MULTIVARIATE CALCULUS
END SEMESTER EXAM

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ABSTRACT. This is a closed-book test with no cheat sheets allowed. There are nine problems in total and one section devoted to the notations and the definitions of essential concepts; the problems are numbered while the sections that define a concept are not. You have precisely 2 hours to finish the test.

NOTATIONS AND CONVENTIONS

Throughout this article the symbol $\mathbb{F}$ will denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. By a *scalar* we shall always mean an element of the *scalar field* $\mathbb{F}$. All linear spaces (or vector spaces) in this set of notes are assumed to be over the scalar field $\mathbb{F}$.

Suppose $\mathcal{V}$ is a linear space over the scalar field $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, $\mathcal{V}$ is called a *real linear space*; similarly if $\mathbb{F} = \mathbb{C}$, we speak of *complex linear spaces*. Any statement about linear spaces in which the scalar field is not explicitly mentioned is to be understood to apply to both of these cases.

Definition 0.1. Suppose $\mathcal{V}$ and $\mathcal{W}$ are linear spaces over the scalar field $\mathbb{F}$. By a *linear transformation* from $\mathcal{V}$ to $\mathcal{W}$, we mean a mapping (or function) $T : \mathcal{V} \to \mathcal{W}$ from $\mathcal{V}$ to $\mathcal{W}$ such that

$$T(\alpha x + \beta y) = \alpha T x + \beta T y$$

whenever $x, y \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{F}$.

Solution set for the end-semester examination

1. PROBLEM

Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function defined via

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le x < 1 \end{cases}.$$

What is the value of $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$? Show your work.

[10 points]

For any $y \in [0,1]$, we have

$$\int_0^1 f(x,y)\,dx = \int_{x=0}^{x=1/2} f(x,y)\,dx + \int_{x=1/2}^{x=1} f(x,y)\,dx$$

$$= 0 + \int_{x=1/2}^{x=1} 1\,dx$$

$$= [x]_{1/2}^{1}$$

$$= 1 - 1/2 = 1/2.$$

(5 points)

Consequently,

$$\int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x,y)\,dx\,dy = \int_{y=0}^{y=1} 1/2\,dy$$

$$= 1/2 \text{ ans.}$$

(5 points)

2. Problem

Let $y = f(x)$ be a real-valued function of single variable such that every point of the form $(x, f(x))$ is equidistant from the point $P = (0, 1)$ and the line $y = -1$.

(a) Find $f(x)$, its domain and range.

(b) For any point $(x_0, f(x_0))$, show that the tangent to the curve at the point $(x_0, f(x_0))$ makes equal angles with the vectors $0\vec{i} + 1\vec{j}$ and $x_0\vec{i} + (f(x_0) - 1)\vec{j}$. (Hint: You may want to find the equation of the tangent to the curve at x_0.)

[20 points]

Question credit: Mudit Aggarwal

Solution – provided by him.

3. Problem

Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that

(i) f is continuous at every point except at $x = -1$ and $x = 3$;

(ii) $g(x) = \begin{cases} x^2 + 1 & \text{if } x > 0 \\ x - 3 & \text{if } x \leq 0 \end{cases}$.

At what points will the function $f(g(x))$ be continuous?

[10 points]

Question credit: Mudit Aggarwal

Solution – provided by him.

4. Problem

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be functions given by

$$f(x,y) = x^4 + y^4 \qquad \text{and} \qquad g(x,y) = x^4 - y^4.$$

(a) Verify that the origin $(0,0) \in \mathbb{R}^2$ is a critical point of domains of both functions f and g.

(b) Do both functions f and g simultaneously have

  (i) local maximum at $(0,0)$; or,

  (ii) local minima at $(0,0)$; or,

  (iii) saddle point at $(0,0)$?

(c) What does this tell us about the second derivative test?

[10 points]

Question credit: Mudit Aggarwal

Solution - provided by him.

