4 (a). $X_i \sim B(3, \theta)$.

$$P_\theta(X_i = x_i) = \binom{3}{x_i} \theta^{x_i} (1-\theta)^{3-x_i}$$

$$L(\theta) = \prod_{i=1}^{n} P_\theta(X_i = x_i)$$

$$= \prod_{i=1}^{4} \binom{3}{x_i} \theta^{x_i} (1-\theta)^{3-x_i}$$

$$l(\theta) = \log L(\theta)$$

$$= \sum_{i=1}^{n} \log C + x_i \log \theta + (3 - x_i) \log(1-\theta)$$

for given data $X = (1, 3, 2, 2)$.

$$l(\theta) = \log C + 8 \log \theta + 4 \log(1-\theta)$$

$$\frac{dl}{d\theta} = \frac{8}{\theta} + \frac{-4}{1-\theta} = 0$$

$$\Rightarrow \frac{8}{\theta} = \frac{4}{1-\theta}$$

$$\Rightarrow \hat{\theta} = 2/3$$

$$\frac{d^2 l}{d\theta^2} = \cancel{\frac{8}{\theta}} - \frac{8}{\theta^2} + \frac{(-1)4}{(1-\theta)^2}$$

$$= -\frac{8}{\theta^2} - \frac{4}{(1-\theta)^2} \Big|_{\theta = \hat{\theta}}$$

$$= -54 < 0.$$

$\Rightarrow \hat{\theta} = 2/3$ is the MLE estimate of θ for observed data.

4 b. $f_\theta(x_i) = \theta e^{-\theta x_i}$

$\cancel{L_\theta(x)} = L(\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \Sigma x_i}$

$l(\theta) = n \ln \theta - \theta \Sigma x_i$

for observed data

$l(\theta) = n \ln \theta - \theta (1.23 + 3.32 + 1.98 + 2.12)$

$l(\theta) = n \ln \theta - \theta (8.65)$

$\frac{dl}{d\theta} = \frac{n}{\theta} - 8.65$

$\frac{dl}{d\theta} = 0 \Rightarrow \frac{n}{\theta} = 8.65$

$\Rightarrow \hat{\theta} = \frac{n}{8.65}$

$\Rightarrow \hat{\theta} = \frac{4}{8.65}$ $\cancel{(\text{as } n = 4)}$

$= \cancel{2.165} =$

$\frac{d^2 l}{d\theta^2} = -\frac{n}{\theta^2} - 8.65 \Big|_{\theta = \hat{\theta}} < 0.$

$= -\frac{4}{(2.165)^2} - 8.65$

$\cancel{= -0.8534 - 8.65 < 0}$

$\cancel{= -9.5034 < 0}$

$\hat{\theta} = 4/8.65$

$\Rightarrow \cancel{\hat{\theta} = 2.165}$ is ML Estimate of θ.

5 $X_i \overset{iid}{\sim} N(0,\theta)$.

$\hat{\theta} = \frac{1}{n} \Sigma X_i^2$.

$E(\hat{\theta}) = E\left[\frac{1}{n} \Sigma X_i^2\right]$

$= \frac{1}{n} E(\Sigma X_i^2)$

Now evaluating exp. of ΣX_i^2 depends on the dist^n. of ΣX_i^2

Recall $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$

here $X_i \overset{iid}{\sim} N(\mu,\sigma^2)$; both μ, σ^2 are unknown

But in this eg. $\mu = 0$ so μ is known. Thus

$U = \frac{ns^2}{\sigma^2} \sim \chi^2_{(n)}$ & $s^2 = \frac{1}{n} \Sigma X_i^2$.

$\Rightarrow E(U) = n$; $Var(U) = n$.

$E\left[\frac{ns^2}{\sigma^2}\right] = n$ $\quad V\left[\frac{ns^2}{\sigma^2}\right] = 2n$

$\Rightarrow E[s^2] = \sigma^2$ $\quad \Rightarrow V(s^2) = \frac{2\sigma^4}{n}$

$= \theta$ (here)

$= \frac{2\theta^2}{n}$ (here)

Thus, s^2 is unbiased est^n of θ.

