Game Theory: Assignment 2

Total points: 75 | Due Date: 18/09/2025

Contribution to grade: 7.5% | Due time: 11:59 PM

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1. **Group Project (CO2, CO3).** A group project consists of $i \in \{1, 2\}$ students. Each student exerts and effort level e_i, and the aggregate effort level is given by $E = e_1 + e_2$. A students' grade function g_i is given by

$$g_i(e_i, e_j) = (t_i - e_i)\sqrt{mE}$$

where t_i is the total time available to player i and m (marks) is the aggregate returns on effort.

(a) Find the best response functions (BRF). Assume $t_1 = t_2$. (5)

Answer. To find the optimal effort level, we first need the best response function for best response function for every student i. $\frac{\partial g_i(e_i,e_j)}{\partial e_i}$ yields

$$-\sqrt{m(e_i + e_j)} + \frac{m(t_i - e_i)}{2\sqrt{m(e_i + e_j)}} = 0 \tag{E1}$$

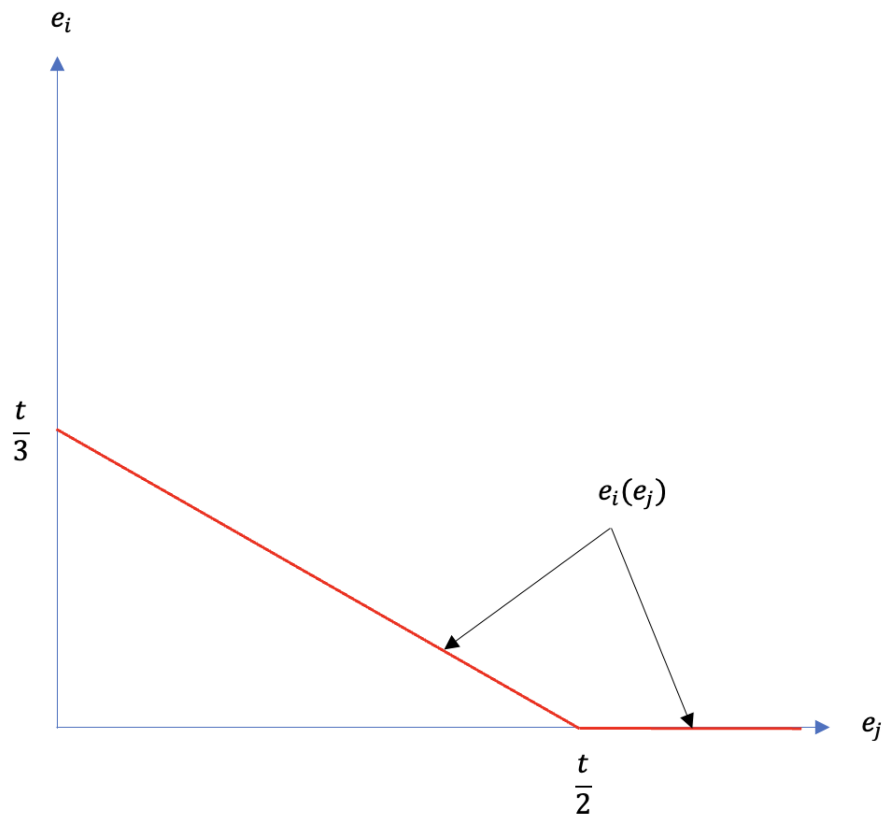
Plugging in $t_1 = t_2 = t$ and solving yields a BRF of

$$e_i(e_j) = \begin{cases} \frac{t}{3} - \frac{2}{3}e_j & \text{if } e_j < \frac{t}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

****The BRF is not correct without the "0 otherwise" condition**

(b) Show the the BRF on a graph. Is this a game of strategic substitutes of complements? Explain. Assume $t_1 = t_2$. (5)

Answer.



The slope of the BRF indicates that it is a hame of strategic substitutes. The figure is only complete with the flat section.

(c) Find the pure strategy Nash equilibrium. Assume $t_1 = t_2$. (5)

Answer. Solving the BRFs simultaneously yields $e_1^* = e_2^* = e^* = \frac{t}{5}$.

(d) Find the pure strategy Nash equilibrium. Assume $t_1 \neq t_2$. (5)

In this case, we have to go back to part (a) and solve (E1) without assuming symmetry on t_i. We first get a best response function of

$$e_i(e_j) = \begin{cases} \frac{t_i}{3} - \frac{2}{3}e_j & \text{if } e_j < \frac{t_i}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and then solving simultaneously yields an equilibrium effort level of $e_i^* = \frac{3t_i - 2t_j}{5}$. This is positive as long as $t_i > \frac{2}{3}t_j$ and 0 otherwise. What you basically find is that the more time you have, the more you contribute, but the more time your group mate has, the less you contribute. This is a variation of what is known as a public goods game. Please read about it.

