* What is a signal?

- Signal is a function of one or more independent variables

- eg. Speech → Volume as a function of time

  Image → Colours as a function of space

  Video → Colours as a function of time and space

  Control Signals → Instructions as a function of time.

  etc.

* Signal processing
  - Task of extracting 'meaningful information' from the signals.

    eg., If speech is noisy, remove the noise

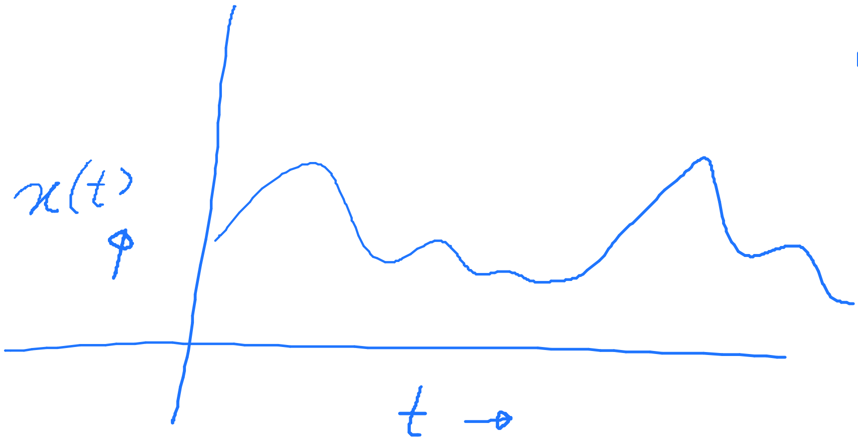
    • Music - Separate or combine the sounds of different instruments.

* Types of signals

  Variable (refer to it by 'time')
  - Discrete
  - Continuous

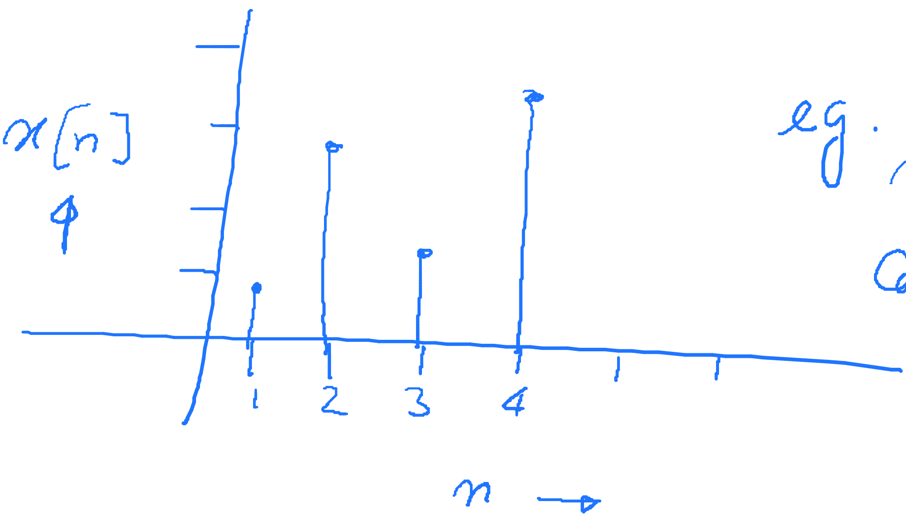
  Function values (refer to it as 'amplitude')
  - Discrete
  - Continuous

Analog Signal
- Continuous time
- Continuous amplitude



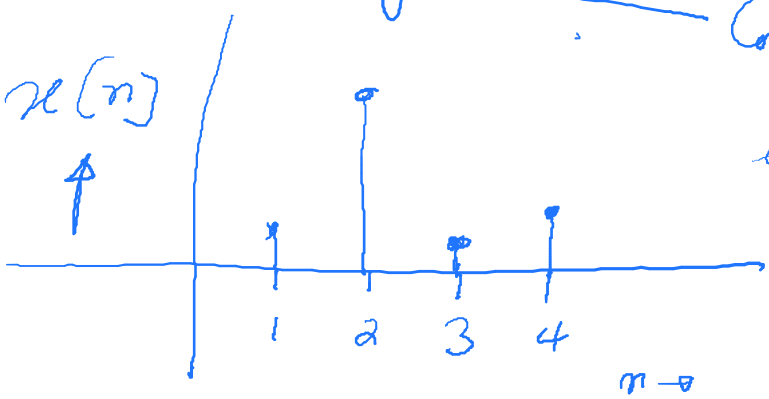
eg. Speech

Digital Signal
- Discrete time
- Discrete amplitude



eg. Reading a memory device, Camera image

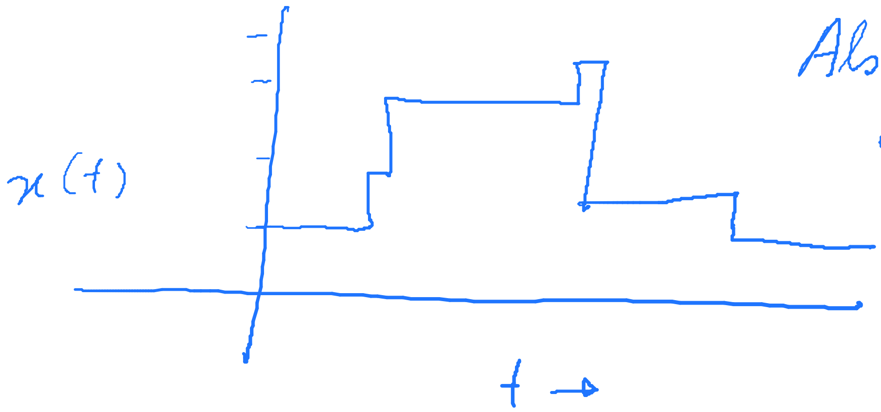
Discrete time signal
- Discrete time
- Continuous amplitude



eg. Sampled Signal, Signal received by a wake-up receiver

Continuous time, discrete amplitude signals.

Also called quantised boxcar symbol.

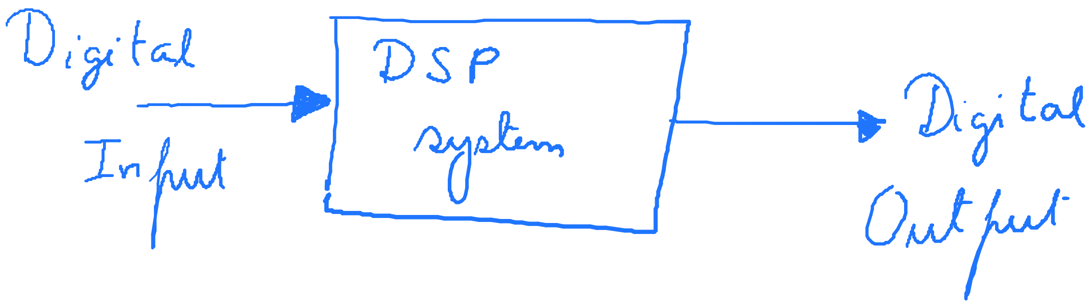


This course :-

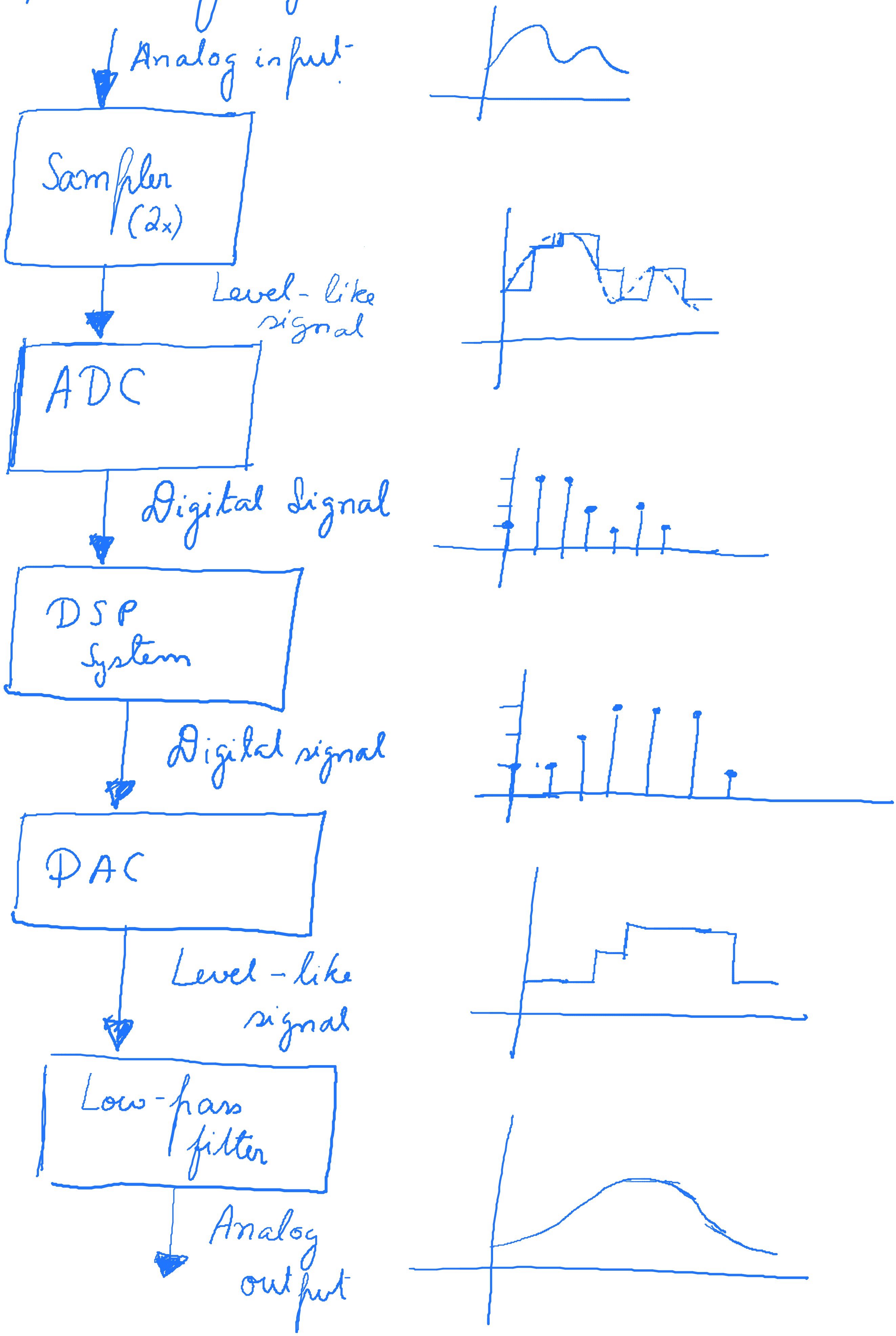
Digital + Discrete time signals.

* Block diagram of basic DSP system with analog and digital signals

i) Digital signals



ii) Analog signals



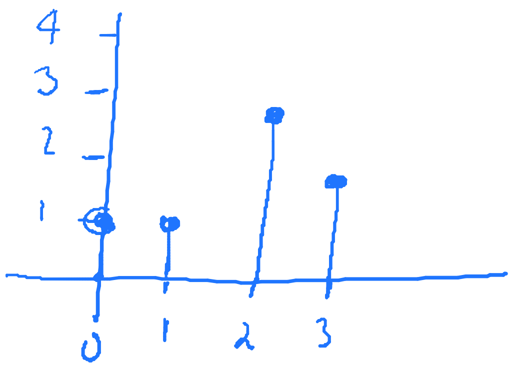
* Multidimensional & Complex signals

Signal $\longrightarrow$ Function of independent variables like time.

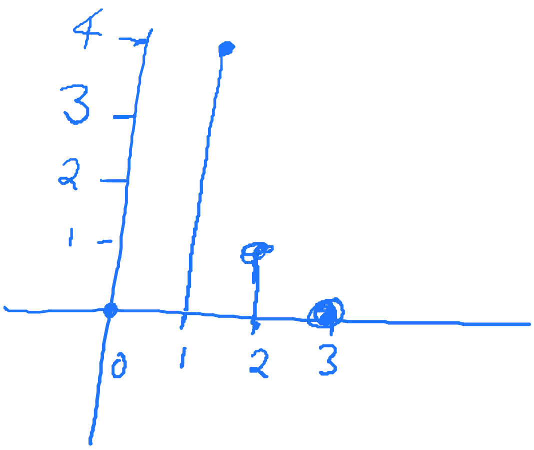
Multidimensional signal

$\longrightarrow$ Vector valued functions of independent variables

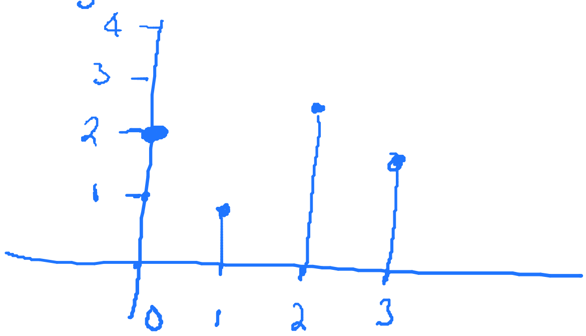
e.g. Let $x_1[n]$



$x_2[n]$



$x_3(n)$



Define $x[n] = (x_1[n], x_2[n], x_3[n])$

i.e. $x[0] = (1, 0, 2)$

$x[1] = (1, 4, 1)$

$x[2] = (3, 1, 3)$

$x[3] = (2, 0, 2)$

Thus, $x[n]$ is a 3-dimensional signal.

2-dimensional signals are also called complex signals since they can be represented using complex numbers.

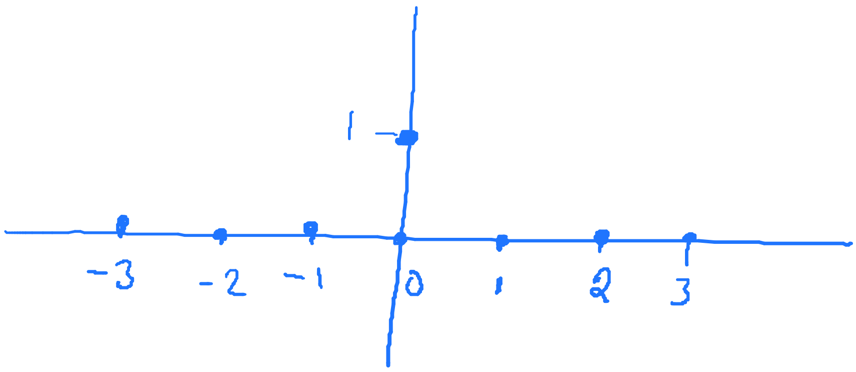
Let $x[n] = (x_1[n], x_2[n])$

Then, we can also represent $x[n]$ as a complex-valued signal, where $x[n] = x_1[n] + j x_2[n]$.

* Some example discrete-time signals

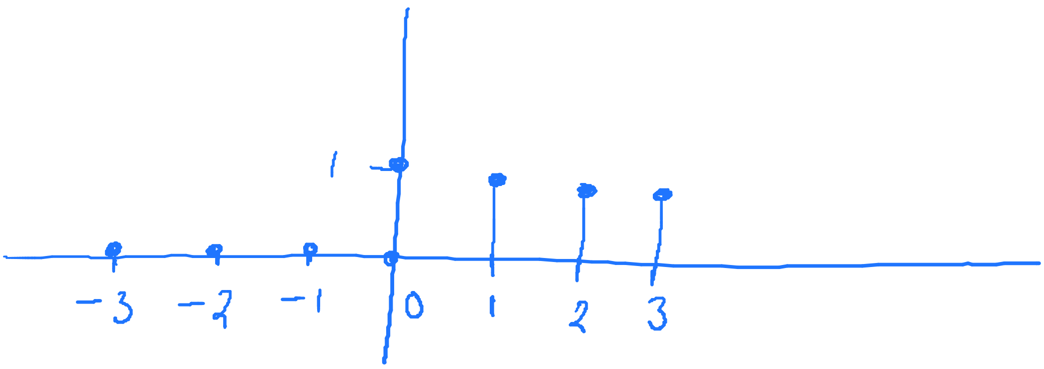
i) Unit impulse

$$\delta[n] = 1, \quad n = 0$$
$$= 0, \quad n \neq 0$$



ii) Unit step

$$u[n] = 1, \quad n \geq 0$$
$$= 0, \quad n < 0$$



iii) Discrete sinusoid

$$x[n] = a\sin(2\pi f n + \theta), \quad \text{where } a \in \mathbb{R}$$

<u>Definition</u>:- A discrete time signal $x[n]$ is periodic, if there exists some $N \in \mathbb{N} \setminus \{0\}$, s.t. $\forall n \in \mathbb{Z}$, $x[n+N] = x[n]$.

The period of $x[n]$ is the smallest N that satisfies the above criteria.

If no such N exists, then $x[n]$ is said to be aperiodic.

- <u>Lemma</u>:- The discrete-time sinusoid is periodic iff f is rational.

Proof:- For $x[n]$ to be periodic, we must have that there exists N such that for all n

$$a \sin(2\pi f n + 2\pi f N + \theta) = a \sin(2\pi f n + \theta)$$

This is possible iff $2\pi f N = 2\pi k$ for some $k \in \mathbb{N} \setminus \{0\}$.

Thus, $f = \frac{k}{N}$.

Since both $k, N \in \mathbb{N} \setminus \{0\}$, f is rational.

$\boxtimes$

iv) Discrete exponential

$$x[n] = a^n.$$

If $a \in \mathbb{R}$, then $x[n]$ is a real signal.

If a is complex, let $a = re^{j\theta}$.

Then $x[n]$ is a complex signal where

$$x[n] = r^n e^{jn\theta}$$

$$= r^n \cos n\theta + j r^n \sin n\theta$$

* Signal energy and power

Let $x[n]$ be a real or complex valued signal.

Signal energy $\boxed{E = \sum_{n=-\infty}^{\infty} |x[n]|^2}$

eg. The energy of the unit impulse is $\sum_{n=-\infty}^{\infty} (\delta[n])^2 = 1$.

Note that all the other examples have infinite energy.

Signal power

i) For periodic signals with period N, the signal power is the 'normalised energy' over a period, i.e.,

$$\boxed{P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2}.$$

eg. Consider the sinusoid

$$x[n] = a \sin\left(\frac{2\pi n}{N} + \theta\right).$$

and observe that N is its period. Thus, signal power

$$P = \frac{1}{N} \sum_{n=0}^{N-1} a^2 \sin^2\left(\frac{2\pi n}{N} + \theta\right)$$

$$= \frac{a^2}{N} \sum_{n=0}^{N-1} \sin^2\left(\frac{2\pi n}{N} + \theta\right)$$

$$= \frac{a^2}{2N} \sum_{n=0}^{N-1} \left(1 - \cos\left(\frac{4\pi n}{N} + 2\theta\right)\right)$$

$$= \frac{a^2}{2} - \frac{a^2}{2N} \sum_{n=0}^{N-1} \cos\left(\frac{4\pi n}{N} + 2\theta\right)$$

Now, note that

$$\sum_{n=0}^{N-1} e^{j\left(\frac{4\pi n}{N} + 2\theta\right)}$$

$$= \frac{e^{j2\theta}\left(e^{j\frac{4\pi}{N}\cdot N} - 1\right)}{e^{j\frac{4\pi}{N}} - 1}$$

$$= \frac{e^{j2\theta}\cdot(1-1)}{e^{j\frac{4\pi}{N}} - 1} = 0$$

Thus, $\displaystyle\sum_{n=0}^{N-1} \cos\left(\frac{4\pi n}{N} + 2\theta\right) + j\sum_{n=0}^{N-1} \sin\left(\frac{4\pi n}{N} + 2\theta\right) = 0$

$$\Longrightarrow \sum_{n=0}^{N-1} \cos\left(\frac{4\pi n}{N} + 2\theta\right) = 0$$

Hence, $P = \frac{a^2}{2}$.

ii) For aperiodic signals, we define the signal power as follows.

→ Consider the energy over $2N+1$ instances at $n = -N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N$

$$E_N = \sum_{n=-N}^{N} |x[n]|^2$$

→ Normalise E_N to get the power of these $2N+1$ instances.

$$P_N = \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

→ Take the limit as $N \to \infty$ to cover the entire signal,

$$\boxed{P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2}$$

Consider the unit step function.

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[u[n]\right]^2$$

$$= \lim_{N\to\infty} \frac{N+1}{2N+1}$$

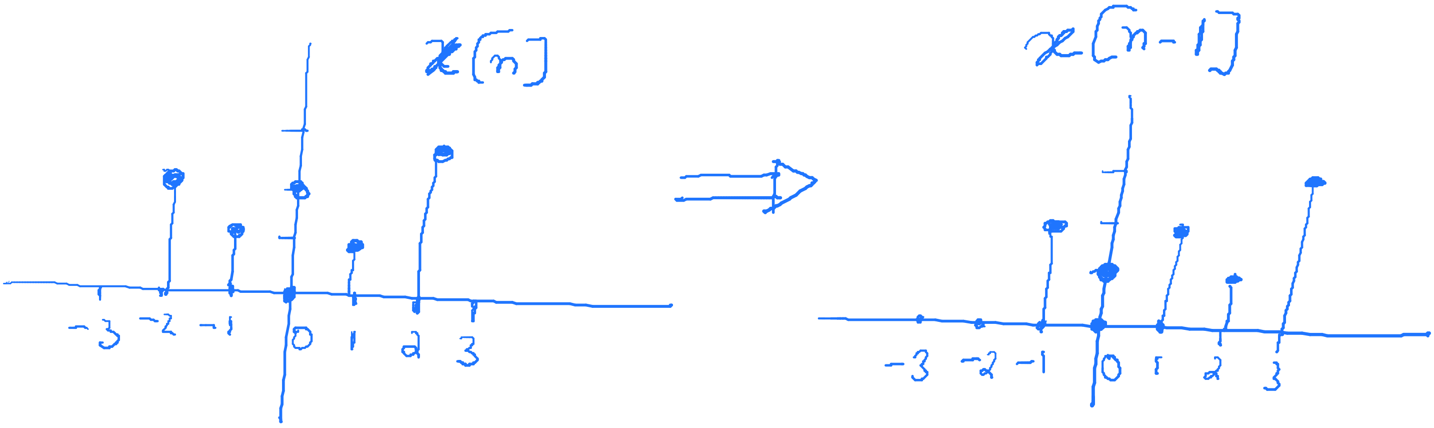
$$= \frac{1}{2}.$$

Finally, note that there are signals with infinite power such as the discrete exponential $x[n] = a^n$, if $|a|>1$.

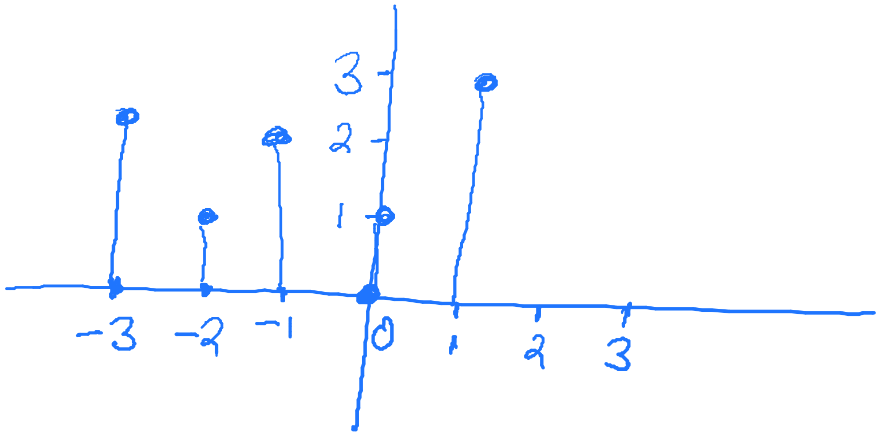
* Elementary operations on signals

- Time shift

$$x[n] \longrightarrow x[n-k], \; k \in \mathbb{N}$$

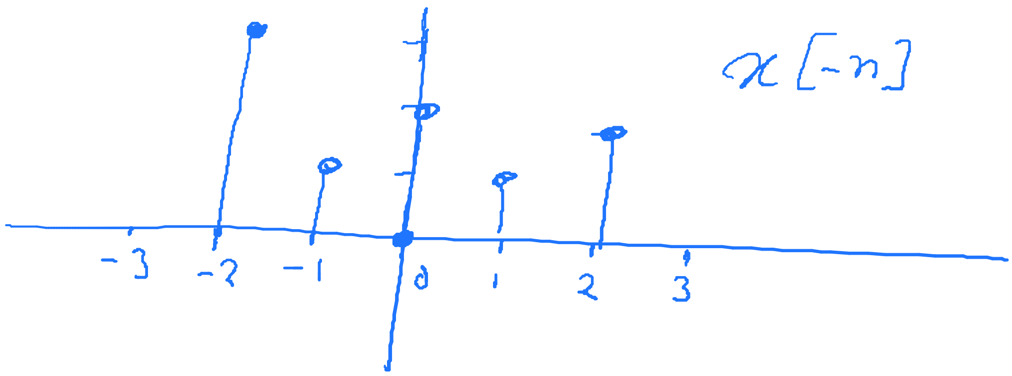


$x[n+1]$



- Reflection

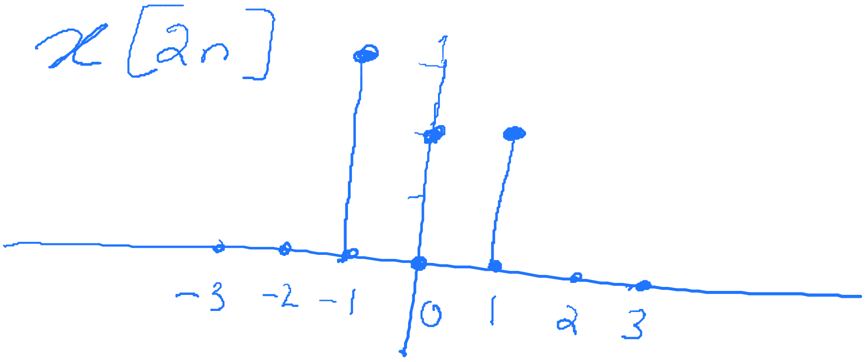
$x[n] \longrightarrow x[-n]$



- Time scaling / Down sampling [Since you are dropping samples]

$x[n] \longrightarrow x[kn]$, $k \in \mathbb{N}$

$x[2n]$



- Amplitude scaling

$$x[n] \rightarrow a x[n], \quad a \in \mathbb{R}$$

- Signal addition

$$x[n] + y[n]$$

- Signal product

$$x[n] \cdot y[n]$$

* Discrete time systems

Note:- All digital systems are discrete time systems.

$\mathcal{T} \rightarrow$ Transfer function

$$y[n] = \mathcal{T}[x[n]]$$

eg. $y[n] = a x[n]$

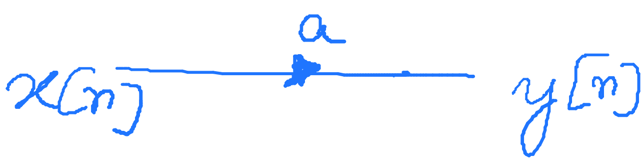
$y[n] = x_1[n] + x_2[n]$

$y[n] = x[n-k]$ etc.

* Block diagram representation of discrete time systems

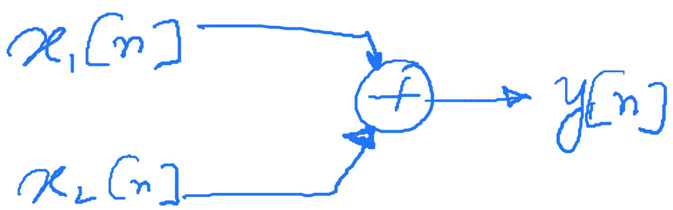
- Scaling block

$$y[n] = a \cdot x[n]$$

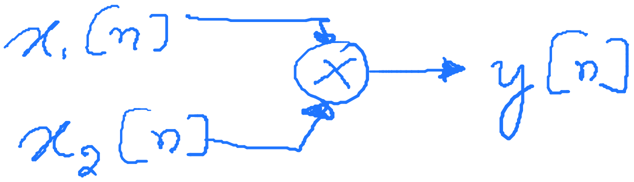


- Addition block

$$y[n] = x_1[n] + x_2[n]$$

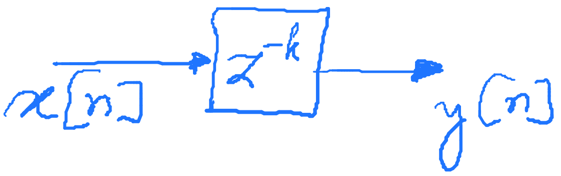


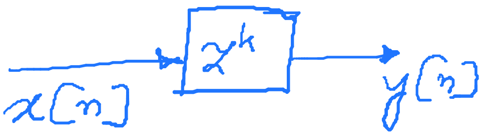
- Product block $y[n] = x_1[n] \cdot x_2[n]$



- Time shift block

$$y[n] = x[n-k] \qquad\qquad y[n] = x[n+k]$$

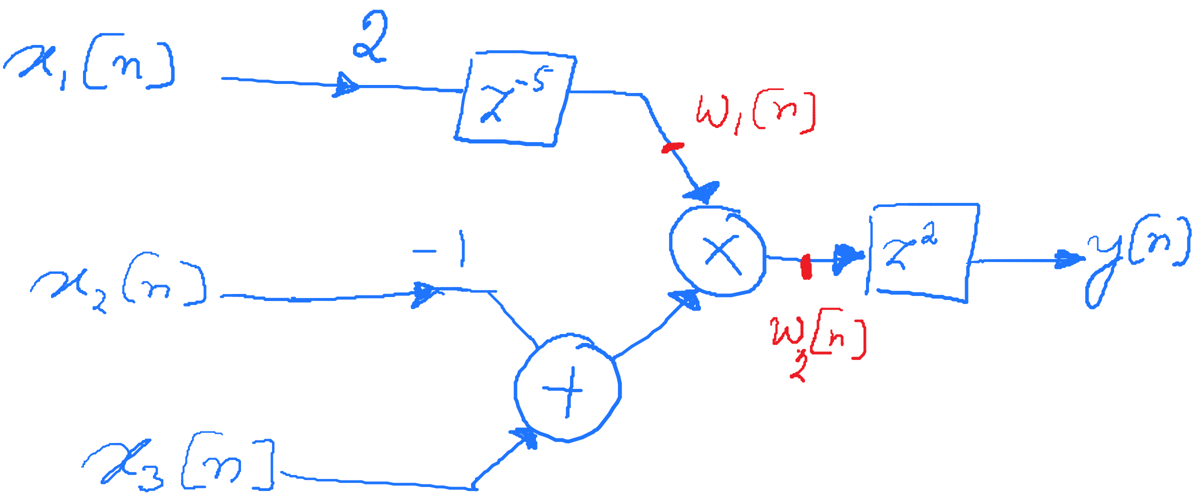




The reason we use z will be apparent later when we discuss z-transforms.

→ Worked out example :-

Find the transfer function of the following system:



$$y[n] = w_2[n+2]$$

Now, $w_2[n] = w_1[n]\left(x_3[n] - x_2[n]\right)$

$$w_1[n] = 2x_1[n-5]$$

Thus, $w_2[n] = 2x_1[n-5]\, x_3(n)$
$\qquad - 2x_1[n-5] \cdot x_2(n)$

Then,

$$y[n] = w_2[n+2]$$
$$= 2x_1[n-3] \cdot x_3[n+2]$$
$$- 2x_1[n-3] \cdot x_2[n+2]$$

* Static (memoryless) versus dynamic systems

Static $\longrightarrow$ $y[n]$ depends on $x[n]$ and $x[n]$ only.
Hence processing can be instantaneous and no memory is required.

eg. $y[n] = x[n]$ $\qquad y[n] = x^2[n]$
$y[n] = n + x[n]$

Dynamic $\longrightarrow$ $y[n]$ depends on $x[n]$ as well as at least some $x[n-k]$, $k \in \mathbb{Z} \setminus \{0\}$.

eg. $y[n] = x[n-k]$

$y[n] = x[n] + x[n-2]$

$y[n] = x[2n]$

* Time invariant system

A system $\mathcal{T}$ is time invariant if for any $k \in \mathbb{Z}$, $y[n-k] = \mathcal{T}[x[n-k]]$ where $y[n] = \mathcal{T}[x[n]]$.

Examples :- i) $\mathcal{T}(x[n], w[n]) = x^3[n] + w[n] = y[n]$

$$y[n-k] = x^3[n-k] + w[n-k]$$

$$\mathcal{T}(x[n-k], w[n-k])$$

$$= x^3[n-k] + w[n-k]$$

Thus $y[n-k] = \mathcal{T}[x[n-k], w[n-k]]$.

Thus the system is time invariant

ii) $\tau(x[n]) = nx[n]$

Let $y[n] = nx[n]$

$\therefore y[n-k] = nx[n-k] - kx[n-k]$

$\tau(x[n-k]) = nx[n-k]$

Thus, $y[n-k] \neq \tau[x[n-k]]$.

Hence τ is time variant

* Linear system

A system τ is said to be linear if

for any $a_1, a_2 \in \mathbb{R}$, $\tau[a_1 x_1[n] + a_2 x_2[n]]$

$= a_1 y_1[n] + a_2 y_2[n]$, where

$y_1[n] = \tau[x_1[n]]$ and $y_2[n] = \tau(x_2[n])$

Examples:-

i) $\mathcal{T}(x[n]) = x[n^2]$

Choose any a_1, a_2

$$y_1[n] = \mathcal{T}[x_1[n]] = x_1[n^2]$$

$$y_2[n] = \mathcal{T}[x_2[n]] = x_2[n^2]$$

$$\mathcal{T}\left[a_1 x_1[n] + a_2 x_2[n]\right]$$

$$= a_1 x_1[n^2] + a_2 x_2[n^2]$$

$$= a_1 y_1[n] + a_2 y_2(n)$$

Thus $\mathcal{T}$ is linear.

ii) $\mathcal{T}[x[n]] = x^2[n]$

Choose $a_1 = a_2 = 1$.

$$y_1[n] = x_1^2[n]$$

$$y_2[n] = x_2^2[n]$$

$$\mathcal{T}\left[x_1[n]+x_2[n]\right] = \left(x_1[n]+x_2[n]\right)^2$$

$$= x_1^2[n] + x_2^2[n] + 2x_1[n]x_2[n]$$

$$= y_1[n] + y_2[n] + 2x_1[n]x_2(n)$$

$$\neq y_1[n] + y_2[n]$$

Thus, $\mathcal{T}$ is not linear.

* Causal systems

A system $\mathcal{T}$ is causal if the output depends on past samples only.

Note that static systems are by default causal.

Examples :-

i) $\mathcal{T}[x[n]] = \sum_{k=-\infty}^{n} x[k]$

This is causal since only past samples are needed.

ii) $\mathcal{T}[x[n]] = x[n^2]$

Here, note that

$$\mathcal{T}[x[2]] = x[4].$$

Thus future samples are needed to compute $\mathcal{T}[x[n]]$.

Thus this is non-causal.

* Stable systems

- Definition :- A signal $x[n]$ is said to be bounded if there exists some constant M such that

$$\boxed{|x[n]| \le M, \quad -\infty < n < \infty.}$$

A system $\mathcal{T}$ is said to be stable if for every bounded input the output is bounded.

Stable systems are also called BIBO or Bounded Input Bounded Output systems.

Examples :- i) $\mathcal{T}(x[n]) = x^3[n]$

Note that if $|x[n]| \le M, \forall n,$

then $|x^3[n]| \le M^3, \forall n.$

Thus, $\mathcal{T}$ is stable.

ii) $\mathcal{T}[x[n]] = \sum_{k=-\infty}^{n} x[k]$

Choose $x[n] = u[n]$.

Note that $|u[n]| \leq 1$, and thus it is bounded.

However, $\mathcal{T}[x[n]]$

$$= \sum_{k=-\infty}^{n} u[k]$$

$$= \begin{cases} n+1, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0. \end{cases}$$

Thus, $\mathcal{T}[x[n]]$ is unbounded.

Hence $\mathcal{T}$ is unstable.

* Linear Time Invariant (LTI) systems

→ Why LTI?

The transfer function is easy to analyse and design.

→ Signal as a sum of impulses

Observe, $\delta[n-k] = 1 \;,\; n=k$
$\qquad\qquad = 0 \;,\; \text{o.w.}$

Then, for any signal $x[n]$

$$x[k]\,\delta[n-k] = x(k)\;,\; n=k$$
$$\qquad\qquad = 0\;,\; \text{o.w.}$$

Hence, $$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$

→ Impulse response and output of an LTI system

For any LTI system τ, define its impulse response as

$$\boxed{h[n] \triangleq \tau\big[\delta[n]\big]}$$

Thus for any signal $x[n]$ as input to an LTI system with impulse response $h[n]$, we have

$$\mathcal{T}\left[x[n]\right] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,\mathcal{T}\left[\delta[n-k]\right]$$

[$\because$ Linear]

$$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

[$\because$ Time invariant]

Thus,

$$\boxed{\mathcal{T}\left[x[n]\right] = \underbrace{\sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]}_{\text{This operation is called convolution, denoted by } x[n] * h[n]}}$$

* Properties of convolution

i) Commutativity

$$x[n] * h[n] = h[n] * x[n]$$

Proof :- $x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$= \sum_{m=-\infty}^{\infty} x[n-m] \cdot h[m]$$

[Substituting $m = n-k$]

$$= h[n] * x[n]$$

$\blacksquare$

ii) Associativity

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$

Proof :- $(x[n] * h_1[n]) * h_2[n]$

$$= \sum_{k=-\infty}^{\infty} (x[k] * h_1[k]) \cdot h_2[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] \cdot h_1[k-m]\, h_2[n-k]$$

$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{k=-\infty}^{\infty} h_1[k-m] \cdot h_2[n-k]$$

$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{t=-\infty}^{\infty} h_1[t] \cdot h_2[n-t-m]$$

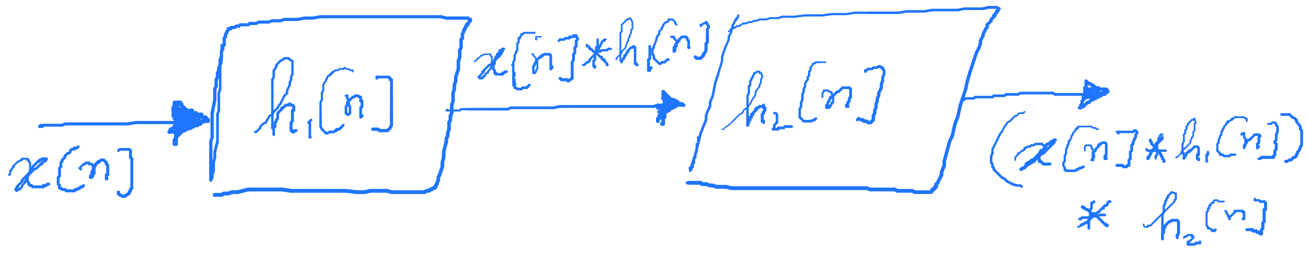
[Substitution
$t = k-m$]

$$= \sum_{m=-\infty}^{\infty} x[m] \left(h_1[n-m] * h_2[n-m]\right)$$

$$= x[n] * (h_1[n] * h_2[n])$$

$\blacksquare$

Physical significance of associativity of convolution



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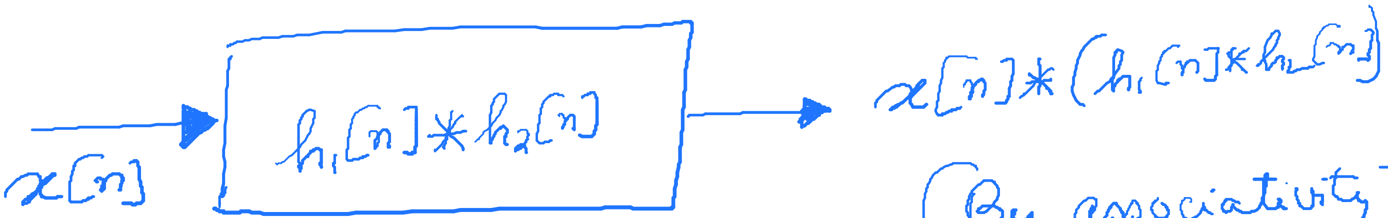
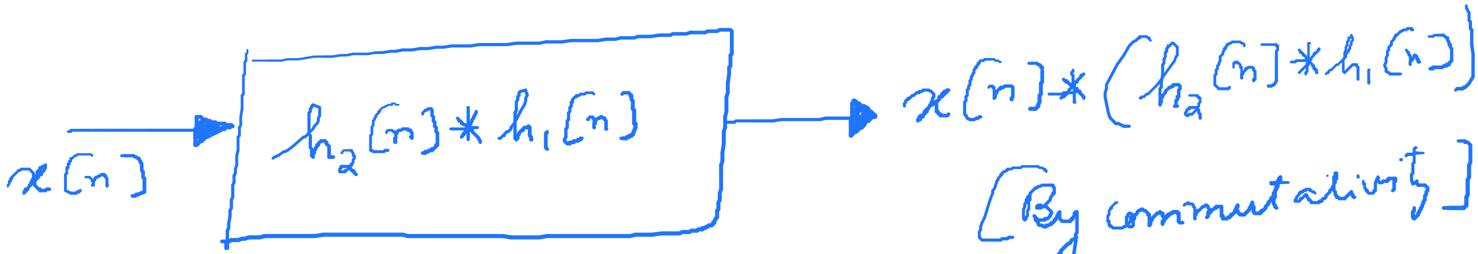


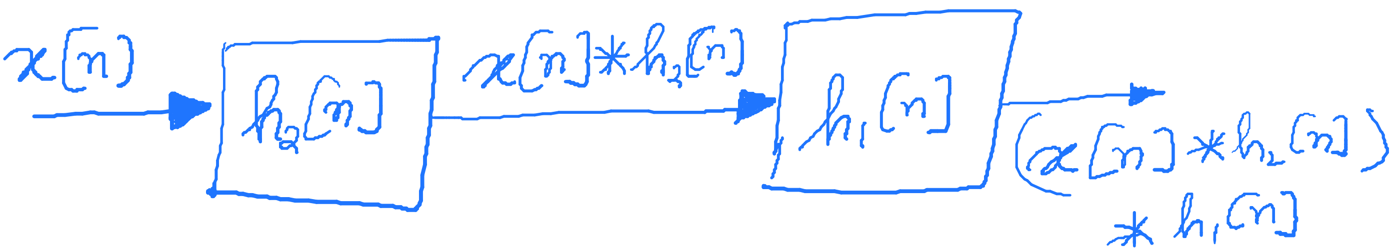
[By associativity]

$\equiv$



[By commutativity]

$\equiv$



[By associativity]

In general, this changing of orders of discrete time systems is not valid for non-LTI systems.

Example:-

Let $T_1[x[n]] = x[n]+5$

$T_2[x[n]] = x^2[n]$

Then $T_2[T_1[x[n]]] = T_2[x[n]+5]$

$= x^2[n] + 10x[n] + 25$

$T_1[T_2[x[n]]] = T_1[x^2[n]]$

$= x^2[n] + 5$

Thus $T_1 T_2 \neq T_2 T_1$.

iii) Distributivity

$x_1[n] * (h_1[n] + h_2[n])$

$= x[n] * h_1[n] + x[n] * h_2[n]$

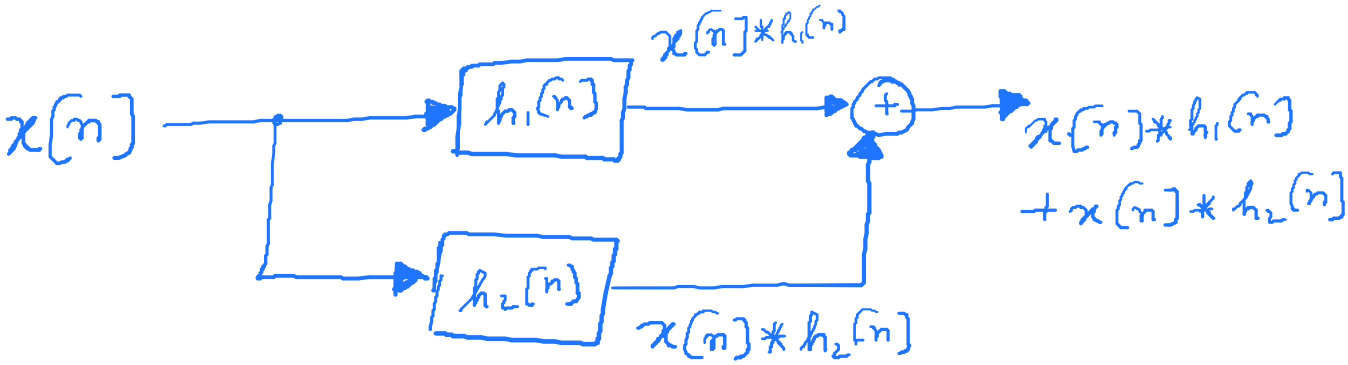
Proof:- $x[n] * (h_1[n] + h_2[n])$

$$= \sum_{k=-\infty}^{\infty} x[k]\left(h_1[n-k] + h_2[n-k]\right)$$

$$= \sum_{k=-\infty}^{\infty} x[k].h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k].h_2[n-k]$$

$$= x[n] * h_1[n] + x[n] * h_2[n]$$

Physical significance of distributivity



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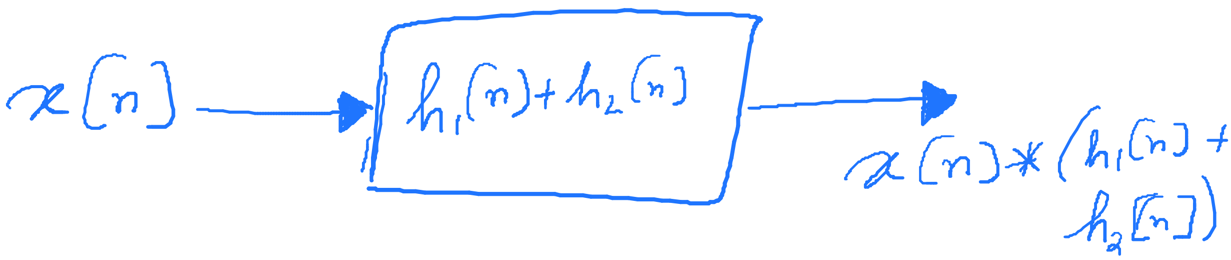


[By distributivity]

* Computing convolution

Let $y[n] = x[n] * h[n]$

Fix some n_0.

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n_0 - k]$$

Steps to compute $y[n_0]$

→ Fix k.

→ Fold $h[k]$ around 0 to obtain $h[-k]$

→ Shift $h[-k]$ right (left) by n_0 units if n_0 is positive (negative) to obtain $h[n_0-k]$

→ Multiply by $x[k]$ to obtain $x[k] \cdot h[n_0-k]$

→ Add up all $x[k] \cdot h[n_0-k]$ terms.

Example :- Let $x[n] = u[n]$ and

$h[n] = a^n u[n]$

Find $x[n] * h[n]$.

Fix $n_0 \geq 0$.

Fix any $0 \leq k \leq n_0$

$h[n_0 - k] = a^{(n_0-k)} u[n_0 - k]$ → Fold + Shift

$= a^{n_0 - k}$

Thus $x[k] \cdot h[n_0 - k] = a^{n_0 - k}$

If $k < 0$, then $x[k] = 0$

and hence $x[k] \cdot h[n_0 - k] = 0$

If $k > n_0$, then $h[n_0 - k] = 0$,

and hence $x[k] \cdot h[n_0 - k] = 0$

Thus $\sum_{k=-\infty}^{\infty} x[k] \cdot h[n_0 - k]$ ← Multiply

$= \sum_{k=0}^{n_0} x[k] \cdot h[n_0 - k]$

$$= \sum_{k=0}^{n_0} a^{n_0-k} = a^{n_0}\left(\frac{a^{-n_0-1}-1}{a^{-1}-1}\right)$$

$$= \frac{a^{-1}-a^{n_0}}{a^{-1}-1}$$

$$= \frac{1-a^{n_0+1}}{1-a}$$

Next, fix $n_0 < 0$.

Fix any $k > n_0$

$$h[n_0-k] = a^{(n_0-k)} u[n_0-k]$$

$$= 0$$

If $k \leq n_0 < 0$,

then $x[k] = 0$

Thus $x[k] \cdot h[n_0-k] = 0$, $\forall k$

Hence $\sum_{k=-\infty}^{\infty} x[k] \cdot h[n_0-k] = 0$.

Thus, $x[n] * h[n] = \frac{1-a^{n+1}}{1-a}$, if $n \geq 0$

$= 0$, o.w.

* Causality of LTI systems

$$x[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$
$$= \sum_{k=-\infty}^{n} x[k] h[n-k] + \sum_{k=n+1}^{\infty} x[k] h[n-k]$$

Thus, the system is causal iff

$$h[n-k] = 0, \quad \forall k > n$$

i.e. $h[n] = 0, \quad \forall n < 0.$

* Stability of LTI systems

Let $y[n] = x[n] * h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k] \cdot h[k]$$

$$\Rightarrow |y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k] h[k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |x[n-k] h[k]|$$

[Triangle inequality]

$$= \sum_{k=-\infty}^{\infty} |x[n-k]| \, |h(k)|$$

Thus, if $x[n]$ is bounded, say

$$|x[n]| \leq M,$$

we have $|y[n]| \leq M \cdot \sum_{k=-\infty}^{\infty} |h(k)|$

Thus, an LTI system is stable

if $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$.

What if $\sum_{k=-\infty}^{\infty} |h(k)| = \infty$?

We show that then the system is unstable.

Consider the input

$$x[n] = \frac{h^*[-n]}{|h[-n]|}, \text{ if } h[-n] \neq 0$$

$$= 0 \quad , \text{ o.w.}$$

Then $|x[n]| \leq 1$, and hence $x[n]$ is bounded.

Now, let $y[n] = x[n] * h[n]$

Thus, $y[0] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[0-k]$

$$= \sum_{k=-\infty}^{\infty} \frac{h^*[-k]}{|h[-k]|} \cdot h[-k]$$

$$= \sum_{k=-\infty}^{\infty} |h(-k)| = \infty.$$

Thus bounded input produced an unbounded output if $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$

Thus, the system is unstable.

In other words, an LTI system is stable iff $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

* Checking stability of causal LTI systems

Since the system is causal, $h[n] \neq 0$ only if $n \geq 0$.

Thus, to check stability we need to see if $\sum_{n=0}^{\infty} |h[n]| = \infty$ or

$\sum_{n=0}^{\infty} |h[n]| < \infty$. There are many ways to do this, we discuss two of them below :-

a) Ratio test

Let $a_n = \left| \frac{h[n+1]}{h[n]} \right|$

Evaluate

$$L = \lim_{n \to \infty} a_n$$

If $L > 1$, then $\sum_{n=0}^{\infty} |h[n]| = \infty$

If $L < 1$, then $\sum_{n=0}^{\infty} |h[n]| < \infty$

If $L = 1$, then the test is inconclusive.

b) Root test

Take $a_n = (|h[n]|)^{1/n}$ and let

$$L = \lim_{n \to \infty} a_n$$

If

$L > 1$, then $\sum_{n=0}^{\infty} |h[n]| = \infty$

$L < 1$, then $\sum_{n=0}^{\infty} |h[n]| < \infty$

$L = 1$, then the test is inconclusive.

* FIR versus IIR LTI systems

- FIR $\rightarrow$ Finite impulse response, i.e. only a finite number, say M, points of $h[n]$ are non-zero.

Thus, to compute the output for any input $x[n]$, at any time n,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$= \sum_{k : h[n-k] \neq 0} x[k] \cdot h[n-k] .$$

In other words, only M units of memory are required to compute $y[n]$.

- IIR $\longrightarrow$ Infinite impulse response.

  i.e. $h[n] \neq 0$ at infinitely many points.

Thus, given a signal $x[n]$ with infinitely many non-zero points, infinite memory will be required if one directly wishes to implement an IIR system.

So, how does one design an IIR system?

Answer:- Difference equations.

