Worksheet-10
Course Name: Math-III (Section-A)
Total marks = 20
Date: 30/11/2022

1. Consider the force field $\mathbf{F}(x, y, z) = (e^y, xe^y + e^z, ye^z)$. (2+2=4 marks)

   (a) Show that $\int_C \mathbf{F} \cdot \overrightarrow{dr}$ is independent of the path, and find a potential function f for $\mathbf{F}$.

   (b) Determine the work done by moving along C arc of the ellipse $x^2 + 4y^2 = 16$ from (0,2,0) to (-4,0,0).

2. Given the vector force field $\mathbf{F}(x, y, z) = (4x - y, 3 - x, 2 - 4z)$. (2+2+2=6 marks)

   (a) Determine if $\mathbf{F}$ is conservative.

   (b) Find its potential function.

   (c) Find the line integral for any path from (0,1,0) to (2,4,1).

3. Compute the vector field $\mathbf{F}(x, y, z)$ associated with the potential function $\phi(x, y, z) = 5x^2 + 3y^2 + 10xyz^2$ and compute the work integral $\int_C \mathbf{F} \cdot \overrightarrow{dr}$ where C is the path given by $\overrightarrow{r(t)} = (2t - 5)\overrightarrow{i} + (3t + 4)\overrightarrow{j} + (4t - 6)\overrightarrow{k}$ over the interval $1 \leq t \leq 2$. (3 marks)

4. Give an example of a non-zero force field $\mathbf{F}$ and a path $\mathbf{r}(t)$ for which the total work done moving along the path is zero. (3 marks)

5. How are the constants a, b and c related if the following differential form

   $(ay^2 + 2czx)dx + y(bx + cz)dy + (ay^2 + cx^2)dz$

   is exact? (2 marks)

6. For what values of b and c will

   $\mathbf{F} = (y^2 + 2czx)\hat{i} + y(bx + cz)\hat{j} + (y^2 + cx^2)\hat{k}$

   be a gradient field? (2 marks)

A few extra Practice problems (NOT TO BE SUBMITTED)

7. Consider the vector field $\mathbf{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$.

   (a) Find a function f such that $\mathbf{F} = \nabla f$ and $f(0, 0, 0) = 0$.

   (b) Use part (a) to compute the work done by $\mathbf{F}$ on a particle moving along the curve C given by $\overrightarrow{r(t)} = (1 + 4\sin t)\overrightarrow{i} + (1 + 5\sin^2 t)\overrightarrow{j} + (1 + 5\sin^3 t)\overrightarrow{k}$, $0 \leq t \leq \pi/2$.

8. Let $\mathbf{F}(x, y) = (y^2 - x^2)\hat{i} + (x^2 + y^2)\hat{j}$ and let C be a triangle bounded by $y = 0, x = 3$ and $y = x$ oriented in the counterclockwise direction. Find the outward flux of $\mathbf{F}$ through C.

Rubric + Solution : Worksheet – 10

Q.1. Given $F(x,y,z) = (e^y,\ xe^y + e^z,\ ye^z)$

$M = e^y$, $N = xe^y + e^z$, $P = ye^z$

(a) $\frac{\partial P}{\partial y} = e^z = \frac{\partial N}{\partial z}$; $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$; $\frac{\partial N}{\partial x} = e^y = \frac{\partial M}{\partial y}$.

$\Rightarrow$ F is Conservative.

1 $\Rightarrow \int_C \vec{F} \cdot d\vec{r}$ is independent of path.

$\because$ F is conservative, this gives there is a function f with $\nabla f = F$.

$\therefore$ we have, $\frac{\partial f}{\partial x} = e^y$; $\frac{\partial f}{\partial y} = xe^y + e^z$; $\frac{\partial f}{\partial z} = ye^z$.

Now, $\frac{\partial f}{\partial x} = e^y$

Integrating, we get,

$f(x,y,z) = xe^y + g(y,z)$

$\frac{\partial f}{\partial y} = xe^y + e^z = xe^y + \frac{\partial g}{\partial y}(y,z)$

$\therefore \frac{\partial g}{\partial y}(y,z) = e^z$.

$\Rightarrow g(y,z) = ye^z + h(z)$

$\therefore f(x,y,z) = xe^y + ye^z + h(z)$.

Now again, $\frac{\partial f}{\partial z} = ye^z = ye^z + \frac{dh}{dz}$

$\Rightarrow \frac{dh}{dz} = 0 \Rightarrow h = C$ (constant)

1 $\therefore f(x,y,z) = xe^y + ye^z + C$.

