Game Theory: Final Exam (Solutions)

Total points: 30 | Due Date: 11/12/2021
Contribution to grade: 30% (3xx); 30% (5xx) | Due time: 3:45 PM

- Show all steps, as it can help you get partial credit.
- Any sign of copying will be taken very seriously.
- **Answer any 3 out of 4. 10 Marks each**

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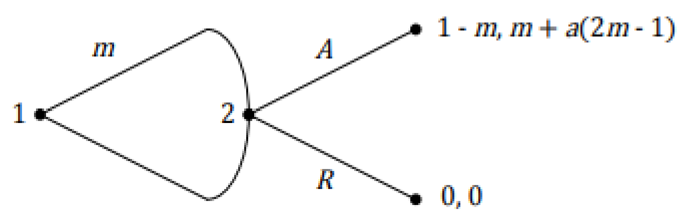
1. **Modified bargaining game.** Consider an ultimatum bargaining game where the responder's payoff function is given by $y + a(y - z)$, where y is the money the responder gets, z is the money the proposer gets, and a is some positive constant. Basically, the responder cares how much money he gets and he cares about relative monetary amounts (the difference between the money he gets and the money the other player gets). Assume that the proposer's payoff is as in the basic model. For simplicity, assume that the total amount being split is \$1.

   (a) Represent the game in extensive form, writing payoffs in terms of parameters m, the monetary offer of the proposer, and the parameter a. *Hint: This will mean the amounts for each player are split into m and $1 - m$.* (2)

   Answer. First, let us write the payoff for the responder in terms of m and a.

$$y + a(y - z) = m + a[m - (1 - m)] = m + a(2m - 1)$$

   This version of the ultimatum bargaining game can be represented by the following figure:



   where A is accept and R is reject.

   Marks breakdown:

   - 0.5 mark for correct figure (with order).
   - 1 mark each for correct payoff of responder.
   - 0.5 marks for correct payoff of proposer.

(b) Find and report the SPNE and show how a affects equilibrium share and payoffs. *Hint: Use payoff function in terms of m and a.* (5)

Answer. Player 2 accepts an offer if $m + a(2m - 1) \geq 0$, which implies that player 2 will accept an offer if

$$m \geq \frac{a}{1+2a}$$

Player 1 maximizes his payoff by offering exactly $m = \frac{a}{1+2a}$. To understand how a affects payoffs and equilibrium share, we need to calculate payoffs. Payoffs are

$$\left(\underbrace{1 - \frac{a}{1+2a}}_{\text{Proposer's payoff}}, \underbrace{\frac{a}{1+2a} + a\left[\frac{a}{1+2a} - 1\right]}_{\text{Responder's payoff}} \right) = \left(\frac{1+a}{1+2a}, 0 \right)$$

The proposer's equilibrium share and payoff is decreasing in a, since the derivatine w.r.t. $a < 0$. The responder's equilibrium share is increasing in a, but payoff remains 0.

Marks breakdown:

- 1 mark for correct m.
- 1 mark each for each payoff.
- 1 mark for how a affects payoffs and shares (0.5×2) **for each player**. (total 2 marks).

(c) What is the equilibrium monetary split as a goes to infinity? Explain why this is the case. (3)

Answer. Taking the limit,

$$\lim_{a \to \infty} \frac{a}{1+2a} = \frac{1}{2}$$

Thus, as inequality has a higher effect on the responder, the closer the payoffs will have to be in order for him to accept. At the extreme, the payoffs will have to be identical and the inequality eliminated completely.

Marks breakdown:

- Setup of equation is 1 mark.
- Solving is 1 mark.
- Explanation (intuition) is 1 mark.

