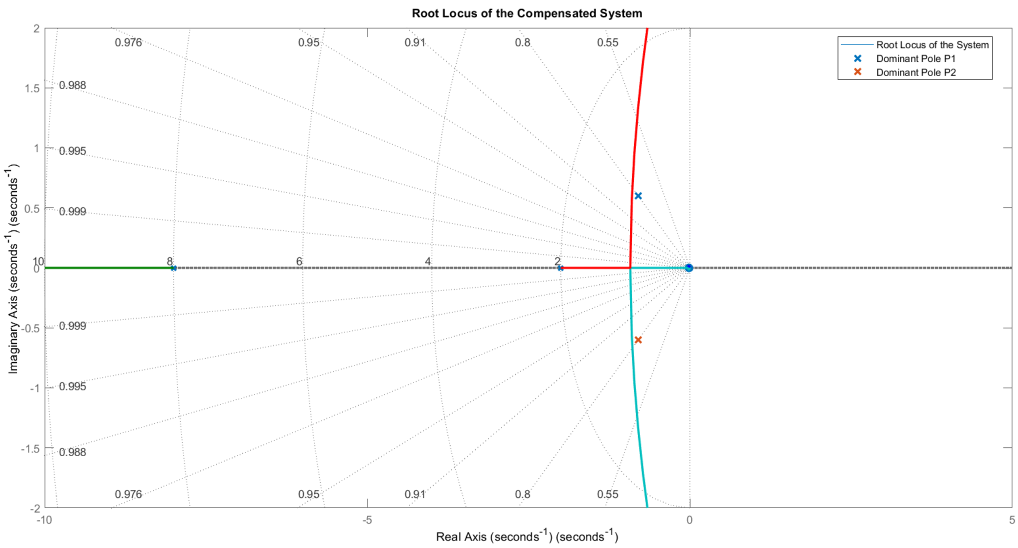

Figure 4: Root Locus of the Compensated System

Inference : Root locus plot gives the trajectory of poles of the system's closed loop Transfer function Gain varies. The root locus plot shows that the dominant poles of the uncompensated system is left of the Root locus which says that the damping ratio of the function is greater than 0.6 .

Result

Transfer Function of the Lag Compensator

$$\frac{s + 0.015}{s + 0.0009626}$$

Open Loop T.F. of the Compensated System with Gain K

$$\frac{15.5815s + 0.2337}{s^4 + 10.0010s^3 + 16.0096s^2 + 0.0154s}$$

Closed Loop T.F. of the Compensated System with Gain K

$$\frac{15.5815s + 0.2337}{s^4 + 10.0010s^3 + 16.0096s^2 + 15.5969s + 0.2337}$$

Velocity Error Constant Kv after Lag Compensation

$$K_v = 8.00\,s^{-1}$$

Unit Step Response of the Lag Compensated System

$$\text{Settling Time obtained through Matlab} = 4.9186\,\text{s}$$

$$\text{Peak Overshoot obtained through Matlab} = 9.7474\%$$

Damping Ratio

$$M_p\% = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \times 100 = 9.7474$$

$$\zeta = 0.6022$$

Hence, the design specifications are met.