The value of the integral is :

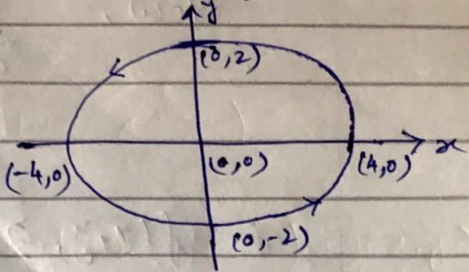
$$f(-4,0,0)-f(0,2,0) = -4+c-2-c = -6.$$

$$\therefore f(x,y,z) = xe^y + ye^z - 6.$$

(b) Work done :

$$x^2+4y^2=16.$$

$$\Rightarrow \frac{x^2}{4^2}+\frac{y^2}{2^2}=1.$$



$$\int_C \vec{F}\cdot d\vec{r}$$

2

$$= \int_{(0,2,0)}^{(-4,0,0)} \nabla f \cdot d\vec{r}$$

$$= f(-4,0,0)-f(0,2,0) = -4-6-(2-6) = -6.$$

Q.2. $F(x,y,z) = (4x-y,\ 3-x,\ 2-4z)$

$M = 4x-y$, $N = 3-x$, $P = 2-4z$.

(a) $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$; $\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$; $\frac{\partial N}{\partial x} = -1 = \frac{\partial M}{\partial y}$.

$\Rightarrow$ F is conservative. 2

(b) $\because$ F is conservative, so there is a potential function f such that $F = \nabla f$.

$$\Rightarrow \frac{\partial f}{\partial x} = 4x-y \ ; \ \frac{\partial f}{\partial y} = 3-x \ ; \ \frac{\partial f}{\partial z} = 2-4z.$$

Integrating $\frac{\partial f}{\partial x} = 4x-y$.

0.5 $\Rightarrow f(x,y,z) = 2x^2 - xy + g(y,z)$

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Now, $\frac{\partial f}{\partial y} = -x + \frac{\partial g}{\partial y} = 3 - x$.

$\therefore \frac{\partial g}{\partial y} = 3$.

$\Rightarrow g(y,z) = 3y + h(z)$ 0.5

~~Now,~~ $\therefore f(x,y,z) = 2x^2 - xy + 3y + h(z)$

Now, $\frac{\partial f}{\partial z} = \frac{dh(z)}{dz} = 2 - 4z$.

$\therefore h(z) = 2z - 2z^2 + c$. 0.5

$\therefore f(x,y,z) = 2(x^2 - z^2) - xy + 3y + 2z + c$

0.5

(c) $\int_{(0,1,0)}^{(2,4,1)} \vec{F} \cdot d\vec{r} = \int_{(0,1,0)}^{(2,4,1)} \nabla f \cdot d\vec{r} = f(2,4,1) - f(0,1,0)$

$= 2(4-1) - 8 + 12 + 2 + c - \{3 + c\}$ 2

$= 6 + 6 + c - 3 - c = 9$

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Q.4. There are many possible examples of a non-zero force field F, which is conservative and a path $r(t)$ (Not a point, but a closed loop/curve) for which the total work done $= \oint_C \vec{F} \cdot d\vec{r} = 0$, moving along the path.

One such example is $\rightarrow$

3

Let, $\vec{F}(x,y) = y\hat{i} + x\hat{j}$

Then $M = y$, $N = x$.

So, $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x} \Rightarrow F$ is conservative.

Let, $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$; $0 \le t \le 2\pi$

(closed circle of radius 1)

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$\therefore$ Work done $= \int_0^{2\pi} \vec{F} \cdot d\vec{r}$

$= \int_0^{2\pi} \left(\sin t\, \hat{i} + \cos t\, \hat{j} \right) \left(-\sin t\, \hat{i} + \cos t\, \hat{j} \right) dt$

$= \int_0^{2\pi} \cos 2t \; dt$

$= \left[\frac{\sin 2t}{2} \right]_0^{2\pi}$

$= 0.$

(A force is conservative if the work it does around any closed path is zero
$W_{\text{closed path}} = \oint \vec{F}_{\text{Conservative}} \cdot d\vec{r} = 0$).

Q.5. Since $(ay^2 + 2czx)dx + y(bx + cz)dy + (ay^2 + cx^2)dz$ is exact.

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$; $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$; $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

1

Where $M = ay^2 + 2czx$, $N = bxy + cyz$, $P = ay^2 + cx^2$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy \Rightarrow c = 2a \quad \forall y$

$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$

1

& $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2ay \Rightarrow 2a = b . \quad \forall y.$

$\therefore b = 2a$ & $c = 2a.$

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Q.6. $F = (y^2 + 2czx)\hat{i} + y(bx + cz)\hat{j} + (y^2 + cx^2)\hat{k}$

Given,
~~Now~~, $F = \nabla f$; a gradient field.

2

so, $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2y = cy \Rightarrow c = 2$.

$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$

& $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2y \Rightarrow b = 2$.

$\therefore$ The values of b & c are 2 for that F will be a gradient field.