5. Problem

Evaluate

$$\int_0^\infty e^{-x^2}\,dx.$$

Hint: Use I to denote the value of the above integral. First compute its square, that is, I^2, and then deduce the value of I.

[10 points]

Let $I = \int_0^\infty e^{-x^2}\,dx.$

First we compute I^2:

$$I^2 = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-x^2}dx\right)$$

$$= \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)$$

$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$\boxed{3 \text{ points}}$ for getting I^2

$$= \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta.$$

$\boxed{2 \text{ points}}$ for conversion in polar coordinates

Let $u = r^2$. Then $du = 2r\,dr$;
$u = 0$ when $r = 0$,
$u = \infty$ when $r = \infty$, so that

$\boxed{2 \text{ points}}$

$$I^2 = \int_0^{\pi/2} \left(\frac{1}{2} \int_0^{\infty} e^{-u}\, du \right) d\theta$$

for correct u-substitution.

$$= \int_0^{\pi/2} \left(\frac{1}{2} \left(\underbrace{\lim_{b \to \infty} \left[\frac{-1}{e^u} \right]_0^b}_{=1} \right) \right) d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\, d\theta$$

$$= \left[\frac{\theta}{2} \right]_0^{\pi/2} = \underline{\pi/4}.$$

Since $I^2 = \pi/4$, we have

$$I = \frac{\sqrt{\pi}}{2} \quad \underline{\text{ans.}}$$

$\boxed{\text{3 points}}$ for Computation & getting I.

6. Problem

Let D be the region in xyz-space defined by the inequalities

$$1 \le x \le 2, \qquad 0 \le xy \le 2, \qquad 0 \le z \le 1.$$

Evaluate

$$\iiint_D (x^2y + 3xyz)\,dx\,dy\,dz$$

by applying the transformation

$$u = x, \qquad v = xy, \qquad w = 3z$$

and integrating over an appropriate region G in uvw-space.

[*10 points*]

Step I: make x, y and z the subject of the equations and express these in terms of u, v and w only.

2 points

$$\left[x = u \;;\; y = \frac{v}{x} = \frac{v}{u} \;;\; z = \frac{w}{3} \right]$$

Step II:

2 points

$$\begin{cases} 1 \le x \le 2 \Rightarrow 1 \le u \le 2 \;; \\ 0 \le xy \le 2 \Rightarrow 0 \le v \le 2 \;; \\ 0 \le z \le 1 \Rightarrow 0 \le w \le 3 \end{cases}$$

Step III: Compute the Jacobian $J(u, v, w)$.

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & 1/3 \end{vmatrix} = \frac{1}{3u}.$$

[3points]

Step IV: Compute the integral.

[3points]

$$\iiint_D (x^2y+3xyz)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^2\int_0^1 \left(u^2\cdot\frac{v}{u}+3u\cdot\frac{v}{u}\cdot\frac{w}{3}\right)\cdot\frac{1}{3u}\,du\,dv\,dw$$

$$= \int_0^3\int_0^2\int_0^1 \frac{v}{3}+\frac{vw}{3u}\,du\,dv\,dw$$

$$= 2+\ln 8.$$ ans

7. Problem

Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$ and suppose that

$$\frac{\partial g_2}{\partial x} = \frac{\partial g_1}{\partial y}.$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \int_0^x g_1(t, 0)\, dt + \int_0^y g_2(x, t)\, dt.$$

Show that

$$\frac{\partial f(x, y)}{\partial x} = g_1(x, y).$$

[20 points]

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}\int_0^x g_1(t,0)\,dt + \frac{\partial}{\partial x}\int_0^y g_2(x,t)\,dt$$

[5 points]

$$= g_1(x,0) + \int_0^y \frac{\partial}{\partial x} g_2(x,t)\,dt$$

[10 points]

$$\overset{\star}{=} g_1(x,0) + \int_0^y \frac{\partial}{\partial t} g_1(x,t)\,dt$$

[5 points]

$$= g_1(x,0) + \Big[g_1(x,t)\Big]_{t=0}^{t=y}$$

$$= g_1(x,0) + g_1(x,y) - g_1(x,0)$$

$$= g_1(x,y)$$

Note that in the highlighted step — step ★ —

$\frac{\partial}{\partial x} g_2(x,t)$ has been replaced by $\frac{\partial}{\partial t} g_1(x,t)$; not by $\frac{\partial}{\partial y} g_1(x,t)$. (Needless to say, the expression $\frac{\partial}{\partial y} g_1(x,t)$ doesn't make any sense.)