Transient response of the Compensated System

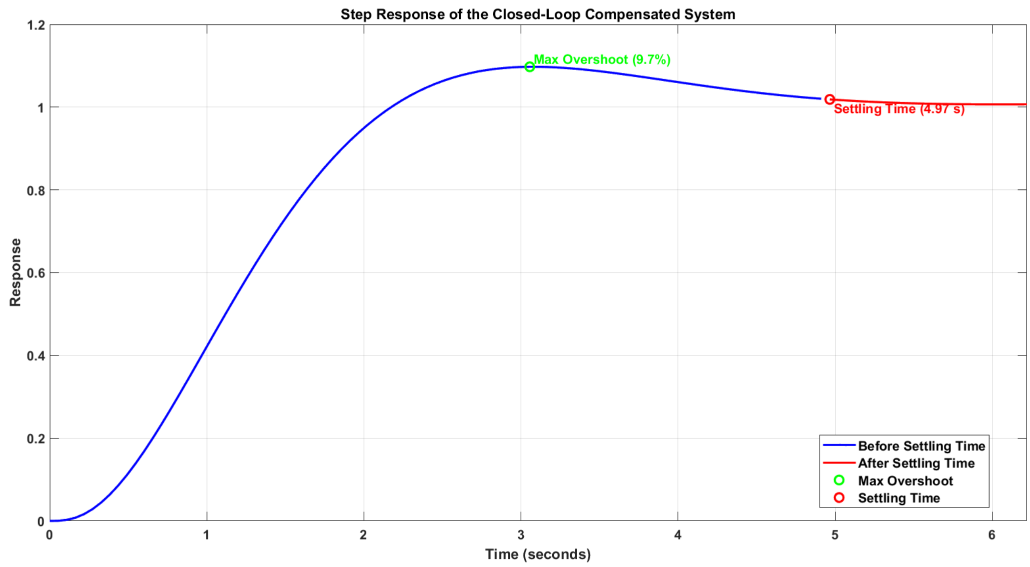

Figure 5: Transient Response of Compensated System

Problem 2

Designing a lag compensator using Bode plot for the following system with open loop transfer function

$$G(s) = \frac{1}{s(s+1)(0.5s+1)}$$

The required specifications are:

- **Phase margin,** $\Phi_m \leq 40°$
- **Gain Margin** $\Phi_m \geq 10$ **dB**
- **Velocity Error constant** $K_v \leq 5$ **rad/s.**

Procedure

Determining the value of K from the desired Velocity error constant, K_v

Assuming the phase lag compensator has the following form:

$$G_c(s) = K\left(\frac{s+z}{s+p}\right)$$

$$G_c(s) = K\left(\frac{z}{p}\right)\left(\frac{\frac{s}{z}+1}{\frac{s}{p}+1}\right)$$

$$K_1 = K\left(\frac{z}{p}\right)$$

The value of K_1 can be found from the desired velocity error constant.
Velocity error constant, K_v:

$$K_v = \lim_{s\to 0} sG_c(s)G(s)$$

$$K_v = \lim_{s\to 0} K_1 \cdot \left(\frac{\frac{s}{z}+1}{\frac{s}{p}+1}\right) \cdot \frac{1}{s(s+1)(0.5s+1)}$$

$$K_v = K_1$$

$$K_1 = 5$$

Using the value of K_1, the open loop transfer function of the uncompensated system is:

$$K_1G(s) = \frac{5}{s(s+1)(0.5s+1)}$$

Uncompensated system with the Gain K_1

- Phase Margin $= -13°$
- $\omega_{gc} = 1.8$ rad/s
- Gain Margin $= -4.44$ dB
- $\omega_{pc} = 1.41$ rad/s

