MTH 377/577 Convex Optimization
Mid-semester 2024-25

Instructions: Q4(b) is only for MTH377 students. Q4(c) is only for MTH577 students. All other questions are for all students. Max. marks: 20. Time: 1 hour.

1. Consider the following optimization problem:

$$\begin{array}{rc} \textit{maximize} & 7x + 3y \\ \textit{subject to} & x + y \leq 2 \\ & 3x + y \leq 4 \\ & x, y \geq 0 \end{array}$$

   (a) Re-write the above problem in standard form. (1)

Ans.
$$\begin{array}{rc} \textit{maximize} & 7x + 3y \\ & x + y + s_1 = 2 \\ & 3x + y + s_2 = 4 \\ & x, y, s_1, s_2 \geq 0 \end{array}$$

All correct then only 1 marks

   (b) Using Farkas Lemma comment on whether or not a non-negative solution for the above system exists. Show all the steps. (2)

Ans. Re-write the original problem with surplus variables in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

For Farkas' alternative:

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

you have to show all necessary steps

and

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} < 0$$

$yA \geq 0$ and $yb < 0$ above imply the following system of equalities:

$$\begin{gathered} y_1 + 3y_2 \geq 0 \\ y_1 + y_2 \geq 0 \\ y_1, y_2 \geq 0 \\ 2y_1 + 4y_2 < 0 \end{gathered}$$

All eqn are correct +1

since $y_1, y_2 \geq 0$, then $2y_1 + 4y_2 < 0$ is not possible. Therefore no such y exists. Therefore, $Ax = b$ (primal) has a non-negative solution. For Farkas lemma explanation +1

(c) Find the dual of the given problem.(2)

Ans. Dual:

$$\begin{aligned} \textit{minimize} \quad & 2y_1 + 4y_2 \\ & y_1 + 3y_2 \geq 7 \\ & y_1 + y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Minimize eqn +1

All eqn other than min +1

or students can use the standard form derived in (a) also, the answer will be the same.

2. State whether the following is true or false and provide a brief formal argument/explanation for your answer.

(a) The dual of a proper cone is a subset of its convex hull. (2)

Ans. False. The dual contains vectors orthogonal to those in the given cone. Convex hull does not contains these orthogonal vectors. [Write definitions/show diagram].

(b) The partial sum of two convex sets is a convex set. (2)

Ans. True. Let S_1, S_2 be two convex sets. Partial sum is the set $S = \{(x, y_1 + y_2) | (x, y_1) \in S_1, (x, y_2) \in S_2\}$. Let $a, b \in S$:

$$a = (x, a_1 + a_2), b = (y, b_1 + b_2)$$

where $(x, a_1), (y, b_1) \in S_1$ and $(x, a_2), (y, b_2) \in S_2$. Now, for any $\theta \in [0, 1]$, $(\theta x + (1 - \theta)y, \theta a_1 + (1 - \theta)b_1) \in S_1$ since S_1 is convex. Similarly, $(\theta x + (1 - \theta)y, \theta a_2 + (1 - \theta)b_2) \in S_2$. By definition of partial sum, $(\theta x + (1 - \theta)y, \theta(a_1 + a_2) + (1 - \theta)(b_1 + b_2)) \in S$. Therefore, S is convex.

3. Let $f : R^2 \to R$ be a function that takes values $f(x) = x_1^2 + x_2^2$ for $(x_1, x_2) \in R^2$. Write down a generic sub-level set for f. Is f quasiconvex? Provide a formal explanation/proof or counterexample in support of your answer. (4)

Ans. A generic sublevel set for f is $S_\alpha = \{(x_1, x_2) \in R^2 | x_1^2 + x_2^2 \leq \alpha\}$ for $\alpha \in R$. Since each sub level set is convex, f is quasiconvex. Indicative logic of the proof: note that every sublevel set consists of points in a closed circle which is convex. (show on your own using definition of convex sets).

4. (a) Let K be the cone generated by the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Let K^* be the smallest proper cone containing K. Suppose $\succeq_{K^*}$ is the generalized inequality defined with respect to K^*. Can you rank the following elements using $\succeq_{K^*}$: $(2,0,1), (0,1,0), (1,1,1)$? Explain your answer. (3)

Ans. K^* consists of all vectors that are convex combinations of the column vectors in A. Let A' denote the set of vectors in A K^* is convex since it is proper. By the definition of generalized inequality, for any x, y, $y \succeq_{K^*} x \iff y - x \in K^*$. Note that $A' \subset K^*$. Consider $x = (2,0,1)$ and $y = (0,1,0)$. Note that $y - x \notin K^*$ and $x - y \notin K^*$. Therefore, these cannot be ranked using $\succeq_{K^*}$. Therefore, the given set of vectors also cannot be ranked, since ranking each element is not possible with the given inequality.

