Q1(a) $X_1, X_2 \rightarrow$ are independent and identical distributed random variable (IIDs)

$$PMF = f_{x_i}(x_i) = \frac{1}{2^{K/2}\Gamma(K/2)} x^{(K/2-1)} e^{-x/2}$$

$$MGF_{x_i}(s) = E[e^{x_i s}]$$

$$= \int_0^\infty e^{s x_i} \frac{1}{2^{K/2}\Gamma(K/2)} x_i^{K/2-1} e^{-x_i/2} dx_i$$

$$= \cancel{\frac{1}{2^{K/2}\Gamma(K/2)} \int_0^\infty x_i^{K/2-1} e^{-1/2(1-2s)x_i}} dx$$

$$= \frac{1}{2^{K/2}\Gamma(K/2)} \int_0^\infty e^{s x_i} \times x_i^{K/2-1} e^{-x_i/2} dx_i$$

$$= \frac{1}{2^{K/2}\Gamma(K/2} \times \cancel{x_i} \int_0^\infty x_i^{K/2-1} e^{-1/2(1-2s)x_i} dx_i$$

Let $\frac{1}{2}(1-2s)x_i = t$

$$dt = \frac{1}{2}(1-2s)\, dx_i$$

Date / /
Page No.

$$M(s) = \frac{1}{2^{K/2}\Gamma(K/2)} \int_0^\infty e^{-t} \times \left(\frac{t}{1/2 - s}\right)^{K/2 - 1} \frac{dt}{\frac{1}{2}(1-2s)}$$

$$= \frac{1}{2^{K/2}\Gamma(K/2)\left[\frac{1}{2} - s\right]^{K/2}} \left[\int_0^\infty t^{K/2-1} e^{-t} dt\right]$$

we know that

$$\Gamma[K] = \int_0^\infty t^{K-1} e^{-t} dt$$

$$\therefore \Gamma[K/2] = \int_\infty^\infty t^{K/2-1} e^{-t} dt$$

$$= \frac{1}{2^{K/2}\cancel{\Gamma[K/2]}\left[\frac{1}{2} - s\right]^{K/2}} \times \cancel{\Gamma[K/2]}$$

$$= \frac{1}{2^{K/2}\left(\frac{1}{2} - s\right)^{K/2}}$$

$$= \cancel{\left(\frac{1/2}{1/2 - s}\right)^{K/2}}$$

Date / /
Page No.

$\therefore$ $$\boxed{MGF = \frac{1}{2^{k/2}\left(\frac{1}{2}-s\right)^{k/2}}}$$

$$\Rightarrow \boxed{MGF = \frac{1}{(1-2s)^{k/2}} \quad [s < 1/2]}$$

(b) $S_n = X_1 + X_2 + \_\_\_ X_n$

$X_i \rightarrow$ iid $\rightarrow$ ~~for~~ independent and identically distributed

to find S_n distribution we need to find MGF of S_n as MGF uniquely identifies the distribution

$\therefore$

$$M_{S_n}(s) = E[e^{sS_n}]$$

$$= E[e^{s[X_1 + X_2 + X_3 - - X_n]}]$$

$$= E[e^{sx_1} \cdot e^{sx_2} - - - e^{sx_n}]$$

$$= \prod_{i=1}^{n} E[e^{sx_i}] \qquad \text{since independent}$$

$$= \prod_{i=1}^{n} M_{x_i}(s) \qquad \text{By defination of mgf}$$

$\therefore$ $\cancel{A}\,[M\_X(s)]^{\wedge}n$, since identically distributed.

$$M_{GF}(s) = \left(\frac{1}{2^{k/2}\left[\frac{1}{2}-s\right]^{k/2}}\right)^{n}$$

$M\_S\_n(s)$

$$= \left(\frac{1/2}{1/2-s}\right)^{nk/2} = \frac{1}{(1-2s)^{nk/2}}$$

Q2.

Sol a) (i) The messages are sent in a discrete manner. Hence, the time 't' is discrete.

(ii) Either a message is received successfully by the pager or not, i.e.,

$$Z_t = \begin{cases} 0 & \text{if message is not received} \\ 1 & \text{if message is received with probability 'p'} \end{cases}$$

where Z_t is a Bernoulli RV.

