Solution to Problem 1.

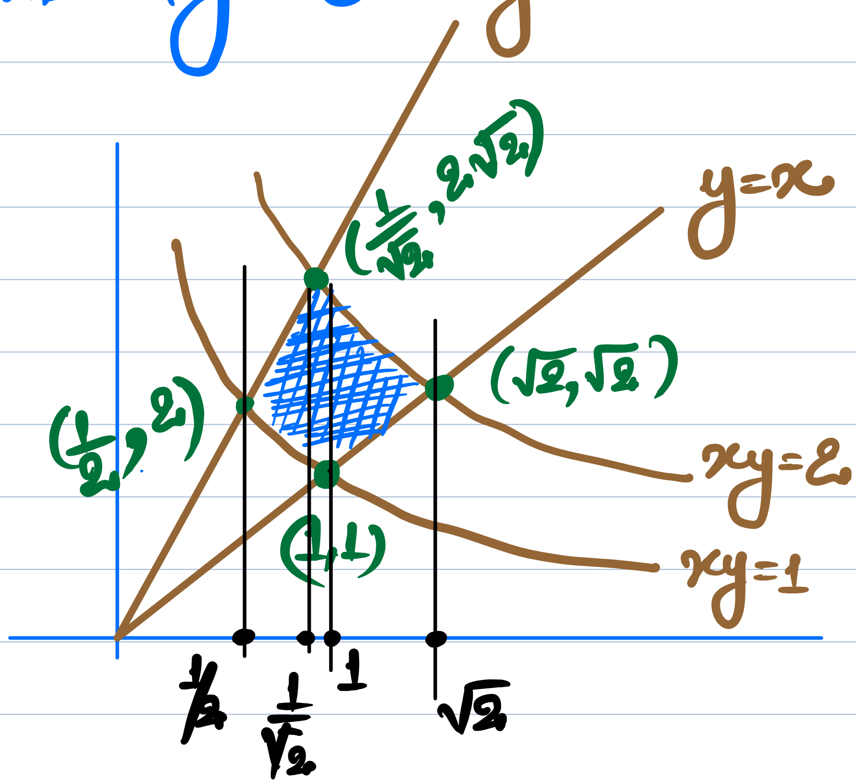
Need to compute $\iint_S x^2 y^2 \, dA$.

where S is given in the question.

I wish to explain the solution of this problem in detail. TAs should keep in mind that this amount of explanation is not expected from the students, and hence the grading/marking should not penalise them for not explaining the way it has been done in this Solution!

Solution

Plotting the region S =



Now, if you want to be adamant in computing this using Cartesian coordinates, you have to be a tad extra cautious. <u>Convince yourself that</u> first considering a vertical line and then moving that vertical line from left to right would be an effective way to cover the region!

<u>Vertical line</u> $\boxed{\text{means}}$ x- constant and therefore integrate <u>wrt</u> "y".

<u>Then traversing the vertical line from left to right</u>

$\boxed{\text{means}}$

integrating w.r.t "x".

$\therefore$ We need vertical lines in such a way that between any two consecutive vertical lines there are only two curves to deal with!

The vertical lines are

$x = \frac{1}{2}$, $x = \frac{1}{\sqrt{2}}$, $x = 1$, $x = \sqrt{2}$

Thus the limits will be

$$\int_{x=1/2}^{x=\frac{1}{\sqrt{2}}} \int_{y=\frac{1}{x}}^{y=4x} x^2 y^2 \, dy \, dx$$

$$+ \int_{x=1/\sqrt{2}}^{x=1} \int_{y=\frac{1}{x}}^{y=2/x} x^2 y^2 \, dy \, dx$$

$$+ \int_{x=1}^{x=\sqrt{2}} \int_{y=x}^{y=2/x} x^2 y^2 \, dy \, dx .$$

→ [getting till this stage should be enough to get 6 points; 2 for each term]

→ Next compute each of these (1 point for each)

→ 1 point for adding all of them.

[Apologies to the TAs for not providing you all with the final answer. I am certain that you all can rely on a few students for the final solution!]

Solution to Problem 2:

Try and recall that for the field F,

F_1 is the component in the direction of x-axis and F_2 is the component along Y-axis.

