# Worksheet 4 Solution

**Problem 1:** $x^2y'' - 3xy' + 10y = 0$

Assume $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

Substituting into the ODE, we get

$$x^2(m(m-1)x^{m-2}) - 3x(mx^{m-1}) + 10x^m = 0$$

$$m(m-1) - 3m + 10 = 0$$

$$m^2 - 4m + 10 = 0 \Rightarrow m = 2 \pm i\sqrt{6}$$

$$\boxed{y = x^2\left(c_1\cos(\sqrt{6}\log x) + c_2\sin(\sqrt{6}\log x)\right)}$$

**Problem 2:** $x^2y'' + xy' + 9y = 0$

Assume $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

Substituting into the ODE, we get

$$x^2 m(m-1)x^{m-2} + x\,mx^{m-1} + 9x^m = 0$$

$$\Rightarrow m(m-1) + m + 9 = 0$$

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

$$\boxed{y = c_1\cos(3\log(x)) + c_2\sin(3\log(x))}$$

Given that $y(1) = 0$, $y'(1) = 2.5$

$$y(1) = c_1 = 0$$

$$y'(x) = -c_1\sin(3\log x)\frac{3}{x} + c_2\cos(3\log x)\cdot\frac{3}{x}$$

$$y'(1) = 3c_2 = 2.5 \quad \Rightarrow \quad c_2 = 0.834$$

Hence the solution is

$$\boxed{y(x) = 0.834 \sin(3 \log x)}$$

<u>Problem 3</u>:

(a) $$y'' + y' + \left(\pi^2 + \frac{1}{4}\right) y = e^{-x/2} \sin \pi x$$

Solution of the homogenous problem:

$$\lambda^2 + \lambda + \pi^2 + \frac{1}{4} = 0$$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4\pi^2 - 1}}{2} = \frac{-1}{2} \pm i\pi$$

$$y_h = e^{-x/2}\left(c_1 \cos \pi x + c_2 \sin \pi x\right)$$

$$y_h = c_1 e^{-x/2} \cos \pi x + c_2 e^{-x/2} \sin \pi x$$

We try a particular solution of the form

$$y_p = A x e^{-x/2} \cos(\pi x) + B x e^{-x/2} \sin(\pi x)$$

$$y_p' = \frac{1}{2} e^{-x/2}\left[\cos(\pi x)(2\pi Bx - A(x-2)) - \sin(\pi x)(2\pi Ax + B(x-2))\right]$$

$$y_p'' = \frac{1}{4} e^{-x/2}\Big[\sin(\pi x)(4\pi A(x-2) + B(-4\pi^2 x + x - 4)) +$$
$$\cos(\pi x)(A(-4\pi^2 x + x - 4) - 4\pi B(x-2))\Big]$$

We now substitute in the original equation,

$$y'' + y' + \left(\pi^2 + \frac{1}{4}\right)y = 2\pi e^{-x/2}\left(B\cos\pi x - A\sin(\pi x)\right)$$

$$= e^{-x/2}(\sin\pi x)$$

from where $B=0$ and $A = \frac{-1}{2\pi}$

So, $y_p = \frac{-1}{2\pi} x e^{-x/2} \cos(\pi x)$

The general solution is

$$\boxed{y = y_h + y_p = e^{-x/2}\left(\left(c_1 - \frac{x}{2\pi}\right)\cos\pi x + c_2 \sin\pi x\right)}$$

(b) $8y'' - 6y' + y = 6\cosh x$

Solution of the homogenous problem

$$8\lambda^2 - 6\lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{4}, \frac{1}{2}$$