Note: In the equilibrium solution I am only looking for e_i^* and the positivity condition. However, for completeness, $e_j^* = 0$, you have to check if e_i^* changes in response to the rival choosing 0. This is because of the asymmetry between t_i and t_j. For the main solution, I will not cut your marks for this, but if you did this, you will receive a bonus mark.

(e) Are equilibrium strategies Pareto efficient? Show for the case $t_1 = t_2$. (5)

Answer. For this question we again assume $t_i = t_j = t$. Let us check the effort levels if both students jointly maximize effort.

$$\max_{e_i, e_j \geq 0} (t - e_i)\sqrt{mE} + (t - e_j)\sqrt{mE}$$

Simplifying yields,

$$\max_{E \geq 0} (2t - E)\sqrt{mE}$$

where $E = e_i + e_j$. Since it is a joint maximization, we treat E as the choice variable. Taking the derivative wrt E and solving yields

$$-\sqrt{mE} + \frac{m(2t - E)}{2\sqrt{mE}}$$

We get $E^{**} = \frac{2t}{3}$ and $e^{**} = \frac{t}{3}$. To see Pareto efficiency, we have to compare how outcomes affect the payoffs. We can show each person is better off in this case by simply subtracting

$$g_i(e_i = e^{**}, e_j = e^{**}) - g_i(e_i = e^*, e_j = e^*) \approx 0.038t\sqrt{mt} > 0$$

Therefore, equilibrium strategies are not Pareto efficient.

2. **AI Innovation Game (CO2, CO3).** Consider firms $i \in \{1, 2\}$ simultaneously choosing an investment level $x_i \in \{0, 1, ..., k\}$ in R&D to find an AI driven solution to

identify rash driving, where $k > 2$. The winner gets a contract from the government to implement their product, while the loser gets noting. The payoff function is given by:

$$\pi_i(x_i, x_j) = \begin{cases} R_i - x_i & \text{if if } x_i > x_j \text{, and} \\ \\ -x_i & \text{otherwise.} \end{cases}$$

Note that $R_i > k$.

(a) Find a minimum of one pure strategy Nash Equilibrium, unless you find that there are no psNE. (5)

Answer. There does not exist a Nash equilibrium in pure strategies. Following is the explanation.

i. For $x_i = x_j = 0$, both firms can gain by deviating with a higher investment and winning the patent.

ii. For $x_i = x_j > 0$, both firms get a negative payoff and are better off not participating.

iii. $x_i < x_j < k$. Firm i can do better, s not an NE.

iv. $x_i < x_j = k$. i would be better off just not participating.

v. $0 = x_i < x_j = k$. This is the closest thing to what looks like an NE, but of course, j can do better by investing less and thus getting a higher profit.

(b) If I told you that one possible msNE was a case in which there was a positive distribution of probabilities across all values of x_i, can you state what the expected payoff for each strategy in support of the msNE would be? (10)

Answer. This is actually very easy. Since $x_i = 0$ is also in support of the msNE, the expected payoff for this is 0. For a mixed strategy to sustain a Nash equilibrium, all expected payoffs must be same, so all expected payoffs are 0.

3. N **player snob effect game (CO2, CO3).** Consider the following game with two possible actions for every player $i \in \{1, 2...., N\}$. Actions are x and y. The payoff from choosing x for a player i is a iff only if she is the only one choosing

x. Otherwise the payoff is b. On the other hand, the payoff from choosing y is c regardless of others' choices. $a > c > b$. Find the mixed strategy Nash equilibrium *(Hint: You will have to find some optimal p which will be a function of N).* (20)

Answer. This may not have been an easy question, had we not done a very similar one in class. It is quite straightforward if you follow that method. But I hope you did it for practice anyway!

$$EU_i(x) = a(1-p)^{N-1} + b\left[1 - (1-p)^{N-1}\right]$$

$$EU_i(y) = c$$

MSNE will be given by solving

$$a(1-p)^{N-1} + b\left[1 - (1-p)^{N-1}\right] = c \Rightarrow p^* = 1 - \left(\frac{c-b}{a-b}\right)^{\frac{1}{N-1}}$$

4. **Mixed Strategy Nash Equilibrium (CO2).** Consider the following normal form game:

1\2	L	R
L	1,1	1,0
M	4,0	-4,1
R	2,0	2,0

(a) Find the pure strategy Nash equilibrium, if any. (5)

Answer. To do this, we first eliminate L for player 1 as it is strongly dominated R. We can then find the PSNE easily, which is given by $\{R, R\}$.