(b) [**Only for MTH377**] Suppose C is a convex set and $D = \{x_1, x_2, \ldots, x_n\}$ for some $x_1, x_2, \ldots, x_n \notin C$. Does there exist a separating hyperplane between C and any x_k, $1 \leq k \leq n$? What about between C and D? Provide a formal explanation for your answer. (4)

(c) [**Only for MTH 577**] Let A be an $m \times n$ real matrix and $F = \{x \in R^n : Ax \leq 0\}$. Let $c \in R^n$ and $G = \{x \in R^n : cx \leq 0\}$. Use Farkas lemma to prove that $F \subseteq G$ if and only if there exists $y \in R_+^m$ such that $c = yA$. (4)

Convex opt. Sol^n

+1 (correct answer true or false)

+1,(correct explanation

No marks if explanation isn't provided.

Q2

Sol^n : a) **Dual cone :** for cone K, dual cone K^*

$$K^* = \{ y \in \mathbb{R}^n \mid \underbrace{x^T y}_{\langle x, y \rangle} \geq 0 \ \forall \ x \in K \}$$

proper cone : cone $K \subseteq \mathbb{R}^n$ is called a proper cone iff:

(i) K is convex

(ii) K is closed

(iii) K is solid ; non-empty interior (contains open ball)

(iv) K is pointed : $K \cap -K = \{0\}$

conv (K) : convex hull of cone K, is the smallest convex set containing K.

$$\text{conv}(K) = \left\{ \sum_{i=1} \lambda_i x_i \mid x_i \in K, \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

since 'K' is convex : $\lambda x + (1-\lambda) y \in K \ \forall \ x, y \in K$, $\lambda \in [0,1]$

$\Rightarrow$ $\text{conv}(K) = K$

Eg: $K = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0 \}$

$K^* = \{ y \in \mathbb{R}^2 \mid x^T y \geq 0 \ \forall\, x \in K \}$

$K^* = \{ (y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_2 \geq 0 \}$

$(x_1, x_2)^T (y_1, y_2) \geq 0$

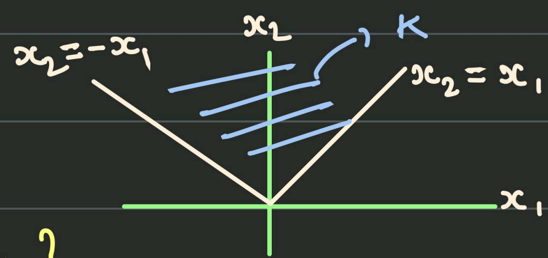
$\Rightarrow \quad x_1 y_1 + x_2 y_2 \geq 0$

since $x_1, x_2 \geq 0$

$\Rightarrow \quad y_1, y_2 \geq 0$

Here $K = K^*$

$\therefore \ K^* = \text{conv}(K)$



let $K = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq |x_1| \}$

$K^* = \{ y \in \mathbb{R}^2 \mid x^T y \geq 0 \ \forall\, x \in K \}$

$$(x_1 \ \ x_2)^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \geq 0$$

along $x_2 = x_1$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

along $x_2 = -x_1$: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

also $x_2 \geq |x_1|$

$\Rightarrow \quad x_1 y_1 + x_2 y_2 \geq 0$

$\Rightarrow \quad y_1 + y_2 \geq 0 \ , \quad y_1 - y_2 \geq 0$

$\Rightarrow \quad y_1 \geq -y_2 \ , \quad y_1 \geq y_2$

$\therefore \ K^* = \{ y \in \mathbb{R}^2 \mid y_1 \geq |y_2| \}$

for $(1,0)$: $1 \geq |0|$

$\Rightarrow (1,0) \in K^*$

But $(1,0) \notin K$ as $0 \ngeq |1|$

$\therefore K^* \nsubseteq K$

$\therefore$ given statement is false, dual of a proper cone is not necessarily a subset of its convex hull.

+2 for correct Explanation

b) A, B convex sets

partial sum: $A+B = \{ a+b \mid a \in A, b \in B \}$

let $x_1, x_2 \in A+B$ and $\lambda \in [0,1]$

$x_1 = a_1 + b_1$, $x_2 = a_2 + b_2$

$$\lambda x_1 + (1-\lambda) x_2 = \lambda (a_1 + b_1) + (1-\lambda)(a_2 + b_2)$$

$$= \underbrace{(\lambda a_1 + (1-\lambda) a_2)}_{\in A} + \underbrace{(\lambda b_1 + (1-\lambda) b_2)}_{\in B}$$

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in A+B$

$\therefore$ Partial sum of convex sets is convex.