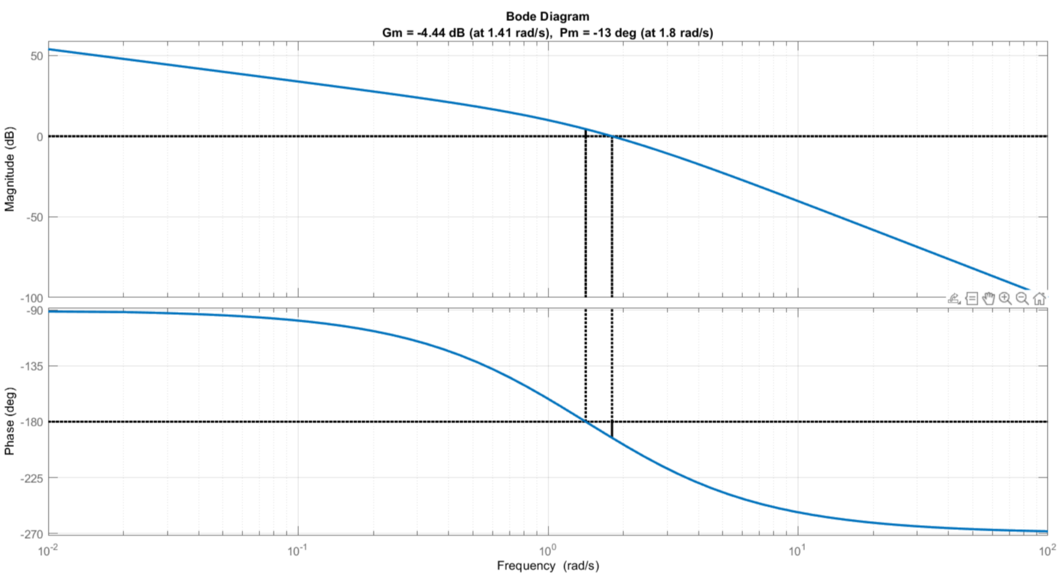

Figure 6: Bode plot of the Uncompensated system with Gain K_1

Designing the Phase-lag compensator

If we place the center of the lag compensator at ω_{gc}, the compensator will give an additional negative phase at ω_{gc} making the phase margin more negative. So we will place the new gain cross-over frequency at a frequency which gives a phase margin of 45°.

- $\omega_m = 0.5621\,\text{rad/s}$
- Phase $= -135°$
- Gain $= 17.6$ dB
- $\Phi_m = 45°$

The zero of the lag compensator can be placed a decade below the new gain cross-over frequency, $\omega_m = 0.5621\,\text{rad/s}$

$$z = \frac{\omega_m}{10} = 0.05621\,\text{rad/s}$$

At this frequency (ω_m) the system should have 0dB gain. But the system has 17.6 dB gain. We can make the gain 0dB by choosing an appropriate value for α.

$$20\log(\alpha) = 17.6$$

$$\alpha = 7.52$$

Therefore by placing a pole at

$$p = \frac{z}{\alpha} = 0.007475\,\text{rad/s}$$

Having determined the value of the pole and zero of the phase lag compensator, we can now find the value of the gain of the compensator.

$$K_1 = K \cdot \frac{z}{p} = K \cdot \alpha$$

$$K = \frac{K_1}{\alpha}$$

$$K = 0.6649$$

The open loop transfer function of the phase lead compensator is

$$G_c(s) = K \cdot \left(\frac{s+z}{s+p}\right)$$

$$G_c(s) = 0.6649\left(\frac{s+0.05621}{s+0.007475}\right)$$

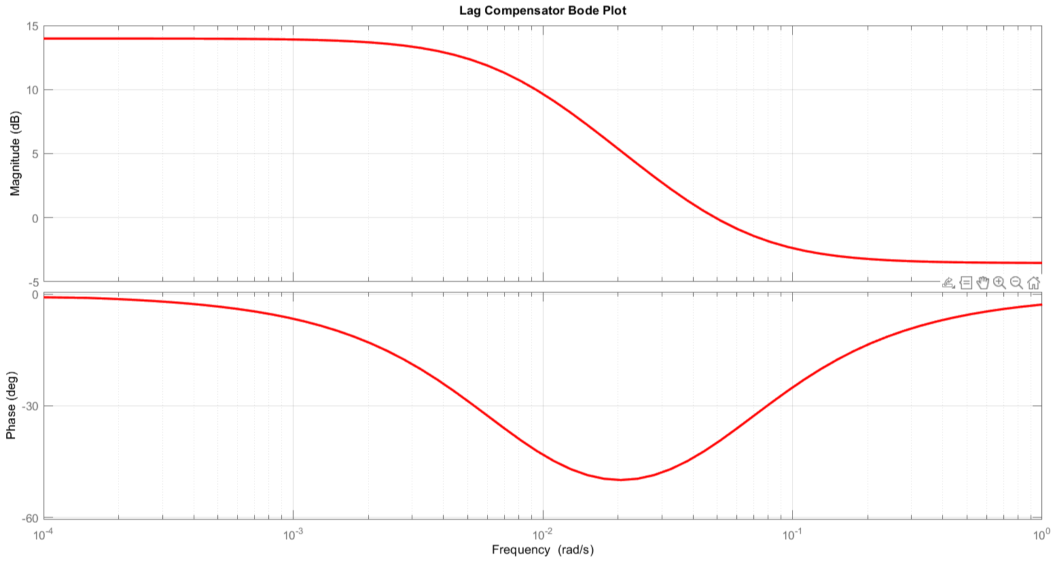

Figure 7: Bode plot of Phase-lag compensator

Compensated System

The compensated system open loop transfer function is

$$G_c(s) \cdot G(s) = 0.6649 \cdot \frac{s + 0.05621}{s + 0.007475} \cdot \frac{1}{s(s+1)(0.5s+1)}$$

