I. (a) $P\{X > 15\} = e^{-15\lambda} = e^{-3/2} \approx 0.220$

(b) Normal distribution; mean=0 var=sum (i,j=1,n)k_ij/n

© hA=(1,1,1)

4. $2\frac{1}{2}$

. $N_1(t) \sim \text{Poisson}(\mu_1 t)$; $N_2(t) \sim \text{Poisson}(\mu_2 t)$

$N_1(t) \perp\!\!\!\perp N_2(t)$. $\Rightarrow N(t) \sim \text{Poisson}((\mu_1+\mu_2)t)$

(a) $N_1(t) \mid N(t) \stackrel{d}{=} ??$ $N(t) =$

$$P(N_1(t) \mid N(t)) = P(N_1(t) = x \cap N(t) = y) / P(N(t) = y)$$

$$= P[N_1(t) = x \cap \{N_1(t) + N_2(t)\} = y] / P(N(t) = y)$$

$$= P[N_1(t) = x \cap N_2(t) = y - x] / P(N(t) = y)$$

$$= P[N_1(t) = x] \cdot P(N_2(t) = y - x) / P(N(t) = y)$$

as $N_1(t) \perp\!\!\!\perp N_2(t)$

$$\cancel{\Rightarrow P[N_1(t) \mid N(t)] = \frac{e^{-\mu_1 t}(\mu_1 t)^x}{x!} \frac{e^{-\mu_2 t}(\mu_2 t)^{y-x}}{(y-x)!}}$$

$$\cancel{= \frac{e^{-(\mu_1+\mu_2)t} \left(\frac{\mu_1 t}{\mu_2 t}\right)^x (\mu_2 t)^y}{x!\,(y-x)!}}$$

$$P[N_1(t) \mid N(t)] = \frac{\dfrac{e^{-\mu_1 t}(\mu_1 t)^x}{x!} \; \dfrac{e^{-\mu_2 t}(\mu_2 t)^{y-x}}{(y-x)!}}{\dfrac{e^{-(\mu_1+\mu_2)t}((\mu_1+\mu_2)t)^y}{y!}}$$

$$= \frac{y!}{x!\,(y-x)!} \left(\frac{\mu_1 t}{(\mu_1+\mu_2)t}\right)^x \left(\frac{\mu_2 t}{(\mu_1+\mu_2)t}\right)^{y-x}$$

$\Rightarrow P[N_1(t) = x \mid N(t) = y] = \binom{y}{x} \left(\frac{\mu_1}{\mu_1+\mu_2}\right)^x \left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{y-x}$

: this is Binomial density (Clearly)

$\Rightarrow N_1(t) \mid N(t) \sim$ Binomial $\left(n = y;\ p = \frac{\mu_1}{\mu_1+\mu_2}\right)$.

(see $q = 1-p = \frac{\mu_2}{\mu_1+\mu_2}$).

Q 4/2

(5). $P(N_1(t) = 1 \mid N(t) = 1) = \frac{P(N_1(t) = 1 \cap N(t) = 1)}{P(N(t) = 1)}$

$= \frac{P(N_1(t) = 1 \cap (N_1(t) + N_2(t) = 1))}{P(N(t) = 1)}$

$= \frac{P(N_1(t) = 1 \cap N_2(t) = 0)}{P(N(t) = 1)}$

$= \frac{P(N_1(t) = 1)\ P(N_2(t) = 0)}{P(N(t) = 1)}$

$N_1(t)$ & $N_2(t)$ are indep.

and note $N(t) = N_1(t) + N_2(t) \sim$ Poisson $(\mu_1+\mu_2)t$.

$\Rightarrow P(N_1(t) = 1 \mid N(t) = 1) = \frac{e^{-\mu_1 t} (\mu_1 t)\ e^{-\mu_2 t}}{e^{-(\mu_1+\mu_2)t}\ (\mu_1+\mu_2)t}$

$$= \frac{\mu_1}{\mu_1 + \mu_2}$$

(3) $\{W(t) ; t \geq 0\}$; $E(W(t)) = 0$; $Cov(W(t), W(s)) = \min(s, t)$

$var(W(t)) = t$.

(4). $P(0 < W(1) + W(2) < 2 ; 3W(1) - 2W(2) > 0)$

let's write (or express in matrix notation).

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} W(1) \\ W(2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} W(1) + W(2) \\ 3W(1) - 2W(2) \end{bmatrix}$$

$\Rightarrow \underset{\sim}{X} = B \underset{\sim}{W}$.

Here, ~~$W \sim N(0, [K])$~~ $W(1)$, $W(2)$ both are normally distributed with $E(W(1)) = 0$; $E(W(2)) = 0$

$var[W(1)] = 1$; $var[W(2)] = 2$; $Cov(W(1), W(2)) = 1$

Thus. $[K] = Cov[W(1), W(2)] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

Now, as $\underset{\sim}{X} = B\underset{\sim}{W}$ thus we can write $\underset{\sim}{X}$ as a linear combination of std. normal. So $\underset{\sim}{X}$ follows Gaussian jointly. ; $\underset{\sim}{X}$ is a Gaussian vector with

$E(\underset{\sim}{X}) = E(B\underset{\sim}{W}) = B\, E(\underset{\sim}{W}) = \underset{\sim}{0}$.

