$$\int \frac{1}{\sqrt{1+x^2}}dx \tag{1}$$

を解きたい．

1　解法 1

$x = \tan\theta\,(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$ と変換．$\frac{dx}{d\theta} = \frac{1}{\cos^2\theta}$ となる．

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{1}{\cos\theta}d\theta \tag{2}$$

$$= \int \frac{\cos\theta}{1-\sin^2\theta}d\theta \tag{3}$$

$s = \sin\theta$ と変換．$\frac{ds}{d\theta} = \cos\theta$ となる．

$$\int \frac{\cos\theta}{1-\sin^2\theta}d\theta = \int \frac{1}{1-s^2}ds \tag{4}$$

$$= \frac{1}{2}\int\left(\frac{1}{1-s} + \frac{1}{1+s}\right)ds \tag{5}$$

$$= \frac{1}{2}\log\left|\frac{1+s}{1-s}\right| + C \tag{6}$$

$s = \sin\theta = \frac{x}{\sqrt{1+x^2}}$ だから，

$$\int \frac{1}{\sqrt{1+x^2}}dx = \frac{1}{2}\log\left|\frac{1+s}{1-s}\right| + C \tag{7}$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right| + C \tag{8}$$

$$= \log\left(x + \sqrt{1+x^2}\right) + C \tag{9}$$

2　解法 1.5

$s = \frac{x}{\sqrt{1+x^2}} (= \sin\theta)$ と変換．$x^2 = \frac{s^2}{1-s^2}$ となる．

$$\frac{ds}{dx} = \frac{1}{1+x^2}\left(\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}\right) \tag{10}$$

$$(1+x^2)\frac{ds}{dx} = \frac{1}{\sqrt{1+x^2}} \tag{11}$$

$$\frac{1}{1-s^2}\frac{ds}{dx} = \frac{1}{\sqrt{1+x^2}} \tag{12}$$

したがって，

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{1}{1-s^2}ds \tag{13}$$

$$= \frac{1}{2}\log\left|\frac{1+s}{1-s}\right| + C \tag{14}$$

$$= \log\left(x + \sqrt{1+x^2}\right) + C \tag{15}$$

3　解法 2

$x = \sinh t$ と変換．$1 + x^2 = 1 + \sinh^2 t = \cosh^2 t$ であり，$\frac{dx}{dt} = \cosh t$ である．

$$\begin{aligned} \int \frac{1}{\sqrt{1+x^2}} dx &= \int \frac{\cosh t}{\sqrt{1+\sinh^2 t}} dt && (16) \\ &= \int dt && (17) \\ &= t + C && (18) \end{aligned}$$

$\sinh t = \frac{e^t - e^{-t}}{2}$ だから，

$$\begin{aligned} x &= \frac{e^t - e^{-t}}{2} && (19) \\ 2xe^t &= e^{2t} - 1 && (20) \\ e^{2t} - 2xe^t - 1 &= 0 && (21) \\ e^t &= x + \sqrt{x^2+1} && (22) \end{aligned}$$

となる．最後の行では，$e^t > 0$ を用いた．したがって，

$$\begin{aligned} \int \frac{1}{\sqrt{1+x^2}} dx &= t + C && (23) \\ &= \log\left(x + \sqrt{1+x^2}\right) + C && (24) \end{aligned}$$

4　解法 2.5

$u = x + \sqrt{x^2+1} (= e^t)$ と変換．

$$\begin{aligned} \frac{du}{dx} &= 1 + \frac{x}{\sqrt{x^2+1}} && (25) \\ \frac{du}{dx} &= \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} && (26) \\ \frac{1}{u}\frac{du}{dx} &= \frac{1}{\sqrt{x^2+1}} && (27) \end{aligned}$$

だから，

$$\begin{aligned} \int \frac{1}{\sqrt{1+x^2}} dx &= \int \frac{1}{u} du && (28) \\ &= \log|u| + C && (29) \\ &= \log\left(x + \sqrt{1+x^2}\right) + C && (30) \end{aligned}$$