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Tutorial 3

Point Estimation → MoM, MLE

MoM → equate population moments & sample moments
→ 'p' population parameters (unknown) ⇒ solving 'p' ~~the~~ such eqns.

MLE → maximize likelihood function / log likelihood fn.

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

Q1. $X_1, X_2 \ldots X_n \sim \text{Poisson}(\lambda)$

a) estimate $\hat{\lambda}$ using MoM

First sample moment $m_1 = \frac{1}{n} \Sigma X_i$

First popn. moment : $E[X] = \lambda$

$$\hat{\lambda}_{mom} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}$$

$$\boxed{\hat{\lambda}_{mom} = \overline{X}}$$

b) Using MLE

$$f(x) = \frac{e^{-\lambda} \lambda^{x_i}}{(x_i)!}$$

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{(x_i)!}$$

$$L(\lambda) = \frac{e^{-n\lambda} \lambda^{\Sigma x_i}}{\prod_{i=1}^{n} (x_i)!}$$

$$\log L(\lambda) = \log\left(e^{-\lambda n} \cdot \lambda^{\Sigma x_i}\right) - \log\left(\prod_{i=1}^{n} (x_i)!\right)$$

$$= -n\lambda + \Sigma x_i \log \lambda - \sum_{i=1}^{n} \log((x_i)!)$$

$$\frac{d}{d\lambda} \log L(\lambda) = -n + \frac{\Sigma x_i}{\lambda} = 0$$

$$\frac{dl}{d\lambda} = 0$$

$$\hat{\lambda} = \frac{\Sigma x_i}{n} = \bar{x}$$

$$\frac{d^2 l}{d\lambda^2} = -\frac{\Sigma x_i}{\lambda^2} \longrightarrow ; \lambda = \bar{x}$$

$$= \frac{-n\bar{x}}{\bar{x}^2} < 0$$

Q2. $X_1, \dots X_n$ are samples from the pdf $f_\theta(x)$

$$f_\theta(x) = \theta x^{-2} \qquad 0 < \theta \le x$$

a) Find suff. stats for θ; we will use the factorization theorem

$$f(x_1 \dots x_n / \theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^2} \cdot I(x_i \ge \theta)$$

$$= \underbrace{\frac{1}{\prod(x_i)^2}}_{h(x)} \times \underbrace{\theta^n \cdot I(x_i \ge \theta)}_{g(T(x)/\theta)}$$

& $T(X) = X_{(1)}$

b) estimate $\hat{\theta}$ using MoM

$$E(X) = \int_\theta^\infty x f(x) \cdot dx$$

$$= \int_\theta^\infty x \times \frac{\theta}{x^2} \cdot dx$$

$$= \theta \log x \Big|_\theta^\infty$$

$$= \infty$$

$\therefore$ MoM estimator ~~doesnt~~ doesn't exist here.

So, $\hat{\theta}$ can't be estimated using MoM.

Q3. $X_1, X_2 \ldots X_n \sim \text{Binomial}(n, p)$

need to find MoM estimates for n, p

$$E(X) = np \qquad Var(X) = npq = np(1-p)$$

Now, there are 2 unknown parameters n & p that we need to estimate;

so we will use the first 2 sample moment & form 2 eqns.

$$m_1 = \frac{1}{n}\sum X_i = E(X) \quad - (1)$$

$$m_2 = \frac{1}{n}\sum X_i^2 = E(X^2) \quad - (2)$$

$$\bar{X} = np \; ; \; \frac{1}{n}\sum X_i^2 = Var(X) + [E(X)]^2$$

$$= np(1-p) + (np)^2$$

$$\frac{1}{n}\sum X_i^2 = np(1-p) + (\bar{X})^2$$

$$\frac{1}{n}\sum X_i^2 - (\bar{X})^2 = np(1-p)$$

pop. variance $\leftarrow S^2 = \frac{1}{n}\sum (X_i - \bar{X})^2$

$$= \frac{1}{n}\left(\sum (X_i^2 - 2X_i\bar{X} + (\bar{X})^2)\right) \quad \because n\bar{X} = \sum X_i$$

$$= \frac{1}{n}\left(\sum X_i^2 - 2n(\bar{X})^2 + n(\bar{X})^2\right)$$

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$$\frac{1}{n}\sum X_i^2 - (\bar{X})^2 = S^2$$