$$G_c(s) \cdot G(s) = \frac{0.6649s + 0.0374}{0.5s^4 + 1.5037s^3 + 1.0112s^2 + 0.0075s}$$

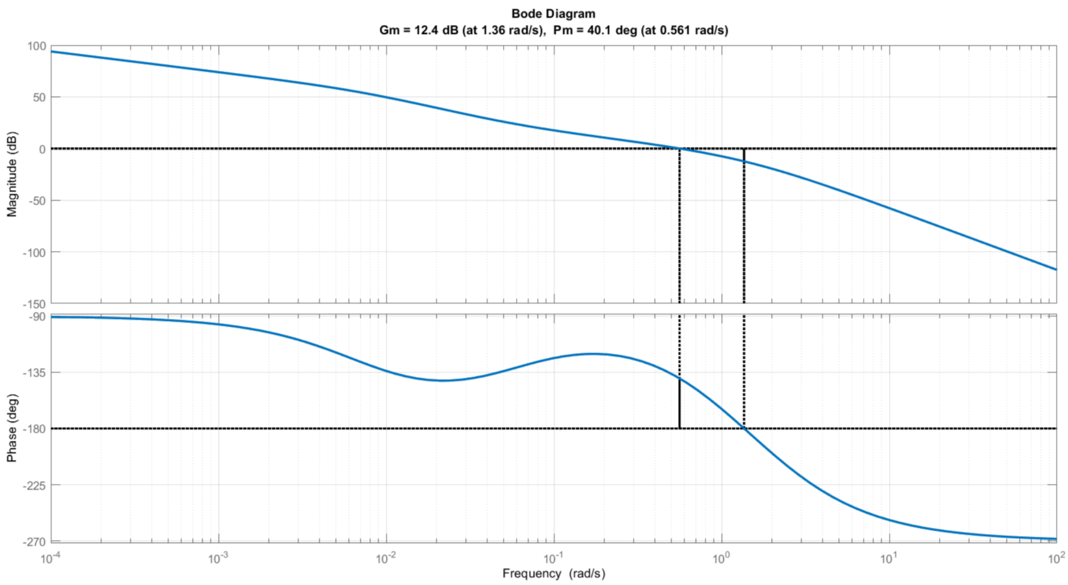

Figure 8: Bode plot of the Compensated system

- Phase Margin, PM = 40.1°
- Gain cross-over frequency, $\omega_{gc} = 0.561$ rad/s

- Gain Margin, GM = 12.4 dB
- Phase cross-over frequency, $\omega_{pc} = 1.36$ rad/s

The phase margin and gain margin of the final compensated system satisfy the required conditions. We also need to verify whether the system satisfies the required velocity error constant, K_v. This can be found easily by finding the DC gain of the compensated system from MATLAB. Velocity error constant of the compensated system:

$$K_v = 5\,\mathrm{s}^{-1}$$

It can be seen that all the three design requirements have been met by the compensated system.

