Assignment 3 Solutions:

Question 1:

(a)

The amount x represents the offer that player 1 gives to player 2, and y is the amount of money player 2 gives to player 1 in return. Note that the utility function of the first mover is $u_1(x, y) = 10 - x + y$, since he retains $10 - x$ dollars after giving x dollars to player 2, and afterwards receives y dollars from player 2 (and we are assuming no discounting). The utility function of the second mover is $u_2(x, y) = 3x - y$, given that for every unit given up by player 1, x , player 2 gets $3x$ and gives y units to player 1.

Working by backwards induction, we need to find the optimal amount of y that player 2 will decide to give back to player 1 at the end of the game:

$$\max_{y \geq 0} u_2(x, y) = 3x - y$$

Obviously, since y enters negatively on player 2's utility function, the value of y that maximizes $u_2(x, y)$ is $y^*(x) = 0$ for any amount of money, x, received from player 1 which can be interpreted as player 2's best response function.

Player 1 can anticipate that, in the subgame originated after his decision, player 2 will respond with $y^*(x) = 0$. Hence, player 1 maximizes:

$$\max_{x} u_1(x, y) = 10 - x + y$$

$$s.t. y^*(x) = 0$$

That is, maximize:

$$\max_{x} u_1(x, y) = 10 - x$$

By a similar argument, since x enters negatively into player 1's utility function, the value of x that maximizes $u_1(x, y)$ is $x^* = 0$. Therefore, the unique SPNE is $(x^*, y^*) = (0,0)$ and a unique SPNE payoff vector $({u_1}^*, {u_2}^*) = (0,0)$.

(b)

Player 2 should always respond with $y^*(x) = 0$ along the equilibrium path for any offer x chosen by player 1 in the first period, i.e., responding with $y = 0$ is a strictly dominant strategy for player 2. Hence, there is no credible threat that could induce player 1 to deviate from $x = 0$ in the first period. Therefore, the unique NE strategy profile (and outcome) is the same as in the SPNE described above.

(c)

Operating by backwards induction, the altruistic player 2 chooses the value of y that maximizes the weighted sum of both players' utilities where δ denotes player 2's concern about player 1's utility.

$$\max_{y} u_2(x, y) + \delta u_1(x, y)$$

That is,

$$\max_{y} (3x - y) + \delta(10 - x + y)$$

Taking first order condition with respect to y and equating to 0,

$$\delta - 1 \leq 0 \Leftrightarrow \delta \leq 1$$

and, in an interior solution, where $y^* > 0$, we then have $\delta = 1$. (Otherwise, i.e., for all$\delta < 1$; we are at a corner solution where $y^* = 0$). Therefore, the only way to justify that player 2 would ever give some positive amount of y back to player 1 is if and only if $\delta \geq 1$. That is, if and only if player 2 cares about player 1's utility at least as much as he cares about his own.

Question 2:

(a)

Consider a market that consists of two firms offering differentiated products. The inverse demand that each firm faces is:

$p_i = a - bq_i - dq_j$ where $0 \leq d \leq b$

Consider the following two stage game. In the first stage each firm can commit either to supply a certain quantity $\overline{q_i}$ or to set a price $\overline{p_i}$. In the second stage, the remaining variables that were not chosen in the first stage are determined in order to clear the market (prices, if quantities were chosen in the first stage; or quantities, if prices were selected in the first stage). Show that a commitment to quantity in the first stage is a dominant strategy for each firm.

Operating by backward induction, let us first analyse the second stage of the game.

<u>Second Stage</u>

Part (a)

If both firms commit to quantities in the first stage, then firm $i's$ market demand is $p_i = a - bq_i - dq_j$

Part (b)

If both firms commit to prices in the first stage, then market demand is found by solving for q_i in $p_i(q_i, q_j)$, as follows:

$$q_i = \frac{a(b-d) - bp_i + dp_j}{(b-d)(b+d)}$$

Part (c)

If one firm commits to prices, $\overline{p_i}$, and the other to quantities, $\overline{q_j}$, then market demand for firm i is

$$\overline{p_i} = a - bq_i - d\overline{q_j} \Rightarrow q_i = \frac{a - d\overline{q_j} - \overline{p_i}}{b}$$

and that for firm j is

$$\overline{q_j} = \frac{a(b-d) - b\overline{p_i} - dp_j}{(b-d)(b+d)} \Rightarrow p_j = a - b\overline{q_j} - d\left(\frac{a - d\overline{q_j} - \overline{p_i}}{b}\right)$$

