Game Theory: End Semester Exam (Solutions)

Total points: 35 | Date: 22/09/2025

Contribution to grade: 35% | Time: 3:00 - 4:00 PM

- Show all steps, as it can help you get partial credit.
- For Part I, do 4/5 questions. Each question is worth 2.5 marks. If you submit all 5, the first 4 you did will be graded (approx 15 mins).
- For Part I, do 1/2 questions. It is worth 10 marks. If you submit both, the first one you did will be graded (approx 15 mins).
- Part III is compulsory, and is worth 15 marks (approx 30 mins).

Part I.

1. Consider a 2 player Second Price Auctions (SPA) with complete information, i.e., player 1 and 2 observe each other's valuations for the good (given by v_1 and v_2). The amount the winner pays is defined by the rules of the SPA, and is given by x_i where $i \in \{1, 2\}$. Player 1's utility if he wins is $v_1 - x_1$ and 0 if he loses. Player 2's utility on the other hand is $v_2 - x_2$ when he wins, but $-(v_1 - x_1)$ when he loses, i.e., he is spiteful/jealous. $v_1 > v_2$. Bids are given by b_i. What is the PSNE of this game?

   Answer. The PSNE is $b_1 = b_2 \in [\frac{v_1+v_2}{2}, v_1]$. (2)

   - In this set of profiles, if player 1 bids lower, he gets 0 and if he gets higher, he gets the same payoff.
   - Player 2 is faced with a more interesting problem. Up until $\frac{v_1-v_2}{2}$, he is willing to go over P1 and take a loss as it prevents P1 from winning. However, beyond that, it is no longer worth it and he just bids as high as possible to minimize P1's payoff without actually winning the good.

1 mark for explanation. It is anyway not possible to arrive at this answer without some explanation.

2. Fundamentally, what is the difference between the way we solve a finitely repeated prisoner's dilemma and an infinitely repeated one? (Hint: The difference is what allows us to sustain cooperation under some conditions).

   Answer. If you said the difference is δ, that is wrong, as δ in a finitely repeated game would not help us sustain cooperation. The answer is that we do not solve an infinitely repeated game by backward induction, as a result of which we do not start from the last round, where cheating is always optimal, and work backwards. Here, to characterize equilibrium, we consider a stream of payoffs from the current point, and show that the expected value of the stream remains identical regardless of where we star the game assuming the same set of actions.

3. Suppose we consider something like a "Last Price Auction"instead of the highest bidder, the lowest bidder wins the game, and pays her bid only if she wins (this is typical of tenders). How does the optimization problem look in general form (no need to solve it, just show me the final optimization problem just before the FOC. N players, a symmetric bidding function $b_i :\to \mathbb{R}_+$ function mapping her valuation $v_i \in [0, 1]$ to a positive monetary amount. Player i's bid is given by x_i I'm not looking for a lot of details and steps. Just setup the problem and define anything you assume.

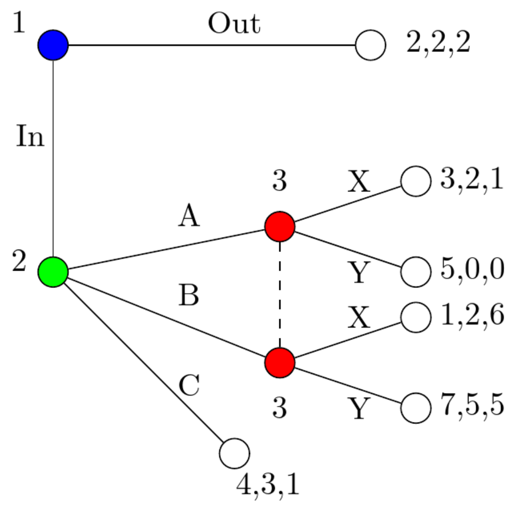
   Answer. The weakly dominant strategy is to bid 0. The symmetric bidding function is $b(v_i) = 0 \cdot v_i$. However, if you were unable to arrive at this and set it up in the way it is done in the notes, that is enough.

4. Why are beliefs about off-the-equilibrium-path actions essential in defining a Perfect Bayesian Equilibrium? Give a short intuitive explanation of what goes wrong if we ignore them.

   Answer. They are sometimes needed to help understand what a player would do if they saw a deviation from the proposed equilibrium. We saw this in some pooling equilibrium questions.

Part II.

1. Find the SPNE of the following game (Don't worry about any MSNE)



(a) Represent the game in normal form. Show all the NE. *Hint: Draw 2 tables, one for each of player 1's actions.*

Answer.

