Modern Algorithm Design : Quiz 2
Full Marks : 25
Time : 45 minutes

8/11/2024

Problem 1. (5 points) Suppose you have n letters labeled $1, 2, \cdots n$ and n envelopes also labeled from $1, 2, \cdots n$. However, the envelopes are shuffled according to an uniformly random permutation - say π denote this ordering. Now we simply place letter i in to the envelope $\pi(i)$. What is the expected number of letters that find their correct envelope ?

Solution. Let us define a random variable $X_i \in \{0, 1\}$ such that $X_i = 1$ if the ith letter goes in to the ith envelope and 0 otherwise. Let Y be the random variable that counts the number of letters that get to the correct envelope. Hence $Y = \sum_{i=1}^{n} X_i$. Now $\mathbf{E}[X_i] = \mathbf{Pr}[X_i = 1] = 1/n$ (the chance that the ith envelope occupies the ith position in the random permutation. Hence **E[Y] = 1**.

Problem 2. (10 points) Let $x_1, x_2, \cdots x_n$ be a random permutation of the numbers $[n] := \{1, 2, \cdots, n\}$. You scan the numbers from left to right. You have a buffer B initially containing ∞. When you see x_i , if B contains a number bigger than x_i , then drop the number in B, and insert x_i to B. What is the expected total number of insertions to B during the scan?

Solution There are two main observations. You will add x_i to the buffer if and only if x_i is the *largest* all the numbers $x_1, x_2, \cdots x_i$. Since you are seeing the numbers according to a uniformly random permutation, the chance of this happening is exactly $1/i$. Thus the expected number of insertion is $\sum_{i=1}^{n} 1/i = \mathcal{O}(\log n)$.

Problem 3 (10 points) An airplane in Politesville has n seats, and n passengers assigned to these seats. The first passenger to board gets confused, and sits down at a uniformly random seat. The rest of the passengers do the following: when they board, if their assigned seat is free they sit in it, else (being too polite) they choose a uniformly random empty seat and sit in it. Show that the last person to board sits in their assigned seat with probability $1/2$. (Hint: there is NO calculation required in this problem. Just some observations)

Solution. This one is more of a logical argument question. The point is, when the last person boards, there can only be one of two possibilities empty - either the seat assigned to the first person or the seat actually assigned to the last one. Why ? Well, suppose that there is some other seat empty and this seat belongs to someone who is neither the first one nor the last one but some ith person. But then this person should have occupied this seat when he/she/they boarded ! So these are the only options for the last person and the process being completely symmetric for all seats, the chance that the last seat left is the right one is exactly 1/2.