W6 Solutions

Problem 1: The characteristic equation of the ODE is

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow \quad \lambda = -2, -1, 1$$

The general solution is

$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x}$$

To convert the ODE into an ODE system, we do the following change of variables

$$y_1 = y$$

$$y_2 = y_1' = y'$$

$$y_3 = y_2' = y''$$

the original ODE can be written as

$$y_3' + 2y_3 - y_2 - 2y_1 = 0$$

$$\Rightarrow \quad y_3' = -2y_3 + y_2 + 2y_1$$

Altogether the ODE system is

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$y' = Ay$$

The eigenvalues and eigenvectors of A are

$$\lambda_1 = -2, \quad V_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$\lambda_2 = -1, \quad V_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 1, \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The general solution of the ODE system is

$$y = c_1 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{-2x} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-x} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{x}$$

$$= \begin{bmatrix} c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} \end{bmatrix}$$

$$\begin{bmatrix} -2c_1e^{-2x} - c_2e^{-x} + c_3e^{x} \\ 4c_1e^{-2x} + c_2e^{-x} + c_3e^{x} \end{bmatrix}$$

finally, remind that $y = y_1$, so we are mostly interested in its first component that is

$$y = c_1e^{-2x} + c_2e^{-x} + c_3e^{x}$$

That is, the same result as we obtained by the direct method.

Problem 2: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

The characteristic equation of the system matrix is

$$\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 2, -2$$

The eigenvector of $\lambda_1 = 2$ is

$$(A - 2I)x = 0$$

solving it, we get $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigenvector of $\lambda_2 = -2$ comes from

$$(A+2I)x=0$$

$\Rightarrow$ we get $x = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Finally the solution of the differential equation is

$$y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} = \begin{bmatrix} c_1 e^{2t} - c_2 e^{-2t} \\ c_1 e^{2t} + 3 c_2 e^{-2t} \end{bmatrix}$$

Problem 3: We can model the system with the following differential equations:

$$y_1' = -\frac{y_1}{200}(16) + \frac{y_2}{200}(4) + (0)(12)$$

$$y_2' = \frac{y_1}{200}(16) - \frac{y_2}{200}(4+12)$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} \frac{-16}{200} & \frac{4}{200} \\ \frac{16}{200} & \frac{-16}{200} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The eigenvalues and eigenvectors of this matrix are

$$\lambda_1 = \frac{-3}{25}, \quad v_1 = \begin{bmatrix} -\frac{1}{2} \\ \ldots \end{bmatrix}$$

$$\lambda_2 = \frac{-1}{25}, \quad V_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

The general solution of the ODE system is

$$y = c_1 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-3/25\, t} + c_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{-1/25\, t}$$

At $t=0$, we have

$$y(0) = \begin{bmatrix} 100 \\ 200 \end{bmatrix} = c_1 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 200 \end{array}$$

So, the solution is

$$y = 200 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{-t/25}$$

<u>Problem 4</u>: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

The characteristic equation of the matrix is

$\lambda^2 - 2\lambda - 3 = 0$

$p = 2 > 0, \quad q = -3 < 0$

So, it is a saddle point, always unstable.

Problem 5: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Its critical point is $(y_1, y_2) = (0,0)$.
Eigenvalues of the matrix are 1 and 2, that is, it is an unstable node (because $p = \lambda_1 + \lambda_2 > 0$)

If we do the change of variable $\tau = -t$ then

$$\frac{dy_i}{d\tau} = \frac{dy_i}{dt}\frac{dt}{d\tau} = -\frac{dy_i}{dt}$$

So, the equation system becomes

$$\begin{bmatrix} \frac{dy_1}{d\tau} \\ \frac{dy_2}{d\tau} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

That is, the direction of motion changes, and the two eigenvalues become negative.

Then we have $p = \lambda_1 + \lambda_2 < 0$ and $q = \lambda_1 \lambda_2 > 0$, consequently a stable node.

Problem 6 : $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

The characteristic equation of the matrix is

$$\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$$

Eigenvector corresponding to $\lambda_1 = 3i$ is $\begin{bmatrix} -\frac{1}{3} i \\ 1 \end{bmatrix}$

Eigenvector corresponding to $\lambda_2 = -3i$ is $\begin{bmatrix} \frac{1}{3} i \\ 1 \end{bmatrix}$

To find the real solution, we construct the functions

$$y_1 = \begin{bmatrix} -\frac{1}{3} i \\ 1 \end{bmatrix} e^{i3x}, \quad y_2 = \begin{bmatrix} \frac{1}{3} i \\ 1 \end{bmatrix} e^{-i3x}$$

$$\tilde{y}_1 = \frac{y_1 + y_2}{2} = \begin{bmatrix} \frac{1}{3}\sin 3x \\ \cos 3x \end{bmatrix}$$

$$\tilde{y}_2 = \frac{y_1 - y_2}{2i} = \begin{bmatrix} -\frac{1}{3}\cos 3x \\ \sin 3x \end{bmatrix}$$

The General real solution is given by

$$y = c_1 \tilde{y}_1 + c_2 \tilde{y}_2 = \begin{bmatrix} \frac{c_1}{3}\sin 3x - \frac{c_2}{3}\cos 3x \\ c_1 \cos 3x + c_2 \sin 3x \end{bmatrix}$$

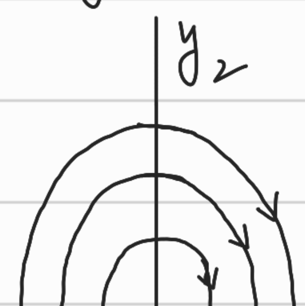
To sketch the trajectories $\rightarrow$

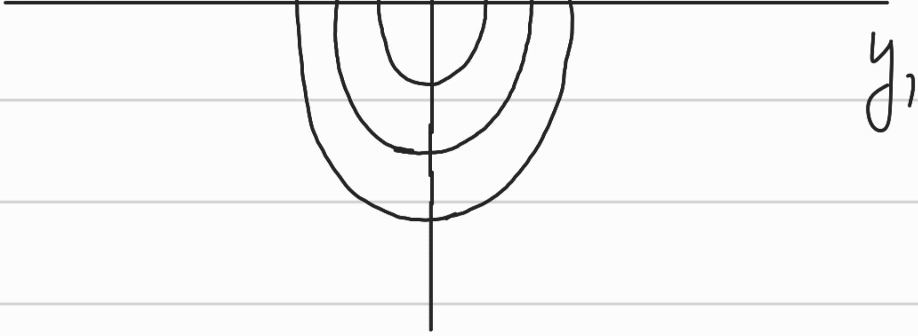
$$\frac{dy_1}{dx} = y_2 \quad , \quad \frac{dy_2}{dx} = -9y_1$$

$$\frac{dy_1}{dy_2} = -\frac{y_2}{9y_1} \quad , \quad 9y_1 dy_1 = -y_2 dy_2$$

$$\Rightarrow \quad 9y_1^2 + y_2^2 = c$$

This is a family of ellipses.

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