Worksheet 5 Solutions

<u>Problem 1</u>: Since μ and λ are roots of the polynomial $\lambda^2 + a\lambda + b$ and we know that

$$(D^2 + aD + bI)\, e^{\mu x} = 0$$

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let us check whether the function $y = \dfrac{e^{\mu x} - e^{\lambda x}}{\mu - \lambda}$ is a solution of the ODE $(D^2 + aD + bI)y = 0$

$$(D^2 + aD + bI)\left(\frac{e^{\mu x} - e^{\lambda x}}{\mu - \lambda}\right)$$

$$= \frac{1}{\mu - \lambda}(D^2 + aD + bI)\, e^{\mu x} - \frac{1}{\mu - \lambda}(D^2 + aD + bI)\, e^{\lambda x}$$

$$= 0$$

So, y is a solution.

<u>Problem 2</u>: (a) This basis is the basis of solutions of the Euler - Cauchy ODE with a double root at $m = 2$.

So, the Euler - Cauchy auxiliary equation

$m^2 + (a-1)m + b = 0$ must be equal to

$(m-2)^2 = 0 \implies a = -3,\ b = 4$

The corresponding ODE is

$$x^2y'' - 3xy' + 4y = 0$$

(b) To show that the two functions are independent, we calculate their wronskian

$$\begin{vmatrix} x^2 & x^2\log(x) \\ 2x & 2x\log(x)+x \end{vmatrix} = x^3\begin{vmatrix} 1 & \log x \\ 2 & 2\log x+1 \end{vmatrix} = x^3 \neq 0$$

Hence, x^2 and $x^2\log x$ are linearly independent functions.

(c) The solution of the initial value problem must be of the form

$$y_p = c_1x^2 + c_2x^2\log x$$

$$y_p(1) = 4 = c_1$$

$$y_p' = 2c_1x + 2c_2x\log x + c_2x$$

$$y_p'(1) = 6 = 2c_1 + c_2 = 8 + c_2$$

$$\Rightarrow \quad c_2 = -2$$

So, the solution is

$$y_p = x^2(4 - 2\log x)$$

Problem 3 : let us find the solution to the homogenous problem. We need the roots of the characteristic polynomial $\lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda = -3, -3$

so, the homogenous solution is

$$y_h = c_1 y_1 + c_2 y_2 = c_1 e^{-3x} + c_2 x e^{-3x}$$

The Wronskian of the y_1 and y_2 functions is

$$W = \begin{vmatrix} e^{-3x} & x e^{-3x} \\ -3e^{-3x} & -3x e^{-3x} + e^{-3x} \end{vmatrix} = (e^{-3x})^2 \begin{vmatrix} 1 & x \\ -3 & -3x+1 \end{vmatrix} = e^{-6x}$$

So, the particular solution to the non-homogenous problem is given by

$$y_p = -y_1 \int \frac{y_2 h}{W} dx + y_2 \int \frac{y_1 h}{W} dx$$

$$= -e^{-3x} \int \frac{x e^{-3x} \cdot 16 \frac{e^{-3x}}{x^2+1}}{e^{-6x}} dx + x e^{-3x} \int \frac{e^{-3x} \cdot 16 \frac{e^{-3x}}{x^2+1}}{e^{-6x}} dx$$

$$= -16 e^{-3x} \int \frac{x}{x^2+1} dx + 16 x e^{-3x} \int \frac{1}{x^2+1} dx$$

$$= -8e^{-3x} \log(x^2+1) + 16 x e^{-3x} \tan^{-1}(x^2+1)$$

The general solution is

$$y = y_h + y_p = (c_1 - 8\log(x^2+1))e^{-3x} + (c_2 + 16\tan^{-1}(x^2+1))xe^{-3x}$$

Problem 4: $y''' + 2y'' + 5y' = 0$

The characteristic equation is $\lambda^3 + 2\lambda^2 + 5\lambda = 0$

$$\lambda(\lambda^2 + 2\lambda + 5) = 0$$

$$\lambda = 0, \quad \lambda^2 + 2\lambda + 5 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

So, $y = 1, e^{-x}\cos 2x, e^{-x}\sin 2x$ are solutions of the ODE.

To see if they are linearly independent, we calculate their wronskian

$$W = \begin{vmatrix} 1 & e^{-x}\cos 2x & e^{-x}\sin 2x \\ 0 & -e^{-x}(\cos 2x + 2\sin 2x) & -e^{-x}(\sin 2x - 2\cos 2x) \\ 0 & e^{-x}(-3\cos 2x + 4\sin 2x) & -e^{-x}(3\sin 2x + 4\cos 2x) \end{vmatrix}$$

$$= e^{-2x}\left(10 - \frac{3}{2}\sin 4x\right) \neq 0 \ \forall x$$

Hence, $1, e^{-x}\cos 2x, e^{-x}\sin 2x$ form a basis on any interval.