Player 1 Plays In

Player 2 \ Player 3	X	Y
A	**3**,2,**1**	**5**,0,0
B	1,2,**6**	**7**,**5**,5
C	**4**,**3**,**1**	**4**,3,**1**

Player 1 Plays Out

Player 2 \ Player 3	X	Y
A	2,**2**,**2**	2,**2**,**2**
B	**2**,**2**,**2**	2,**2**,**2**
C	2,**2**,**2**	2,**2**,**2**

The 2 NEs are (In,C,X) and (Out,B,X).

(b) Which of the above equilbria are SPNE. Show how.

Answer. As a first step, we have to isolate the smallest proper subgame for backward induction. In this case, it may be a bit confusing. A subgame has to start at a node. If we consider it to start at the point where player 2 plays, we get

		Player 3	
		X	Y
Player 2	A	2,**1**	0,0
	B	2,**6**	**5**,5
	C	**3**,**1**	3,**1**

. The NE of this simultanous move game is (C,X). Thus, if player 1 chooses In she gets 4, and if she chooses out she gets 2. The SPNE path is thus (In, C) with off the equilibrium strategy of X by player 3. From your normal Form game, In,C,X was an NE and is also SPNE.

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The alternate way is not the right way to approach it as a subgame has to include all the paths coming out of its first node. However, I will consider it.

Suppose you isolate the simultaneous move part. This is when player 2 chooses A or B. This has to be solved as to a 2 X 2 simultaneous move game.

		Player 3	
		X	Y
Player 2	A	**2**,**1**	0,0
	B	**2**,**6**	**5**,5

The two possible NE yields payoffs of 2 for player 2. Player 2 can instead choose C, which will yield 3. Thus, if player 1 chooses In she gets 4, and if she chooses out she gets 2. The SPNE path is thus (In, C) with off the equilibrium strategy of X by player 3. From your normal Form game, In,C,X was an NE and is also SPNE.

2. Consider the linear Cournot model we often discuss in class. The two firms 1 and 2, simultaneously choose the quantities they will sell on the market, q_1 and q_2. The price each receives for each unit given these quantities is

$$P(q_1, q_2) = a - b(q_1 + q_2).$$

Now, however, suppose that each firm has probability μ of having unit costs of c_L and $(1-\mu)$ of having unit costs of c_H, where $c_H > c_L$. Solve for the Bayesian Nash equilibrium.

Answer. A firm of type $i = H$ or L will maximize its expected profit, taken as given that the other firm will supply q_H or q_L depending whether this firm is of type H or L. A type $i \in \{H, L\}$ firm 1 will maximize:

$$\max_{q_i^1 \geq 0} (1-\mu)\left[(a - b(q_i^1 + q_H^2) - c_i)q_i^1\right] + \mu\left[(a - b(q_i^1 + q_L^2) - c_i)q_i^1\right]$$

The FOC yields:

$$(1-\mu)\left(a - b(2q_i^1 + q_H^2) - c_i\right) + \mu\left(a - b(2q_i^1 + q_L^2) - c_i\right) = 0$$

In a symmetric Bayesian Nash equilibrium:

$$q_H^1 = q_H^2 = q_H \quad \text{and} \quad q_L^1 = q_L^2 = q_L$$

Plugging this into the F.O.C we get the following two equations:

$$(1-\mu)\left[a - 3bq_H - c_H\right] + \mu\left[a - b(2q_H + q_L) - c_H\right] = 0$$
$$(1-\mu)\left[a - b(q_H + 2q_L) - c_L\right] + \mu\left[a - 3bq_L - c_L\right] = 0$$

Therefore, we obtain that

$$q_H = \left[a - c_H + \frac{\mu}{2}(c_L - c_H)\right]\frac{1}{3b},$$

$$q_L = \left[a - c_L + \frac{1-\mu}{2}(c_H - c_L)\right]\frac{1}{3b}.$$

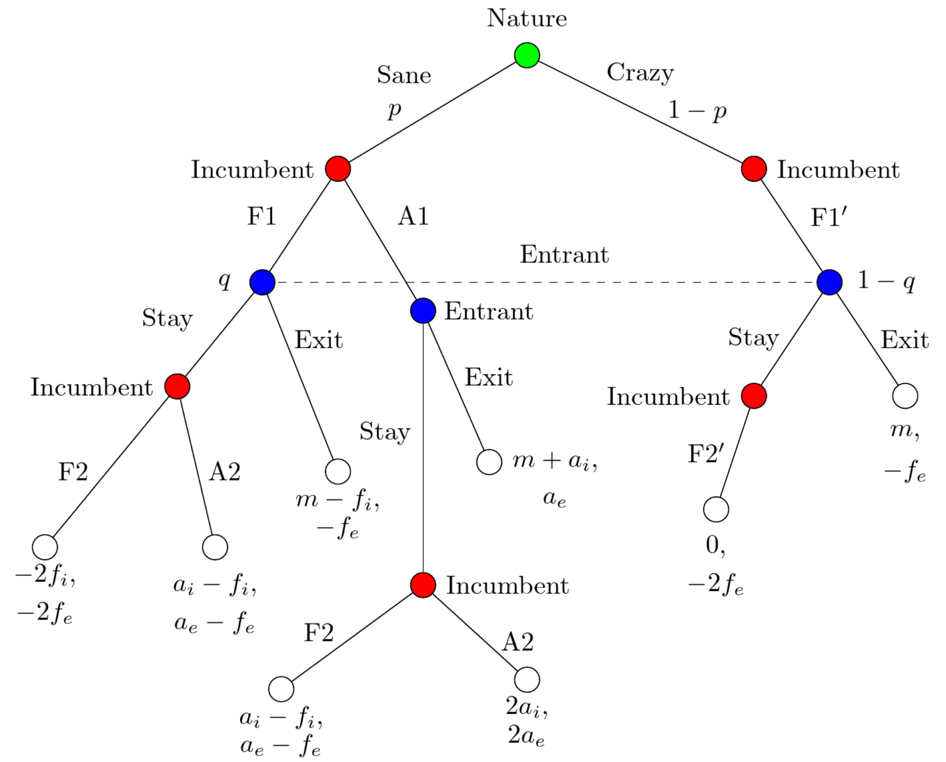
Part III.

