

# Game Theory: Mid Semester Exam

Total points: 30

Due Date: 29/09/2021

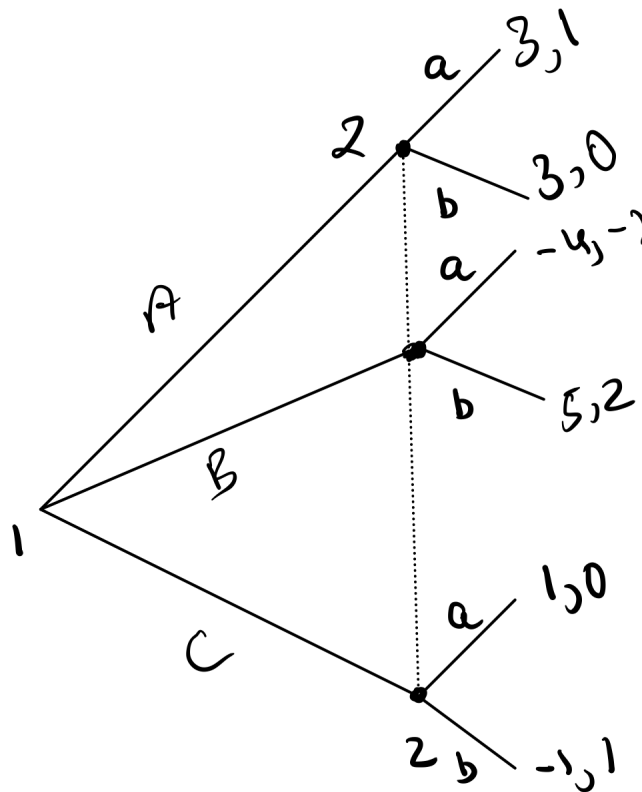
Contribution to grade: 30% (3xx); 20% (5xx)

Due time: 4:15 PM

- Show all steps, as it can help you get partial credit.
- Any sign of copying will be taken very seriously.
- Answer any 3 out of 4. 10 Marks each

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1. **Nash Equilibrium** Consider the following extensive form game, and answer the following questions:



- (a) Identify all proper subgames one by one, and explain why they are proper subgames. (3)

*Answer.* There is only one proper subgame, and it is the whole game. This is because we cannot start from any other node and create a sub-game without breaking an indifference set.

- (b) Find all Nash equilibria (pure and mixed), and show the BRFs (7)

*Answer.* First, we need to write this in the normal form.

		Player 2	
		$a$	$b$
Player 1	$A$	3, 1	3, 0
	$B$	-4, -1	5, 2
	$C$	1, 0	-1, 1

Normal form representation of the above game.

The natural next step is to use IESDS to eliminate action  $A$  for player 1, as it is strictly dominated by  $C$ . This leaves us with

		Player 2	
		$a$	$b$
Player 1	$A$	<u>3</u> , <u>1</u>	3, 0
	$B$	-4, -1	<u>5</u> , <u>2</u>

Normal form representation of the above game after IESDS.

It is clear that  $(A, a), (B, b)$  are psNE of the game. We now look for msNE. For this, we assign probability  $p$  to player 1 choosing  $B$  and  $q$  to player 2 choosing  $a$ . For indifference, we need

$$EU_1(A) = EU_1(B)$$

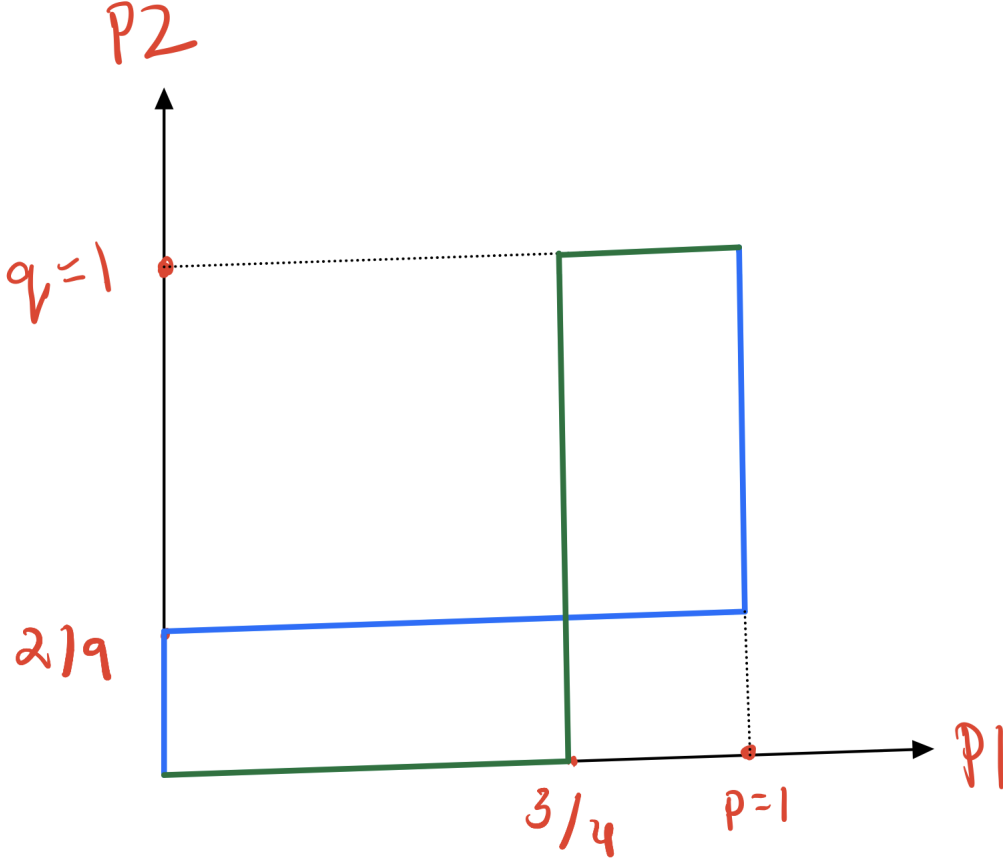
$$\Rightarrow 3q + (3)(1 - q) = (-4)q + 5(1 - q) \Rightarrow q = \frac{2}{9}$$