Problem 5 : $(D^3 - 9D^2 + 27D - 27I)y = 27 \sin 3x$

The homogenous problem has a characteristic equation

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0$$

$\Rightarrow (\lambda - 3)^3 = 0 \implies \lambda = 3,3,3$

So, the homogenous solution is given by

$$y_h = (c_1 + c_2 x + c_3 x^2) e^{3x}$$

To find a particular solution for non-homogenous problem we try a solution of the form

$$y_p = A \cos 3x + B \sin 3x$$

$$y_p' = -3A \sin 3x + 3B \cos 3x$$

$$y_p'' = -9A \cos 3x - 9B \sin 3x$$

$$y_p''' = 27A \sin 3x - 27B \cos 3x$$

Substituting in the ODE, we get

$$\cos(3x)(-27B + 81A + 81B - 27A) + \sin 3x(27A + 81B - 81A - 27B)$$
$$= 27 \sin 3x$$

$\Rightarrow \cos 3x(54A + 54B) + \sin 3x(-54A + 54B) = 27 \sin 3x$

$$\Rightarrow \quad A + B = 0 \, , \quad -2A + 2B = 1$$

$$\Rightarrow \quad A = -1/4 \, , \quad B = 1/4$$

so, the general solution is

$$y = (c_1 + c_2 x + c_3 x^2) e^{3x} - \frac{1}{4} \cos 3x + \frac{1}{4} \sin 3x$$

Problem 6 : $x^3 y''' + x y' - y = x^2$

The homogenous ODE is an Euler cauchy equation, so we try with a solution of the form

$$y = x^m$$

$$y' = m x^{m-1} \, , \quad y'' = m(m-1) x^{m-2}$$

$$y''' = m(m-1)(m-2) x^{m-3}$$

Substituting in the equation, we get

$$m(m-1)(m-2) + m - 1 = 0$$

$$(m-1)\left(m(m-2) + 1 \right) = 0$$

$$(m-1)^3 = 0 \Rightarrow m = 1, 1, 1$$

so, $y_h = c_1 x + c_2 x \log x + c_3 x (\log x)^2$

For finding a particular solution of the non-homogenous problem we use the method of variation of parameter whose solution is

$$y_p = \sum_{k=1}^{3} y_k \int \frac{W_k}{W} r \, dx$$

where y_k are the 3 homogenous solutions and W and W_k are following

$$W = \begin{vmatrix} x & x\log x & x(\log x)^2 \\ 1 & 1+\log x & 2\log x + (\log x)^2 \\ 0 & 1/x & 2/x + 2/x \log x \end{vmatrix} = 2$$

$$W_1 = \begin{vmatrix} 0 & x\log x & x(\log x)^2 \\ 0 & 1+\log x & 2\log x + (\log x)^2 \\ 1 & 1/x & 2/x + 2/x \log x \end{vmatrix} = x(\log x)^2$$

$$W_2 = \begin{vmatrix} x & 0 & x(\log x)^2 \\ 1 & 0 & 2\log x + (\log x)^2 \\ 0 & 1 & 2/x + 2/x \log x \end{vmatrix} = -2x\log x$$

$$W_3 = \begin{vmatrix} x & x\log x & x(\log x)^2 \\ 1 & 1+\log x & 2\log x + (\log x)^2 \\ 0 & 1/x & 2/x + 2/x \log x \end{vmatrix} = x$$

Write the ODE in the standard form

$$x^3y''' + xy' - y = x^2 \Rightarrow y''' + \frac{y'}{x^2} - \frac{y}{x^3} = \frac{1}{x}$$

Now calculate the integrals (with $r = 1/x$)

$$\int \frac{W_1}{W} r\, dx = \int \frac{x(\log x)^2}{2} \cdot \frac{1}{x}\, dx = \frac{x((\log x)^2 - 2\log x + 2)}{2}$$

$$\int \frac{W_2}{W} r\, dx = \int \frac{-2x \log x}{2x}\, dx = -x(\log x - 1)$$

$$\int \frac{W_3}{W} r\, dx = \int \frac{x}{2} \cdot \frac{1}{x}\, dx = \frac{x}{2}$$

The particular solution is

$$y_p = y_1 \int \frac{W_1}{W} r\, dx + y_2 \int \frac{W_2}{W} r\, dx + y_3 \int \frac{W_3}{W} r\, dx$$

$$y_p = x^2$$

General solution $\rightarrow$

$$y = c_1 x + c_2 x \log x + c_3 x (\log x)^2 + x^2$$

Imposing the initial conditions, we get

$$c_1 = 0, \quad c_2 = 3, \quad c_3 = \frac{11}{2}$$

So, the solution is

$$y = 3x\log x + \frac{11}{2}x(\log x)^2 + x^2$$