① $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$; $\underset{\sim}{\theta} = (\mu, \sigma^2)$.

$$f_{\underset{\sim}{\theta}}(x) = \cancel{\frac{1}{(2\pi)^{n/2}}} \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right]$$

We can find MSS for $\underset{\sim}{\theta}$ by using the th^m.

$$\frac{f_{\underset{\sim}{\theta}}(x)}{f_{\underset{\sim}{\theta}}(\underset{\sim}{y})} = \frac{\frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right]}{\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right]}$$

$$= \exp\left[-\frac{1}{2\sigma^2}\left(\sum x_i^2 - \sum y_i^2\right) + \frac{2\mu}{2\sigma^2}\left(\sum x_i - \sum y_i\right)\right]$$

The ratio does not depend on $\underset{\sim}{\theta}$ iff
$\sum x_i = \sum y_i$ & $\sum x_i^2 = \sum y_i^2$

Thus $T(X) = \left(\sum X_i, \sum X_i^2\right)$ are MSS for $\underset{\sim}{\theta} = (\mu, \sigma^2)$.

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(2) $X_i \overset{iid}{\sim} U(\theta, \theta+1)$

$$f_\theta(x_i) = \begin{cases} 1 & \text{if } \theta < x_i < \theta+1 \\ 0 & \text{o/w} \end{cases}$$

w.r.t. Indicator f^n method,

$$I_\theta(x_i) = \begin{cases} 1 & \text{if } \theta < x_i < \theta+1 \\ 0 & \text{o/w.} \end{cases}$$

$$\Rightarrow L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$= \prod_{i=1}^{n} 1 \cdot I_\theta(x_i)$$

$$= \prod_{i=1}^{n} I_\theta(x_i)$$

$$= I_\theta(x_{(1)})\, I_\theta(x_{(2)}) \cdots I_\theta(x_{(n)})$$

$$= I_\theta(x_{(1)})\, I_{(x_{(1)}, x_{(2)})}(x_{(2)}) \cdots I_\theta(x_{(n)})$$

$$= I_\theta(x_{(1)})\, I_\theta(x_{(n)}).$$

Now to find MSS, using th^m.

$$\frac{f_\theta(x)}{f_\theta(y)} = \frac{I_\theta(x_{(1)})\, I_\theta(x_{(n)})}{I_\theta(y_{(1)})\, I_\theta(y_{(n)})}$$

This ratio does not depend on θ iff
$X_{(1)} = Y_{(1)}$ & $X_{(n)} = Y_{(n)}$
(In fact, the ratio then is 1).

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Thus $T(X) = (X_{(1)}, X_{(n)})$ are MSS ~~for~~ θ.

Notice, here dim of $\theta = 1$ (i.e. one unknown parameter) but " " $T = 2$ (i.e. 2 statistic req. to study θ.)

(5). $X_i \overset{iid}{\sim} B(p)$. ; $T(X) = \Sigma X_i$

We know that here,

$T(X) = \Sigma X_i \sim$ Binomial (n, p). To verify if it is complete we will use the def^n of completness.

Thus, we have to evaluate

$$E_\theta[g(t)] = 0 \qquad \forall\, \theta.$$

or $$E_p[g(t)] = 0. \qquad \forall\, p.$$

$$\Rightarrow \cancel{g(t)} \sum_{t=0}^{n} g(t) \binom{n}{t} p^t q^{n-t} = 0. \qquad \forall\, p.$$

$$\Rightarrow g(0)\binom{n}{0} p^0 q^{n-0} + g(1)\binom{n}{1} p^1 q^{n-1} + \dots + g(n)\binom{n}{n} p^n q^{n-n} = 0$$

$$\Rightarrow q^n \left[\sum_{t=0}^{n} \binom{n}{t} g(t) \left(\frac{p}{q}\right)^t \right] = 0. \qquad \forall\, p.$$

$$\Rightarrow \sum_{t=0}^{n} g(t)\binom{n}{t} \left(\frac{p}{q}\right)^t = 0. \qquad \forall\, p.$$

The LHS could be zero iff all the co-effi of $\left(\frac{p}{q}\right)$ are zero

$$\Rightarrow g(t)\binom{n}{t} = 0 \qquad \forall\, t.$$

$$\Rightarrow g(t) = 0 \qquad \forall\, t$$

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or $\quad g(t) = 0 \quad$ a.e.

Thus $T = \sum X_i$ is a complete statistic or equivalently for $T = \sum X_i \sim B(n, p)$; $0 < p < 1$. This family of distributions is complete.

