Game Theory: Mid Semester Exam

Total points: 35 — Date: 22/09/2025

Contribution to grade: 35% — Time: 3:00 - 4:00 PM

- Show all steps, as it can help you get partial credit.
- For Part I, do 4/5 questions. Each question is worth 2.5 marks. If you submit all 5, the first 4 you did will be graded.
- For Part II, do 1/2 questions. Each question is worth 10 marks. If you submit both, the first one you did will be graded.
- Part III is compulsory, and is worth 15 marks.

Part I.

1. Consider a symmetric 2 player game where the payoff function for every player $i \in \{1, 2\}$ is given by $v_i(x_i, x_j) = x_i x_j$ where $i \neq j$. The choice variable for every player i is $x_i \in (0, 1)$. How many Nash equilibria does this game have? Identify them. *Update from exam: Only check for PSNE.*

   Answer. There are no Nash Equilibrium here in mixed or pure strategies. However, since, in the exam, I only asked for pure strategies, it is enough to say there are no pure strategy Nash Equilibrium *(1 mark).* The action set does not include its boundaries, as a result of which one can always choose a better response to their rival's choices (a higher number in the action set). This is clear in pure strategies. Though not asked in the exam, for mixed strategies, an extra step is needed. You have to show that for any mixed strategy over the action set, one can obtain a deviation with a higher expected payoff (closer to 1). This can be shown formally as well. *(1.5 marks)*

2. Suppose we modify the standard Bertrand dupoly game sightly to say that if $p_1 = p_2$, player 1 keeps all the profits, does that affect the pure strategy Nash Equilibrium? Explain. *Update from the exam: I clarified that in case of a tie, one player gets 0 (not $-c$) while the other gets all the profits. This is actually clear from the fact that player 1 only keeps all the profits, not all the revenue.*

   Answer. No it does not *(1 mark)*, as, in the event that prices are equal, player 2 still has the incentive to undercut by some $\epsilon > 0$. The only PSNE is $p_1 = p_2 = c$. *(1.5 marks)*

3. There is a concept called "partial cross ownership" in economics. The simplest interpretation of this is that firms have a share in each other's profits. However, firms cannot directly affect each other's decisions. Suppose we have a standard symmetric Cournot duopoly setting. Does partial cross ownership make collusion more or less likely? Why?

   Answer.

   - More likely. *(1 mark).*
   - Since a firm gains some portion of their rival's profits, they see some value to their rival doing well as well. Note that for a completely cooperative outcome, they would need 50% ownership of each other's profits. In general, they collude more as they own more in each other's profits.*(1.5 marks)*

4. Is it possible to arrive at a Nash equilibrium through IESDS by eliminating a "Pareto superior" outcome. Explain. *Update from the exam: I clarified that we can think of "possible" outcomes instead of just outcomes, to make the question simpler*

   Answer. Yes, it is. *(1 mark)* You cannot eliminate a Nash equilibrium by IESDS, but you can eliminate a Pareto superior outcome. A simple example of this is prisoner's dilemma, where not confessing for both is Pareto superior but gets eliminated. *(1.5 marks).*

5. Suppose a player i competing in a game has strategies $\{A, B, C\}$ at their disposal. The game has only one (mixed strategy) Nash equilibrium profile. In this MSNE,

the expected payoff from playing $A = B = 0.3$. C is not in support of the MSNE strategy for i. Given that the game is in MSNE, what can you then say about the payoff from playing pure strategy C?

Answer. The payoff has to be less than or equal to 0.3. Otherwise deviation would be optimal *(2.5 marks)*.

Part II.

1. **Anti coordination game.** Consider the following game where $c \in (0, 1)$

		Player 2: A	Player 2: B
Player 1	A	$1, 1$	$1 - c, 3$
	B	$3, 1 - c$	$0, 0$

   (a) Find the best response function of each player *(1 mark)*

   Answer. The BRF is given by

   - Player 2 chooses A: Player 1 prefers B ($3 > 1$).
   - If Player 2 chooses B: Player 1 prefers A ($1 - c > 0$).
   - By symmetry, we can write player 2's best responses are the same.