Results

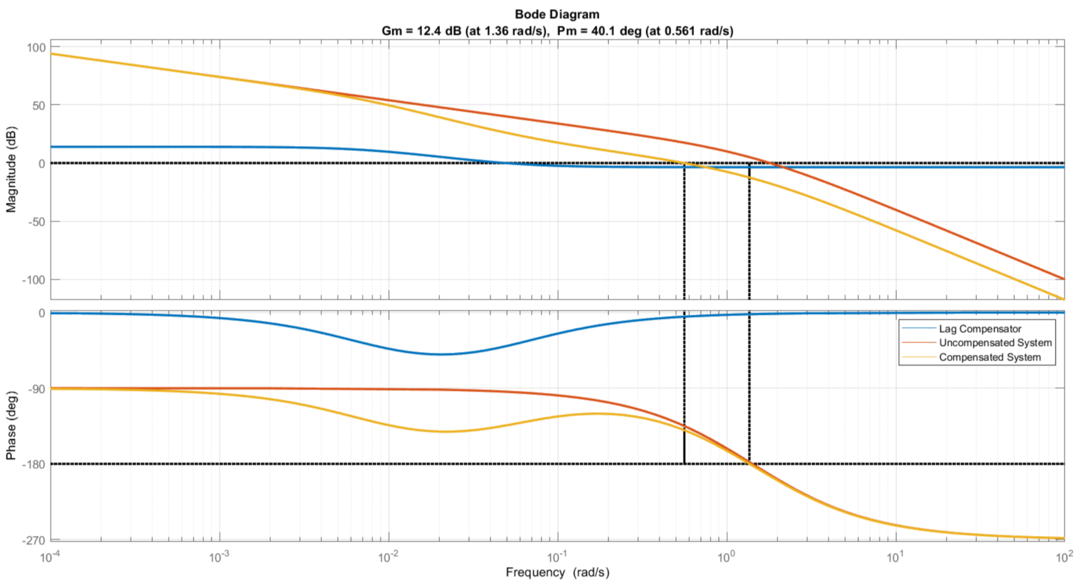

Figure 9: Bode plots of the Uncompensated System, Lag Compensator and the Compensated system

System Parameters

1. The compensator transfer function,

$$G_c(s) = 0.6649\frac{s + 0.05621}{s + 0.007475}$$

2. The compensated system transfer function,

$$G_c(s) \cdot G(s) = \frac{0.6649s + 0.0374}{0.5s^4 + 1.5037s^3 + 1.0112s^2 + 0.0075s}$$

3. The closed loop transfer function of the system

$$T(s) = \frac{0.6649s + 0.0374}{0.5s^4 + 1.5037s^3 + 1.0112s^2 + 0.6724s + 0.0374}$$

4. For the compensated system:
   - The phase margin, PM = $40.1°$
   - Gain Margin, GM = 12.4 dB
   - The Velocity error constant $K_v = 5$
   - Gain cross-over frequency $\omega_{gc} = 0.561$ rad/s
   - Phase cross-over frequency, $\omega_{pc} = 1.36$ rad/s

Effect of lag compensator on steady state error of the system

Lag compensators work by adding a small amount of delay to the system. This delay reduces the system's gain at high frequencies, which makes the system less responsive to high-frequency disturbances. This helps to reduce the system's tendency to oscillate around the desired output.

Maximum lag provided by a lag compensator

$$G_c(s) = K \cdot \frac{s+z}{s+p}$$

$$G_c(s) = K \cdot \frac{jw+z}{jw+p}$$

Phase provided by lag compensator:

$$\Phi = \tan^{-1}\left(\frac{\omega}{z}\right) - \tan^{-1}\left(\frac{\omega}{p}\right)$$

$$\text{At } \omega = \sqrt{zp}$$

$$\Phi = \tan^{-1}\left(\sqrt{\frac{p}{z}}\right) - \tan^{-1}\left(\sqrt{\frac{z}{p}}\right)$$

$$\Phi = \tan^{-1}(\frac{\sqrt{\frac{p}{z}} - \sqrt{\frac{z}{p}}}{2})$$

$$\Phi = \tan^{-1}\left(\frac{1-\alpha^2}{\sqrt{\alpha}}\right)$$

Hence the maximum lag that can be provided by the compensator is $\Phi = 90°$ when $\alpha = 0$. One way to provide a lag beyond the maximum range is by increasing the gap between the poles and zero, that is by placing a pole toward $+\infty$, so that the value of α becomes zero. But this would make the system unstable. Instead of using a single compensator, we can also cascade two or more lag compensators, so that the phase lag is beyond the maximum range. Another way is to use a lag-lead compensator. A lag-lead compensator is a type of compensator that can provide both phase lag and phase lead. This can be useful for systems that require a high phase lag at low frequencies and a high phase lead at high frequencies.