Explanation:

In the hypothesis of the question when it is told that

$$\frac{\partial}{\partial x} g_2(x,y) = \frac{\partial}{\partial y} g_1(x,y),$$

then, in plain simple English, we mean to say that

the partial derivative of g_2 w.r.t. the first variable

is equal to

the partial derivative of g_1 w.r.t. the second variable.

8. Problem

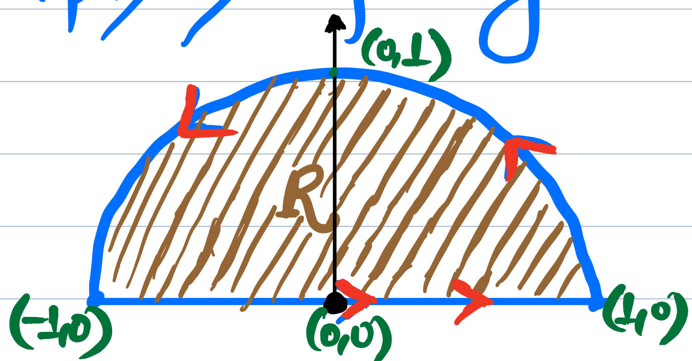
Apply Green's theorem to evaluate the following integral.

$$\oint_C y^2\,dx + 3xy\,dy$$

where C is the boundary of upper half of the unit disc traversed in the anticlockwise sense (that is, $C = C_1 \cup C_2$, where C_1 is parametrized by $r_1(t) = (2t-1)\vec{i} + 0\vec{j}$, $t \in [0,1]$ and C_2 is parametrized by $r_2(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, $t \in [0,\pi]$).

[10 points]

Note that C represents the following curve in anticlockwise direction.



Let us use R to denote the area enclosed by this simple closed curve C. If we use γ to denote the parameterization of the curve C, then by Green's theorem, we have

$$\oint_\gamma F\cdot ds = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy.$$

We need to solve this question by using Green's theorem to convert line integral into a double integral.

Now the solution:

[3 points]

The integrand:

Clearly, the vector field above in the question is

$$F(x,y) = (y^2, 3xy) = y^2\vec{i} + 3xy\vec{j}.$$

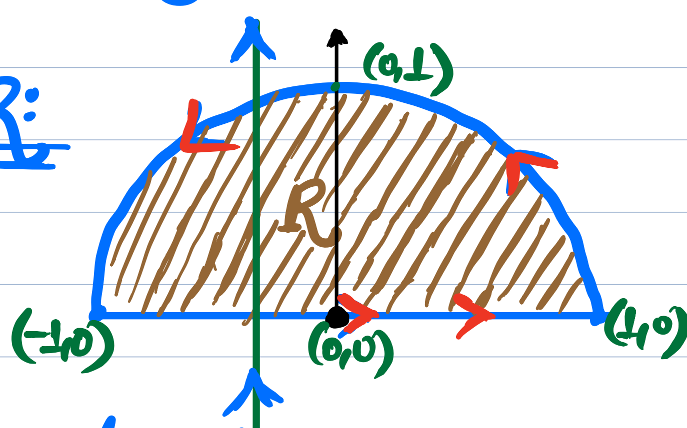
Consequently, $F_1(x,y) = y^2$ and
$F_2(x,y) = 3xy$

$$\Rightarrow \quad \frac{\partial F_2}{\partial x} = 3y \quad \text{and} \quad \frac{\partial F_1}{\partial y} = 2y$$

$$\Rightarrow \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 3y - 2y = y.$$

[3 points]

The region R:



for a fixed x, the vertical line enters the region $y=0$ and leaves at $y = \sqrt{1-x^2}$.

$$\therefore \quad 0 \leq y \leq \sqrt{1-x^2}.$$

Clearly, then $-1 \leq x \leq 1$.

