MTH 102: Probability and Statistics

Mid Semester Exam

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No books, notes, or devices are allowed. Just a pen and eraser. Any exchange of information related to the quiz with a human or machine will be deemed as cheating. Institute rules will apply. Explain your answers. Show your steps. Approximate calculations are fine as long as the approximations are reasonable. You have about 60 minutes.

Question 1. 80 **marks** The state government has come up with a scheme to encourage use of buses for travel. It creates a large bin consisting of n identities, where n is the total number of people in the state. Every day, the bin containing n identities is mixed thoroughly and m identities are drawn from it without replacement. The chosen m people are given a free bus pass for the day.

For a total of 30 marks, answer the following questions that involve a straightforward application of the chapter on discrete random variables.

(a) (3 marks) Derive the probability that any person in the city is chosen for a free pass for the day.

(b) (9 marks) Derive the distribution (PMF) of the random number of days a person must wait to get his first free pass.

(c) (9 marks) Derive the distribution of the random number of days a person must wait to get his k^{th}, $k > 1$, free pass.

(d) (9 marks) Derive the distribution of the number of free passes a person obtains over a month of 30 days.

For a total of 50 marks, answer the following questions that require more than a straightforward application of the chapter on discrete RVs.

Suppose we monitor allocations of free passes to two randomly chosen people in the state. Let X_i and Y_i, respectively, count the number of free passes received by the two individuals on day i of a 30 day month. We have $S_{X_i} = S_{Y_i} = \{0, 1\}$. Also $i \in \{1, 2, \ldots, 30\}$. Let $X = \sum_{i=1}^{30} X_i$ and $Y = \sum_{i=1}^{30} Y_i$. Define $Z = \sum_{i=1}^{30} X_i Y_i$ and let $Z_i = X_i Y_i$, for any i.

(aa) (4 marks) Derive $P[Z_i = 1]$, for any i. Keep calm and work out the probability from first principles.

(bb) (12 marks) Derive the conditional PMF of X given $Z = 5$.

(cc) (12 marks) Derive the conditional PMF of X given $Z = 5$ and $Y = 10$.

(dd) (12 marks) Derive the conditional PMF of X_{15} given $Z = 5$.

(ee) (5 marks) Derive $E[X_{15}|Z_{15} = 1]$.

(ff) (5 marks) Derive $E[X_{15}|Z_{15} = 0]$.

Question 2. 10 **marks** You arrive at a bus stop knowing that the time you must wait for a bus is distributed as an exponential RV with PDF $f_X(x) = \lambda e^{-\lambda x}, x \geq 0$. The bus hasn't arrived for the first s seconds of your wait. Derive the conditional CDF of X.

Repeat the above under the assumption that you must wait for a bus for a time that is uniformly distributed over $(0, 120)$. Note that $s \in (0, 120)$.

Question 3. 10 **marks** You want to approximate the exponentially distributed random variable X with PDF $f_X(x) = (1/100)e^{-x/100}, x \geq 0$, using a Gaussian RV Y with the same mean and variance as that of X. Derive $P[Y > 120]$ in terms of the CDF $\Phi_Z(z)$ of the standard normal.

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(a) $\dfrac{{}^{n-1}C_{m-1}}{{}^{n}C_{m}} = \dfrac{m}{n}$

${}^{n-1}C_{m-1}$ ← Ways of choosing given that a certain person is selected.

${}^{n}C_{m}$ ← Total nos. of ways of selecting

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$$\frac{\lfloor n-1}{\lfloor m-1 \, \lfloor n-m} \quad \frac{\lfloor m \, \lfloor n-m}{\lfloor n} = \frac{m}{n}$$

Let $q = \dfrac{m}{n}$.

(b) The number of days a person must wait is geometric(q) = geometric$\left(\dfrac{m}{n}\right)$

Let N be the no. of days.

$$P[N=1] = q = \frac{m}{n}$$

$$P[N=2] = (1-q)\,q = \left(1-\frac{m}{n}\right)\left(\frac{m}{n}\right)$$

$\vdots$

$$P[N=k] = \begin{cases} (1-q)^{k-1} q = \left(1-\dfrac{m}{n}\right)^{k-1} \dfrac{m}{n}, & k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

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(c) Let N_k be the no. of days a person must wait to get his k^{th} pass.

$$S_{N_k} = \{k, k+1, k+2, \ldots\}$$

$$P[N_k = k] = P[\text{Person gets a pass every day}]$$

$$= \left(\frac{M}{n}\right)^k$$

$$P[N_k = k+1] = P[\text{Person gets } (k-1) \text{ passes in } k \text{ days } \& \text{ the } k^{th} \text{ pass on the } k^{th} \text{ day}]$$

$$= \left({}^{k}C_{k-1} \left(\frac{M}{n}\right)^{k-1} \left(1 - \frac{M}{n}\right)\right) \left(\frac{M}{n}\right)$$

$$P[N_k = j] = \begin{cases} {}^{j-1}C_{k-1} \left(\frac{m}{n}\right)^{k-1} \left(1-\frac{m}{n}\right)^{j-k} \left(\frac{m}{n}\right), & j = k, k+1, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

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(d) Let N' be the no. of passes one obtains in 30 days.