—— x —— x —— x ——

[Alternative to Q.3 solution of worksheet - 10]

$F = \nabla\phi$

$= \nabla(5x^2+3y^2+10xyz^2)$

$= (10x+10yz^2)\hat{i} + (6y+10xz^2)\hat{j} + 20xyz\hat{k}$

0.5

$M \quad N \quad P$

$\frac{\partial P}{\partial y} = 20xz = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 20yz = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 10z^2 = \frac{\partial M}{\partial y}$

$\therefore$ F is conservative.

Now, work done

$= \int_C \vec{F}\cdot d\vec{r} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$

$= \int_C \nabla\phi\cdot\frac{d\vec{r}}{dt}dt$.

0.5

$F_x = 10x+10yz^2 = 10(2t-5)+10(3t+4)(4t-6)^2$

$= 20t-50+(30t+40)(16t^2-48t+36)$

$= 20t-50+480t^3-1440t^2+1080t$
$+640t^2-1920t+1440$

$= 480t^3-800t^2-820t+1390$.

$\int_C F_x\cdot r_{tx} = \int_C (\qquad)\times 2$

$= \int_C 20(48t^3-80t^2-82t+139)$

$$= 20\int_1^2 (48t^3 - 80t^2 - 82t + 139)\,dt$$

$$= 20\left[\cancel{\frac{48}{4}}^{12}(2^4 - 1^4) - \frac{80}{3}(2^3 - 1^3) - \frac{82}{2}(2^2 - 1^2) + 139(2-1) \right]$$

$$= 20\left[12 \times 15 - \frac{80 \times 7}{3} - 41 \times 3 + 139 \right]$$

$$= 20\left[196 - \frac{560}{3} \right]$$

0.5

$$= \frac{20 \times 28}{3} = \frac{560}{3}.$$

$$F_y = 6y + 10xz^2$$

$$= 6(3t+4) + 10(2t-5)(4t-6)^2$$

$$= 18t + 24 + (20t - 50)(16t^2 - 48t + 36)$$

$$= 18t + 24 + 320t^3 - 960t^2 + 720t - 800t^2 + 2400t - 1800$$

$$= 320t^3 - 1760t^2 + 3138t - 1776.$$

$$F_y \cdot r_{ty} = (320t^3 - 1760t^2 + 3138t - 1776) \times 3.$$

$$\int_1^2 = \cancel{\frac{320 \times 3}{4}}^{80}(2^4 - 1^4) - \frac{1760 \times \cancel{3}}{\cancel{3}}(2^3 - 1^3) + \frac{3138 \times 3}{2}(2^2 - 1^2)$$

$$- 1776 \times 3(2-1)$$

$$= 3600 - 12320 + 14,121 - 5328$$

0.5

$$= 73.$$

$$F_z = 20xyz = 20(2t-5)(3t+4)(4t-6)$$

$$F_z \cdot r_{tz} = [20(6t^2 - 7t - 20)(4t-6)] \times 4$$

$$= 80[24t^3 - 28t^2 - 80t - 36t^2 + 42t + 120]$$

$$\int_1^2 = 80\int_1^2 (24t^3 - 64t^2 - 38t + 120)\,dt.$$

$$= 80 \times \left[\frac{\overset{6}{\cancel{24}}}{\cancel{4}}(2^4-1^4) - \frac{64}{3}(2^3-1^3) - \frac{38}{2}(2^2-1^2) + 120\times 1\right].$$

$$= 80\left[6\times15 - \frac{64\times 7}{3} - 19\times 3 + 120\right]$$

$$= 80\left[153 - \frac{448}{3}\right]$$

0.5

$$= \frac{80\times 11}{3} = \frac{880}{3}.$$

$$\therefore \int_1^2 = \frac{560}{3} + 73 + \frac{880}{3}.$$

$$= \frac{1440}{3} + 73.$$

$$= 480 + 73$$

$$= 553.$$

0.5

— x —

$x = -1, 3$

$y = 3t + 4$

$y = 10, 7$

$z = 4t - 6$

$z = 2, -2$

$x = 2t - 5$

$\frac{x+5}{2} = t$

$x = 4 - 5 = -1$

$\frac{x+5}{2} = 1 \Rightarrow x = -3$

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Q.3. Work done (Corrected Answer)

$$= \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla\phi \cdot d\vec{r} \quad 1$$

$$= \phi(2) - \phi(1) \quad 1$$

$$= \Big[5(2t-5)^2 + 3(3t+4)^2 + 10(2t-5)(3t+4)(4t-6)^2\Big]_1^2$$

$$= 5\times(-1)^2 + 3(10)^2 + 10\times(-1)\times(10)\times(2)^2 - \{5\times(-3)^2 + 3\times(7)^2 + 10\times(-3)\times(7)\times(-2)^2\}$$

$$= 5 + 300 - 400 - 45 - 147 + 840$$

$$= 553 \text{ (Ans)} \quad 1$$