* Recursive system and difference equations

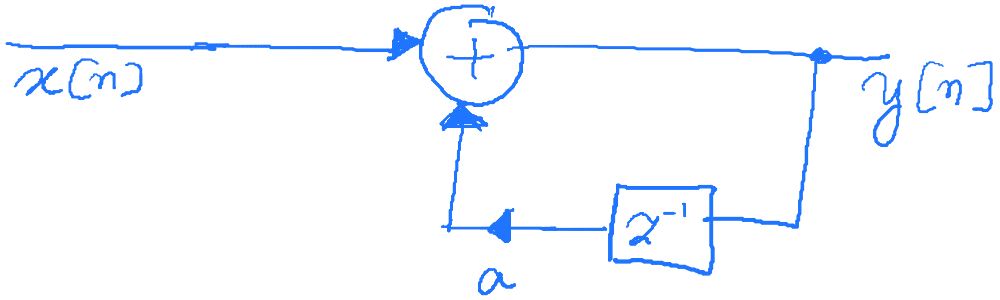
A system is said to be recursive if its output can be written in form of a difference equation

$$y[n] = \sum_{k=1}^{N} a_k \cdot y[n-k] + \sum_{k=0}^{M} b_k x(n-k)$$

Observe that this system can be implemented with M+N memory. The number max{M, N} is called the order of the difference equation.

Example :-

$$y[n] = a\,y[n-1] + x[n]$$



We shall next show that the above system is LTI with IIR.

Consider the LTI system with $h[n] = a^n u[n]$. It is obvious that the system is IIR.

Now, for any given input $x[n]$,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} a^{n-k}\,x[k]\cdot u[n-k]$$

$$= \sum_{k=-\infty}^{n} a^{n-k}\cdot x[k]$$

$$y[n-1] = \sum_{k=-\infty}^{n-1} a^{n-1-k} x[k]$$

Thus $y[n] - a \cdot y[n-1]$

$$= \sum_{k=-\infty}^{n} a^{n-k} \cdot x[k] - \sum_{k=-\infty}^{n-1} a^{n-k} \cdot x[k]$$

$$= a^{n-n} \cdot x[n]$$

$$= x[n]$$

Thus, the recursive system with the difference equation $y[n] = a\,y[n-1] + x[n]$ is the IIR LTI system with $h[n] = a^n u[n]$, and this can be implemented with only 1 unit of memory.

In general, a linear difference equation

$$y[n] = \sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

describes a linear system.

The system is also time invariant as long as a_k, b_k are constants, i.e., they do not depend on n.

Implementation of the LTI system based on the difference equation is easy.

But, how does one get a mathematical description of $y(n)$

* Zero-state response versus zero-input response

Consider a recursive LTI system described by the linear difference equation

$$y[n] = \sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

For all practical purposes we shall assume that the input signal $x[n]$ is causal, i.e.

$x[n] = 0$, if $n < 0$.

Thus, since the difference equation is order N, and $x[n]$ is causal, $y[n]$ will depend on the values of $y[-N], y[-N+1], \ldots, y[-1]$, along with $x[0], \ldots, x[n]$.

- Zero state response :- When we assume the initial condition that $y[-N] = \cdots = y[-1] = 0$, the solution of the difference equation is called the zero-state response, denoted by $y_{zs}[n]$.

- Zero input response :- Here, we assume $x[n] = 0$,

$\forall n$, and compute the solution of $y[n]$ based on the initial conditions on $y[-N], \ldots, y[-1]$. This solution is called the zero input response, denoted by $y_{zI}[n]$.

Since the equation is linear, we have

$$y[n] = y_{zs}[n] + y_{zI}[n].$$

Example:- Let us reconsider

$$y[n] = a\,y[n-1] + x[n]$$

We have already seen the zero-state response

$$y_{zs}[n] = \sum_{k=0}^{n} a^{n-k} x[k]$$

Let us obtain the zero-input response. Fix $x[n]=0$.

Note that $y[0] = a\,y[-1]$

$$y[1] = a\,y[0] = a^2 y[-1]$$

Then assuming $y[k] = a^{k+1} y[-1]$, we have

$$y[k+1] = a\,y[k] = a^{k+2} y[-1].$$

Thus, by induction, we have

$$y[n] = a^{n+1} y[-1].$$

In other words $y_{ZI}[n] = a^{n+1} y[-1]$.

So, $y[n] = y_{ZS}[n] + y_{ZI}[n]$

$$= \sum_{k=0}^{n} a^{n-k} x[k] + a^{n+1} y[-1].$$

* Moving to 'frequency' domain

Why — Eases calculation and hence both analysis and design.

Key idea — Find a basis (i.e. a set of signals so that every possible signal can be expressed as a weighted sum of those signals)

These basis signals should have useful properties which simplify analysis.

* Discrete time Fourier series for periodic signals

Let $x[n]$ be a signal of period N, i.e. $x[n+N] = x[n]$.

Choice of basis :-

$$\boxed{s_k[n] = e^{j\frac{2\pi}{N}kn}, \quad k = 0, 1, \dots, N-1}$$

So, $$\boxed{x[n] = \sum_{k=0}^{N-1} c_k s_k[n]}$$

Lemma:

$$\boxed{\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn} = N, \text{ if } n = 0, \pm N, \pm 2N, \dots \qquad = 0, \text{ o.w.}}$$

Proof:

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi kn}{N}} = \frac{1 - e^{j2\pi n}}{1 - e^{j\frac{2\pi}{N}n}}$$

$$= 0 \quad \text{if } n \neq 0, \pm N, \pm 2N, \dots$$

If $n = \pm mN$, then

$$e^{j\frac{2\pi}{N}kn} = 1 \qquad 0 \le k \le N-1.$$

Then,

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi kn}{N}} = N$$

$\therefore$ ∎

Then,

$$\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} c_m e^{j\frac{2\pi}{N}mn} e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} c_m e^{j\frac{2\pi}{N}n(m-k)}$$

$$= \sum_{m=0}^{N-1} c_m \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n(m-k)}$$

$$= N \cdot C_k \quad \text{[By lemma]}$$

Thus,
$$\boxed{C_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}}$$

• **Lemma:-** $C_k, s_k[n]$ are periodic with period N.

Proof:- $s_{k+N}[n] = e^{j\frac{2\pi}{N}(k+N) \cdot n}$

$$= e^{j\frac{2\pi kn}{N}} \cdot e^{j2\pi n}$$

$$= e^{j\frac{2\pi}{N}kn}$$

$$= s_k[n]$$

$$C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(k+N)n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn} \cdot e^{-j2\pi n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= C_k.$$

... ▨

→ Signal power

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, x^*[n]$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} C_m \cdot e^{j\frac{2\pi}{N}mn} \cdot \sum_{l=0}^{N-1} C_l^* e^{-j\frac{2\pi}{N}ln} \right)$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} C_m \cdot C_l^* \cdot \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(m-l)n}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} C_m \cdot C_m^* \cdot N \quad \text{[By lemma]}$$

$$= \sum_{m=0}^{N-1} |C_m|^2 .$$

$$\boxed{P = \sum_{m=0}^{N-1} |C_m|^2} \longrightarrow \text{Parseval's theorem}$$

→ Fourier series coefficients for real signals

$$C_k^* = \sum_{n=0}^{N-1} x^*[n] \cdot e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} x(n) e^{j\frac{2\pi}{N}nk} \cdot e^{-j2\pi n} \quad [\because x[n] = x^*(n)]$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(N-k)n}$$

$$= c_{N-k}$$

Thus, $\boxed{c_k^* = c_{N-k} \text{ if the signal is real}}$

* Discrete time Fourier transform for aperiodic signals

Let $x[n]$ be aperiodic.

Choice of basis

$$\boxed{\frac{1}{2\pi} e^{j\omega n}, \quad \omega \in [-\pi, \pi]}$$

Define the coefficients as follows

$$\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}}$$

The coefficients $X(\omega)$ are called the discrete time Fourier transform of $x[n]$.

Note:- For the time being, assume that the series $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ converges for all $\omega \in [-\pi, \pi]$.

If the sum indeed converges, we can verify that $X(\omega)$ are indeed the coefficients. To see this,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cdot e^{j\omega n}\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty} x[m]\cdot e^{-j\omega m}\cdot e^{j\omega n}\cdot d\omega$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\int_{-\pi}^{\pi} e^{j\omega(n-m)}\cdot d\omega$$

[The interchange of the sum and integral are possible if $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ converges for all ω]

$$= \frac{1}{2\pi} \sum_{m \neq n} x[m] \left[\frac{e^{j(n-m)\pi} - e^{-j(n-m)\pi}}{j(n-m)} \right]$$

$$+ \frac{1}{2\pi} x[n] \cdot 2\pi$$

$$= x[n]$$

Thus, $\boxed{x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \cdot d\omega}$

→ A note on convergence

i) Convergence proper

$\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ converges if

$$\sum_{n=-N}^{N} x[n] e^{-j\omega n} \xrightarrow[N \to \infty]{} X(\omega)$$

To guarantee convergence, we need

$$\left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| < \infty$$

Now, if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, then, $\sum_{n=-\infty}^{\infty} |x[n]e^{-j\omega n}| < \infty$

and hence by the triangle inequality

$$\left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| < \infty$$

Hence, convergence is guaranteed if

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty}$$

ii) Mean-square convergence

Some signals $x[n]$ which do not satisfy $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ can still be expressed in the form

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega .$$

In such cases,

$$\sum_{n=-N}^{N} x[n]e^{-j\omega n} \not\to X(\omega).$$

but

$$\boxed{\lim_{N\to\infty} \int_{-\pi}^{\pi} \left| \sum_{n=-N}^{N} x[n]e^{-j\omega n} - X(\omega) \right|^2 \cdot d\omega = 0}$$

<u>Example:-</u>

Let $x[n] = \dfrac{\omega_c}{\pi} \dfrac{\sin n\omega_c}{n\omega_c}, \quad n \neq 0$

$$= \frac{\omega_c}{\pi}, \quad n = 0.$$

Define $X(\omega) = 1, \quad |\omega| \leq \omega_c$

$$= 0, \quad \omega_c < |\omega| \leq \pi$$

Let us evaluate $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \cdot d\omega.$

When $n \neq 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n} \cdot d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} \cdot d\omega$$

$$= \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n}$$

$$= \frac{2j \sin \omega_c n}{2\pi j n}$$

$$= \frac{1}{\pi} \frac{\sin \omega_c n}{n}$$

$$= \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{n \omega_c}$$

$$= x(n)$$

When $n=0$, $\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega$$

$$= \frac{\omega_c}{\pi}$$

$$= x[0].$$

Thus, $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) . e^{j\omega n} . d\omega.$

→ Signal energy in terms of discrete time Fourier transform.

Let $x[n]$ be a finite energy signal.

Then, $$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} x[n] . x^*[n]$$

$$= \sum_{n=-\infty}^{\infty} \frac{x[n]}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) e^{-j\omega n} . d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left(\sum_{n=-\infty}^{\infty} x[n] . e^{-j\omega n} \right) . d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) . X(\omega) . d\omega$$

$$\Rightarrow \boxed{E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 . d\omega}$$

→ Parseval's theorem

* Properties of the Fourier transform for discrete time signals

Notation :- If $X(\omega)$ is the Fourier transform of $x[n]$, then we write $x[n] \xleftrightarrow{F} X(\omega)$

i) $\boxed{X(\omega) \text{ is periodic with a period of } 2\pi}$

$$X(\omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n} \cdot e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= X(\omega).$$

ii) $\boxed{X(\omega) = X^*(-\omega) \text{ if } x[n] \text{ is real.}}$

$$X^*(-\omega) = \sum_{n=-\infty}^{\infty} x^*[n] e^{j(-\omega)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad [\because x[n] \text{ is real}]$$

$$= X(\omega)$$

iii) $\boxed{\text{Fourier transform is a linear operation}}$

Let $x_1[n] \xleftrightarrow{\mathcal{F}} X_1(\omega)$

$x_2[n] \xleftrightarrow{\mathcal{F}} X_2(\omega)$

Then $a_1 X_1(\omega) + a_2 X_2(\omega)$

$$= \sum_{n=-\infty}^{\infty} a_1 x_1[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} a_2 x_2[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(a_1 x_1[n] + a_2 x_2[n]\right) e^{-j\omega n}$$

Thus, $a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{\mathcal{F}} a_1 X_1(\omega) + a_2 X_2(\omega)$

iv) Time shift

$$\boxed{\begin{aligned} x[n] &\xleftrightarrow{\mathcal{F}} X(\omega) \\ x[n-k] &\xleftrightarrow{\mathcal{F}} e^{-j\omega k} X(\omega) \end{aligned}}$$

$$x[n-k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \cdot e^{j\omega(n-k)} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[X(\omega) \cdot e^{-j\omega k}\right] \cdot e^{j\omega n} \cdot d\omega$$

v) Time reversal

$$\boxed{\begin{array}{c} x[n] \xleftrightarrow{\mathcal{F}} X(\omega) \\ x[-n] \xleftrightarrow{\mathcal{F}} X(-\omega) \end{array}}$$

$$x[-n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \cdot e^{-j\omega n} \cdot d\omega$$

$$= -\frac{1}{2\pi} \int_{\pi}^{-\pi} X(-\omega) e^{j\omega n} \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(-\omega) \cdot e^{j\omega n} \cdot d\omega$$

vi) Convolution

$$\boxed{x_1[n] * x_2[n] \xleftrightarrow{\mathcal{F}} X_1(\omega) \cdot X_2(\omega)}$$

$$\sum_{n=-\infty}^{\infty} (x_1[n] * x_2[n]) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k] \right) \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] \cdot e^{-j\omega k} \cdot x_2[n-k] e^{-j\omega(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \cdot e^{-j\omega k} \left(\sum_{n=-\infty}^{\infty} x_2[n-k] \cdot e^{-j\omega(n-k)} \right)$$

$$= X_2(\omega) \cdot \sum_{k=-\infty}^{\infty} x_1[k] \cdot e^{-j\omega k}$$

$$= X_1(\omega) \cdot X_2(\omega)$$

- Example:-

$$u[n] \longrightarrow \boxed{\begin{array}{l} h[n] \\ = a^n u[n], \\ \quad |a|<1 \end{array}} \longrightarrow y[n].$$

Find $y[n]$

We know that

$$y[n] = u[n] * h[n]$$

Then, $Y(\omega) = U(\omega) \cdot H(\omega)$

Now $U(\omega) = \sum_{n=-\infty}^{\infty} u[n] e^{-j\omega n}$

$$= \sum_{n=0}^{\infty} e^{-j\omega n} = \frac{1}{1 - e^{-j\omega}}$$

$$H(\omega) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

Then, $U(\omega) \cdot H(\omega)$

$$= \frac{1}{(1-ae^{-j\omega})(1-e^{-j\omega})}$$

$$= \frac{1}{(a-1)} \cdot \frac{(a-1)}{(1-ae^{-j\omega})(1-e^{-j\omega})}$$

$$= \frac{1}{(a-1)} \left[\frac{a}{1-ae^{-j\omega}} - \frac{1}{1-e^{-j\omega}} \right]$$

$$= -\frac{a}{1-a} \cdot \frac{1}{1-ae^{-j\omega}} + \frac{1}{(1-a)} \cdot \frac{1}{1-e^{-j\omega}}$$

Now, noting that

$$u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}$$

and $$h[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

by linearity we have

$$y[n] = \frac{1}{(1-a)}\left[u[n] - a\cdot h[n]\right]$$

$$= \frac{1}{(1-a)}\left(1-a^{n+1}\right)\cdot u[n]$$

Vii) Cross correlation and cross energy spectral density

$$\boxed{\begin{array}{l} r_{xy}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)\, Y^*(\omega) \\ \text{where } r_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot y^*[k-n] \end{array}}$$

Here, $r_{xy}(n)$ is called the cross correlation between $x(n)$ and $y(n)$.

$r_{xx}(n)$, in particular, is called the auto correlation of $x(n)$.

Thus, the above property says

$$\boxed{r_{xx}[n] \overset{\mathcal{F}}{\longleftrightarrow} \underbrace{|X(\omega)|^2}_{\text{Energy spectral density}}}$$

→ Wiener-Khinchine theorem

To see the general version,

$$\sum_{n=-\infty}^{\infty} r_{xy}(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] \cdot y^*[k-n] \cdot e^{-j\omega k} \cdot e^{-j\omega(n-k)}$$

$$= \sum_{n=-\infty}^{\infty} X(\omega) \cdot y^*[k-n] e^{-j\omega(k-n)}$$

$$= X(\omega) \cdot Y^*(\omega)$$

viii) Frequency shifting

$$\boxed{\begin{aligned} x[n] &\stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) \\ e^{j\omega_0 n}x[n] &\longleftrightarrow X(\omega-\omega_0) \end{aligned}}$$

$$\sum_{n=-\infty}^{\infty} x(n)e^{j\omega_0 n}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n}$$

$$= X(\omega-\omega_0)$$

ix) Modulation theorem

$$\boxed{\begin{aligned} x[n] &\stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) \\ x[n]\cos\omega_0 n &\longleftrightarrow \frac{1}{2}\left[X(\omega+\omega_0)+X(\omega_0-\omega_0)\right] \end{aligned}}$$

$$\cos\omega_0 n = \frac{1}{2}\left[e^{j\omega_0 n}+e^{-j\omega_0 n}\right]$$

Thus by linearity

$$x[n]\cos\omega_0 n = \frac{x[n]}{2}\cdot e^{j\omega_0 n} + \frac{x[n]}{2}\cdot e^{-j\omega_0 n}$$

$$\xleftrightarrow{\mathcal{F}} \frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$$

X) Plancherel's theorem

$$\boxed{\sum_{n=-\infty}^{\infty} x_1[n]\,x_2^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)\cdot X_2^*(\omega)\cdot d\omega}$$

$$\sum_{n=-\infty}^{\infty} x_1[n]\cdot x_2^*[n]$$

$$= \sum_{n=-\infty}^{\infty} \frac{x_1(n)}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)\,e^{-j\omega n}\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)\cdot \sum_{n=-\infty}^{\infty} x_1[n]\cdot e^{-j\omega n}\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)\cdot X_1(\omega)\cdot d\omega$$

This is the generalised version of Parseval's theorem

xi) Frequency convolution

$$\boxed{x_1[n]\cdot x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\cdot X_2(\omega-\lambda)\cdot d\lambda}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\cdot X_2(\omega-\lambda)\cdot d\lambda\right)\cdot e^{j\omega n}\cdot d\omega$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi} X_1(\lambda)\cdot \int_{-\pi}^{\pi} X_2(\omega-\lambda)\, e^{j\omega n}\, d\omega\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\cdot e^{j\lambda n}\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(\omega-\lambda)\, e^{j(\omega-\lambda)n}\cdot d\omega\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\, e^{j\lambda n}\frac{1}{2\pi}\int_{-\pi-\lambda}^{\pi-\lambda} X_2(\tau)\, e^{j\tau n}\cdot d\tau\, d\lambda$$

$[\tau \triangleq \omega-\lambda]$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\, e^{j\lambda n}\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(\tau)\, e^{j\tau n}\, d\tau\, d\lambda$$

[$\because$ $X_2(\tau)e^{j\tau n}$ is periodic with period 2π]

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\, e^{j\lambda n}\cdot x_2[n]\, d\lambda$$

$$= x_1[n]\cdot x_2[n]$$

xii) Frequency differentiation

$$\boxed{n x[n] \xleftrightarrow{\mathcal{F}} j \frac{dX(\omega)}{d\omega}}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega} \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right)$$

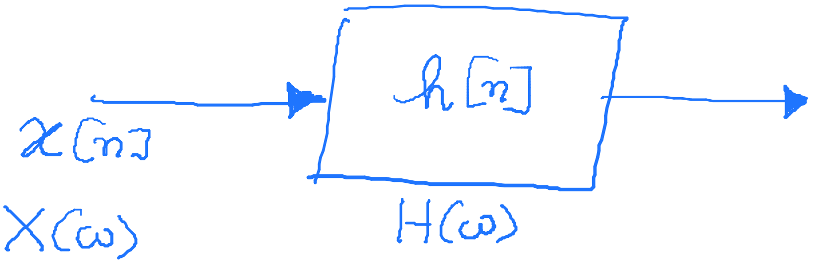
$$= \sum_{n=-\infty}^{\infty} \frac{d}{d\omega} \left(x[n] e^{-j\omega n} \right)$$

[The interchange of derivative and sum is okay provided $X(\omega)$ exists]

$$= \sum_{n=-\infty}^{\infty} -x[n] jn e^{-j\omega n}$$

$$\Rightarrow j \frac{dX(\omega)}{d\omega} = \sum_{n=-\infty}^{\infty} n x[n] e^{-j\omega n}$$

* LTI systems in frequency domain



$y[n] = x[n] * h[n]$

$Y(\omega) = X(\omega) \cdot H(\omega)$

Output energy

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(\omega)|^2 \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega)|^2 \cdot |X(\omega)|^2 \cdot d\omega$$

→ Frequency response of LTI system

i) Excitation by unit exponential

let $x[n] = e^{j\omega_0 n}$

$$y[n] = x(n) * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]\, h(k)$$

$$= \sum_{k=-\infty}^{\infty} e^{j\omega_0 (n-k)} \cdot h(k)$$

$$= e^{j\omega_0 n}\left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}\right)$$

$$= H(\omega_0)\cdot e^{j\omega_0 n}$$

$$\boxed{y[n] = H(\omega_0)\cdot e^{j\omega_0 n}}$$

ii) Excitation by sinusoids [Assume $h[n]$ is real]

Let $x(n) = \cos(\omega_0 n + \theta)$

Then $x[n] = \frac{1}{2}\left[e^{j(\omega_0 n+\theta)} + e^{-j(\omega_0 n+\theta)}\right]$

Then by linearity property of the LTI system,

$$y[n] = \frac{e^{j\theta}}{2} H(\omega_0) e^{j\omega_0 n} + \frac{e^{-j\theta}}{2} H(-\omega_0) e^{-j\omega_0 n}$$

$$= \frac{1}{2}\left[H(\omega_0) e^{j(\omega_0 n+\theta)} + H^*(\omega_0) e^{-j(\omega_0 n+\theta)}\right]$$

$$= \mathrm{Re}\left[H(\omega_0) e^{j(\omega_0 n+\theta)}\right]$$

$$= |H(\omega_0)|\cdot \cos(\omega_0 n + \theta + \angle H(\omega_0))$$

i.e. $$\boxed{\begin{aligned} y[n] &= |H(\omega_0)| \cdot \cos(\omega_0 n + \theta + \angle H(\omega_0)) \\ \text{where } \angle H(\omega_0) &= \tan^{-1} \frac{Im(H(\omega_0))}{Re(H(\omega_0))} \end{aligned}}$$

Thus, a single frequency signal passed through an LTI system develops a phase lag of $\angle H(\omega_0)$.

Another way to look at it is as follows

$$y[n] = |H(\omega_0)| \cos\left(\omega_0\left(n + \frac{\angle H(\omega_0)}{\omega_0}\right) + \theta\right)$$

Thus the LTI system introduces a 'time delay' of $-\angle \frac{H(\omega_0)}{\omega_0}$.

This is called the phase delay at ω_0.

$$\boxed{\tau_p(\omega_0) = -\frac{\angle H(\omega_0)}{\omega_0}}$$

iii) Group delay

What happens when we pass not a sinusoid but a general signal containing many frequencies?

They might have different delays.

If $\angle H(\omega) = -C\omega$, then

phase delay at frequency ω

is $-\frac{\angle H(\omega)}{\omega} = C$

Thus, for linear $\angle H(\omega)$ all the frequencies in the signal are time-delayed by C.

In general, define group delay

$$\boxed{\tau_g(\omega) = -\frac{d\angle H(\omega)}{d\omega}}$$

Group delay can be intepreted as the phase delay of the narrow band of frequencies from $\omega - d\omega$ to $\omega + d\omega$.

* Z-transform

Some basic signals like $u[n]$ doesn't have a Fourier transform. Thus we use a more general transform called the z-transform

The z-transform of $x[n]$ is given by

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad , z \in \mathbb{C}}$$

Finite duration signals can be immediately computed after looking at their z-transforms.

eg. Let $X(z) = z^3 + 2z^2 - z + 2 + z^{-1} + 4z^{-2}$

Then, we immediately have the following

$x[-3]=1$ $\quad x[1]=1$

$x[-2]=2$ $\quad x[2]=0$

$x[-1]=-1$ $\quad x[n]=0$, o.w.

$x[0]=2$

Also, note that $X(z)$ exists for all $z\neq 0$.

* ROC of z-transform

Let $z=re^{j\theta}$

$$|X(z)|=\left|\sum_{n=-\infty}^{\infty} x[n].r^{-n}e^{jn\theta}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| \quad \text{[By } \Delta \text{ inequality]}$$

Thus $X(z)$ exists if $\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$.

We define ROC of $X(z)$ to be the set of all z s.t. $\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$.

Note:- $X(z)$ might still exist even if $z\notin$ ROC.

eg. $x[n]=\frac{-1}{n}u[n-1]$, where ROC is $|z|>1$, but $X(z)$ exists at $z=-1$.

$$= \sum_{n=-\infty}^{-1} |x[n] r^{-n}| + \sum_{n=0}^{\infty} |x[n] r^{-n}|$$

$$= \sum_{n=1}^{\infty} |x[-n] r^{n}| + \sum_{n=0}^{\infty} |x[n] r^{-n}|$$

For the $z \in ROC$, we need both sums $\sum_{n=1}^{\infty} |x[-n] r^{n}| < \infty$ and

$$\sum_{n=0}^{\infty} |x[n] r^{-n}| < \infty .$$

Consider first $\sum_{n=1}^{\infty} |x[-n] r^{n}|$.

Let for some r we have $\sum_{n=1}^{\infty} |x[-n]| |r^{n}| < \infty$

Then, for every $r' < r$ we will have

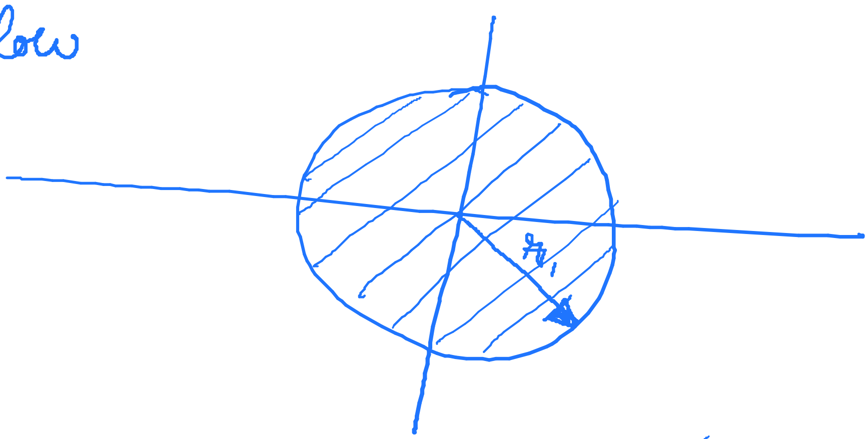
$$\sum_{n=1}^{\infty} |x[-n]| |r'^{n}| < \sum_{n=1}^{\infty} |x[-n]| |r^{n}| < \infty$$

Let $r_1 = \sup\{r : \sum_{n=1}^{\infty} |x[-n]| |r^{n}| < \infty\}$

Then, for $|z| < r_1$, $\sum_{n=-\infty}^{-1} x[n] z^{-n} < \infty$

and for $|z| > r_1$, $\sum_{n=-\infty}^{-1} x(n) z^{-n}$ is not summable.

Thus, $\sum_{n=-\infty}^{-1} x[n] z^{-n}$ converges for the region below



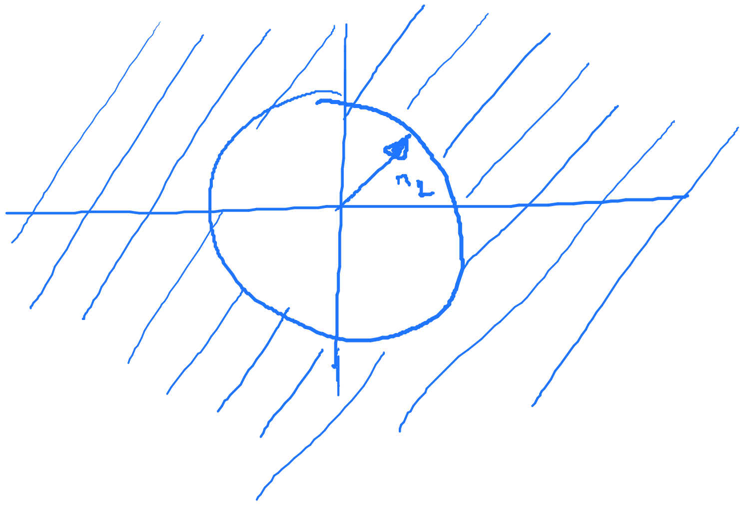
Similarly, note that if $\sum_{n=0}^{\infty} |x[n]| |r^{-n}|$ converges for r, then for all $r' > r$

we have $$\sum_{n=0}^{\infty} |x[n]| |r'^{-n}| < \sum_{n=0}^{\infty} |x[n]| |r^{-n}| < \infty.$$

Let $r_2 = \inf\{r : \sum_{n=0}^{\infty} |x[n]| |r^{-n}| < \infty\}$

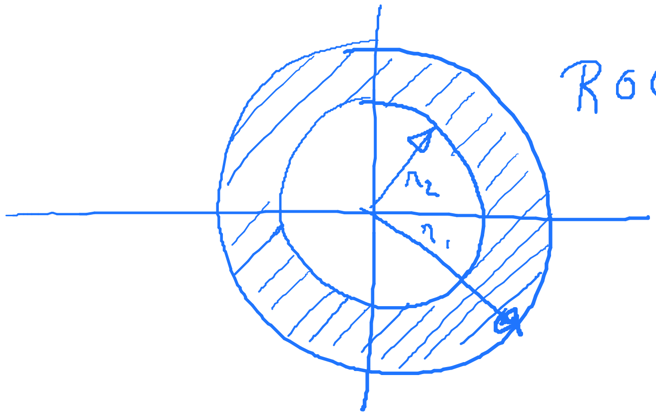
Then, for $|z| > r_2$, the sum $\sum_{n=0}^{\infty} x[n] z^{-n}$ exists, but for $|z| < r_2$, the sum $\sum_{n=0}^{\infty} x[n] z^{-n}$ doesn't converge.

Thus, the sum $\sum_{n=0}^{\infty} x[n] z^{-n}$ converges for the region below



Thus, if $r_2 > r_1$, the $X(z)$ cannot exists.

On the other hand, if $X(z)$ does exist then $r_2 \le r_1$, and the region of convergence is always annular.



ROC of $X(z)$.

* Uniqueness of Z-transform

Example :-

i) $x(n) = \alpha^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

$$= \frac{1}{1-\frac{\alpha}{z}} \quad \text{if} \quad \left|\frac{z}{\alpha}\right| > 1$$

$$= \frac{z}{z-\alpha}, \quad \text{when } |z| > |\alpha|$$

ii) $x[n] = -\alpha^n u[-n-1]$

i.e. $x[n] = -\alpha^n$, if $n \leq -1$

$= 0, \quad$ o.w.

Then, $X(z) = \displaystyle\sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1] z^{-n}$

$$= \sum_{n=-\infty}^{-1} -\alpha^n \cdot z^{-n}$$

$$= -\sum_{n=1}^{\infty} \left(\frac{z}{\alpha}\right)^n$$

$$= -\frac{z/\alpha}{1 - z/\alpha}, \quad \text{when } |z| < |\alpha|$$

$$= \frac{z}{z-\alpha}, \quad |z| < |\alpha|$$

Note that $X(z)$ is the same for examples (i) and (ii).

However the ROC are different.

The z-transforms alone are hence not unique, and the ROC must always be specified.

* Properties of z-transform

i) $\boxed{X(\omega) = X(z)\Big|_{z=e^{j\omega}}}$

Obvious from the definition.

eg. $x[n] = u[n]$

Then $X(\omega)$ doesn't exist for all ωs.

But, $X(z) = \sum_{n=-\infty}^{\infty} u(n) \cdot z^{-n}$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1}{1-z^{-1}}, \quad |z|>1$$

$$= \frac{z}{z-1}$$

We can by analytic continuation define $X(z) = \frac{z}{z-1}$, except when $z=1$.

Then, $X(\omega) = \dfrac{e^{j\omega}}{e^{j\omega}-1}$, except at $\omega = 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$

$$= \frac{e^{j\omega/2}}{2j\sin(\omega/2)}$$

ii) Linearity

$$\boxed{\begin{gathered} x_1[n] \xleftrightarrow{Z} X_1(z) \\ x_2[n] \xleftrightarrow{Z} X_2(z) \\ a_1x_1[n]+a_2x_2[n] \xleftrightarrow{Z} a_1X_1(z)+a_2X_2(z) \end{gathered}}$$

$$\sum_{n=-\infty}^{\infty}\left(a_1x_1[n]+a_2x_2[n]\right)z^{-n}$$

$$= a_1\sum_{n=-\infty}^{\infty}x_1[n]z^{-n} + a_2\sum_{n=-\infty}^{\infty}x_2[n]z^{-n}$$

$$= a_1X_1(z) + a_2X_2(z)$$

iii) Time shift

$$\boxed{x[n-k] \xleftrightarrow{Z} z^{-k}X(z)}$$

$$z^{-k}X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n-k}$$

$$= \sum_{m=-\infty}^{\infty} x[m-k] z^{-m}$$

$$[m = n+k]$$

This is why we represent k units of delay by the block $\boxed{z^{-k}}$

To obtain the ROC, note that if $X(z)$ exists, then only way $z^{-k}X(z)$ doesn't exist is if $k>0$ and $z=0$.

Thus the ROC is same when $k<0$ and ROC doesn't include $z=0$ if $k>0$.

iv) z-Scaling

$$a^n x(n) \overset{Z}{\longleftrightarrow} X(a^{-1}z),$$

If ROC of $x(n)$ is $r_2 < |z| < r_1$, then ROC of $X(a^{-1}z)$ is $|a|r_2 < |z| < |a|r_1$.

$$\sum_{n=-\infty}^{\infty} a^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(a^{-1}z\right)^{-n}$$

$$= X\left(a^{-1}z\right)$$

The ROC is $r_2 < |a^{-1}z| < r_1$ i.e.

$$|a| r_2 < |z| < |a| r_1.$$

v) Time reversal

$$\boxed{\begin{array}{c} x[-n] \overset{Z}{\longleftrightarrow} X(z^{-1}) \\ \text{If ROC of } x(n) \text{ was } r_2 < |z| < r_1 \\ \text{then ROC of } x[-n] \text{ is } \frac{1}{r_1} < |z| < \frac{1}{r_2} \end{array}}$$

$$\sum_{n=-\infty}^{\infty} x[-n] z^{-n} = \sum_{m=-\infty}^{\infty} x[m] \cdot \left(z^{-1}\right)^{-m} \quad [m=-n]$$

$$= X(z^{-1})$$

The ROC follows by noting

$$|z^{-1}| < r_1 \Rightarrow |z| > \frac{1}{r_1}, \quad |z^{-1}| > r_2 \Rightarrow |z| < \frac{1}{r_2}$$

vi) Differentiation in z-domain

$$\boxed{\begin{array}{c} n\,x[n] \xleftrightarrow{Z} -z\dfrac{dX(z)}{dz} \\ \text{with the same ROC as } X(z) \end{array}}$$

$$-z\frac{dX(z)}{dz} = -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n]\,z^{-n}$$

$$= -z\sum_{n=-\infty}^{\infty} \frac{d}{dz}\left(x[n]\cdot z^{-n}\right)$$ [The interchange is valid as long as z lies in ROC]

$$= -z\sum_{n=-\infty}^{\infty} -n\,x[n]\,z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} n\,x[n]\,z^{-n}$$

Note that ROC remains same since we only assumed (during the summation differentiation interchange) that z lies in ROC of X(z).

vii) Convolution

$$\boxed{x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z) \cdot X_2(z) \text{ and ROC includes the intersection of } X_1(z) \text{ and } X_2(z).}$$

$$\sum_{n=-\infty}^{\infty} (x_1[n] * x_2[n]) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] \cdot z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \sum_{n=-\infty}^{\infty} x_2[n-k] \cdot z^{-(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]\, z^{-k}\, X_2(z)$$

$$= X_1(z)\, X_2(z)$$

Also, we only need that z lies in ROC of both $X_1(z)$ and $X_2(z)$. Thus ROC includes the intersection of their ROCs.

viii) Initial value theorem

$$\boxed{\text{If } x[n] \text{ is causal (i.e. } x[n]=0 \text{ for } n<0) \text{ then } x[0] = \lim_{z\to\infty} X(z)}$$

Proof:- $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} x[n] \cdot z^{-n} \quad [\text{Causality}]$$

Then $$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \lim_{z\to\infty} x[n] \cdot z^{-n}$$

[Interchange possible since all large enough zs belong to ROC]

$$= x[0]$$

. . . ▨

* Z-transforms of some basic signals

i) $\boxed{\delta[n] \xleftrightarrow{Z} 1 \qquad ROC: \mathbb{C}}$

$$\sum_{n=-\infty}^{\infty} \delta[n] . z^{-n} = \delta[0] = 1 .$$

Holds for all $z \in \mathbb{C}$.

ii)

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} \qquad ROC: |z| > |a|$$

$$-a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} \qquad ROC: |z| < |a|$$

We have seen this when we looked at uniqueness of z-transform

iii)

$$n a^n u[n] \xleftrightarrow{Z} \frac{az^{-1}}{(1-az^{-1})^2} \qquad ROC: |z| > |a|$$

$$-n a^n u[-n-1] \xleftrightarrow{Z} \frac{az^{-1}}{(1-az^{-1})^2} \qquad ROC: |z| < |a|$$

Let $S = \sum_{n=-\infty}^{\infty} n a^n u[n] z^{-n} = \sum_{n=1}^{\infty} n(az^{-1})^n$

$$az^{-1}S = \sum_{n=1}^{\infty} n(az^{-1})^{n+1}$$

Then $S(1-az^{-1}) = \sum_{n=1}^{\infty} n(az^{-1})^n - \sum_{n=1}^{\infty} n(az^{-1})^{n+1}$

$$= \sum_{n=1}^{\infty} n(az^{-1})^n - \sum_{n=2}^{\infty} (n-1)(az^{-1})^n$$

$$= az^{-1} + \sum_{n=2}^{\infty} (az^{-1})^n$$

$$= \sum_{n=1}^{\infty} (az^{-1})^n$$

$$= \frac{az^{-1}}{1-az^{-1}}, \quad \text{if } |az^{-1}|<1$$

$$\Rightarrow |z| > |a|$$

Thus $S = \dfrac{az^{-1}}{(1-az^{-1})^2}$

Similarly, let $S' = \sum_{n=-\infty}^{\infty} -na^n u[-n-1] \cdot z^{-n}$

$$= -\sum_{n=-\infty}^{1} n(az^{-1})^n$$

$$\Rightarrow S' = \sum_{n=1}^{\infty} n(za^{-1})^n$$

Thus, similar to the previous summation we have

$$S' = \frac{za^{-1}}{(1-za^{-1})^2} \quad , \text{for } |za^{-1}| < 1$$
$$\Rightarrow |z| < |a|$$

$$= \frac{az}{(a-z)^2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}$$

iv)

$$\boxed{\begin{aligned} a^n e^{j\omega_0 n} u[n] &\xleftrightarrow{Z} \frac{1}{1-az^{-1}e^{j\omega_0}} \\ &\qquad \text{ROC: } |z| > |a| \\ a^n e^{-j\omega_0 n} u[n] &\xleftrightarrow{Z} \frac{1}{1-az^{-1}e^{-j\omega_0}} \\ &\qquad \text{ROC: } |z| > |a| \end{aligned}}$$

$$\sum_{n=-\infty}^{\infty} a^n e^{j\omega_0 n} u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(a z^{-1} e^{j\omega_0}\right)^n$$

$$= \frac{1}{1 - a z^{-1} e^{j\omega_0}}, \quad \text{for } \left|a z^{-1} e^{j\omega_0}\right| < 1$$

$$\Rightarrow |z| > |a|$$

$$\boxed{\begin{array}{l} a^n \cos\omega_0 n \, u[n] \xleftrightarrow{Z} \dfrac{1 - a z^{-1}\cos\omega_0}{1 - 2a z^{-1}\cos\omega_0 + a^2 z^{-2}} \\ \qquad \text{ROC: } |z| > |a| \\ a^n \sin\omega_0 n \, u[n] \xleftrightarrow{Z} \dfrac{a z^{-1}\sin\omega_0}{1 - 2a z^{-1}\cos\omega_0 + a^2 z^{-2}} \\ \qquad \text{ROC: } |z| > |a| \end{array}}$$

$$Z\left[a^n \cos\omega_0 n \, u[n]\right]$$

$$= Z\left[\frac{1}{2} a^n e^{j\omega_0 n} u[n] + \frac{1}{2} a^n e^{-j\omega_0 n} u[n]\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-az^{-1}e^{j\omega_0}} + \frac{1}{1-az^{-1}e^{-j\omega_0}}\right],$$

for $|z|>|a|$

[By linearity]

$$= \frac{1}{2}\left[\frac{2-2az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}\right]$$

$$= \frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$$

Similarly,

$$\mathcal{Z}\left[a^n \sin\omega_0 n\, u[n]\right]$$

$$= \mathcal{Z}\left[\frac{1}{2j}a^n e^{j\omega_0 n}u[n] - \frac{1}{2j}a^n e^{-j\omega_0 n}\cdot u[n]\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-az^{-1}e^{j\omega_0}} - \frac{1}{1-az^{-1}e^{-j\omega_0}}\right]$$

for $|z|>|a|$

$$= \frac{1}{2j}\left[\frac{az^{-1}(e^{j\omega_0}-e^{-j\omega_0})}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}\right]$$

$$= \frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$$

* Rational z-transforms — Poles and zeros

By rational, we mean z-transforms which can be expressed as a ratio of polynomials.

$$X(z) = \frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+\cdots+a_Nz^{-N}}$$

$$= G\,\frac{\prod_{l=1}^{M}(1-z^{-1}z_l)}{\prod_{l=1}^{N}(1-z^{-1}p_l)}$$

[By fundamental theorem of algebra]

The points $z_1, \ldots, z_M$ are called zeros of $X(z)$

since $X(z_\ell) = 0$, $\cancel{1} \le \ell \le M$.

The points $p_1, \ldots, p_N$ are called poles of $X(z)$

and $X(z)$ is not defined at these points.

Note that neither the zeros, nor the poles

are unique.

→ Signal behaviour based on pole location

i) Poles on real axis

An example is $X(z) = \dfrac{1}{1 - az^{-1}}$,

$|z| > |a|$

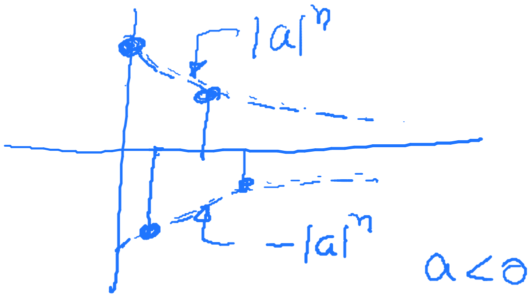
Here the pole is at $z = a$, and let

a be real.

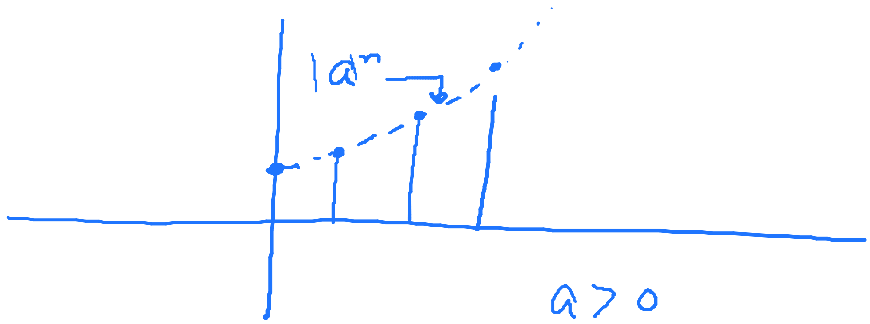
Note that $a^n u[n] \xleftrightarrow{Z} \dfrac{1}{1 - az^{-1}}$,

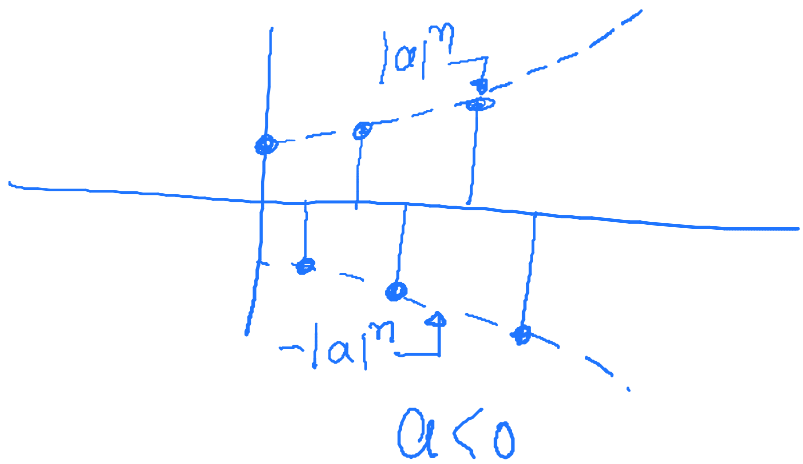
$|z| > |a|$

Thus, if pole is within the unit circle, i.e. $|a| < 1$, the signal is decaying



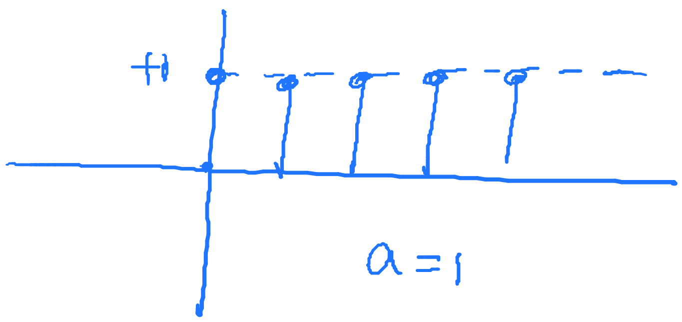
If pole is outside the unit circle, i.e. $|a| > 1$, the signal is growing



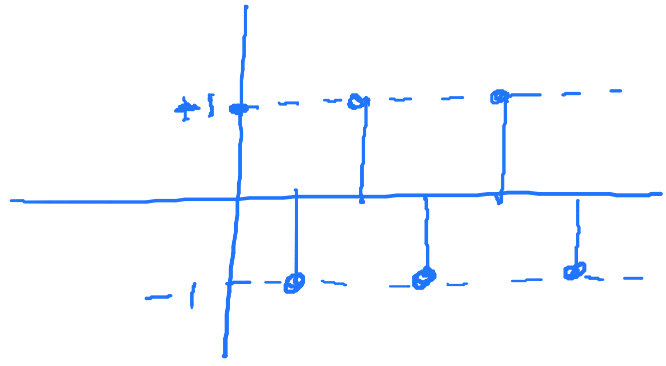


If pole is on the unit circle, i.e. $a = \pm 1$

then the signal is constant magnitued



$a = 1$



$a = -1$

ii) Pair of complex conjugate poles

An example

$$X(z) = \frac{1 - r z^{-1} \cos\omega_0}{1 - 2 r z^{-1} \cos\omega_0 + r^2 z^{-2}}, \quad r \in \mathbb{R}^+, \quad |z| > |r|$$

$$= \frac{1 - r z^{-1} \cos\omega_0}{\left(1 - r z^{-1} e^{j\omega_0}\right)\left(1 - r z^{-1} e^{-j\omega_0}\right)}$$

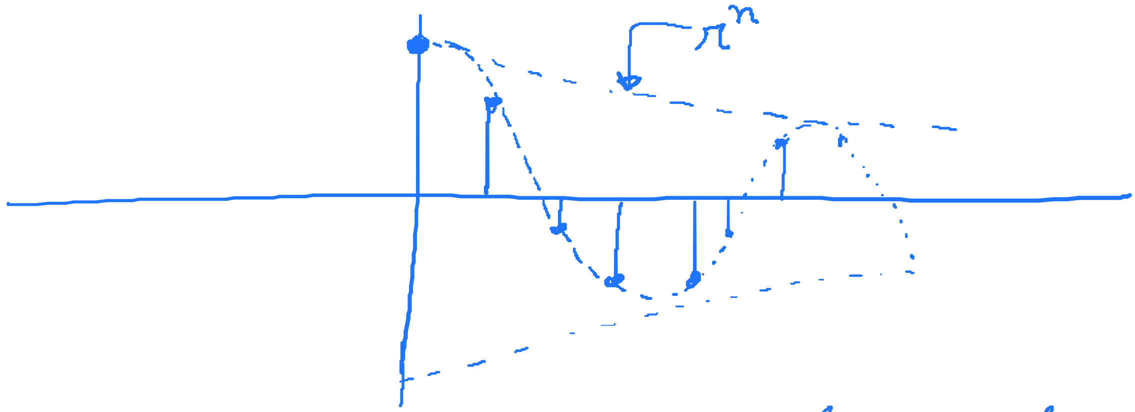
Thus, the poles are

$$z = r e^{j\omega_0}, \; r e^{-j\omega_0}$$

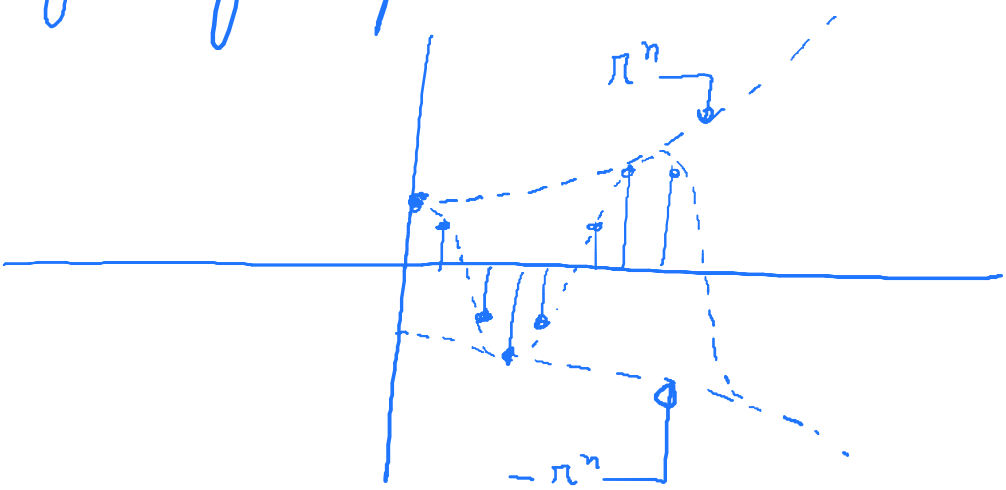
We know that

$$x[n] = r^n \cos \omega_0 n \, u[n]$$

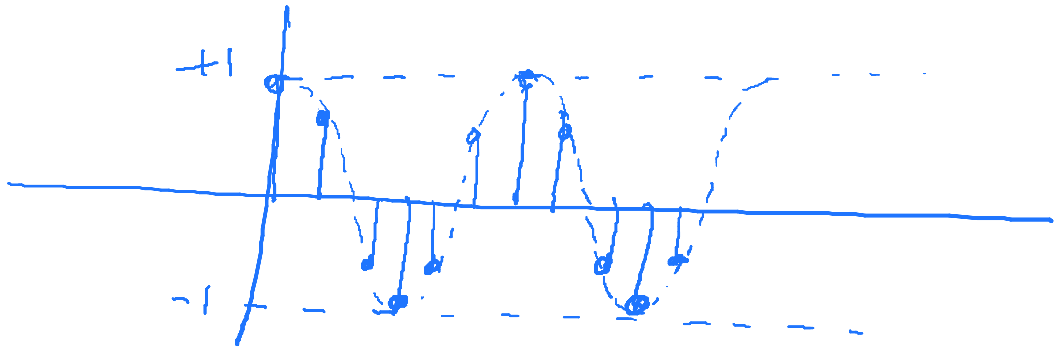
Thus, if poles are within the unit circle, then the signal is a sinusoid with decaying amplitude



If the poles are outside the unit circle then the signal is a sinusoid with a growing amplitude



If the poles are on the unit circle, i.e. $r=1$, then the signal is a constant magnitude sinusoid



* Inverse z-transform

The inverse z-transform for general signals is a complicated operation which involves solving a 'contour integral' most often through a result called the 'residue theorem'.