Thus,
$$F_1(x,y) = \frac{-y}{x^2+y^2}; \quad (x,y) \neq (0,0)$$

and
$$F_2(x,y) = \frac{x}{x^2+y^2}; \quad (x,y) \neq (0,0)$$

$$\boxed{\text{(a)} \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}}$$

[5 points]

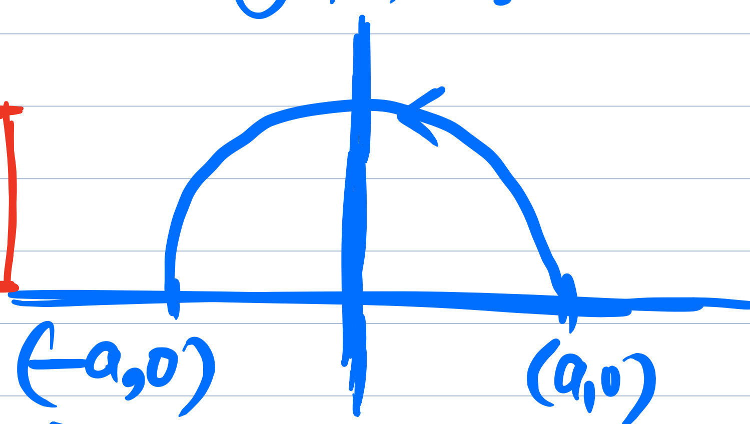
Note to students:

One would be tempted to infer that the vector field F is conservative since $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, and thus integral should be independent of the path chosen.

However, that's NOT the case, as can be seen from the following parts of the question.

$\boxed{(b)-(i)}$

5 points



$$\sigma(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$$

$$t \in [0,\pi]$$

$$\therefore d\sigma/dt = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$$

Also, $F(r(t)) = F(a\cos t, a\sin t)$

$$\frac{-y}{x^2+y^2} \approx \frac{-a\sin t}{a^2\cos^2 t + a^2\sin^2 t} = \frac{-\sin t}{a}$$

& $$\frac{x}{x^2+y^2} \approx \frac{a\cos t}{a^2} = \frac{\cos t}{a}$$

$\therefore$ $$F(r(t)) = \frac{-\sin t}{a}\vec{i} + \frac{\cos t}{a}\vec{j}$$

$$\Rightarrow F(r(t)) \cdot \frac{dr}{dt} =$$

$$\left(\frac{-\sin t}{a}\right) \cdot (-a\sin t) + \left(\frac{\cos t}{a}\right) \cdot (a\cos t)$$

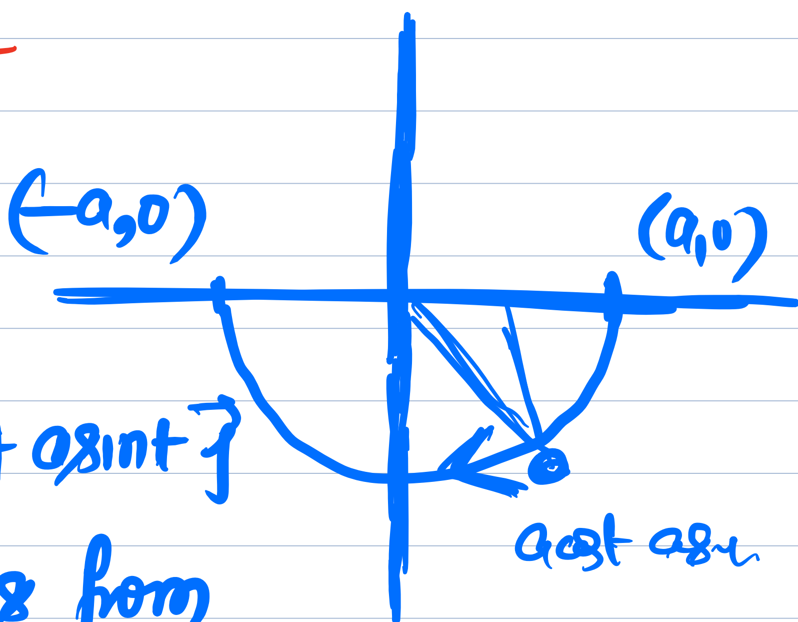
$$= \sin^2 t + \cos^2 t = 1$$

$$\therefore \int_{t=0}^{\pi} F(\sigma(t)) \cdot \frac{d\sigma}{dt}\, dt = \int_{0}^{\pi} dt = [t]_0^{\pi} = \pi \ \underline{\underline{\text{ans}}}$$

$\boxed{\text{(b) — (ii)}}$

5 points

In here,



$\sigma(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

but t starts from

$t=0$ and goes to $t=-\pi$.

