Game Theory: Assignment 4 (Solutions)

Total points: 75 | Due Date: 17/11/2025

Contribution to grade: 7.5% | Due time: 11:59 PM

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1. **Bayesian Nash Equilibrium.** Consider two candidates fighting an election, each with a privately known personal cost for entering the race. The probability of having a low entry cost, f_L, is p and the probability of having a high entry cost, f_H, is $1-p$. A candidate's payoff depends on whether she enters the race and whether the other enters or not as well. Let v_2 be a candidate's payoff when she enters and the candidate does as well, v_1 be a candidate's payoff when she enters and the other candidate does not, and 0 be the payoff when she does not enter (regardless of the other candidate's decision). Assume that

$$\begin{aligned} v_1 > v_2 &> 0 \\ f_H > f_L &> 0 \\ v_2 - f_L > 0 &> v_2 - f_H \\ v_1 - f_H &> 0 \end{aligned}$$

   (a) Show the conditions for when it is a symmetric BNE for a candidate to enter only when she has a low cost from doing so. (15)

   Answer. It is a symmetric BNE for a candidate to enter if she has low personal cost iff, for a low type

$$p(v_2 - f_L) + (1-p)(v_1 - f_L) \geq 0$$

   Rearranging,

$$pv_2 + (1-p)v_1 \geq f_L \tag{1}$$

Similarly, when a candidate has high cost, she will not enter if

$$0 \geq p(v_2 - f_H) + (1 - p)(v_1 - f_H)$$

which yields

$$f_H \geq pv_2 + (1 - p)v_1 \tag{2}$$

Combining (1) and (2), the condition for a candidate to enter only when she has a low personal cost is

$$f_H \geq pv_2 + (1 - p)v_1 \geq f_L$$

Marks breakdown:

- 1 marks for equation (1). 6 marks Partial credit for setup as well.
- 1 marks for equation (2). 6 marks
- 3 mark for final combined result.

(b) Show the conditions for when it is a symmetric BNE for a candidate to enter for sure when she has a low cost (i.e., with probability 1) and to enter with some probability between 0 and 1 when she has a high cost (mixed strategy). *Hint: Assume that if she has high cost she enters with probability q. This problem is a bit difficult so all you need to do is set it up the type-wise conditions and find the expression for q for full marks. You do not have to compare all conditions to get the final answer.* (15)

Answer. The conditions are that if she has low cost, she enters but if she has high cost, she enters with probability q. For a low type,

$$(p + (1 - p)q](v_2 - f_L) + (1 - p)(1 - q)(v_1 - f_L) \geq 0$$

The first expression counts the cases when the rival is a low type (probability p) or a high type who chooses to enter (probability $(1 - p)q$. The second case considers a situation in which the rival does not enter (probability $(1-p)(1-q)$. Rearranging gives

$$[p + (1-p)q]v_2 + (1-p)(1-q)v_1 \geq f_L \tag{3}$$

And for a high type,

$$0 = [p + (1-p)q](v_2 - f_H) + (1-p)(1-q)(v_1 - f_H)$$

Since the high type is randomizing, the condition has to be equal to 0 as he has to be indifferent between participating and not participating (and not participating yields 0).

$$q = \frac{(1-p)(v_1 - v_2) + v_2 - f_H}{(1-p)(v_1 - v_2)} \tag{4}$$

Not graded after this point.