Instead, we shall focus on an easier method, called the partial fractions method, which covers the case of all signals with rational z-transforms.

Key idea:- 1. Break $X(z)$ into a linear combination of 'known z-transforms'

$$X(z) = \alpha_1 X_1(z) + \cdots + \alpha_k X_k(z).$$

This is where the 'rational' nature of $X(z)$ is needed.

2. Since $X_i(z)$ are known z-transforms, we know the signals $x_i(n)$ s.t. $x_i[n] \overset{Z}{\longleftrightarrow} X_i(z)$.

3. Then, by linearity,

$$x[n] = \sum_{i=1}^{k} \alpha_i x_i(n)$$

with ROC being the intersections of the ROCs of $x_i(n)$, $1 \le i \le k$.

So, WLOG, assume

$$X(z) = \frac{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}}{1 + b_1 z + \cdots + b_N z^{-N}}$$

If $M \geq N$, then, we can rewrite $X(z)$ in the form

$$X(z) = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)}$$
$$+ \frac{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)}}{1 + b_1 z^{-1} + \cdots + b_N z^{-N}}$$

[This simply follows by division]

Now, it is easy to see that the signal $x'[n]$ given by

$$x'[0] = c_0$$
$$x'[1] = c_1$$
$$\vdots$$
$$x'[M-N] = c_{M-N}$$

and $x'[n] = 0$, otherwise

possesses the z-transform

$$c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)}$$

with ROC $\mathbb{C} \setminus \{0\}$.

Thus, all that is needed is to find the inverse transform of

$$\frac{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)}}{1 + b_1 z^{-1} + \cdots + b_N z^{-N}}$$

Hence, from now on, we are only going to consider signals whose z-transform is of the form

$$X(z) = \frac{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}}{1 + b_1 z^{-1} + \cdots + b_N z^{-N}},$$

where $b_N \neq 0$ and $M < N$.

$$\frac{X(z)}{z} = \frac{a_0 z^{N-1} + a_1 z^{N-2} + \cdots + a_M z^{(N-M-1)}}{z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N}$$

$$= \frac{C \prod_{l=1}^{N-M-1} (z - z_l)}{\prod_{l=1}^{N} (z - p_l)}$$

[By the fundamental theorem of algebra]

To proceed, we divide the problem in two cases

i) $p_1, \ldots, p_N$ are distinct.

Claim:- $\dfrac{X(z)}{z} = \dfrac{A_1}{z-p_1} + \cdots + \dfrac{A_N}{z-p_N}$

where $A_i = (z-p_i) \cdot \dfrac{X(z)}{z} \Big|_{z=p_i}$

Proof:- The fact that $N > M$ means $\dfrac{X(z)}{z}$ can be expressed as $\dfrac{A_1}{z-p_1} + \cdots + \dfrac{A_N}{z-p_N}$.

Now,

$$(z-p_i)\frac{X(z)}{z} = \sum_{j=1}^{N} \frac{(z-p_i)\cdot A_j}{(z-p_j)}$$

$$= A_i + \sum_{j\neq i} \frac{A_j(z-p_i)}{(z-p_j)}$$

Then, $(z-p_i)\cdot\frac{X(z)}{z}\Big|_{z=p_i} = A_i + 0$

$$= A_i$$

$$\blacksquare$$

Thus, by the claim, we have

$$X(z) = \sum_{j=1}^{N} \frac{z A_j}{z-p_j}$$

$$= \sum_{j=1}^{N} \frac{A_j}{1-p_j z^{-1}}$$

Then, by noting the z-transforms for known functions, we can conclude that

$$x(n) = \sum_{j=1}^{N} A_j p_j^{\,n} u(n),$$

with ROC being

$$|z| > \max_{1 \le j \le N} |p_j|.$$

ii) Poles with multiplicity k

Let there exists k poles which are the same, and none of the other $N-k$ poles have the same value.

Then, WLOG, we assume

$p_1 = p_2 = \cdots = p_k = p$ and hence

$$\frac{X(z)}{z} = \frac{c \prod_{j=1}^{N-M-1} (z - z_e)}{(z-p)^k \prod_{j=k+1}^{N} (z - p_j)}$$

where $p_{k+1}, \ldots, p_N$ need not be distinct

but $p_i \neq p$, $k+1 \le i \le N$.

In such a case, one can expand $\frac{X(z)}{z}$ as

$$\frac{X(z)}{z} = \frac{A_1}{(z-p)} + \frac{A_2}{(z-p)^2} + \cdots + \frac{A_k}{(z-p)^k} + \text{Other terms}$$

where 'Other terms' are a summation of terms like $\frac{C}{(z-q)^l}$, where there are at least l p_i in $k+1 \le i \le N$, such that $p_i = q$.

Note that thus $p \neq q$.

<u>Claim</u> :- $A_k = (z-p)^k \frac{X(z)}{z}\Big|_{z=p}$

$$A_i = \frac{d^{k-i}}{dz^{k-i}} \left(\frac{(z-p)^k}{(k-i)!} \frac{X(z)}{z} \right)\Bigg|_{z=p}$$

$1 \le i \le k-1$

Proof :- The proof for A_k is similar to the distinct poles case.

To prove the case with $1 \le i \le k-1$

we first prove the following result.

For $m > i-1$

$$\frac{d^i}{dz^i}\left[\frac{(z-p)^k}{(z-q)^l}\right],$$

$$= \frac{(z-p)^{k-i}\, A(z)}{(z-q)^{2^i l}}$$

where $A(z)$ is some polynomial in z including the zero polymial.

To see this, we use induction.

Consider the case $i = 2$.

Then
$$\frac{d}{dz}\frac{(z-p)^k}{(z-q)^l}$$

$$= \frac{2(z-p)^{k-1}(z-q)^l - l(z-q)^{l-1}(z-p)^k}{(z-q)^{2l}}$$

$$= \frac{(z-p)^{k-1}\cdot\left[2(z-q)^l - l(z-q)^{l-1}(z-p)\right]}{(z-q)^{2l}}$$

Thus, the base case is true.

Assume the induction hypothesis that

$$\frac{d^{i-1}}{dz^{i-1}}\left[\frac{(z-p)^{k}}{(z-q)^{\ell}}\right] = \frac{(z-p)^{k-i+1}A(z)}{(z-q)^{2^{i-1}\ell}}$$

Then

$$\frac{d^{i}}{dz^{i}}\left[\frac{(z-p)^{k}}{(z-q)^{\ell}}\right]$$

$$= \frac{d}{dz}\left[\frac{d^{i-1}}{dz^{i-1}}\frac{(z-p)^{k}}{(z-q)^{\ell}}\right]$$

$$= \frac{d}{dz}\left[\frac{(z-p)^{k-i+1}\cdot A(z)}{(z-q)^{2^{i-1}\ell}}\right]$$

$$= \frac{(k-i+1)(z-p)^{k-i}A(z)(z-q)^{2^{i-1}\ell} + (z-p)^{k-i+1}A'(z)(z-q)^{2^{i-1}\ell} - 2^{i-1}\ell(z-q)^{2^{i-1}\ell-1}(z-p)^{k-i+1}A'(z)}{(z-q)^{2^{i}\ell}}$$

$$= \frac{(z-p)^{k-i}B(z)}{(z-q)^{2^{i}\ell}}$$

Thus, we have by induction, that

$$\frac{d^i}{dz^i}\left(\frac{(z-p)^k}{(z-q)^l}\right) = \frac{(z-p)^{k-i}\,A(z)}{(z-q)^{2^i l}}$$

for some polynomial $A(z)$.

With this in hand, note that since 'other terms' are a sum of terms of the form $\frac{C}{(z-q)^l}$, $p \neq q$, $l \geq 1$,

we have
$$\left.\frac{d^{k-i}}{dz^{k-i}}\left(\frac{(z-p)^k}{(k-1)!}\frac{C}{(z-q)^l}\right)\right|_{z=p}$$

$$= \left.\frac{C(z-p)^i A(z)}{(k-1)!\,(z-q)^{2^{(k-i)}l}}\right|_{z=p}$$

$$= 0, \text{ since } p \neq q.$$

Next, consider terms of the form

$$\frac{A_j}{(z-p)^j}$$

Here $\frac{d^{k-i}}{dz^{k-i}}\left[(z-p)^k \frac{A_j}{(z-p)^j}\right]$

$$= \frac{d^{k-i}}{dz^{k-i}}\left[(z-p)^{k-j} A_j\right]$$

$$= \begin{cases} (k-j)(k-j-1)\cdots(i-j+1)(z-p)^{i-j} A_j, & \text{if } i>j \\ 0, & \text{if } i<j \\ (k-i)!\cdot A_i, & \text{if } i=j \end{cases}$$

Thus, $\frac{d^{k-i}}{dz^{k-i}}\left[\frac{(z-p)^k}{(k-i)!}\frac{X(z)}{z}\right]\Bigg|_{z=p}$

$$= \frac{1}{(k-1)!}(k-1)!\, A_i$$

$$= A_i$$

. . . . . $\blacksquare$

Next, note that we already know the inverse of $\frac{A_1}{(z-p)}\cdot z$.

We need to compute the inverse of

$$\frac{zA_i}{(z-p_i)^i} \quad \text{for } i > 1.$$

To do so, we need to build on the differentiation property.

Firstly, note that

$$z^l \frac{d^l X(z)}{dz^l} = z^l \frac{d^l}{dz^l} \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= z^l \sum_{n=-\infty}^{\infty} x(n) \frac{d^l}{dz^l} z^{-n}$$

$$= z^l \sum_{n=-\infty}^{\infty} x[n] (-1)^l n(n+1) \cdots (n+l-1) z^{-n-l}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^l n(n+1) \cdots (n+l-1) x[n] z^{-n}$$

Thus, $n(n+1)\cdots(n+l-1)x[n] \xleftrightarrow{Z} (-1)^l z^l \dfrac{d^l X(z)}{dz^l}$.

With this result in hand, observe that

$$\frac{z A_i}{(z-p)^i}$$

Now, $$Z[p^n u[n]] = \frac{1}{1-pz^{-1}} = \frac{z}{z-p}$$

$$\frac{d}{dz} Z[p^n u(n)] = \frac{z-p-z}{(z-p)^2} = -\frac{p}{(z-p)^2}$$

Assume $$\frac{d^{i-1}}{dz^{i-1}} Z[p^n u[n]] = \frac{(-1)^{i-1} p (i-1)!}{(z-p)^i}$$

Then, $$\frac{d^i}{dz^i}\left[Z[p^n u(n)]\right]$$

$$= \frac{d}{dz}\left[\frac{(-1)^{i-1}}{(z-p)^i} p (i-1)!\right]$$

$$= (-1)^{i-1} p (i-1)! \left[\frac{-i}{(z-p)^{i+1}}\right]$$

$$= \frac{(-1)^i \cdot p\, i!}{(z-p)^{i+1}}$$

Thus, by induction, we have shown that $\frac{d^i}{dz^i}\left[Z[p^n u(n)]\right] = \frac{(-1)^i i!\, p}{(z-p)^{i+1}}$

Thus, $(-1)^{i-1} z^{i-1} \frac{d^{i-1}}{dz^{i-1}}\left[Z[p^n u(n)]\right] = \frac{z^{i-1} p\,(i-1)!}{(z-p)^i}$

Then, by the differentiation property we derived, we have that

$$n(n+1)\cdots(n+i-2)\, p^n u[n] \xleftrightarrow{Z} \frac{z^{i-1} p\,(i-1)!}{(z-p)^i}$$

$$\Rightarrow \frac{n(n+1)\cdots(n+i-2)}{(i-1)!} p^{n-1} u[n] \xleftrightarrow{Z} \frac{z^{i-1}}{(z-p)^i}$$

$$= z^{i-2} \cdot \frac{z}{(z-p)^i}$$

[By linearity]

$$\Rightarrow \frac{(n-i+2)(n-i+1)\cdots n}{(i-1)!} p^{n-i+1} u[n-i+2]$$

$$\xleftrightarrow{Z} \frac{z}{(z-p)^i}$$ [By time shift]

Note that in the derivation of the differentiation property the ROC did not change.

Next, for the time shift, noting that since the ROC did not contain 0 to begin with, ROC remains the same.

Hence, to summarise, we have

$$\boxed{\frac{zA_i}{(z-p)^i} \overset{Z}{\longleftrightarrow} \frac{A_i n(n-1)\dots(n-i+2)}{(i-1)!} p^{n-i+1} u[n-i+1], \quad \text{with ROC } |z|>|p|}$$

* LTI systems

By the convolution property.

$$x[n] * h[n] \longleftrightarrow X(z) \cdot H(z).$$

Thus, to obtain the output of LTI systems

i) Obtain z-transforms to get $X(z)$, $H(z)$.

ii) By convolution property, note that

$$Y(z) = H(z) \cdot X(z)$$

iii) Take inverse transform to obtain $y[n]$.

→ Causality

Recall that LTI system is called causal if $y[n]$ depends only on past samples.

We have already derived that this is equivalent to saying $h[n] = 0$ if $n < 0$.

Thus, ROC of $H(z)$ must be the exterior of a circle in z-plane.

In other words,

$$\boxed{\text{Causal} \Longleftrightarrow \text{ROC of } H(z) \text{ is of the form } |z| > r}$$

→ Stability of LTI systems

Recall that an LTI system is stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Now, $H(z) = \sum_{n=-\infty}^{\infty} h[n] \cdot z^{-n}$

Then for $|z| = 1$,

$$\sum_{n=-\infty}^{\infty} |h[n] z^{-n}| = \sum_{n=-\infty}^{\infty} |h[n]|$$

Then $\sum_{n=-\infty}^{\infty} |h[n]| < \infty \Longleftrightarrow |z| = 1$ belongs to ROC

Thus,

$$\boxed{\text{Stable system} \Longleftrightarrow |z| = 1 \text{ belongs to ROC}}$$

* Recursive LTI

Consider the recursive LTI system given by the Nth order linear difference equation

$$y[n] = -\sum_{k=1}^{N} a_k \, y[n-k] + \sum_{k=0}^{M} b_k \, x[n-k]$$

Then taking z-transforms and noting the linearity and time shift properties, we have

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} \cdot Y(z) + \sum_{k=0}^{M} b_k z^{-k} \cdot X(z)$$

$$\Longrightarrow \quad Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$\Longrightarrow \quad \boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}}$$

* Filters

Idea

- To 'filter out' some frequencies.

  These may be frequencies affected by noise for example.

Things to keep in mind.

- Filter must be stable, i.e. $H(z)$ should contain the unit circle in its ROC.

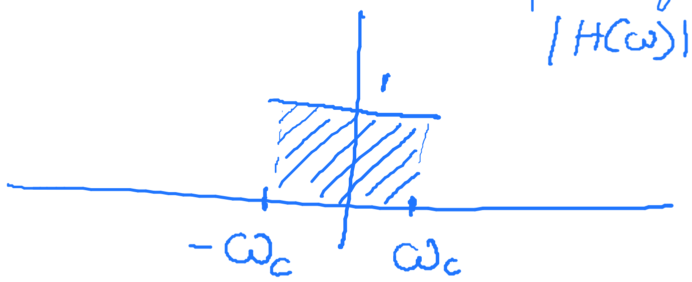
  Sanity check :- All poles must be within the unit circle.

- Filter must be causal

  Side effect :- Ideal filters cannot be designed.

Example :-

Consider the ideal low pass filter



Then, as we saw earlier,

$$h[n] = \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}, \quad n \neq 0$$

$$= \frac{\omega_c}{\pi}, \quad n = 0$$

Thus $h[n] \neq 0$ when $n < 0$.

and hence this is not causal.

- Complex poles and zeros must appear together with its conjugates to ensure filter coefficients are real.

* FIR Low pass filter or moving average filter

$$H(z) = \frac{1}{2}(1 + z^{-1})$$

i.e. $h[n] = \frac{1}{2}$, $n = 0$

$= \frac{1}{2}$, $n = 1$

Then, $h[n] * x[n]$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \frac{1}{2} x[n] + \frac{1}{2} x[n-1]$$

Hence the name moving average.

Now, $H(e^{j\pi}) = \frac{1}{2}(1 - 1) = 0$

$H(e^{j0}) = \frac{1}{2}(1 + 1) = 1$

Thus, the filter is 'passing' the low frequencies, $\omega = 0$, but shutting off the high frequency, $\omega = \pi$.

Give python demo at this point.

The 3-dB cutoff frequency of the filter is the frequency ω_c where the filter halves the input signals' energy spectral density.

Recall, $|Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$

Thus, we need $|H(\omega_c)|^2 = \frac{1}{2}$ or

$$|H(\omega_c)| = \frac{1}{\sqrt{2}}$$

Now, $H(\omega_c) = \frac{1}{2}(1 + e^{-j\omega_c})$

$$= \left(\frac{1}{2} + \frac{1}{2}\cos\omega_c\right) - j\,\frac{1}{2}\sin\omega_c$$

Thus, $|H(\omega_c)|^2 = \frac{1}{4}(1+\cos\omega_c)^2 + \frac{1}{4}\sin^2\omega_c$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{2}\cos\omega_c$$

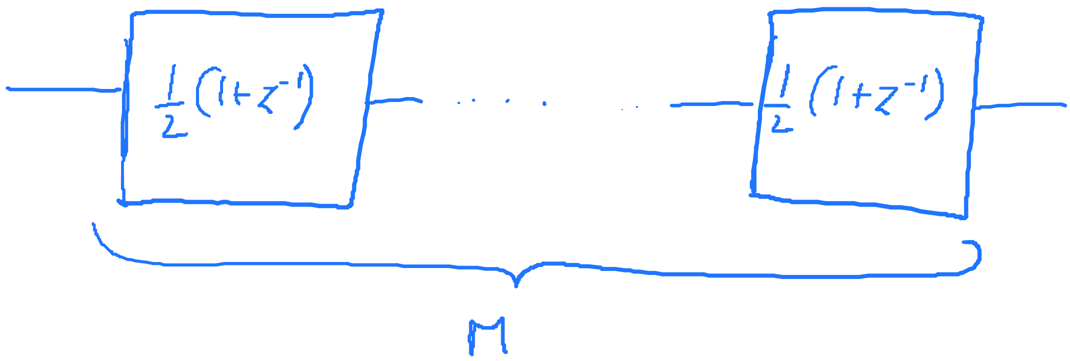
$$= \frac{1}{2}(1+\cos\omega_c)$$

Thus $\cos\omega_c = 0$, $\Rightarrow$ $\boxed{\omega_c = \frac{\pi}{2}}$

Issues :- • Pass band has non-constant gain.

• Cut off frequency cannot be controlled.

• Cascade of FIR low pass filter.



Thus, $H(z) = \frac{1}{2^M}(1+z^{-1})^M$

Again, $H(e^{j0}) = \frac{1}{2^M}(1+1)^M = 1$

$H(e^{j\pi}) = \frac{1}{2^M}(1-1)^M = 0.$

Show magnitude response in python.

For the cutoff frequency, note that

$$|H(\omega_c)|^2 = \frac{1}{2^{2M}} \left(|1+e^{-j\omega_c}|^2 \right)^M$$

$$= \frac{1}{2^{2M}} \left(2(1+\cos\omega_c) \right)^M$$

$$= \left(\frac{1+\cos\omega_c}{2} \right)^M$$

Thus, $(1+\cos\omega_c)^M = 2^{M-1}$

$$\Rightarrow \quad 1+\cos\omega_c = 2^{\frac{M-1}{M}} = 2 \cdot 2^{-1/M}$$

$$\Rightarrow \quad 2\cos^2 \omega_c/2 = 2 \cdot 2^{-1/M}$$

$$\Rightarrow \quad \cos\frac{\omega_c}{2} = 2^{-\frac{1}{2M}}$$

$$\Rightarrow \quad \boxed{\omega_c = 2 \cdot \cos^{-1}\left(2^{-1/2M}\right)}$$

Thus, by controlling M, we can control ω_c.

* Order

The order of a filter is the number of delay elements needed to implement the transfer function.

For IIR filter, order is the order of the difference equation of the filter.

eg. The previous low pass filter has order M.

* FIR High Pass Filter

$$H(z) = \frac{1}{2}(1 - z^{-1})$$

Thus, $H(e^{j0}) = \frac{1}{2}(1-1) = 0$

$$H(e^{j\pi}) = \frac{1}{2}(1+1) = 1$$

To obtain the cutoff frequency,

$$|H(\omega)|^2 = \frac{1}{4}\left|1 - e^{-j\omega}\right|^2$$

$$= \frac{1}{4}\left|1 - \cos\omega + j\sin\omega\right|^2$$

$$= \frac{1}{2}(1 - \cos\omega)$$

Thus, $\frac{1}{2}(1 - \cos\omega_c) = \frac{1}{2}$

$$\Rightarrow \cos\omega_c = 0 \Rightarrow \boxed{\omega_c = \frac{\pi}{2}}$$

Again to control the cutoff, we can cascade M high pass filters

Thus,

$$H(z) = \frac{1}{2^M}(1 - z^{-1})^M.$$

To find ω_c, note that

$$|H(\omega)|^2 = \frac{1}{2^{2M}}\left|1 - \cos\omega + j\sin\omega\right|^{2M}$$

$$= \frac{1}{2^{2M}}(2 - 2\cos\omega)^M$$

$$= \frac{1}{2^M} 2^M \sin^{2M}\frac{\omega}{2} = \sin^{2M}\frac{\omega}{2}$$

Thus, we have,

$$\sin^{2M} \frac{\omega_c}{2} = \frac{1}{2}$$

$$\Rightarrow \boxed{\omega_c = 2 \sin^{-1}\left(2^{-\frac{1}{2M}}\right)}$$

Show python demo

* IIR Low Pass filter & High Pass filter

Idea :- To control the cut off frequency, add a pole at $z = \alpha$, where $0 < \alpha < 1$. Note that $\alpha < 1$ is needed for stability.

$$\boxed{H(z) = \frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-z^{-1}\alpha}}$$

Note :- The added $1-\alpha$ factor is used to ensure that the maximum value of $|H(\omega)|$ is 1.

$$|H(e^{j0})| = \frac{1-\alpha}{2} \cdot \frac{(1+1)}{(1-\alpha)} = 1$$

$$|H(e^{j\pi})| = \frac{1-\alpha}{2} \frac{(1-1)}{(1+\alpha)} = 0$$

Also, to see that $|H(\omega)|$ is indeed maximised at $\omega = 0$, note that

$$|H(\omega)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{|1+e^{-j\omega}|^2}{|1-\alpha e^{-j\omega}|^2}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \cdot \frac{2(1+\cos\omega)}{|1-\alpha\cos\omega + j\alpha\sin\omega|^2}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \frac{2(1+\cos\omega)}{1+\alpha^2 - 2\alpha\cos\omega}$$

Now, $\frac{d}{d\omega} \frac{(1+\cos\omega)}{(1+\alpha^2 - 2\alpha\cos\omega)}$

$$= \frac{-\sin\omega\,(1+\alpha^2 - 2\alpha\cos\omega) - \sin\omega\,(2\alpha(1+\cos\omega))}{(1+\alpha^2 - 2\alpha\cos\omega)^2}$$

$$= \frac{-\sin\omega\,(1+\alpha)^2}{(1+\alpha^2-2\alpha\cos\omega)^2}$$

$$< 0, \qquad 0 < \omega \leq \pi$$

Hence, $|H(\omega)|^2$ is a decreasing function and its maxima is at $\omega = 0$.

Finally, to obtain the cutoff frequency

$$\left(\frac{1-\alpha}{2}\right)^2 \frac{2(1+\cos\omega_c)}{(1+\alpha^2-2\alpha\cos\omega_c)} = \frac{1}{2}$$

$$\Rightarrow (1-\alpha)^2(1+\cos\omega_c) = 1+\alpha^2-2\alpha\cos\omega_c$$

$$\Rightarrow \cos\omega_c\,(1+\alpha^2) = 1+\alpha^2-1-\alpha^2+2\alpha$$

$$\Rightarrow \boxed{\omega_c = \cos^{-1}\frac{2\alpha}{1+\alpha^2}}$$

Increasing α thus decreases ω_c.

Show python demo.

Similarly

$$H(z) = \left(\frac{1-\alpha}{2}\right) \cdot \frac{(1-z^{-1})}{(1+\alpha z^{-1})}$$

is the basic IIR high pass filter.

Observe that

$$H(e^{j0}) = \left(\frac{1-\alpha}{2}\right) \cdot \frac{(1-1)}{1+\alpha} = 0$$

$$H(e^{j\pi}) = \left(\frac{1-\alpha}{2}\right) \cdot \left(\frac{1+1}{1-\alpha}\right) = 1$$

Also, $$|H(\omega)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \cdot \frac{|1-e^{-j\omega}|^2}{|1+\alpha e^{-j\omega}|^2}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \frac{2(1-\cos\omega)}{|1+\alpha\cos\omega - j\alpha\sin\omega|^2}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \cdot \frac{2(1-\cos\omega)}{1+\alpha^2+2\alpha\cos\omega}$$

Thus, to get the cutoff frequency,

$$\left(\frac{1-\alpha}{2}\right)^2 \cdot \frac{2(1-\cos\omega_c)}{(1+\alpha^2+2\alpha\cos\omega_c)} = \frac{1}{2}$$

$$\Rightarrow (1-\alpha)^2 \cdot (1-\cos\omega_c) = 1+\alpha^2+2\alpha\cos\omega_c$$

$$\Rightarrow \cos\omega_c\,(1+\alpha^2) = -2\alpha$$

$$\Rightarrow \omega_c = \cos^{-1}\left(-\frac{2\alpha}{1+\alpha^2}\right)$$

$$\Rightarrow \boxed{\omega_c = \pi - \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)}$$

Show python demo.

* IIR Bandpass Filter

$$\boxed{H(z) = \left(\frac{1-\alpha}{2}\right)\frac{(1-z^{-2})}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \qquad |\alpha|<1,\ |\beta|<1}$$

Notice that there are two zeros at $z = 1, -1$.

Thus, $H(e^{j0}) = H(e^{j\pi}) = 0$

Next,

$$|H(\omega)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{|1-e^{-2j\omega}|^2}{|1-(1+\alpha)\beta e^{-j\omega}+\alpha e^{-2j\omega}|^2}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \frac{\left[(1-\cos 2\omega)^2 + \sin^2 2\omega\right]}{\left(1-(1+\alpha)\beta\cos\omega+\alpha\cos 2\omega\right)^2 + \left((1+\alpha)\beta\sin\omega-\alpha\sin 2\omega\right)^2}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \frac{2(1-\cos 2\omega)}{1+(1+\alpha)^2\beta^2+\alpha^2-2(1+\alpha)\beta\cos\omega + 2\alpha\cos 2\omega - 2\alpha(1+\alpha)\beta\cos\omega\cdot\cos 2\omega - 2\alpha(1+\alpha)\beta\sin\omega\cdot\sin 2\omega}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \frac{2(1-\cos 2\omega)}{\left[1+\alpha^2+(1+\alpha)^2\beta^2-2\beta(1+\alpha)\cos\omega(1+\alpha)+2\alpha\cos 2\omega\right]}$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \frac{2(1-\cos 2\omega)}{\left[1+\alpha^2+(1+\alpha)^2\beta^2 - 2\beta(1+\alpha)^2\cos\omega + 2\alpha\cos 2\omega\right]}$$

Plugging in $\omega = \cos^{-1}\beta$, ie. $\cos 2\omega = 2\beta^2 - 1$, we have

$$|H(\cos^{-1}\beta)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \cdot \frac{2(1-2\beta^2+1)}{\left[1+\alpha^2+(1+\alpha)^2\beta^2 - 2\beta^2(1+\alpha)^2 + 4\alpha\beta^2 - 2\alpha\right]}$$

$$= \frac{(1-\alpha)^2 \cdot (1-\beta^2)}{(1-\alpha)^2 - \beta^2(1+\alpha)^2 + 4\alpha\cdot\beta^2}$$

$$= \frac{(1-\alpha)^2(1-\beta^2)}{(1-\alpha)^2 - \beta^2(1-\alpha)^2} = 1$$

It can be shown that the maximum value of $|H(\omega)|^2$ is 1.

Thus, $|H(\omega)|^2$ reaches its maximum at $\boxed{\omega_0 = \cos^{-1}\beta}$ also referred to as the central frequency.

Show python demo.

The 3dB bandwidth of the filter is then given by

$$\boxed{\omega_{c_2} - \omega_{c_1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)}$$

* IIR Bandstop / Notch Filter

$$\boxed{H(z) = \left(\frac{1+\alpha}{2}\right) \cdot \frac{(1-2\beta z^{-1} + z^{-2})}{1-\beta(1+\alpha)z^{-1} + \alpha z^{-2}}, \quad |\alpha|<1, \quad |\beta|<1}$$

Observe here that,

$$H(e^{j0}) = \left(\frac{1+\alpha}{2}\right) \frac{(1-2\beta+1)}{1-\beta(1+\alpha)+\alpha}$$

$$= \left(\frac{1+\alpha}{2}\right) \cdot \frac{2(1-\beta)}{(1-\beta)(1+\alpha)} = 1$$

$$H(e^{j\pi}) = \left(\frac{1+\alpha}{2}\right) \cdot \frac{(1+2\beta+1)}{1+\beta(1+\alpha)+\alpha}$$

$$= \left(\frac{1+\alpha}{2}\right) \cdot \frac{2(1+\beta)}{(1+\alpha)(1+\beta)} = 1$$

Again it can be shown that

$|H(\omega)|^2$ takes a maximum value of 1,

and thus $\omega = 0, \pi$ form the peaks.

Next, consider the numerator

$N(z) = 1 - 2\beta z^{-1} + z^{-2}$.

Note that $|H(\omega)|^2 = 0$, whenever

$|N(\omega)|^2 = 0$.

Now, $|N(\omega)|^2 = \left|1 - 2\beta e^{-j\omega} + e^{-2j\omega}\right|^2$

$$= (1 - 2\beta\cos\omega + \cos 2\omega)^2$$
$$+ (2\beta\sin\omega - \sin 2\omega)^2$$

$$= 1 + 4\beta^2 + 1 + 2\cos 2\omega - 4\beta\cos\omega$$
$$- 4\beta\cos\omega \cdot \cos 2\omega - 4\beta\sin\omega\sin 2\omega$$

$$= 2 + 4\beta^2 + 2\cos 2\omega - 8\beta\cos\omega$$

$$= 4\cos^2\omega - 8\beta\cos\omega + 4\beta^2$$

$$= 4(\cos\omega - \beta)^2$$

Thus, $|H(\omega)|^2 = |N(\omega)|^2 = 0$

at $\boxed{\omega_0 = \cos^{-1}\beta}$ → Notch frequency

Show python demo.

Again, it can be shown that the 3-dB bandwidth is given by

$$\boxed{\omega_{c_2} - \omega_{c_1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)}$$

* Filters with multiple pass and stop bands / Comb filters

Any filter with an $H(\omega)$ periodic with period $\frac{2\pi}{L}$, $L > 1$ is a comb filter.

Let $G(z)$ be any filter.

Then $H(z) = G(z^L)$ is its associated comb filter, with $H(\omega)$ having a period of $\frac{2\pi}{L}$.

This is because,

$$
\begin{aligned}
H\left(\omega + \frac{2\pi}{L}\right) &= G(L\omega + 2\pi) \\
&= G(L\omega) \quad [\text{DTFT is periodic with the period } 2\pi] \\
&= H(\omega)
\end{aligned}
$$

Also, $G(\omega) = H\left(\frac{\omega}{L}\right), \; -\pi \le \omega \le \pi$

Thus, if $G(\omega)$ is a filter with a peak at ω_0

Then $H(\omega)$ has L peaks with at

$$\boxed{\frac{\omega_0}{L} + \frac{2\pi k}{L}, \quad 0 \le k \le L-1}$$

Similarly, if $G(\omega)$ has a zero at ω_0, then

$H(\omega)$ has zeros at $\frac{\omega_0}{L} + \frac{2\pi k}{L}$,

$0 \leq k \leq L-1$

Note :- $\frac{\omega_0}{L} + \frac{2\pi k}{L}$ may be higher than π. In that case, peaks/zeros appear at corresponding frequencies in $[-\pi, \pi]$.

* All Pass Filter

$$\boxed{H(z) = \frac{d_M + d_{M-1} z^{-1} + \dots + d_1 z^{-(M-1)} + z^{-M}}{1 + d_1 z^{-1} + \dots + d_{M-1} z^{-(M-1)} + d_M z^{-M}}}$$

Catch :- Poles should be within the unit circle.

The numerator is a 'mirror image' of the denominator.

Let $D_M(z) = 1 + d_1 z^{-1} + \cdots + d_M z^{-M}$

$$= z^{-M}\left(d_M + d_{M-1} z + \cdots + d_1 z^{M-1} + z^M\right)$$

Thus, $$\boxed{H(z) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)}}$$

Now, $$|H(\omega)|^2 = H(\omega) \cdot H^*(\omega)$$

$$= \frac{e^{-j\omega M} \cdot D_M(e^{-j\omega})}{D_M(e^{j\omega})} \cdot \frac{e^{j\omega M} D_M(e^{j\omega})}{D_M(e^{-j\omega})}$$

$$= \frac{|D_M(e^{j\omega})|^2}{|D_M(e^{j\omega})|^2}$$

$$= 1$$

Thus $$\boxed{|H(\omega)|^2 = 1, \ \forall \omega}$$

This why we call it all pass.

Question :- Why use this filter?

Answer :- This is used in cascade with other filter to correct the phase of various frequency components.

Show demo including phase response.

Now, recall that group delay is defined as

$$\tau_g = -\frac{d}{d\omega}(\Theta_u(\omega)), \text{ where}$$

$\Theta_u(\omega)$ is the unwrapped phase.

Then, total phase lag between from $\omega=0$ to $\omega=\pi$,

$$\int_0^{\pi} \tau_g(\omega)\cdot d\omega = -\int_0^{\pi} d(\Theta_u(\omega))$$

$$= -\Theta_u(e^{j\pi}) + \Theta_u(e^{j0})$$

Now, $H(e^{j0}) = \dfrac{d_M + d_{M-1} + \cdots + d_1 + 1}{1 + d_1 + d_2 + \cdots + d_{M-1} + d_M} = 1$

Thus, $\Theta_u(e^{j0}) = 0$

On the other hand,

$$H(e^{j\pi}) = e^{-j\pi M} \cdot \frac{D_M(e^{-j\pi})}{D_M(e^{j\pi})} = e^{-j\pi M}$$

Thus, $\Theta_u(e^{j\pi}) = -M\pi$

Thus, total phase lag,

$$\boxed{\int_0^{\pi} \tau_g(\omega) \cdot d\omega = M\pi}$$

Note :- Notice the effect of unwrapping. If we mistakenly wrapped the phase, the answer would have been 0 if M was even, and π if M was odd.

Finally, we shall show that $\tau_g(\omega)$ is always positive for an all pass filter.

Firstly, note that if z_1 is a root of $D_M(z)$, then z_1^{-1} is a root of $D_M(z^{-1})z^{-M}$.

Hence, assuming $D_M(z)$ has $2N$ complex-conjugate zeros and $M-2N$ real zeros, we can rewrite the all pass transfer function in the form

$$\boxed{H(z) = \pm \prod_{i=1}^{N} \frac{(z^{-1}-\beta_i)(z^{-1}-\beta_i^*)}{(1-z^{-1}\beta_i)(1-z^{-1}\beta_i^*)} \prod_{l=1}^{M-2N} \frac{(z^{-1}-\alpha_l)}{(1-z^{-1}\alpha_l)}}$$

where $\alpha_1, \ldots, \alpha_{M-2N}$ are the real roots of $D_M(z)$ and $(\beta_1, \beta_1^*), (\beta_2, \beta_2^*), \ldots, (\beta_N, \beta_N^*)$ are the $2N$ complex-conjugate zeros of $D_M(z)$.

Next, consider the term of the form

$$\left. \frac{z^{-1}-\beta_i^*}{(1-z^{-1}\beta_i)} \right|_{z=j\omega} = \frac{e^{-j\omega} - re^{-j\theta}}{1-e^{-j\omega}re^{j\theta}}$$

[where we have expressed $\beta_i = re^{j\theta}$]

$$= \frac{e^{-j\omega}(1 - re^{j(\omega-\theta)})}{(1 - re^{-j(\omega-\theta)})}$$

Then $\angle \dfrac{e^{-j\omega}(1-re^{j(\omega-\theta)})}{(1-re^{-j(\omega-\theta)})}$

$$= -\omega + \tan^{-1}\frac{-r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}$$

$$- \tan^{-1}\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}$$

$$= -\omega - 2\tan^{-1}\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}$$

$$-\frac{d}{d\omega}\angle\frac{e^{-j\omega}(1-re^{j(\omega-\theta)})}{(1-re^{-j(\omega-\theta)})}$$

$$= 1 + \frac{2(1-r\cos(\omega-\theta))^2}{\left(\left(1-r\cos(\omega-\theta)\right)^2 + r^2\sin^2(\omega-\theta)\right]}\cdot$$

$$\frac{(1-r\cos(\omega-\theta))r\cos(\omega-\theta) - r^2\sin^2(\omega-\theta)}{(1-r\cos(\omega-\theta))^2}$$

$$= 1 + \frac{2r\cos(\omega-\theta) - 2r^2}{1+r^2-2r\cos(\omega-\theta)}$$

$$= \frac{1-r^2}{1+r^2-2r\cos(\omega-\theta)}.$$

> 0 [$\because$ $|r|<1$, as we are considering only stable systems]

Next, note that $H(\omega)$ consist of products of terms of the form

$$\frac{e^{-j\omega} - re^{-j\theta}}{1-e^{-j\omega}re^{j\theta}}.$$

(Basically we take the complex conjugates ones in cross i.e, $\frac{z^{-1}-\beta_i^*}{1-z^{-1}\beta_i}$, whereas, for the real roots $\alpha_i^* = \alpha_i$ (i.e, $\theta=0$), so $\frac{z^{-1}-\alpha_i}{1-z^{-1}\alpha_i}$ suffices]

Thus $\angle H(\omega) = \sum \angle \dfrac{e^{-j\omega} - re^{-j\theta}}{1 - e^{-j\omega} re^{j\theta}}$

Thus, $\tau_g(\omega) = -\dfrac{d}{d\omega} \angle H(\omega)$

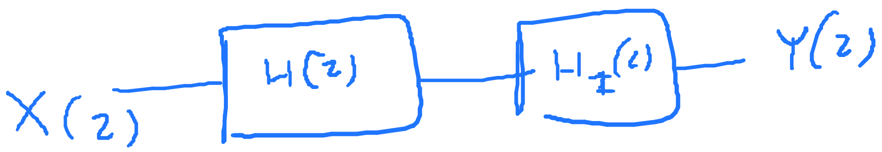
$$= \sum -\frac{d}{d\omega} \angle \frac{e^{-j\omega} - re^{-j\theta}}{1 - e^{-j\omega} re^{j\theta}}$$

> 0 [$\because$ Each term is positive]

* Inverse system / Equaliser

For any system $H(z)$, the system $H_I(z) = \dfrac{1}{H(z)}$ is called its inverse.

This is because



Then $Y(z) = X(z)$

Note that if $H_I(z)$ and $H(z)$ both are stable, then both zeros and poles of $H(z)$ must be within the unit circle.

* Minimum Phase, Max Phase, Min Phase system

Stable systems with stable inverse

$\equiv$ Systems with poles and zeros within unit circle

$\triangleq$ Min Phase system.

These systems are called min-phase since it can be shown that they exhibit the least delay among all systems with same gain response.

On the other hand, systems with all poles inside the unit circle and all zeros outside the unit circle (eg. stable all pass filters) are called maximum phase system.

Again it can be shown that these are the systems with the highest phase lag among all systems with same magnitude response.

Let us take an example:-

$$H_1(z) = \frac{1+bz^{-1}}{1+az^{-1}}, \quad H_2(z) = \frac{b+z^{-1}}{1+az^{-1}}, \quad |a|<1, \ |b|<1$$

$H_1(z) \downarrow$ Min Phase, $H_2(z) \downarrow$ Max Phase

Note that $|1+be^{-j\omega}|^2 = 1+b^2+2b\cos\omega$

$$|b+e^{-j\omega}|^2 = b^2+1+2b\cos\omega$$

Thus, $|H_1(\omega)| = |H_2(\omega)|$.

But let us look at their phases plotted.

Show python.

Finally, stable systems with some of its zeros in the unit circle, and some outside the unit circle are called mixed phase systems.

Any mixed phase system can be written as the product of a min-phase system and an all pass system.

To see this, consider the mixed phase system with

N_1 real zeros outside the unit circle

$\alpha_1, \dots, \alpha_{N_1}$

N_2 real zeros inside the unit circle

$\beta_1, \dots, \beta_{N_2}$

$2M_1$ complex-conjugate zeros outside the unit circle, $(\gamma_1, \gamma_1^*), (\gamma_2, \gamma_2^*), \ldots, (\gamma_{M_1}, \gamma_{M_1}^*)$

$2M_2$ complex-conjugate zeros inside the unit circle, $(\delta_1, \delta_1^*), (\delta_2, \delta_2^*), \ldots, (\delta_{M_2}, \delta_{M_2}^*)$.

Then,
$$H_{mixed}(z) = \frac{\prod_{i=1}^{N_1}(1-z^{-1}\alpha_i)\prod_{i=1}^{N_2}(1-z^{-1}\beta_i)\prod_{i=1}^{M_1}(1-z^{-1}\gamma_i)(1-z^{-1}\gamma_i^*)\prod_{i=1}^{M_2}(1-z^{-1}\delta_i)(1-z^{-1}\delta_i^*)}{A(z)}$$

where $A(z)$ has roots inside the unit circle to ensure stability.

Then,
$$H_{mixed}(z) = \left(\frac{\prod_{i=1}^{N_2}(1-z_i^{-1}\beta_i)\prod_{i=1}^{M_2}(1-z_i^{-1}\delta_i)(1-z_i^{-1}\delta_i^*)\prod_{i=1}^{N_1}(1-z^{-1}\alpha_i^{-1})\prod_{i=1}^{M_1}(1-z^{-1}\gamma_i^{-1})(1-z^{-1}\gamma_i^{*-1})}{A(z)}\right.$$

$$\left.\prod_{i=1}^{N_1}(-\alpha_i)\prod_{i=1}^{M_1}|\gamma_i|^2\right) \cdot \left(\frac{\prod_{i=1}^{N_1}(z^{-1}-\alpha_i^{-1}) \cdot \prod_{i=1}^{M_1}(z^{-1}-\gamma_i^{-1})(z^{-1}-\gamma_i^{*-1})}{\prod_{i=1}^{N_1}(1-z^{-1}\alpha_i^{-1})\prod_{i=1}^{M_1}(1-z^{-1}\gamma_i^{-1})(1-z^{-1}\gamma_i^{*-1})}\right)$$

$$= H_{min}(z) \cdot H_{ap}(z)$$

* Linear Phase system

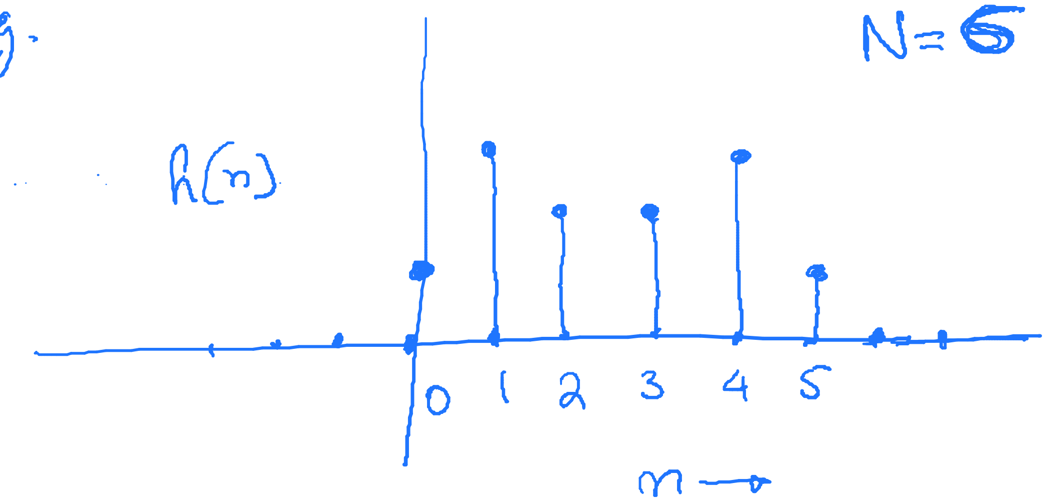
These are systems with linear phase response.
i.e., constant group delay.

$$h[n] = h[N-n-1], \quad n = 0, 1, \ldots, \frac{N-1}{2} \quad \text{if } N \text{ odd}$$

$$n = 0, 1, \ldots, \frac{N}{2} - 1, \quad \text{if } N \text{ even}$$

$$= 0 \quad , \text{ o.w.}$$

eg. $N = 6$



To see that the system has linear phase, observe

$$H(\omega) = \sum_{n=0}^{N-1} h[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-1}{2}-1} h[n]\left(e^{-j\omega n} + e^{-j\omega N + j\omega n + j\omega}\right)$$

$$+ h\left[\frac{N-1}{2}\right] e^{-j\omega\left(\frac{N-1}{2}\right)}, \quad N \text{ odd}$$

$$\sum_{n=0}^{N/2-1} h[n]\left(e^{-j\omega n} + e^{-j\omega N + j\omega n + j\omega}\right) \quad N, \text{ even}$$

$$= \sum_{n=0}^{\frac{N-1}{2}-1} h[n] e^{-j\omega\left(\frac{N-1}{2}\right)}\left(e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} + e^{j\omega\left(\frac{N-1}{2}\right) - j\omega n}\right)$$

$$+ h\left[\frac{N-1}{2}\right] e^{-j\omega\left(\frac{N-1}{2}\right)}, \quad N \text{ odd}$$

$$\sum_{n=0}^{\frac{N}{2}-1} h[n] e^{-j\omega\left(\frac{N-1}{2}\right)}\left(e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} + e^{j\omega\left(\frac{N-1}{2}\right) - j\omega n}\right)$$

N even

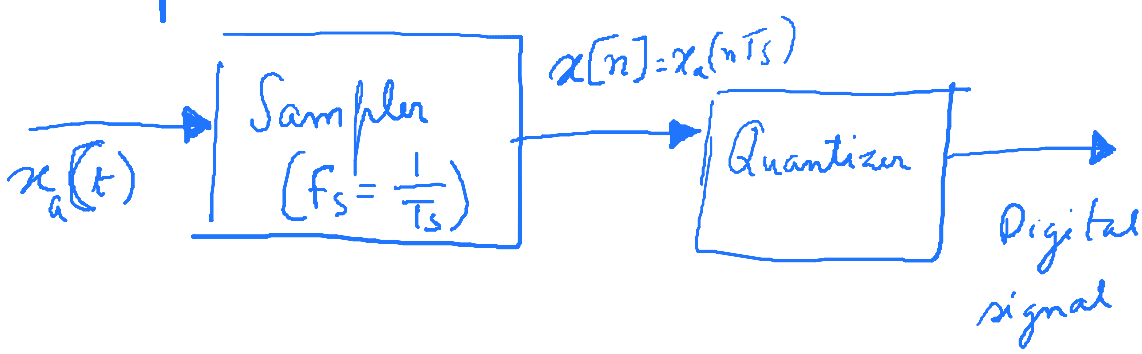
$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\underbrace{\left[\sum_{n=0}^{\frac{N-1}{2}-1} 2h[n]\cos\left(\left(n-\left(\frac{N-1}{2}\right)\right)\omega\right) + h\left[\frac{N-1}{2}\right]\right]}_{A} \quad N \text{ odd}$$

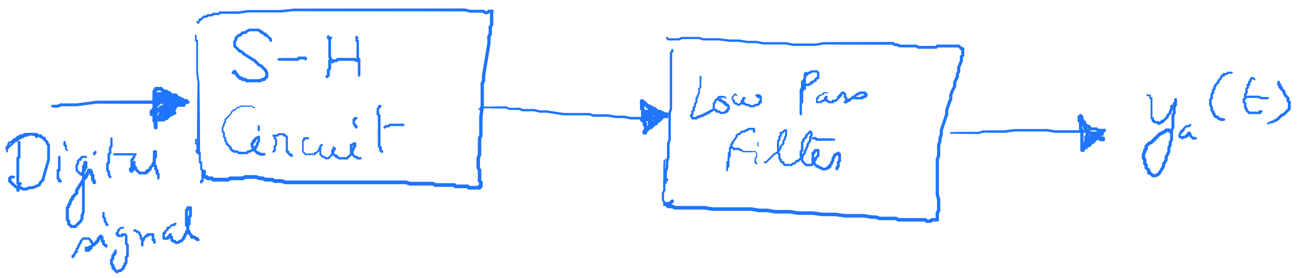
$$e^{-j\omega\left(\frac{N-1}{2}\right)}\underbrace{\left[\sum_{n=0}^{\frac{N}{2}-1} 2h[n]\cos\left(\omega\left(n-\left(\frac{N-1}{2}\right)\right)\right)\right]}_{A} \quad N \text{ even}$$

Thus,

$$\boxed{\begin{aligned} \Theta(\omega) &= -\,\omega\left(\frac{N-1}{2}\right) - \pi\,\mathbb{1}\{A<0\} \\ \tau_g(\omega) &= \left(\frac{N-1}{2}\right) \end{aligned}}$$

* Recap:-

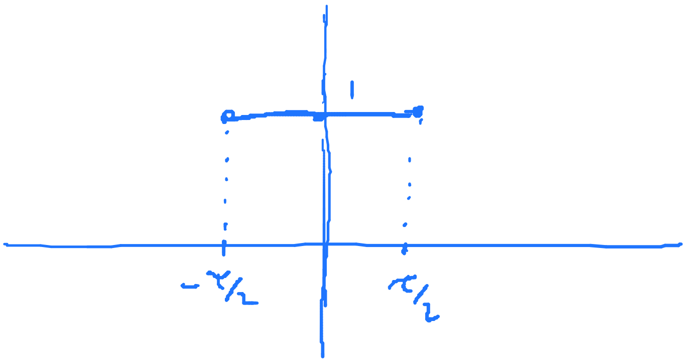




* Recap :- An important result from continuous time Fourier Transform (CTFT)

Define $rect\left(\frac{t}{\tau}\right) = 1, \quad |t| \le \tau/2$

$= 0, \quad o.w.$



Then, $\mathcal{F}\left[rect\left(\frac{t}{\tau}\right)\right] = \int_{-\infty}^{\infty} rect\left(\frac{t}{\tau}\right) e^{-j2\pi f t}.dt$

$$= \int_{-\tau/2}^{\tau/2} e^{-j2\pi f t}.dt$$

$$= \frac{-2j \sin(2\pi f \tau/2)}{-j2\pi f}$$

$$= \tau \, Sinc(\pi f \tau).$$

Then, by the duality of CTFT, we have

$$\boxed{\mathcal{F}\left[F \operatorname{Sinc}(\pi t F)\right] = \operatorname{rect}\left(\frac{f}{F}\right)}$$

* Relationship between CTFT and DTFT

Consider the discrete time signal $x[n]$ and its 'continuous time variant'

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t - nT_s),$$

i.e. we represent the discrete time signal as a sum of scaled continuous time impulses.

Now

$$X(F) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi F n T_s}$$

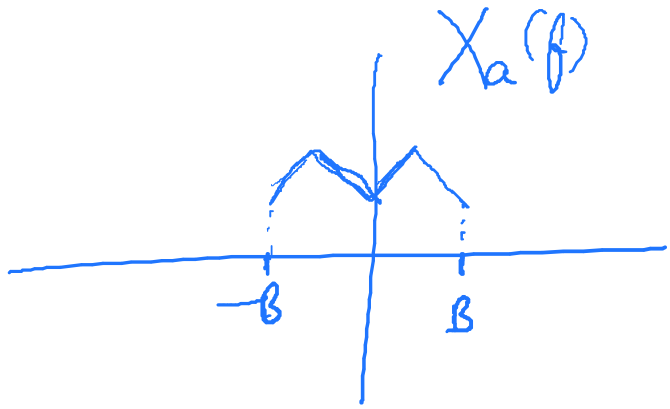
$$\Rightarrow \boxed{X(F) = X_{DTFT}(FT_s)}$$

We shall use F to represent CTFT frequencies, f to represent DTFT frequencies.

$$\boxed{f = \frac{F}{F_s}}$$

* Sampling of Bandlimited signal

Input :- Bandlimited continuous time signal $x_a(t)$ with maximum frequency B.