(iii) $Z_t \; \forall t$ are independent and identically distributed.

Hence, the given process can be modeled as Bernoulli Process.

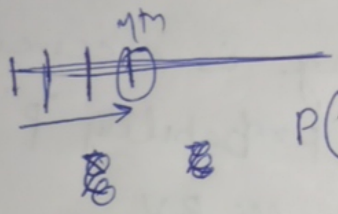
(b) Let X be the number of times the system sends the same message. We know that a message will be repeatedly transmitted until there is a successful reception.

So, probability of successful reception in x^{th} attempt is $P(X=x) = \begin{cases} (1-p)^{x-1} p, & x = 1,2,3 \\ 0 & \text{Otherwise} \end{cases}$

(c) Let N_{10} be the number of successes in 10 attempts. N_{10} will follow a Binomial distribution ($\because N_{10}$ is the sum of Bernoulli RVs.)

$\therefore P(N_{10} = 1) = {}^{10}C_1 (1-P)^9 (P)^1$

(d) The time till the first success is nothing but the interarrival time. In case of Beronlli process, the interarrival time follows geometric distribution.



Let X_1 be the interarrival time.

$$P(X_1 > 3) = 1 - P(X_1 \leq 3)$$

$$= 1 - \big(P(X_1 = 1) + P(X_1 = 2) + P(X_1 = 3)\big)$$

$$= 1 - \left(p + (1-p)p + (1-p)^2 p\right)$$

$$= 1 - p - (1-p)p - (1-p)^2 p$$

$$= 1 - p - p + p^2 - (1 + p^2 - 2p)p$$

$$= 1 - 2p + p^2 - p - p^3 + 2p^2$$

$$= 1 - 3p + 3p^2 - p^3$$

Page No.
Date

Ques-3. (a) A Renewal Process is an arrival process for which the sequence of interarrival times is a sequence of IID random variables.

In the question, it is given that N (the number of vehicles) arriving is a POISSON PROCESS. Hence, by the ~~prop~~ definition of Poisson process we can say the process starts at $t=0$ and multiple arrivals are not occurring simultaneously. So, it is an ~~ordering~~ arrival process.

Also, the inter-arrival times in a poisson process have an exponential distribution and it is IID.

Hence, the process is a Renewal process.
Also, by the definition of Poisson Process we know that it is a renewal process.

(b) P(2 vehicles arriving in (0,2] minutes) = ?

(Given) ⇒ N ⇒ the number of vehicles that arrive is a poisson process with $\lambda = 2.8$ vehicles/minute.

$N(t)$ = the number of vehicles arriving in time interval $(0,t]$.

$N(t) \sim \text{Poisson}(\lambda t)$

Page No.
Date

$$\boxed{\text{formula} \Rightarrow P(N(t)=n) = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}}$$

$$P(N(2)=2) = \frac{e^{-2.8\times 2} \times (2.8\times 2)^2}{2!}$$

$$= \frac{e^{-5.6} \times (5.6)^2}{2}$$

$$\boxed{P(N(2)=2) = 0.0579}$$

(c) $P($atleast 2 vehicles arrived in $(3,5]$ minutes and utmost 1 vehicle arrived in $(5,8]) = ?$

$\Rightarrow P(\tilde{N}(3,5) \geq 2 \cap \tilde{N}(5,8) \leq 1)$

(Since, $(3,5]$ and $(5,8]$ are non-overlapping intervals we can use Independent Increment property as,)

$$= P(\tilde{N}(3,5) \geq 2) \cdot P(\tilde{N}(5,8) \leq 1)$$

(By Stationary Increment Property,)

$$= P(N(5)-N(3) \geq 2) \cdot P(N(8)-N(5) \leq 1)$$

$$= P(N(2) \geq 2) \cdot P(N(3) \leq 1)$$

$$= (1 - P(N(2) < 2)) \cdot P(N(3) \leq 1)$$

$$= \left(1 - \left(\frac{e^{-2.8\times 2}(2.8\times 2)^0}{0!} + \frac{e^{-2.8\times 2}(2.8\times 2)^1}{1!}\right)\right) \times$$