+2 for correct Explanation

Marking scheme:

sub- level set for f written properly +1.5

Q 3

if mentioned it is quasiconvex +0.5

$f: \mathbb{R}^n \to \mathbb{R}$

Soln: **sublevel subsets:** $S_\alpha = \{x \in \text{dom} f \mid f(x) \le \alpha\}$

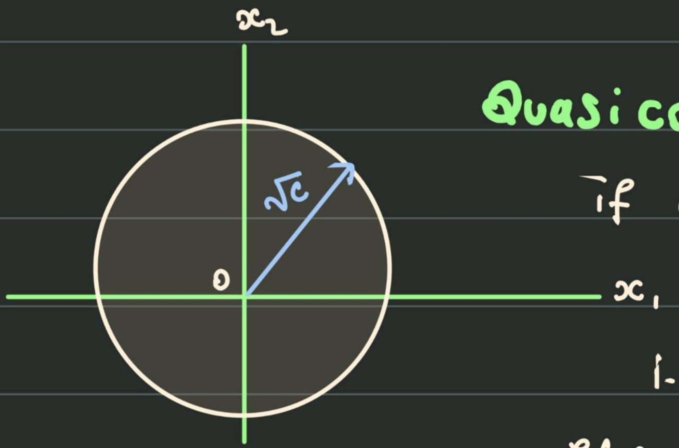
$\alpha \in \mathbb{R}$

$f(x) = x_1^2 + x_2^2, \quad x \in \mathbb{R}^2$

$S_\alpha = \{(x_1, x_2) \in \text{dom} f \mid f(x) \le \alpha\}$

$S_\alpha = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le \alpha\}$

$\Rightarrow +1.5$



Quasiconvex: 'f' is Quasiconvex if all its sub-level sets are convex.

i.e.

$f(\lambda x + (1-\lambda) y) \le \max(f(x), f(y))$

$\forall \lambda \in [0,1]$

let $z = \lambda x + (1-\lambda) y = (\lambda x_1 + (1-\lambda) y_1, \lambda x_2 + (1-\lambda) y_2)$

$f(z) = (\lambda x_1 + (1-\lambda) y_1)^2 + (\lambda x_2 + (1-\lambda) y_2)^2$

given mathematical proof +2Marks

$= \lambda^2 x_1^2 + \lambda^2 x_2^2 + (1-\lambda)^2 y_1^2 + (1-\lambda)^2 y_2^2$

If theoretically proved with

$+ 2\lambda(1-\lambda)(x_1 y_1 + x_2 y_2)$

expressions then marks will

be given based on accuracy

$\le \lambda f(x) + (1-\lambda) f(y)$

also $\lambda f(x) + (1-\lambda) f(y) \le \max(f(x), f(y))$

$f(z) \le \max(f(x), f(y))$

$\therefore$ 'f' is Quasiconvex.

Q4) (a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

$\therefore v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(0.5 mark for calculating v_1, v_2, v_3).

$K = \{\theta_1 v_1 + \theta_2 v_2 + \theta_3 v_3 \mid \theta_1, \theta_2, \theta_3 \geq 0\}$ —— (1)

$K \subseteq K^*$ —— (2)

$v_1' \succeq_{K^*} v_2 \iff v_1' - v_2' \in K^*$ —— (3)

(1 mark if all eq^n mentioned, 0.5 mark if atleast 1 of the above eq^n mentioned, else 0).

Solving:

$$v_1' - v_2' \notin K^* \iff v_2' - v_1' \notin K^*$$
$$v_2' - v_3' \notin K^* \iff v_3' - v_2' \notin K^*$$
$$v_3' - v_1' \notin K^* \iff v_1' - v_3' \notin K^*$$

$v_1' = [2\ 0\ 1]^T,\ v_2' = [0\ 1\ 0]^T,\ v_3' = [1\ 1\ 1]^T$

(1 mark if adequate solving is done).

Final Verdict : Ranking not possible (0.5 mark).

(b) (i) For C & x_k ~~→ Hyperplan~~

Strict Hyperplane exists (0.5 mark)

Correct Explanation (1.5 mark)

(ii) For C & D

Strict Hyperplane exists (0.5 mark)

Correct Explanation (1.5 mark)

Explanation (a) : If $\{x_k\} \notin C$ Then, $\{x_k\} \cap C = \phi$, and since both, $\{x_k\}$ & C would be convex, hyperplane would exist.

Explanation (b) : $D = \{x_1, x_2, x_3 \ldots x_n\}$. We know for $1 \le k \le n$, $x_k \notin C$, so take any $\{x_k\}$ that is closest to C in distance which will be used to separate C from D. (Theorem is not directly applicable here).