Goal :- Find a sampling rate $F_s = \frac{1}{T_s}$, such that $x_a(t)$ can be completely recovered from the samples.

The sample signal $x[n] = x_a(nT_s)$

Now, $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{DTFT}(\omega) \cdot e^{j\omega n} d\omega$$

$$= \int_{-1/2}^{1/2} X_{DTFT}(f) e^{j2\pi f n} df$$

[Note :- We do not write
$X_{DTFT}(2\pi f)$ and simply $X_{DTFT}(f)$
because $X_{DTFT}(f)$ is itself
obtained by replacing ω by $2\pi f$ in
$X_{DTFT}(\omega)$]

On the other hand,

$$x_a(nT_s) = \int_{-\infty}^{\infty} X_a(F') \cdot e^{j2\pi F' n T_s} \cdot dF'$$

$$= \sum_{k=-\infty}^{\infty} \int_{(k-\frac{1}{2})F_s}^{(k+\frac{1}{2})F_s} X_a(F') \cdot e^{j2\pi F' n T_s} \cdot dF'$$

Substituting $F = F' - \ell F_s$ in the ℓth term,

we get

$$x_a(nT_s) = \sum_{k=-\infty}^{\infty} \int_{-F_s/2}^{F_s/2} X_a(F + kF_s)\, e^{j2\pi n (F+kF_s)\cdot T_s} \cdot dF$$

$$= \sum_{k=-\infty}^{\infty} \int_{-F_s/2}^{F_s/2} X_a(F + kF_s)\, e^{j2\pi n \frac{F}{F_s}} \cdot dF$$

$$\left[\because e^{j2\pi n k F_s T_s} = e^{j2\pi n \ell} = 1\right]$$

$$= \sum_{k=-\infty}^{\infty} \int_{-F_s/2}^{F_s/2} X_a(F - kF_s)\, e^{j2\pi n F/F_s}\, dF$$

[Substituting $k = -\ell$]

$$= \int_{-F_s/2}^{F_s/2} \sum_{k=-\infty}^{\infty} X_a(F - kF_s)\, e^{j2\pi n F/F_s} \cdot dF$$

$$= \int_{-1/2}^{1/2} F_s \sum_{k=-\infty}^{\infty} X_a(f F_s - kF_s)\, e^{j2\pi n f} \cdot df$$

Since $x_a(nT_s) = x(n)$, we must have

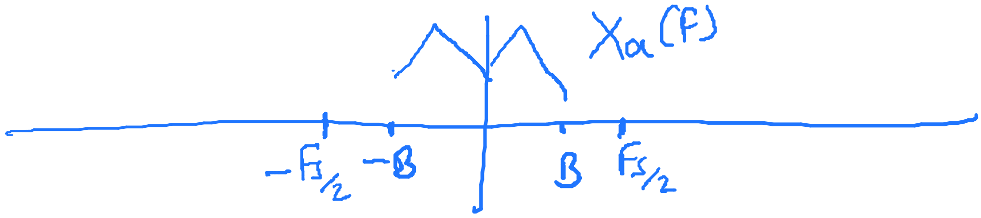
$$X_{DTFT}(f) = F_s \sum_{k=-\infty}^{\infty} X_a(fF_s - kF_s)$$

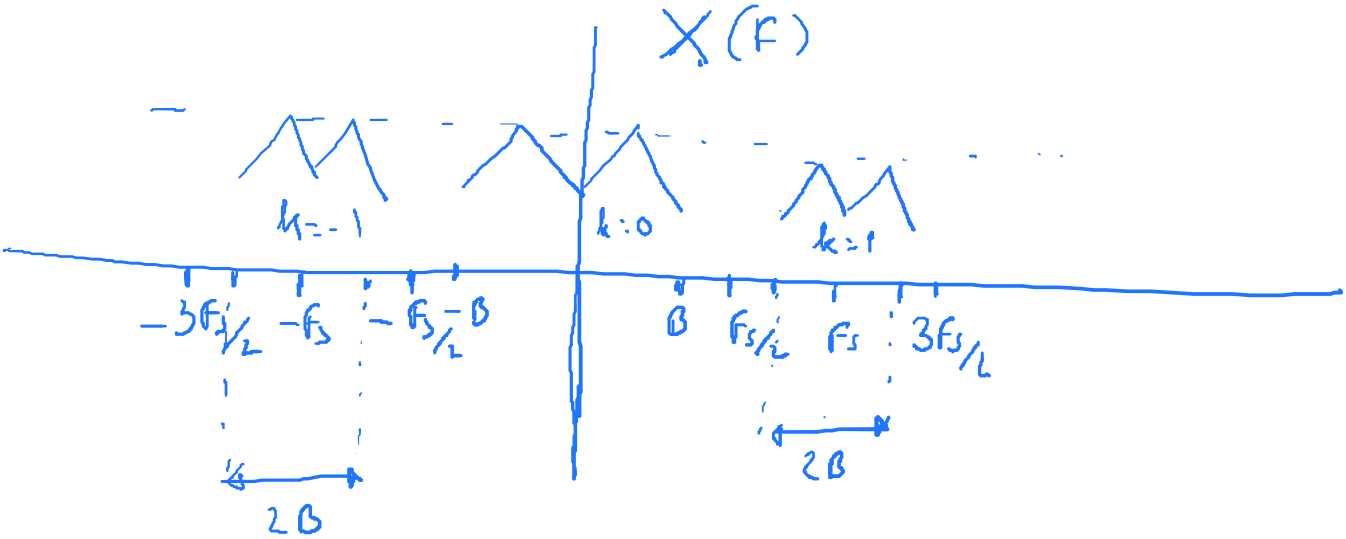
$$\Rightarrow \quad X(fF_s) = F_s \sum_{k=-\infty}^{\infty} X_a(fF_s - kF_s)$$

$$\Rightarrow \quad \boxed{X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s)}$$

Now assume $F_s \geq 2B$

Then, let us draw $X(F)$

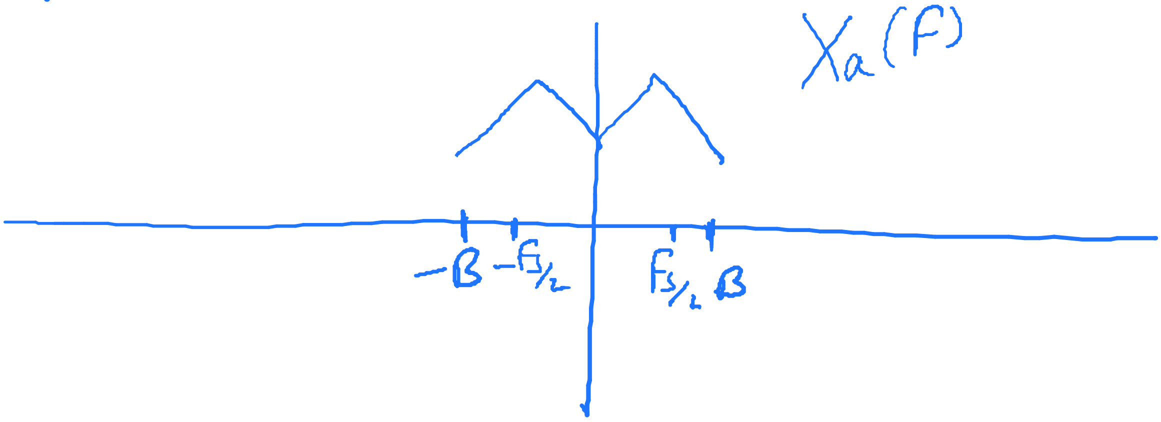


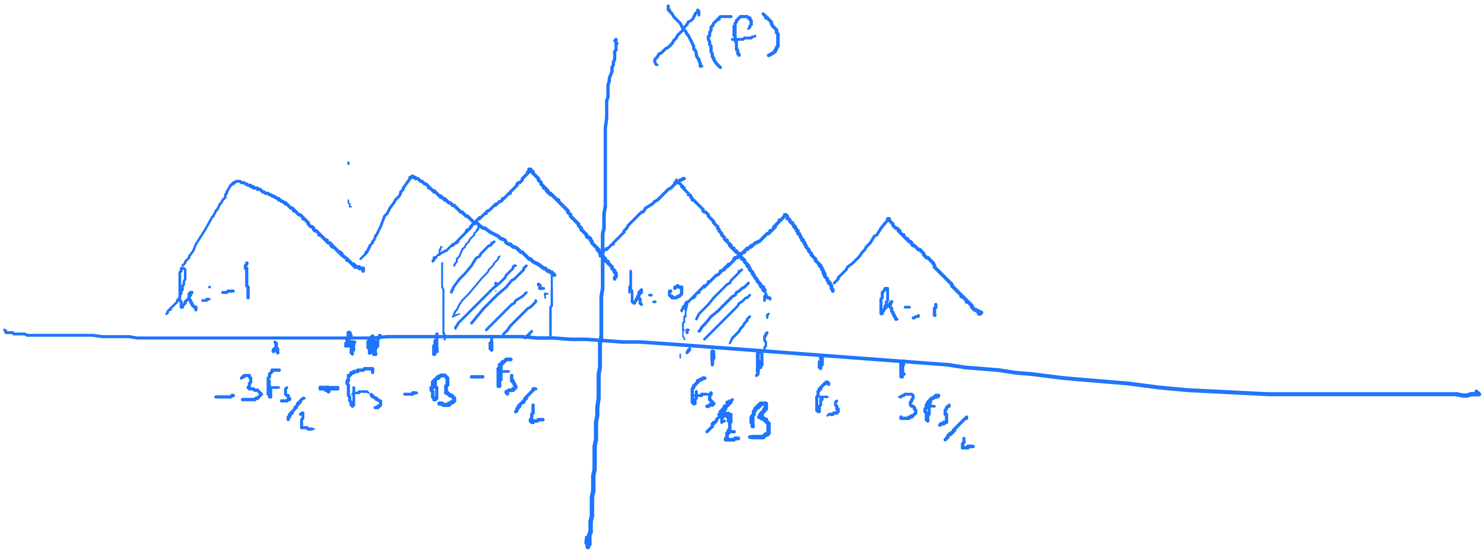


In other words,

$$X(F) = F_s X_a(F) \quad , \quad F \in \left[-\frac{F_s}{2}, \frac{F_s}{2}\right]$$

If however $F_s < 2B$, then





Thus, we see the part of $X(F)$ due to $k=0$, is now corrupted by parts due to $k=1$, and $k=-1$.

This phenomenon is known as aliasing.

In this case, the perfect recovery of $x_a(t)$ is impossible.

Note:- In reality continuous time signals are time limited, and hence they have high frequency components. Thus, to ensure negligible aliasing, a continuous time low pass filter (called the anti-aliasing filter) is applied before sampling.

Next we turn to signal recovery.

Recall that if $F_s \geq 2B$, then

$$X(F) = F_s X_a(F), \text{ for } F \in \left[-\frac{F_s}{2}, \frac{F_s}{2}\right]$$

Thus, $\frac{X(F)}{F_s} \cdot \mathrm{rect}\left(\frac{F}{F_s}\right) = X_a(F)$.

In other words, $x_a(t) = x_{cont}(t) * \mathrm{Sinc}(\pi t F_s)$,

where $x_{cont}(t)$ is the 'continuous variant' of $x[n]$.

The only issue is we only have $x[n]$.
and not $x_{cont}(t)$, but this doesn't matter
as we see below.

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t} \cdot dF$$

$$= \int_{-F_s/2}^{F_s/2} \frac{X(F)}{F_s} e^{j2\pi F t} \cdot dF \quad \left[\because X(F) = F_s \cdot X_a(F)\right.$$

for $F \in \left(-\frac{F_s}{2}, \frac{F_s}{2}\right]$

and

$X_a(F) = 0$, if

$F \notin \left[-\frac{F_s}{2}, \frac{F_s}{2}\right]$

$\left. \text{if } F_s \geq 2B \right]$

$$= \int_{-F_s/2}^{F_s/2} \frac{X_{DTFT}(F/F_s)}{F_s} e^{j2\pi F t} \cdot dF$$

$$= \int_{-F_s/2}^{F_s/2} \frac{1}{F_s} \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi \frac{F}{F_s} n} e^{j2\pi F t} \cdot dt$$

$$= \frac{1}{F_s} \sum_{n=-\infty}^{\infty} x[n] \int_{-F_s/2}^{F_s/2} e^{j2\pi F(t - nT_s)} \cdot dt$$

$$= \frac{1}{F_s} \sum_{n=-\infty}^{\infty} \frac{x[n]}{j2\pi(t-nT_s)} \sin\left(2\pi \frac{F_s}{2}(t-nT_s)\right) j2$$

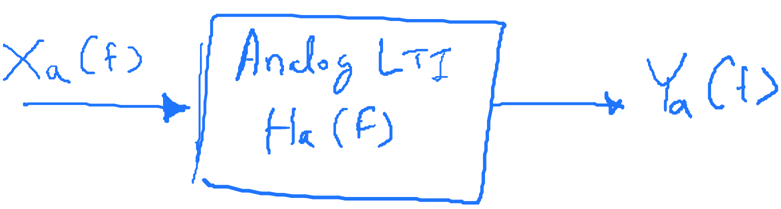
$$= \sum_{n=-\infty}^{\infty} x[n] \operatorname{Sinc}\left(\pi F_s (t-nT_s)\right)$$

i.e. $$\boxed{x_a(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \operatorname{Sinc}\left(\pi F_s (t-nT_s)\right)}$$

The function $\operatorname{Sinc}(\pi F_s t)$ is called the interpolating function.

* Discrete time processing of band limited signals

versus



We assume that both $X_a(F)$ is bandlimited to B.

Question:- Can we design a discrete time system $H(F)$ such that $\hat{Y}_a(t) = Y_a(t)$?

Note:-

not LTI

To proceed, choose $F_s = 2B$.

Then, let $X(F) = \mathcal{F}[x_{cont}(t)]$, and

$$X(F) = F_s X_a(F), \quad \forall F \in [-B, B],$$

as $F_s = 2B$

The output of the filter is thus

$X_{DTFT}(f) \cdot H_{DTFT}(f)$, since the filter is still LTI.

The D/A converter is assumed to be an interpolation by $\operatorname{Sinc}(\pi F_s t)$ of the incoming discrete signal.

Call the incoming discrete signal as $y(n)$.

Thus,
$$\hat{y}_a(t) = \sum_{n=-\infty}^{\infty} y[n] \cdot \operatorname{Sinc}\left(\pi F_s (t - nT_s)\right)$$

Thus,
$$Y_a(F) = \frac{1}{F_s} \sum_{n=-\infty}^{\infty} y[n] \cdot \operatorname{rect}\left(\frac{F}{F_s}\right) e^{-j2\pi n T_s F}$$

(By shifting property of CTFT)

$$= \frac{1}{F_s} Y_{DTFT}(F T_s) \operatorname{rect}\left(\frac{F}{F_s}\right)$$

$$= \frac{1}{F_s} X_{DTFT}(F T_s)\, H_{DTFT}(F T_s) \cdot \operatorname{rect}\left(\frac{F}{F_s}\right)$$

$$= \frac{1}{F_s} X(F)\, H(F) \cdot \operatorname{rect}\left(\frac{F}{F_s}\right)$$

Now, we design the DSP filter $H_{DTFT}(f)$ as follows,

$$\boxed{H_{DTFT}(f) = H(F) = H_a(F) = H_a(f F_s), \quad -\tfrac{1}{2} \le f \le \tfrac{1}{2}}$$

Then, noting that $X(F)\,\mathrm{rect}\left(\frac{f}{F_s}\right) = F_s X_a(F)$

since $X_a(F)$ is bandlimited to $B \le \frac{F_s}{2}$,

we have,

$$\hat{Y}_a(F) = X_a(F) \cdot H_a(F) = Y_a(F).$$

* Sampling Band Pass Signals

- Integer Band Positioning

Consider a bandpass signal $x_a(t)$, with its passband being from F_L to F_H, and $B = F_H - F_L$

One can always sample the signal at $2F_H$.

But this is not needed and $2B$ suffices.

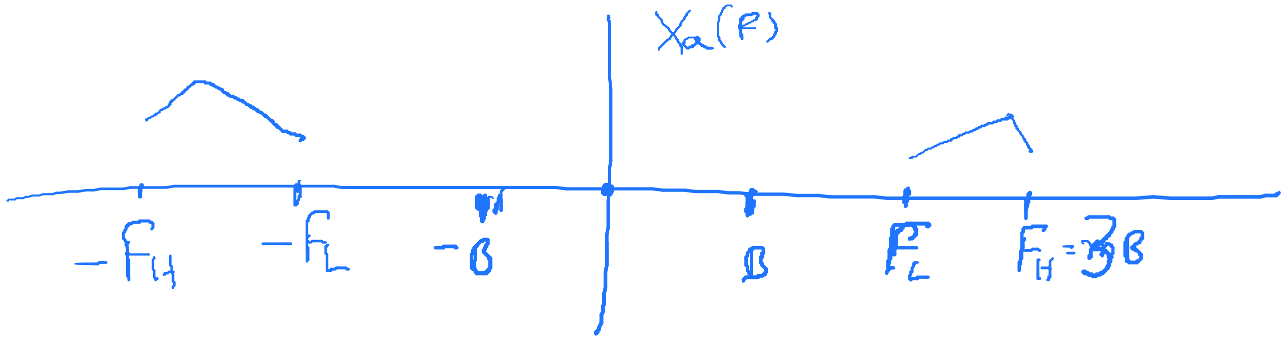
In the integer band positioning scheme, we assume that $F_H = mB$, for some $m \in \mathbb{N}$.

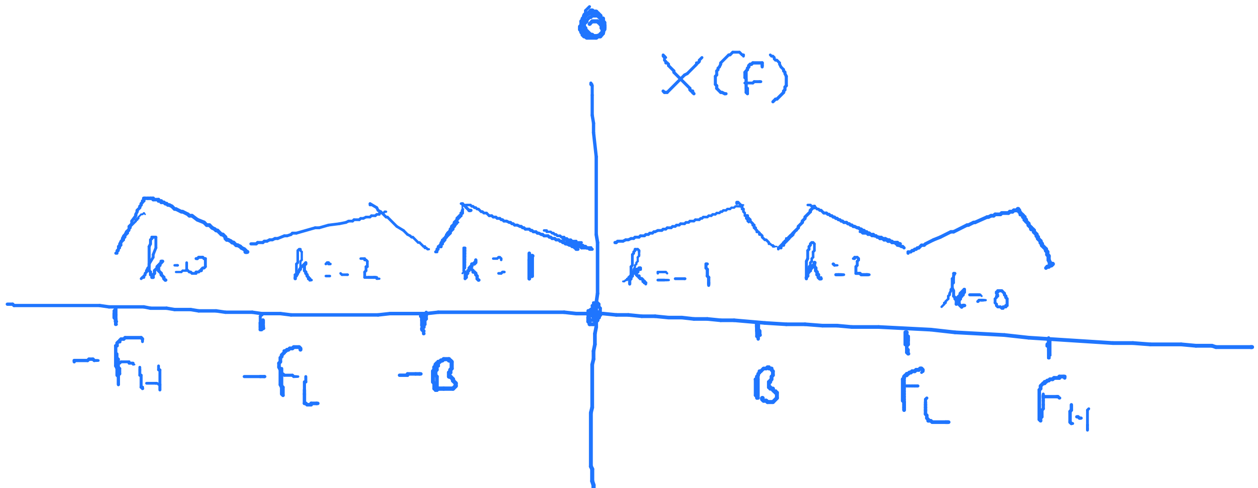
We sample at $F_s = 2B$.

As shown earlier, we have

$$X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

Let us now draw $X_a(F)$ and $X(F)$





Notice that there is no overlap.

More formally, consider the band between

$F_L = (m-1)B$ and $F_H = mB$.

For each k, the entire spectrum shift left or right by $F_s = 2B$.

Then, for $k = (m-1)$, we see that the negative part stretches from

$$-F_H + (m-1)F_s = -mB + 2(m-1)B =$$
$$= mB - 2B = F_L - B$$

to $-F_L + (m-1)F_s = F_L - B + B = F_L$

For $k = m$, the negative part now stretches from

$$-F_H + mF_s = -mB + 2mB = mB = F_H$$

to $-F_L + mF_s = F_H + B$.

Thus spectra from F_L to F_H is left unaltered.

Similarly, one can show that the spectra from $-F_H$ to $-F_L$ is also unaltered.

Next, again we can use the previous intuition for reconstruction.

Note that if $F_s = 2B$, then

$$X_a(F) = \frac{1}{F_s} \cdot X(F), \quad F \in [F_L, F_H] \cup [-F_H, -F_L]$$

$$= 0, \quad \text{o.w}$$

Then,

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t} \cdot dF$$

$$= \int_{-F_H}^{-F_L} \frac{X(F)}{F_s} \cdot e^{j2\pi F t} dF + \int_{F_L}^{F_H} \frac{X(F)}{F_s} \cdot e^{j2\pi F t} \cdot dF$$

$$= \sum_{n=-\infty}^{\infty} \frac{x[n]}{F_s} \left(\int_{-F_H}^{-F_L} e^{j2\pi F t} e^{-j2\pi \frac{F}{F_s} n} \, dF + \int_{F_L}^{F_H} e^{j2\pi F\left(t - \frac{n}{F_s}\right)} \, dF \right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{x[n]}{F_s} \frac{1}{j2\pi(t-nT_s)} \left[e^{-j2\pi F_L (t-nT_s)} - e^{-j2\pi F_H (t-nT_s)} + e^{j2\pi F_H (t-nT_s)} - e^{-j2\pi F_L (t-nT_s)} \right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{x[n]}{F_s \pi (t-nT_s)} \left(\sin 2\pi F_H (t-nT_s) - \sin 2\pi F_L (t-nT_s) \right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{x[n]}{F_s \pi (t-nT_s)} \left(\sin\left(2\pi \left(F_c + \tfrac{B}{2}\right)(t-nT_s)\right) - \sin 2\pi \left(F_c - \tfrac{B}{2}\right)(t-nT_s) \right)$$

where $\boxed{F_c = \frac{F_L + F_H}{2}}$

$$= \sum_{n=-\infty}^{\infty} \frac{x(n)}{2B\pi(t-nT_s)} \, 2 \sin \pi B (t-nT_s) \cdot \cos 2\pi F_c (t-nT_s)$$

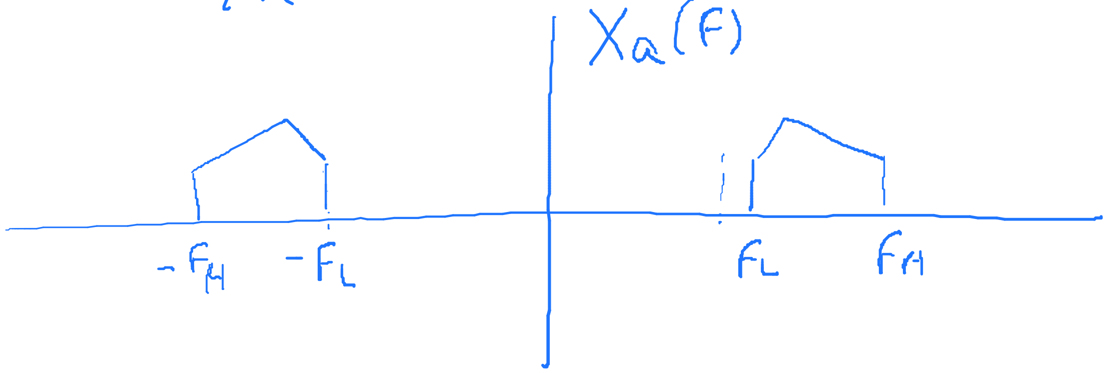
i.e.

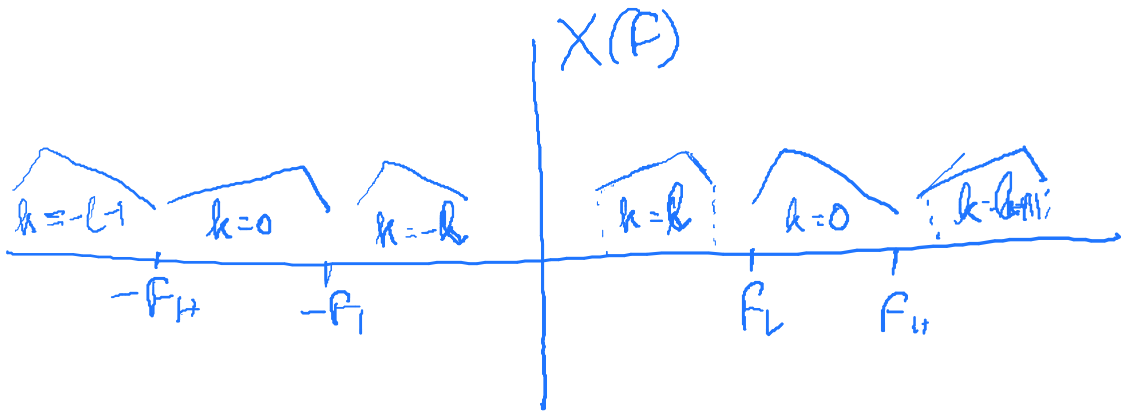
$$\boxed{x_a(t) = \sum_{n=-\infty}^{\infty} x[n]\, \mathrm{Sinc}\left(\pi B (t - nT_s)\right) \cos 2\pi F_c (t - nT_s) \\ \text{where } F_c = \frac{F_L + F_H}{2}}$$

* Bandpass sampling - Arbitrary Band Positioning

Again, note that

$$X(F) = F_s \cdot X_a(F - kF_s).$$





Again, we note that the band from F_L to F_H is not affected if $\exists$ a l, s.t. the shifted spectra of negative part for $k=l$ finishes before F_L and the shifted spectra of negative part for $k=l+1$ starts after $k=l+1$.

Thus, $-F_L + l F_S \leq F_L$

$$-F_H + (l+1)F_S \geq F_H$$

$$\Rightarrow \quad l F_S \leq 2F_L$$

$$(l+1)F_S \geq 2F_H$$

$$\Rightarrow \quad \boxed{\frac{2F_H}{l+1} \leq F_S \leq \frac{2F_L}{l}}$$

Similarly, for keeping the negative part of the spectra unaffected, we need that there exists an l such that

$$F_L - l F_S \geq -F_L \quad \Rightarrow \quad 2F_L \geq l F_S$$

$$F_H - (l+1)F_S \leq -F_H \quad \Rightarrow \quad 2F_H \leq (l+1)F_S$$

We thus get back the same relationship as before.

Next to fix F_S we follow the following procedure.

Note that

$$(l+1)F_S \geq 2F_H$$

$$l F_S \leq 2F_L$$

$$\Rightarrow \quad \frac{l+1}{l} \geq \frac{F_H}{F_L}$$

$$\Rightarrow \quad \frac{1}{l} \geq \frac{B}{F_L} \quad \Rightarrow \quad l \leq \frac{F_L}{B}$$

Define $\boxed{l_{max} := \left\lfloor \frac{F_L}{B} \right\rfloor}$.

Then, for any $l = 1, \dots, l_{max}$,

$$\boxed{\text{Choose any } F_s \text{ such that } \quad \frac{2F_H}{l+1} \leq F_s \leq \frac{2F_L}{l}}$$

Guard Bands:- To increase the available range for F_s, guard bands are introduced.

Essentially, this means ensuring that instead of using F_H and F_L, we use in our calculations $\boxed{F_H' = F_H + \frac{\Delta B}{2}}$ and

$$\boxed{F_L' = F_L - \frac{\Delta B}{2}}, \text{ and } \boxed{B' = F_H' - F_L' = B + \Delta B}$$

i.e. we assume that the signal has a larger bandwidth, and hence prevent aliasing over a larger band.

The reason for this is because F_s can shift due to device errors.

There are more complicated bandpass sampling techniques such as Interleaved Sampling.

* Resampling Discrete time signals

Why? – Different digital systems have different sampling rates and hence there is a need to convert from one to other.

eg. CDs and digitised audio signals have different sampling rates.

Assume $x[n]$ is the sampled version of $x_a(t)$ and sampling rate $F_s = \frac{1}{T_s}$.

Assume $X_a(F)$ is bandlimited to $B < \frac{F_s}{2}$

We shall down sample $x[n]$ by a factor of D, i.e. for every Dth sample, we shall drop the previous $(D-1)$ samples, i.e.

$$x_d[n] = x[nD]$$

The sampling rate is thus $\frac{F_s}{D}$, and

we must have $\boxed{F_s \geq 2DB}$ to ensure

proper reconstruction.

Now,

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a(f - kF_s)$$

and

$$X_D(f) = \frac{F_s}{D} \sum_{k=-\infty}^{\infty} X_a\left(f - k\frac{F_s}{D}\right)$$

The reconstruction is also as before

$$\boxed{x_a(t) = \sum_{m=-\infty}^{\infty} x_d[m] \cdot \text{Sinc}\left(\pi \frac{F_s}{D}(t - mDT_s)\right)}$$

On the other hand, to recover $x[n]$, from $x_d[m]$, note that $x_a(nT_s) = x[n]$.

Thus,

$$x[n] = \sum_{m=-\infty}^{\infty} x_d[m] \, \text{Sinc}\left(\pi \frac{F_s}{D}(nT_s - mDT_s)\right)$$

$$\Rightarrow \boxed{x[n] = \sum_{m=-\infty}^{\infty} x_d[m] \operatorname{sinc}\left(\frac{\pi}{D}(n-mD)\right)}$$

* Quantization and coding

Goal:- Convert the discrete time signal into a digital signal.

Divide the real line into L intervals

$$I_i = [x_i, x_{i+1}], \quad 1 \leq i \leq L$$

where $x_1, \ldots, x_{L+1}$ are called decision levels.

Generally $x_1 = -\infty$, $x_{L+1} = \infty$.

Next consider L points

$$\hat{x}_i \in [x_i, x_{i+1}], \quad 1 \leq i \leq L.$$

These points, $\hat{x}_1, \ldots, \hat{x}_L$ are called quantization levels.

A quantiser works as follows.

If the sample magnitude lies in $[x_i, x_{i+1}]$, then assign to it the value $\hat{x}_i$.

In general, we shall be using the uniform quantiser, where

$$x_{i+1} - x_i = \Delta, \quad 2 \leq i \leq L-1$$

$$x_1 = -\infty$$

$$x_{L+1} = \infty$$

and $\hat{x}_i = x_i + \frac{\Delta}{2}, \quad 2 \leq i \leq L$

and $\hat{x}_1 = x_2 - \frac{\Delta}{2}$

We shall call this uniform quantiser a midtread quantiser if $\exists\, i$ s.t. $\hat{x}_i = 0$.

The full scale range of the quantizer is defined as,

$$\boxed{\begin{aligned} FSR &= (x_L + \Delta) - (x_2 - \Delta) \\ &= x_L - x_2 + 2\Delta = L\Delta \end{aligned}}$$

Next, note that the quantization error of any sample is bounded by $\frac{\Delta}{2}$ in magnitude.

$$\boxed{|e_Q[n]| \leq \frac{\Delta}{2}}$$

Now, once a sample is quantized to some $\hat{x}_i$, it needs to be given a digital equivalent.

Since there are L levels, the number of bits needed for L levels is $\lceil \log_2 L \rceil$.

Then suppose FSR is R, and the quantiser is $b+1$ bits then

$$\boxed{\begin{aligned}\Delta &= \frac{R}{L} \\ &= \frac{R}{2^{b+1}}\end{aligned}}$$

Tradeoff :- i) Low Δ $\Rightarrow$ Better $e_Q[n]$
but more bits

ii) High Δ $\Rightarrow$ Fewer bits but
bad $e_Q[n]$

Coding – Process of representing the L quantization levels using $\lceil \log_2 L \rceil$ bits.

One commonly used method is the 2s complement coding.

Choose $L = 2^{b+1}$, and use a uniform midtread quantiser with

$$\boxed{\hat{x}_i = (i - 1 - 2^b)\Delta, \quad 1 \le i \le L.}$$

Then, for levels $i = 2^b + 1 + j$, $0 \le j \le \frac{L}{2} - 1$, use the coding

0.(b bit representation of j).

For levels $i = 2^b + 1 - j$, $1 \le j \le \frac{L}{2}$,

use the representation

$1 \cdot$ (complement of b bit representation of $j-1$)

* Quantisation error

There are two kinds of error

i) Granular noise – Pure quantisation error resulting from quantising any value within $[x_i, x_{i+1}]$ to $\hat{x}_i$.

ii) Overload noise – Resulting from input amplitude going beyond the range of the quantizer.

We shall only consider granular noise.

Overload noise has no particular remedy the best one can do is scale down the signal amplitude.

To compute the effect of quantisation noise, we shall be using a statistical method.

We make the following assumptions:-

i) $e_Q(n) \sim \text{unif}\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$

ii) $e_Q[n]$ is white, i.e. $e_Q[n]$ & $e_Q[m]$ are uncorrelated if $n \neq m$.

iii) $e_Q[n]$ is uncorrelated with $x[n]$

iv) $x[n]$ is zero mean and wide sense stationary (WSS)

Then $P_{sig} = \mathbb{E}\left[x^2[n]\right] \longrightarrow$ Doesn't depend on n since WSS

$= \sigma_x^2$

$$\sigma_e^2 \triangleq P_{Q\,noise} = \mathbb{E}\left[e_Q^2[n]\right] = \int_{-\Delta/2}^{\Delta/2} \frac{e_Q^2[n]}{\Delta} \cdot d\,e_Q[n]$$

$$\Rightarrow \boxed{\sigma_e^2 = \frac{\Delta^2}{12}}$$

Then, signal to quantisation noise ratio is given by

$$SQNR = 10\log_{10}\frac{\sigma_x^2}{\sigma_e^2}$$

$$= 20\log_{10}\frac{\sigma_x}{\sigma_e}$$

$$= 20\log_{10}\frac{2\sqrt{3}\sigma_x}{\Delta}$$

$$\boxed{SQNR = 20\log_{10}\frac{2\sqrt{3}\sigma_x}{FSR}\cdot 2^{b+1}}$$

where we are using a $b+1$ bit quantiser.

Notice the SQNR depends on $\dfrac{FSR}{\sigma_x}$.

For specifying the SQNR of a generic A/D converter to be used for any signal, the following is done.

It is assumed that the signal is Gaussian.

Since the signal is zero mean. we know that the signal with high probability stays between $\pm 3\sigma_x$.

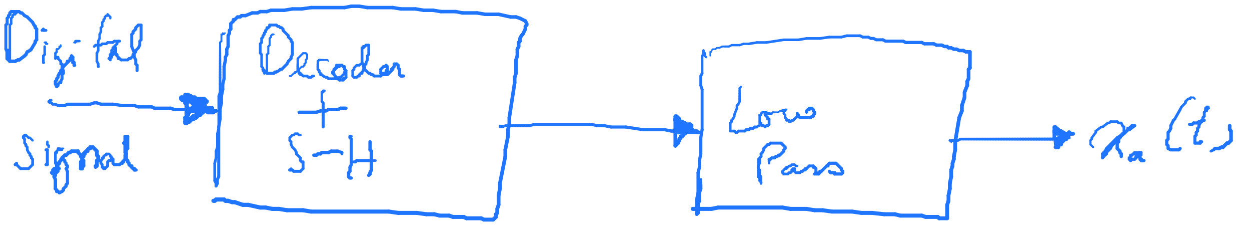
Choose $FSR = 6\sigma_x$.

Then SQNR for a $b+1$ bit quantiser is specified as

$$SQNR = 20b \log_{10} 2 + 20 \log_{10} \frac{2}{\sqrt{3}}$$

$$\Rightarrow \boxed{SQNR = 6.02b + 1.25}$$

* D/A converters



We assume that the original signal was bandlimited to B, and we used a sampling frequency

$$F_s \geq 2B.$$

The decoder reads a digital input and convert it to a voltage level corresponding to the correct quantization level.

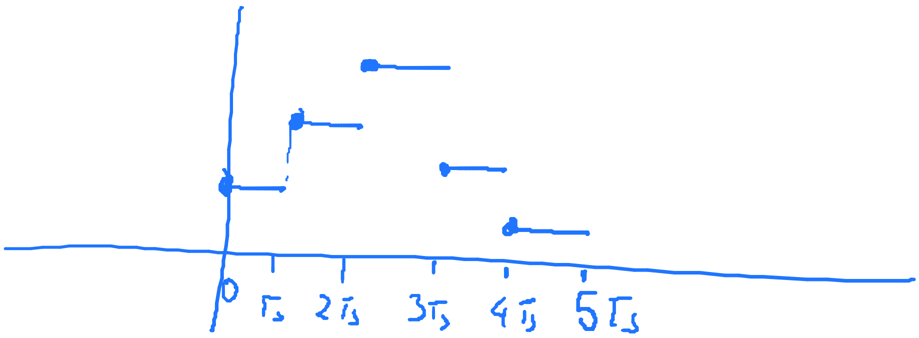
The S-H hold circuit then holds this voltage level till the next sample arrives, i.e. for $T_s = \frac{1}{F_s}$

S-H thus performs two functions

i) Converts the discrete time signal to a continuous time one.

ii) Ensures that any transients resulting from the decoder is kept in check.

The output of the S-H is thus as follows



More formally

$$\boxed{x_{S\text{-}H}(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \text{rect}\left(\frac{t - (n+\frac{1}{2})T_s}{T_s}\right)}$$

Then,

$$X_{S\text{-}H}(F) = \sum_{n=-\infty}^{\infty} x[n]\, T_s \,\text{Sinc}(\pi F T_s)\, e^{-j2\pi \frac{F}{F_s}(n+\frac{1}{2})}$$

$$= X(F) \cdot T_s \,\text{Sinc}(\pi F T_s)\, e^{-j\pi \frac{F}{F_s}}$$

Now noting that the signal $x_a(t)$ was bandlimited and $F_s \geq 2B$, we have

$$X(F) = F_s X_a(F), \quad |F| \leq F_s$$

Then, by using an analog low pass filter with transfer function

$$\boxed{H_{L-P}(F) = \frac{\pi F T_s}{\sin(\pi F T_s)} e^{j\pi F/F_s} \cdot \text{rect}\left(\frac{F}{F_s}\right)}$$

we have that

$$\boxed{X_{S-H}(F) \cdot H_{L-P}(F) = X_a(F)}$$

* Discrete Fourier Transform

- Digital computers cannot handle DTFT since the DTFT $X(\omega)$ is a continuous function of ω.
- Idea:- What if we also sample from $X(\omega)$?

Note :- $X(\omega)$ is periodic with a period 2π.

Let us take N uniformly placed samples

$$\boxed{\omega_k = \frac{2\pi}{N}k, \quad k = 0, 1, \dots, N-1}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \dots + \sum_{n=-N}^{-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$+ \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$+ \sum_{n=N}^{2N-1} x[n] e^{-j\frac{2\pi}{N}kn} + \dots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n-lN] e^{-j\frac{2\pi}{N}k(n-lN)}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n-lN] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left(\sum_{l=-\infty}^{\infty} x[n-lN] \right) e^{-j\frac{2\pi}{N}kn}$$

Define $x_p[n] = \sum_{l=-\infty}^{\infty} x[n-lN]$

Note that

$$x_p[n+N] = \sum_{l=-\infty}^{\infty} x[n+N-lN]$$

$$= \sum_{l=-\infty}^{\infty} x[n - (l-1)N]$$

$$= \sum_{l=-\infty}^{\infty} x[n-lN]$$

Thus $x_p[n]$ is periodic with period N.

Then, we can expand $x_p[n]$ using a DT Fourier series.

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$$

where $$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p[n] e^{-j\frac{2\pi}{N}kn}$$

Thus, $N c_k = X\left(\frac{2\pi}{N}k\right)$.

and $$\boxed{x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn}}$$

Thus, we can reconstruct $x_p[n]$ from

$$X\left(\frac{2\pi}{N}k\right),\quad k = 0, 1, \dots, N-1$$

How does $x_p[n]$ relate to $x[n]$??

In general $x_p[n]$ and $x[n]$ have no specific relationship.

However, assume $x[n]$ was time limited to only N samples at max.

Then, $$x_p[n] = \sum_{l=-\infty}^{\infty} x[n-lN]$$

If $0 \le n \le N-1$, then

$$x_p[n] = x[n]$$

Thus, if $x[n]$ has at most N non-zero points, then

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \ldots, N-1$$

$\longrightarrow$ DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn}, \quad n = 0, 1, \ldots, N-1$$

$\longrightarrow$ IDFT

Next, we show how to recover $X(\omega)$ from $X\left(\frac{2\pi}{N}k\right)$, $k = 0, 1, \ldots, N-1$, if $x(n)$ is time-limited to N.

$$X(\omega) = \sum_{n=0}^{N-1} x_p(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

[$\because x_p(n) = x(n)$ as $x(n)$ is time-limited to N]

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn} e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) \cdot \left(\sum_{n=0}^{N-1} \frac{1}{N} e^{-j\left(\omega - \frac{2\pi}{N}k\right)n} \right)$$

$$= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) \frac{1}{N} \frac{\left(1 - e^{-j\left(\omega - \frac{2\pi}{N}k\right)N}\right)}{\left(1 - e^{-j\left(\omega - \frac{2\pi}{N}k\right)}\right)}$$

$$= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) \frac{1}{N} \frac{\sin\left(\left(\omega - \frac{2\pi}{N}k\right)\frac{N}{2}\right)}{\sin\left(\left(\omega - \frac{2\pi}{N}k\right)/2\right)} e^{-j\left(\omega - \frac{2\pi}{N}k\right)\frac{(N-1)}{2}}$$

Thus, $$X(\omega) = \sum_{n=0}^{N-1} X\left(\frac{2\pi}{N}k\right) \cdot P\left(\omega - \frac{2\pi}{N}k\right),$$

if $x[n]$ is time limited to N

where, $$P(\omega) = \frac{\sin\left(\frac{\omega N}{2}\right)}{N \sin\left(\frac{\omega}{2}\right)} \cdot e^{-j\omega\left(\frac{N-1}{2}\right)}$$

* DFT as a linear transform

Define the matrix

$$(W_N)_{kn} = e^{-j\frac{2\pi}{N}nk}, \quad n = 0, 1, \ldots, N-1$$
$$k = 0, 1, \ldots, N-1$$

Then, $$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{n=0}^{N-1} x[n] \cdot (W_N)_{kn}$$

Thus, $X = \left[X(0) \;\; X(\tfrac{2\pi}{N}) \;\; \cdots \;\; X(\tfrac{2\pi}{N}(N-1))\right]^T$

and $x = \left[x[0] \;\; x[1] \;\; \cdots \;\; x[N-1]\right]^T$

are related as

$$\boxed{X = W_N \cdot x}$$

Next, one can show that W_N is invertible

[To show this observe that W_N is a Vander Monde matrix, and none of its 'common factors' are the same.]

Then, $x = W_N^{-1} X$.

On the other hand,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) (W_N)^*_{kn}$$

i.e. $$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k \cdot (W_N^{*})_{kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_k \cdot (W_N^{H})_{nk}$$

Then $$\boxed{x = \frac{1}{N} W_N^{H} \cdot X}$$

Comparing we get,

$$\frac{1}{N} W_N^{H} = W_N^{-1}$$

$$\Rightarrow \boxed{W_N^{H} W_N = N I_N}$$

Thus, the DFT matrix is orthogonal.

i) Periodicity

$$X(k+N) = X(k)$$

Follows by noting $e^{-j\frac{2\pi}{N}(k+N)n} = e^{-j\frac{2\pi}{N}kn}$

ii) Linearity

$$x_1[n] \longleftrightarrow X_1(k)$$

$$x_2[n] \longleftrightarrow X_2(k)$$

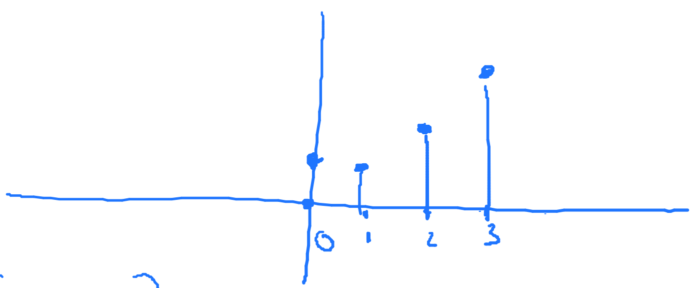
Then $\alpha x_1[n] + \beta x_2[n] \longleftrightarrow \alpha X_1(k) + \beta X_2(k)$

Obvious from the definitions

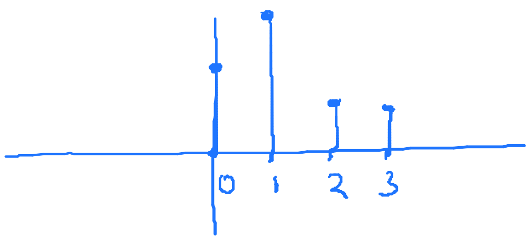
iii) Circular shift $x((n-k)_N)$

eg. let $N=4$, $k=2$

$x[n]$



$x((n-k)_N)$



* Circular convolution

Let $x_1[n] \longleftrightarrow X_1(k)$

$x_2(n) \longleftrightarrow X_2(k)$

What is the IDFT of $X_1(k) \cdot X_2(k)$?

Let $X_3(k) = X_1(k) \cdot X_2(k)$.

Then,

$$x_3[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2(k) e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} x_1[m] e^{-j\frac{2\pi}{N}km} \cdot \sum_{\ell=0}^{N-1} x_2[\ell] e^{-j\frac{2\pi}{N}k\ell} \, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{\ell=0}^{N-1} x_1[m] \cdot x_2[\ell] \cdot \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m-\ell)}$$

Now, fix m and ℓ. Then,

if $n-m-\ell \neq 0 \mod N$,

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m-\ell)} = \frac{1-e^{j2\pi(n-m-\ell)}}{1-e^{j\frac{2\pi}{N}(n-m-\ell)}} = 0$$

if $n-m-\ell = 0 \mod N$, then

$$e^{j\frac{2\pi}{N}k(n-m-\ell)} = 1, \quad \forall k: 0, 1, \cdots, N-1.$$

Thus, $\displaystyle\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m-\ell)} = N.$

Thus, $x_3[n] = \sum_{m=0}^{N-1} x_1[m] \cdot x_2[(n-m)_N]$

[where $(p)_N \triangleq p \bmod N$]

Thus,

$$\boxed{\sum_{m=0}^{N-1} x_1(m)\, x_2[(n-m)_N] \longleftrightarrow X_1(k) \cdot X_2(k)}$$

The operation $\sum_{m=0}^{N-1} x_1(m) \cdot x_2[(n-m)_N]$

is called the circular convolution

between $x_1[n]$ and $x_2[n]$ and is

denoted by $x_1[n] \; \text{Ⓝ} \; x_2[n]$.

Example :- $N = 4$.

$x_1[n] = \{1, 1, 2, 2\}$

$x_2[n] = \{1, 3, 2, 1\}$

$x_1[n]$: 1, 1, 2, 2

$x_2[n]$: 1, 3, 2, 1

$x_2[(-n)_4]$ = 1, 1, 2, 3

$x_2[(1-n)_4]$ = 3, 1, 1, 2

$x_2[(2-n)_4]$: 2, 3, 1, 1

$x_2[(3-n)_4]$ = 1, 2, 3, 1

Then, let $x_3[n] = x_1[n] \; 4 \; x_2[n]$

$x_3[0] = 1 + 1 + 4 + 6 = 12$

$x_3[1] : 3 + 1 + 2 + 4 = 10$

$x_3[2] = 2 + 3 + 2 + 2 = 9$

$x_3[3] = 1 + 2 + 6 + 2 = 11$

* Further properties of DFT

i) Time reversal

$$\boxed{\begin{array}{c} x[n] \longleftrightarrow X(k) \\ x[(-n)_N] \longleftrightarrow X((-k)_N) \end{array}}$$

To see this

$$\text{DFT}\left[x[N-n]\right](k)$$

$$= \sum_{n=0}^{N-1} x[N-n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{m=0}^{N-1} x[m] \cdot e^{-j\frac{2\pi}{N}k(N-m)}$$

$$[m = N-n]$$

$$= \sum_{m=0}^{N-1} x[m] e^{j\frac{2\pi}{N}km}$$

$$= \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}(N-k).m}$$

$$= X\left((-k)_N\right)$$

ii) Circular shift

$$\boxed{\begin{array}{c} x[n] \longleftrightarrow X(k) \\ x\left[(n-l)_N\right] \longleftrightarrow X(k)\, e^{-j\frac{2\pi}{N}kl} \end{array}}$$

To see this,

$$\mathrm{DFT}\left[x\left[(n-l)_N\right]\right](k)$$

$$= \sum_{n=0}^{N-1} x\left[(n-l)_N\right] e^{-j\frac{2\pi}{N}kn}$$

Let $m = n - l \mod N$

Then, $n = m + l \mod N$

Then, $n = m + l + pN$, for some $p \in \mathbb{Z}$

Also, as n varies from 0 to $N-1$

$(n-l)_N$ also varies from 0 to $N-1$.

Thus,

$$\text{DFT}\left[x\left[(n-l)_N\right]\right](k)$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}k(m+l+pN)}$$

$$= \sum_{m=0}^{N-1} x[m] \cdot e^{-j\frac{2\pi}{N}km} e^{-j\frac{2\pi}{N}kl}$$

$$= X(k) \cdot e^{-j\frac{2\pi}{N}kl}$$

iii) Circular frequency shift

$$\boxed{\begin{array}{c} x[n] \longleftrightarrow X(k) \\ x[n] e^{j\frac{2\pi}{N}ln} \longleftrightarrow X((k-l)_N) \end{array}}$$

$$\mathrm{DFT}\left[x[n]e^{j\frac{2\pi}{N}\ell n}\right](k)$$

$$= \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi}{N}\ell n} e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}n(k-\ell)}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}n\left[(k-\ell) \bmod N\right]}$$

$\Big[\because (k-\ell) \bmod N =$

$k-\ell+pN$, for some $p \in \mathbb{Z}$.

and $e^{-j\frac{2\pi}{N}n(k-\ell+pN)}$

$= e^{-j\frac{2\pi}{N}n(k-\ell)}\Big]$

$$= X\left((k-\ell)_N\right)$$

iv) Complex conjugate properties

$$\boxed{\begin{aligned} x[n] &\longleftrightarrow X(k) \\ x^*[n] &\longleftrightarrow X^*((-k)_N) \end{aligned}}$$

To see this

$$DFT\left[x^*[n]\right](k)$$

$$= \sum_{n=0}^{N-1} x^*[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x^*[n] e^{j\frac{2\pi}{N}(N-k)n}$$

$$= \left(\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(N-k)\cdot n}\right)^*$$

$$= \left(X((-k)_N)\right)^*$$

$$= X^*((-k)_N)$$

v) Multiplication in time

$$x_1[n] \longleftrightarrow X_1(k)$$

$$x_2[n] \longleftrightarrow X_2(k)$$

Then $\quad x_1[n]\, x_2[n] \longleftrightarrow \frac{1}{N} X_1(k) \, (N) \, X_2(k)$

To see this

$$\text{IDFT}\left[\frac{1}{N} X_1(k) \, (N) \, X_2(k)\right][n]$$

$$= \frac{1}{N^2} \sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} X_1(m) \cdot X_2((k-m)_N) \right) e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N^2} \sum_{m=0}^{N-1} X_1(m) \cdot \sum_{k=0}^{N-1} X_2((k-m)_N) \, e^{j\frac{2\pi}{N}kn}$$

Define $l = k - m \bmod N$

Then as k varies from

0 to N-1, l varies from 0 to N-1 as well.