For this to be an equilibrium we need, $0 < q < 1$. So

$$0 < \frac{(1-p)(v_1 - v_2) + v_2 - f_H}{(1-p)(v_1 - v_2)} < 1$$

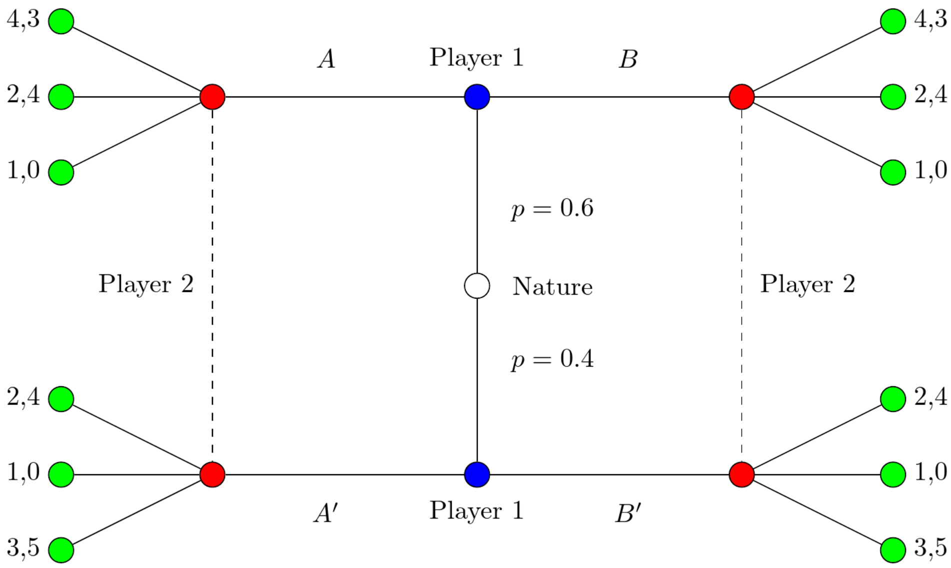
$$pv_2 + (1-p)v_1 > f_H \tag{5}$$

Using (3), (4) and (5), you will get $f_H \geq f_L$ (true by definition), and the only binding condition here is the expression for q, i.e., (4)

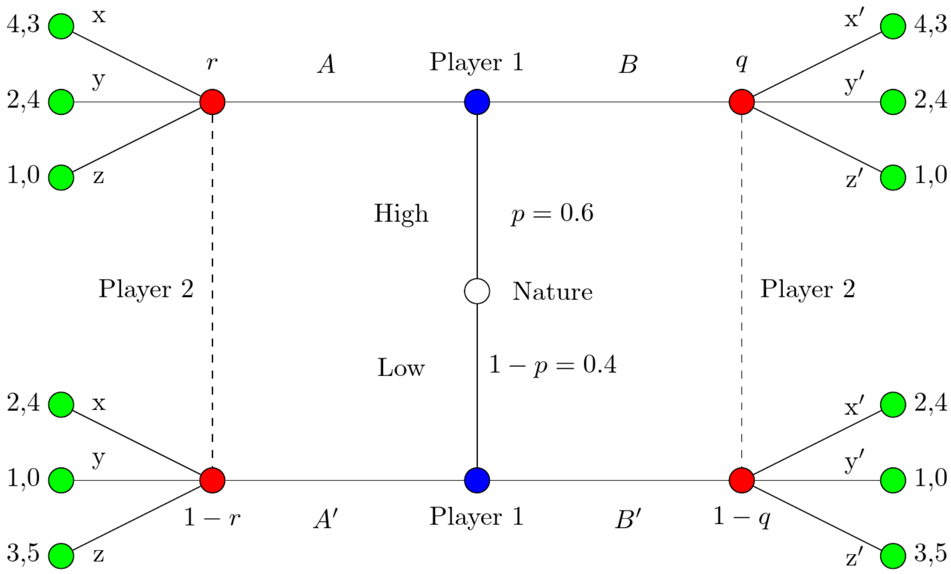
Marks breakdown:

- 5 marks for equation (3).
- 10 marks for equation (4). 7 marks Partial credit for setup as well.

2. **Perfect Bayesian Equilibrium.** Check all pooling and separating NE for the following game *(5 marks for each equilibrium check)*



Answer. I am going to use an updated tree with better labeling, but you can answer as per the tree above with your own labels wherever needed. The above figure is



symmetric in terms of payoffs (regardless of player 1's type), so really, we can check for just 1 pooling and one separating equilibrium, and then invoke symmetry.

Checking for Separating with A, B'

For this to be sustained, $r = 1$ and $q = 0$ needs to hold. Thus, upon observing action A, player 2 chooses y, and upon observing action B', player 2 chooses z′. If high type of player 1 deviates, she would get a payoff of 0, which is lower than 2.

If low type of player 2 deviates, she would get a payoff of 1, which is lower than 3. Hence, this separating PBE is sustained. By symmetry, the separating PBE B, A' is also sustained.

Checking for Pooling with A, A'

For this to be sustained, $r = 0.6$ and $1 - r = 0.4$ needs to hold, as player two has no way of inferring what type player 1 will be, relying purely in the underlying distribution that nature acts on. For player 2, if the pooling equilibrium sustains:

$$EU(\text{x}) = 3 \times 0.6 + 3 \times 0.4 = 3.4$$

$$EU(\text{y}) = 4 \times 0.6 + 0 \times 0.4 = 2.4$$

$$EU(\text{z}) = 0 \times 0.6 + 5 \times 0.4 = 2$$

Implying that player 2 chooses y. Upon observing B, we need to specify off the equilibrium path beliefs of Player 2, given by q. For the high type, player 1 has no incentive to deviate anyway, as they cannot get a higher payoff regardless of player 2's choices. However, the Low type can benefit from player 2 choosing z$'$ off the equilibrium. Player 2 will not choose z$'$ as long as $\mathbb{E}(\text{x}'|q) \geq \mathbb{E}(\text{z}'|q)$ or $\mathbb{E}(\text{y}'|q) \geq \mathbb{E}(\text{z}'|q)$ are both satisfied.

