Quiz 3

Time : 40 minutes Full Marks :20

Problem 1. (5 points) In the *maximum-weight* bipartite matching problem, the input is a bipartite graph $G = (L \cup R, E)$ with a nonnegative weight w_e per edge, and the goal is to compute a matching M that *maximizes* $\sum_{e \in M} w_e$. Give a *linear-time* reduction of an instance of this problem to an instance of minimum cost perfect matching such that Hungarian algorithm is directly applicable. Clearly state the reduction. A formal proof is *not required*.

Solution. The reduction will be as follows :

1. For every edge e, define cost function $c_e = -w_e$. (Since we will minimize the sum of negated costs, this will maximize the sum of original weights) (Rubric : $+2.5$)
2. Add $\Delta = \min_e c_e$ to every c_e to make all costs non-negative. (Rubric : $+2.5$)
3. Add sufficient number of dummy vertices to make the graph balanced and dummy edges of cost ∞ to make sure there exists a perfect matching (**this step might take more than linear time in the size of the given graph** and hence it is fine if you had skipped this)

Problem 2. (5+5+5 points) Let $G = (V, E)$ be an undirected graph. The *vertex cover* problem is defined as follows - pick a *minimum sized* subset $S \subseteq V$ such that for any edge u, v, at least one of u and v is included in S. Also recall that a matching M in a graph is defined as a set of edges such that no two edges in M share a common vertex.

(a) Prove that the size of the maximum matching in any graph cannot exceed the size of a minimum vertex cover. (Hint : Just use the definitions)

Solution. Suppose there exists a matching M and a vertex cover V' such that $|M| > |V'|$. Then, by PHP, there exists some vertex in V' which is 'covering' more than one edge in M. But then M cannot be a matching leading to a contradiction.

Rubric : Pretty much subjective to my judgement.

(b) Write a linear program whose *integer solutions* will exactly correspond to a vertex cover. Clearly state the variables and constraints.

Solution. We will have a variable x_v for each vertex $v \in V$. Then the LP is

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v \\ \text{s.t.} \quad & x_u + x_v \ \geq \ 1 \qquad \forall\, (u, v) \in E, \\ & x_v \ \geq \ 0 \qquad \forall\, v \in V. \end{aligned}$$

Rubric : +1 for correct variables, +1 for correct objective, +2 for first set of constraints , +1 for last set of constraints

(c) Show an example where the value of the optimal solution to the above linear program is strictly smaller than the actual minimum vertex cover of the graph. (Hint: You may have seen this example is a related context)

Solution. Consider a triangle with vertices v_1, v_2, v_3. An optimal LP solution will simple put $1/2$ on each vertex and satisfy all constraints, leading to a value of $3/2$. But any integral solution needs to pick 2 edges.

Rubric : +2 for the correct example, +1.5 each for arguing what the LP and integral solutions are.