Also, $k = m + l \mod N$

Thus,

$$\text{IDFT}\left[\frac{1}{N} X_1(k) \, N \, X_2(k)\right][n]$$

$$= \frac{1}{N^2} \sum_{m=0}^{N-1} X_1(m) \sum_{l=0}^{N-1} X_2(l)\, e^{j\frac{2\pi}{N} n ((m+l)_N)}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} X_1(m) \cdot \frac{1}{N} \sum_{l=0}^{N-1} X_2(l)\, e^{j\frac{2\pi}{N} n (m+l)}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} X_1(m) \cdot x_2[n] \cdot e^{j\frac{2\pi}{N} n \cdot m}$$

$$= x_2[n] \cdot x_1[n]$$

* Plancherel's Theorem

$$x[n] \longleftrightarrow X(k)$$

$$y[n] \longleftrightarrow Y(k)$$

then

$$\sum_{n=0}^{N-1} x[n] \cdot y^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot Y^*(k)$$

In particular,

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Proof:

$$\frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn} \right) \cdot \left(\sum_{m=0}^{N-1} y^*[m]\, e^{j\frac{2\pi}{N}km} \right)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[n] y^*[m] \cdot \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(n-m)k}$$

Now, if $n-m = 0 \bmod N$,

then $e^{-j\frac{2\pi}{N}(n-m)k} = 1$

So, $\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(n-m)k} = N$

On the other hand, if $n-m \neq 0 \bmod N$

$$\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}(n-m)k} = \frac{1-e^{-j2\pi(n-m)}}{1-e^{-j\frac{2\pi}{N}(n-m)}}$$

$$= 0$$

Next, for any $n = 0, 1, \ldots, N-1$, the only $m \in \{0, 1, \ldots, N-1\}$ satisfying

$n-m=0 \mod N$ is $m=n$.

Thus,

$$\frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot Y^*(k) = \sum_{n=0}^{N-1} x[n]\cdot y^*[n]$$

. . . . $\boxed{\text{ES}}$

* Linear Filtering using DFT

Note:- If $x[n]$ is limited to N samples at $n=0,1,\dots,N-1$, then DFT followed IDFT recovers $x[n]$.

Assume a causal FIR LTI system

i.e. $h[n]=0,\ n<0$
$=0,\ n\geq M$.

Assume an input $x[n]$ which is causal and limited to L samples,

i.e. $x[n] = 0, \quad n < 0$
$= 0, \quad n \geq L$

Then $y[n] = h[n] * x[n]$ is atmost $M+L-1$ sequence long.

Zero pad both sequences to obtain $N = M+L-1$ length sequences $h[n]$ and $x[n]$.

Now, notice that

$$Y(\omega) = X(\omega) \cdot H(\omega), \text{ and}$$

$$y[n] = 0, \quad n < 0 \text{ and } n \geq \underbrace{M+L-1}_{=N}.$$

Thus, the N-point DFT,

$Y(k)$, $k = 0, 1, \ldots, N-1$ is

sufficient to recover $y[n]$ using

IDFT.

Now, as $Y(\omega) = X(\omega) H(\omega)$

$$Y\left(\frac{2\pi}{N}k\right) = X\left(\frac{2\pi}{N}k\right) H\left(\frac{2\pi}{N}k\right), \quad k = 0, 1, \ldots, N-1$$

i.e. $Y(k) = X(k) \cdot H(k)$

Thus, to compute the linear convolution between causal $x[n]$ and $h[n]$ of duration L and M respectively:-

1. $N = M + L - 1$
2. Zero pad $x[n]$ by $L-1$ zeros
3. Zero pad $h[n]$ by $M-1$ zeros
4. Obtain N-point DFTs
   $X(k), H(k)$
5. $Y(k) = X(k) \cdot H(k)$
6. $y[n] = IDFT[Y(k)][n]$

* Linear Filtering Long Sequences

Assume causal $x(n)$ and $h(n)$ of lengths L' and M, where $L' >> M$.

To get the output $y(n)$, we need to compute $N' = M + L' - 1$ point DFT.

Issue :- If N' is too larg, the memory might not be enough to compute N' point DFT - IDFT.

To bypass this, we split the entire N' length sequence into blocks of length N in such a manner, that we can take DFT and IDFT on these small blocks and 're forge' them together to obtain $y[n]$

i) Overlap and save method

Step 1 :- Segment the N' length zero padded input $x[n]$ into blocks of length L. (zero pad further if L does not divide N'). Now zero pad $M-1$ zeros in the beginning of the sequence.

Step 2:- Let $x_i[n] \triangleq 0, \quad n<0 \;\&\; n \geq N$

$$= x[(i-1)L+n-M+1], \quad 0 \leq n \leq N-1,$$

where $N = L+M-1$, when $i>1$.

$$x_1[n] = 0, \quad 0 \leq n \leq M-2$$

$$= x[n-M+1], \quad M-1 \leq n \leq L-1$$

Zero pad $h[n]$ to length N as well.

Step 3:- Compute N-point DFT $X_i(k)$ and $H(k)$ and obtain

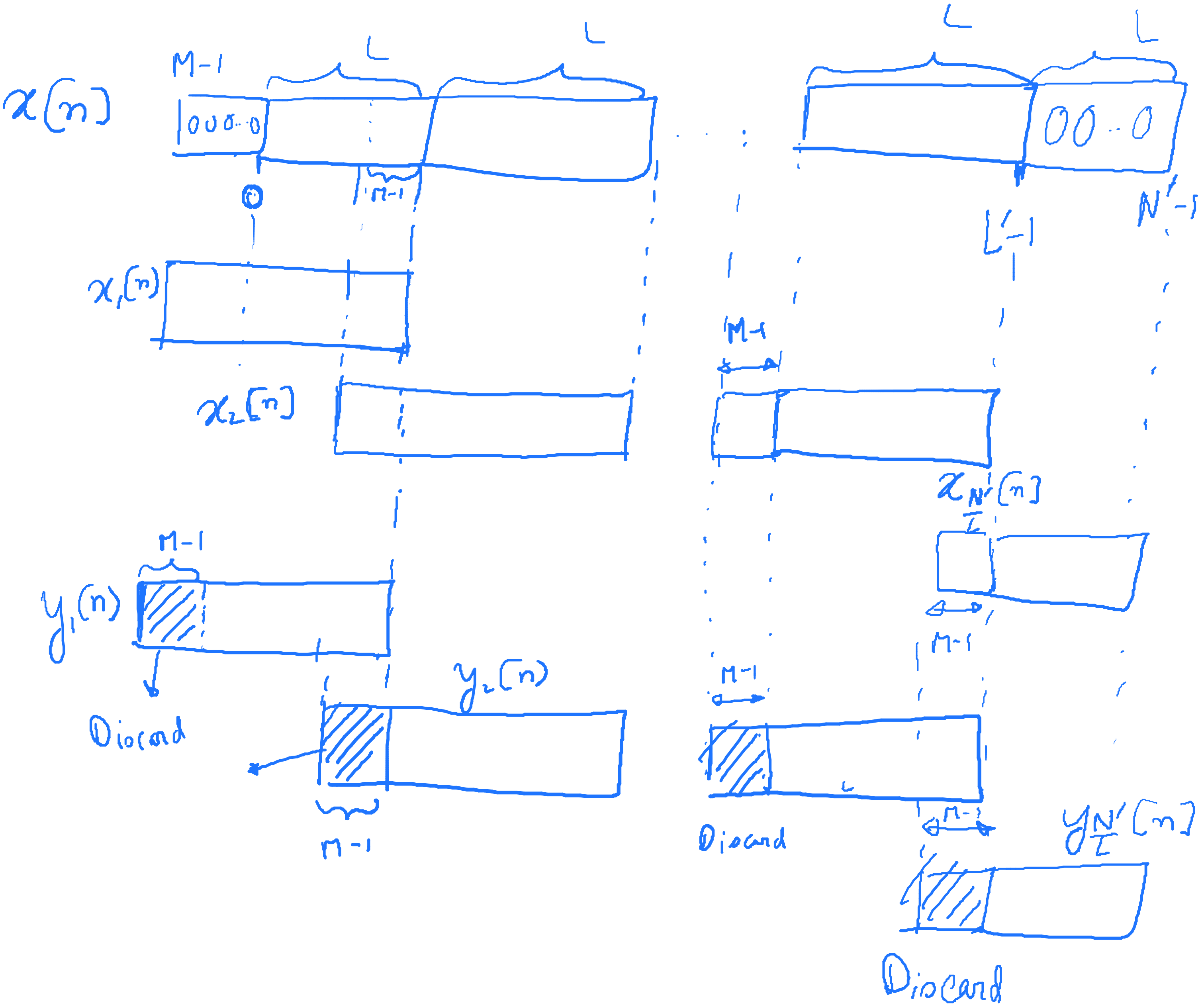
$$Y_i(k) = X_i(k) \cdot H(k), \quad k = 0, 1, \ldots, N-1$$

Take IDFT to obtain $y_i[n]$.

Step 4 :- Output $y[n]$, where

$$y[(i-1)L+n] = y_i[M-1+n], \quad 0 \le n \le L-1.$$

and $y[n] = 0, \quad n<0 \;\&\; n \ge N'$



Why does this work?

Let us take $y_i[n]$.

Recall from the definitions of DFT and IDFT that

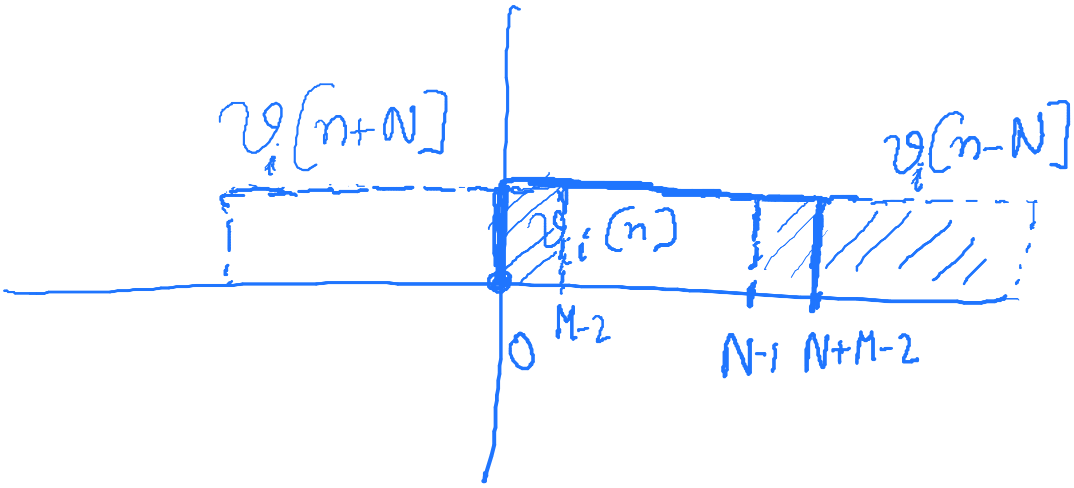
$$y_i[n] = \sum_{\ell=-\infty}^{\infty} v_i[n-\ell N]$$

where $v_i[n] = x_i[n] * h[n]$

Thus $v_i[n]$ is a $N+M-1$ length sequence.

Then, notice that for $0 \le n \le N-1$,

$v_i[n+N]$ is going to alias the first $M-1$ samples of $y_i[n]$, and no other $v_i[n-\ell N]$, $\ell \neq 0,-1$, affects $y_i[n]$.



Notice that these are exactly the samples from $y_i[n]$ which are discarded.

Next consider any one of the non discarded samples

$$y_i[M-1+n] \quad , \quad 0 \le n \le L-1$$

$$= \sum_{m=0}^{M-1} h[m]\, x_i[M-1+n-m]$$

$$= \sum_{m=0}^{M-1} h(m]\, x[(i-1)L+M-1+n-m-M+1]$$

(By defn of $x_i(n]$)

$$= \sum_{m=0}^{M-1} h[m] \cdot x[(i-1)L+n-m]$$

$$= v[(i-1)L+n],$$

where $v[n] = x[n] * h[n]$

Thus, by defn of $y_i[n]$ and $y[n]$,

we have $y[(i-1)L+n] = v[(i-1)L+n]$,

$$0 \le n \le L-1,$$

for all $\cancel{1} \le i \le \frac{N'}{L}$, as required.

ii) Overlap and add method

Step 1:- Segment the L' length zero padded input $x[n]$ into blocks of length L.

Take $L > M$.

Step 2:- Let $x_i[n] \triangleq 0, \quad n<0 \ \& \ n \geq L$

$$= x[(i-1)L+n]$$

Zero pad $x_i(n)$ and $h[n]$ to length N, where

$$N = L+M-1.$$

Step 3:- Compute N-point DFT.

$X_i(k)$ and $H(k)$ and obtain

$$Y_i(k) = X_i(k) \cdot H(k),$$

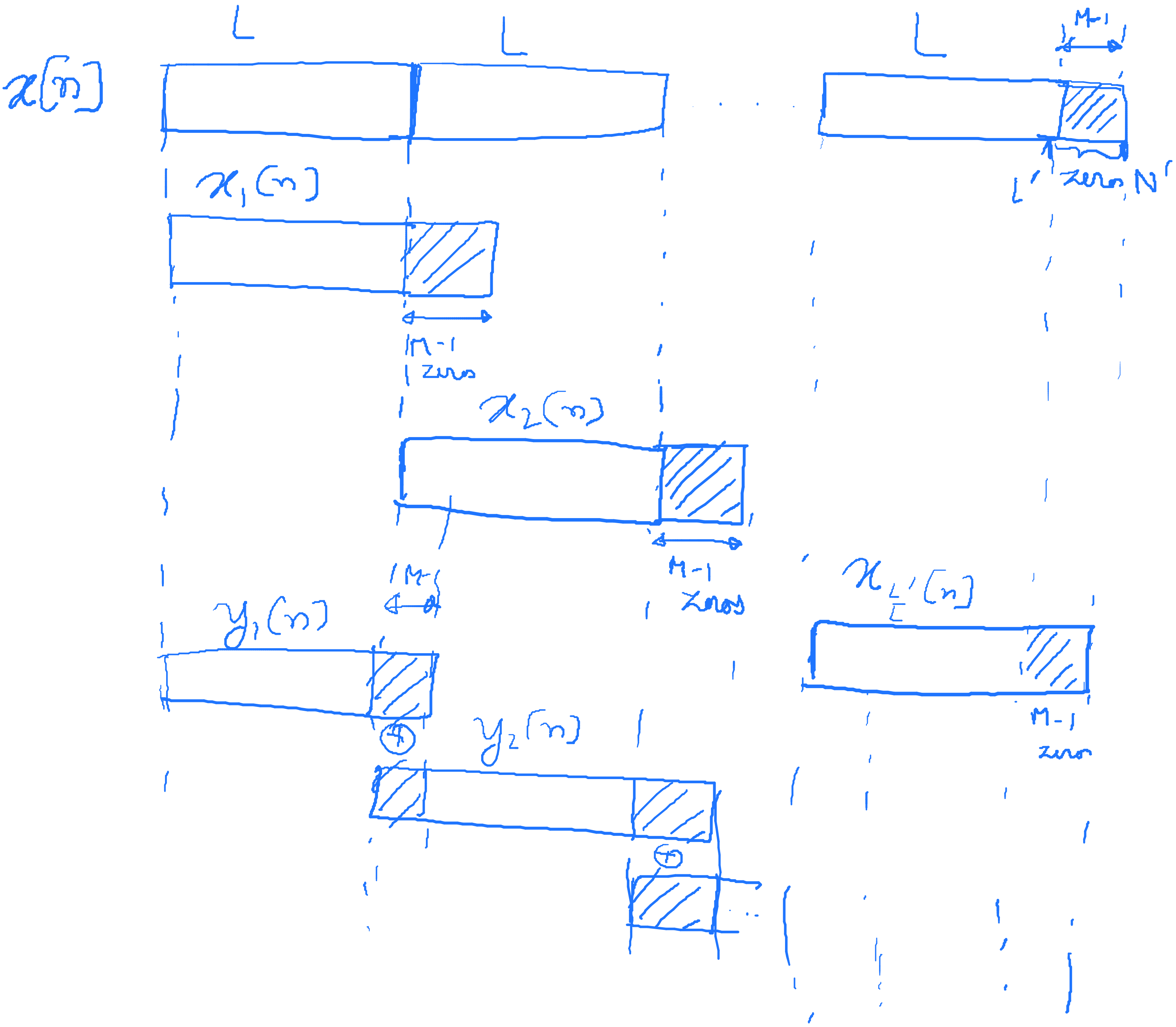
$$k = 0, 1, \ldots, N-1$$

Take IDFT to obtain $y_i[n]$.

Step 4:- Output $y[n]$, where

$$y[(i-1)L+n] = y_i[n] + y_{i-1}[L+n], \quad 0 \le n \le M-2$$

$$= y_i[n], \quad M-1 \le n \le L-1$$



Why does this work?

Consider $y_i[n]$, and recall that

$$y_i[n] = \sum_{l=-\infty}^{\infty} v_i[n-lN], \text{ where}$$

$$v_i[n] = h[n] * x_i[n]$$

Now, each $v_i[n]$ are N length sequences since $x_i[n]$ are L length and $h[n]$ is M-length.

Thus, for $0 \leq n \leq N-1$,

$$y_i[n] = v_i[n]$$

Hence for $0 \leq n \leq M-2$

$$y[(i-1)L+n] = y_i[n] + y_{i-1}[L+n]$$

$$= v_i[n] + v_{i-1}[L+n]$$

$$= \sum_{m=0}^{M-1} h[m] x_i[n-m] + \sum_{m=0}^{M-1} h[m] x_{i-1}[L+n-m]$$

$$= \sum_{m=0}^{n} h[m] x_i[n-m] + \sum_{m=n+1}^{M-1} h[m] x_{i-1}[L+n-m]$$

$[\because x_i[l] = 0,\ l < 0 \text{ and } l \geq L]$

$$= \sum_{m=0}^{n} h[m]\, x[(i-1)L+n-m] + \sum_{m=n+1}^{M-1} h[m]\, x[(i-2)L+L+n-m]$$

$$= \sum_{m=0}^{M-1} h[m] \cdot x[(i-1)L+n-m]$$

$$= v[(i-1)L+n],$$

where $v[n] = h[n] * x[n]$

On the other hand, when

$M-1 \leq n \leq L-1,$

$$y[(i-1)L+n] = y_i[n]$$

$$= v_i[n]$$

$$= \sum_{m=0}^{M-1} h[m] \cdot x_i[n-m]$$

$$= \sum_{m=0}^{M-1} h[m] x[(i-1)L+n-m]$$

[By defn of $x_i[n]$]

$$= v[(i-1)L+n],$$

where $v[n] = h[n] * x[n].$

* Time truncation and leakage

In real life we do not see the entire signal for all n, but only finitely many, say L samples.

More precisely, define

$$\boxed{\begin{aligned} w[n] &= 1, \quad 0 \leq n \leq L-1 \\ &= 0, \quad o.w. \end{aligned}}$$

Then, for any signal $x[n]$, we observe

$$\boxed{\hat{x}[n] = x[n]w[n]}$$

This time truncation leads to increasing the frequency spectrum of $x[n]$.

To see this, first note that

$$W(\omega) = \sum_{n=-\infty}^{\infty} w[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{L-1} e^{-j\omega n}$$

$$= \frac{1-e^{-j\omega L}}{1-e^{-j\omega}}, \quad \omega \neq 0$$

$$= e^{-j\omega(L-1)/2} \cdot \frac{2j \sin\left(\frac{\omega L}{2}\right)}{2j \sin\left(\frac{\omega}{2}\right)}, \omega \neq 0$$

$$= \frac{\sin\left(\frac{\omega L}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega(L-1)/2}, \text{ if } \omega \neq 0$$

$$L, \text{ if } \omega = 0$$

Now, consider signal

$$x[n] = \cos \omega_0 n.$$

Then $\hat{X}(\omega) = \sum_{n=0}^{L-1} \cos \omega_0 n . e^{-j\omega n}$

$$= \sum_{n=0}^{L-1} \frac{1}{2}\left[e^{-j(\omega-\omega_0)n} + e^{-j(\omega+\omega_0)n}\right]$$

$$= \frac{1}{2}\left(\frac{1-e^{-j(\omega-\omega_0)L}}{1-e^{-j(\omega-\omega_0)}}\right)$$

$$+ \frac{1}{2}\left(\frac{1-e^{-j(\omega+\omega_0)L}}{1-e^{-j(\omega+\omega_0)}}\right)$$

$$= \frac{1}{2}\left[W(\omega-\omega_0) + W(\omega+\omega_0)\right]$$

We plot the N-point DFT of $\hat{x}[n]$, with $N >> L$. We take $\omega_0 = \frac{\pi}{4}$,

Notice that the spectrum has leaked beyond $\omega = \frac{\pi}{4}$ and $\omega = -\frac{\pi}{4}$ (same as $\omega = \frac{7\pi}{4}$).

This phenomenon is called leakage.

Notice that the main lobe of the spectrum $\hat{X}(\omega)$ is centred at $\omega = \frac{\pi}{4}$ and $\frac{7\pi}{4}$, corresponding to the original frequency $\omega_0 = \frac{\pi}{4}$

Next note that $W(\omega)$ has its first zero crossing at $\omega = \pm\frac{2\pi}{L}$, and hence the width of the main lobe is $\frac{4\pi}{L}$.

Thus, the main lobe of stretches from

$$\omega_0 - \frac{2\pi}{L} \text{ to } \omega_0 + \frac{2\pi}{L}.$$

In other words, if we had another frequency component $\omega_1 \in \left(\omega_0 - \frac{\pi}{4}, \omega_0 + \frac{\pi}{4}\right)$, then we won't be able to resolve

ω_1 and ω_0.

To illustrate this, we consider

$$x[n] = \cos\omega_0 n + \cos\omega_1 n$$

with $\omega_0 = \frac{\pi}{4}$, $\omega_1 = 0.22\pi$, and

plot the N-point DFT $\hat{X}(k)$ for

$L = 25, 50, 100$.

Notice that $L = 25$ the frequencies are not resolvable $L = 50$ the frequencies are just resolvable and $L = 100$ frequencies can be resolved much better.

* Computation complexity of DFT and IDFT

Define $W_N \triangleq e^{-j\frac{2\pi}{N}}$.

Assume with full generality that $x[n]$ is complex.

Then

$$X(k) = \sum_{n=0}^{N-1} x[n] \cdot W_N^{kn}, \quad k = 0, 1, \ldots, N-1$$

Thus, an N-point DFT, if computed directly requires

N^2 complex multiplications [N for each k]

$N^2 - N$ complex additions [N-1 for each k]

Notice that IDFT computations are similar since,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn},$$

$$n = 0, 1, \ldots, N-1$$

and require similar number of additions and multiplications.

. . .

* Divide and conquer algorithm for reducing complexity

→ General outline of divide and conquer algos :-

1. Break problem into smaller subproblems.
2. Solve the subproblems
3. Combine the solutions of the subproblems to obtain the result of the main problem

→ Assumption :-

N can be factored into

$$N = LM.$$

If N is prime, zero pad to make it

composite.

→ Ingredients for the divide step –

The storage scheme

The goal is to store $x[n]$, $0 \le n \le N-1$ in an $L \times M$ matrix.

Let $0 \le l \le L-1$, $0 \le m \le M-1$.

Then, call the map

$n = lM + m$ the row-wise storage map, since this stores the entries $x[n]$ in the following fashion. The first M entries occupy the first row. The next M entries occupy the second row, and so on.

$l \backslash m$	0	1		$M-1$
0	$x[0]$	$x[1]$	$\cdots$	$x[M-1]$
1	$x[M]$	$x[M+1]$	$\cdots$	$x[2M-1]$
	$\vdots$	$\vdots$	$\vdots \; \cdots$	$\vdots$
$L-1$	$x[M(L-1)]$	$x[M(L-1)+1]$		$x[ML-1]$

Alternately, there is the columnwise storage map $n = l + mL$, i.e. first L entries are stored in the first column, next L entries in the second column and so on.

$x[0]$	$x[L]$	$\cdots$	$x[L(M-1)]$
$x[1]$	$x[L+1]$		$x[L(M-1)+1]$
		$\vdots$	
$x[L-1]$	$x[2L-1]$		$x[ML-1]$

→ The divide and conquer algorithm

We shall store $x[n]$ columnwise and $X(k)$ row wise., i.e.,

$$x(l,m) = x[mL+l], \quad 0 \le m \le M-1, \quad 0 \le l \le L-1$$

$$X(p,q) = X(Mp+q), \quad 0 \le p \le L-1, \quad 0 \le q \le M-1$$

Now,

$$X(p,q) = \sum_{n=0}^{N-1} x[n] W_N^{(Mp+q)n}$$

$$= \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x[mL+l] . W_N^{(Mp+q)(mL+l)}$$

$$= \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x[mL+l] W_N^{Npm+qmL+Mpl+ql}$$

$$= \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x(l,m) . W_M^{qm} W_L^{pl} . W_N^{ql}$$

$$\left[\because W_N^{Npm} = 1 \right.$$

$$W_N^{qmL} = W_M^{qm}$$

$$\left. W_N^{Mpl} = W_L^{lp} \right]$$

$$= \sum_{l=0}^{L-1} \left(W_N^{ql} \cdot \sum_{m=0}^{M-1} x(l,m) W_M^{qm} \right) \cdot W_L^{lp}$$

This then gives us the divide and conquer methodology.

- Step 1 :- Organise the input columnwise

  i.e. $x(l,m) = x[mL + l]$

- Step 2 :- Compute M-point DFT of each row, and store it in a new grid,

$$F(l,q) = \sum_{m=0}^{M-1} x(l,m) \cdot W_M^{qm},$$

$$0 \leq l \leq L-1, \quad 0 \leq q \leq M-1$$

- Step 3:- Update the new grid as follows.

$$G(l,q) = W_N^{lq} \cdot F(l,q)$$

$$0 \le l \le L-1, \quad 0 \le q \le M-1$$

- Step 4:- Calculate the L point DFT of each of the columns of $G(l,q)$ and this gives the matrix $X(p,q)$ which corresponds to $X(k)$ stored row wise, i.e., $X(p,q) = X(pM+q)$.

This follows by noting

$$X(p,q) = \sum_{l=0}^{L-1} G(l,q) \cdot W_L^{lp}$$

→ Computational Complexity

Step 2 computes L M-point DFTs

So, complex $\times \rightarrow LM^2$

complex $+ \rightarrow LM^2 - LM$

Step 3 computes LM complex multiplications

Step 4 computes M L-point DFTs,

So, complex $\times \rightarrow ML^2$

complex $+ \rightarrow ML^2 - ML$

Thus,

$$\boxed{\begin{aligned} &\text{Total complex } \times \rightarrow LM^2 + LM + ML^2 \\ &\qquad = ML(M+L+1) \\ &\qquad = N(M+L+1) \\ &\text{Total complex } + \rightarrow LM^2 - LM + ML^2 - ML \\ &\qquad = ML(L+M-2) \\ &\qquad = N(L+M-2) \end{aligned}}$$

→ Remarks

i) For $M \geq 2$, $L \geq 2$, $ML \geq M+L$.

Thus the number of computation reduce.

ii) If N has a large number of prime factors then this divide and conquer methodology can be subdivided further.

* Radix 2-FFT- Decimation in time

Let $N = 2^{\nu}$. Otherwise zero pad sequences to make $N = 2^{\nu}$.

Apply the divide and conquer approach with $L = 2$ and $M = \frac{N}{2} = 2^{\nu - 1}$

In other words

Let $f_1(n) = x[2n], \quad n = 0, \ldots, \frac{N}{2} - 1$

$f_2(n) = x[2n+1], \quad n = 0, \ldots, \frac{N}{2} - 1$

Now, observe that

$$X(k) = \sum_{n=0}^{N-1} x(n]\, W_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} f_1(n) \cdot W_N^{2kn} + W_N^k \sum_{n=0}^{N/2-1} f_2(n) \cdot W_N^{2kn}$$

$$= \sum_{n=0}^{N/2-1} f_1(n) \cdot W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} f_2(n)\, W_{N/2}^{kn}$$

$$\left[\because W_{N/2}^{kn} = W_N^{2kn}\right]$$

Thus, $\boxed{X(k) = F_1(k) + W_N^k F_2(k)}$

where $F_i(k)$ is $\frac{N}{2}$-point DFTs of $f_i(n)$, $i = 1, 2$

$k = 0, 1, 2, \ldots, N-1$

Next, note that since F_1 and F_2 are $\frac{N}{2}$-point DFTs and hence

$$\boxed{F_i\left(k + \frac{N}{2}\right) = F_i(k), \quad k = 0, 1, \ldots, \frac{N}{2} - 1}$$

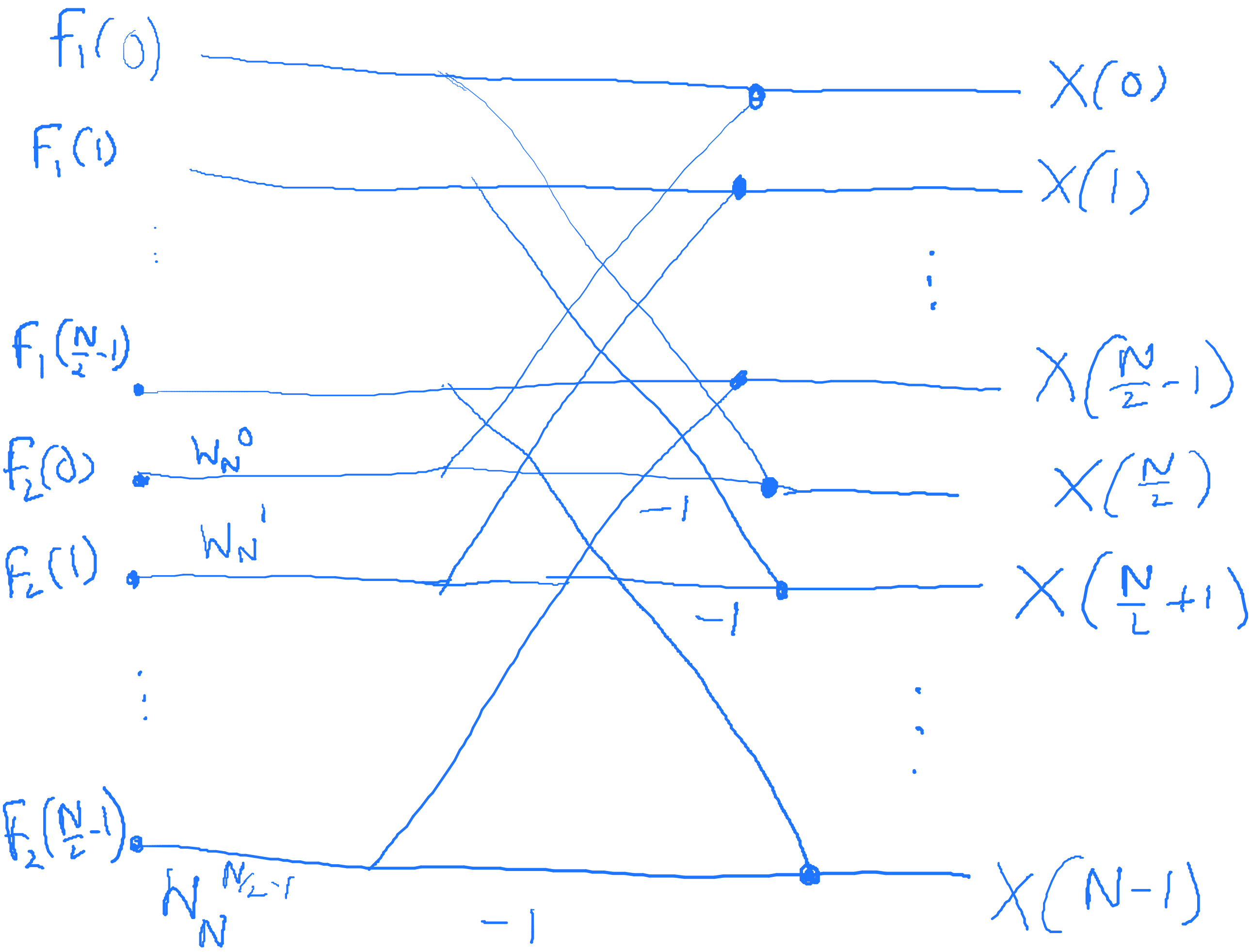
On the other hand,

$$\boxed{\begin{aligned} W_N^{k+\frac{N}{2}} &= e^{-j\frac{2\pi}{N}k} e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}} \\ &= -e^{-j\frac{2\pi}{N}\cdot k} \\ &= -W_N^k, \quad k = 0, 1, \ldots, \frac{N}{2} - 1 \end{aligned}}$$

Thus,

$$\boxed{\begin{aligned} X(k) &= F_1(k) + W_N^k F_2(k), \\ X\left(k+\tfrac{N}{2}\right) &= F_1(k) - W_N^k F_2(k), \\ k &= 0, 1, \ldots, \tfrac{N}{2}-1 \end{aligned}}$$

We can draw the computations so far as follows



Here each • (in bold) represents a complex addition.

Note that the

$$\boxed{\begin{aligned} \#\text{ multiplications} &= \frac{N}{2} + \#\text{ multiplications to get } \frac{N}{2}\text{-point DFTs} \\ \#\text{ additions} &= N + \#\text{ additions to get the } \frac{N}{2}\text{-point DFTs} \end{aligned}}$$

Next, since $\frac{N}{2} = 2^{\nu-1}$, we can again use divide and conquer to compute both $F_1(k)$ and $F_2(k)$.

In the same way as before

let $v_{11}[n] = f_1[2n]$, $n = 0, 1, \dots, \frac{N}{4} - 1$

$v_{12}[n] = f_1[2n+1]$, $n = 0, 1, \dots, \frac{N}{4} - 1$

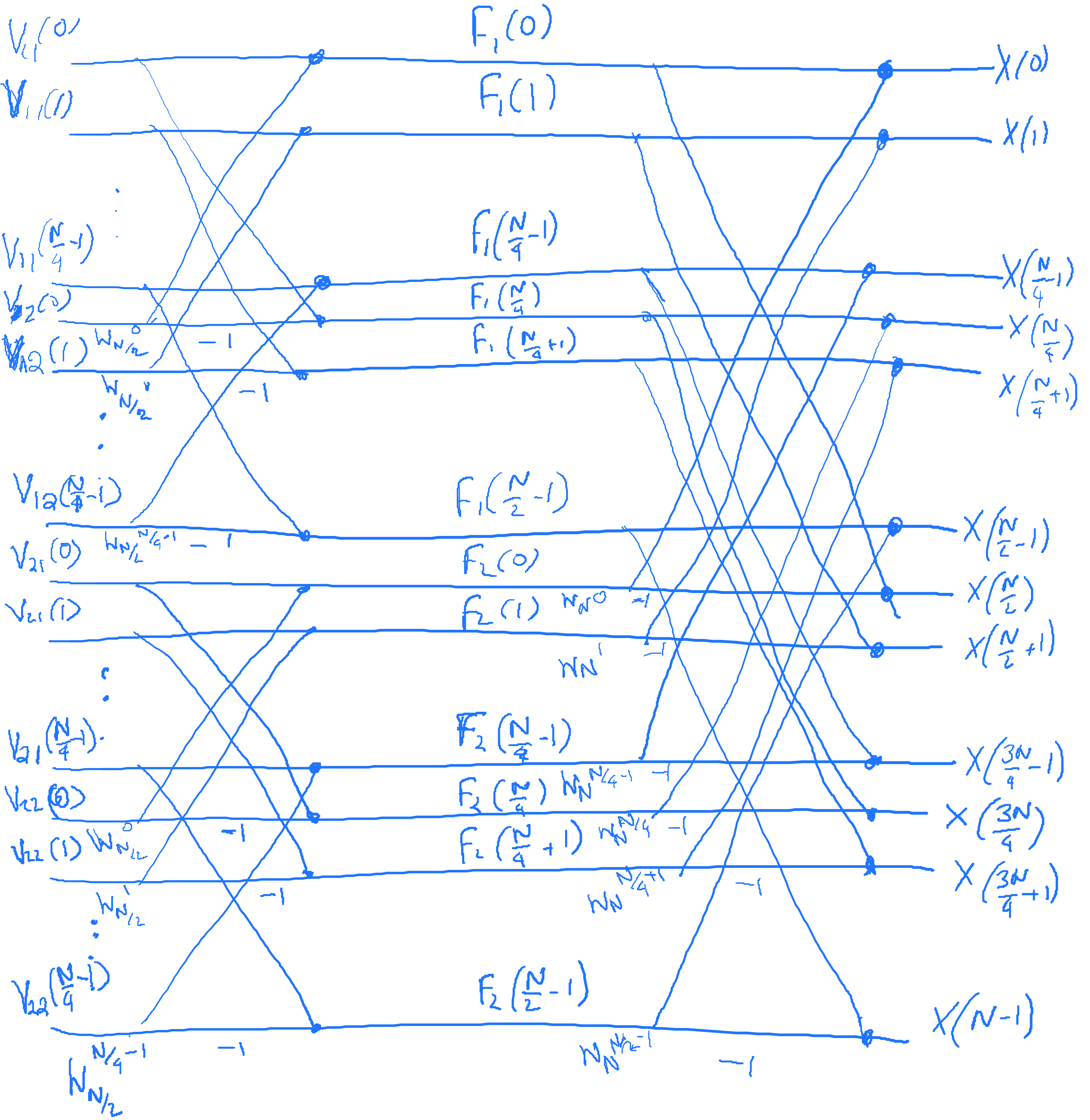
$v_{21}[n] = f_2[2n]$, $n = 0, 1, \dots, \frac{N}{4} - 1$

$v_{22}[n] = f_2[2n+1]$, $n = 0, 1, \dots, \frac{N}{4} - 1$

Then, arguing as before we can obtain the following,

$$\boxed{\begin{aligned} F_1(k) &= V_{11}(k) + W_{N/2}^{k} V_{12}(k), \\ F_1\left(k + \frac{N}{4}\right) &= V_{11}(k) - W_{N/2}^{k} V_{12}(k), \\ F_2(k) &= V_{21}(k) + W_{N/2}^{k} V_{22}(k) \\ F_2\left(k + \frac{N}{4}\right) &= V_{21}(k) - W_{N/2}^{k} V_{22}(k) \\ k &= 0, 1, \dots, \frac{N}{4} - 1 \end{aligned}}$$

Let's draw the computations so far



So now we have two stages,

Each stage contains $\frac{N}{2}$ multiplication

and N additions.

So far

$$\boxed{\begin{aligned}\#\text{multiplications} &= 2\cdot\frac{N}{2} + \#\text{multiplications to compute the } \tfrac{N}{4}\text{-point DFTs}\\ \#\text{additions} &= 2\cdot N + \#\text{additions to compute the } \tfrac{N}{4}\text{-point DFT}\end{aligned}}$$

We can reapply divide and conquer again to compute these

4 $\frac{N}{4}$-point DFTs, $V_{11}(k)$, $V_{12}(k)$, $V_{21}(k)$, $V_{22}(k)$; and increase

another stage.

So, if i stages are used,

we have as inputs the outputs

of the 2^i $\frac{N}{2^i}$-point DFTs.

As usual, each stage will have

$\frac{N}{2}$ multiplications, and N additions.

Thus

$$\boxed{\begin{aligned} \#\text{multiplications} &= \frac{N}{2}i + \#\text{multiplication needed for } \frac{N}{2^i}\text{-point DFTs} \\ \#\text{additions} &= Ni + \#\text{addition needed for } \frac{N}{2^i}\text{-point DFTs} \end{aligned}}$$

Note that this process can be continued till $\frac{N}{2^i} = 1$, i.e., $i = \nu = \log_2 N$

However, note that a 1-point DFT requires no additions or multiplications.

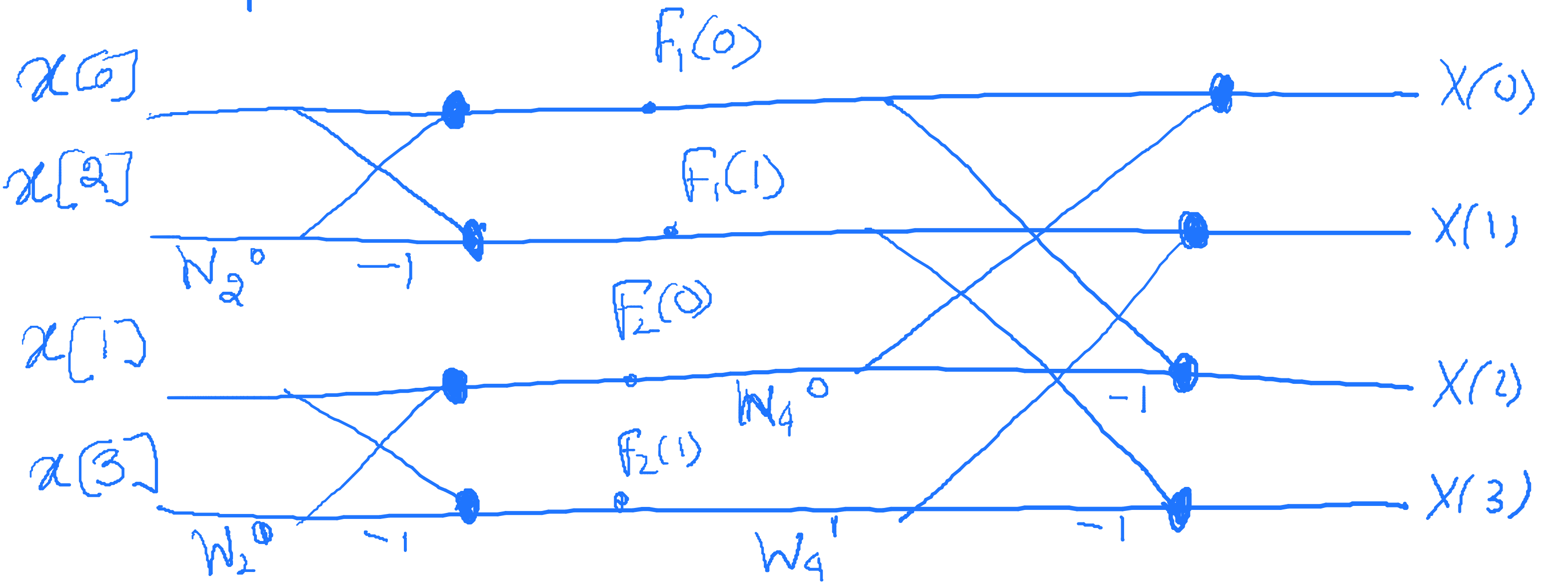
Thus,

$$\boxed{\begin{aligned} \#\,\text{multiplications} &= \frac{N}{2}\log_2 N \\ \#\,\text{additions} &= N\log_2 N \end{aligned}}$$

→ Remarks

This method is called decimation in time since the order of the inputs (i.e., the time part) is shuffled.

To see this, let $N = 2^2 = 4$. Then, we shall have two stages and the computation diagram is



In general, the input has to be arranged in the bit-reversed order, i.e., the position of the input is obtained by bit reversing the ν-bit binary representation of its natural location.

* Radix-2 FFT- Decimation in frequency

This is the divide and conquer approach with $L = \frac{N}{2}$, $M = 2$, and $N = 2^{\nu}$.

Here,

$$X(k) = \sum_{n=0}^{N/2-1} x[n]\, W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n+\frac{N}{2}\right] \cdot W_N^{k\left(n+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{N/2-1} x[n]\, W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{N/2-1} x\left[n+\frac{N}{2}\right] W_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} \left(x[n] + (-1)^k x\left[n+\frac{N}{2}\right]\right) \cdot W_N^{kn}$$

$$\left[\because W_N^{kN/2} = e^{-j\frac{2\pi}{N}\cdot k \cdot \frac{N}{2}} = (-1)^k\right]$$

Decimating in frequency, we have

$$X(2k) = \sum_{n=0}^{N/2-1} \left(x[n] + x\left[n+\tfrac{N}{2}\right]\right) \cdot W_N^{2kn},$$

$$k = 0, 1, \ldots, N/2 - 1$$

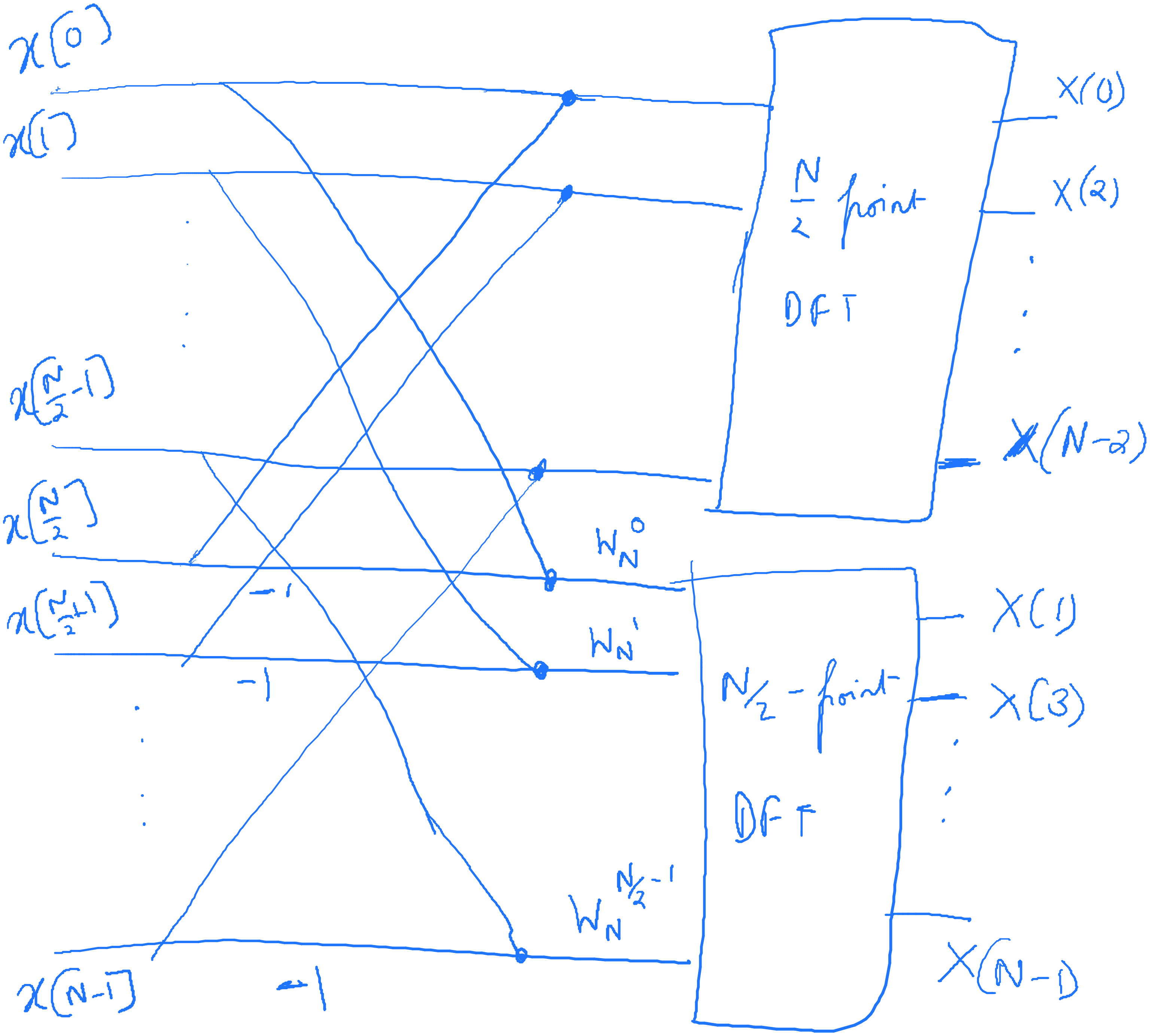
$$= \sum_{n=0}^{N/2-1} \left(x[n] + x\left[n+\tfrac{N}{2}\right]\right) W_{N/2}^{kn}$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} \left[\left(x[n] - x\left[n+\tfrac{N}{2}\right]\right) \cdot W_N^{n}\right] \cdot$$

$$W_{N/2}^{kn}$$

$$k = 0, 1, \ldots, \tfrac{N}{2} - 1$$

Thus, the butterfly diagram at this stage looks like



Thus, number of computations so far

$\#$ multiplication $= \frac{N}{2} + \#$ multiplications needed for the $\frac{N}{2}$-point DFT

$\#$ additions $= N + \#$ additions needed for the $\frac{N}{2}$-points DFTs

The two $\frac{N}{2}$-point DFTs can be again subjected to decimation in frequency divide-and-conquer.

Let

$$g_1[n] = x_1[n] + x_1\left[n+\frac{N}{2}\right]$$

$$g_2[n] = \left(x_1[n] - x_1\left[n+\frac{N}{2}\right]\right)\cdot W_N^{n},$$

$$n = 0, 1, \ldots, \frac{N}{2}-1$$

Then, just as in the previous part, we can get

$$G_1(2k) = \sum_{n=0}^{N/4-1}\left(g_1[n] + g_1\left[n+\frac{N}{4}\right]\right)\cdot W_{N/4}^{kn}$$

$$G_1(2k+1) = \sum_{n=0}^{N/4-1}\left[\left(g_1[n] - g_1\left[n+\frac{N}{4}\right]\right)\cdot W_{N/2}^{n}\right]\cdot W_{N/4}^{kn}$$

$$k = 0, 1, \ldots, \frac{N}{4}-1$$

$$G_2(2k) = \sum_{n=0}^{N/4-1} \left(g_2[n] + g_2\left[n+\frac{N}{4}\right] \right) \cdot W_{N/4}^{kn}$$

$$G_2(2k+1) = \sum_{n=0}^{N/4-1} \left[\left(g_2[n] - g_2\left[n+\frac{N}{4}\right] \right) \cdot W_{N/2}^{k} \right] \cdot W_{N/4}^{kn}$$

$$k = 0, 1, 2, \ldots, \frac{N}{4} - 1$$

We see that in this stage we again require $2 \cdot \frac{N}{4} = \frac{N}{2}$ complex multiplications and $4 \cdot \frac{N}{4} = N$ complex additions.

Following this we will still need to perform 4 $\frac{N}{4}$-point DFTs.

Note that we can continue the divide-and-conquer strategy for $\nu = \log_2 N$ stages, after which we end up with N 1-point DFTs, which require no calculations.

Since each stage requires $\frac{N}{2}$ multiplications and N-additions, we have

$$\boxed{\begin{aligned} \#\text{ multiplications} &= \frac{N}{2}\log_2 N \\ \#\text{ additions} &= N\log_2 N \end{aligned}}$$

* Practical Filters

i) Ideal filters are non-causal.

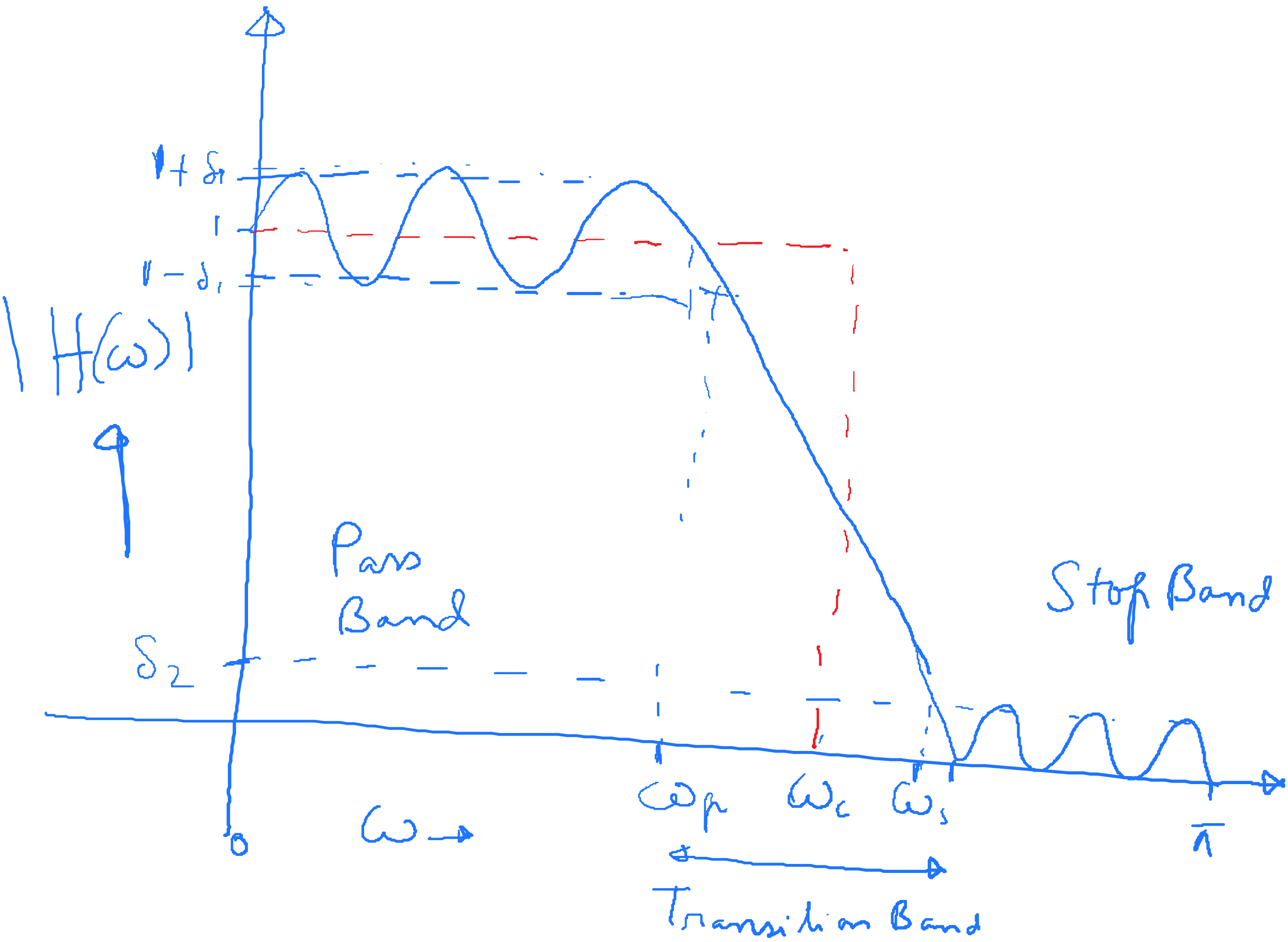
ii) Causal $\Rightarrow$ There are no intervals $[\omega_1, \omega_2]$ for which $|H(\omega)| = 0$.