$$EU(\text{x}'|q) = 4 - q$$

$$EU(\text{y}'|q) = 4q$$

$$EU(\text{z}'|q) = 5 - 5q$$

$\mathbb{E}(\text{x}'|q) \geq \mathbb{E}(\text{z}'|q) \Rightarrow q \geq 0.25$ and $\mathbb{E}(\text{y}'|q) \geq \mathbb{E}(\text{z}'|q) \Rightarrow q \geq \frac{5}{9} \approx 0.56$. THus, as long as $q \geq 0.25$, there is no incentive for either type to deviate, as they will not get a better payoff. Hence, the poolsing can be sustained with $q \geq 0.25$. By symmetry, we get the same result for BB'

3. **Perfect Bayesian Equilibrium.** Answer the following:

   (a) What is a separating equilibrium, and how can it be useful for policy makers? (5)

   Answer. A separating equilibrium is a PBE in which different types take different actions (thus revealing their type by virtue of their action). This can be very useful for policy makers in cases where the policy maker wants the each type to take a specific decision, eg., they may want an environmentally friendly firm to produce, but a polluting firm to not produce, or efficient mergers to exist, but inefficient mergers to not exist.

   (b) In addition to sequential rationality in SPNE, PBE has another layer of sequential rationality. What is it? (5)

   Answer. Players also update their beliefs about the opponent's type, or any private information based on the actions they observe.

4. **Cournot competition.** Consider a setting in which two firms compete in quantities. They have symmetric marginal costs, $c = 1$, but they have asymmetric information about market demand. Firm 2 does not know the state of market demand, but think is it is $p(Q) = 10 - Q$ with probability 0.5 and $p(Q) = 5 - Q$ with probability 0.5. Firm 1 has full information. Find the BNE. (15)

   Answer.

Firm 1

First, let us focus on Firm 1, the informed player in this game, as we usually do when solving for the BNE of games of incomplete information. When Firm 1 observes a high demand market its profits are:

$$\text{Profits}_1 = 10q_1^H - (q_1^H)^2 - q_2 q_1^H - 1 \cdot q_1^H.$$

Differentiating with respect to q_1^H, we obtain Firm 1's best response function when experiencing low costs:

$$10 - 2q_1^H - q_2 - 1 = 0 \quad \Rightarrow \quad q_1^H(q_2) = 4.5 - \frac{q_2}{2}.$$

On the other hand, when Firm 1 observes a low demand market, its profits are:

$$\text{Profits}_1 = (5 - q_1^L - q_2)q_1^L - q_1^L.$$

Differentiating with respect to q_1^L gives the best response function:

$$5 - 2q_1^L - q_2 - 1 = 0 \quad \Rightarrow \quad q_1^L(q_2) = 2 - \frac{q_2}{2}.$$

Firm 2

Let us now analyze Firm 2 (the uninformed player in this game). Its profits must be expressed in expected terms, since Firm 2 does not know whether market demand is high or low:

$$\text{Profits}_2 = \tfrac{1}{2}\Big[(10 - q_1^H - q_2)q_2 - 1 \cdot q_2\Big] + \tfrac{1}{2}\Big[(5 - q_1^L - q_2)q_2 - 1 \cdot q_2\Big].$$

Rearranging and differentiating with respect to q_2 yields Firm 2's best response:

$$\tfrac{1}{2}\Big[(10 - q_1^H - 2q_2) - 1\Big] + \tfrac{1}{2}\Big[(5 - q_1^L - 2q_2) - 1\Big] = 0$$

which simplifies to

$$q_2(q_1^H, q_1^L) = \frac{13 - q_1^H - q_1^L}{4}.$$

After substituting the BRFs of Firm 1 into this expression:

$$q_2 = 3.25 - 1.625 + 0.25q_2 \quad \Rightarrow \quad q_2 = 2.167.$$

With this value we can compute the particular production levels:

$$q_1^L(q_2) = 2 - \frac{2.167}{2} = 0.916,$$

$$q_1^H(q_2) = 4.5 - \frac{2.167}{2} = 3.417.$$

Therefore, the Bayesian Nash equilibrium (BNE) of this oligopoly game with incomplete information about market demand prescribes the following production levels:

$$(q_1^H, q_1^L, q_2) = (3.42,\ 0.92,\ 2.17).$$

**5 points for each equilibrium value, partial credit for BRFs.