1) ⇒ $u = x^3 - 3xy^2$

for a function to be harmonic, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ — ① mark $\qquad \frac{\partial u}{\partial y} = -6xy$ — ① mark

$\frac{\partial^2 u}{\partial x^2} = 6x$ — ① mark $\qquad \frac{\partial^2 u}{\partial y^2} = -6x$ — ① mark

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\quad \therefore$ function is harmonic

— ① mark

b) Using Cauchy-Riemann eqⁿ

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$v(x,y) = \int 3x^2 - 3y^2 \, dy$

$= 3x^2y - y^3 + C(x)$ — ① mark

$$\frac{\partial v}{\partial x} = 6xy$$

$$6xy + c'(x) = 6xy$$

$\therefore c'(x) = 0$ — ① mark

$f(z) = x^3 - 3x^2y + i(3x^2y - y^3)$ — ① mark

if added c, also give 1 mark

$f(z) = z^3$ — ② marks for solving & showing answer

2) a) $r = f(t)\hat{i} + g(t)\hat{j}$

$r = x\hat{i} + y\hat{j}$

$v = x'\hat{i} + y'\hat{j}$ — ① mark

$|v| = \sqrt{x'^2 + y'^2}$ — ① mark

$$T = \frac{v}{|v|} = \frac{x'\hat{i} + y'\hat{j}}{\sqrt{x'^2 + y'^2}}$$ — ① mark

$$\frac{dT}{dt} = \frac{y'(y'x'' - x'y'')\hat{i} + x'(x'y'' - y'x'')}{(x'^2 + y'^2)^{3/2}}$$ — ② marks if properly solved

① mark if partial solved & correct answer

$$\left|\frac{dT}{dt}\right| = \frac{|y'x'' - x'y''|}{|x'^2 + y'^2|}$$ — ① mark

$$K = \frac{1}{|v|}\left|\frac{dT}{dt}\right| = \frac{|x''y' - x'y''|}{(x'^2+y'^2)^{3/2}}$$ —— ① mark

Award full marks if done using V, a for finding K correctly and done cross product determinant correctly and for other steps done clearly

b) i) $r(t) = \tan^{-1}(\sinh t)\,\hat{i} + \ln(\cosh t)\,\hat{j}$

$x = \tan^{-1}(\sinh t)$

$y = \ln(\cosh t)$

$x' = \operatorname{sech} t$

$y' = \tanh t$

① mark for x' & y'

$x'' = -\operatorname{sech} t \tanh t$

$y'' = \operatorname{sech}^2 t$

① mark for x'' & y''

$K = |\operatorname{sech} t|$ —— ① mark for answer

3) i) $F = (-x^2 - 4xy)\hat{i} - 6yz\,\hat{j} + 12z\,\hat{k}$

$\nabla \cdot F = -2x - 4y - 6z + 12$ —— ① mark

$$\text{Flux} = \int_0^a \int_0^b \int_0^1 (-2x - 4y - 6z + 12)\,dz\,dy\,dx$$

$= ab(-a - 2b + 9)$ —— ③ marks (each mark for solving correct integral)

$$\frac{\partial Flux}{\partial a} = -2ab - 2b^2 + 9b$$ — ① mark

$$\frac{\partial Flux}{\partial b} = -a^2 - 4ab + 9a$$ — ① mark

$$\frac{\partial Flux}{\partial a} = 0, \quad \frac{\partial Flux}{\partial b} = 0 \longrightarrow a = 0 \text{ or } a + 4b = 9$$ — ① mark

$\downarrow$

$b = 0, \ 2a + 2b = 9$ — ① mark

Solving above eqⁿ

$a = 3, b = 3/2$ — ① mark for a & b

Max. flux = $27/2$ — ① mark

If done using other method, give marks accordingly

Sol^n ⇒

a) ⇒ $x - y = u \quad , \quad v = 2x + y$

$$x = \frac{1}{3}(u+v)$$

$$y = \frac{(-2u+v)}{3}$$

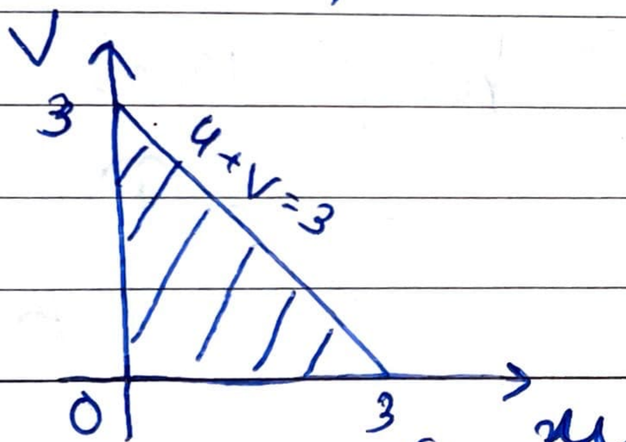
① mark for x & y

① mark for writing determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} = 3/9 = 1/3$$