Page No.	
Date	

$$\left(\frac{e^{-2.8\times 3}(2.8\times 3)^0}{0!} + \frac{e^{-2.8\times 3}(2.8\times 3)^1}{1!} \right)$$

$$\Rightarrow \left(1-\left(e^{-5.6}\times 6.6\right)\right) \times \left(e^{-8.4}\times 9.4\right)$$

$$= (0.9756)(0.002113)$$

$$= \boxed{2.061\times 10^{-3} = 0.002061}$$

(d) P(time taken to arrival of two vehicles > 5 minutes)

$$= P(S_2 > 5) = ?$$

($S_n \Rightarrow$ time taken to arrival of n vehicles).

By property of Poisson Process,

$$P(S_n \le t) = P(N(t) \ge n)$$

Also, $P(S_n > t) = P(N(t) < n)$

So, $P(S_2 > 5) = P(N(5) < 2)$

$$= P(N(5)=0) + P(N(5)=1)$$

$$= \frac{e^{-14}(14)^0}{0!} + \frac{e^{-14}(14)^1}{1!}$$

$$= \boxed{0.000012472 = 1.247\times 10^{-5}}$$

Solution 4

X_i follows Gaussian
and $E[X_i] = 0$
$var[X_i] = 1$

(a) $M_{Y_n}[s] = E[e^{Y_n s}]$

given $Y_n = X_n + X_{n-1}$

$= E[e^{s[X_n + X_{n-1}]}]$

$= E[e^{sX_n}] E[e^{sX_{n-1}}]$ X_i is independent

$= M_{X_n}(s) M_{X_{n-1}}(s)$

$= [M_{X_n}(s)]^2$ as X_i is identically distributed

$= e^{s^2/2 \times 2}$

therefore Y_n follows Gaussian distribution with mean $= E[Y_n] = 0$ and
$Var[Y_n] = 2$

(b) As, $Y_n = X_n + X_{n-1}$

$\therefore$ $E[Y_n] = E[X_n] + E[X_{n-1}]$

as $E(A+B) = E[A] + E[B]$

& since $E[X_n] = 0 = E[X_{n-1}]$ and as X_i is identically distributed

therefore $E[Y_n] = 0$

(c) As $E[X_n]=0$ and $Var[X_n]=1$

$$C_x[m,k] = cov(X_m, X_k)$$
$$= E[X_m X_k] - E[X_m]E[X_k]$$
$$= E[X_m X_k] - 0$$

$$\Rightarrow C_x[m,k] = E[X_m X_k]$$

so,

$$C_x[m,k] = \begin{cases} 1 & \text{if } m=k \\ 0 & \text{else} \end{cases}$$

therefore, when

$m=k$, $C_x[m,k] = E[X_m^2] = Var[X_m] = 1.$

$m \neq 1$, $\quad = E[X_m].E[X_n] = 0$

as X_i is iid

Now

$$C_y[m,k] = E[Y_m, Y_k] - E[Y_m]E[Y_k]$$
$$= E[Y_m Y_k] - 0$$

$$C_y[m,k] = E[(X_m + X_{m-1})(X_k + X_{k-1})]$$
$$= E[X_m X_k + X_m X_{k-1} + X_{m-1} X_k + X_{m-1} X_{k-1}]$$
$$= E[X_m X_k] + E[X_m X_{k-1}] + E[X_{m-1} X_k] + E[X_{m-1} X_{k-1}]$$
$$= C_x(m,k) + C_x[m, k-1] + C_x[m-1, k] + C_x[m-1, k-1] \quad \text{(1)}$$

when $m=k$, putting in eq (1)

we get

$$\Rightarrow 1+0+0+1$$

$$\therefore C_y[m,k] = 2.$$

Date :

Page No.

when $m = k-1$, putting in eq^n ①

$\Rightarrow 0+1+0+0$

$C_Y[m,k] = 1$

when $m = k+1$, putting in eq^n ①

$\Rightarrow 0+0+1+0$

$C_Y[m,k] = 1$

$$C_Y[m,k] = \begin{cases} 2 & \text{if, } m=k \\ 1 & \text{else if } m=k-1 \text{ or } m=k+1 \\ 0 & \text{else} \end{cases}$$