<u>First Stage</u>

If both firms commit to quantities, then from the demand function in point (a), we obtain the Cournot outcome. In particular, taking first order conditions with respect to q_i in:

$$\max_{q_i} \ (a - bq_i - dq_j)q_i$$

And we obtain, $a - 2bq_i - dq_j = 0$ which yields a best response function of

$$q_i(q_j) = \frac{a}{2b} - \frac{d}{2b}q_j$$

Simultaneously solving we get $q_i = q_j = \frac{(2b-d)a}{4b^2-2d^2}$ thus entailing equilibrium profits of $\Pi_i^C \equiv \frac{ba^2}{(2b+d)^2}$ for each firm.

If both firms commit to prices, from point (b), we obtain the Bertrand outcome. In particular, every firm i chooses its price level p_i that maximizes:

$$\max_{p_i} p_i \left(\frac{a(b-d) - bp_i + dp_j}{(b-d)(b+d)} \right)$$

Taking first order conditions with respect to p_i yields

$$\frac{a(b-d) - 2bp_i + dp_j}{b^2 - d^2} = 0$$

Solving for p_i:

$$p_i = \frac{a(b-d) + dp_j}{2b}$$

The price of firm j is symmetric. Hence, plugging these results into firm i's profit function, we obtain firm i's equilibrium profits for every firm:

$$\Pi_i^B \equiv \frac{(b-d)ba^2}{(b+d)(2b-d)^2}$$

Where $\Pi_i^B < \Pi_i^C$ for all parameter values.

From (c), we obtain a hybrid outcome, which can be found by maximizing firm i's profits:

$$\max_{\overline{p_i}} \left(\frac{a - d\overline{q}_j - \overline{p_i}}{b} \right) \overline{p_i}$$

Taking first-order conditions with respect to pi, since firm i commits in using prices, we obtain:

$$\frac{a - d\overline{q}_j - 2\overline{p}_i}{b} = 0 \ \rightarrow \overline{p_i}(\overline{q}_j) = \frac{a}{2} - \frac{d}{2}\overline{q}_j$$

Similarly, firm j, which commits to using quantities, maximizes profits

$$\max_{q_j}[a - b\overline{q}_j - d\left(\frac{a - d\overline{q}_j - \overline{p_i}}{b} \right)]\overline{q}_j$$

Taking first-order conditions with respect to $\overline{q}_j$ we find

$$a - 2b\overline{q}_j - \ d\left(\frac{a - 2d\overline{q}_j - \overline{p}_i}{b} \right) = 0 \ \rightarrow \overline{q}_j(\overline{p}_i) = \ \left(\frac{a(b-d) + d\overline{p}_i}{2(b^2 - \ d^2)} \right)$$

Substituting the expression, we found for $\bar{p}_i$ into $\bar{q}_j$, we have

$$\bar{q}_j = \left(\frac{a(b-d)+d(\frac{a-d\bar{q}_j}{2})}{2(b^2-\ d^2)}\right) \Rightarrow \ \bar{q}_j = \frac{2ab-ad}{4b^2-\ 3d^2}$$

Plugging the equilibrium quantity of firm j, q_j into $\bar{p}_i(\bar{q}_j) = \frac{a}{2} - \frac{d}{2}\bar{q}_j$

$$\bar{p}_i = \frac{a}{2} - \overbrace{\frac{d}{2}\left(\frac{2ab-ad}{4b^2-\ 3d^2}\right)}^{\bar{q}_j} = \frac{a(b-d)(2b+d)}{4b^2-\ 3d^2}$$

We can now find q_i by using the demand function firm i faces, $\bar{p}_i = a - b\bar{q}_i - \ d\bar{q}_j$, and the expressions for p_i and q_j we found above. In particular

$$\underbrace{\frac{a(b-d)(2b+d)}{4b^2-\ 3d^2}}_{\bar{p}_i} = a - bq_i - d\underbrace{\left(\frac{2ab-ad}{4b^2-\ 3d^2}\right)}_{\bar{q}_j}$$

