Modern Algorithm Design : Quiz 1
Full Marks : 25

27/8/2024

Write solutions in the space provided. NO extra pages will be provided. Write brief and precise solutions. Meaningless rambles fetch negative credits.

Problem 1. (10 points) Prove the *cycle rule* for minimum spanning trees - given a weighted graph $G = (V, E)$, consider the heaviest edge $e^\star$ on any cycle C. Then $e^\star$ is not in the minimum spanning tree of T. (you may assume distinct edge weights)

Proof. We prove it by contradiction. Consider T to be a minimum spanning tree of G. For contradiction, let $e^\star$ be the heaviest edge of cycle C, which is in T. **(+1 for this)**

Observe that removing an edge from T results in exactly two connected components. Each of the two components is a sub-tree of T. So, removing edge $e^\star$ from T creates two components, say L and R **(+2 for identifying this cut)**. Since C is a cycle, two edges of C cross the cut corresponding L and R **(+3 for identifying this property)**. One of the edges is $e^\star$. Let the other edge be $e' \in C$. We remove $e^\star$ from T and add e' to T, which results in a new spanning tree T' **(+2 for correctly identifying the edge e' and then creating T')**. Considering all edge weights are distinct, we have $w(e') < w(e^\star)$. Hence, we conclude $w(T') < w(T)$, which contradicts the minimality of T **(+2 for the final argument)**. □

Problem 2. (15 points) A graph is called *planar* if there exists a drawing of the graph in the 2D plane such that no edge cross another one. For instance, a triangle is planar, the complete graph on 4 vertices is planar, but the complete graph on 5 vertices is not. Further, it is known that the number of edges in a planar graph with n vertices is at most $3n - 6$. Show that Boruvka's algorithm runs in $O(n)$ time on such graphs. (You *do not* need to change the algorithm in ANY way)

Proof. Here, we use the following observation: Planarity is preserved under contraction. The easiest way to see this is to start with a planar drawing of G and observe that contraction cannot introduce any new crossings. **(+6 for this property)**

As discussed in the lecture, Boruvka's algorithm starts with n vertices and each phase decreases the number of vertices by half (here vertices are defined as a subset of the vertices in the original graph contracted to a single vertex). Using the observation above and the property stated in the problem statement, the number of edges after ith phase is at most $3\left(\frac{n}{2^i}\right), i = 0, 1, 2, \cdots$ **(+4 for deriving this upper bound on number of vertices in each round)**. Using the fact that each phase of Boruvka's algorithm requires $O(m_i)$ time, where m_i is the number of remaining edges after phase $i - 1$, we can upper bound the total runtime as follows **(+2 for writing this argument)**

$$\sum_{i=0}^{\log n} 3\frac{n}{2^i} < 3n \sum_{i=0}^{\infty} \frac{1}{2^i} = O(n)$$

(+3 for final calculation)

□