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3

let $X_i \overset{iid}{\sim} N(\alpha\theta, \theta^2)$ α: known,

$$f_\theta(x_i) = \frac{1}{\sqrt{2\pi}\,\theta} \exp\left[-\frac{1}{2}\frac{(x-\alpha\theta)^2}{\theta^2}\right]$$

$$L(\theta) = \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\alpha\theta}{\theta}\right)^2\right]$$

$$= \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left[-\frac{1}{2}\frac{\Sigma x_i^2}{\theta^2} - n\alpha + \alpha\sum_{i=1}^{n} x_i\theta\right]$$

$$= \exp\left[n\alpha\right] \frac{1}{(2\pi\theta^2)^{n/2}} \exp\left[-\frac{1}{2}\frac{\sum_{i=1}^{n} x_i^2}{\theta^2} + \alpha\theta\sum_{i=1}^{n} x_i\right]$$

$$= h(x)\; g_\theta\left[T(x)\right]$$

Using factorisation Th^m

$T(x) = (\Sigma X_i^2, \Sigma X_i)$ for θ (as α is known)

Note here dim. of $\theta = 1$ while dim. of $T(x) = 2$.

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$\frac{\sum x_i^2}{n}$

Here $T = (T_1, T_2)$

So, if T is not complete then

$E[g(T)] = 0 \quad \forall \theta \not\Rightarrow g(t) = 0 \ \forall t$

Now, $T_1 = \sum X_i$, $E(T_1) = \sum_{i=1}^{n} E(X_i) = n\alpha\theta$

$T_2 = \sum X_i^2$; $E(T_2) = \sum_{i=1}^{n} E(X_i^2) = \sum_{i=1}\{V(X_i) + (E(X_i))^2\}$

$= \sum_{i=1}^{n} \theta^2 + (\alpha\theta)^2$

$= n(\theta^2 + \alpha^2\theta^2)$

$= n\theta^2(1+\alpha^2)$

$E(T_1) = n\alpha\theta$

$E(T_2) = n\theta^2(1+\alpha^2)$

Now, $T_1^2 = (\sum X_i)^2 \Rightarrow E(T_1^2) = E[(\sum X_i)^2] = E[(n\bar{X})^2]$

$= n^2 E[(\bar{X})^2]$

if $X_i \sim N(\mu, \sigma^2) \Rightarrow \bar{X} \sim N(\mu, \sigma^2/n)$ so here

$E(\bar{X}) = \alpha\theta$; $V(\bar{X}) = \theta^2/n \Rightarrow E(\bar{X}^2) = \frac{\theta^2}{n} + \alpha^2\theta^2$

$\Rightarrow E(T_1^2) = n^2\left(\frac{\theta^2}{n} + \alpha^2\theta^2\right) = \theta^2(n + n^2\alpha^2)$

Our goal is to create $g(T) = g(T_1, T_2)$ s.t. $E(g(T)) = 0 \ \forall \theta$

$E\left[\frac{T_1^2}{n+n^2\alpha^2} - \frac{T_2}{n(1+\alpha^2)}\right] = \theta^2 - \theta^2 = 0 \quad \forall \theta$

$\forall \theta$

$\Rightarrow E[g(T)] = 0$

$g(T) = \frac{T_1^2}{n+n^2\alpha^2} - \frac{T_2}{n(1+\alpha^2)}$

But $g(T) \neq 0 \quad \forall t$

$\Rightarrow T = (\sum X_i, \sum X_i^2)$ is not complete.

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$X_i \overset{iid}{\sim} P(\lambda)$. $\lambda > 0$

Step 1. $T(X) = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\lambda)$.

Step 2. Using def^n of completeness.

$E_\theta[g(t)] = 0 \quad \forall \theta \Rightarrow$

$g(t) = 0$ a.e.

Thus, $E_\theta[g(t)] = 0$

$\Rightarrow E_\lambda[g(t)] = 0$

$\Rightarrow \sum_{t=0}^{\infty} g(t) \frac{e^{-n\lambda}(n\lambda)^t}{t!} = 0$

$\Rightarrow \sum_{t=0}^{\infty} \frac{g(t)}{t!} (n\lambda)^t = 0$

$\Rightarrow g(0) + \frac{g(1)}{1!}(n\lambda) + \frac{g(2)}{2!}(n\lambda)^2 + \dots = 0.$

as $n\lambda > 0$

$\Rightarrow g(t) = 0 \quad \forall t$.

(for the sum of the series to be zero).

$\Rightarrow g(t) = 0$ a.e.