(Although it doesn't make too much sense to write $t \in [0, -\pi]$, because $-\pi$ is smaller than 0, but in integrating things, we are not wrongly placed)

So, keeping everything same, we reach

$$\int_0^{-\pi} dt = -\int_{-\pi}^{0} dt$$

$$= -\left[t\right]_{-\pi}^{0} = -(0-(-\pi)) = -\pi.$$

Note to students:

The integral, as you can see, depends on path chosen!

So, clearly F is not conservative!

(C) — 5 points

It is, indeed, not incorrect!
(That's why the word "Seemingly")

But the nagging question remains "what's going on here?"

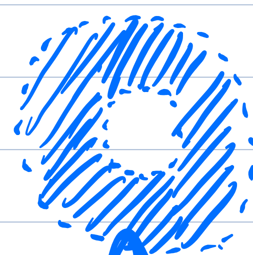
The domain of the vector field is $\mathbb{R}^2 \setminus \{(0,0)\}$ which is not simply connected — there is a "hole" in the domain of the vector field which prevents it from being "simply connected".

This is where I would like to revisit this concept. Let me elaborate on it! (For students)

Definition: An open set G in $\mathbb{R}^2$ is said to be "simply connected" if given any simple closed curve γ in G, the inside of the γ lies wholly in G.

Alternatively, an open set G in $\mathbb{R}^2$ is said to be "simply connected" if it has no "holes".
This serves to be an informal definition of simply connected regions.

Simply Connected



Not simply Connected. (there is a hole)

Simply connected

Not Simply Connected.

Solution to Problem 3:

The domain of F, in this problem, happens to be the open set in $\mathbb{R}^2$ given by $\{(x,y) \in \mathbb{R}^2 : x > 0\}$. Pictorially, this is the set of all points in $\mathbb{R}^2$ lying on the right side of the Y-axis (excluding the Y-axis).

Therefore, the region is "Simply Connected" since there is $\boxed{\text{no hole in the region.}}$

Consequently, the Vector field $\boxed{\text{F must be conservative.}}$

For TAs —— (give 1 point for simply saying that Yes, F is conservative)

For TAs —— (1 point for mentioning that there is no "hole," or that, the point (0,0) is no longer into consideration)

For TAs — 8 points for computing the potential function $f(x,y)$.

Finding the potential function:

Clearly $\exists$ a potential function $f(x,y)$.

And,

$$\frac{\partial f}{\partial x} = F_1(x,y) = \frac{-y}{x^2+y^2}; \quad \text{---} \; \mathrm{I}$$

$$\frac{\partial f}{\partial y} = F_2(x,y) = \frac{x}{x^2+y^2}. \quad \text{---} \; \mathrm{II}$$

Integrating II w.r.t. "y" we get

$$f(x,y) = x \int \frac{dy}{x^2+y^2}$$

$$= (x) \frac{1}{x} \tan^{-1}(y/x) + g(x)$$

$$= \tan^{-1}(y/x) + g(x). \quad \text{---} \; \mathrm{III}$$

Partially differentiating III w.r.t. "x"

We get

$$\frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \left(\frac{-y}{x^2}\right) + g'(x)$$

$$= \frac{x^2}{x^2+y^2} \cdot \left(\frac{-y}{x^2}\right) + g'(x)$$

$$= \frac{-y}{x^2+y^2} + g'(x)$$, and then

equating this to (I), we have

$g'(x) = 0$ which yields $g(x) = C \in \mathbb{R}$.

$\therefore f(x,y) = \tan^{-1}(y/x) + C$. ans.

Solution to Problem 4:

Area bdd. by $\gamma(t) = (\sin 2t, 2\sin t)$ $t \in [0, \pi]$ using Green's theorem.

$\boxed{\text{Digression (for students)}}$

1. **Can you plot the curve $\gamma(t)$?**

Hint: Use the values of "t" to be

$t=0$; $t=\pi/4$; $t=\pi/2$; $t=\frac{3\pi}{4}$; $t=\pi$.

This will help you "see" the curve and its counterclockwise orientation!

2. Recall Green's theorem:

$$\oint_{\gamma} F \cdot T \, ds = \iint_{R} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

3. Typically, we use the Green's thm as an alternative way to compute a line integral.

That is, Given a simple closed curve $\gamma(t)$ in $\mathbb{R}^2$, and a nice Vector field $F(x,y)$, we use Green's thm to convert the problem of computing line integral into a problem of computing a double integral.

$$\oint_{\gamma} F \cdot ds = \iint_{R} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

where $R \equiv$ the region enclosed by $\gamma(t)$.