This follows from the Paley-Wiener Theorem.

Paley-Wiener Theorem :- If $h[n]$ is causal with finite energy, then

$$\int_{-\pi}^{\pi} \left| \ln |H(\omega)| \right| d\omega < \infty$$

As a result, a practical filter's $|H(\omega)|$ the following shape.



- - - - Corresponding ideal filter

$\omega_p \rightarrow$ Pass band edge

$\omega_s \rightarrow$ Stop band edge

$|\omega_s - \omega_p| \rightarrow$ Transition band length

$\left.\begin{array}{l}\delta_1 \rightarrow \text{Pass band ripple} \\ \delta_2 \rightarrow \text{Stop band ripple}\end{array}\right] \rightarrow$ Often specified in dB.

* FIR Filters and Linear Phase systems

Key idea :- To use linear phase systems to ensure constant group delay. i.e., all frequency components are time delayed by the same amount.

- Type I and II Linear Phase systems

  M taps s.t.

  i) $h[n] = h[M-1-n]$, $0 \le n \le \frac{M-3}{2}$

  M is odd

  — Type I

  ii) $h[n] = h[M-1-n]$, $0 \le n \le \frac{M-2}{2}$,

  M is even

  — Type II

Recall that

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \left[h\left[\frac{M-1}{2}\right] + \sum_{n=0}^{\frac{M-3}{2}} 2h[n] \cdot \cos\left(\omega\left(n - \frac{M-1}{2}\right)\right) \right]$$

— Type I

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \sum_{n=0}^{\frac{M}{2}-1} 2h[n] \cos\left(\omega\left(n - \frac{M-1}{2}\right)\right)$$

— Type II

Note that in type II $\left(n - \frac{M-1}{2}\right)$ is a multiple of $\frac{1}{2}$, and hence at $\omega = \pi$, $H(\omega) = 0$.

Thus type II can never be used for high pass filters.

There are also Type III and IV linear phase systems, where

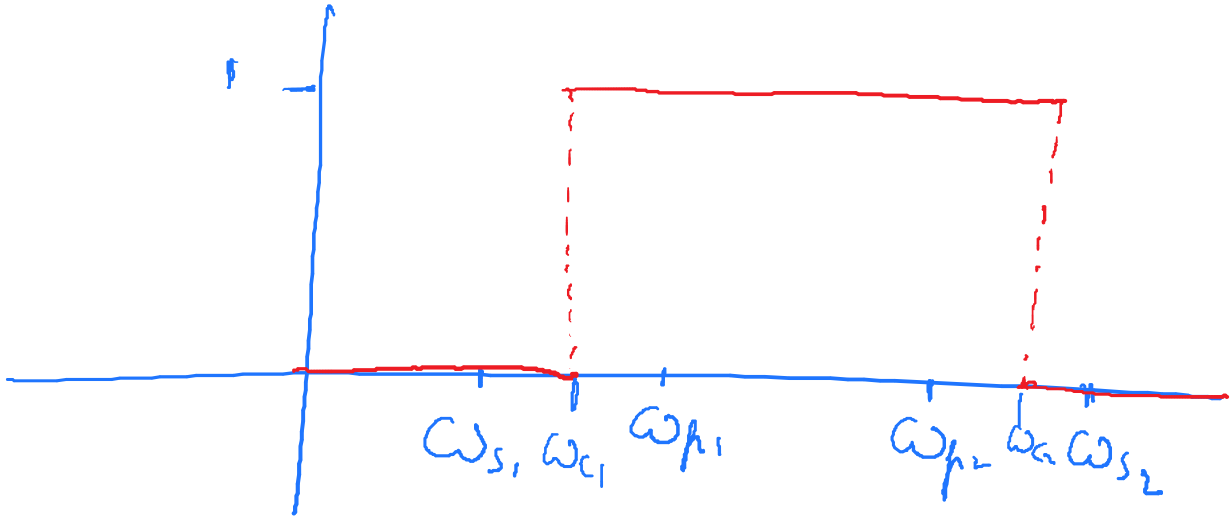
$$h[n] = -h[M-1-n].$$

We shall not need these for the filter design methods in this course.

* Method of Windows

- Ideal filter consists only of $H(\omega) = 1, 0$. Generally if Passband and stop band edges are specified, we take the centre/notch/cut off frequencies in the middle of the transition band.

Example:- BPF $\omega_{s_1}, \omega_{p_1}, \omega_{p_2}, \omega_{s_2}$



$$\omega_{c_1} = \frac{\omega_{s_1}+\omega_{p_1}}{2}, \quad \omega_{c_2} = \frac{\omega_{p_2}+\omega_{s_2}}{2}$$

Thus in this example

$$H_{ideal}(\omega) = 1, \quad \omega \in [\omega_{c_1}, \omega_{c_2}]$$
$$= 0, \quad \omega \in [0, \omega_{c_1}) \cup (\omega_{c_2}, \pi]$$

- Rectangular window

$h_{ideal}[n]$ will be a phase-shifted Sinc, which is non-causal.

Truncate & shift h_{ideal} to samples $0, 1, \dots, M-1$, and make all remaining samples to be 0. This gives us a new filter

$$h[n] = h_{ideal}\left[n - \frac{M-1}{2}\right], \quad n = 0, 1, \dots, M-1$$
$$= 0, \quad \text{o.w.}$$

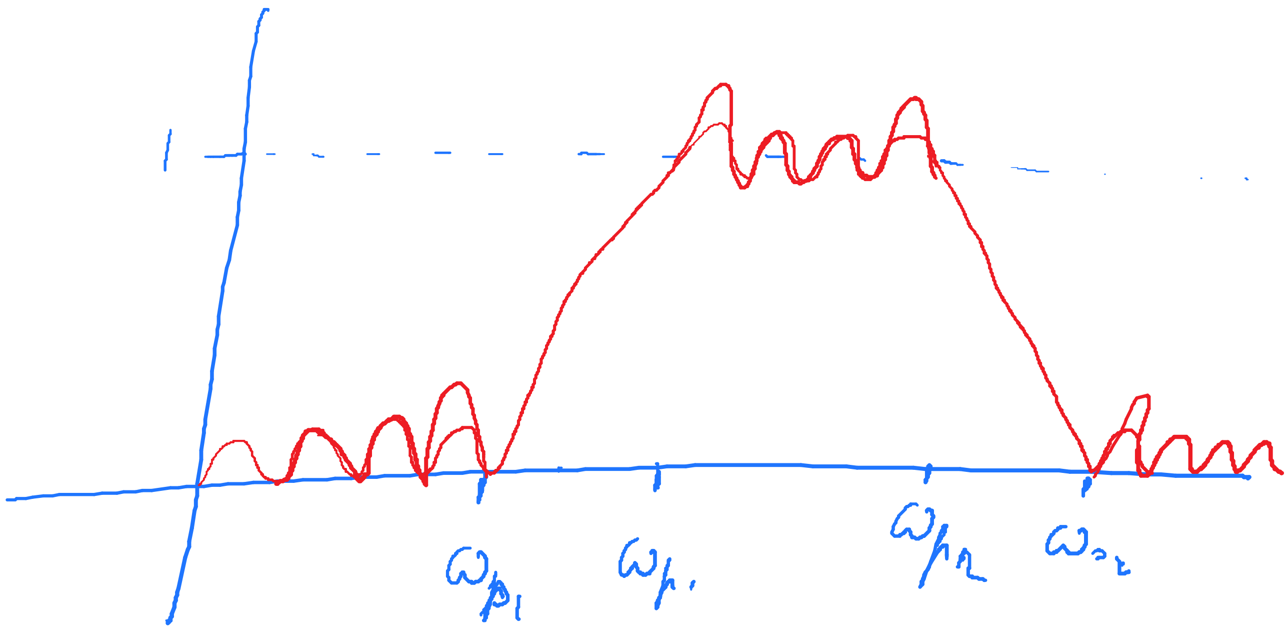
In other words, it is as if $h_{ideal}[n]$ has been multiplied by a rectangular window, i.e.,

$$\boxed{\begin{aligned} h[n] &= h_{ideal}\left[n - \frac{M-1}{2}\right] \cdot w_{rect}[n] \\ \text{where } w_{rect}[n] &= 1, \quad n = 0, 1, \dots, M-1 \\ &= 0, \quad \text{o.w.} \end{aligned}}$$

Thus, we have the frequency response of the resulting filter being given by,

$$H(\omega) = \frac{1}{2\pi} \cdot H_{ideal}(\omega) \cdot e^{-j\omega\left(\frac{M-1}{2}\right)} * W_{rect}(\omega)$$

Since $W_{rect}(\omega)$ is a sinc function, $|H(\omega)|$ will have the following shape



Issues of rectangular window:-

i) Very high ripples near band edges

ii) Overall area under the stopband

ripples does not change with M.

- Goal - Come up with better windows such that

  i) Windows are linear-phase to ensure the final filter is linear phase, since ideal filter has $H(\omega)$ to be real.

  ii) High ripples at band edges can be avoided.

There are example window functions such at Bartlett, Hanning, Hamming etc.

All other windows will have a larger transition band than rectangular window

We want the 'best' possible window, i.e., the one with the transition band being larger than only the rectangular window, with the least possible ripples.

The Kaiser window is a window which is 'almost best', which we study below.

- Kaiser Window

$$\boxed{\begin{aligned} w_K[n] &= \frac{I_0\left(\beta\sqrt{1-\left(\frac{n-\frac{M-1}{2}}{\frac{M-1}{2}}\right)^2}\right)}{I_0(\beta)}, \quad 0 \le n \le M-1 \\ &= 0 \quad , \text{o.w.} \end{aligned}}$$

where $I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta} \cdot d\theta$,

is called the zeroth order modified Bessel function of the first kind

Now, note that

$$w_K[M-1-n]$$

$$= \frac{1}{I_0(\beta)} \cdot I_0\left(\beta\sqrt{1-\left(\frac{\frac{M-1}{2}-n}{\frac{M-1}{2}}\right)^2}\right)$$

$$= w_K[n].$$

Thus, $w_K[n]$ is linear phase.

To design a filter with

$\delta_1, \delta_2, \omega_{s1}, \omega_{p1}, \omega_{p2}, \omega_{s2}, \omega_{s3}, \omega_{p3}, \dots$

let $\boxed{\Delta\omega = \min_i \left[|\omega_{s_i} - \omega_{p_i}| \right] \rightarrow \text{Minimum transition band}}$

Also, let $\boxed{\delta = \min\{\delta_1, \delta_2\}}$, and

$$\boxed{A = -20 \log_{10} \delta.}$$

Then take

$$\boxed{\begin{aligned} M &\simeq 1 + \frac{A-8}{2.285\,\Delta\omega} \\ \beta &\simeq 0.1102(A-8.7), \quad A > 50 \\ &\simeq 0.5842(A-21)^{0.4} + 0.07886(A-21), \quad 21 \le A \le 50, \\ &\simeq 0, \quad A < 21 \end{aligned}}$$

Example design using python :- [Show python script side by side]

Design a BPF with

$\Omega_{s_1} = 20$ rad/s $\qquad \delta_2 \leq -35$ dB

$\Omega_{p_1} = 40$ rad/s $\qquad \delta_1 > \delta_2$

$\Omega_{p_2} = 60$ rad/s $\qquad \Omega_s = 200$ rad/s

$\Omega_{s_2} = 80$ rad/s $\qquad$ ↳ Sampling angular frequency

Step 1:- We need to get M, β for the Kaiser window.

This is done using the function 'kaiserord'.

kaiserord requires two arguments ripple $A = -35$ dB and width of the transition band.

Note here that $\Delta\omega$ is the discrete-time angular frequency measured in radians/sample.

Here we have continuous-time angular frequencies in rad/s.

Note that

$$f = \frac{F}{F_s} = \frac{\Omega}{\Omega_s}$$

$$\Rightarrow \omega = 2\pi \frac{\Omega}{\Omega_s}$$

$$\therefore \Delta\omega = 2\pi \cdot \frac{\Delta\Omega}{\Omega_s}$$

Now,

$$\Delta\Omega = \min\{\Omega_{p_1} - \Omega_{s_1}, \Omega_{s_2} - \Omega_{p_2}\}$$

Finally, kaiserord requires width normalized between 0 to 1, as opposed to 0 to π.

Thus width $= \dfrac{\Delta\omega}{\pi}$

$$= 2\frac{\Delta\Omega}{\Omega_s}$$

kaiserord return M, β.

Use 'windows.kaiser(M,β)' to get $w_K[n]$.

Now, to get the filter $h[n] = w_K[n] \cdot h_{ideal}\left[n - \left(\frac{M-1}{2}\right)\right]$

we shall use 'firwin' function.

'firwin' needs the following arguments

M - We already have this

Normalized ideal cutoff frequencies of $h_{ideal}[n]$

Window type — We are going to use Kaiser with β which we already have

Pass_zero = False [$\because \omega = 0$ belongs in a stop band.

Now, to get the normalized cutoff frequencies of the ideal filter, first note that the cutoff frequencies are

$$\Omega_{c_1} = \frac{\Omega_{p_1} + \Omega_{s_1}}{2}$$

$$\Omega_{c_2} = \frac{\Omega_{p_2} + \Omega_{s_2}}{2}$$

Thus, $f_{c_1} = \frac{\Omega_{c_1}}{\Omega_s}$, $f_{c_2} = \frac{\Omega_{c_2}}{\Omega_s}$

Now, note that discrete-time frequency runs from 0 to $\frac{1}{2}$.

But 'firwin' wants it scaled to 0 to 1

Thus, $\text{normalized\_}f_{c_1} = 2 \cdot f_{c_1} = 2 \cdot \frac{\Omega_{c_1}}{\Omega_s}$

$\text{normalized\_}f_{c_2} = 2 f_{c_2} = 2 \frac{\Omega_{c_2}}{\Omega_s}$

With these inputs, 'firwin' outputs the required filter $h(n)$.

- Disadvantage of windows

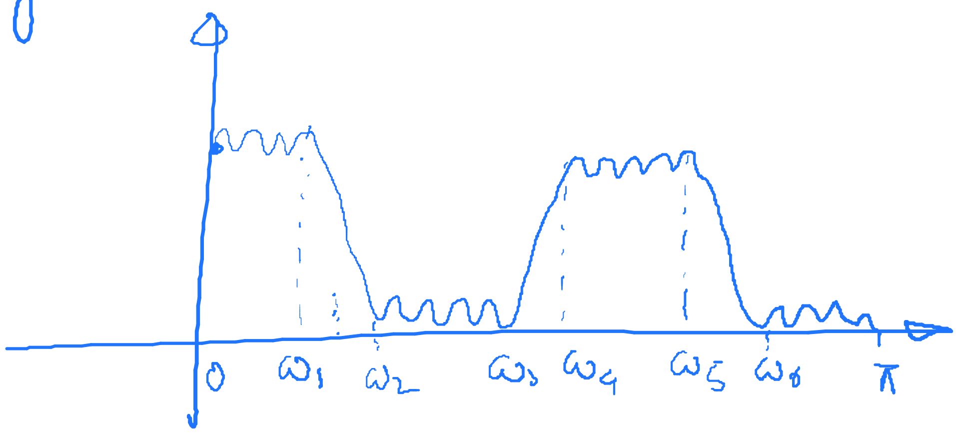
a) Ripples are of different size

b) They are not optimum, i.e., for a given M, the filters with the least ripple δ.

* Optimum FIR Filters
– Equiripple Approximation.

Let $S_1, S_2, \ldots, S_\ell$ be the various pass and stop bands.

eg.



Here $\ell = 4$

$S_1 = [0, \omega_1]$

$S_2 = [\omega_2, \omega_3]$

$S_3 = [\omega_4, \omega_5]$

$S_4 = [\omega_6, \pi]$

Let each band have ripples $\delta_1, \delta_2, \dots, \delta_L$.

Consider a weight function $W(\omega)$ given

as $W(\omega) = \frac{\delta_i^*}{\delta_i}$, $\omega \in S_i$, where $\delta_i^* = \max_i \{\delta_i\}$

Now consider the ideal filter $H_{ideal}(\omega)$

such that $H_{ideal}(\omega) = 1$, if ω is in

a pass band, $H_{ideal}(\omega) = 0$, if ω is

in a stop band.

In the example,

$$H_{ideal}(\omega) = 1, \quad \omega \in S_1, S_3$$
$$= 0, \quad \omega \in S_2, S_4.$$

Now, consider approximating this ideal

filter with a linear phase system $H(\omega)$

Define error

$$E(\omega) = W(\omega)\left[H_{ideal}(\omega) - H(\omega)\right]$$

Notice that if $H(\omega)$ meets the ripple criteria in each of the bands

$S_1, S_2, \ldots, S_\ell$,

then $|E(\omega)| \leq \delta_i^*$, if $\omega \in S_1 \cup S_2 \cup \cdots \cup S_\ell$

The goal is then to find coefficients of $H(\omega)$ and the minimum M to ensure

$|E(\omega)| \leq \delta_i^*$.

Alternately, one choose to fix M, and find the filter coefficients that minimizes

$$\max_{\omega \in S_i} |E(\omega)|.$$

This second procedure can be solved by the Remez Algorithm, and gives an FIR filter whose ripples are minimum among all other FIR linear phase filters with same M.

Moreover the ripples in different bands obey the ratio specified by $W(\omega)$.

* Equiripple filter design example using Scipy

Scipy has a built-in Remez algorithm solver and we shall use that.

Firstly note that while Remez gives the minimum ripple possible for a fixed M, a filter design problem on the other hand will specify ripples (generally δ_1 and δ_2 for each pass and stop bands respectively) and not M.

To calculate an initial guess for M, use an empirical formula given by Kaiser

$$\boxed{M = \left\lceil \frac{-20\log_{10}\sqrt{\delta_1 \delta_2} - 13}{14.6\,\Delta f} \right\rceil + 1, \qquad \Delta f = \min_i \frac{|\omega_{s_i} - \omega_{p_i}|}{2\pi}}$$

If this M fails to meet the specified ripple, increase M by 1 and retry.

We now show the exact design for the example studied before.

Design specifications:-

$\Omega_{s_1} = 20$ rad/s $\qquad \delta_1 \le -24.5$ dB

$\Omega_{p_1} = 40$ rad/s $\qquad \delta_2 \le -35$ dB

$\Omega_{p_2} = 60$ rad/s $\qquad \Omega_o = 200$ rad/s

$\Omega_{s_2} = 80$ rad/s

- Step 1:- Initial guess for M.

To make the initial guess, we note that

$$-20\log\sqrt{\delta_1\delta_2} = \frac{1}{2}\left(-20\log\delta_1 - 20\log\delta_2\right)$$

$$= \frac{1}{2}\left(24.5+35\right)$$

Now to get Δf, recall that

$$f = \frac{\Omega}{\Omega_s}.$$

Thus, $$\Delta f = \min_i \left\{ \frac{|\Omega_{si} - \Omega_{pi}|}{\Omega_s} \right\}$$

- Step 2 :- Use scipy.signal.remez to get the filter coefficients

Remez requires the following inputs :-

M — Already calculated

The Bands —

Desired — An array indicating which as pass and which are stop bands

The weights —

Here the bands are

$$[0, \Omega_{s_1}], [\Omega_{p_1}, \Omega_{p_2}], [\Omega_{s_2}, \frac{\Omega_s}{2}]$$

Thus, in terms of f, the bands are

$$\left[0, \frac{\Omega_{s_1}}{\Omega_s}\right], \left[\frac{\Omega_{p_1}}{\Omega_s}, \frac{\Omega_{p_2}}{\Omega_s}\right], \left[\frac{\Omega_{s_2}}{\Omega_s}, \frac{1}{2}\right]$$

So we need to input the array

$$\left[0, \frac{\Omega_{s_1}}{\Omega_s}, \frac{\Omega_{p_1}}{\Omega_s}, \frac{\Omega_{p_2}}{\Omega_s}, \frac{\Omega_{s_2}}{\Omega_s}, \frac{1}{2}\right]$$

Desired means for stop bands the array should indicate 0, and one for pass.

Since in this example, the bands are Stop, pass, Stop, the desired array is

$$[0, 1, 0]$$

For the weights, let $\delta^* = \max\{\delta_1, \delta_2\}$

These being expressed as numbers not dB.

Then the weights are

$$\left[\frac{\delta^*}{\delta_a}, \frac{\delta^*}{\delta_1}, \frac{\delta^*}{\delta_2} \right].$$

In numbers, not dB.

With these outputs, remez will return the desired coefficients.

* IIR Filter design

Recipe :-

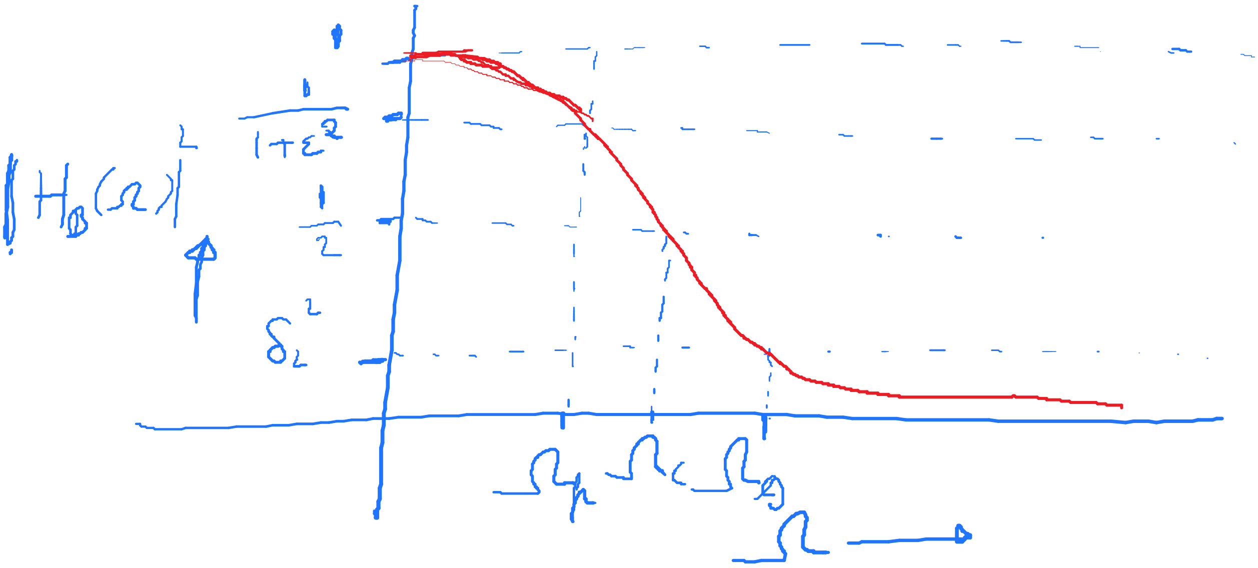
1. Design Analog lowpass filter
2. Use Frequency transformation to convert it to a high pass, band pass, bandstop filter

3. Apply Bilinear transform to get a digital filter

* Analog Low Pass Filters

i) Butterworth Filters

$$|H_B(\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}} = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}}$$



So ε is determined from the pass band ripple. In particular,

$$\boxed{(1-\delta_1)^2 = \frac{1}{1+\varepsilon^2}}$$

Ω_c – 3-dB cutoff frequency, i.e. $|H_B(\Omega_c)|^2 = \frac{1}{2}$

Ω_p is the pass band edge | Easy to see that $|H_B(\Omega_p)|^2 = \frac{1}{1+\varepsilon^2}$

Ω_s is the stop band edge

N is determined as follows.

At $\Omega = \Omega_s$, $|H_B(\Omega)| = \delta_2$

Thus,

$$\frac{1}{1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}} = \delta_2^2$$

$$\Rightarrow \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{1-\delta_2^2}{\delta_2^2\varepsilon^2}$$

$$\Rightarrow N \cdot \log\left(\frac{\Omega_s}{\Omega_p}\right) = \log \frac{\sqrt{1-\delta_2^2}}{\delta_2\varepsilon}$$

$$\Rightarrow \boxed{N = \frac{\log\left(\frac{\sqrt{1-\delta_2^2}}{\delta_2\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

Finally, to obtain Ω_c,

we plug in $\Omega = \Omega_p$ to obtain that

$$\varepsilon^2 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}$$

$$\Rightarrow \frac{\Omega_p}{\Omega_c} = \varepsilon^{1/N}$$

$$\Rightarrow \boxed{\Omega_c = \Omega_p \cdot \varepsilon^{-1/N}}$$

- Design example :-

Design a low pass Butterworth filter with $\delta_1 \leq -24.5\,dB$,

$\delta_2 \leq -35\,dB$, $\Omega_p = 20\,rad/s$

$\Omega_s = 40\,rad/s$

We are going to use scipy.signal.butter

butter needs two inputs, Ω_c and N, which we can calculate from $\Omega_p, \Omega_s, \delta_1, \delta_2$. In addition we need to set 'analog = True' in butter.

butter outputs the filter numerator and denominator polynomials of

$$H_B(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n}$$

where $s = R + j\Omega$ is the Laplace transform variable.

Once b, a are obtained using butter, scipy.signal.freqs can be used to obtain $|H_B(\Omega)|$ from b, a.

ii) Chebyshev - Type I Filter

Define the Chebyshev Polynomial

$$\boxed{\begin{aligned} T_N(x) &= \cos\left(N \cos^{-1} x\right), \quad |x| \leq 1 \\ &= \cosh\left(N \cosh^{-1} x\right), \quad |x| > 1 \end{aligned}}$$

Note that

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

While $\cosh$ is not an injective function, as $\cosh(x) = \cosh(-x)$, we define $\cosh^{-1}(x)$ to be the positive argument y satisfying $\cosh(y) = x$.

Then,

$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right).$$

T_N satisfies the following properties

i) $|T_N(x)| \leq 1, \quad |x| \leq 1.$

Follows by noting $|\cos(\cdot)| \leq 1$

ii) $T_N(1) = 1$

Follows by noting

$$\cos(N \cos^{-1} 1) = \cos(0) = 1$$

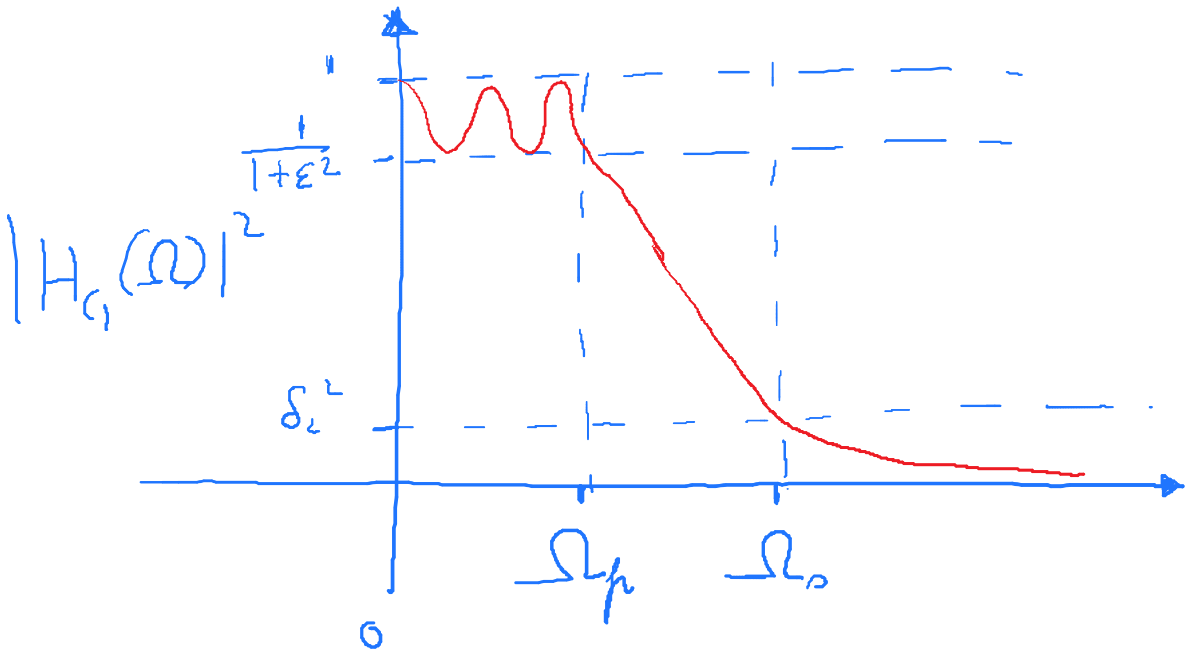
iii) All roots of the polynomial $T_N(x)$ lie in $|x| \leq 1.$

Follows by noting

$$\cosh(\cdot) > 0$$

The type I Chebyshev filter is given by

$$\boxed{|H_{c_1}(\Omega)|^2 = \frac{1}{1 + \varepsilon^2 T_N^2(\Omega/\Omega_p)}}$$



Here $\Omega_p \rightarrow$ Pass band edge

$\Omega_s \rightarrow$ Stop band edge

Note that as long as $\Omega \leq \Omega_p$,

$$T_N^2\left(\frac{\Omega}{\Omega_p}\right) \leq 1$$

Thus, $|H_{c_1}(\Omega)|^2 \geq \frac{1}{1+\varepsilon^2}$

We get the ripples in passband, since roots of $T_N(x)$ lie in $|x| \leq 1$.

Now, it is easy to see that

$$(1-\delta)^2 = \frac{1}{1+\varepsilon^2}$$

$$\Rightarrow \boxed{\varepsilon = \sqrt{\frac{1}{(1-\delta)^2} - 1}}$$

Now, since $|H_{C_1}(\Omega)|^2 = \delta_2^2$ at

$\Omega = \Omega_s$, we have,

$$\frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} = \delta_2^2$$

$$\Rightarrow \varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{1}{\delta_2^2} - 1 = \frac{1-\delta_2^2}{\delta_2^2}$$

$$\Rightarrow T_N\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{\sqrt{1-\delta_2^2}}{\delta_2\,\varepsilon}$$

Now $\Omega_s > \Omega_p$ and hence

$$T_N\left(\frac{\Omega_s}{\Omega_p}\right) = \cosh\left(N \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)$$

$$= \cosh\left(N \ln \underbrace{\left(\frac{\Omega_s}{\Omega_p} + \sqrt{\frac{\Omega_s^2}{\Omega_p^2} - 1}\right)}_{= y}\right)$$

$$= \frac{e^{N \ln y} + e^{-N \ln y}}{2}$$

$$= \frac{y^N + y^{-N}}{2}$$

Thus,

$$y^N + y^{-N} = \frac{2\sqrt{1-\delta_2^2}}{\delta_2 \varepsilon}$$

$$\Rightarrow y^{2N} - \frac{2\sqrt{1-\delta_2^2}}{\delta_2 \varepsilon} y^N + 1 = 0$$

$$\Rightarrow y^N = \frac{\sqrt{1-\delta_2^2}}{\delta_2 \varepsilon} \pm \sqrt{\frac{1-\delta_2^2(1+\varepsilon^2)}{\delta_2^2 \varepsilon^2}}$$

Since $y > 1$, $y^N > 1$.

Thus $y^N = \dfrac{\sqrt{1-\delta_2^2} + \sqrt{1-\delta_2^2(1+\varepsilon^2)}}{\delta_2 \varepsilon}$

$$\Rightarrow N \ln y = \ln\left(\frac{\sqrt{1-\delta_2^2}+\sqrt{1-\delta_2^2(1+\varepsilon^2)}}{\delta_2 \varepsilon}\right)$$

$$\Rightarrow \boxed{N = \frac{\ln\left(\frac{\sqrt{1-\delta_2^2}+\sqrt{1-\delta_2^2(1+\varepsilon^2)}}{\delta_2 \varepsilon}\right)}{\ln\left(\frac{\Omega_s}{\Omega_p}+\sqrt{\frac{\Omega_s^2}{\Omega_p^2}-1}\right)}}$$

- Design example :-

Design a low pass Chebyshev I filter with $\delta_1 \leq -24.5\,dB$, $\delta_2 \leq -35\,dB$, $\Omega_p = 20$ rad/s, $\Omega_s = 40$ rad/s

We are going to use scipy.signal.cheby1

cheby1 takes three arguments

$N \rightarrow$ Already calculated

$r_p = -20\log_{10}(1-\delta_1)$

$\hookrightarrow$ number, not dB

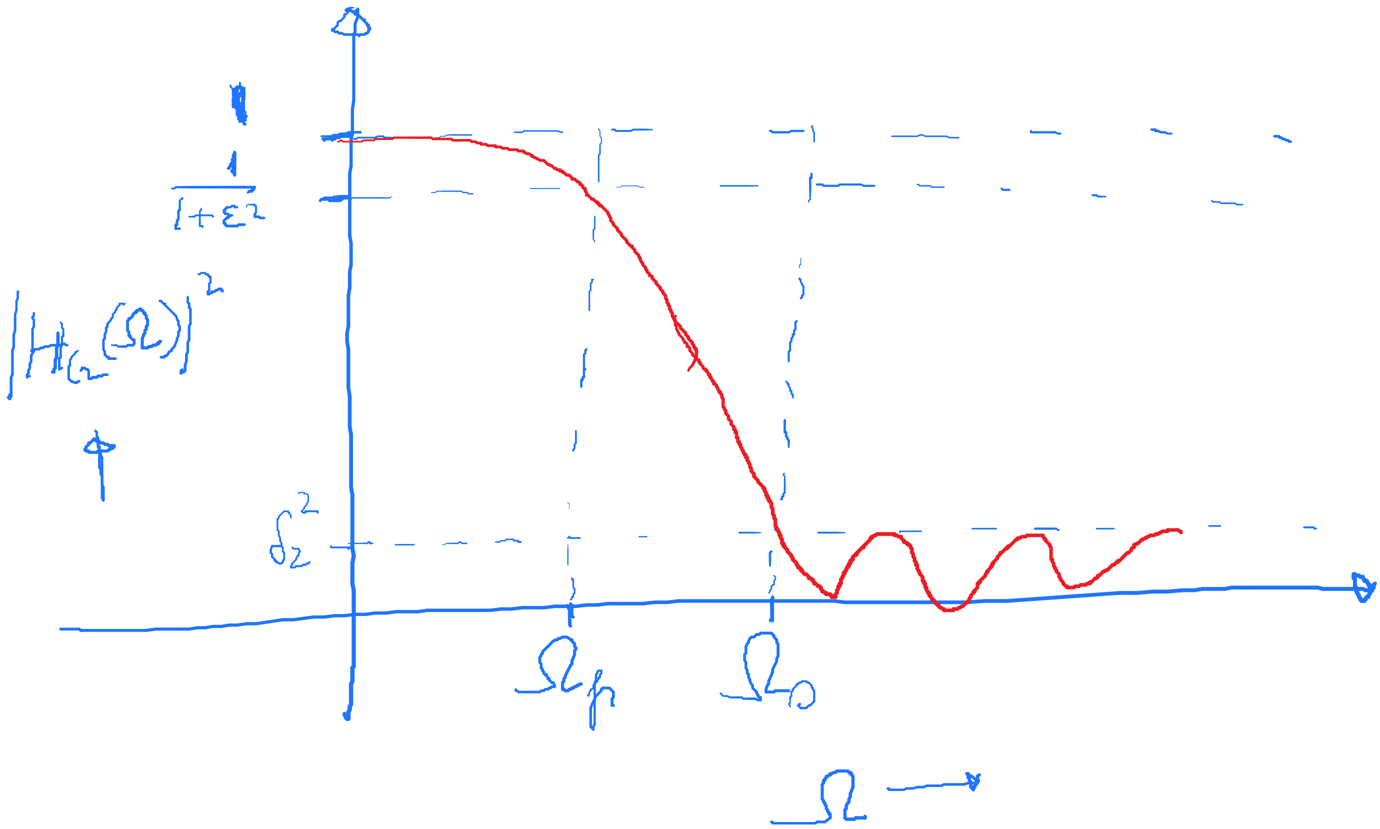
Ω_p

In addition one needs to specify

analog = True.

iii) Chebyshev Type II filter

$$\boxed{|H_{c_2}(\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[T_N^2(\Omega_s/\Omega_p)/T_N^2(\Omega_s/\Omega)\right]}}$$



Here,

$\Omega_p \rightarrow$ Pass band edge

$\Omega_s \longrightarrow$ Stop band edge

As before,

$$\boxed{\varepsilon = \sqrt{\frac{1}{(1-\delta_1)^2} - 1}}$$

Next, to obtain N,

we plug in $\Omega = \Omega_0$ to obtain

$$\delta_2^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega_0/\Omega_p)/T_N^2(1)}$$

$$= \frac{1}{1+\varepsilon^2 T_N^2(\Omega_0/\Omega_p)}$$

This is exactly the same equation as the Chebyshev I filter, and hence

$$\boxed{N = \frac{\ln\left(\dfrac{\sqrt{1-\delta_1^2}+\sqrt{1-\delta_2^2(1+\varepsilon^2)}}{\delta_2\,\varepsilon}\right)}{\ln\left(\dfrac{\Omega_0}{\Omega_p}+\sqrt{\dfrac{\Omega_0^2}{\Omega_p^2}-1}\right)}}$$

: Design example :-

Design a lowpass Chebyshev II filter with $\delta_1 \leq -24.5$ dB,

$\delta_2 \leq -35$ dB, $\Omega_p = 20$ rad/s

$\Omega_s = 40$ rad/s

Here, we shall use scipy.signal.cheby2.

Cheby2 requires three argument

$N \to$ Calculated already

$rs = -20 \log_{10} \delta_2$

Ω_s

In addition, one needs to specify

analog = True

* Frequency Transformation

Let $H_{LP}(s)$ be any analog low pass filter with pass band edge Ω_p

- Low pass $\rightarrow$ High pass

Consider $\boxed{H_{HP}(s) = H_{LP}\left(\frac{\Omega_p \Omega_p'}{s}\right)}$

Let $0 \leq \Omega \leq \Omega_p'$ and $s = j\Omega$

Then, $\frac{\Omega_p \Omega_p'}{s} = -j\frac{\Omega_p \Omega_p'}{\Omega}$

Note that since $\frac{\Omega_p'}{\Omega} > 1$

$$\frac{\Omega_p \Omega_p'}{\Omega} > \Omega_p$$

Thus $\frac{\Omega_p \Omega_p'}{\Omega}$ corresponds to the transition band / stop band of H_{LP}.

On the other hand,

if $\Omega_p' \leq \Omega$, and $s = j\Omega$, then

$$\frac{\Omega_p' \Omega_p}{\Omega} = -j \frac{\Omega_p' \Omega_p}{\Omega}$$

and $\frac{\Omega_p' \Omega_p}{\Omega} < \Omega_p$.

Thus, $\frac{\Omega_p' \Omega_p}{\Omega}$ lies in pass band of

$H_{LP}(s)$.

Hence $H_{HP}(s)$ is thus a high pass filter

with pass band edge Ω_p', and of course

same pass and stop band ripples.

This transformation places the stop

band edge of the high pass filter

at $\boxed{\Omega_s' = \frac{\Omega_p \cdot \Omega_p'}{\Omega_s}}$

- Lowpass $\longrightarrow$ Bandpass

$$\boxed{H_{BP}(s) = H_{LP}\left(\Omega_p \frac{(s^2 + \Omega_l \Omega_u)}{s(\Omega_u - \Omega_l)}\right)},$$

where $\Omega_u > \Omega_l$

For $s = j\Omega$,

$$\Omega_p \frac{(s^2 + \Omega_l \Omega_u)}{s(\Omega_u - \Omega_l)} = -j \frac{\Omega_p(\Omega_l \Omega_u - \Omega^2)}{\Omega(\Omega_u - \Omega_l)}$$

If $\Omega > \Omega_u$, then

$\Omega_l \Omega_u - \Omega^2 < 0$. By symmetry

since, $H_{LP}(j\Omega) = H_{LP}(-j\Omega)$,

we consider

$$\frac{\Omega_p(\Omega^2 - \Omega_l \Omega_u)}{\Omega(\Omega_u - \Omega_l)}$$

Now, consider

$$g(\Omega) = \Omega^2 - \Omega(\Omega_u - \Omega_\ell) - \Omega_u \Omega_\ell$$

$$= \Omega(\Omega - \Omega_u) + \Omega_\ell(\Omega - \Omega_u)$$

$$= (\Omega - \Omega_u)(\Omega + \Omega_\ell)$$

Thus, $g(\Omega) \geq 0$, if $\Omega \geq \Omega_u$.

Hence, $\dfrac{\Omega^2 - \Omega_\ell \Omega_u}{\Omega(\Omega_u - \Omega_\ell)} \geq 1$, if $\Omega \geq \Omega_u$

Hence,

$$\Omega_p \frac{(\Omega^2 - \Omega_\ell \Omega_u)}{\Omega(\Omega_u - \Omega_\ell)} \geq \Omega_p \text{, if } \Omega \geq \Omega_u$$

Thus when $\Omega > \Omega_u$,

$\dfrac{\Omega_p(\Omega^2 - \Omega_\ell \Omega_u)}{\Omega(\Omega_u - \Omega_\ell)}$ falls in the stop/transition band of H_{LP}.

Similarly one can show that if $\Omega < \Omega_l$, then the corresponding frequency falls in the stop/transition band of H_{LP}.

Now, consider $\Omega \in [\Omega_l, \sqrt{\Omega_l \Omega_u}]$

Thus, the corresponding frequency of H_{LP} is

$$\Omega_p \cdot \frac{(\Omega_l \Omega_u - \Omega^2)}{\Omega(\Omega_u - \Omega_l)}$$

Consider, $g(\Omega) = \Omega^2 + \Omega(\Omega_u - \Omega_l) - \Omega_l \Omega_u$

$$= (\Omega + \Omega_u)(\Omega - \Omega_l)$$

Thus, $g(\Omega) > 0$, if $\Omega > \Omega_l$.

Then, $\dfrac{\Omega_l \Omega_u - \Omega^2}{\Omega(\Omega_u - \Omega_l)} < 1$, if $\Omega > \Omega_l$

$$\Rightarrow \Omega_p \frac{(\Omega_u\Omega_\ell - \Omega^2)}{\Omega(\Omega_u - \Omega_\ell)} < \Omega_p,$$

and hence the corresponding frequency lies in the pass band of H_{LP}.

Similarly, one can show that if $\Omega \in [\sqrt{\Omega_\ell\Omega_u}, -\Omega_u]$, then the corresponding frequency lies in the passband of H_{LP}.

In other words, $H_{BP}(s)$ is a bandpass filter with passband edges Ω_ℓ, Ω_u, and exactly the same pass and stop band ripples.

The corresponding stop band edges are then calculated as follows.

First consider the upper stop band edge Ω_{u}. Since $\Omega_{us} > \Omega_{u}$, we have

$$\Omega_s = \frac{\Omega_p(\Omega_{us}^2 - \Omega_u \Omega_l)}{\Omega_{us}(\Omega_u - \Omega_l)}$$

Similarly, we have

$$\Omega_s = \frac{\Omega_p(\Omega_u \Omega_l - \Omega_{ls}^2)}{\Omega_{ls}(\Omega_u - \Omega_l)}$$

The corresponding stop band edges can then be obtained as follows.

$$\boxed{\begin{aligned}\Omega_{us} &= \frac{\Omega_s(\Omega_u-\Omega_l)+\sqrt{\Omega_s^2(\Omega_u-\Omega_l)^2+4\Omega_p^2\Omega_u\Omega_l}}{2\Omega_p}\\[2ex] \Omega_{ls} &= \frac{-\Omega_s(\Omega_u-\Omega_l)+\sqrt{\Omega_s^2(\Omega_u-\Omega_l)^2+4\Omega_p^2\Omega_u\Omega_l}}{2\Omega_p}\end{aligned}}$$

- Lowpass $\longrightarrow$ Bandstop

$$\boxed{H_{BS}(s) = H_{LP}\left(\frac{\Omega_p s(\Omega_u-\Omega_l)}{s^2+\Omega_u\Omega_l}\right)},$$

where Ω_u and Ω_l are the upper and lower pass band edges.

The fact that this transformation indeed gives a bandstop filter with same pass band and stop-band ripple as the original H_{LP} follows by noting that $\dfrac{s(\Omega_u-\Omega_l)}{s^2+\Omega_l\Omega_u}$ is just

the reciprocal of $\dfrac{s^2+\Omega_l\Omega_u}{s(\Omega_u-\Omega_l)}$.

The lower and upper stop band edge frequencies are given below w.r.t. the stop band edge Ω_s of H_{LP}.

$$\boxed{\begin{aligned}
\Omega_{l_s} &= \frac{-\Omega_p(\Omega_u-\Omega_l)+\sqrt{\Omega_p^2(\Omega_u-\Omega_l)^2+4\Omega_s^2\Omega_u\Omega_l}}{2\Omega_s} \\
\Omega_{u_s} &= \frac{\Omega_p(\Omega_u-\Omega_l)+\sqrt{\Omega_p^2(\Omega_u-\Omega_l)^2+4\Omega_s^2\Omega_u\Omega_l}}{2\Omega_s}
\end{aligned}}$$

* Bilinear Transform

Let s and z be the Laplace and z-transform variables

Then the Bilinear Transform is a map from $z \to s$ defined as

$$\boxed{B_{i_T}(z) = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Now,

$$\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{2}{T}\left(\frac{1-\frac{1}{r}e^{-j\omega}}{1+\frac{1}{r}e^{-j\omega}}\right)$$

$$= \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

$$= \frac{2}{T} \frac{(re^{-j\omega}+1)(re^{j\omega}-1)}{(1+r^2+2r\cos\omega)}$$

$$= \frac{2}{T} \frac{r^2-1+2jr\sin\omega}{1+r^2+2r\cos\omega}$$

Thus, if $|z|=1$, then $r=1$ and

$$\boxed{B_{i_T}(z) = j\frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = j\frac{2}{T}\tan\frac{\omega}{2}}$$

Thus $B_{i_T}(z)$ maps the unit circle in z-plane to the imaginary axis in s-plane.

Now, note that $1+r^2+2r\cos\omega \geq 0$.

Then if $r>1$, i.e. $|z|>1$, $Re(B_{i_T}(z))>0$

if $r<1$, i.e., $|z|<1$, $Re(B_{i_T}(z))<0$.

Thus, $B_{i_T}(z)$ maps the interior of the unit circle of the z-plane to the left half s-plane. The exterior of the unit circle in the z-plane gets mapped to the right half of the z-plane.

Suppose now we want a digital filter with critical frequencies ω_i.

Where by critical frequencies, we mean pass and stop-band frequencies.

Then, by bilinear transform, we will end up with an analog filter with critical frequencies

$$\boxed{\Omega_i = \frac{2}{T}\tan\frac{\omega_i}{2}}.$$

Also, noting that tan is an increasing function in $[0, \frac{\pi}{2}]$, and $\omega_i \in [0, \pi]$,

Ω_is will appear in exactly the same order as ω_is, i.e., if $\omega_i > \omega_j$, then, $\Omega_i > \Omega_j$.

Furthermore, let $H(\omega)$ be a digital filter where $\omega \in [\omega_i, \omega_{i+1}]$, and $|H(\omega)|$ satisfies some ripple property, i.e.,

$||H(\omega)| - 1| \leq \delta_1$, if (ω_i, ω_{i+1}) is a pass band

$|H(\omega)| \leq \delta_2$, if $[\omega_i, \omega_{i+1}]$ is a stop band.

Now, consider the analog filter

$$H_a(s) = H(Bi_+(z))$$

Then, $\Omega \in [\Omega_i, \Omega_{i+1}]$ will have the same 'ripple characteristics'.

In other words, a digital filter can be designed from an analog filter by using a bilinear transform.

* IIR Filter design example

Consider the old example for a bandpass filter design with

$\Omega_{s_1} = 20 \text{rad/s}$ $\quad \Omega_{s_2} = 80 \text{rad/s}$ $\quad \delta_2 \leq -35 \text{dB}$

$\Omega_{p_1} = 40 \text{rad/s}$ $\quad \Omega_s = 200 \text{rad/s}$

$\Omega_{p_2} = 60 \text{rad/s}$ $\quad \delta_1 \leq -24.5 \text{dB}$

→ Step 1. Obtain the corresponding analog bandpass filter

Note that

$$\omega = 2\pi \frac{\Omega}{\Omega_s}.$$

This transformation gives us

$\omega_{s_1}, \omega_{p_1}, \omega_{p_2}, \omega_{s_2}.$

Next, the built in bilinear transform of scipy uses $T=1$.

So, the corresponding frequency specifications of the equivalent analog filter is obtained using the transformation

$\Omega_a = 2\tan\frac{\omega}{2}$.

This way we get the analog filter specifications,

$$\Omega_{a_{s1}}, \Omega_{a_{p1}}, \Omega_{a_{p2}}, \Omega_{a_{s2}}.$$

→ Step 2:- Choosing the Ω_p and Ω_s of the analog low pass filter which will be converted using frequency transformation to the filter with $\Omega_{a_{s1}}, \Omega_{a_{p1}}, \Omega_{a_{p2}}, \Omega_{a_{s2}}$

- Note:- Since our filter is band pass it suffices if we design a filter with upper stop band frequency lower than

$\Omega_{a_{s_2}}$ and lower stop band edge frequency higher than $\Omega_{a_{s_1}}$.

We may not be able to design for exactly $\Omega_{a_{s_1}}$ and $\Omega_{a_{s_2}}$ since frequency transformations do not give precise control over stop band edges.

Now firstly note that we will be using scipy.signal.lp2bp to make this transformation, which by default assumes an initial H_{LP} with $\Omega_p = 1$.

Recall that

$$H_{BP}(s) = H_{LP}\left(\frac{s^2+\Omega_{a_{p_1}}\Omega_{a_{p_2}}}{s\left(\Omega_{a_{p_2}}-\Omega_{a_{p_1}}\right)}\right)$$

for $\Omega_{a_{s_2}}$, we shall require the LPF to have stop band edge

$$\Omega_s' = \frac{\Omega_{a_{s_2}}^2-\Omega_{a_{p_1}}\Omega_{a_{p_2}}}{\Omega_{a_{s_2}}\left(\Omega_{a_{p_2}}-\Omega_{a_{p_1}}\right)}$$

Now consider

$$g(\Omega) = \frac{\Omega^2 - \Omega\cdot B}{\Omega B}$$

$$g'(\Omega) = \frac{2\Omega^2 B - \Omega B^2 - B\Omega^2 + \Omega B^2}{\Omega^2 B^2}$$

$$= \frac{\Omega^2 B}{\Omega^2 B^2} = \frac{1}{B} > 0$$

Thus g is increasing and hence increasing Ω_o' will increase $\Omega_{a_{s_2}}$.