Solving for q_i we obtain an equilibrium output of $q_i = \frac{a(b-d)(2b+d)}{b(4b^2-\ 3d^2)}$. Hence, the profits of the firm that committed to prices (firm i) are

$$\Pi_i^P = \bar{p}_i\,.\,q_i = \frac{a(b-d)(2b+d)}{(4b^2-\ 3d^2)}\cdot\frac{a(b-d)(2b+d)}{b(4b^2-\ 3d^2)} = \frac{a^2(b-d)^2(2b+d)^2}{b[(4b^2-\ 3d^2)]^2}$$

Similarly, regarding firm j, we can obtain its equilibrium price by using the demand function firm j faces, $p_j = a - b\bar{q}_j - dq_i$, and the expressions for q_j and q_i found above. Specifically,

$$p_j = a - b\left[\frac{a(2b-d)}{4b^2-3d^2}\right] - d\left[\frac{a(b-d)(2b+d)}{b(4b^2-3d^2)}\right] - \frac{a(b-d)(2b-d)(b+d)}{b(4b^2-3d^2)}$$

Thus, the profits of the firm that commits to quantities (firm j) are

$$\Pi_j^Q = p_j\bar{q}_j = \left[\frac{a(b-d)(2b-d)(b+d)}{b(4b^2-3d^2)}\right]\left(\frac{2ab-ad}{4b^2-3d^2}\right) = \frac{a^2(b-d)(2b-d^2)(b+d)}{b(4b^2-3d^2)^2}$$

Summarizing, the payoff matrix that firms face in the first period game is:

		Firm *j*	
		Prices	**Quantities**
Firm *i*	**Prices**	$\mathbf{\Pi_i^B, \Pi_j^B}$	$\mathbf{\Pi_i^P, \Pi_j^Q}$
	Quantities	$\mathbf{\Pi_i^Q, \Pi_j^P}$	$\mathbf{\Pi_i^C, \Pi_j^C}$

If firm j chooses prices (fixing our attention on the left column), firm i's best response is to select quantities since $\Pi_i^Q > \Pi_i^B$ given that d>0 by assumption. Similarly, if firm j chooses quantities (in the right column), firm i's best responds with quantities since $\Pi_i^C > \Pi_i^P$ given that d>0. Since payoffs are symmetric, a similar argument applies to firm j. Therefore, committing to quantities is a strictly dominant strategy for both firms, and the unique Nash equilibrium is for both firms to commit in quantities.

Question 3:

Consider two firms competing a la Cournot, and facing linear inverse demand $p(Q) = 100 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. For simplicity, assume that firms face a common marginal cost of production $c = 10$.

Unrepeated game. Find the equilibrium output each firm produces when competing a la Cournot (that is, when they simultaneously and independently choose their output levels) in the unrepeated version of the game (that is, when firms interact only once). In addition, find the profits that each firm earns in equilibrium.

Firm i's profit function is given by:

$$\pi_i(q_i, q_j) = [100 - (q_i + q_j)]q_i - 10q_i \tag{7}$$

Differentiating with respect to q_i yields,

$$100 - 2q_i - q_i - 10 = 0$$

Solving for q_i, we find

$$q_i(q_j) = 45 - \frac{q_j}{2} \qquad \text{(BRF}_7\text{)}$$

Since it is a symmetric game, we invoke symmetry in (BRF$_7$) and get,

$$q_i = 45 - \frac{q_i}{2}$$

Solving this yields $q_i^* = q_1^* = q_2^* = 30$. Substituting these into $\pi_1(q_1, q_2)$ and $\pi_2(q_1, q_2)$, we get $\pi_i(q_i^*, q_j^*) = \pi_1(q_1^*, q_2^*) = \pi_2(q_1^*, q_2^*)$.

$$\pi_i(q_i^*, q_j^*) = \pi_i(30{,}30) = [100 - (30 + 30)]30 - 10 \times 30 = \$900$$

This yields a profit of \$900 for each firm and a combined profit 0f \$900+\$900=\$1800.