Similarly,

$$EU_2(a) = EU_2(b)$$

$$\Rightarrow (1)p + (-1)(1-p) = 0p + 2(1-p) \Rightarrow p = \frac{3}{4}$$

Blue: BRF 1; Green: BRF 2



Thus, in addition to the two psNE (visible from the graph as well), we have an msNE where  $\left\{\left(\frac{3}{4}A, \frac{1}{4}B\right), \left(\frac{2}{9}a, \frac{7}{9}b\right)\right\}$ .

2. **Manager and Employee** Consider a manager interacting with an employee in the following strategic setting. The choice set available to the employee is  $S_c = \{0, 1\}$ . The manager wants to influence the employee's choice over  $S_c$ . In an attempt to do so, the manager reveals, before the employee selects an action, a monetary transfer rule  $t : S_c \rightarrow \mathbb{R}$  that will be implemented after the employee has made a decision. The monetary transfers induced by  $t$  (i.e., the values of  $t(s_c) \forall s_c \in S_c$ ) can take either of two values: zero and one. The manager's objective is to maximize  $\pi_f(s_c, t) = 2s_c - t$ ,

while the employee's is to maximize  $\pi_e(s_c, t) = t - c(s_c)$ , where  $c(s_c)$  is the monetary cost of its action.  $c(0) = 0$  and  $c(1) = \frac{1}{2}$ .

(a) Represent the game in extensive form. (7)

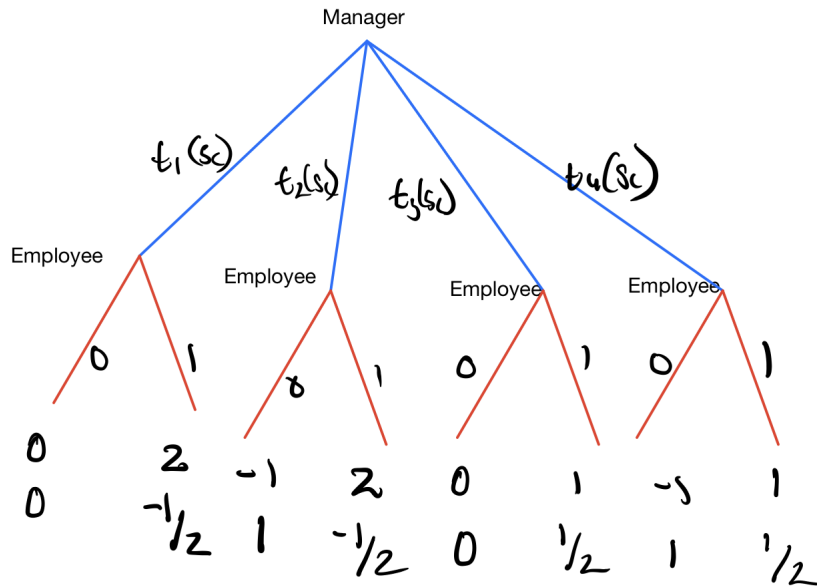
*Answer.* There are 4 options available to the manager in terms of the transfer rule.

$t_1(s_c)$ . Transfer 0 for any  $S_c$ .

$t_2(s_c)$ . Transfer 1 if  $S_c = 0$  and 0 if  $S_c = 1$ .