$\Rightarrow MSE = \frac{2\theta^2}{n}$

Tut IV Solutions

(1) $X_i \overset{iid}{\sim}$ Binomial (n, p). ; $\underset{\sim}{\theta} = (n, p)$.

$$P(X_i = x_i) = \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i} \quad ; x_i = 0, 1, \ldots, n.$$

Here two parameters are n & p thus we need two eq^n.s.
Using method of moment estimation method.

$$m_1 = \bar{X} = \frac{1}{n} \Sigma x_i \, ; \quad m_2 = \frac{1}{n} \Sigma x_i^2$$

$$E(X) = \mu = np \quad \text{(mean of Binomial Dist}^n\text{.)}$$

$$E(X^2) = V(X) + (E(X))^2$$
$$= npq + (np)^2$$

Now, using MOM

$$\bar{X} = np \quad \Rightarrow \quad \hat{p}_{MOM} = \bar{X}/\hat{n}_{MOM}$$

& $$\frac{1}{n}\sum X_i^2 = npq + (np)^2$$

$$\frac{1}{n}\sum X_i^2 = n\frac{\bar{X}}{n}\left(1-\frac{\bar{X}}{n}\right) + n^2\frac{\bar{X}^2}{n^2}$$

$$\Rightarrow \frac{1}{n}\sum X_i^2 = \bar{X}\left(1-\frac{\bar{X}}{n}\right) + \bar{X}^2$$

$$\Rightarrow \frac{1}{n}\sum X_i^2 - \bar{X}^2 = \bar{X} - \frac{\bar{X}^2}{n}$$

$$\Rightarrow \quad S^2 = \bar{X} - \frac{\bar{X}^2}{n}$$

$$\Rightarrow \frac{\bar{X}^2}{n} = \bar{X} - S^2$$

$$\Rightarrow \hat{n}_{MOM} = \frac{\bar{X}^2}{\bar{X} - S^2}. \quad \text{Thus } \hat{p}_{MOM} = \frac{\bar{X}}{\hat{n}_{MOM}}$$

[here using $S^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$

$$= \frac{1}{n}\sum(X_i^2 + \bar{X}^2 - 2X_i\bar{X})$$

$$= \frac{1}{n}\left(\sum X_i^2 + n\bar{X}^2 - 2\bar{X}(n\bar{X})\right)$$

$$= \frac{1}{n}\left(\sum X_i^2 - n\bar{X}^2\right)$$].

$\hat{n}_{MOM} = \frac{\bar{X}^2}{\bar{X} - S^2}$; if $\bar{X} < S^2$ then $\hat{n}_{MOM}$ will be negative. But $n = 0, 1, 2, \ldots$ in true parameter value.

Thus the estimator can take negative value though the true value of the parameter can be only positive. Thus $\hat{n}_{MOM}$ is not a good estimator of n.

Similarly, $\hat{p}_{MOM} = \frac{\bar{X}(\bar{X} - S^2)}{\bar{X}^2}$. Here $\hat{p}_{MOM}$ can be negative. But in real $0 \leq p \leq 1$.

So, $\hat{p}_{MOM}$ is also not a good estimator of p.

2

$V(X) = E(X^2) - (E(X))^2$ (using formula of var)

$\sigma^2 = E(X^2) - 0$

(a)

as $E(X) = 0$ given.

$\Rightarrow E(X^2) - \sigma^2 = 0$

$\Rightarrow W(X) = X^2$ is an unbiased est^{r} of $\sigma^2 = \theta$

(b) $L = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\frac{x^2}{\sigma^2}\right]$

$l = \log L = -\log\sigma - \frac{1}{2}\frac{x^2}{\sigma^2} + C$

$\frac{\partial l}{\partial \sigma} = -\frac{1}{\sigma} + \frac{2}{2}\frac{x^2}{\sigma^3} = 0$

$\Rightarrow \frac{1}{\sigma} = \frac{x^2}{\sigma^3}$

$\Rightarrow \hat{\sigma} = \sqrt{x^2} = x$

But as $\sigma > 0$ always thus

$\hat{\sigma} = X$ s.t $X > 0$.

or $\hat{\sigma} = |X|$

$\frac{\partial^2 \log l}{\partial \sigma^2} = \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4} \Big|_{\sigma = \hat{\sigma}}$

$$\Rightarrow \frac{d^2 \log L}{d\sigma^2} = \frac{1}{x^2} - \frac{3}{x^2} < 0$$

$$\Rightarrow \hat{\sigma} = |x| \quad \text{is MLE of } \sigma$$

Or
take $\theta = \sigma^2$.

