Q1

ans) Given $X_1, X_2 \overset{iid}{\sim}$ Poisson (λ), $\lambda > 0$

Also,

$$T = X_1 + 2X_2$$

We need to check if T is sufficient for λ

The formal defn of sufficiency is

$$P(\underset{\sim}{X} = \underset{\sim}{x} \mid T(x) = t) : \text{does not depend on } \lambda$$

For values $X_1 = 0$, $X_2 = 1$;

$$P(X_1 = 0, X_2 = 1 \mid T = X_1 + 2X_2 = 2)$$

$$= \frac{P(X_1 = 0, X_2 = 1 \cap \overbrace{X_1 + 2X_2 = 2}^{\text{redundant}})}{P(X_1 + 2X_2 = 2)}$$

$$= \frac{P(X_1 = 0, X_2 = 1)}{P(X_1 + 2X_2 = 2)}$$

$$= \frac{P(X_1 = 0) \cdot P(X_2 = 1)}{P(X_1 + 2X_2 = 2)} \quad \text{(i.i.d)}$$

$$= \frac{P(X_1=0)\cdot P(X_2=1)}{P(X_1=0, X_2=1) + P(X_1=2, X_2=0)}$$

[Break the den^r into sep. probs by setting each to 0]

$$= \frac{(e^{-\lambda})(\lambda e^{-\lambda})}{(e^{-\lambda})(\lambda e^{-\lambda}) + (e^{-\lambda})\left(\frac{e^{-\lambda}\lambda^2}{2}\right)}$$

$$= \frac{\lambda}{\lambda + \frac{\lambda^2}{2}}$$

$$= \frac{1}{1 + \frac{\lambda}{2}}$$

$\therefore$, it does depend on $\theta = \lambda$

Hence, $T = X_1 + 2X_2$ is not sufficient for λ

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Q-2

a) For a distribution to belong to exponential family of distributions, the likelihood fn should be of the form

$$L(\theta) = \prod_{i=1}^{n} \{h(x_i)\} [c(\theta)]^n \exp\Big[\omega_1(\theta) \sum_{i=1}^{n} t_1(x_i)$$

$$+ \omega_2(\theta) \sum_{i=1}^{n} t_2(x_i) + \dots \omega_k(\theta) \sum_{i=1}^{n} t_k(x_i$$

Given: pdf of X from Gamma(α, β) given by:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad 0 < x < \infty, \ \alpha > 0, \ \beta > 0$$

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$: \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} (x_i)^{\alpha-1} e^{-\beta x_i}$$

$$= \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \prod_{i=1}^{n} (x_i)^{\alpha-1} e^{-\beta \Sigma x_i}$$

$$= \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \prod_{i=1}^{n} \frac{1}{x_i} \prod_{i=1}^{n} (x_i)^{\alpha} e^{-\beta \Sigma x_i}$$

$$= \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\right]^n \prod_{i=1}^{n} \frac{1}{x_i} \prod_{i=1}^{n} e^{\ln (x_i)^\alpha} \cdot e^{-\beta \Sigma x_i}$$

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$$= \left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right]^{n} \prod_{i=1}^{n} \frac{1}{x_i} \, e^{\alpha \Sigma \ln x_i - \beta \Sigma x_i}$$

Comparing it to exponential family distribution $L(\theta)$ we get,

$$C(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \quad \prod_{i=1}^{n} h(x_i) = \prod_{i=1}^{n} \frac{1}{x_i}$$

$$w_1(\theta) = \alpha \quad \Sigma t_1(x_i) = \Sigma \ln x_i$$

$$w_2(\theta) = \beta \quad \Sigma t_2(x_i) = \Sigma x_i$$

$\Rightarrow$ gamma distribution belongs to exponential family of distributions

b) Using factorization theorem, we can find sufficient statistic for exponential family of distributions

$$T(X) = [T_1(X), T_2(X) \ldots T_n(X)]$$

where in this case $T_1(X) = \sum_{i=1}^{n} \ln X_i$ and $T_2(X) = \sum_{i=1}^{n} X_i$

Therefore $T = \{T_1(X), T_2(X)\}$

$= \{\sum_{i=1}^{n} \ln X_i, \sum_{i=1}^{n} X_i\}$ are jointly

sufficient statistic for $\theta = (\alpha, \beta)$.

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Q.3)

For a random variable X_i the pdf is

$$f_\theta(x_i) = \begin{cases} 1 & \text{if } \theta - \frac{1}{2} < x_i < \theta + \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

The corresponding liklihood f^n is

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$= 1 \cdot \prod_{i=1}^{n} I_\theta(x_i)$$

where $I_\theta(x_i) = \begin{cases} 1 & \text{if } \theta - \frac{1}{2} < x_i < \theta + \frac{1}{2} \\ 0 & \text{else} \end{cases}$

Let us take ordered random variable X_i

i.e $\theta - \frac{1}{2} < x_{(1)} < x_{(2)} \ldots < X_{(n)} < \theta + \frac{1}{2}$

Again defining indicator function

$$I_\theta(x_{(1)}) = \begin{cases} 1 & \text{if } \theta - \frac{1}{2} < x_{(1)} < x_{(2)} \\ 0 & \end{cases}$$

$$I_\theta(x_{(2)}) = \begin{cases} 1 & \text{if } x_{(1)} < x_{(2)} < x_{(3)} \\ 0 & \text{else} \end{cases}$$

$\vdots$

$$I_\theta(x_{(n)}) = \begin{cases} 1 & \text{if } x_{(n-1)} < x_{(n)} < \theta + \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

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From the indicator functions, if one of them is zero then the likelihood will be zero

So $I_\theta(x_{(2)}), \ldots, I_\theta(x_{(n-1)}) = 1$

hence,

$$L(\theta) = 1 \cdot I_\theta(x_{(1)}) \cdot I_\theta(x_{(n)})$$

Applying factorization theorem

if $L(\theta) = g_\theta(T(x)) \cdot h(x)$ then $T(x)$ is sufficient for θ.

& taking $h(x) = 1$

$$g(\theta, T(x)) = I_\theta(x_{(1)}) \cdot I_\theta(x_{(n)})$$

So $T(\cancel{X}) = (\min(X_1, \ldots X_n), \max(X_1 \ldots X_n))$

are jointly sufficient statistic for θ.