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MTH203 - Multivariate Calculus
End Semester Exam

Section - B
Total Marks - 50 (Including 10 Bonus)

12th December 2024

Instructions

1. The exam duration is **2 hours.**
2. This end-semester exam accounts for **40% of the total course grade.**
3. The question paper contains **5 questions**. Certain questions have sub-parts marked with **, indicating bonus marks.
4. To be eligible for bonus marks, you must first attempt all the mandatory parts of the corresponding question.
5. **Bonus parts worth up to 10 additional marks.**
6. You are allowed to bring **two A4-sized cheat sheets**, which may only include formulas.
   - **Important:** If solutions are found on your cheat sheets, it will be considered academic dishonesty and may result in strict disciplinary action.

Problem - 1

$$u = x^3 - 3xy^2$$

a. Show that the given function is harmonic.

[5 marks]

b. Find the harmonic conjugate of $u(x, y)$.

[3 marks]

c. ** Find the analytic function $f(z) = u(x, y) + iv(x, y)$ where v is harmonic conjugate of u in terms of z instead of x. and y

[2 marks]

Problem - 2

a. Show that the curvature of a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ defined by twice-differentiable functions $x = f(t)$ and $y = g(t)$ is given by the formula:

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

[7 marks]

b. ** Apply the formula to find the curvatures of the curve: $\mathbf{r}(t) = [\tan^{-1}(\sinh t)]\mathbf{i} + (\ln \cosh t)\mathbf{j}$.

[3 marks]

Problem - 3

Along all rectangular solids defined by the inequalities

$$0 \le x \le a, \quad 0 \le y \le b, \quad 0 \le z \le 1,$$

find the values of a and b for which the total flux of

$$\mathbf{F} = (-x^2 - 4xy)\,\mathbf{i} - 6yz\mathbf{j} + 12z\mathbf{k}$$

outward through the six sides is greatest. Also, what is the greatest flux?

[10 marks]

Problem - 4

a. Solve the system

$$u = x - y,$$
$$v = 2x + y,$$

for x and y in terms of u and v. Then, find the value of the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

[3 marks]

b. Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices $(0,0)$, $(1,1)$, and $(1,-2)$ in the xy-plane. Sketch the transformed region in the uv-plane.

[4 marks]

c. ** Use the above transformation to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\,dx\,dy$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$. Sketch the figure and label the sides.

[5 marks]

Problem - 5

Find the point on the plane $x + 2y + 3z = 13$ closest to the point (1, 1, 1). (Do it using Lagrange multiplier)

[8 marks]