Game Theory: Assignment 3

Total points: 75

Contribution to grade: 7.5%

Due Date: 17/10/2025

Due time: 11:59 PM

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1. **Moral Hazard in the Workplace.** A firm offers either Contract 1 to a worker, which guarantees him a wage of $w = 26$ regardless of the outcome of the project or Contract 2, which gives him a salary of 36 when the outcome of the project is good (G) and 9 if the outcome is bad (B), i.e., $w_G = 36$ and $w_B = 9$. The worker can exert two levels of effort, high (e_H) or low (e_L). Probabilities $f(G|e_H) = 0.9$ and $f(B|e_H) = 0.1$. Similarly $f(G|e_L) = 0.5$ and $f(B|e_L) = 0.5$.

   The worker's utility function is $U_w(w, e) = \sqrt{w} - l(e)$, where $\sqrt{w}$ reflects the utility from the salary he receives, and $l(e)$ is the disutility of effort. $l(e_H) = 1$ and $l(e_L) = 0$.

   The payoff function for the firm is $90 - w$ when the outcome of the project is good, but decreases to $30 - w$ when the outcome is bad.

   (a) Represent the game tree of this sequential-move game. Wherever the is a random outcome, the player can be labeled as "Nature". (15)

   Solution.

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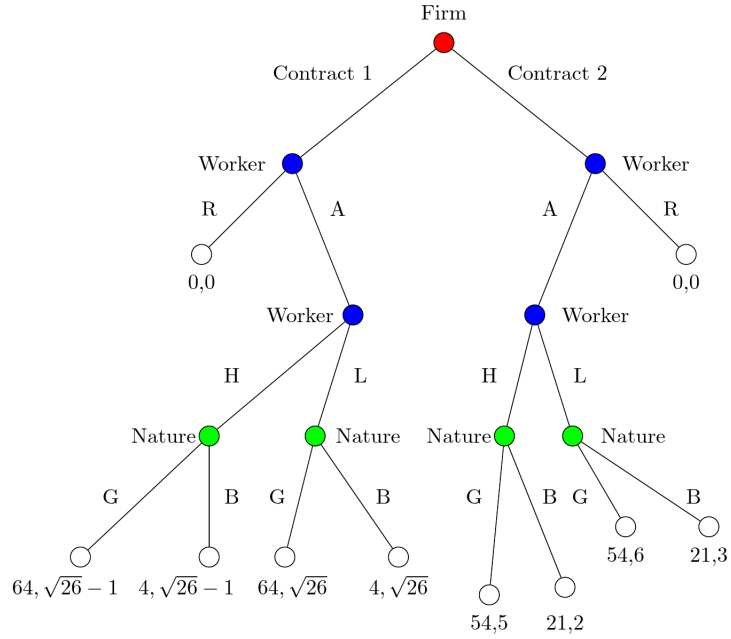
**You need not show the "R", i.e., Reject branches on the extreme left and right. You are required to show some work for the above solution (regarding how you arrived at the payoffs, but not a lot of detail, as most of it is relatively clear. The tree is worth 12 points and some explanation is worth 3 points. Note that you must incude nature in the tree. You must not reduce it to expected payoffs and end the game at the green dot.

(b) Find the SPNE of the game. (10)

Solution.

Let's first find the expected utility the worker obtains from exerting a high effort level.

$$\text{EU}_w(H \,|\, \text{Contract 1}) = f(G \,|\, e_H)(\sqrt{26} - 1) + f(B \,|\, e_H)(\sqrt{26} - 1)$$
$$= 0.9(\sqrt{26} - 1) + 0.1(\sqrt{26} - 1) = \sqrt{26} - 1$$

And the expected utility of exerting a low effort level under Contract 1 is

$$\mathrm{EU}_w(L \,|\, \text{Contract } 1) = 0.5\sqrt{26} + 0.5\sqrt{26} - 0 = \sqrt{26}$$