① mark for answer

b) ⇒



— ① mark for figure

$(0,0)$ to $(1,1)$, $y = x$

$\boxed{\therefore u = 0}$ — ① mark

$(1,1)$ to $(1,-2)$, $x = 1$

$x-y+2x+y=3$

$\Rightarrow \boxed{u+v=3}$ —— ① mark

$(0,0)$ to $(1,-2)$, $2x+y=0$

$\Rightarrow \boxed{v=0}$ —— ① mark

c) $\Rightarrow$ $\iint_R (2x^2-xy-y^2)\,dx\,dy$

$y=-2x+4$, $y=-2x+7$, $y=x-2$, $y=x+1$

$$\iint_R (x-y)(2x+y)\,dx\,dy = \iint_G uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

$$= \iint_G \frac{uv}{3}\,du\,dv$$ —— ① mark for convert to u,v

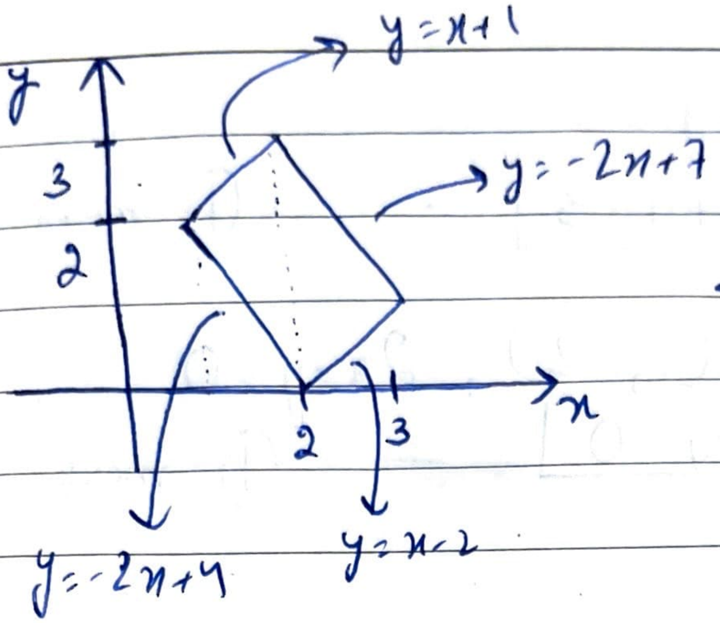
$$\frac{1}{3}\int_{-1}^{2}\int_{4}^{7} uv\,du\,dv = \frac{1}{3}\int_{-1}^{2} u\left[\frac{v^2}{2}\right]_4^7 du$$

① mark for u limit

① mark for v limit

$$= \frac{11}{2}\int_{-1}^{2} u\,du = \frac{11}{2}\left(\frac{u^2}{2}\right)_{-1}^{2}$$

$$= 33/4$$ —— ① mark for answer



— ① mark for figure

Sol^n) $g(x,y,z) = x + 2y + 3z - 13$

$\nabla g = \hat{i} + 2\hat{j} + 3\hat{k}$ — ① mark

$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$

$\nabla f = 2(x-1)\hat{i} + 2(y-1)\hat{j} + 2(z-1)\hat{k}$ — ① mark

$$\nabla f = \lambda \nabla g$$

$2(x-1)\hat{i} + 2(y-1)\hat{j} + 2(z-1)\hat{k} = \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$ — ① mark

$2(x-1) = \lambda$

$2(y-1) = 2\lambda$

$2(z-1) = 3\lambda$

① mark for relation

$x = \dfrac{y+1}{2}$

$z = 3\left(\dfrac{y+1}{2}\right) - 2 = \dfrac{3y-1}{2}$ — ① mark for relation

$$x + 2y + 3z = 13$$

$$\frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) = 13$$

$$\Rightarrow y = 2 \quad \text{———} \quad 1 \text{ mark}$$

$$(x, y, z) = \left(3/2, 2, 5/2\right) \quad \text{———} \quad 1 \text{ mark for each } x \,\&\, z$$

$\therefore$ It is closest to the point $(1, +1, 1)$