ⓓ Yes, we can find the distribution of $\underline{Y} = (Y_1, Y_2, Y_3 \cdots Y_n)$

As X_i follows with $E[X_i]=0$ and $Var[X_i]=1$ therefore,

$X_i \sim N(0,1)$

And $Y_i = X_i + X_{i-1}$, Y_i is linear combination of the finite set of iid gaussian Random variables with mean=0 and variance=1

$\therefore$ set is $\{X_0, X_1, X_2 \cdots X_n\}$.

Hence $\underline{Y} = (Y_1, Y_2 \cdots Y_n)$ is a Gaussian zero mean and therefore, Also jointly gaussian

Date :
Page No.

$E[\underset{\sim}{Y}]=0$, As $E[Y_i]=0$

And $var[\underset{\sim}{Y}] = \begin{bmatrix} var(Y_1) & cov(Y_1, Y_2) & \cdots & Cov(Y_1, Y_n) \\ cov(Y_2, Y_1) & \vdots & & \vdots \\ cov(Y_3, Y_1) & \vdots & & \vdots \\ \vdots & & & \\ cov(Y_n, Y_1) & \cdots & \cdots & Var(Y_n) \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & & \\ 0 & 1 & 2 & 1 & \\ 0 & & 1 & 2 & \\ \vdots & & & & \\ 0 & & & & 2 \end{bmatrix}$$

~~[illegible]~~:

Multivariate Gaussian is explained by its mean and cov. matrix.

Hence we have the distribution of $\underset{\sim}{Y} = (Y_1, \cdots Y_n)$.

DATE: / /
PAGE

Soln → 5, 6

Soln 5 X : time required to repair a machine

X : exponential (λ)

Mean $= 1/\lambda = 1/2$

$\lambda = 2$

As we know

$$P(X > t+n \mid X > t) = \frac{P((X > t+n) \wedge P(X > t))}{P(X > t)}$$

$$= \frac{P(X > t+n)}{P(X > t)}$$

$$P(X > 12.5 \mid X > 12) = \frac{P(X > 12.5)}{P(X > 12)}$$

$$= \frac{1 - P(X \le 12.5)}{1 - P(X \le 12)}$$

$$= \frac{1 - F_X(12.5)}{1 - F_X(12)}$$

$$= \frac{1 - [1 - e^{-2 \times 12.5}]}{1 - [1 - e^{-2 \times 12}]}$$

$$= e^{-1}$$

alternatively :-

we can use memoryless property of expo. distribution from that we can simply

GOOD WRITE

write LHS = $P(X >= 0.5)$ & then solve it.

Ans6 Car arrive according to poisson process with $\lambda = 2$ per hour

$N(t)$ - No. of arrivals in time t
$N(t) \sim$ Poisson$(2t)$

we have to find
$Cov(N(5) - 2N(6), 3N(10))$

Using, $Cov(X+Y, Z) = Cov(X, Z) + Cov(Y, Z)$
we get-
$= Cov(N(5), 3N(10)) - Cov(2N(6), 3N(10))$

Using, $Cov(ax, by) = ab\,Cov(x, y)$
$= 3Cov(N(5), N(10)) - 6Cov(N(6), N(10))$ — (1)

Now
$Cov(N(5), N(10)) = Cov[N(5), (N(5) + \tilde{N}(5,10))]$
$= Cov[N(5), N(5)] + Cov[\underbrace{N(5), \tilde{N}(5,10)}_{0}]$
$= Var[N(5)] + 0$ (∵ non overlapping intervals)
$= \lambda \times 5$ — (2)

DATE: ___/___/___
PAGE ______

$Cov(N(6), N(10)]$

$\Downarrow$

$Cov[N(6), (N(6) + \tilde{N}(6,10))]$

(same as above)

$= Var[N(6)$

$= \lambda \cdot 6$ — 3

Putting 2 and 3 in 1 we get

$= 3 \times \lambda \times 5 - 6 \times \lambda \times 6$

$= 30 - 72$

$= -42$