Similarly, for $\Omega_{a_{s_1}}$, we shall require LPF to have stop band edge

$$\Omega_o'' = \frac{\Omega_{a_{p_1}}\Omega_{a_{p_2}} - \Omega_{a_{s_1}}^2}{\Omega_{a_{s_1}}(\Omega_{a_{p_2}} - \Omega_{a_{p_1}})}$$

Here, again one can argue similarly that increasing Ω_o'' decreases $\Omega_{a_{s_1}}$.

Since $\Omega_{a_{s_1}}$ can be higher and $\Omega_{a_{s_2}}$ can be lower than the calculated value, Ω_o' and Ω_o'' can both be lower than the calculated value.

We thus choose

$$\Omega_s = \min(\Omega_s', \Omega_s'')$$

→ Step 3 :-

Design Analog LPF with $\delta_1, \delta_2, \Omega_s, \Omega_p = 1$ using Butterworth / Chebyshev I / Chebyshev II.

scipy will return the numerator denominator polynomials of $H_{LP}(s)$.

→ Step 4 :-

We shall obtain numerator and denominator polynomials of $H_{BP}(s)$ using scipy.signal.lp2bp lp2bp requires 4 inputs

- numerator and denominator polynomials of $H_{LP}(s)$.
- $W_0 = \sqrt{\Omega_{a_{p1}} \Omega_{a_{p2}}}$
- $bw = \Omega_{a_{p2}} - \Omega_{a_{p1}}$

lp2bp will return the numerator and denominator polynomials of $H_{BP}(s)$.

→ Step 5 :- We shall use scipy.signal.bilinear to obtain the numerator and denominator polynomials of the required digital filter.

bilinear requires only two inputs.

the numerator and denominator

polynomials of $H_{BP}(s)$.

* Realization of FIR systems

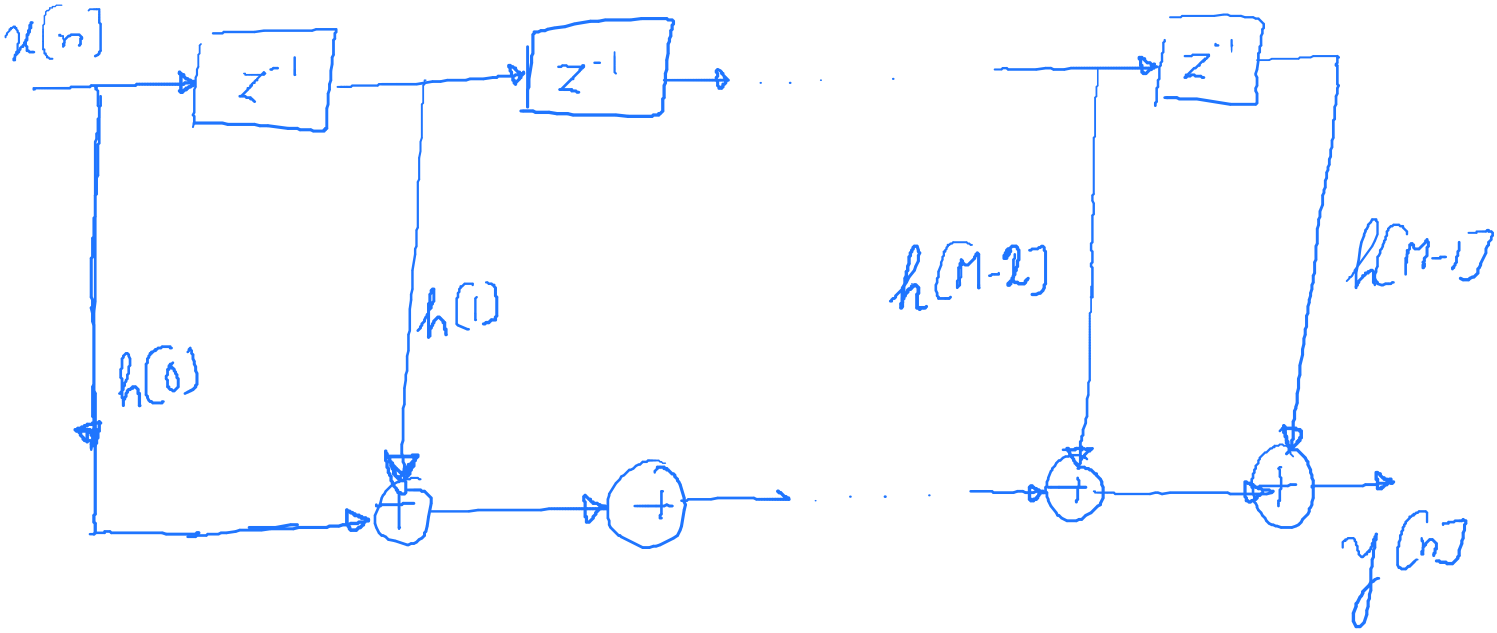
— Direct form

Let $h(n)$ have M taps.

Then, $$y[n] = \sum_{n=0}^{M-1} h[k] \cdot x[n-k]$$

The direct form realization is

drawn below.



\# of memory units $= M-1$.

\# of $\times$ $= M$

\# of $+$ $= M-1$

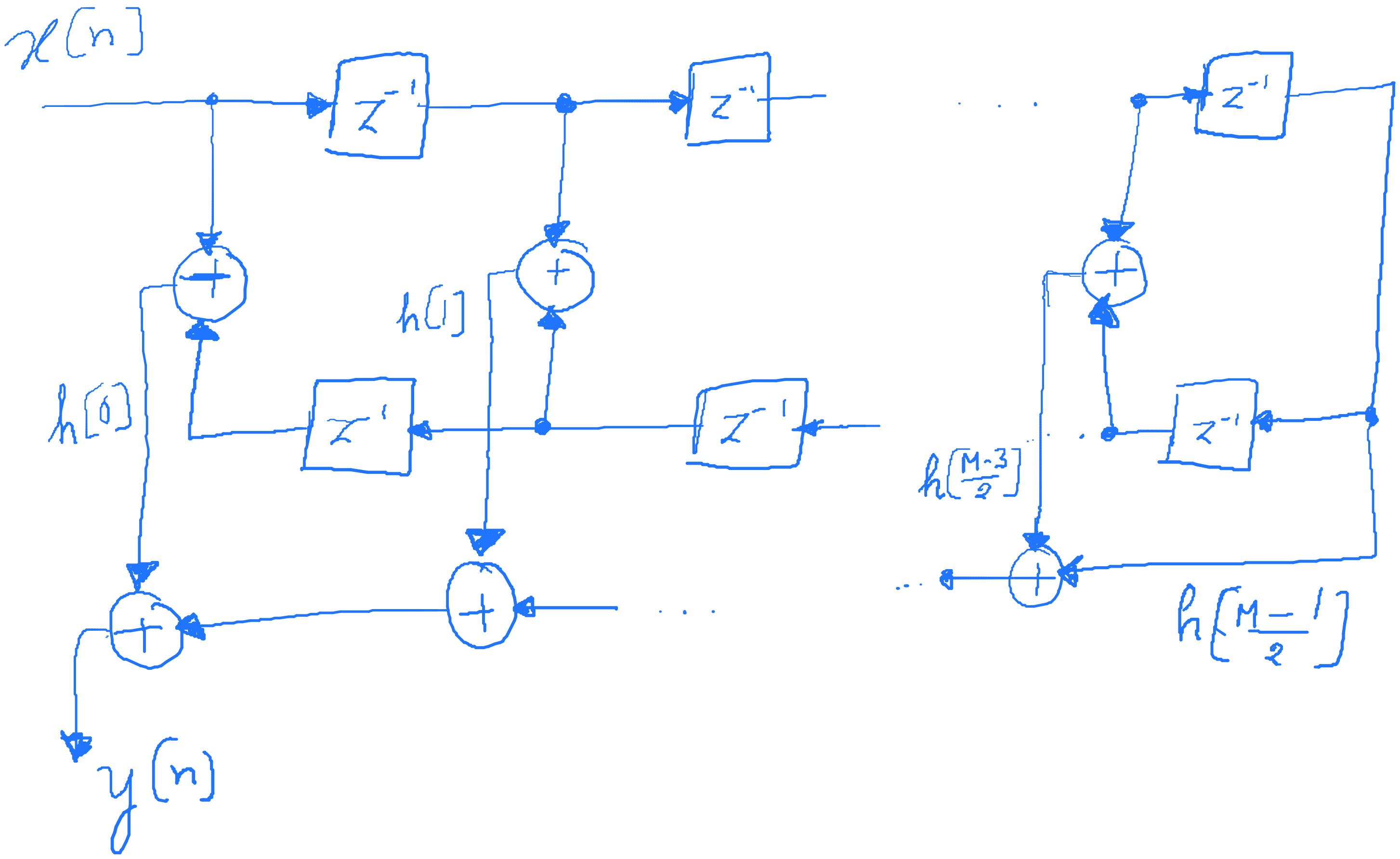
If $h[n]$ is a linear phase system, i.e.

$$h[n] = \pm h[M-1-n],$$

then we can reduce the number of multiplications.

Realization for Type I system, i.e.

M odd and $h[n] = h[M-1-n]$.



# of memory units $= M-1$

of $+$ $= \frac{M-1}{2} + \frac{M-1}{2} = M-1$

of $\times$ $= \frac{M-1}{2}$

* Cascade form FIR

$$H(z) = \prod_{k=1}^{M-1} (1 - z_k z^{-1})$$

Break this product into product of pair of zeros (bunch complex conjugate ones together).

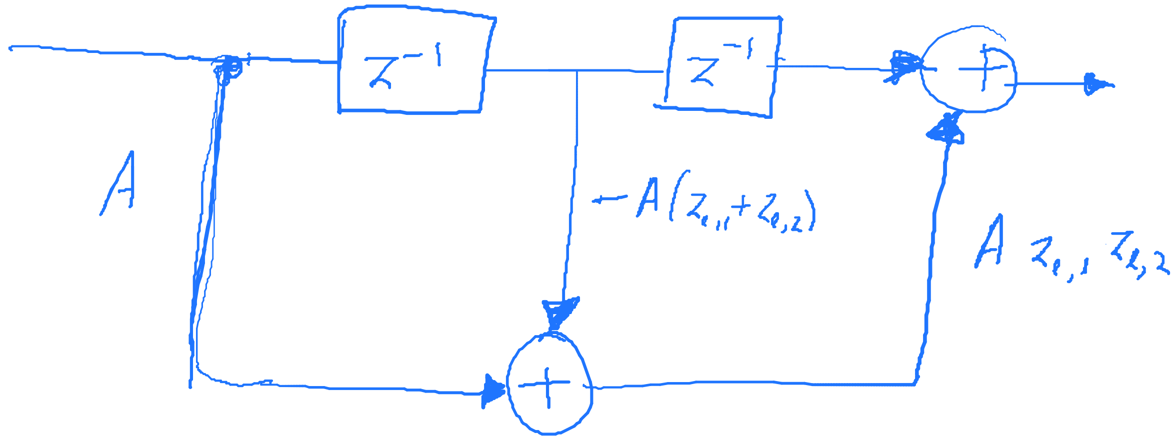
Thus $H(z) = \prod_{l=1}^{\lceil \frac{M-1}{2} \rceil} H_l(z)$

where $H_l(z) = (1 - z_{l,1} z^{-1})(1 - z_{l,2} z^{-1}) \cdot A$

$$= A - z^{-1}(z_{l,1} + z_{l,2})A + A z_{l,1} \cdot z_{l,2} z^{-2}$$

Now, $h_l[n] = [\boxed{A}, -(z_{l,1} + z_{l,2})A, z_{l,1} z_{l,2} A]$

So, $h_l[n]$ can be designed in the direct form as



Memory = 2

+ = 2

X = 3

Simply cascade $H_1(z), H_2(z), \ldots, H_{\lceil \frac{M-1}{2} \rceil}(z)$

to get $H(z)$.

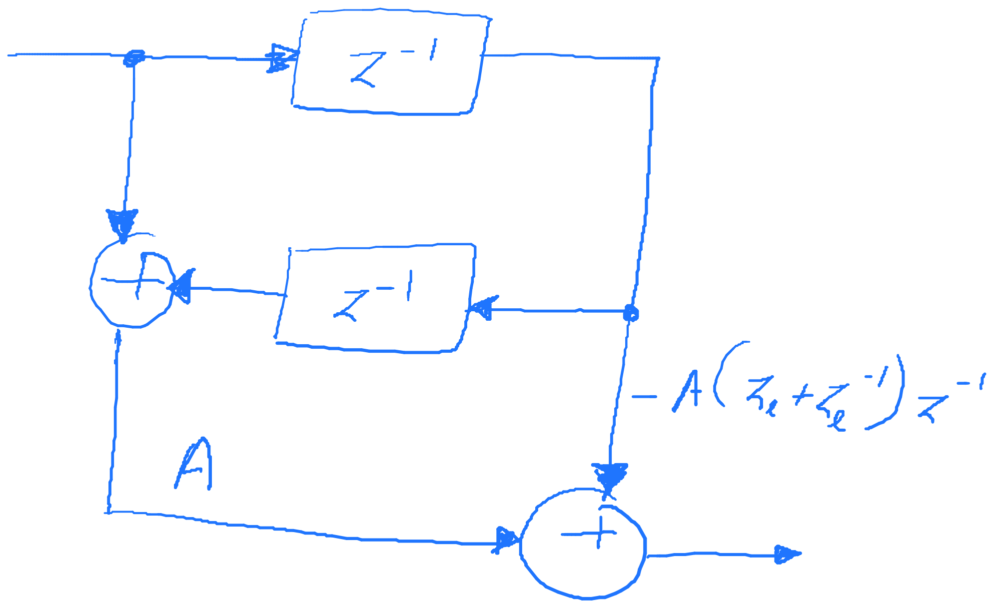
If the system is linear phase, then it can be shown that a zero at z_1 implies a zero at z_1^{-1}!

Thus, while we bunch pairs of zeros, we will bunch reciprocal zeros.

Hence, $H_\ell(z) = A(1 - z_\ell z^{-1})(1 - z_\ell^{-1} z^{-1})$

$$= A - A(z_\ell + z_\ell^{-1})z^{-1} + Az^{-2}$$

This can thus be realized as



So we reduce #X by 1.

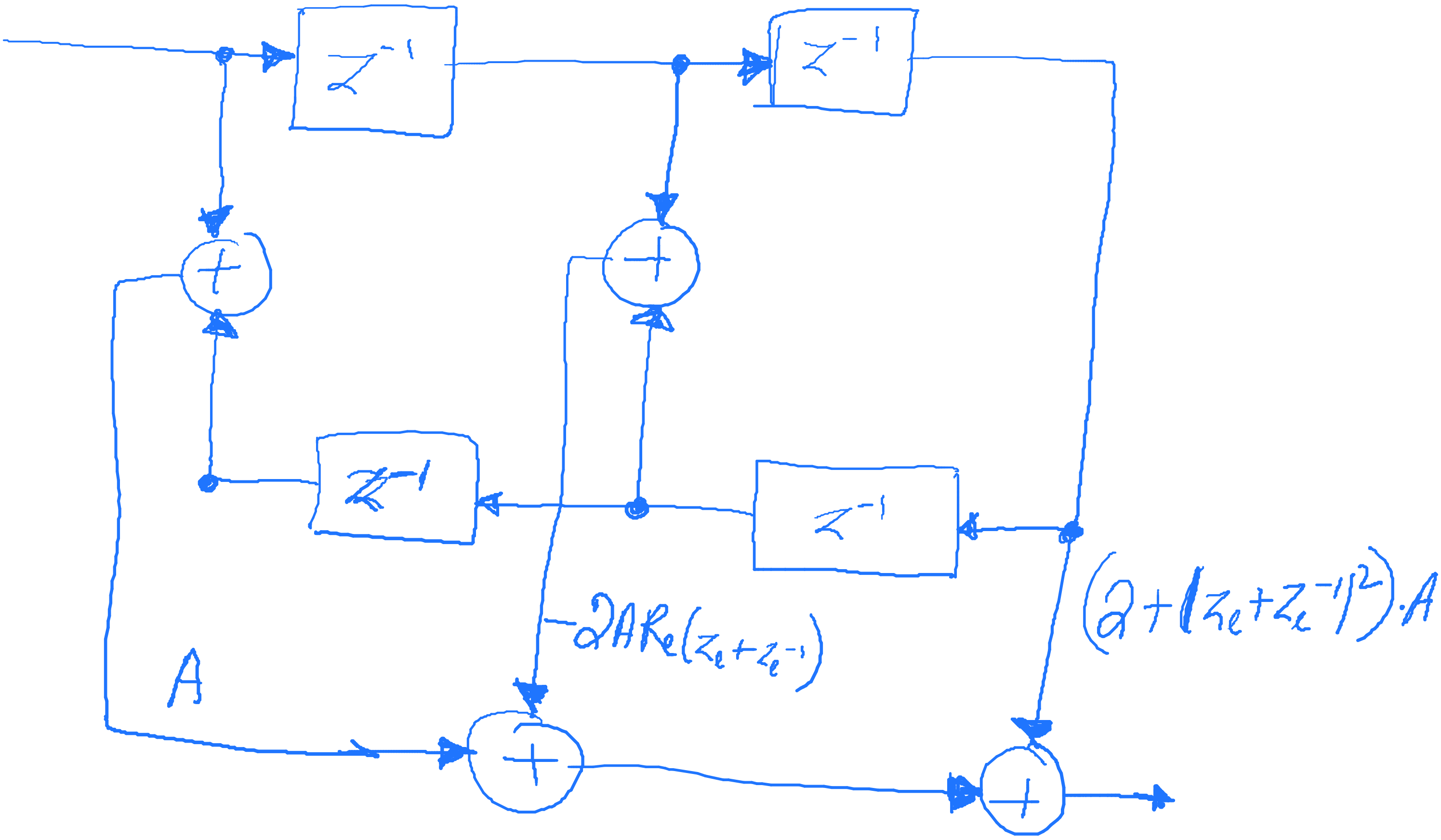
For complex conjugate zeros, bunch four of them, i.e., conjugate and their reciprocals.

$$H_\ell(z) = A\left(1 - z_\ell z^{-1}\right)\cdot\left(1 - z_\ell^* z^{-1}\right)\left(1 - z_\ell^{-1} z^{-1}\right)\left(1 - z_\ell^{*-1} z^{-1}\right)$$

$$= A\left(1 - \left(z_\ell + z_\ell^{-1}\right) z^{-1} + z^{-2}\right)\cdot\left(1 - \left(z_\ell + z_\ell^{-1}\right)^* z^{-1} + z^{-2}\right)$$

$$= A\left(1 - 2\,\mathrm{Re}\left(z_\ell + z_\ell^{-1}\right) z^{-1} + \left(2 + \left|z_\ell + z_\ell^{-1}\right|^2\right) z^{-2} - 2\,\mathrm{Re}\left(z_\ell + z_\ell^{-1}\right) z^{-3} + z^{-4}\right)$$

$$= A - 2A\,\mathrm{Re}\left(z_\ell + z_\ell^{-1}\right) z^{-1} + A\left(2 + \left|z_\ell + z_\ell^{-1}\right|^2\right) z^{-2} - 2A\,\mathrm{Re}\left(z_\ell + z_\ell^{-1}\right) z^{-3} + A z^{-4}$$



If not bunched together, these would have constituted two different units taking 6 multiplications.

$\# \times$ is reduced to 3 from 6.

* IIR system — Direct form

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = H_1(z) \cdot H_2(z)$$

where $H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$

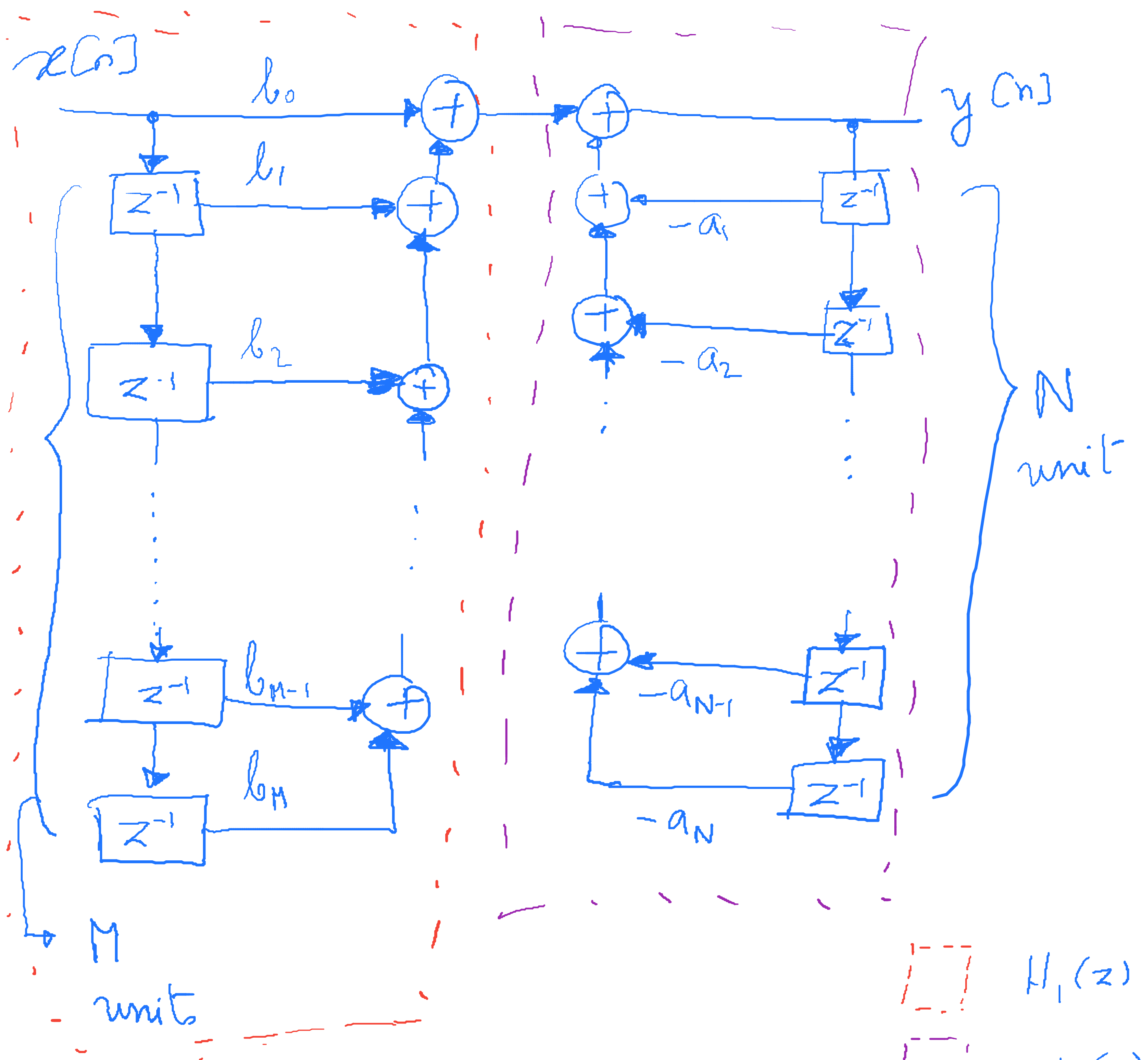
$$H_2(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$H_1(z)$ can be implemented as FIR direct form.

For $H_2(z)$, recall that this can be implemented using the difference equation

$$y[n] = x[n] - \sum_{k=1}^{N} a_k y[n-k]$$

We shall implement this.



memory $M+N$

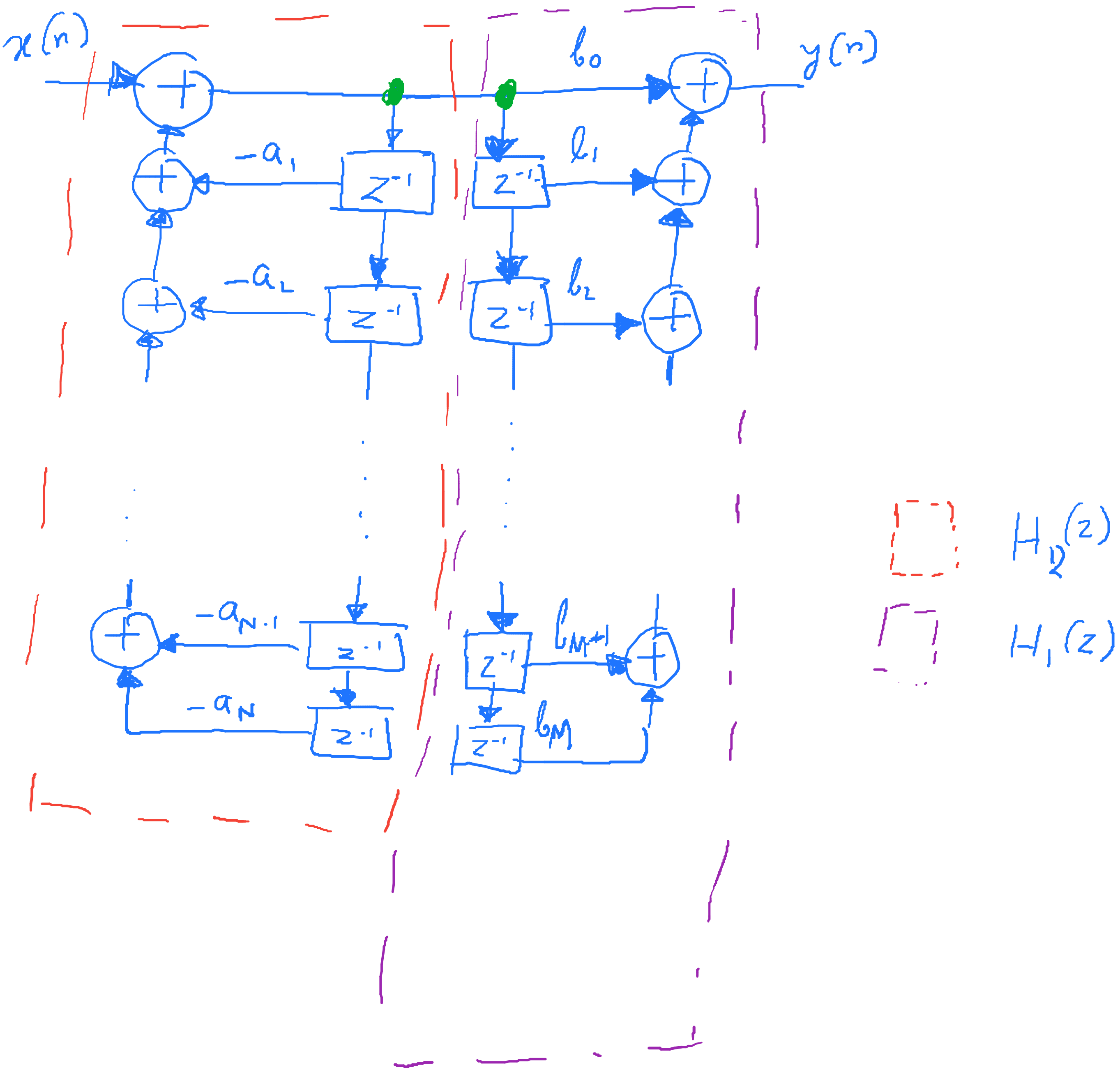
multiplications $M+N+1$

additions $M+N$

We have a more efficient implementation by switching the order.

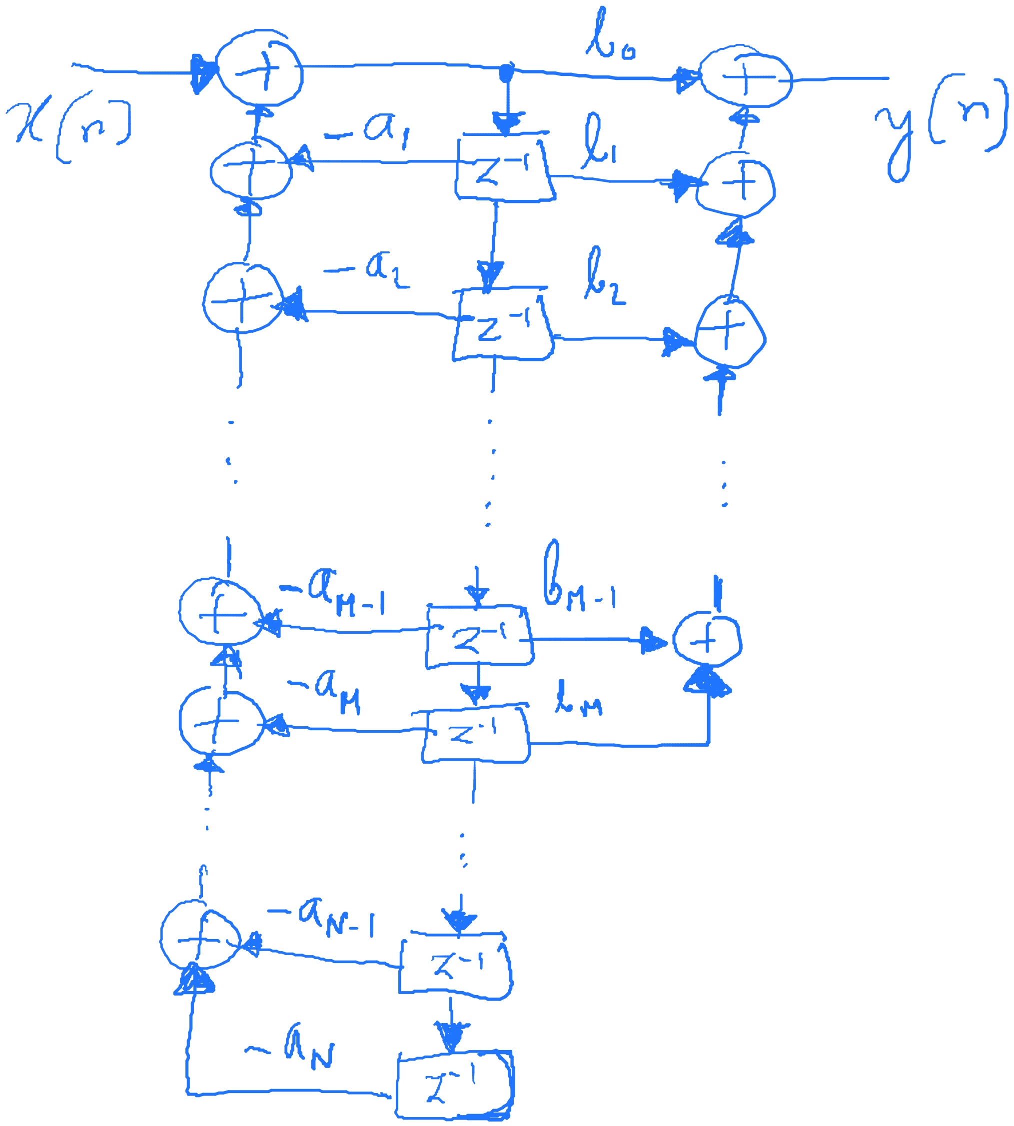
$$H(z) = H_2(z) \cdot H_1(z)$$

Applying the direct forms directly, we have



Notice that the signals at the • points are the same, and we are unnecessarily using extra memory units.

Thus combining the two • points and noting that $M \leq N$ in general, we have the following implementation.



While number of additions and multiplications remain unchanged, number of memory units reduce from $M+N$ to $\max\{M,N\}$.

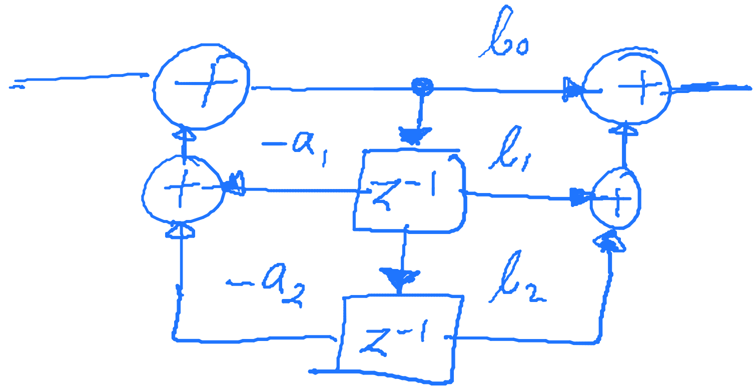
* Cascade form realization of IIR

Break $H(z) = \prod H_k(z)$

where $H_k(z) = \dfrac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$

i.e., two pole - two zero systems, and then place $H_k(z)$ in series.

Individual $H_k(z)$ can be implemented in the direct form.



$H_k(z)$

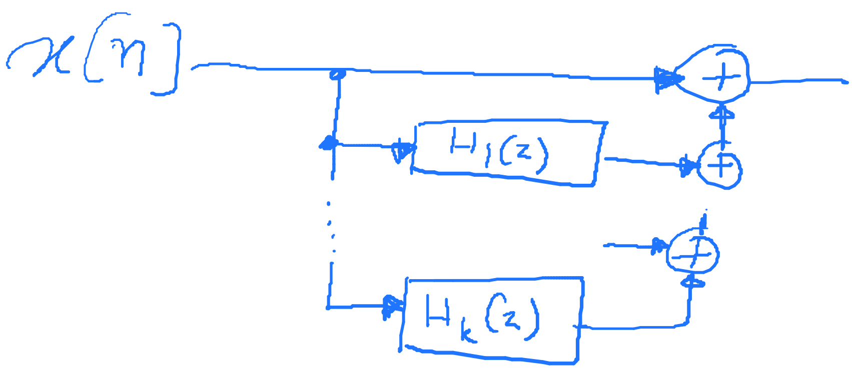
* Parallel form realization of IIR systems

Expand $H(z)$ in partial fractions.

$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

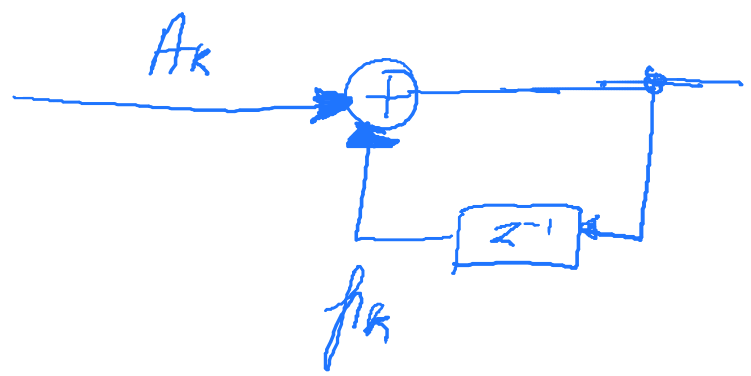
$$= C + \sum_{k=1}^{N} H_k(z)$$

This can be implemented as



Now, note that $H_k(z)$ can be expressed via the difference equation

$$y[n] = p_k\, y[n-1] + A_k x[n]$$



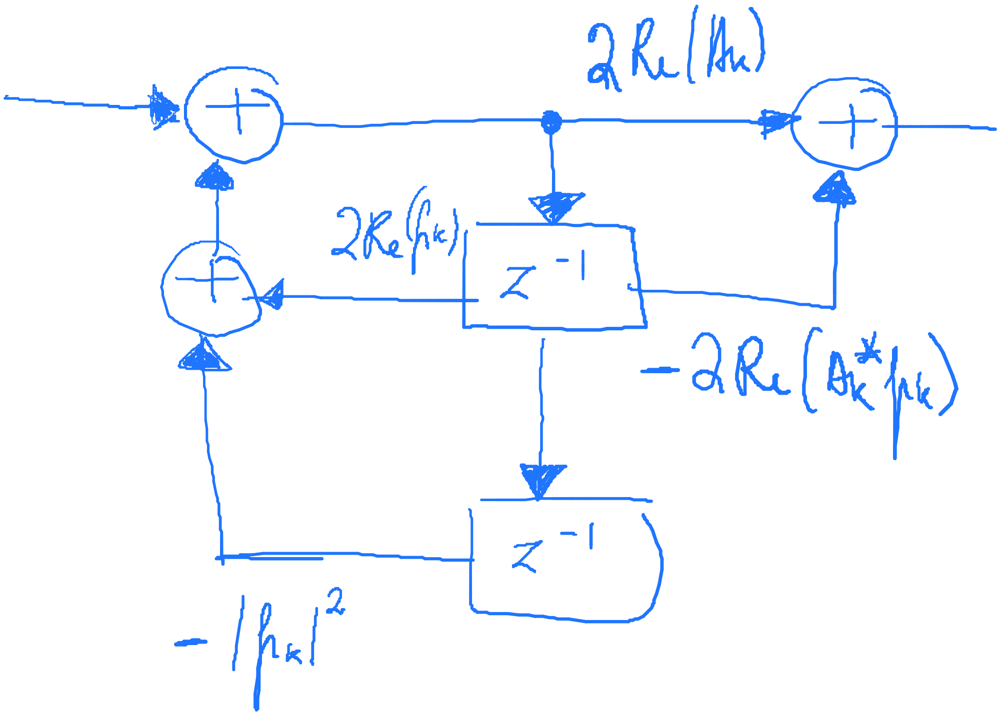
In case of complex conjugate poles, we shall combine them in two.

Then, $$H_k(z) = \frac{A_k}{1 - p_k z^{-1}} + \frac{A_k^*}{1 - p_k^* z^{-1}}$$

[The numerators have to be complex conjugates as otherwise $h[n]$ will be complex]

$$= \frac{2\left(Re(A_k) - Re(A_k^* p_k) z^{-1}\right)}{1 - 2Re(p_k) + |p_k|^2 \cdot z^{-2}}$$

This can be implemented in direct form as



* Signal flow graphs and transpose form

Every structure can be equivalently modelled through a signal flow graph.

Steps :-

1. Convert every addition to a node
2. Convert every $-\boxed{z^{-1}}-$ to a link with multiplier z^{-1}.
3. Remove redundant links.

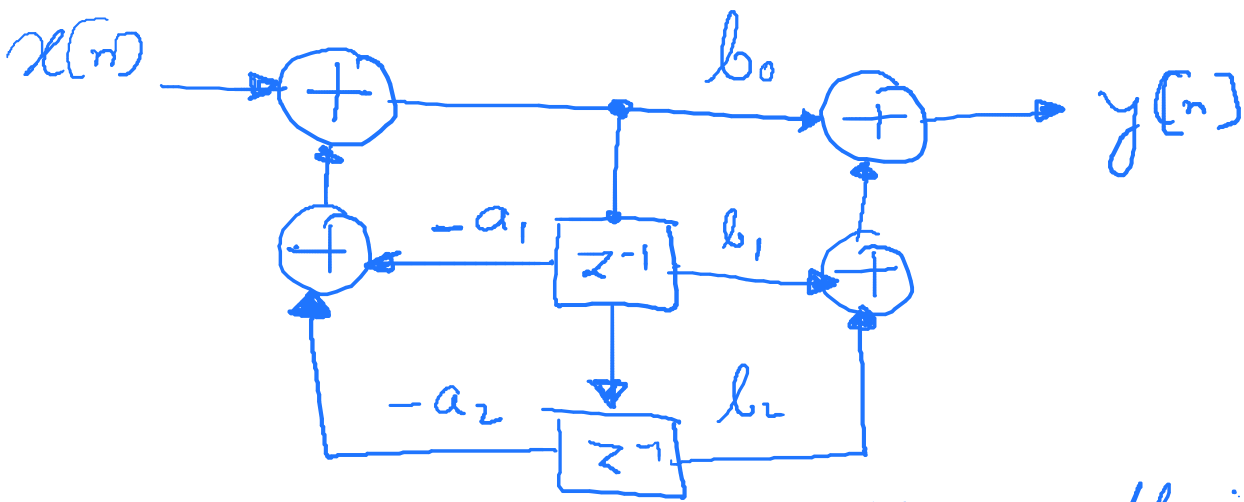
In a signal flow graph, any signal flowing through a link gets multiplied by a link multiplier.

At every node, the signal on the links with outgoing edges is the sum of the signals at the incoming edges.

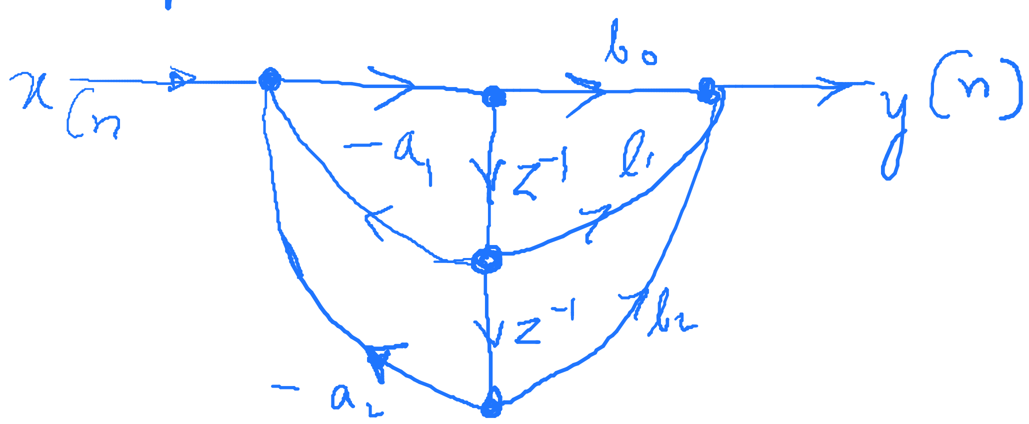
Example:- Consider the system

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}.$$

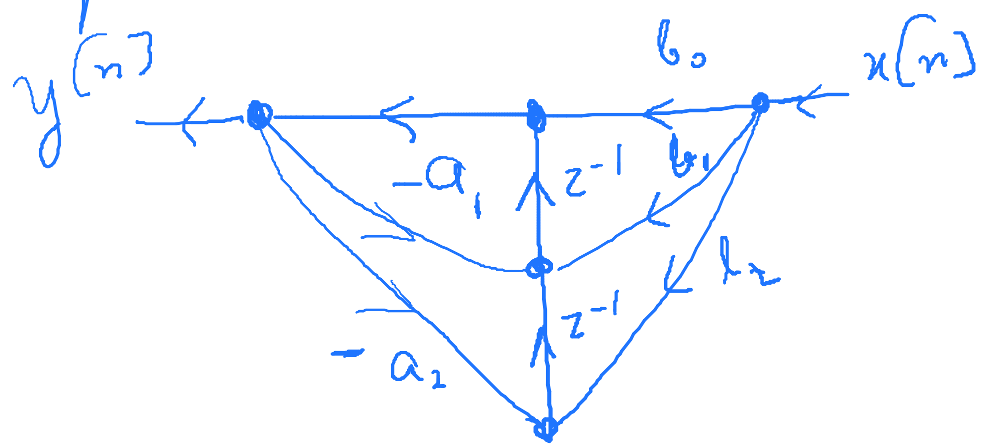
Its corresponding direct form implementation is



The corresponding signal flow graph is

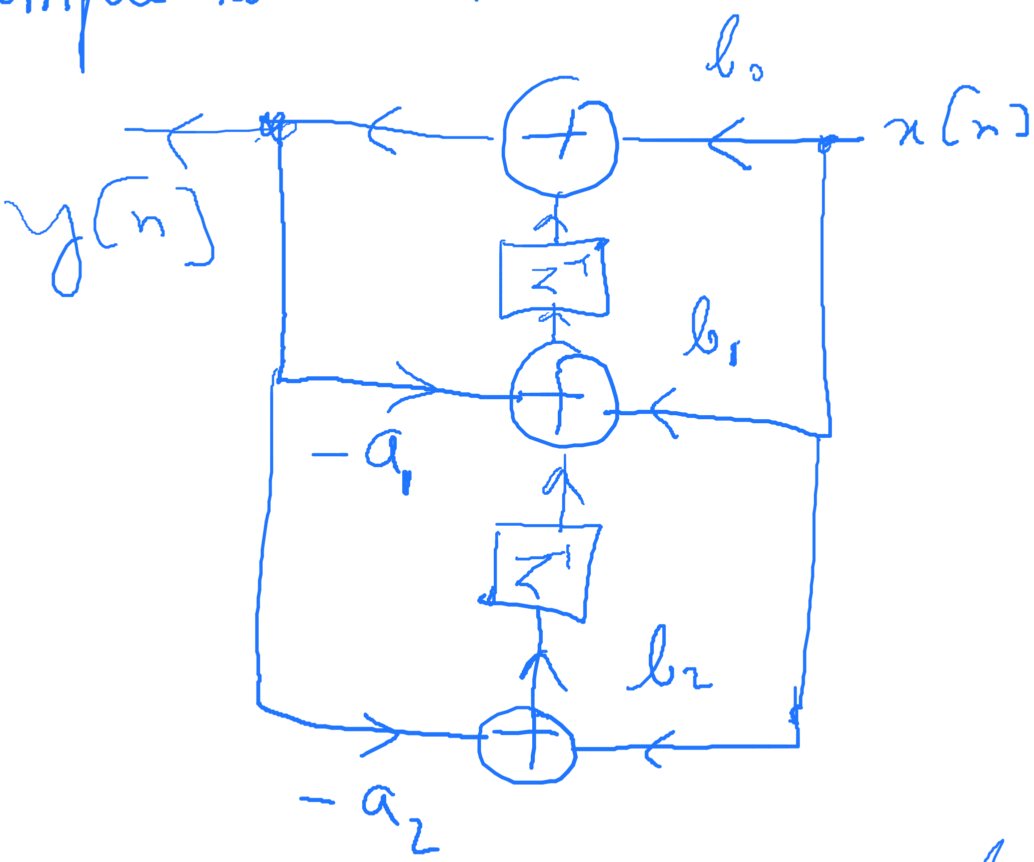


Now, the transposition theorem for signal flow graphs state that if arrows on links are reversed and the input and output are interchanged, the signal flow graph still implements the same transfer function. So, for example the previous signal flow graph becomes



We can convert the transposed signal flow graph into a representation involving delays and adders.

The one corresponding to the running example is below.



This form is known as the transpose form.

* Lattice-Ladder form

This implementation is used primarily for linear predictor systems where multiple 'step-by step' outputs are needed from the system.

We will not deal with such systems in this course, but we shall learn the implementation.

— FIR systems

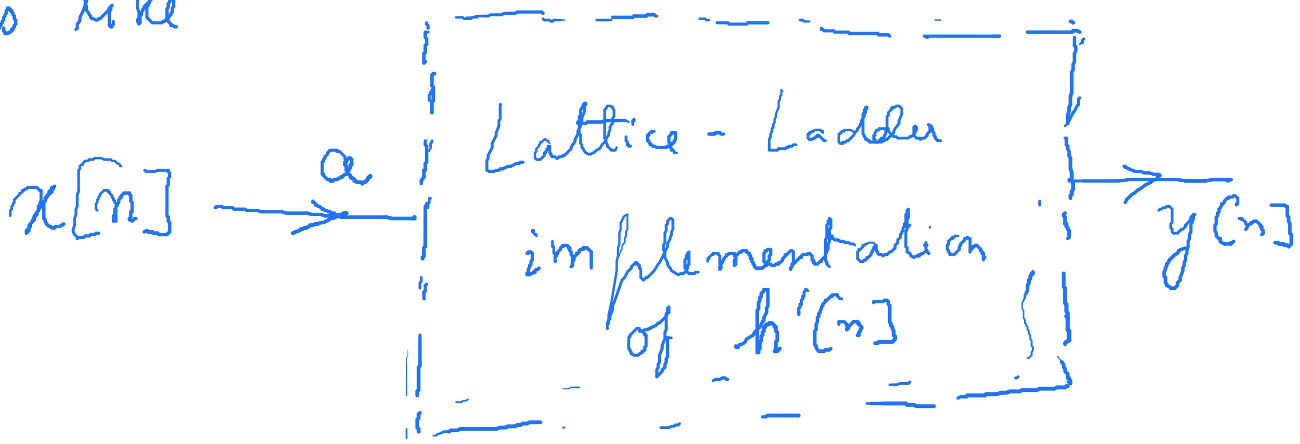
We are going to assume

$$\boxed{\begin{gathered} y[n] = \sum_{k=0}^{M-1} \alpha_M[k]\, x[n-k] \\ \text{where} \quad \alpha_M[k] = h[k] \text{ and} \\ \alpha_M(0) = h[0] = 1 \end{gathered}}$$

Note :- If $h[0] = a \neq 1$, we simply use a new LTI system with

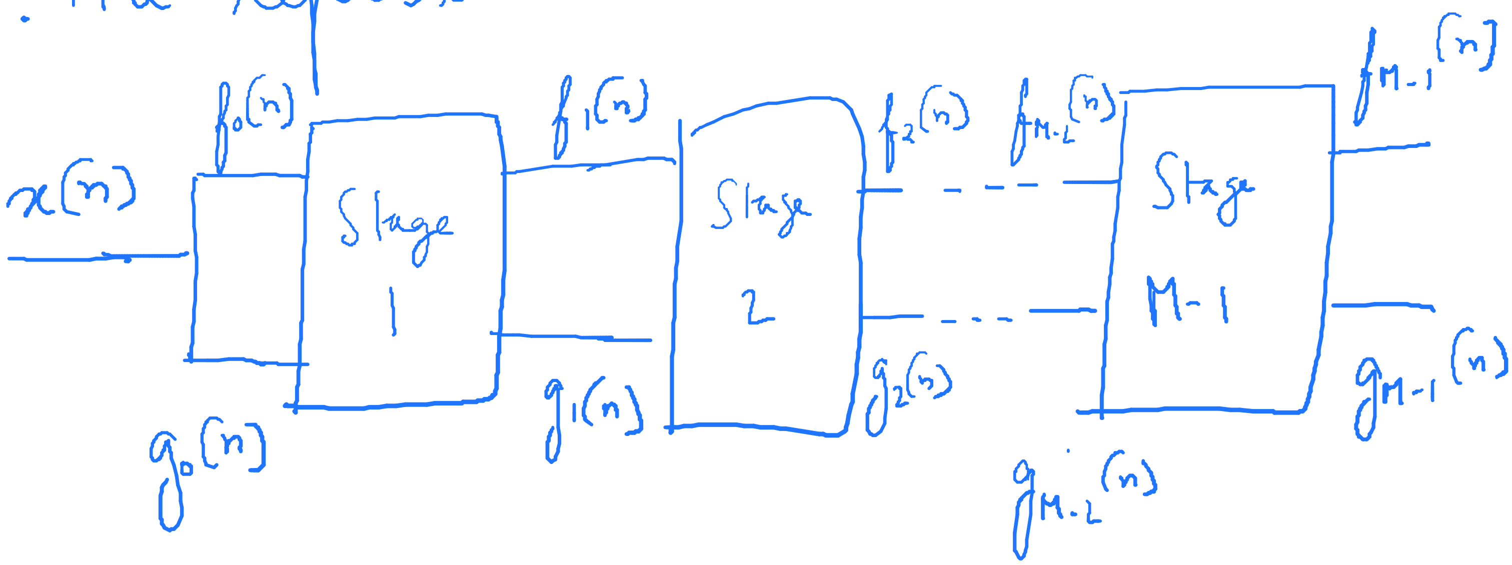
$$h'[n] = \frac{1}{a} h[n],$$

and scale the input by a. The implementation looks like

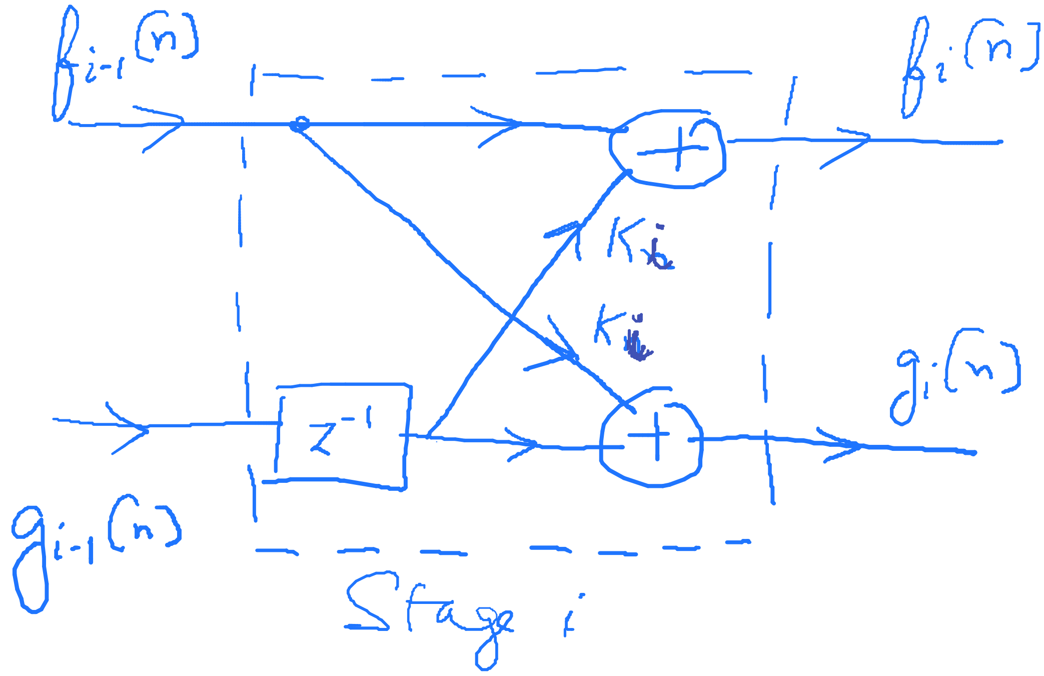


From here onwards, we therefore assume $h[0] = 1$.