$$L = \frac{1}{\sqrt{2\pi\theta}} \exp\left[-\frac{1}{2} \frac{x^2}{\theta}\right]$$

$$\ln L = c - \frac{1}{2} \ln\theta - \frac{1}{2} \frac{x^2}{\theta}$$

$$\frac{d \ln L}{d\theta} = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2} = 0$$

$$\Rightarrow \frac{1}{2\theta} = \frac{x^2}{2\theta^2}$$

$$\Rightarrow \hat{\theta} = x^2$$

$$\frac{d^2 \log L}{d\theta^2} = \frac{1}{2\theta^2} - \frac{2x^2}{2\theta^3} \Big|_{\theta = \hat{\theta}}$$

$$= \frac{1}{2x^2} - \frac{1}{x^4} = \frac{1}{x^2}\left[\frac{1}{2} - \frac{1}{x^2}\right] < 0$$

Thus $\hat{\sigma}^2_{MLE} = X^2$ by invariance property of MLE

$\sigma = \sqrt{\sigma^2} = \tau(\theta)$

$\Rightarrow \hat{\sigma} = \sqrt{\hat{\sigma}^2_{MLE}}$

$= \sqrt{x^2}$

$\Rightarrow \hat{\sigma} = |X|$ as $\hat{\sigma} > 0$ always

(c). $E(X) = \mu = 0$ (here)

So the first order moment won't give us any result

Let's see what does 2^{nd} order moments give.

$E(x^2) = V(X) = \sigma^2$

On equating $E(x^2) = \frac{1}{n} \Sigma X_i^2$

$\Rightarrow \hat{\sigma}^2 = \frac{1}{n} \Sigma X_i^2$

Thus, $\hat{\sigma}^2_{MME} = \frac{1}{n} \sum_{i=1}^{n} X_i^2 = X^2$ (as only one obs.)

& $\hat{\sigma}_{MME} = \sqrt{X}$

or $\hat{\sigma}_{MME} = |X|$ as $\hat{\sigma} > 0$

Thus in this case MME = MLE for σ.

Q 3 $f_\theta(x_i) = 1 \qquad \theta \le x_i \le \theta + 1$

$$\Rightarrow L(\theta) = \begin{cases} 1 & \text{if } \theta \le x_i \le \theta+1 \quad \forall i \\ 0 & \text{o/w} \end{cases}$$

$\theta \le X_i \Rightarrow \theta \le X_{(1)} < X_{(2)} < \dots < X_{(n)}$

& $X_i \le \theta + 1 \Rightarrow X_i - 1 \le \theta$

$\Rightarrow$ ~~order~~ $(X_i - 1) \le \theta$

$\Rightarrow \max(X_1, \dots X_n) - 1 \le \theta$

Thus $L(\theta) = 1$ is max when

$\max(X_1, \dots, X_n) - 1 \le \theta \le \min(X_1, \dots, X_n)$

$\Rightarrow \max(X_1 \dots X_n) - 1 \le \theta \le X_{(1)}$

Thus mle of θ i.e. ~~$\hat{\theta}_{MLE} \in [$~~

$\hat{\theta}_{MLE} \in [\max(X_1 \dots X_n) - 1, X_{(1)}]$.

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Tut II

5 b $X_i \overset{iid}{\sim} P(\lambda)$

$$L(\lambda) = \prod_{i=1}^{n} P(X_i = x_i)$$

$$= \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$$

$$= \frac{e^{-n\lambda} \lambda^{\Sigma x_i}}{\Pi x_i!}$$

$$l(\lambda) = \log L(\lambda) = -n\lambda + \Sigma x_i \log \lambda - \Sigma \log x_i!$$

$$\frac{\partial l}{\partial \lambda} = -n + \frac{\Sigma x_i}{\lambda} = 0$$

$$\Rightarrow \hat{\lambda} = \bar{X}$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{\Sigma x_i}{\lambda^2} \Big|_{\hat{\lambda}}$$

$$= \frac{-n\bar{x}}{\bar{x}^2} < 0.$$

Thus $\hat{\lambda}_{MLE} = \bar{X}$.