Repeated game - Collusion. Assume now that the CEOs from both companies meet to discuss a collusive agreement that would increase their profits. Set up the maximization problem that firms solve when maximizing their *joint* profits (that is, the sum of profits for both firms). Find the output level that each firm should select to maximize joint profits. In addition, find the profits that each firm obtains in this collusive agreement.

In this case, each firm maximizes the sum $\pi_1(q_1, q_2) + \pi_2(q_1, q_2)$. Thus, each firm $i, j \in \{1,2\}$ maximizes,

$$\{[100 - (q_i + q_j)]q_i - 10q_i\} + \{[100 - (q_i + q_j)]q_j - 10q_j\} \tag{8}$$

Differentiating with respect to q_i, yields

$$100 - 2q_i - q_j - 10 - q_j = 0$$

Solving for q_i, we obtain

a) $q_i(q_j) = 45 - q_j$

Since the costs here are symmetric, we can invoke symmetry to get

b) $q_i = 45 - q_i$

Solving this gives $q_i^* = q_1^* = q_2^* = \frac{45}{2} = 22.5$.

This yields each firm a profit of

$$\pi_i(q_i^*, q_j^*) = \pi_i(22.5,22.5)$$
$$= \{[100 - (22.5 + 22.5)]22.5 - (10 \times 22.5)\}$$
$$+ \ \{[100 - (22.5 + 22.5)]22.5 - (10 \times 22.5)\} = \$1012.5$$

Thus, each firm makes a profit of 1012.5, which yields a joint profit of 2025.

Repeated game – Permanent punishment. Consider a grim-trigger strategy in which every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. Otherwise, every firm deviates to the Cournot equilibrium thereafter (that is, every firm produces the Nash equilibrium of the unrepeated game found in part a forever). In words, this says that the punishment of deviating from the collusive agreement is *permanent*, since firms never return to the collusive outcome. For which discount factors this grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

For this, let us first list all the payoffs from each decision. First, cooperation yields a payoff of 1012.5 for firm i. Defecting when firm j chooses to cooperate $(q_j = 22.5)$, by choosing Cournot output $(q_i = 30)$ will yield a profit of 1125. Finally, Cournot competition yields a profit of 900. The decision for the firm is as follows. The firm can either cooperate fir the entire period and get a stream of intermediate levels of profit, or defect and get a very high profit in that period. However, defection will lead to punishment in future periods, leading to low profits in subsequent periods. One has to keep in mind that a firm will always value those periods close to the present by a factor of δ. Thus, for cooperation to be sustained:

$$1012.5 + 1012.5\delta + 1012.5\delta^2 \ldots\ldots.. \geq 1125 + 900\delta + 900\delta^2 \ldots\ldots$$

$$\Rightarrow 1012.5(1 + \delta + \delta^2 \ldots..) \geq 1125 + 900\delta(1 + \delta + \delta^2 \ldots..)$$

$$\Rightarrow \frac{1012.5}{1-\delta} \geq 1125 + \frac{900\delta}{1-\delta}$$

$$\delta \geq 0.5$$

Thus, for cooperation to be sustainable in a permanent punishment game, the discount factor δ has to be at least 0.5.

Repeated game – Temporary punishment. Consider now a "modified" grim-trigger strategy. Like before, every firm starts colluding in period 1, and it keeps doing so as long as both firms colluded in the past. However, if a deviation is detected by either firm, every firm deviates to the Cournot equilibrium during only 1 period, and then every firm returns to cooperation (producing the collusive output). Intuitively, this implies that the punishment of deviating from the collusive agreement is now *temporary* (rather than permanent) since it lasts only one period. For which discount factors this "modified" grim-trigger strategy can be sustained as the SPNE of the infinitely-repeated game?

The set up now becomes such that cooperation is possible if:

$$1012.5 + 1012.5\delta + 1012.5\delta^2 \ldots. \geq 1125 + 900\delta + 1012.5\delta^2 \ldots.$$

The stream of payoffs after the punishment period (in this case, a one period punishment) returns to cooperation. Hence, the part after the punishment is equal on both sides of the equation and cancels out. We are left with,

$$\Rightarrow 1012.5 + 1012.5\delta \ldots. \geq 1125 + 900\delta$$

$$\Rightarrow \delta \geq 1$$

Thus, in this case cooperation is only possible if discount factor δ is at least 1. Since we generally define δ to be strictly between 0 and 1, cooperation is not possible with just single period punishment.