$$S^2 = np(1-p)$$

Now $\bar{X} = np$

$$S^2 = \bar{X}(1-p)$$

$$1 - \frac{S^2}{\bar{X}} = p$$

$$\boxed{\hat{p}_{mom} = \frac{\bar{X} - S^2}{\bar{X}}}$$

$$\bar{X} = \hat{n}_{mom}\,\hat{p}_{mom}$$

$$\hat{n}_{mom} = \frac{\bar{X}}{\left(\frac{\bar{X} - S^2}{\bar{X}}\right)} \Rightarrow \boxed{\frac{\bar{X}^2}{(\bar{X} - S^2)} = \hat{n}_{mom}}$$

Q5 $X_1, X_2 \cdots X_n \sim U(\theta, \theta+1)$

$f_\theta(x) = \frac{1}{b-a}$ or $\frac{1}{(\theta+1)-\theta} = 1$

$$L(\theta) = \begin{cases} 1 & \text{if } \theta \le x_i \le \theta+1 \\ 0 & \text{o/w} \end{cases}$$

Use MLE to estimate θ

$\theta \le x_i$ & $x_i \le \theta + 1$

$\Rightarrow \theta \le X_{(1)} < X_{(2)} \cdots X_{(n)}$ or $\min(X_1, X$ ①

$x_i \le \theta + 1 \Rightarrow x_i - 1 \le \theta$

$\Rightarrow \max(\cancel{X_{(i)}} X_1, X_2 \cdots X_n) - 1 \le \theta$ ②

$\therefore$ from ① & ②

$\max(X_1 \cdots X_n) - 1 \le \hat{\theta}_{MLE} \le \min(X_1, X_2 \cdots X_n)$

Q6. a) $X_i \sim \text{Binomial}(3, \theta)$

$$(X_1 = 1,\ X_2 = 3,\ X_3 = 2,\ X_4 = 2) \qquad n = 4$$

need to estimate θ using MLE based on the above observations

constant (not depending on θ)

$$L(\theta) = \prod_{i=1}^{4} {}^{3}C_{x_i}\ \theta^{x_i} (1-\theta)^{3-x_i}$$

$$\log L(\theta) = \sum_{i=1}^{4} \left(x_i \log \theta + (3 - x_i)\log(1-\theta) + \log C \right)$$

$$\frac{d \log L(\theta)}{d\theta} = \sum_{i=1}^{4} \frac{x_i}{\theta} - \left(\frac{3 - x_i}{1-\theta}\right) + 0$$

$$= \frac{1}{\theta} - \frac{2}{1-\theta} + \frac{3}{\theta} - 0 + \frac{2}{\theta} - \frac{1}{1-\theta} + \frac{2}{\theta} - \frac{1}{1-\theta}$$

$$= \frac{8}{\theta} - \frac{4}{1-\theta}$$

$$\frac{d \log(L(\theta))}{d\theta} = 0$$

$$\Rightarrow \frac{8}{\theta} = \frac{4}{1-\theta} \Rightarrow 2 - 2\theta = \theta$$

$$\boxed{\hat{\theta} = 2/3}$$

check $\frac{d^2 l}{d\theta^2} < 0$

b) $X_i \sim \text{exponential}(\theta)$

$(X_1 = 1.23, \; X_2 = 3.32, \; X_3 = 1.98, \; X_4 = 2.12)$

$$f_\theta(x_i) = \theta e^{-\theta x_i}$$

$$L(\theta) = \prod_{i=1}^{4} \theta e^{-\theta x_i} = \theta^4 e^{-\theta \sum_{i=1}^{4} x_i}$$

$$\log L(\theta) = \cancel{\sum_{i=1}^{4}} \; 4 \log \theta - \theta \sum_{i=1}^{4} x_i$$

$$\frac{d \log L(\theta)}{d\theta} = \frac{4}{\theta} - \sum_{i=1}^{4} x_i$$

$$\frac{d \log L(\theta)}{d\theta} = 0 = \frac{4}{\theta} - (1.23 + 3.32 + 1.98 + 2.1$$

$$\boxed{\hat{\theta} = \frac{4}{8.65} = 0.46}$$

check $\frac{d^2 l}{d\theta^2} < 0$