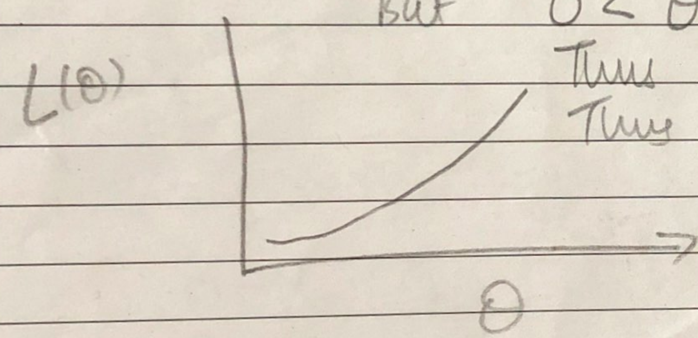
6

θ^n is an increasing f^n in $\theta \Rightarrow$ we want to take max. value of θ to max. $L(\theta)$

But $0 < \theta \leq X_{(1)} < X_{(2)} \cdots < X_{(n)}$

Thus θ can go max to $X_{(1)}$

Thus $\hat{\theta}_{MLE} = X_{(1)}$



(Note, the term $\frac{1}{\prod x_i^2}$ does not need to be max. w.r.t. θ).

(8) $X_i \overset{iid}{\sim} B(p)$

$T(X) = \Sigma X_i$ is sufficient for p.

$T(X) = \Sigma X_i$ " complete for p.

$E(\Sigma X_i) = np$

$\Rightarrow E(\bar{X}) = p$

$\Rightarrow \bar{X}$ is unbiased estimator of p

As, by Lehman-Scheffe thm (on UMVUE),

if T is a complete & sufficient statistic for p

then if $E(\phi(T)) = \tau(\theta)$ then $\phi(T)$ is UMVU

So, $\bar{X}$ is UMVUE of p

(7)

$$E(s^2) = E\left[\frac{1}{n-1} \Sigma (x_i - \bar{x})^2\right]$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} E(x_i - \bar{x})^2$$

$$= \frac{1}{n-1} \Sigma E(x_i^2 + \bar{x}^2 - 2x_i\bar{x})$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} E(x_i^2 + \bar{x}^2 - 2n\bar{x}^2)$$

$$= \frac{1}{n-1} \Sigma E(x_i^2 - n\bar{x}^2)$$

Now, if $x_i \overset{iid}{\sim} P(\lambda) \Rightarrow E(x_i) = V(x_i) = \lambda$

$\Rightarrow E(x_i^2) = \lambda + \lambda^2$

& $\Sigma x_i \sim P(n\lambda) \Rightarrow$ if $Y = \Sigma x_i \sim P(n\lambda)$

then $E(Y) = n\lambda = V(Y)$

$\Rightarrow E(Y^2) = n\lambda + n^2\lambda^2$

Thus, $E(s^2) = \frac{1}{n-1}\left[\Sigma E(x_i^2) - nE\left(\frac{Y}{n}\right)^2\right]$

$$= \frac{1}{n-1}\left[n(\lambda + \lambda^2) - \frac{1}{n}(n\lambda + n^2\lambda^2)\right]$$

$$= \frac{1}{n-1}\left[n\lambda + n\lambda^2 - \lambda - n\lambda^2\right]$$

$$= \lambda$$

Thus, s^2 is unbiased estimator of λ.

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(7)

$V(S^2)$ is given by.

$$V(S^2) = \frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}\sigma^4\right]$$

Where, $\sigma^2 = d$ (i.e. variance)

$\mu_4 = E(X-d)^4$ (4th order central moment)

$= d(1+3d)$

Thus, putting in the values,

$$V(S^2) = \frac{1}{n}\left[d(1+3d) - \frac{n-3}{n-1}d^2\right]$$

$$= \frac{d}{n} + \frac{2d^2}{n-1}$$

[redacted] [redacted]

eg. $X_i \overset{iid}{\sim} P(\lambda)$.

$E(X_i) = \lambda \quad ; \quad V(X_i) = \lambda$

$E(\bar{X}) = \lambda \quad ; \quad E(s^2) = \lambda$ (difficult to verify).

$$\left.\begin{aligned} \text{Bias}(\bar{X}) &= E(\bar{X}) - \lambda \\ &= 0 \\ \text{Bias}(s^2) &= E(s^2) - \lambda \\ &= 0 \end{aligned}\right\} \Rightarrow \bar{X}, s^2 \text{ both are unbiased estimators of } \lambda.$$

[redacted] [redacted]

$var(\bar{x})$ & $var(s^2)$ we will try to find.

$V(\bar{X}) = \frac{n}{n^2} V(X_i) = \sigma/n$

[redacted]

In conclusion, here $\bar{X}$ and S^2 both are unbiased estimators of λ.

$V(\bar{X}) \leq V(S^2)$ thus $\bar{X}$ is a better unbiased estimator of λ.

(Actually, $V(S^2) = \frac{\lambda}{n} + \frac{2}{n-1}\lambda^2$

Clearly $V(S^2) > V(\bar{X})$)