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Thus ~~gt~~ $T = \Sigma X_i$ is a complete statistic

& equivalently this family of distribution

(i.e. $X_i \sim P(\lambda)$; $\lambda > 0$) is complete.

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6

For completeness we want $E_\theta[g(t)] = 0 \quad \forall \theta$

$\Rightarrow g(t) = 0 \quad \forall t$

(a) for dist^n f

$E_\theta[g(t)] = \sum_t g(t)\, P(T=t) \qquad 0 < p < 1/4$

$= g(0)p + g(1)3p + (1-4p)\, g(2)$

$\Rightarrow p[g(0) + 3g(1) - 4g(2)] + g(2) = 0$

Clearly when $g(2) = 0$ & $g(0) + 3g(1) - 4g(2) = 0$

i.e $g(2) = 0$ & $g(0) = -3g(1)$

then $g(t) = 0 \quad \forall p$

But $g(t) \neq 0 \quad \forall t$.

it happens only when above conditions are satisfied

Hence, the family is not complete

(b) $E_p[g(t)] = 0$

$\Rightarrow \quad g(0)\,p + g(1)\,p^2 + g(2)\,(1-p-p^2) = 0.$

$g(2) + [g(0) - g(2)]\,p + [g(1) - g(2)]\,p^2 = 0$

$\Rightarrow \quad g(2) = 0 \quad \& \quad g(0) - g(2) = 0 \quad \& \quad g(1) - g(2) = 0$

or $\quad g(2) = 0 \quad \& \quad g(0) = g(2) = 0 \quad \& \quad g(1) = g(2) = 0$

Thus, $E_p[g(t)] = 0 \quad \forall p$

$\Rightarrow \quad g(t) = 0 \quad \forall t.$

Thus, the family is complete.

7

$f_\theta(x_i) = e^{-(x_i - \theta)} \quad \text{if } \theta < x_i < \infty$

$f_\theta(x_i) = e^{-(x_i - \theta)}\, I_{(\theta,\infty)}(x_i)$

(a) $L(\theta) = \prod_{i=1}^{n} e^{-(x_i - \theta)}\, I_{(\theta,\infty)}(x_i)$

st $I_\theta(x_i) = \begin{cases} 1 & \text{if } \theta < x_i < \infty \\ 0 & \text{o/w} \end{cases}$

$$L(\theta) = e^{-\Sigma x_i + n\theta} I_{(\theta,\infty)}(x_{(1)})$$

$$= \underbrace{e^{-\Sigma x_i}}_{h(x)} \underbrace{e^{n\theta} I_{(\theta,\infty)}(x_{(1)})}_{g_\theta(T(x))}$$

$\Rightarrow$ By Factorization thm $T(x) = X_{(1)}$ is suff. for θ

To get MSS.

$$\frac{f_\theta(\underset{\sim}{x})}{f_\theta(\underset{\sim}{y})} = \frac{e^{-\Sigma x_i} e^{n\theta} I_{(\theta,\infty)}(x_{(1)})}{e^{-\Sigma y_i} e^{n\theta} I_{(\theta,\infty)}(y_{(1)})}$$

$$= e^{-(\Sigma x_i - \Sigma y_i)} \frac{I_{(\theta,\infty)}(x_{(1)})}{I_{(\theta,\infty)}(y_{(1)})}$$

Now, this ratio does not depend on θ iff $X_{(1)} = Y_{(1)}$

Thus, $T(x) = X_{(1)}$ is MSS for θ.

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8 $f_\theta(x) = \theta x^{\theta-1} e^{-x^\theta}$ if $x>0$

$Y = \theta \log X \Rightarrow X = \exp(Y/\theta)$ for $x<1$ $y<0$ o/w $y>0$ $\Rightarrow -\infty<y<\infty$

$\frac{dx}{dy} = \exp\left[\frac{y}{\theta}\right]\frac{1}{\theta}$

$f_\theta(y) = \theta\, e^{y/\theta(\theta-1)}\, e^{-(e^{y/\theta})^\theta} \cdot e^{\frac{y}{\theta}} \frac{1}{\theta}$; $-\infty<y<\infty$

$= e^{y} e^{-e^{y}}$; $-\infty<y<\infty$

: pdf does not depend on θ

$\Rightarrow$ But $Y = \theta \log X$ is not a statistic

So Consider $U = \frac{\theta \log X_1}{\theta \log X_2} = \frac{Y_1}{Y_2} = \frac{\log X_1}{\log X_2}$

this will be independent of θ.
Also, the distn will not depend on θ.

Thus $U = \frac{\log X_1}{\log X_2}$ is an ancillary statistic

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