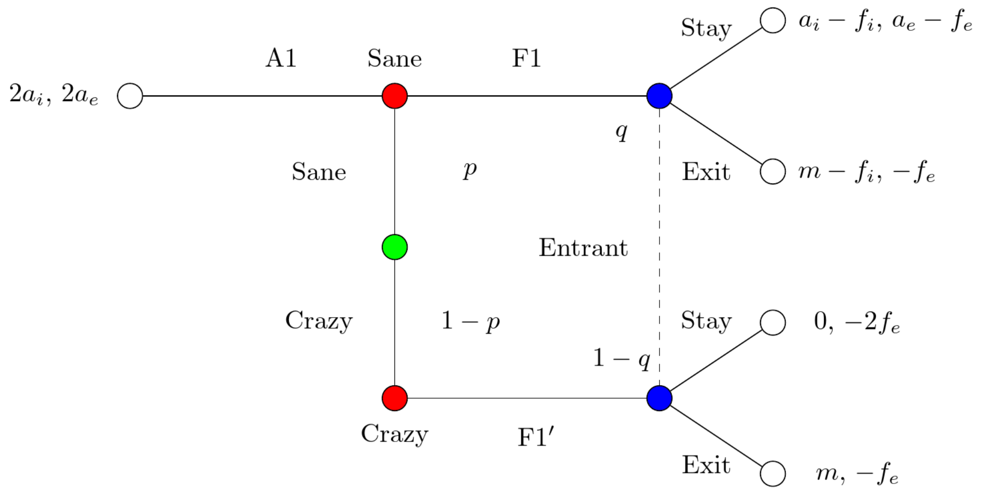
1. **Entry Deterrence Game.** Consider two firms competing in an industry, an incumbent and an entrant. The entrant has already entered the market, and in period 1 the incumbent must decide whether to fight or accommodate; if she fights, the entrant loses $f_e > 0$ (i.e., profit $= -f_e$) and if she accommodates the entrant makes a profit of $a_e > 0$. The incumbent may be either "sane", in which case he loses $f_i > 0$ from fighting, and gains $a_i > 0$ from accommodating; or "crazy", in which case his only available action is to fight. In period 2, the entrant must choose whether to stay in the market or exit; if she exits, the incumbent makes a profit of m; if she stays, the incumbent gets to fight or accommodate again and the payoffs are as described before. Assume that there is no discounting, and that the prior probability of a "sane" incumbent is $0 < p < 1$ (the actual type of the incumbent is private knowledge to the incumbent).

   (a) Draw the game in its extensive form. *(Hint: After drawing it, simplify it by using backward induction wherever possible)*

   Answer. It looks as follows (you can draw it sideways too)

   Doing backward induction wherever possible yields





(b) Show the algebraic condition(s) that sustains a separating equilibrium.

Answer. Only one separating can possible here: A1,F1′. Since this has to be sustained with $q=0$, the entrant will choose to exit. The sane incumbent has no incentive to deviate from A1 in this strategy profile as long as $m-f_i \leq 2a_i$.