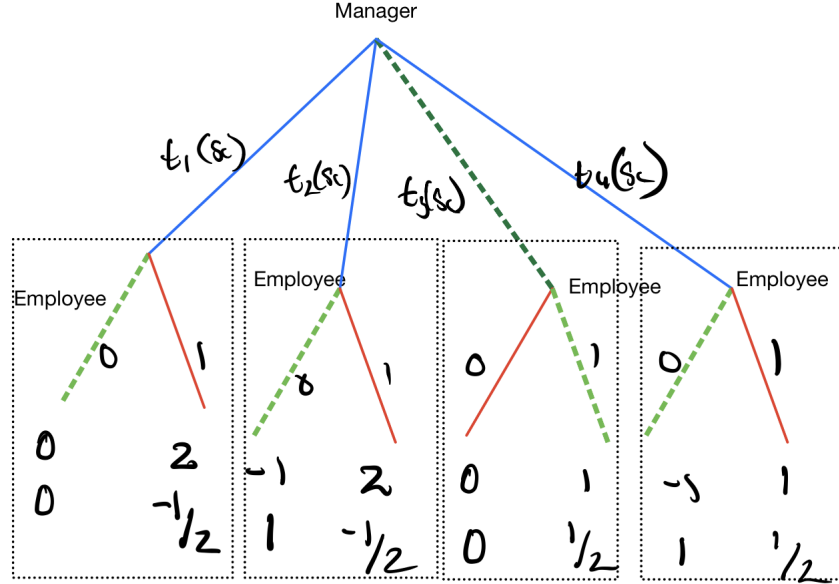
$t_3(s_c)$ . Transfer 0 if  $S_c = 0$  and 1 if  $S_c = 1$ .

$t_4(s_c)$ . Transfer 1 for any  $S_c$ .



(b) Find the SPNE. (3)

*Answer.* The SPNE is found by identifying the proper subgames and subsequent paths. Thus, the employee plays 0 if  $t_1(s_c)$ , 0 if  $t_2(s_c)$ , 1 if  $t_3(s_c)$ , 0 if  $t_4(s_c)$  and



the manager plays  $t_3(s_c)$ .

3. **Covid Vaccine Game.** Consider two firms simultaneously choosing an investment level in R&D to find a vaccine for Covid,  $x_i \in \{0, 1, \dots, k\}$  for every firm  $i$  where  $i \in \{1, 2\}$  and  $k > 2$ . The payoff function is given by:

$$\pi_i(x_i, x_j) = \begin{cases} R_i - x_i & \text{if } x_i > x_j, \text{ and} \\ -x_i & \text{otherwise.} \end{cases}$$

Note that to get the patent, the firm has to beat his rival, and hence, profits are not split in case they invest the same.

- (a) Find a minimum of one pure strategy Nash Equilibrium, unless you find that there are no psNE. (6)

There does not exist a Nash equilibrium in pure strategies. Following is the

explanation.

- i. For  $x_i = x_j = 0$ , both firms can gain by deviating with a higher investment and winning the patent.
  - ii. For  $x_i = x_j > 0$ , both firms get a negative payoff and are better off not participating.
  - iii.  $x_i < x_j < k$ . Firm  $i$  can do better, so not an NE.
  - iv.  $x_i < x_j = k$ .  $i$  would be better off just not participating.
  - v.  $0 = x_i < x_j = k$ . This is the closest thing to what looks like an NE, but of course,  $j$  can do better by investing less and thus getting a higher profit.
- (b) If I told you that one possible mixed strategy NE was a case in which there was a positive distribution of probabilities across all values of  $x_i$ , can you state what the expected payoff for each strategy in support of the msNE would be? (4)

*Answer.* This is actually very easy. Since  $x_i = 0$  is also in support of the msNE, the expected payoff for this is 0. For a mixed strategy to sustain a Nash equilibrium, all expected payoffs must be same, so all expected payoffs are 0.

4. **Firm vs Labor Union.** The world is in an economic downturn, and a job with high wages are hard to come by. Firms are trying to hire labor at an affordable wage, but unions want to protect labourers from underpaid jobs. This leads to a strategic stand-off between unions and firms. In the first stage, the labor union chooses the wage level,  $w$ , that all workers will receive. In the second stage, the firm responds to the wage  $w$  by choosing the number of workers it hires,  $h \in [0, 1]$ . The labor union's payoff function is  $u_L(w, h) = w \cdot h$ , thus being increasing in wages and in the number of workers hired. The firm's profit is

$$\pi(w, h) = \left( h - \frac{h^2}{2} \right) - wh$$

Intuitively, revenue is increasing in the number of workers hired,  $h$ , but at a decreasing

rate (i.e.,  $h - \frac{h^2}{2}$  is concave in  $h$ ), reaching a maximum when the firm hires all workers, i.e.,  $h = 1$  where  $R(1) = \frac{1}{2}$ .

- (a) Applying backward induction, analyze the optimal strategy of the last mover (firm). Find the firm's best response,  $h(w)$ . (5)
- (b) Let us now move on to the first mover in the game (labor union). Anticipating the best response function of the firm that you found in part (a),  $h(w)$ , determine the labor union's optimal wage,  $w^*$ . (5)