1\2	L	R
~~L~~	~~1,1~~	~~1,0~~
M	4,0	-4,1
R	2,0	2,0

(b) Show the process of finding the mixed strategy Nash equilibrium. Draw the BRFs. What do you find? (10)

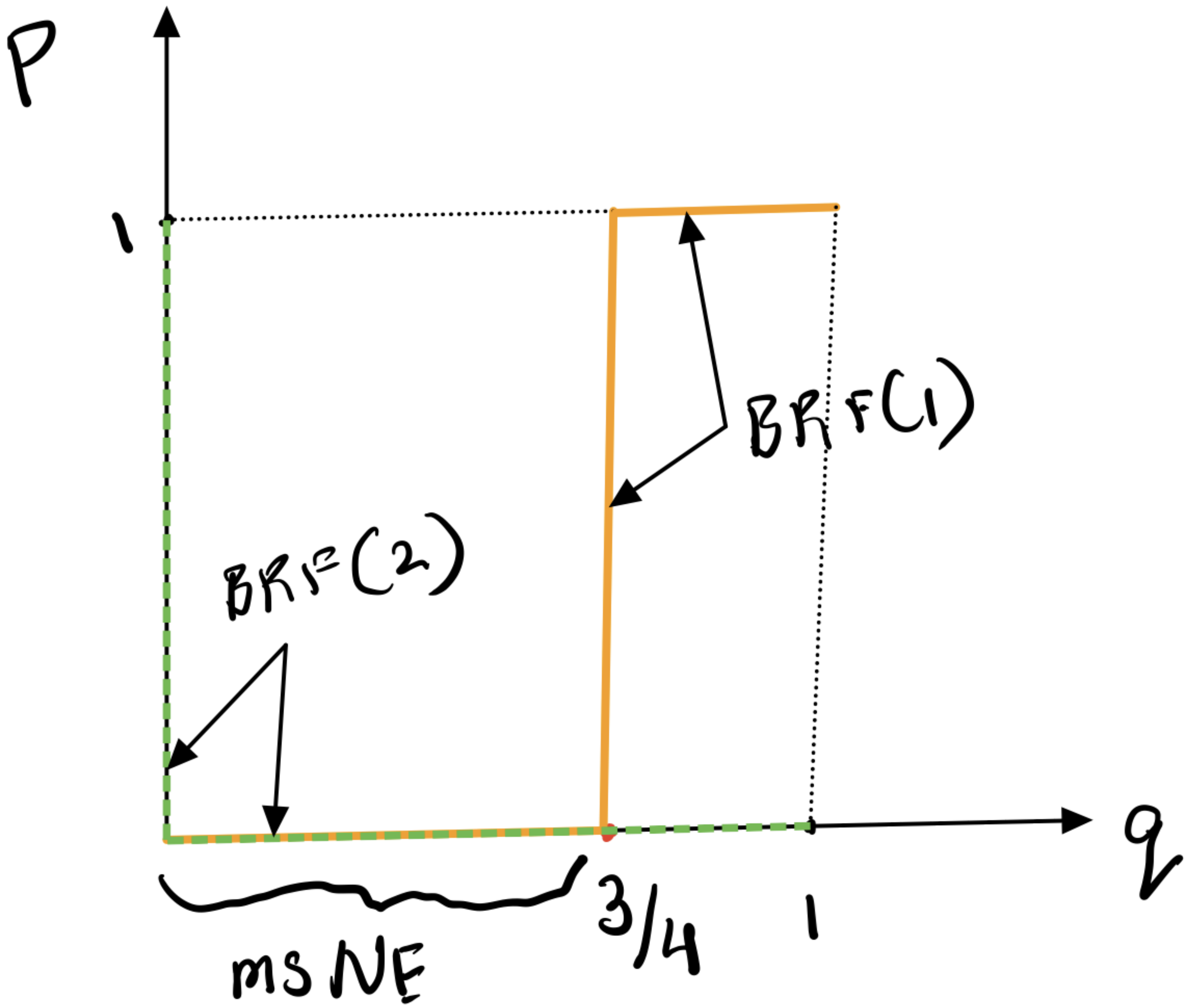
Answer. Let p be the probability of 1 choosing Mn ad q be the probability of 2 choosing L. we need to find the value for which

$$EU_1(M) = EU_1(R)$$

$$4q + (-4)(1-q) = 2q + 2(1-q) \Rightarrow q = \frac{3}{4}$$

$$EU_2(L) = EU_2(R) \Rightarrow 0p + 0(1-p) = p + 0(1-p) \Rightarrow p = 0$$

When $q < \frac{3}{4}$, 1 plays R, if $q = \frac{3}{4}$, 1 is indifferent and if $q > \frac{3}{4}$, 1 plays M. 2 is only indifferent when $p = 0$, and plays R otherwise. Thus, the Nash Equilibrium is player 1 playing R with probability 1 and player 2 playing M with probability $q \leq \frac{3}{4}$. Note that this solution includes the psNE. The BRFs give us.


p
1
BRF(1)
BRF(2)
q
3/4
1
msNE