Can we use Green's theorem to walk the other direction? That is, if we are given a double integral, can we use Green's thm to convert the double integral into a line integral and calculate the line integral?

Suppose we are given a double integral

$$\iint_R f(x,y)\, dA.$$

We can use Green's theorem if we are able to find a vector field $F(x,y)$ such that

$$f(x,y) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{y}.$$

How to find such a vector field??

This question is a special case — — an intrinsically beautiful one!

Can you recall that, if D is a region in $\mathbb{R}^2$, then we have $\text{Area of } D = \iint_D 1\, dA$?

$\boxed{\text{End of digression}}$

Solution: Need to find a vector field F, such that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$.

There are many such F; let's choose

$$\boxed{F(x,y) = \frac{-y}{2}\vec{i} + \frac{x}{2}\vec{j}}$$

Clearly $F_1 = -y/2$ and $F_2 = x/2$

$\therefore \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$

Finding such a field is worth 3 points

Then $\iint_R dx\,dy$ is the required area. Now,

$r(t) = \sin 2t\,\vec{i} + 2\sin t\,\vec{j}$

$\therefore x = \sin 2t \qquad y = 2\sin t$

Since $F = \frac{-y}{2}\vec{i} + \frac{x}{2}\vec{j}$, we get

$$F(\gamma(t)) = -\sin t\,\vec{i} + \frac{\sin 2t}{2}\,\vec{j}$$

and $$\frac{d\gamma}{dt} = 2\cos 2t\,\vec{i} + 2\cos t\,\vec{j}$$

$$\therefore \oint_{\gamma} F \cdot T\, ds = \int_{t=0}^{\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_0^{\pi} \left(-\sin t, \frac{\sin 2t}{2}\right) \cdot \left(2\cos 2t, 2\cos t\right) dt$$

$$= \int_0^{\pi} \left(-\sin t, \sin t\cos t\right) \cdot \left(2\cos^2 t - 2\sin^2 t, 2\cos t\right) dt$$

$$= \int_0^{\pi} \left(\cancel{-2\sin t\cos^2 t} + 2\sin^3 t + \cancel{2\sin t\cos^2 t}\right) dt$$

$$= \int_0^{\pi} 2\sin^3 t\, dt = 2\int_0^{\pi} \sin^3 t\, dt$$

$$= 2 \int_0^{\pi} \sin t\,(1 - \cos^2 t)\, dt$$

$$= 2\int_0^{\pi} \sin t \, dt - 2\int_0^{\pi} \sin t \cos^2 t \, dt$$

$$= -2\left[\cos t\right]_0^{\pi} + \frac{2}{3}\left[\cos^3 t\right]_0^{\pi}$$

$$= -2\{-1-1\} + \frac{2}{3}\{-1-1\}$$

$$= 4 - 4/3 = 8/3 \text{ sq. } \underline{\text{units.}}$$

→ [Find $F(\sigma(t))$, $d\sigma/dt$, and the dot product] – $\underline{\text{3 points}}$

→ [Computing the integral — $\underline{\text{4 points}}$

[Remark:]

Finding better F makes computation easier.

Suppose, $\vec{F}(x,y) = 0\,\vec{i} + x\,\vec{j}$

Then too, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - 0 = 1$

But then

$r(t) = \sin 2t\,\vec{i} + 2\sin t\,\vec{j}\,;\ t \in [0,\pi]$

$\frac{dr}{dt} = 2\cos 2t\,\vec{i} + 2\cos t\,\vec{j}$

& $F(r(t)) = 0\,\vec{i} + \sin 2t\,\vec{j}$

$\therefore F(r(t)) \cdot \frac{dr}{dt} = 0 + 2\cos t \sin 2t$

$\therefore \iint_R dx\,dy = \int_0^{\pi} 2\cos t \sin 2t \, dt$

$$= \int_0^{\pi} 4 \sin t \cos^2 t \, dt.$$

Using substitution like before, we get

$$\left[-\frac{4}{3}\cos^3 t\right]_{t=0}^{\pi} = \left(\frac{4}{3}\right) - \left(-4/3\right)$$

$= 8/3$ sq. units! <u>Ans.</u>