$Cov(\underset{\sim}{X}) = Cov(B\underset{\sim}{W}) = B\, Cov(\underset{\sim}{W}) B^T$

$= B[K]B^T$.

$B[K]B^T = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$

The Matrix multiplication gives,

$B[K]B^T = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

Thus $var(X_1) = 5$; $var(X_2) = 5$; $Cov(X_1, X_2) = 0$.

$\Rightarrow X_1 \sim N(0,5)$; $X_2 \sim N(0,5)$; $X_1 \perp\!\!\!\perp X_2$

where $X_1 = W(1) + W(2)$; $X_2 = 3W(1) - 2W(2)$

Thus $P(0 < X_1 < 2, X_2 > 0)$

$= P(0 < X_1 < 2) \; P(X_2 > 0)$

$= P\left(\frac{0-0}{\sqrt{5}} < \frac{X_1 - 0}{\sqrt{5}} < \frac{2-0}{\sqrt{5}}\right) P\left(\frac{X_2 - 0}{\sqrt{5}} > \frac{0-0}{\sqrt{5}}\right)$

$= P\left(0 < Z < \frac{2}{\sqrt{5}}\right) P(Z > 0)$

$= \left[P(Z < 2/\sqrt{5}) - P(Z < 0)\right] \left[1 - \Phi(0)\right]$

$= \left[\Phi(2/\sqrt{5}) - \Phi(0)\right] \left[1 - \Phi(0)\right]$

where $Z \sim N(0,1)$; $\Phi(z) =$ cdf of Z

$\Rightarrow \left(\Phi(2/\sqrt{5}) - 0.5\right)(0.5)$

③ⓑ $X(t) = \rho W(t) + \sqrt{(1-\rho^2)}\, U(t)$

$E(X(t)) = \rho E(W(t)) + \sqrt{(1-\rho^2)}\, E(U(t))$

$= 0.$ as $E(W(t)) = 0 = E(U(t))$

$Cov[X(t), W(t)] = E[X(t) W(t)] - E(X(t)) E(W(t))$

$= E[(X(t)) W(t)]$

$= E[(\rho W + \sqrt{1-\rho^2}\, U) W]$

$= E[\rho (W(t))^2 + \sqrt{1-\rho^2}\, U(t) W(t)]$

$= \rho E[(W(t))^2] + \sqrt{1-\rho^2}\, E[U(t) W(t)]$

$= \rho \operatorname{var} W(t) + \sqrt{1-\rho^2}\, Cov(U(t), W(t))$

$= \rho t$ (as $U(t), W(t)$ indep. $\Rightarrow Cov = 0$)

Now, $Cor[X(t), W(t)] = \dfrac{Cov(X(t), W(t))}{\sqrt{\operatorname{var} X(t)}\sqrt{\operatorname{var} W(t)}}$

$= \dfrac{\rho t}{t} = \rho.$

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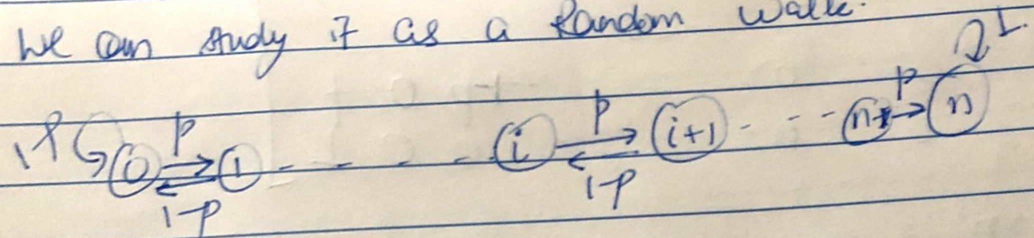
#q steps = n.

$P($Climing up$) = p$; $P($going down$) = 1-p$.

Let the boy (Harry) is at i-th step. If he is at i-th step then he can walk (Climb up or go down) in a random way. We can study it as a Random walk.



Thus if he reaches floor then he is at step 0 and can again start climbing. While if he gets the jar i.e reaches step "n" then the game is over.

Here, $X_t = i$; $i = i$-th step on the ladder.

$t = \{0, 1, 2, \ldots\}$ time space.

(b) Based on (a) the transition prbs matrix will be

$$\begin{array}{c|c}
 & \begin{matrix} 0 & 1 & 2 & \cdots & n-1 & n \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{matrix} &
\begin{bmatrix}
1-p & p & 0 & \cdots & 0 & 0 \\
1-p & 0 & p & \cdots & 0 & 0 \\
0 & 1-p & 0 & p \cdots & 0 & 0 \\
\vdots & & & & & \vdots \\
0 & 0 & 0 & \cdots 1-p & 0 & p \\
0 & 0 & 0 & \cdots 0 & 0 & 1
\end{bmatrix}
\end{array}$$

Clearly 'n' is an absorbing state here.