$$y_h = c_1 e^{x/2} + c_2 e^{x/4}$$

We try a particular solution of the form

$$y_p = A\cosh x + B\sinh x$$

$$y_p' = A\sinh x + B\cosh x$$

$$y_p'' = A\cosh x + B\sinh x$$

Substituting into the differential equation

$$(9A-6B)\cosh x + (9B-6A)\sinh x = 6\cosh x$$

$\Rightarrow$

$$9A-6B = 6 \quad ; \quad 9B-6A=0$$

$$\Rightarrow A = \frac{6}{5}, \quad B = \frac{4}{5}$$

General solution $\rightarrow$

$$y = c_1 e^{x/2} + c_2 e^{x/4} + \frac{6}{5}\cosh x + \frac{4}{5}\sinh x$$

Given that $y(0) = 0.2$, $y'(0) = 0.05$

$$y(0) = c_1 + c_2 + \frac{6}{5} = 0.2$$

$$y'(0) = \frac{1}{2}c_1 + \frac{1}{4}c_2 + \frac{4}{5} = 0.05$$

$$\Rightarrow \quad c_1 = -2, \quad c_2 = 1$$

$$\boxed{y = -2e^{x/2} + e^{x/4} + \frac{6}{5}\cosh x + \frac{4}{5}\sinh x}$$

<u>Problem 4</u> : $y'' + y = \cos \omega t$

solution of the homogeneous problem $\rightarrow$

$$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

$$y_h = c_1 \cos t + c_2 \sin t$$

let $\quad y_p = A\cos\omega t + B\sin\omega t$

$$y_p' = -\omega A \sin\omega t + \omega B \cos\omega t$$

$$y_p'' = -\omega^2 A \cos\omega t - \omega^2 B \sin\omega t$$

Substituting into the differential equation

$$A(1-\omega^2)\cos\omega t + B(1-\omega^2)\sin\omega t = \cos\omega t$$

$$\Rightarrow \quad A = \frac{1}{1-\omega^2} \; ; \quad B = 0$$

So, the general solution is

$$\boxed{y = c_1 \cos t + c_2 \sin t + \frac{1}{1-\omega^2}\cos\omega t}$$

Problem 5 : $\quad y'' + 5y = \cos\pi t - \sin\pi t \, , \; y(0) = 0, \; y'(0) = 0$

Homogenous solution :

$$\lambda^2 + 5 = 0 \quad \Rightarrow \lambda = \pm\sqrt{5}\, i$$

$$y_h = c_1 \cos\sqrt{5}\, t + c_2 \sin\sqrt{5}\, t$$

let $\quad y_p = A\cos\pi t + B\sin\pi t$

$$y_p' = -\pi A \sin\pi t + \pi B \cos\pi t$$

$$y_p'' = -\pi^2 A \cos\pi t - \pi^2 B \sin\pi t$$

Substituting, $A\cos\pi t(-\pi^2+5) + B\sin\pi t(-\pi^2+5)$
$$= \cos\pi t - \sin\pi t$$

$$A(5-\pi^2) = 1 \quad ; \quad B(5-\pi^2) = -1$$

$$A = \frac{1}{5-\pi^2} \quad ; \quad B = \frac{-1}{5-\pi^2}$$

$$y_p = \frac{\cos\pi t - \sin\pi t}{5-\pi^2}$$

General solution :

$$y = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t + \frac{\cos\pi t - \sin\pi t}{5-\pi^2}$$

$$y(0)=0 \Rightarrow c_1 + \frac{1}{5-\pi^2} = 0 \quad ; \quad c_1 = \frac{1}{\pi^2-5}$$

$$y'(0) = 0 \Rightarrow \sqrt{5}c_2 - \frac{\pi}{5-\pi^2} = 0 \, ; \quad c_2 = \frac{\pi}{\sqrt{5}(5-\pi^2)}$$

$$\boxed{y = \frac{1}{\pi^2-5}\left(\cos\sqrt{5}t - \frac{\pi}{\sqrt{5}}\sin\sqrt{5}t - \cos\pi t + \sin\pi t\right)}$$

<u>Problem 6</u> : To model the circuit we sum the drops of voltage along the RC loop

$$E(t) - IR - \frac{Q}{C} = 0$$

$$E(t) - IR - \frac{1}{C}\int I(t)\,dt = 0$$

Differentiating, $E'(t) - I'R - \frac{1}{C} I = 0$

$$I' + \frac{1}{RC} I = \frac{E'}{R}$$

If $E$ is constant, then $E(t) = E_0$, then $E' = 0$

$$I' + \frac{1}{RC} I = 0$$

Solution is $I(t) = A e^{-\frac{1}{RC} t}$

at $t = 0$, we have $I(0)$, then

$$A = I(0)$$

$$\boxed{I(t) = I(0) e^{\frac{-1}{RC} t}}$$

(b) Given that $E = E_0 \sin \omega t$, we have

$$I' + \frac{1}{RC} I = \frac{E_0 \omega \cos \omega t}{R}$$

$$I(t)\, e^{\frac{t}{RC}} = \int \frac{E_0 \omega \cos\omega t}{R} \cdot e^{\frac{t}{RC}}\, dt + C_1$$

$$I(t)\, e^{\frac{t}{RC}} = \frac{E_0 \omega}{R} \frac{e^{\frac{t}{RC}}}{\left(\frac{1}{R^2C^2} + \omega^2\right)} \left[ \frac{1}{RC} \cos\omega t + \omega \sin\omega t \right] + C_1$$

$$\boxed{I(t) = C_1 e^{\frac{-t}{RC}} + \frac{\omega E_0 C}{1 + (\omega RC)^2} \left(\cos\omega t + \omega RC \sin\omega t\right)}$$