N' is Binomial$\left(30, \frac{m}{n}\right)$.

$$P[N' = n] = \begin{cases} {}^{30}C_n \left(\frac{m}{n}\right)^n \left(1-\frac{m}{n}\right)^{30-n}, & n = 0, 1, \ldots, 30. \\ 0 & \text{otherwise.} \end{cases}$$

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(aa) $P[Z_i=1] = P[X_i=1, Y_i=1]$

$$= P[Y_i=1 \mid X_i=1]\, P[X_i=1]$$

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$$= \left(\frac{{}^{n-2}C_{m-2}}{{}^{n-1}C_{m-1}}\right)\left(\frac{m}{n}\right)$$

[Ways that include the other person getting the pass]

[Total no. of ways, given that one individual got the pass]

$$= \frac{(n-2)!}{(m-2)!\,(n-m)!}\;\frac{(n-m)!\,(m-1)!}{(n-1)!}\;\frac{m}{n}$$

$$= \left(\frac{m-1}{n-1}\right)\left(\frac{m}{n}\right) \quad //$$

(bb) $$P[X=x \mid Z=5] = \begin{cases} ? & x=5,6,\dots \\ 0 & \text{otherwise.} \end{cases}$$

$$P[X=5 \mid Z=5] = P\left[\sum_{i=1}^{30} X_i = 5 \;\middle|\; \sum_{i=1}^{30} X_i Y_i = 5\right]$$

$$= \frac{P[X=5, Z=5]}{P[Z=5]}$$

$$= \frac{P[\text{One got 5 passes \& on the days even the other got passes}]}{P[Z=5]}$$

$$= \frac{{}^{30}C_5 \left(\left(\frac{M-1}{n-1}\right)\left(\frac{M}{n}\right)\right)^5 \left(1 - \frac{M}{n}\right)^{25}}{{}^{30}C_5 \left(\left(\frac{M-1}{n-1}\right)\left(\frac{M}{n}\right)\right)^5 \left(1 - \left(\frac{M-1}{n-1}\right)\left(\frac{M}{n}\right)\right)^{25}}$$

$$= \frac{\left(1 - \frac{M}{n}\right)^{25}}{\left(1 - \left(\frac{M-1}{n-1}\right)\left(\frac{M}{n}\right)\right)^{25}}$$

More generally,

$$P[X=x \mid Z=5] = \frac{P[X=x,\ Z=5]}{P[Z=5]}$$

6 for this calc

6 for this.

$$= \begin{cases} \dfrac{{}^{30}C_x \, {}^{x}C_5 \left(\left(\frac{M-1}{n-1}\right)\left(\frac{M}{n}\right)\right)^5 \left(\frac{M}{n}\right)^{x-5}\left(1-\frac{M}{n}\right)^{30-x}}{{}^{30}C_5 \left(\left(\frac{M-1}{n-1}\right)\left(\frac{M}{n}\right)\right)^5\left(1-\left(\frac{M-1}{n-1}\right)\left(\frac{M}{n}\right)\right)^{25}} & x=5,6,\dots \\[2em] 0 & \text{otherwise} \end{cases}$$

(cc) $P[X=x \mid Z=5, Y=10]$

$$= \begin{cases} ? & x = 5, 6, \dots \\ 0 & \text{otherwise.} \end{cases}$$

$$P[X=x \mid Z=5, Y=10] = \frac{P[X=x, Z=5, Y=10]}{P[Z=5, Y=10]}$$

$P[X=x, Z=5, Y=10]$

= P[One gets a pass on x days, on 5 of the x days even the other gets a pass, there are 5 days such that one doesn't get a pass but the other does]

$$= {}^{30}C_x \, {}^{x}C_5 \, {}^{30-x}C_5 \left(\left(\frac{m-1}{n-1}\right)\left(\frac{m}{n}\right)\right)^5 \left(P[X_i=1, Y_i=0]\right)^{(x-5)} \left(P[X_i=0, Y_i=1]\right)^5 \left(P[X_i=0]\right)^{30-x-5}$$

$$= {}^{30}C_x \, {}^{x}C_5 \, {}^{30-x}C_5 \left(\left(\frac{m-1}{n-1}\right)\left(\frac{m}{n}\right)\right)^5 \left(P[X_i=1, Y_i=0]\right)^{(x-5)} \left(P[X_i=0, Y_i=1]\right)^5 \left(P[X_i=0]\right)^{30-x-5}$$