   Note that I did not ask for a graph, I asked for a function as above (defined relatively loosely). Functions are not always continuous. You can write it more formally as well, for example, $BR_1(A) = B$ and so on.

   (b) Find the Pure strategy Nash equilibria *(2 marks)*

   Answer. The PSNE are $\{\text{A}, \text{B}\}$ and $\{\text{B}, \text{A}\}$.

   (c) Find the Mixed strategy Nash equilibrium *(5 marks)*

   Answer. Let player 1 assign probability p to A and player 2 assign probability q to B.

$$EU_1(A) = q \cdot 1 + (1 - q)(1 - c) = 1 - c + cq$$

$$EU_1(G) = 3q$$

For indifference:

$$1 - c + cq = 3q \quad \Rightarrow \quad q = \frac{1-c}{3-c}.$$

which is above 0 since $1 - c > 0$ and less than 1 since $1 - c > 3 - c$. By symmetry, $p = \frac{1-c}{3-c}$. Calculate $1 - p$ and $1 - q$ to get the MSNE

$$\left\{\left(\frac{1-c}{3-c}A, \frac{2}{3-c}B\right), \left(\frac{1-c}{3-c}A, \frac{2}{3-c}B\right)\right\}$$

Note: You do not really need the graph here as you know $\{A, B\}$ and $\{B, A\}$ are PSNE already, so the intersection at the middle will be one more MSNE (like battle of the sexes), but to be safe you can draw it.

(d) If $c \in [0, 1]$, how are your results in part (c) affected by an increase in parameter c? *(2 marks)*

Answer. Since $\frac{\partial p}{\partial c} = -\frac{2}{(3-c)^2}$, as c increases, both players assign a lower probability to A. In the case that $c = 1$, we get a PSNE (B,B).

Note: Derivative not compulsory but easiest way to show it.

2. **Cost externalities.** Two firms ($N = 2$) exploit a common pool resource of initial stock S. Each firm i chooses appropriation $q_i \geq 0$. Firms take prices as given (normalized to 1), and $\theta \in [0, 1]$ denotes severity of cost externality. Total cost of production of q_i units is given by

$$C_i(q_i, q_j) = \frac{q_i(q_i + \theta q_j)}{S}, \quad \theta \in [0, 1].$$

(a) Construct the objective function of the firm i (think along the lines of Revenue - Costs) *(1.5 marks)*

Answer.

$$\max_{q_i \geq 0} \ \pi_i = q_i - \frac{q_i(q_i + \theta q_j)}{S}.$$

(b) Find the best response function of firm i and mention whether this is a problem of strategic substitutes or complements. *(2 marks)*

Answer. Differentiating w.r.t. q_i yields

$$1 - \frac{2q_i + \theta q_j}{S} = 0 \quad \Rightarrow \quad q_i(q_j) = \frac{S}{2} - \frac{\theta}{2} q_j.$$

Thus appropriation levels are strategic substitutes.

(c) How do S and θ affect the best response? Is there a possible parameter value for which this problem loses its game theoretic flavor? *(3 marks)*

Answer.

- Larger S leads to higher appropriation independent of q_j.
- Higher θ leads to a steeper best response (stronger negative externality). When $\theta = 0$, the i no longer responds to j's production.

(d) Find the symmetric PSNE q^* *(2 marks)*

Answer. In symmetric equilibrium $q_i = q_j = q^*$. Solving simultaneously yields

$$q^* = \frac{S}{2 + \theta}.$$

(e) What is the effect of *theta* on the q^* *(1.5 marks)*

Answer.

$$\frac{\partial q^*}{\partial \theta} = -\frac{S}{(2 + \theta)^2} < 0.$$

So higher θ reduces appropriation, since costs rise more with the rival's appropriation.

Part III.

1. **SPNE: A model of limit capacity.** This one is a little hard. Think carefully. First of all, consider a simple enough inverse demand function $p = 900 - q_1 - q_2$. In stage 1, firm 1 must decide investment level: N (does not enter market), S (Costs \$50,000, allows a firm to produce upto 100 units or L (costs \$175,000, allows a firm to produce any amount). After observing firm 1's investment decision, firm 2

chooses N,S or L. After they both decide their investment profile, they compete a la Cournot.