(c) Show the condition(s) that sustain a a pooling equilibrium

Answer. Show the algebraic condition(s) that sustain a pooling equilibrium.

$$p(a_e-f_e)+(1-p)(-2f_e) \geq p(-f_e)+(1-p)(-f_e) = -f_e \Rightarrow p \geq \frac{f_e}{f_e+a_e}$$

In such a case, the pooling is sustained as long as the sane type does not have incentive to deviate, i.e., $a_i - f_i \geq 2a_i$. However, this is not possible, meaning this cannot be sustained as a PBE. Alternatively, if $p \leq \frac{f_e}{f_e+a_e}$, the Entrant chooses exit. This is sustainable as a pooling PBE as long as $m - f_i \geq 2a_i$

**You can attribute the "equality" as a tie breaker on either side. I have just written both cases as weak conditions.

2. **Bertrand Collusion with Probability of Being Caught.** Consider two symmetric firms competing in prices and earning a per-period profit $\pi_n = 0$, where subscript n denotes Nash equilibrium. If firms collude, they set a monopoly price $p_m > 0$, which yields a per-period profit of $\pi_m > 0$ for each firm. Firms have a common discount factor $\delta \in (0, 1)$.

   There also exists an antitrust authority, which investigates the industry in every period:

   - If firms collude, the authority will find them guilty with a probability p and will accordingly give them a fine $F > 0$.
   - If firms are found colluding, assume that the authority will prevent them from colluding in the future, earning the Nash profit $\pi_n = 0$ each for all subsequent periods.
   - If firms do not collude, they cannot be fined.

   Consider a simple Grim Trigger Strategy (GTS) in which firms start cooperating, but punish deviation from either firm by reverting to the Nash equilibrium of the stage game forever. Upon deviating, the firm who deviates earns π_d profits, where π_d satisfies $\pi_d > \pi_m > \pi_n = 0$.

   (a) Find the minimum discount factor, δ, sustaining collusion in the Subgame Perfect Equilibrium of the game. [Hint: δ should be a function of π_m, π_d, p, and F, where π_d denotes the deviation profit.]

If the antitrust authority investigates the sector in every period, the present discounted value of collusion is given by

$$V^c = p\left(\pi_m - F + \frac{\delta}{1-\delta}\pi_n\right) + (1-p)(\pi_m + \delta V^c)$$

Since $\pi_n = 0$, this expression simplifies to

$$V^c = p(\pi_m - F) + (1-p)(\pi_m + \delta V^c)$$

The firm's discounted continuation payoff is V^c, since it will collude in the next period (having the probability of being caught once again). Solving for the continuation payoff V^c in the above expression yields

$$V^c = \frac{\pi_m - pF}{1-\delta(1-p)}$$

Therefore, we can express the condition to cheat as follows

$$V^c = \frac{\pi_m - pF}{1-\delta(1-p)} \geq \pi_d$$

Solving for δ, we find that the minimal discount factor sustaining collusion is

$$\delta^* \geq \frac{\pi_d - (\pi_m - pF)}{(1-p)\pi_d}$$

(b) Comparative statics. How do p and F affect δ? Use partial derivatives to be sure!

Differentiating the above cutoff with respect to p, yields

$$\frac{\partial \delta^*}{\partial p} = \frac{\pi_d - \pi_m + F}{(1-p)\pi_d^2}$$

which is positive since $\pi_d > \pi_m$ by assumption.

In addition, differentiating the cutoff with respect to F, we obtain

$$\frac{\partial \delta^*}{\partial F} = \frac{p}{(1-p)\pi_d^2}$$

which is also positive. Therefore, the higher the probability of being caught, p, and the fine if caught, F, the less likely that collusion will be sustained in equilibrium (higher cutoff δ^*).

3. Consider the following all-pay auction with two bidders privately observing their valuation for the object. Valuations are uniformly distributed, i.e., $v_i \sim U[0,1]$. The player submitting the highest bid wins the object, but all players must pay the bid they submitted. Find the optimal bidding strategy, taking into account that it is of the form $b_1(v_1) = m \cdot v_i^2$, where m denotes a positive constant.

Answer.

- Bidder i's expected utility from submitting a bid of x dollars in an all-pay auction is

$$EU_i(x|v_i) = prob(win) \cdot v_i - x$$

where, if winning, bidder i gets the object (which he values at v_i), but he pays his bid, x, both when winning and losing the auction.

- Let us now specify the probability of winning, $prob(win)$. If bidder i submits a bid x using a bidding function $x = m \cdot v_i^2$, we can recover the valuation v_i that generated such a bid, i.e., solving for v_i in $x = m \cdot v_i^2$ we obtain $\sqrt{\frac{x}{m}} = v_i$. Hence, since valuations are distributed according to $v_i \sim U[0,1]$, the probability of winning is given by

$$prob(v_j < v_i) = prob\left(v_j < \sqrt{\frac{x}{m}}\right) = \sqrt{\frac{x}{m}}$$

- Therefore, the above expected utility becomes

$$EU_i(x|v_i) = \sqrt{\frac{x}{m}} \cdot v_i - x$$

Taking first-order conditions with respect to the bid, x, we obtain

$$-1 + \frac{v_i\sqrt{\frac{x}{m}}}{2x} = 0$$

and solving for x, we find bidder i's optimal bidding function in the all-pay auction

$$b_i(v_i) = \frac{1}{4m}v_i^2$$