Hence, the worker exerts a low effort level, e_L if offered contract 1. Similarly.

$$\mathrm{EU}_w(H \,|\, \text{Contract } 2) = 0.9 \cdot 5 + 0.1 \cdot 2 = 4.7, \quad \text{and}$$

$$\mathrm{EU}_w(L \,|\, \text{Contract } 2) = 0.5 \cdot 6 + 0.5 \cdot 3 - 0 = 4.5$$

Hence, the worker exerts a high effort level, e_H if offered contract 2. Overall, the Player accepts either contract, if offered, but exerts low effort for contract 1 and high for contract 2. Now, for the firm,

$$\mathbb{E}\pi(\text{Contract } 1 \mid A, e_L) < \mathbb{E}\pi(\text{Contract } 2 \mid A, e_H), \text{ that is}$$

$$0.5 \cdot 64 + 0.5 \cdot 4 < 0.9 \cdot 54 + 0.1 \cdot 21$$

$$34 < 50.7$$

Hence, the SPNE is for the firm to offer contract 2, while the employer accepts and exerts high effort in equilibrium, but accept and exert low effort if offered contract 1 (off-the-equilibrium).
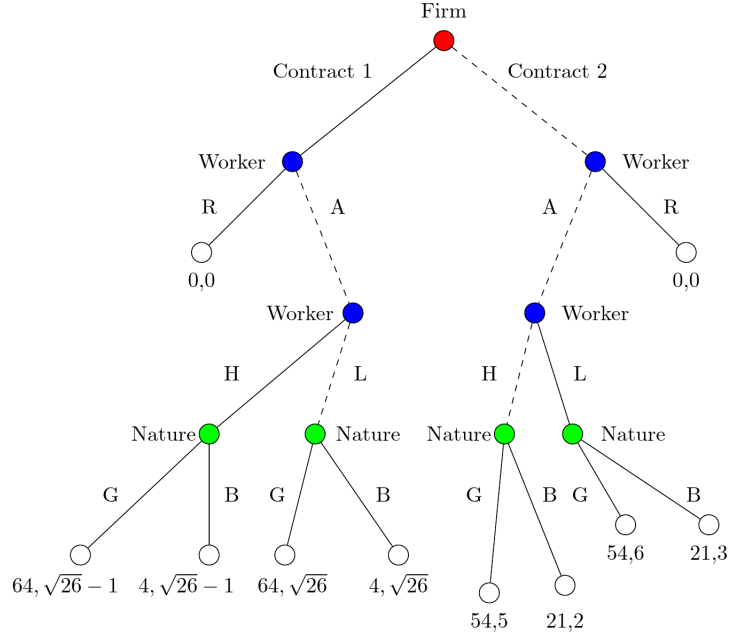
(c) Consider now the existence of a social security payment that guarantees a payoff of x to the worker. Find the value of x for which the worker chooses to exert a low effort level in equilibrium (here you have to consider the possibility that the worker rejects both contracts. We do not want this). Do not consider the social security amount x to be under the root like w. It will complicate things. Consider it to be linear. Note that you only get x if you reject the contract offered to you (as it is a social security payment)! (10)

Solution. In equilibrium, the employer offers Contract 2 and the worker exerts high effort. The social security payment needs to achieve that the worker exerts a low effort, the easiest way to guarantee this is by making the worker reject Contract 2. Anticipating such rejection, the employer offers Contract 1, which implies an associated low level of effort. That is, a social security payment of \$x induces the worker to reject Contract 2 as long as \$x satisfies

$$\text{EU}_w(A\,|\,\text{Contract 2},\, e_H) < \text{EU}_w(R\,|\,\text{Contract 2},\, e_H)$$

$$0.9 \cdot \sqrt{36} + 0.1 \cdot \sqrt{9} - 1 < x$$

which simplifies to $x > 4.7$. That is, the social security payment must be relatively generous. In order to guarantee that the worker still accepts Contract 1, we need that

$$\text{EU}_w(A\,|\,\text{Contract 1, } e_L) > \text{EU}_w(R\,|\,\text{Contract 1, } e_L)$$
$$0.5 \cdot \sqrt{26} + 0.5 \cdot \sqrt{26} > x$$

, thus implying that the social security payment cannot be too generous since otherwise the worker would reject both types of contracts. Hence, when the social security payment has to be in the range $x \in [4.7, \sqrt{26}]$.