The representation has $M-1$ stages



where $f_{M-1}(n) = y(n)$, and stage i has the form,



The goal is thus the following

(i) Given $K_1, \ldots, K_{M-1}$, find $h[n]$, and

(ii) Given an FIR system $h[n]$ with M taps, find $K_1, \ldots, K_{M-1}$ for its lattice representation.

To perform (i) & (ii), let us analyze the implementation.

Define
$$\boxed{A_i(z) = \frac{F_i(z)}{F_0(z)} = \frac{F_i(z)}{X(z)}}$$

Thus,
$$\boxed{F_i(z) = X(z) \cdot A_i(z)}$$

Similarly, define
$$\boxed{B_i(z) = \frac{G_i(z)}{G_0(z)} = \frac{G_i(z)}{X(z)},}$$

thus,
$$\boxed{G_i(z) = B_i(z) X(z)}$$

Next, using the structure of stage i,
$$\boxed{\begin{aligned} F_i(z) &= z^{-1} G_{i-1}(z) \cdot K_i + F_{i-1}(z) \\ G_i(z) &= K_i F_{i-1}(z) + z^{-1} G_{i-1}(z) \end{aligned}}$$

Thus,
$$\boxed{\begin{aligned} A_i(z) &= z^{-1} K_i B_{i-1}(z) + A_{i-1}(z) \\ B_i(z) &= K_i \cdot A_{i-1}(z) + z^{-1} B_{i-1}(z) \end{aligned}}$$

Next observe that $f_i[n]$ and $g_i[n]$ $\forall\ 1 \leq i \leq M-1$ are outputs of FIR LTI systems with input $x[n]$, with the system corresponding to $f_{M-1}[n]$ being the target system $h(n)$.

Let $\alpha_i[n], \beta_i[n]$ be the impulse responses of the LTI systems that output $f_i[n], g_i[n]$ respectively.

Again, note that $\alpha_{M-1}[n] = h[n]$.

We claim the following:

$$\boxed{\begin{aligned} &\underline{\text{Lemma}} :- \quad \alpha_i[0] = 1, \quad \forall\, 1 \le i \le M-1 \\ &\qquad \beta_i[k] = \alpha_i[i-k], \quad \forall\, 0 \le k \le i \end{aligned}}$$

<u>Proof</u> :- We shall prove this by induction on i.

First consider the base case of $i=1$.

Here, $f_1[n] = x[n-1] \cdot K_1 + x[n]$

$$g_1[n] = x[n-1] + K_1 x[n].$$

Thus, $\alpha_1[n] = [1, K_1]$

$$\beta_1[n] = [K_1, 1]$$

which satisfies the induction hypothesis.

Assume the hypothesis holds upto $i-1$.

Then, $f_i[n] = f_{i-1}[n] + K_i g_{i-1}[n-1]$

$$= \sum_{k=0}^{i-1} x[n-k] \cdot \alpha_{i-1}[k]$$

$$+ K_i \cdot \sum_{k=0}^{i-1} x[n-1-k] \beta_{i-1}[k]$$

$$= \sum_{k=0}^{i-1} x[n-k] \cdot \alpha_{i-1}[k]$$

$$+ K_i \sum_{k=1}^{i} x[n-k] \beta_{i-1}[k-1]$$

$$\Rightarrow \sum_{k=0}^{i} \alpha_i[k] \cdot x[n-k]$$

$$= \alpha_{i-1}[0] \cdot x[n]$$

$$+ \sum_{k=1}^{i-1} \left(\alpha_{i-1}[k] + K_i \beta_{i-1}[k-1] \right) x[n-k]$$

$$+ K_i \, x[n-i] \, \beta_{i-1}[i-1]$$

Since the result is true for all $x[n]$, we must have

$$\alpha_i[0] = \alpha_{i-1}[0] = 1$$ $\rightarrow$ and hence we have proved the first part by induction.

$$\alpha_i[k] = \alpha_{i-1}[k] + K_i \beta_{i-1}[k-1], \quad 1 \le k \le i-1$$

$$\alpha_i[i] = K_i \beta_{i-1}[i-1] \quad \ldots (1)$$

Next, similarly note that

$$g_i[n] = K_i f_{i-1}[n] + g_{i-1}[n-1]$$

$$= \sum_{k=0}^{i-1} K_i \cdot \alpha_{i-1}[k] x[n-k] + \sum_{k=0}^{i-1} \beta_{i-1}[k] . x[n-1-k]$$

$$= \sum_{k=0}^{i-1} K_i \cdot \alpha_{i-1}[k] . x[n-k] + \sum_{k=1}^{i} \beta_{i-1}[k-1] x[n-k]$$

$$\Rightarrow \sum_{k=0}^{i} g_i[k]\, x[n-k] = K_i \alpha_{i-1}[0]\, x[n] + \sum_{k=1}^{i-1} \left(K_i \alpha_{i-1}[k] + \beta_{i-1}[k-1] \right) x[n-k] + \beta_{i-1}[i-1] \cdot x[n-i]$$

Thus, noting that the above holds for all $x[n]$, we must have

$$\left.\begin{aligned} \beta_i[0] &= K_i \alpha_{i-1}[0] = K_i \\ \beta_i[k] &= K_i \alpha_{i-1}[k] + \beta_{i-1}[k-1], \quad 1 \le k \le i-1 \\ \beta_i[i] &= \beta_{i-1}[i-1] \end{aligned}\right\} \quad \ldots (2)$$

Thus, for any $1 \le k \le i-1$

$$\beta_i[k] = K_i \alpha_{i-1}[k] + \beta_{i-1}[k-1]$$

$$= K_i \beta_{i-1}[i-1-k] + \alpha_{i-1}[i-k]$$

[By induction hypothesis]

$$= \alpha_i[i-k] \quad [\text{By } (1)]$$

Also, $\beta_i[0] = K_i \cdot \alpha_{i-1}[0]$

$$= K_i \cdot \beta_{i-1}[i-1] \quad [\text{By induction hypothesis}]$$

$$= \alpha_i[i] \quad [\text{By } (1)]$$

and $\beta_i[i] = \beta_{i-1}[i-1]$

$$= \alpha_{i-1}[0]$$

$$= \alpha_i(0) \quad [\text{By } (1)]$$

Thus, we have,

$$\beta_i[k] = \alpha_i[i-k], \quad \forall\, 0 \le k \le i$$

and so the result holds by induction. ☒

- $\boxed{\underline{\text{Lemma}}:-\ B_i(z) = z^{-i} A_i(z^{-1}).}$

Proof: Note that

$$B_i(z) = \sum_{k=0}^{i} \beta_i[k] \cdot z^{-k}$$

$$= \sum_{k=0}^{i} \alpha_i[i-k] \cdot z^{-k} \quad \text{[By lemma]}$$

$$= \sum_{k=0}^{i} \alpha_i[k] \cdot z^{-i+k}$$

$$= z^{-i} \sum_{k=0}^{i} \alpha_i[k] \cdot (z^{-1})^{-k}$$

$$= z^{-i} \cdot A_i(z^{-1})$$

. . . . $\square$

We summarise below the key results needed for Task (i), i.e., given $K_1, .., K_{M-1}$, find $h[n]$.

These are,

$$A_0(z) = 1, \quad B_0(z) = 1$$

$$A_i(z) = z^{-1} K_i B_{i-1}(z) + A_{i-1}(z)$$

and $B_i(z) = z^{-i} A_i(z^{-1})$

We shall use these step-by-step to obtain $A_{M-1}(z)$ which will give us $\alpha_{M-1}[n] = h[n]$. An example follows.

Let $K_1 = 2$, $K_2 = 0.5$, $K_3 = -1$.

Then, $A_1(z) = z^{-1} K_1 B_0(z) + A_0(z)$

$$= 2z^{-1} + 1$$

$$B_1(z) = z^{-1} A_1(z^{-1}) = 2 + z^{-1}$$

$$A_2(z) = z^{-1} K_2 B_1(z) + A_1(z)$$

$$= \frac{1}{2}(2 + z^{-1}) \cdot z^{-1} + 2z^{-1} + 1 = 1 + 3z^{-1} + \frac{1}{2} z^{-2}$$

$$B_2(z) = z^{-2} A_2(z^{-1})$$
$$= z^{-2}\left(1 + 3z + \frac{1}{2} z^{2}\right)$$
$$= \frac{1}{2} + 3z^{-1} + z^{-2}$$

$$A_3(z) = z^{-1} K_3 \cdot B_2(z) + A_2(z)$$
$$= -\left(\frac{1}{2} + 3z^{-1} + z^{-2}\right) \cdot z^{-1} + 1 + 3z^{-1} + \frac{1}{2} z^{-2}$$
$$= 1 + \frac{5}{2} z^{-1} - \frac{5}{2} z^{-2} - z^{-3}$$

Thus $h[n] = \left[1, \frac{5}{2}, -\frac{5}{2}, -1\right]$

We finish by implementing task (ii), i.e. given $h[n]$, find $K_1, \dots, K_{M-1}$.

Firstly note that from the proof of the lemmas we observed that

$$\alpha_i[i] = K_i \, \beta_{i-1}[i-1]$$

and $\beta_{i-1}[i-1] = \alpha_{i-1}[0] = 1$

and so, $\boxed{\alpha_i[i] = K_i}$.

We therefore have

$$A_i(z) = z^{-1} \cdot K_i B_{i-1}(z) + A_{i-1}(z)$$

$$= K_i \left[B_i(z) - K_i \cdot A_{i-1}(z) \right] + A_{i-1}(z)$$

$$\left[\because B_i(z) = z^{-1} B_{i-1}(z) + K_i A_{i-1}(z) \right]$$

$$= (1 - K_i^2) \cdot A_{i-1}(z) + K_i B_i(z)$$

Thus, $$A_{i-1}(z) = \frac{A_i(z) - K_i \cdot B_i(z)}{1 - K_i^2}$$

Hence, we shall use the following formulas to recursively complete task (ii).

$$\boxed{\begin{aligned} A_{M-1}(z) &= \sum_{m=0}^{M-1} h[m] \cdot z^{-m} \\ B_i(z) &= z^{-i} A_i(z^{-1}) \\ A_{i-1}(z) &= \frac{A_i(z) - K_i B_i(z)}{1 - K_i^2} \\ K_i &= a_i[i] \end{aligned}}$$

An example follows.

Let $h[n] = [1, \frac{1}{2}, -\frac{1}{2}, 2]$

Then, $A_{M-1}(z) = 1 + \frac{1}{2} z^{-1} - \frac{1}{2} z^{-2} + 2z^{-3}$.

So $B_{M-1}(z) = z^{-3} A_{M-1}(z^{-1})$

$$= 2 - \frac{1}{2} z^{-1} + \frac{1}{2} z^{-2} + z^{-3}$$

$K_3 = 2$

So, $$A_2(z) = \frac{1 + \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2} + 2z^{-3} - 4 + z^{-1} - z^{-2} - 2z^{-3}}{-3}$$

$$= \frac{-3 + \frac{3}{2}z^{-1} - \frac{3}{2}z^{-2}}{-3}$$

$$= 1 - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2}$$

So, $K_2 = \frac{1}{2}$

Now, $B_2(z) = \frac{1}{2} - \frac{1}{2}z^{-1} + z^{-2}$

So, $$A_1(z) = \frac{1 - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4} + \frac{1}{4}z^{-1} - \frac{1}{2}z^{-2}}{3/4}$$

$$= \frac{\frac{3}{4} - \frac{1}{4}z^{-1}}{3/4}$$

$$= 1 - \frac{1}{3}z^{-1}$$

Thus, $K_1 = -\frac{1}{3}$

Note that the procedure fails if any K_i satisfies $K_i = \pm 1$.

— IIR system

Here, let $H(z) = \frac{C_M(z)}{A_N(z)}$

where $C_M(z) = \sum_{i=0}^{M} c_M[i] \cdot z^{-i}$

$A_N(z) = 1 + \sum_{i=1}^{N} \alpha_N[i] z^{-i}$,

and define $\alpha_N[0] = 1$, which is consistent with $A_N(z)$.

We will assume $M \leq N$. If not, we can do Euclidean division and express $H(z) =$ FIR part with $(M-N)$ taps $+ \dfrac{C_N'(z)}{A_N(z)}$,

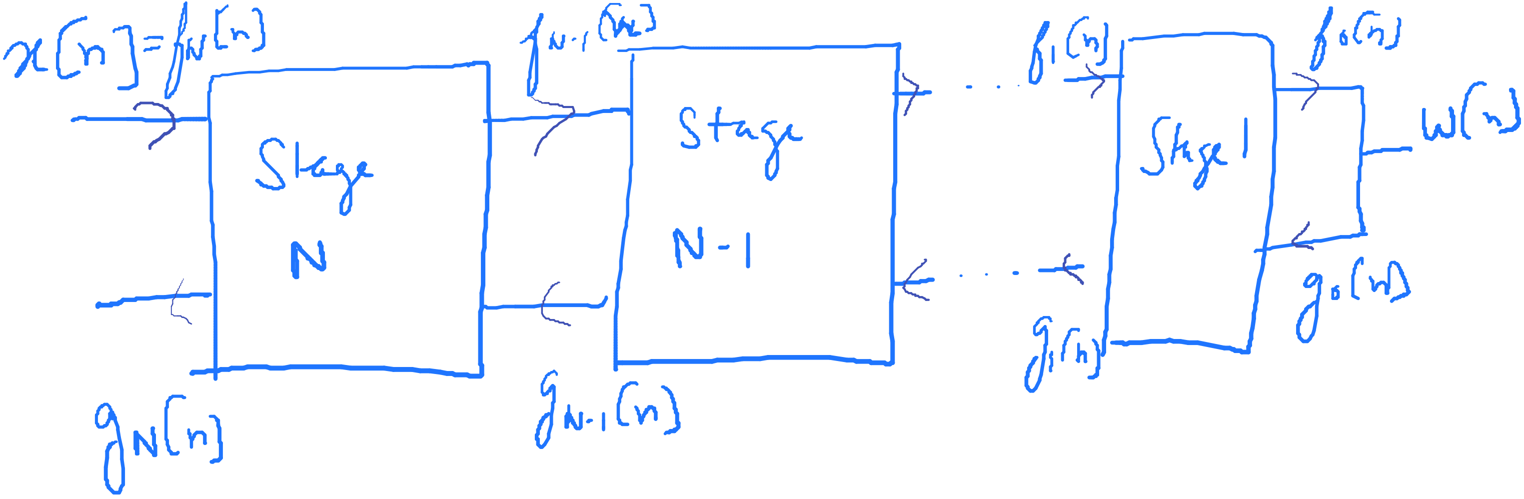
and implement the FIR part in parallel with $\dfrac{C_N'(z)}{A_N(z)}$. Henceforth, we will assume $M \leq N$.

Now $H(z) = \dfrac{1}{A_N(z)} \cdot C_M(z)$.

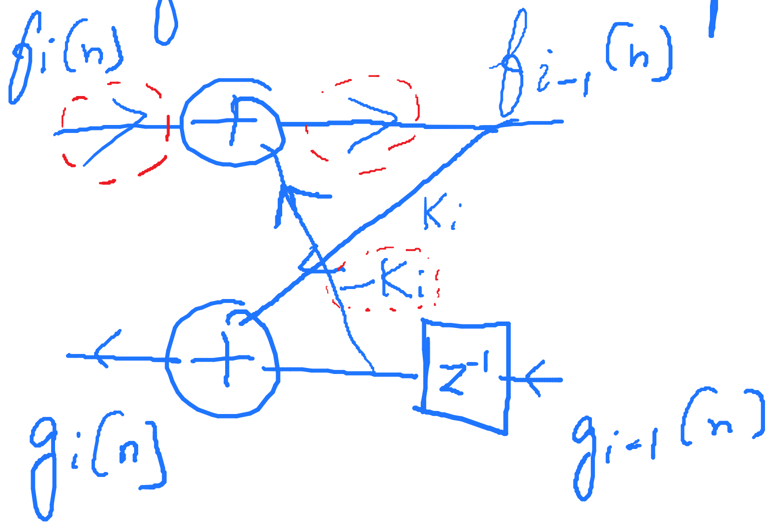
Let $W(z) = \dfrac{X(z)}{A_N(z)}$.

Hence $X(z) = W(z) \cdot A_N(z)$

Thus, we have the following lattice realisation,



where Stage i can be expanded as



[Notice the reversal in signal flow direction from the previous lattice stage used for FIR systems. The reversals are marked in red. This is because $x(n)$ is the input in this case. Also, observe that we have a $-K_i$ in place of K_i from the lower to the higher branch.]

In this case $f_N[n] = x[n]$

Just as in the FIR case, we define

$$\boxed{\begin{aligned} A_i(z) &= \frac{F_i(z)}{F_0(z)} = \frac{F_i(z)}{W(z)} \\ \text{and } B_i(z) &= \frac{G_i(z)}{G_0(z)} = \frac{G_i(z)}{W(z)} \end{aligned}}$$

In this case, the recursion equations become,

$$A_{i-1}(z) = A_i(z) - K_i z^{-1} B_{i-1}(z)$$

$$\Rightarrow A_i(z) = A_{i-1}(z) + K_i z^{-1} B_{i-1}(z)$$

Also, $B_i(z) = A_{i-1}(z) \cdot K_i + z^{-1} B_{i-1}(z)$

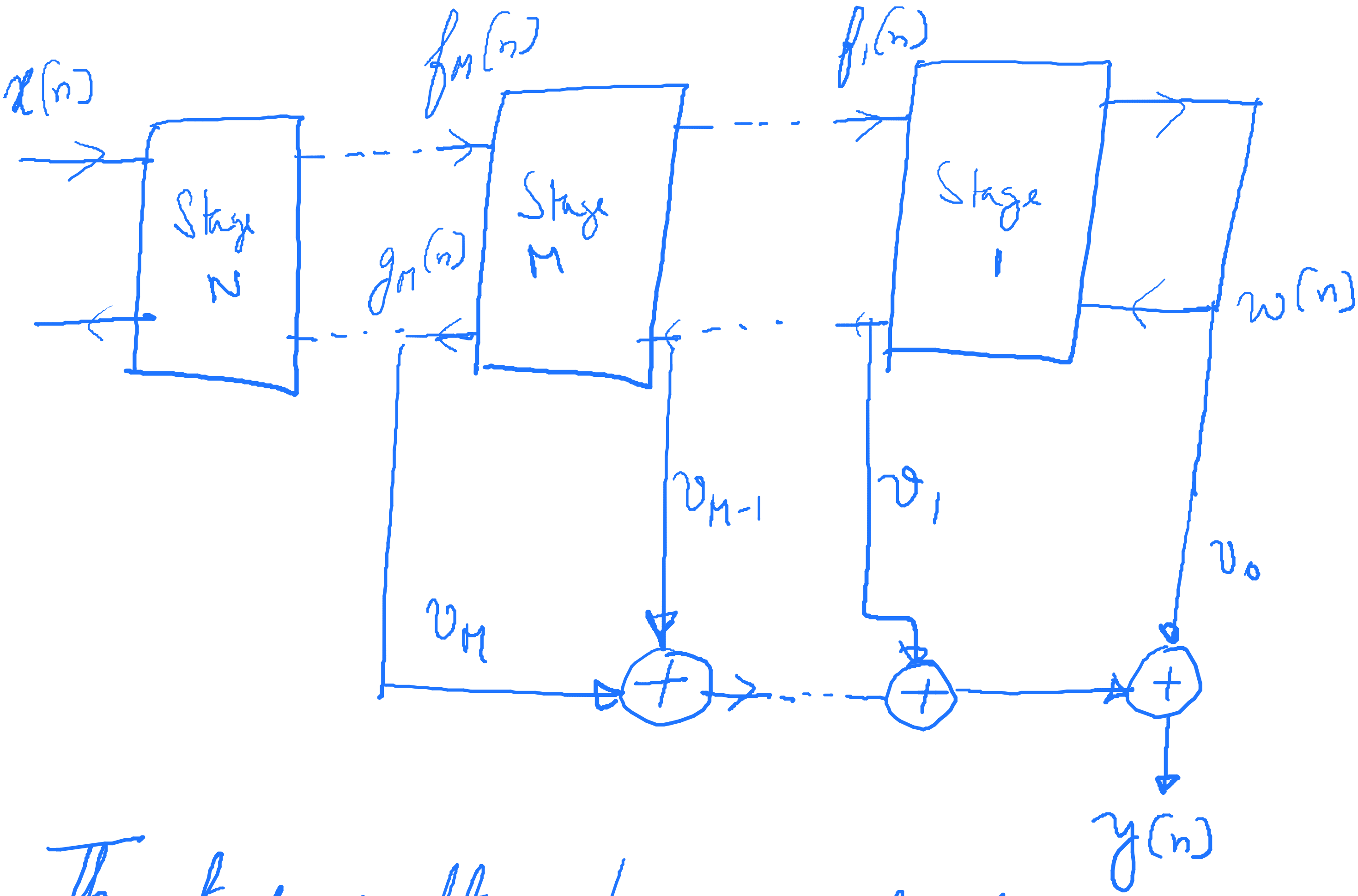
Since the recursion equations are similar we have

$$\boxed{\begin{gathered} \alpha_i[0]=1 \\ \alpha_i[i]=K_i \\ \alpha_i[k]=\beta_i[i-k] \quad \forall 0\le k\le i \\ A_i(z)=A_{i-1}(z)+z^{-1}K_iB_{i-1}(z) \\ B_i(z)=z^{-i}A_i(z^{-1}), \\ A_{i-1}(z)=\frac{A_i(z)-K_iB_i(z)}{1-K_i^2}. \end{gathered}}$$

exactly same as in the FIR case.

Thus, one can convert between the $A_N(z)$ and $K_1,\dots,K_N$ using exactly the same rules..

Next, to implement the $C_M(z)$ part, with $M \leq N$, we utilise the ladder structure below the lattice as shown below.



The task is then to convert between $v_0, \ldots, v_M$ and $C_M(z)$.

To do so, observe that

$$y[n] = \sum_{i=0}^{M} g_i[n] \cdot v_i$$

$$\Rightarrow Y(z) = \sum_{i=0}^{M} v_i \, G_i(z)$$

$$\Rightarrow C_M(z) = \frac{Y(z)}{W(z)} = \frac{Y(z)}{G_0(z)} = \sum_{i=0}^{M} v_i \cdot B_i(z)$$

Define $\boxed{C_i(z) = \sum_{j=0}^{i} v_j \cdot B_j(z), \quad 0 \le i \le M}$

Thus, $\boxed{C_i(z) = C_{i-1}(z) + v_i B_i(z)}$

- $\boxed{\text{Lemma :- } v_i = c_i[i].}$

Proof :- Since $C_i(z) = C_{i-1}(z) + v_i B_i(z)$, we have

$$\sum_{j=0}^{i} c_i[j] z^{-j} = \sum_{j=0}^{i-1} c_{i-1}[j] z^{-j} + \sum_{j=0}^{i} v_i \beta_i[j] z^{-j}$$

Equaling the coefficient of z^{-i}

from both sides we get

$$C_i[i] = v_i \beta_i[i] = v_i \quad [\because \beta_i[i]=1]$$

$\ldots \blacksquare$

With these, we can recursively use

$$c_i[i] = v_i$$

$$C_i(z) = C_{i-1}(z) + v_i B_i(z)$$

to convert between $C_M(z)$ and

$v_0, \ldots, v_M$.

<u>Example</u>:- Give the lattice ladder form of

$$H(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)}$$

Here $M = N = 2$.

$$H(z) = \frac{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}{1 - z^{-1} + \frac{2}{9}z^{-2}}$$

Thus, $A_2(z) = 1 - z^{-1} + \frac{2}{9} z^{-2}$.

So, $K_2 = \frac{2}{9}$

$$B_2(z) = \frac{2}{9} - z^{-1} + z^{-2}$$

Thus, $A_1(z) = \dfrac{A_2(z) - K_2 B_2(z)}{1 - \frac{4}{81}}$

$$= \frac{1 - z^{-1} + \frac{2}{9} z^{-2} - \frac{4}{81} + \frac{2}{9} z^{-1} - \frac{2}{9} z^{-2}}{\frac{77}{81}}$$

$$= 1 - \frac{\frac{7}{9} z^{-1}}{\frac{77}{81}}$$

$$= 1 - \frac{63}{77} z^{-1}$$

Thus, $K_1 = -\frac{63}{77}$

Also, $B_1(z) = -\frac{63}{77} + z^{-1}$

Now, $C_2(z) = 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}$

Thus, $v_2 = \frac{1}{8}$

So, $C_1(z) = C_2(z) - \frac{1}{8}B_2(z)$

$$= 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}$$
$$- \frac{1}{8}\left(\frac{2}{9} - z^{-1} + z^{-2}\right)$$

$$= \left(1 - \frac{1}{36}\right) - \frac{5}{8}z^{-1}$$

$$= \frac{35}{36} - \frac{5}{8}z^{-1}$$

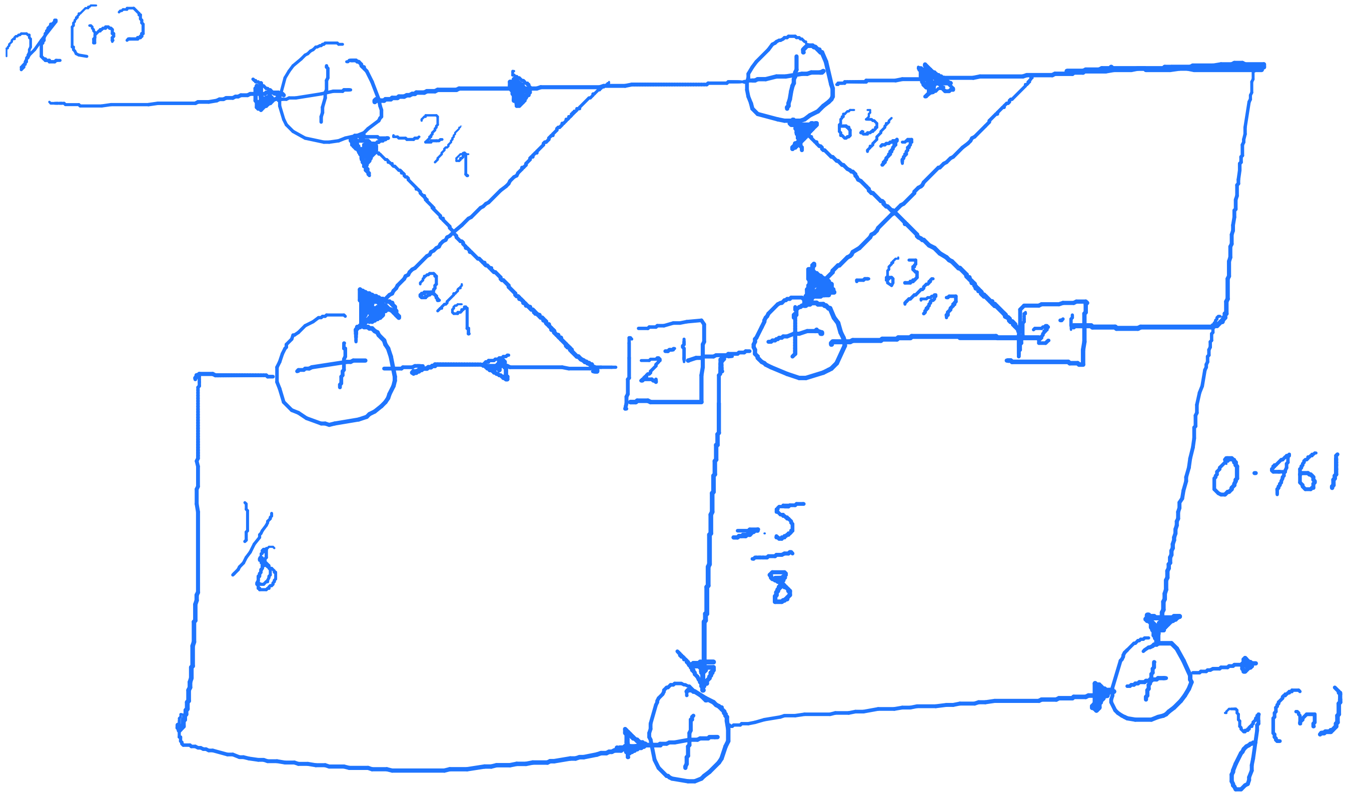
Thus, $v_1 = -\frac{5}{8}$

Now, $C_0(z) = C_1(z) + \frac{5}{8}B_1(z)$

$$= \frac{35}{36} - \frac{5}{8}z^{-1} + \frac{5}{8}\left(-\frac{63}{77} + z^{-1}\right)$$

$$= \frac{35}{36} - \frac{63 \times 5}{8 \times 77} = 0.461$$

Thus, $v_0 = 0.461$.



- **Lemma :-** An IIR system $H(z)$ is stable iff $|K_i| < 1$, $1 \leq i \leq N$.

This result is called the Marden Stability criteria and the proof involves theory of Hurwitz polynomials which is beyond the scope of this course.

* Effects of quantization on IIR filter

Fact :- All real numbers cannot be produced with finite number of bits.

Thus, consider $H(z) = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$.

Here b_k's and a_k's cannot always be implemented, and we take approximations

$\bar{b}_k$, $\bar{a}_k$.

Define $\Delta b_k = \bar{b}_k - b_k$

$\Delta a_k = \bar{a}_k - a_k$

Let us analyse how the poles of the system change as the coefficients change.

Let $1 + \sum_{k=1}^{N} a_k z^{-k} = \prod_{i=1}^{N} (1 - p_k z^{-1})$,

where $p_1, \ldots, p_N$ are the poles.

Note that p_i can be viewed as a

function of $a_1, \ldots, a_N$, i.e.,

$$p_i = p_i(a_1, \ldots, a_N)$$

$\bar{p_i} = p_i(\bar{a}_1, \ldots, \bar{a}_N)$ constitute the poles

of the actual realised system.

$$\Delta p_i = \bar{p_i} - p_i$$

$$= \nabla p_i^T(a_1, \ldots, a_N) \cdot \Delta a \quad \left[\begin{array}{l} a \triangleq (a_1, \ldots, a_N) \\ \Delta a \triangleq (\Delta a_1, \ldots, \Delta a_N) \end{array} \right]$$

$$+ \Delta a^T H(a + \theta \Delta a) \Delta a, \quad 0 \leq \theta \leq 1$$

where H is the Hessian matrix of

p_i w.r.t. a_i s.

Now, $|\Delta a^T H(a+\theta\Delta a)\Delta a| \leq \max\{|\mu_1|, |\mu_N|\} \cdot \|\Delta a\|^2$

where μ_1, μ_N are respectively the

largest and smallest eigen values of

$H(a + \Theta \Delta a)$.

In general, with large enough bits

$$\|\Delta a\|^2 \simeq 0.$$

Hence, we will neglect the second order term and write

$$\Delta p_i \simeq \nabla p_i(a) \cdot \Delta a$$

Now, let $D(z) = 1 + \sum_{k=1}^{N} a_k z^{-k}$

$$= \prod_{k=1}^{N} (1 - p_k z^{-k})$$

$$\frac{\partial D(z)}{\partial a_k} = \sum_{j=1}^{N} \frac{\partial D(z)}{\partial p_j} \cdot \frac{\partial p_j}{\partial a_k}$$

Then, $$\left.\frac{\partial D(z)}{\partial a_k}\right|_{z=p_i} = \sum_{j=1}^{N} \left.\frac{\partial D(z)}{\partial p_j}\right|_{z=p_i} \cdot \frac{\partial p_j}{\partial a_k}$$

Now,

$$\frac{\partial D(z)}{\partial p_j} = -\left(\prod_{k \neq j} (1 - p_k z^{-1})\right) z^{-1}$$

Thus, if $j \neq i$

$$\left(\frac{\partial D(z)}{\partial p_j}\right)_{z = p_i} = 0$$

Hence, we have

$$\left(\frac{\partial D(z)}{\partial a_k}\right)\Bigg|_{z = p_i} = \frac{\partial D(z)}{\partial p_i}\Bigg|_{z = p_i} \frac{\partial p_i}{\partial a_k}$$

Thus

$$\frac{\partial p_i}{\partial a_k} = \frac{\left(\frac{\partial D(z)}{\partial a_k}\right)\Big|_{z = p_i}}{\left(\frac{\partial D(z)}{\partial p_i}\right)\Big|_{z = p_i}}$$

Now $\left(\frac{\partial D(z)}{\partial a_k}\right)\Big|_{z=p_i} = z^{-k}\Big|_{z=p_i} = p_i^{-k}$

$$\left(\frac{\partial D(z)}{\partial p_i}\right)\Bigg|_{z=p_i} = -z^{-1}\prod_{j\neq i}(1-p_j z^{-1})\Bigg|_{z=p_i}$$

$$= -p_i^{-1}\prod_{j\neq i}(1-p_j p_i^{-1})$$

$$= -p_i^{-N}\prod_{j\neq i}(p_i-p_j)$$

Thus, $\frac{\partial p_i}{\partial a_k} = -\frac{p_i^{N-k}}{\prod_{j\neq i}(p_i-p_j)}$

Thus, we have

$$\boxed{\begin{aligned}\Delta p_i &\simeq \nabla p_i^T \cdot \Delta a \\ &= -\sum_{k=1}^{N}\frac{p_i^{N-k}\,\Delta a_k}{\prod_{j\neq i}(p_i-p_j)}\end{aligned}}$$

Thus if p_i is far apart from other poles, i.e., $\prod_{j \neq i} (p_i - p_j)$ is large enough, then Δp_i is small.

For this reason, instead of direct form 2-pole realizations like cascaded form and parallel form are preferred.

* Multirate systems

Let $x[n]$ be sampled from $x(t)$ at rate T_x, where

$$\frac{1}{2T_x} \geq B, \quad \text{where } X(F)=0, \ |F|>B.$$

By sampling theorem and interpolation

$\exists\, g(t)$ s.t.

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} x[n]\, g(t-nT_x)}$$

eg. $g(t) = \text{Sinc}(\pi F_x t)$.

Now suppose we need samples

$$y[m] \triangleq x(mT_y), \text{ where } T_y \leq \frac{1}{2B}.$$

One way is to reconstruct $x(t)$ from $x[n]$ and then resample at

$$F_y = \frac{1}{T_y}.$$

This however requires analog processing.

Instead we will try to do this digitally by directly converting from $x[n]$ to $y[m]$.

Observe that

$$y[m] = x(mT_y)$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot g(mT_y - nT_x)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_x) \cdot g\left(T_x\left(\frac{mT_y}{T_x} - n\right)\right)$$

Let

$$\boxed{\begin{aligned} k_m &\triangleq \left\lfloor \frac{mT_y}{T_x} \right\rfloor \\ \Delta_m &\triangleq \frac{mT_y}{T_x} - \left\lfloor \frac{mT_y}{T_x} \right\rfloor \end{aligned}}$$

Then,

$$y[m] = \sum_{n=-\infty}^{\infty} x(nT_x) \cdot g((k_m + \Delta_m - n) T_x)$$

Define $g_m(t) = g(t + \Delta_m T_x)$

Then,

$$y[m] = \sum_{n=-\infty}^{\infty} x(nT_x)\, g_m((k_m - n) T_x)$$

$$= \sum_{k=-\infty}^{\infty} x((k_m - k) T_x)\, g_m(kT_x)$$

(where $k = k_m - n$)

$$\Rightarrow \boxed{y[m] = \sum_{k=-\infty}^{\infty} x[k_m - k]\, g_m[k]}$$

where $g_m[k] = g_m(kT_x)$.

The above equation is not a convolution as g_m is dependent on m.

In other words, sampling rate conversion involves using linear but time varying systems.

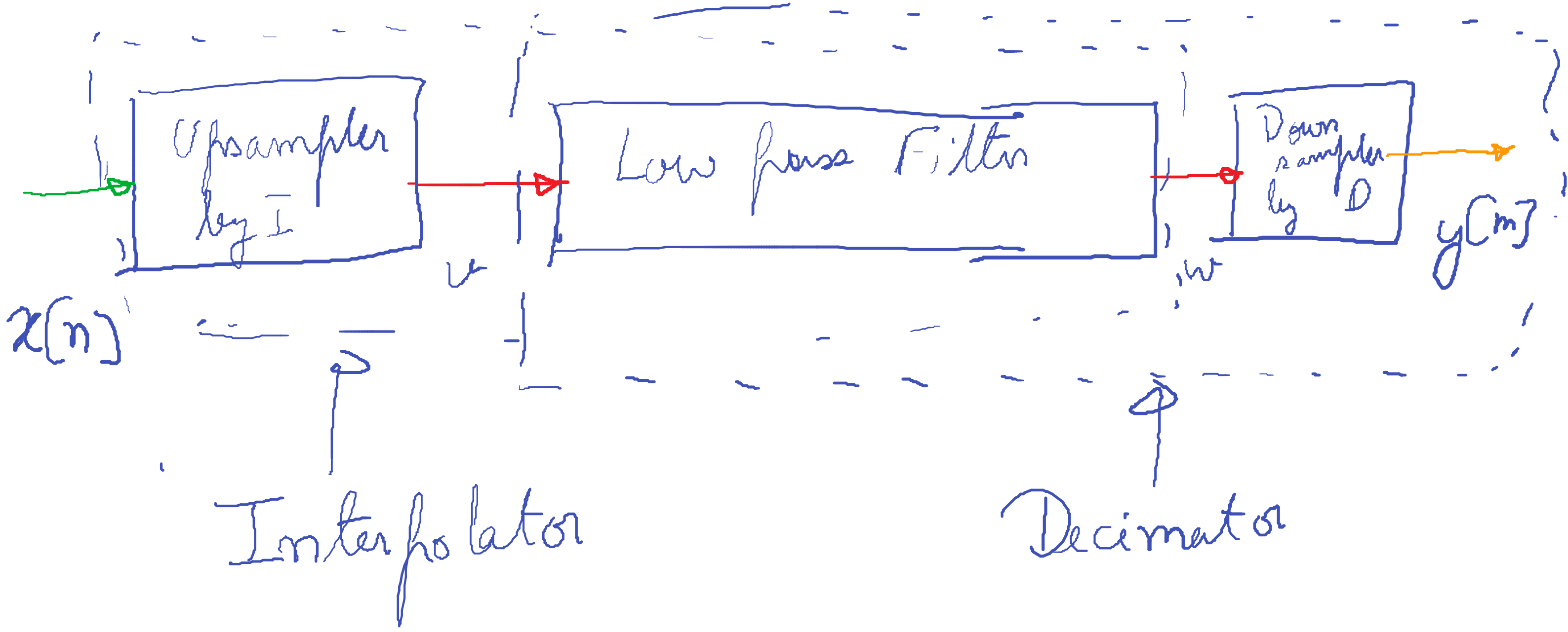
In general, it is not possible to convolve $x[n]$ with a different $g_m[n]$ for every m. However, simplification is possible when $\frac{F_y}{F_x} = \frac{I}{D}$, where $I, D \in \mathbb{N}$.

* Fractional sampling –

Interpolation by I + Decimation by D

Here we assume $\frac{F_y}{F_x} = \frac{I}{D}$ and

$F_y, F_x \geq 2B$.



Signal at sampling rate $F_v = I F_x$

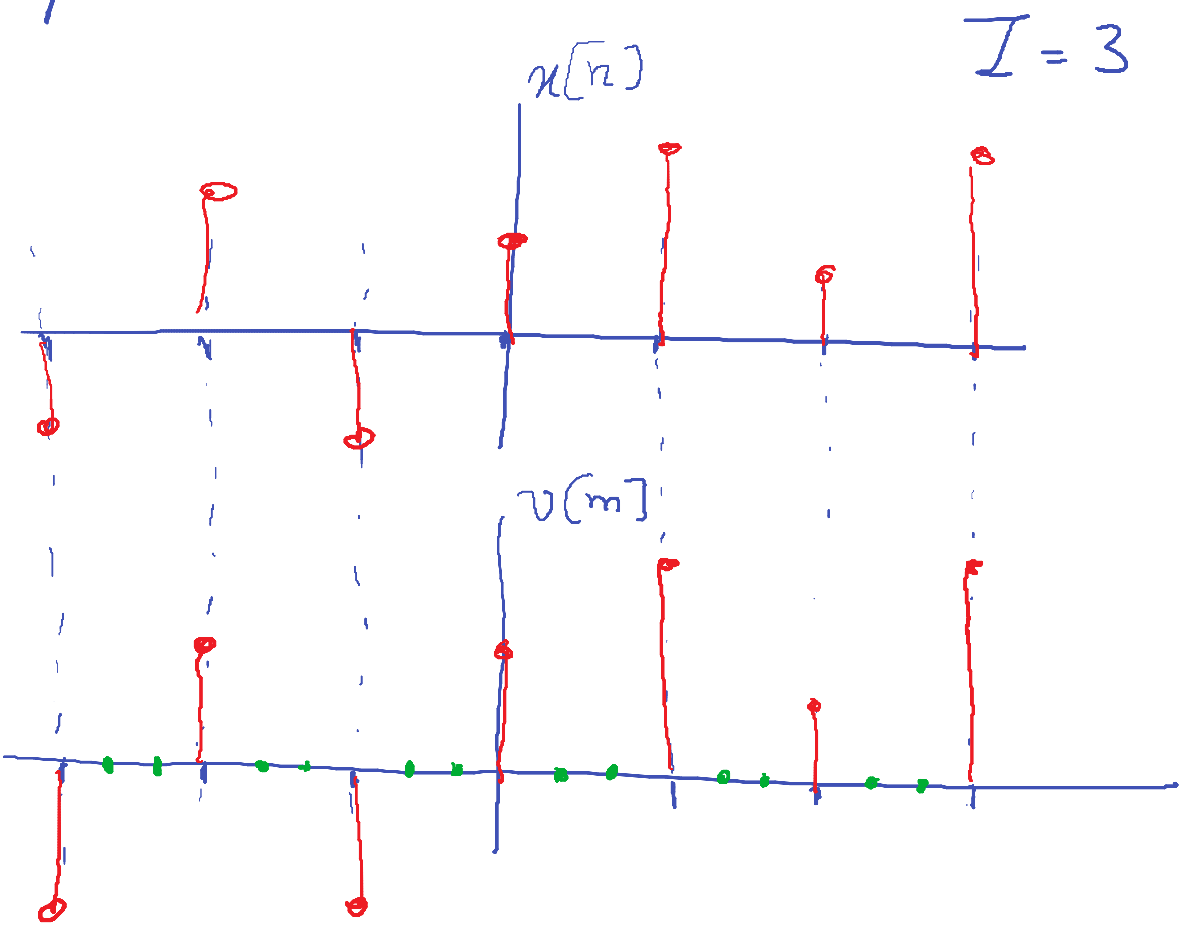
Signal at sampling rate F_x

Signal at sampling rate F_y

The upsampler block outputs $v[m]$ at rate $I F_x$ as follows.

$$\boxed{\begin{aligned} v[m] &= x[m/I], \quad m = 0, \pm I, \pm 2I, \ldots \\ &= 0, \quad \text{o.w.} \end{aligned}}$$

In other words $v[m]$ is just $x[n]$ and additional $(I-1)$ zero valued samples placed between to neighbouring $x[n]$ samples as shown below.



The low pass filter has frequency response

$$\boxed{\begin{aligned} H(\omega_v) &= I, \quad 0 \le |\omega_v| \le \min\{\pi/D, \pi/I\} \\ &= 0, \quad o.w. \end{aligned}}$$

Here ω_v is the angular frequency variable at sampling rate $F_v = IF_x$.

Similarly, we shall use ω_x and ω_y to represent angular frequency variables at rates respectively F_x and F_y.

Since, $F = f F_s = \frac{\omega F_s}{2\pi}$, we have

$$F = \frac{\omega_v F_v}{2\pi} = \frac{\omega_x F_x}{2\pi} = \frac{\omega_y F_y}{2\pi}$$

$$\Rightarrow \boxed{\begin{aligned} I\omega_v &= \omega_x \\ D\omega_v &= \omega_y \\ \text{and } \frac{\omega_y}{\omega_x} &= \frac{D}{I} \end{aligned}}$$

$\left[\because \frac{F_y}{F_x} = \frac{I}{D}\right]$

The downsampler performs the following operation

$$\boxed{y[m] = w[mD]}$$

Claim :- If $F_y, F_x \geq 2B$ then it is enough to show

$$Y_{DTFT}(\omega_y) = \frac{I}{D} X_{DTFT}\left(\frac{I}{D}\omega_y\right)$$

Proof :- If $F_y, F_x \geq 2B$, then by sampling theorem, we must have

$$X_{DTFT}(f_x) = F_x X(F)$$

$$Y_{DTFT}(f_y) = F_y X(F)$$

$$\Leftrightarrow \quad Y_{DTFT}(f_y) = \frac{F_y}{F_x} \cdot X_{DTFT}(f_x)$$

$$\Leftrightarrow \quad Y_{DTFT}(f_y) = \frac{I}{D} X_{DTFT}(f_x) \quad \left[\because \frac{F_y}{F_x} = \frac{I}{D}\right]$$

$$\Leftrightarrow \quad Y_{DTFT}(\omega_y) = \frac{I}{D} X_{DTFT}(\omega_x)$$

$$\Leftrightarrow \quad Y_{DTFT}(\omega_y) = \frac{I}{D} X_{DTFT}\left(\frac{I}{D}\omega_y\right) \quad \left[\because \frac{\omega_x}{\omega_y} = \frac{I}{D}\right]$$

. . . . . (8)

Thus, we need to show that

$$Y_{DTFT}(\omega_y) = \frac{I}{D} X_{DTFT}\left(\frac{I}{D}\omega_y\right)$$

is achieved by the system shown.

To see this, observe that

$$V(\omega_v) = \sum_{k=-\infty}^{\infty} v[k] e^{-j\omega_v k}$$

$$= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega_v k I}$$

$$= X_{DTFT}(\omega_v I)$$

$$= X_{DTFT}(\omega_x)$$

Now, $W(\omega_v) = V(\omega_v) \cdot H(\omega_v)$

$$= I\, X_{DTFT}(\omega_v I), \quad 0 \le |\omega_v| \le \min\left\{\frac{\pi}{D}, \frac{\pi}{I}\right\}$$

$$= 0, \quad \text{o.w.}$$

Now, let $h[m] = \frac{1}{D} \sum_{k=0}^{D-1} e^{j 2\pi k m / D}$

Then, if $m \bmod D \neq 0$

$$h[m] = \frac{1}{D} \sum_{k=0}^{D-1} e^{j 2\pi k m / D}$$

$$= \frac{1}{D} \cdot \frac{\left(e^{j 2\pi k m} - 1\right)}{e^{j 2\pi k m / D} - 1} = 0$$

else, if $m \bmod D = 0$, then

$$p[m] = \frac{1}{D}\sum_{k=0}^{D-1} e^{j2\pi km/D}$$

$$= \frac{1}{D}\sum_{k=0}^{D-1} 1$$

$$= 1$$

Hence, we can write

$$y[m] = w[mD] = w[mD] \cdot p[mD]$$

Then,

$$Y_{DTFT}(\omega_y) = \sum_{m=-\infty}^{\infty} y[m] \cdot e^{-j\omega_y m}$$

$$= \sum_{k=-\infty}^{\infty} w[k]\, p[k]\, e^{-j\omega_y k/D}$$

$$= \sum_{k=-\infty}^{\infty} w[k]\, p[k]\, e^{-j\omega_v k}$$

$$\left[\because \omega_v = \frac{\omega_y}{D}\right]$$

$$= \frac{1}{D} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{D-1} w[k] e^{j 2\pi l k / D} e^{-j\omega_0 k}$$

$$= \frac{1}{D} \sum_{l=0}^{D-1} \sum_{k=-\infty}^{\infty} w[k] e^{-j\left(\omega_0 - \frac{2\pi l}{D}\right)k}$$

$$= \frac{1}{D} \sum_{l=0}^{D-1} W\left(\omega_0 - \frac{2\pi l}{D}\right)$$

$$= \frac{1}{D} \sum_{l=0}^{D-1} V\left(\omega_0 - \frac{2\pi l}{D}\right) \cdot H\left(\omega_0 - \frac{2\pi l}{D}\right)$$

Now, $V\left(\omega_0 - \frac{2\pi l}{D}\right) \cdot H\left(\omega_0 - \frac{2\pi l}{D}\right)$

$\neq 0$, only if

$$-\min\left\{\frac{\pi}{D}, \frac{\pi}{I}\right\} + 2\pi l \leq \omega_0 \leq$$

$$2\pi l + \min\left\{\frac{\pi}{D}, \frac{\pi}{I}\right\}$$

Now, if $|\omega_y| \leq \pi$,

$$|\omega_v| \leq \frac{\pi}{D}$$

Thus, for $l > 0$,

$$-\min\left\{\frac{\pi}{D}, \frac{\pi}{I}\right\} + \frac{2\pi l}{D}$$

$$\geq -\frac{\pi}{D} + \frac{2\pi l}{D}$$

$$= \frac{\pi}{D}(2l-1)$$

$$> \frac{\pi}{D}$$

Thus, we have for $|\omega_y| \leq \pi$,

$$Y_{DTFT}(\omega_Y) = \frac{1}{D} V(\omega_v) \cdot H(\omega_v)$$

$$= \frac{I}{D} X_{DTFT}(\omega_v I),$$

$$|\omega_v| \leq \min\left\{\frac{\pi}{D}, \frac{\pi}{I}\right\}$$

$$= \frac{I}{D} X_{DTFT}\left(\frac{\omega_y \cdot I}{D}\right)$$

[$|\omega_y| \leq \frac{\pi}{I}$ ensures argument of X_{DTFT} remains within π]

Thus, we see that the above system indeed achieves sampling rate conversion.

In time domain, let

$h[k]$ be the impulse response of the filter.

Then,

$$y[m] = w[mD]$$

$$= \sum_{k=-\infty}^{\infty} v[k] h[mD-k]$$

Since $v[k] = 0$, if $k \bmod I \neq 0$,

we have

$$y[m] = \sum_{k=-\infty}^{\infty} x[k]\, h[mD - kI]$$

$$= \sum_{n=-\infty}^{\infty} x\left[\left\lfloor \frac{mD}{I} \right\rfloor - n\right] h\left[mD + nI - \left\lfloor \frac{mD}{I} \right\rfloor \cdot I\right]$$

$$\left[\text{By } n = \left\lfloor \frac{mD}{I} \right\rfloor - k\right]$$

$$= \sum_{n=-\infty}^{\infty} x\left[\left\lfloor \frac{mD}{I} \right\rfloor - n\right] h\left[(mD \bmod I) + nI\right]$$

Let $g(n,m) \triangleq h\left[(mD \bmod I) + nI\right]$

Then

$$\boxed{y[m] = \sum_{n=-\infty}^{\infty} x\left[\left\lfloor \frac{mD}{I} \right\rfloor - n\right] \cdot g(n,m)}$$

In other words, just as we saw before $g(n,m)$ is a discrete time LTV system.

Note that we only need I such systems as

$$g(n, m+kI)$$

$$= h\left[\left(mD + kDI \bmod I\right) + nI\right]$$

$$= h\left[\left(mD \bmod I\right) + nI\right]$$

$$= g(n,m).$$