These equations are enough

Applying Green's theorem and computing the integral:

$$\oint_C y^2\,dx + 3xy\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \int_{x=-1}^{x=1} \left(\int_{y=0}^{y=\sqrt{1-x^2}} y\,dy \right) dx$$

$$= \int_{-1}^{1} \left[\frac{y^2}{2} \right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \frac{1-x^2}{2}\,dx$$

$$= \left[\frac{1}{2}x - \frac{x^3}{6} \right]_{-1}^{1}$$

[4 points]

$$= [1/2 - 1/6] - [-1/2 + 1/6]$$

$$= 1/2 + 1/2 - 1/6 - 1/6 = 1 - 1/3 = \boxed{2/3}$$

9. Problem

Find the area bounded by the curve r parametrized by $r(t) = (\sin 2t, \sin t)$, $t \in [0, \pi]$ using Green's theorem.

[*10 points*]

Recall:

$$\oint_\gamma F \cdot ds = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

where R = the region enclosed by $\gamma(t)$.

Need to find a vector field F, such that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. Let's choose

$$\boxed{F(x,y) = -\frac{y}{2}\vec{i} + \frac{x}{2}\vec{j}}$$

Clearly $F_1 = -y/2$ and $F_2 = x/2$

$$\therefore \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

Then $\iint_R dx\,dy$ is the required area. Now,

$$r(t) = \sin 2t\,\vec{i} + \sin t\,\vec{j}$$

$$\therefore\ x = \sin 2t \text{ and } y = \sin t$$

Since $F = \frac{-y}{2}\vec{i} + \frac{x}{2}\vec{j}$, we get

$$F(r(t)) = -\frac{\sin t}{2}\vec{i} + \frac{\sin 2t}{2}\vec{j}$$

and

$$r'(t) = 2\cos 2t\,\vec{i} + \cos t\,\vec{j}$$

$$\therefore \oint_{r} F\cdot T\,ds = \int_{t=0}^{\pi} F(r(t))\cdot r'(t)\,dt$$

$$= \int_0^{\pi}\left(-\frac{\sin t}{2}, \frac{\sin 2t}{2}\right)\cdot\left(2\cos 2t, \cos t\right)dt$$

$$= \int_0^{\pi}\left(-\frac{\sin t}{2}, \sin t\cos t\right)\cdot\left(2\cos^2 t - 2\sin^2 t, \cos t\right)dt$$

$$= \int_0^{\pi}\left(\cancel{-\sin t\cos^2 t} + \sin^3 t + \cancel{\sin t\cos^2 t}\right)dt$$

$$= \int_0^{\pi}\sin^3 t\,dt = \int_0^{\pi}\sin t\left(1 - \sin^2 t\right)dt$$

$$= \int_0^{\pi} \sin t \, dt - \int_0^{\pi} \sin t \cos^2 t \, dt$$

$$= -\left[\cos t\right]_0^{\pi} + \left[\frac{\cos^3 t}{3}\right]_{t=0}^{\pi}$$

$$= -\{-1-1\} + 1/3\{-1-1\}$$

$$= 2 - 2/3 = 4/3 \text{ square units.}$$

→ [<u>Finding</u> $F(x,y) = -y/2\,\vec{i} + x/2\,\vec{j}$] – 3 points

→ [Find $F(\vec{r}(t))$, $d\vec{r}/dt$, and the dot product] – <u>3 points</u>

→ [Computing the integral — <u>4 points</u>