$$P[X_i=1, Y_i=0] = P[Y_i=0 \mid X_i=1]\, P[X_i=1]$$

$$= \left(1 - \frac{(m-1)}{(n-1)}\right)\left(\frac{m}{n}\right)$$

$$= \left(\frac{n-m}{n-1}\right)\left(\frac{m}{n}\right).$$

$$P[X_i=0, Y_i=1] = P[X_i=0 \mid Y_i=1]\, P[Y_i=1]$$

$$= \left(\frac{n-m}{n-1}\right)\left(\frac{m}{n}\right)$$

$\therefore P[X=x, Z=5, Y=10]$ ⑥ for this

$$= {}^{30}C_x \, {}^{x}C_5 \, {}^{30-x}C_5 \left(\left(\frac{m-1}{n-1}\right)\left(\frac{m}{n}\right)\right)^5 \left(\left(\frac{n-m}{n-1}\right)\left(\frac{m}{n}\right)\right)^x \left(1-\frac{m}{n}\right)^{30-x-5}$$

⑥ for this.

$P[Z=5, Y=10] = P$[The other has 10 passes of which on 5 days even the first one got a pass]

$$= {}^{30}C_{10} \, {}^{10}C_5 \left[\left(\frac{m-1}{n-1}\right)\left(\frac{m}{n}\right)\right]^5 \left(\frac{m}{n}\right)^5 \left(1-\frac{m}{n}\right)^{20}.$$

(dd)

$$P[X_{15} = x \mid Z = 5] = ?$$

$$P[X_{15} = 1 \mid Z = 5]$$

$$= \frac{P[X_{15} = 1,\ Z = 5]}{P[Z = 5]}.$$

12 for the calc of this prob.

$P[X_{15} = 1,\ Z = 5] = P[X_{15} = 1,\ Z = 5,$ On the 15^{th} day the other didn't get a pass$]$

$+\ P[X_{15} = 1,\ Z = 5,$ On the 15^{th} the other also got a pass$]$

$$= {}^{30}C_6\ {}^{6}C_5 \underbrace{\left(\left(\frac{m-1}{n-1}\right)\left(\frac{m}{n}\right)\right)^5}_{\text{Both got a pass}} \underbrace{\left(P[X_i = 1,\ Y_i = 0]\right)}_{\text{The 15th day.}}$$

$$+\ {}^{29}C_4 \underbrace{\left(\left(\frac{m-1}{n-1}\right)\left(\frac{m}{n}\right)\right)^4}_{\text{4 out of 5}} \underbrace{\left(\frac{m-1}{n-1}\right)\left(\frac{m}{n}\right)}_{\text{The 15th day}}$$

(ee) $E[X_{15} | Z_{15} = 1]$

$= P[X_{15} = 1 | Z_{15} = 1]$ 5

$= 1.$

(ff) $E[X_{15} | Z_{15} = 0]$

$= P[X_{15} = 1 | Z_{15} = 0]$

$$= \frac{P[X_{15} = 1, Y_{15} = 0]}{P[X_{15} = 1, Y_{15} = 0] + P[X_{15} = 0, Y_{15} = 1] + P[X_{15} = 0, Y_{15} = 0]}$$ 4

$$= \frac{P[X_{15} = 1, Y_{15} = 0]}{1 - P[X_{15} = 1, Y_{15} = 1]}$$

Find ans 1.

$$= \frac{\left(\frac{n-m}{n-1}\right)\left(\frac{m}{n}\right)}{1 - \frac{(m-1)m}{(n-1)n}} = \frac{(n-m)m}{n(n-1) - m(m-1)}.$$

Q2 $f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0;$

$$P[X > x \mid X > s]$$

$$= \frac{P[X > x, X > s]}{P[X > s]}$$

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$$= \begin{cases} \dfrac{P[X > x]}{P[X > s]} & x > s \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{e^{-\lambda x}}{e^{-\lambda s}} & x > s \\ 0 & \text{otherwise.} \end{cases}$$

$$f_X(x) \sim \text{Unif}(0, 120)$$

$$P[X > x \mid X > s]$$

$$= \begin{cases} \dfrac{P[X > x]}{P[X > s]} & x > s \\ 0 & \text{otherwise} \end{cases}$$

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$$= \begin{cases} \dfrac{(120 - x)/120}{(120 - s)/120} = \dfrac{120 - x}{120 - s} & x > s \\ 0 & \text{otherwise.} \end{cases}$$

(Q3)

$$P[Y > 120]$$

$$= P[Y - E[Y] > 120 - E[Y]]$$

$$= P\left[\frac{Y - E[Y]}{\sigma_Y} > \frac{120 - E[Y]}{\sigma_Y}\right]$$

$$= 1 - \Phi_Z\left(\frac{120 - E[Y]}{\sigma_Y}\right)$$

$$= 1 - \Phi_Z\left(\frac{120 - E[X]}{\sigma_X}\right)$$

$$= 1 - \Phi_Z\left(\frac{120 - 100}{100}\right)$$

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