The row sum has to be 1. and $P(X_{t+1}=1 \mid X_t=0)=p$.

thus $P(X_{t+1}=0 \mid X_t=0)=1-p$.

(from 2 b).

2. i) $P(X_1 = 1 / X_0 = 0) = p_{01} = p$

ii) $\pi^{(0)} = \begin{array}{c} \begin{matrix} 0 & 1 & & & & n \end{matrix} \\ [1 \;\; 0 \; \cdots \cdots \; 0] \end{array}$

$\pi^{(1)} = \pi^{(0)} P$

$= [1-p \quad p \quad 0 \; \cdots \cdots \; 0]$

$\Rightarrow P(X_1 = 2) = 0.$

iii) $P(X_1 = 1, X_2 = 2, X_3 = 3) = P(X_3 = 3 / X_2 = 2, X_1 = 1) \; P(X_2 = 2 / X_1 = 1) \; P(X_1 = 1)$

$= P(X_3 = 3 / X_2 = 2) \; P(X_2 = 2 / X_1 = 1) \; P(X_1 = 1)$

(by Markov property)

$= p_{23} \, p_{12} \; P(X_1 = 1)$

$= p \cdot p \cdot p$ (from $\pi^{(1)}$ & P).

$= p^3$ $\quad 0 \leq p \leq 1$

Clearly there are 2 independent processes. The open periods are iid & so are closed periods.

Let X_i = i-th inter arrival time for open period
Similarly Y_i = i-th " " " closed "

$$E(X_i) = \mu_0 \quad \forall i$$
$$E(Y_i) = \mu_1 \quad \forall i$$

This can be studied as renewal process.

(b). By strong law for renewal process

$$\lim_{t \to \infty} \frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{w.p.1.}$$

$N_1(t)$ = # of times the restaurant remained open.
And by elementary renewal thm.

$$\lim_{t \to \infty} \frac{m(t)}{t} \to \frac{1}{\mu}.$$

$m(t) = E(N(t))$

$$\Rightarrow \lim_{t \to \infty} \frac{m_1(t)}{t} \to \frac{1}{\mu_0}$$

Thus, $m(t) \simeq \frac{t}{\mu_0}$ in the long run.

c) $p = 0.6 \quad q = 0.4$

Step 5, $\Rightarrow X_0 = 5 \qquad A = \{15\}$

P(steal the jar) $\Rightarrow P$ (hitting last step)

$$h_{iA} = \begin{cases} 1 & , \; i \in A \\ \sum_{j \in S} p_{ij} h_{jA} & , \; i \notin A \end{cases}$$

$$h_{iA} = \sum p_{ij} h_{jA}$$

$$= p_{i(i-1)} h_{(i-1)A} + p_{i(i+1)} h_{(i+1)A}$$ {using the transition matrix

$$h_{iA} = p\, h_{(i+1)A} + q\, h_{(i-1)A} \quad \text{——— ①}$$

$p + q = 1$

$$h_{iA} = h_{iA}\, p + h_{iA}\, q \quad \text{——— ②}$$

From ① & ②

$$h_{iA} p + h_{iA} q = p h_{(i+1)A} + q h_{(i-1)A}$$

$$p(h_{(i+1)A} - h_{iA}) = q(h_{iA} - h_{(i-1)A})$$

$$h_{(i+1)A} - h_{iA} = \frac{q}{p}(h_{iA} - h_{(i-1)A})$$

$$= \frac{q}{p} \cdot \frac{q}{p}(h_{(i-1)A} - h_{(i-2)A})$$

$$\vdots$$

$$\Rightarrow h_{(i+1)A} - h_{iA} = \left(\frac{q}{p}\right)^{i-1}(h_{2A} - h_{1A}) = \left(\frac{q}{p}\right)^{i}(h_{1A} - h_{0A})$$

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$h_{0A} = q h_{0A} + p h_{1A}$

$\Rightarrow h_{0A} = h_{1A}$

$\Rightarrow h_{0A} - h_{1A} = 0$

$\Rightarrow h_{i+1A} - h_{iA} = \left(\frac{q}{p}\right)^i (h_{1A} - h_{0A})$

$= 0$

$\Rightarrow h_{i+1A} = h_{iA}$

$h_{AA} = 1$

$\rightarrow$ Hitting Probability ~~for~~ of set A is 1 for all states.

$\Rightarrow$ Harry will be able to steal jar with probability 1.

If the ladder is very large, then also the chance of stealing the jar is 1.

2d) The entire computation remains same as ~~the~~ in 2c, only A becomes {7} here,

$h_{1A} = 1$.