Quiz 2

Time : 40 minutes Full Marks :15

Problem 1. (5 points) In a flow network $G = (V, E)$ with source s and sink t, the capacity of an s–t cut (S, T) is the sum of capacities of all edges directed from S to T, where $s \in S$, $t \in T$, and $S \cup T = V$. The minimum s–t cut value is the smallest capacity over all s–t cuts. We say that the minimum s–t cut is *unique* if there is exactly one partition (S, T) that achieves this minimum capacity.

Consider the following proposed test for uniqueness: compute a minimum cut (S, T) of value F, pick any edge e crossing this cut, increase the capacity of e by 1, and recompute the minimum cut. If the new minimum cut value is $F + 1$, conclude that the original minimum cut was unique; if the new minimum cut value is F, conclude that the minimum cut was not unique.

Either prove that this test is correct or provide a small counterexample network (with explicit capacities) in which the test gives the wrong conclusion.

Solution. The test is incorrect. Consider the following graph - $V = a, b, c, d$ with (directed) edges $E = ab, ac, bd, cd$ - each with capacity 1. Now suppose you compute the min-cut as $S = \{s, a\}$; pick the edge (sd) and increase its capacity by 1. Then the min-cut in the resulting graph with increase by 1. But $\{s, a\}$ is clearly not a unique min-cut.

Rubric : $+5$ for correct conclusion and counter example. No partial marking.

Problem 2. Menger's Theorem (10 points) Let $G = (V, E)$ be a finite directed graph with distinct vertices $s, t \in V$. Prove that the *maximum* number of pairwise edge-disjoint s–t paths equals the *minimum* number of edges whose removal separates s from t. (Hint: Of course this is not a Graph Theory course and this is a quiz about flows. So the task is to design a suitable flow network and use Theorems that you have learnt in lectures to prove the above.)

Solution. We prove this theorem using a combination of Path Decomposition Theorem and the Max-flow/Min-Cut Duality. Construct a flow network with capacity 1 on every edge. Consider any integral $s-t$ flow in this network of value F. Applying the path decomposition theorem, we claim that this flow can be decomposed in to F edge disjoint paths. This can be proved by observing that - (a) The number of paths in the decomposition is at most F - if not, then some path has to carry a fractional flow and (b) The number of paths in the decomposition is at least F - if not, the some path has to carry a flow of 1 unit, which is not possible since the capacity on all edges is 1. Hence, using integrality property of max-flow, the max-flow in the network = maximum number of edge-disjoint s-t paths. Invoking the max-flow = min-cut theorem, this is exactly equal to the min $s-t$ cut which is equal to the minimum number of edges whose removal disconnects s and t (due to all capacities being 1).

Rubric : +2 for realizing this can be done using path decomposition and max-flow min-cut. +2 for setting up the flow network correctly. +2 for properly using path decomposition thm to argue that value of *integral* flow = number of paths. +2 for stating the max-flow is integral given integer capacities. +2 for finishing the argument invoking max-flow = min-cut.