Worksheet 11

Problem 1: Since x^2 is an even function in the domain $-\pi < x < \pi$, we only need to compute the a_0 and a_n terms, since the b_n terms will be 0.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx = \frac{1}{\pi}\left[\frac{(n^2x^2-2)\sin(nx)+2nx\cos(nx)}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{4}{n^2}\cos(\pi n) = (-1)^n\frac{4}{n^2}$$

Finally, the Fourier Series of x^2 between $-\pi < x < \pi$ is

$$\boxed{x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2}\cos(nx)}$$

Problem 2: Clearly the period of the function is 2π, then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\pi\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= -[x]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= -\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{\cos n\pi - 1}{\pi n^2}$$

$$\Rightarrow \quad a_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{-2}{\pi n^2}, & \text{when } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{1-2\cos n\pi}{n}$$

$$\Rightarrow \quad b_n = \begin{cases} -1/n, & \text{when } n \text{ is even} \\ 3/n, & \text{when } n \text{ is odd} \end{cases}$$

Putting the values of a's and b's, we get the required fourier series :

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

$$+ 3\sin x - \frac{1}{2}\sin 2x + \sin 3x - \cdots$$

Problem 3 : We apply the definition of the different coefficients, where $L = \frac{1}{2}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \int_0^{1/2} x\,dx = \left[\frac{x^2}{2}\right]_0^{1/2} = \frac{1}{8}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 2\int_0^{1/2} x\cos(2n\pi x)\,dx$$

$$= 2\left[\frac{2n\pi x\sin(2n\pi x) + \cos(2n\pi x)}{4\pi^2 n^2}\right]_0^{1/2}$$

$$= 2\left[\frac{\cos(\pi n)}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2}\right] = \frac{(-1)^n - 1}{2\pi^2 n^2}$$

$$= \begin{cases} 0 & , \quad n = 2 \\ \dfrac{-1}{\pi^2 n^2} & , \quad n \neq 2 \end{cases}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= 2\int_{0}^{1/2} x\sin(2n\pi x)\,dx$$

$$= 2\left[\frac{\sin(2\pi n x) - 2\pi n x\cos(2\pi n x)}{4\pi^2 n^2}\right]$$

$$= 2\left[\frac{\sin(2\pi n x) - 2\pi n x\cos(2\pi n x)}{4\pi^2 n^2}\right]_0^{1/2}$$

$$= 2\left[\frac{-\pi n\cos(\pi n)}{4\pi^2 n^2}\right] = \frac{(-1)^{n+1}}{2\pi n}$$

The fourier series is, then

$$y(x) = \frac{1}{8} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{2\pi^2 n^2}\cos(2n\pi x) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2\pi n}\sin(2\pi n x)$$

Problem 4:

The general solution of the homogeneous equation is

$$y_h = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

For the particular solution, we look for a solution of the form

$$y_p = a \sin(\alpha t) + b \cos(\alpha t) + A \sin(\beta t) + B \cos(\beta t)$$

$$y_p' = a\alpha \cos(\alpha t) - b\alpha \sin(\alpha t) + A\beta \cos(\beta t) - B\beta \sin(\beta t)$$

$$y_p'' = -a\alpha^2 \sin(\alpha t) - b\alpha^2 \cos(\alpha t) - A\beta^2 \sin(\beta t) - B\beta^2 \cos(\beta t)$$

Substituting into the differential equation, we get

$$\left[-a\alpha^2 \sin(\alpha t) - b\alpha^2 \cos(\alpha t) - A\beta^2 \sin(\beta t) - B\beta^2 \cos(\beta t)\right] +$$
$$\omega^2 \left[a \sin\alpha t + b\cos\alpha t + A \sin\beta t + B \cos\beta t\right] =$$

$$\sin\alpha t + \sin\beta t$$

$$\Rightarrow a(\omega^2-\alpha^2)\sin(\alpha t) + b(\omega^2-\alpha^2)\cos\alpha t + A(\omega^2-\beta^2)\sin\beta t + B(\omega^2-\beta^2)\cos\beta t = \sin\alpha t + \sin\beta t$$

Equating coefficients we find that

$$b = B = 0$$

$$a = \frac{1}{\omega^2-\alpha^2}, \quad A = \frac{1}{\omega^2-\beta^2}$$

The general solution of the equation is

$$y = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{1}{\omega^2-\alpha^2}\sin(\alpha t) + \frac{1}{\omega^2-\beta^2}\sin(\beta t)$$

Problem 5:

1) Change of Variable :

$$x = e^t \Rightarrow t = \log x$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \dot{y}\,\frac{1}{x} = \dot{y}\,e^{-t}$$

Substituting in the differential equation :

$$\frac{d(\dot{y}e^{-t})}{dt}\, e^{-t} + (\lambda+1)\,\frac{y}{e^{3t}} = 0$$

$$\frac{d(\dot{y}e^{-2t})}{dt}\, e^{-t} + (\lambda+1)\, y e^{-3t} = 0$$

$$(\ddot{y}\, e^{-2t} - 2\dot{y}\, e^{-2t})\, e^{-t} + (\lambda+1)\, y e^{-3t} = 0$$

$$\ddot{y} - 2\dot{y} + (\lambda+1)y = 0 \ , \ y(0) = 0 = y(\pi)$$

2) <u>find the general solution:</u>

The general solution of this equation is given by the roots of the polynomial

$$s^2 - 2s + (\lambda+1) = 0 \Rightarrow s = 1 \pm \sqrt{-\lambda}$$

$$y(t) = e^{t}\left(c_1 e^{-\sqrt{-\lambda}\, t} + c_2 e^{\sqrt{-\lambda}\, t}\right)$$

(3) <u>find the eigenvalues and eigenfunctions:</u>

<u>case $\lambda < 0$:</u> If $\lambda < 0$, then $\sqrt{-\lambda} > 0$ and the boundary conditions imply

$$y(0) = c_1 + c_2 = 0$$

$$y(\pi) = e^{\pi}\left(c_1 e^{-\sqrt{-\lambda}\,\pi} + c_2 e^{\sqrt{-\lambda}\,\pi}\right) = 0$$

$$\Rightarrow \quad c_1 = c_2 = 0$$

<u>Case $\lambda = 0$:</u> If $\lambda = 0$, then the general solution is

$$y(t) = e^{t}(c_1 + c_2 t)$$

The boundary conditions imply

$$y(0) = c_1 = 0$$
$$y(\pi) = e^{\pi}(c_1 + c_2 \pi) = 0$$

$\Rightarrow c_1 = c_2 = 0$

<u>case $\lambda > 0$</u>: the general solution becomes

$$y(t) = e^t\left(c_1 \cos(\sqrt{\lambda}\, t) + c_2 \sin(\sqrt{\lambda}\, t)\right)$$

from the boundary conditions, we get

$$y(0) = c_1 = 0$$
$$y(\pi) = e^{\pi}(-c_1) = 0$$

$$\Rightarrow c_1 = 0$$

Consequently, the eigenfunctions are the functions of the form

$$y(t) = e^t \sin(\sqrt{\lambda}\, t)$$

and the associated eigenvalue is λ.
Undoing the change of variable, we get the eigenfunctions

$$\boxed{y(x) = x \sin(\sqrt{\lambda} \log x)}$$