2. **Profitable and Unprofitable Mergers.** Consider an industry with n identical firms competing à la Cournot. Suppose that the inverse demand function is $P(Q) = a - bQ$, where Q is total industry output, and $a, b > 0$. Each firm has marginal costs c, where $c < a$, and no fixed costs.

   (a) **No merger.** Find the equilibrium output that each firm produces at the symmetric Cournot equilibrium. What is the aggregate output and the equilibrium price? What are the profits that every firm obtains in the Cournot equilibrium? (10)

   Solution. Since this is a symmetric Cournot game with n firms, we have that each firm i solves

   $$\max_{q_i} \left(a - bQ_{-i} - bq_i \right) q_i - cq_i$$

   where $Q_{-i} \equiv \sum_{j \neq i} q_j$ denotes the aggregate production of all other $j \neq i$ firms. Taking first-order conditions with respect to q_i, we obtain

   $$a - bQ_{-i} - 2bq_i^* - c = 0.$$

   At the symmetric equilibrium $Q_{-i} = (n-1)q_i^*$. Hence, the above first-order

conditions become

$$a - b(n-1)q_i^* - 2bq_i^* - c = 0, \quad \text{or} \quad a - c = b(n+1)q_i^*.$$

Solving for q_i^*, we find that the individual output level in equilibrium is

$$q_i^* = \frac{a-c}{(n+1)b}.$$

Hence, aggregate output in equilibrium is

$$Q^* = nq_i^* = \frac{n}{n+1}\frac{a-c}{b},$$

while the equilibrium price is

$$p^* = a - bQ^* = a - b\frac{n}{n+1}\frac{a-c}{b} = \frac{a+cn}{n+1}.$$

And equilibrium profits that every firm i obtains are

$$\pi_i^* = (p^* - c)q_i^* = \frac{(a-c)^2}{(n+1)^2 b}.$$

(b) **Merger.** Now let m out of n firms merge. Show that the merger is profitable if and only if it involves a sufficiently large number of firms. (10)

Solution. Assume that m out of n firms merge. While before the merger there are n firms in this industry, after the merger there are $n - m + 1$. [For instance, if $n = 10$ and $m = 7$, the industry now has $n - m + 1 = 4$ firms, i.e., 3 nonmerged firms and the merged firm.] In order to examine whether the merger is profitable for the m merged firms, we need to show that the profit after the merger, π_{n-m+1}, satisfies $\pi_{n-m+1} \geq m\pi_n$,

$$\frac{(a-c)^2}{(n-m+2)^2 b} \geq m\frac{(a-c)^2}{(n+1)^2 b}.$$

Solving for n, we obtain that

$$n < m + \sqrt{m} - 1.$$

(c) Are the profits of the nonmerged firms larger when m of their competitors merge than when they do not? (10)

Solution. The output produced by the merged firms decreases (relative to their output before the merger), implying that each of the *nonmerged firms* earns larger profits after the merger because of $\pi_{n-m+1} \geq \pi_n$. This condition holds for mergers of any size, i.e., both when condition $n < m + \sqrt{m} - 1$ holds and otherwise. This surprising result is often referred to as the "merger paradox."

3. **Representing Feasible, Individually Rational Payoffs.** Consider an infinitely-repeated game where the stage game is depicted below. Players discount the future using a common discount factor δ.

Player 2

		L	R
Player 1	U	9,9	1,10
	D	10,1	7,7

(a) What outcomes in the stage-game are consistent with Nash equilibrium play? (2)

Player 2

|		L	R
Player 1	U	9,9	1,**10**
	D	**10**,1	**7,7**

The Nash equilibrium of the stage game is (D,R).