(a) Construct the game tree with payoffs *(10 marks)*

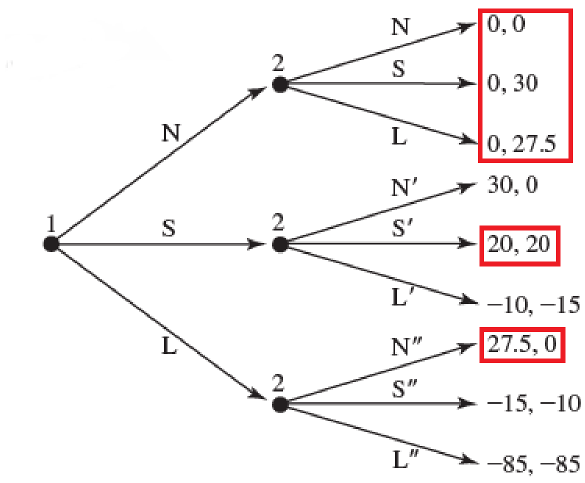
Ansnwer.

- If only firm i enters, its revenue is $(900 - q_i)q_i$, maximized at $q_i = 450$, yielding a revenue of \$202,500.
- Firm i can only produce this much if it had initially invested \$175,000. This leaves a profit of \$27,500.
- If instead, it invested in a small facility, it can produce a maximum of 100 units, which yields a profit of \$$(900 - 100)(100) - 50{,}000 = 30{,}000$.
- Now consider both firms in the industry.
- The revenue function now becomes $(900 - q_i - q_j)q_i$.
- The BRF is given by

$$BR_i(q_j) = 450 - \frac{q_j}{2}$$

- If neither firm has a capacity constraint, $q_1 = q_2 = 300$ and revenue = \$90,000.
- If both have capacity constraints and produce $q_i = 100$, revenue is \$70,000 for each.
- If one is capacity constrained (produces 100), while the other is unconstrained (produces $BR_i(100) = 400$, the constrained firm's revenue is \$40,000 while the unconstrained firm's revenue is \$160,000.
- If
  - Both firms have large facilities: Revenue = \$-85,000.
  - Both firms have small facilities: Revenue = \$20,000.
  - When one has large and the other has a small, the small firm gets \$-10,000 while the large one gets \$-15,000.
- Our analysis so far was for firms after they have both entered.

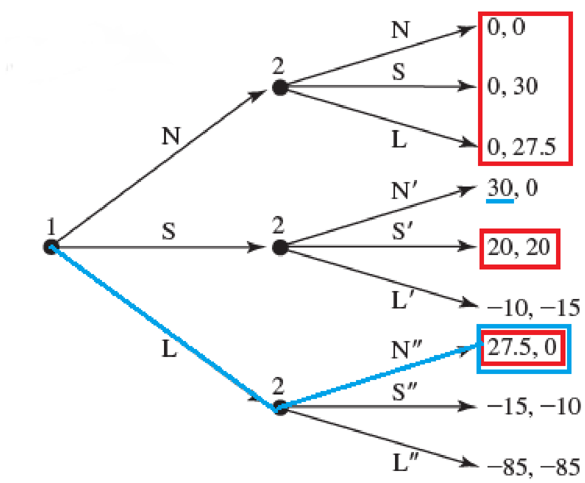
- The next step is to consider their entry decisions.
- The following pay-offs are computed in '000 dollars.
- The following game tree helps us understand how Firm 1 would invest in a small investment facility (and ear more profits), if there wasn't a potential follower.
- Instead, it chooses to invest in a larger facility to keep the follower out of the market.



**Source: Strategy by Joel Watson (Third Edition)

(b) Find the SPNE *(5 marks)*

Answer.



Thus, the SPNE is {L,SSN}.

Note 1 : Since we are yet to get used to SPNE, I will accept an answer that just shows the path. However, ideally, one must show the entire strategy set.

Note 2: The last question (Part III., will be graded relatively leniently, and will focus on checking your understanding and method. Not so for other questions.