Let us now generalize our findings about the length of the temporary punishment facilitating collusive behavior. To do that, let us now consider a more general setting where the temporary punishment lasts $T \geq 1$ periods. Find under which conditions for firms' discount factor (delta) collusion can be sustained as the SPNE of the infinitely repeated game. How are your results affected by an increase in T?

In general, for a punishment of T periods, we have,

$$1012.5 + 1012.5 \sum_{i=1}^{T} \delta^i + 1012.5 \sum_{i=T+1}^{\infty} \delta^i \geq 1125 + 900 \sum_{i=1}^{T} \delta^i + 1012.5 \sum_{i=T+1}^{\infty} \delta^i$$

The stream of payoffs after the punishment period (in this case, a T period punishment) returns to cooperation. Hence, the part after the punishment is equal on both sides of the equation and cancels out. We are left with,

$$\Rightarrow 1012.5 + 1012.5 \sum_{i=1}^{T} \delta^i \geq 1125 + 900 \sum_{i=1}^{T} \delta^i$$

$$\Rightarrow 1012.5 \sum_{i=0}^{T} \delta^i \geq 225 + 900 \sum_{i=0}^{T} \delta^i$$

In the above step, we split 1125 as 900 + 225 so we can put 900 into the summation series. Note that the summation now starts from $i = 0$ instead of $i = 1$ because of this step

$$\Rightarrow 112.5 \sum_{i=0}^{T} \delta^i \geq 225$$

$$\Rightarrow \sum_{i=0}^{T} \delta^i \geq 2$$

Note that $\sum_{i=0}^{T} \delta^i$ is a finite geometric progression, and can be expressed as $\sum_{i=0}^{T} \delta^i = \frac{1-\delta^{T+1}}{1-\delta}$,

$$\Rightarrow \frac{1 - \delta^{T+1}}{1 - \delta} \geq 2$$

W rearrange to get,

$$\delta^{T+1} \leq 2\delta - 1$$

Taking Logs, we get

$$\ln \delta^{T+1} \leq \ln(2\delta - 1)$$

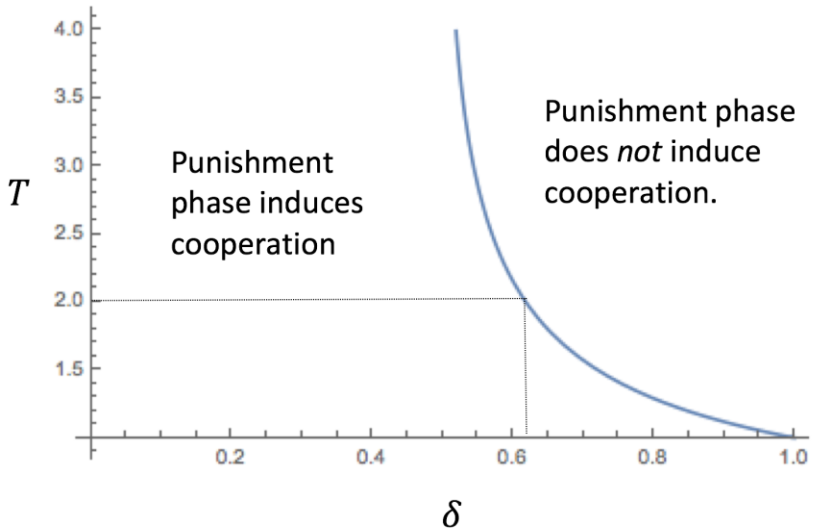
$$\Rightarrow (T + 1) \ln \delta \leq \ln(2\delta - 1)$$

$$\Rightarrow T + 1 \geq \frac{\ln(2\delta - 1)}{\ln \delta}$$

Note that the inequality flips since $\ln \delta$ is a negative number. This is because $\delta \in (0,1)$.

$$T \geq \hat{T} \equiv \frac{\ln 2\delta - 1}{\ln 2\delta} - 1$$

Plotting the above equation for $\delta \in (0,1)$ gives,



We can see from the above graph that as T increases, the δ required to support cooperation decreases. This makes intuitive sense as the more one cares about the future, the less one needs to be punished.