(b) Let v_1 and v_2 be the repeated game payoffs to player 1 and player 2 respectively. Draw the set of feasible payoffs from the repeated game, explaining any normalization you use. (3)

*Solution.* At first one may think the figure should look as follows:
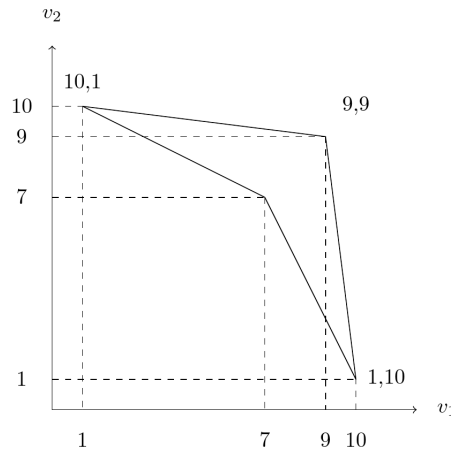
However, the above set of feasible payoffs in the repeated game is not convex. In order to address this non-convexity, we can "convexify" the set of feasible payoffs of the game by adding the green line. It is clear that in a repeated game, one can obtain an average per period payoff within the shaded region below (including the extended region).

Finally, the payoffs you get in each period are technically discounted (for example, if you are getting 9 in every period, your actual payoff is $\frac{9}{1-\delta}$. The value is then multiplied by $1 - \delta$ and we get a per period payoff. This is a footnote in your in class notes, and is the normalization I am referring to. This is only for 0.5 marks, but worth reading for clarity.
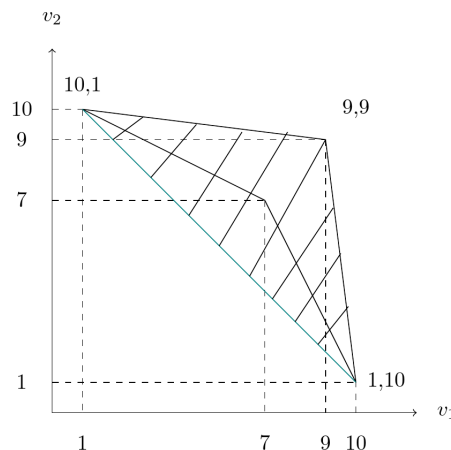
(c) Find the set of individually rational feasible set of payoffs. (2)

Solution. This is represented by the following region:
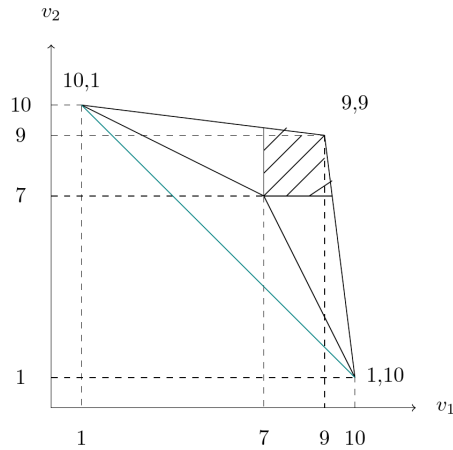
The additional constraints here are is $v_i \geq 7 \; \forall \; i \in \{1, 2\}$

(d) Find a SPNE equilibrium in which the players obtain $(9, 9)$ each period. What restrictions on δ are necessary? (3)

Solution. This is easy. Cooperation yields:

$$9 + 9\delta + 9\delta^2.... = \frac{9}{1 - \delta}$$

while defection yields

$$10 + 7\delta + 7\delta^2.... = 10 + \frac{7\delta}{1 - \delta}$$

Cooperation is thus possible if which yields $\delta \geq \frac{1}{3}$. (Note, we are assuming a grim trigger strategy, which is the harshest. More lenient punishments may be possible, but would need a higher δ).

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