2. **Bayesian Nash Equilibrium.** Consider two candidates fighting an election, each with a privately known personal cost for entering the race. The probability of having a low entry cost, f_L, is p and the probability of having a high entry cost, f_H, is $1-p$. A candidate's payoff depends on whether she enters the race and whether the other enters or not as well. Let v_2 be a candidate's payoff when she enters and the candidate does as well, v_1 be a candidate's payoff when she enters and the other candidate does not, and 0 be the payoff when she does

not enter (regardless of the other candidate's decision). Assume that

$$
\begin{aligned}
v_1 > v_2 &> 0 \\
f_H > f_L &> 0 \\
v_2 - f_L > 0 > v_2 - f_H& \\
v_1 - f_H &> 0
\end{aligned}
$$

(a) Show the conditions for when it is a symmetric BNE for a candidate to enter only when she has a low cost from doing so. (5)

Answer. It is a symmetric BNE for a candidate to enter if she has low personal cost iff, for a low type

$$p(v_2 - f_L) + (1-p)(v_1 - f_L) \geq 0$$

Rearranging,

$$pv_2 + (1-p)v_1 \geq f_L \tag{1}$$

Similarly, when a candidate has high cost, she will not enter if

$$0 \geq p(v_2 - f_H) + (1-p)(v_1 - f_H)$$

which yields

$$f_H \geq pv_2 + (1-p)v_1 \tag{2}$$

Combining (1) and (2), the condition for a candidate to enter only when she has a low personal cost is

$$f_H \geq pv_2 + (1-p)v_1 \geq f_L$$

Marks breakdown:

- 2 marks for equation (1). 1 mark Partial credit for setup as well.
- 2 marks for equation (2). 1 mark Partial credit for setup as well.
- 1 mark for final combined result.

(b) Show the conditions for when it is a symmetric BNE for a candidate to enter for sure when she has a low cost (i.e., with probability 1) and to enter with some probability between 0 and 1 when she has a high cost (mixed strategy). *Hint: Assume that if she has high cost she enters with probability q. This problem is a bit difficult so all you need to do is set it up the type-wise conditions and find the expression for q for full marks. You do not have to compare all conditions to get the final answer.* (5)

Answer. The conditions are that if she has low cost, she enters but if she has high cost, she enters with probability q. For a low type,

$$(p + (1-p)q](v_2 - f_L) + (1-p)(1-q)(v_1 - f_L) \geq 0$$

The first expression counts the cases when the rival is a low type (probability p) or a high type who chooses to enter (probability $(1-p)q$. The second case considers a situation in which the rival does not enter (probability $(1-p)(1-q)$. Rearranging gives

$$[p + (1-p)q]v_2 + (1-p)(1-q)v_1 \geq f_L \tag{3}$$

And for a high type,

$$0 = [p + (1-p)q](v_2 - f_H) + (1-p)(1-q)(v_1 - f_H)$$

Since the high type is randomizing, the condition has to be equal to 0 as he has to be indifferent between participating and not participating (and not participating yields 0).

$$q = \frac{(1-p)(v_1 - v_2) + v_2 - f_H}{(1-p)(v_1 - v_2)} \tag{4}$$

Not graded after this point.

For this to be an equilibrium we need, $0 < q < 1$. So

$$0 < \frac{(1-p)(v_1 - v_2) + v_2 - f_H}{(1-p)(v_1 - v_2)} < 1$$

$$pv_2 + (1-p)v_1 > f_H \tag{5}$$

Using (3), (4) and (5), you will get $f_H \geq f_L$ (true by definition), and the only binding condition here is the expression for q, i.e., (4)

Marks breakdown:

- 2 marks for equation (3).
- 3 marks for equation (4). 1.5 marks Partial credit for setup as well.

3. **Repeated Bertrand Game.** Consider a homogeneous industry where N firms produce at 0 cost and compete in prices for infinite periods. If firms choose the same price, they earn per period profit of $\pi(p) = p\alpha\frac{D(p)}{N}$, where $D(p)$ represents the quantity demanded of the good at the given price and α is the state of demand. If firm i defects and charges some price $p_i < p$ for a period, it earns $\pi(p_i) = p_i\alpha D(p_i)$, and everyone else gets 0 profits. In this game, cooperation means everyone chooses a monopoly price P^m and grim trigger punishment means everyone chooses marginal cost pricing such that profits are 0 (basically an N player Bertrand game). In the current period, $\alpha = 1$, but from the next period $\alpha = \theta > 1$. All firms have same discount factor δ, and all information is common knowledge (no private information). *Hint: Assume that if they do defect, the defection is so small that it does not affect demand or price significantly. They charge some price $p_i = p - \epsilon$ where ϵ is positive but close to 0*

   (a) Show the cutoff value of δ for which it is optimal to sustain cooperation rather than defecting in stage 1 of the game. (6)

Answer. Cooperation yields

$$\frac{\pi(P^m)}{N} + \delta\theta\frac{\pi(P^m)}{N} + \delta^2\theta\frac{\pi(P^m)}{N}... = \left(1 + \frac{\delta\theta}{1-\delta}\right)\frac{\pi(P^m)}{N}$$

Defection yields

$$\pi(P^m) + 0...$$

Cooperation in stage 1 is optimal if

$$\left(1 + \frac{\delta\theta}{1-\delta}\right)\frac{\pi(P^m)}{N} \geq \pi(P^m) + 0$$

$$\delta \geq \frac{n-1}{n-1+\theta} = \delta^* \qquad (6)$$

Marks breakdown:

- 2 marks for equation first expression (cooperation).
- 1 mark for second expression (defection).
- 2 marks for setting up the inequality (third expression).
- 1 mark for δ^*.

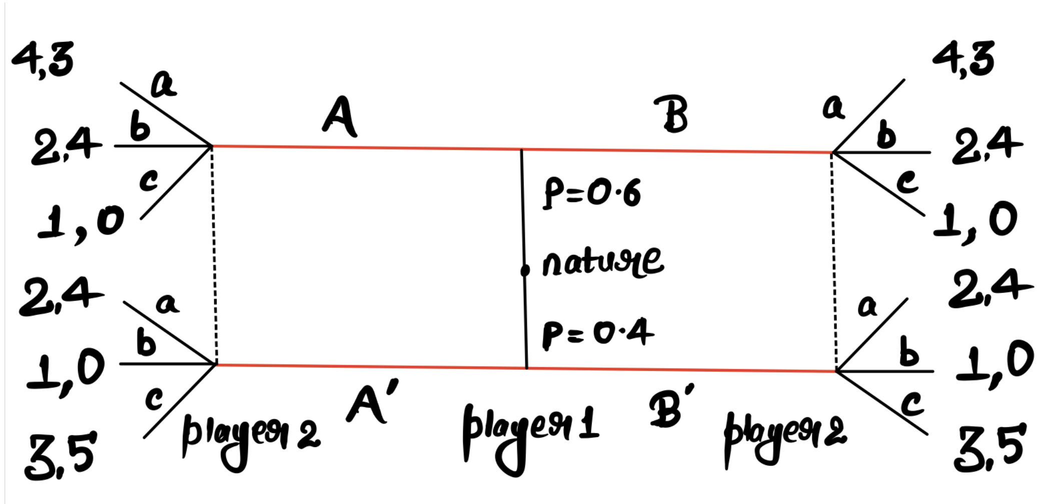
(b) Do you expect this cutoff value to increase or decrease in later stages of the game. Prove it. What about if N increases/decreases? (4)

Answer. This is easy. We just need to take derivatives. The derivative of δ^* with respect to θ is negative. Thus, cooperation is more sustainable in later stages (since $\theta > 1$). The derivative with respect to N is positive. Thus, it is more difficult to sustain collusion as number of firms increase.

Marks breakdown:

- 1 mark for each derivative (θ and N). 2 marks each.
- 1 mark for the relevant conclusion in each case (2 marks total)

4. **Perfect Bayesian Equilibrium.** Check all pooling and separating NE for the following game *(2.5 marks for each equilibrium)* (10)



Answer. This is an easy question. All the 4 can be sustained. You have to show answers in detail and follow steps discussed in class.

Marks breakdown:

- 0.5 mark for each setup in terms of finding updated probabilities (2 marks total).
- 2 marks each for the rest of it for each equilibrium (total 8 marks).