Game Theory: Assignment 2

Total points: 50 — Due Date: 21/09/2021

Contribution to grade: 10% (3xx); 7.5%(5xx) — Due time: 11:59 PM

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1. **IESDS using randomization.** Find the Nash Equilibrium using IESDS in the following game. *You must randomize between strategies as we do in MSNE to eliminate at least on pure strategy.* (5)

		Player 2 L	M	R
	U	$6, 2$	$0, 4$	$1, 0$
Player 1	M	$3, 1$	$3, 0$	$2, 6$
	D	$8, 2$	$4, 4.5$	$3, 5.5$

Answer. Assign probabilities p and $1-p$ to actions M and R for Player 2. To dominate L, we need

$$4p + 0(1-p) > 2 \Rightarrow p > \frac{1}{2}$$

$$0p + 6(1-p) > 1 \Rightarrow p < \frac{5}{6}$$

$$4.5p + 5.5(1-p) > 2 \Rightarrow p < 3.5$$

Which means we can eliminate L, as it is strictly dominated by player 2 randomizing between M and R, with $p \in \left[\frac{1}{2}, \frac{5}{6}\right]$. This is even though L cannot be eliminated right away using pure strategies. Once you keep following IESDS, you will arrive at (D, R).

2. **Mixed Strategy Nash Equilibrium.** Consider the following normal form game:

1\2	L	R
L	1,1	1,0
M	4,0	-4,1
R	2,0	2,0

- Find the pure strategy Nash equilibrium, if any. (2)

  Answer. We start by eliminating the dominated strategies. For player one, L is dominated by R, leaving us with a 2 by 2 matrix. We can easily find the psNe by

1\2	L	R
~~L~~	~~1,1~~	~~1,0~~
M	4,0	-4,1
R	2,0	2,0

  observation, which is $\{R, R\}$.

1\2	L	R
~~L~~	~~1,1~~	~~1,0~~
M	4,0	-4,1
R	2,0	2,0

- Show the process of finding the mixed strategy Nash equilibrium. Draw the BRFs. What do you find? (5)

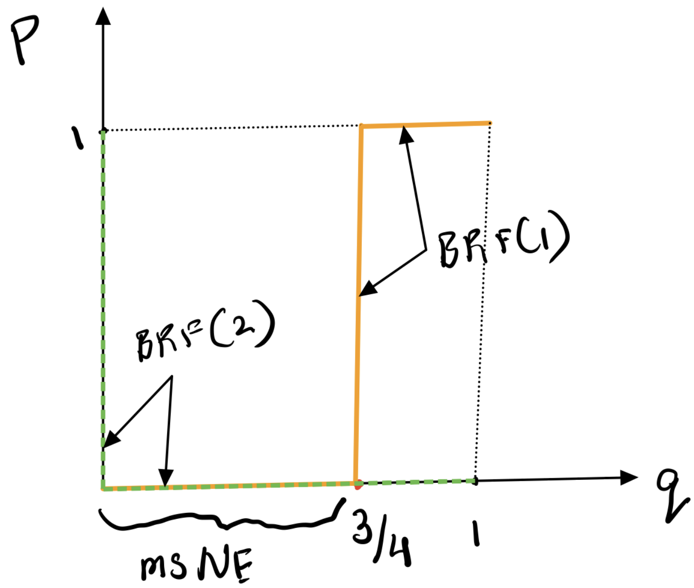
  Answer. To find the msNE, first let p be the probability of 1 choosing M and let q be the probability of 2 choosing L.

$$EU_1(M) = EU_1(R)$$

$$4q + (-4)(1-q) = 2q + 2(1-q) \Rightarrow q = \frac{3}{4}$$

$$EU_2(L) = EU_2(R) \Rightarrow 0p + 0(1-p) = p + 0(1-p) \Rightarrow p = 0$$

  When $q < \frac{3}{4}$, 1 plays R, if $q = \frac{3}{4}$ 1 is indifferent and if $q > \frac{3}{4}$, 1 plays M. 2 is only indifferent when $p = 0$, and plays R otherwise. Thus, the Nash Equilibrium is player 1 playing R with probability 1 and player 2 playing M with probability $q \leq \frac{3}{4}$. Note that this solution includes the psNE. The BRFs give us.



3. **Three player game.** Find the pure strategy Nash Equilibria. Note Player 3's strategies are A and B. (5)

Player 3: Matrix A

		Player 2		
		X	Y	Z
Player 1	A	2,0,4	1,1,1	1,2,3
	B	3,2,3	0,1,0	2,1,0
	C	1,0,2	0,0,3	3,1,1

Player 3: Matrix B

		Player 2		
		X	Y	Z
Player 1	A	2,0,3	4,1,2	1,1,2
	B	1,3,2	2,2,2	0,4,3
	C	0,0,0	3,0,3	2,1,0

Answer. The best responses in each scenario for each player are highlighted in red, and the NE are in blue. Thus, the psNE are $\{(B, X, A), (C, Z, A), (A, Y, B)\}$.

Player 3: Matrix A

		Player 2		
		X	Y	Z
Player 1	A	2,0,4	1,1,1	1,2,3
	B	3,2,3	0,1,0	2,1,0
	C	1,0,2	0,0,3	3,1,1

Player 3: Matrix B

		Player 2		
		X	Y	Z
Player 1	A	2,0,3	4,1,2	1,1,2
	B	1,3,2	2,2,2	0,4,3
	C	0,0,0	3,0,3	2,1,0

4. n**-Player discrete choice game.** Consider a simultaneous move game with n players where $n \geq 2$. Each player has two options, a and b. The payoff function if a player chooses a is

$$2k_a - k_a^2 + 3$$

where k_a represents the number of players choosing a. Similarly, the payoff if one chooses b is

$$4 - k_b$$

where k_b represents the number of players choosing b. Every player must choose at least one of the two, i.e., $k_a + k_b = n$.

(a) For $n = 2$, represent the game in its normal form and find all pure strategy NE (psNE). (5)

Answer. If both players bhoose a, $k_a = 2$ and $k_b = 0$. Thus, the payoffs for both players are:

$$2(2) - (2)^2 + 3 = 3$$

Similarly, if both choose b, the payoffs are

$$4 - (2) = 2$$

If one chooses a and one chooses b, $k_a = k_b = 1$.

$$2(1) - (1)^2 + 3 = 4 \quad \& \quad 4 - (1) = 3$$

Thus, the normal form is

		Player 2	
		a	b
Player 1	a	$\underline{3}, \underline{3}$	$\underline{4}, \underline{3}$
	b	$\underline{3}, \underline{4}$	$2, 2$

The psNE here are $\{(a,a),(b,a),(a,b)\}$.

(b) For $n = 3$, show all psNE. (6)

Answer. Follow similar steps as above to get the following table:

	a	b
a	$0,0,0$	$\underline{3},\underline{3},\underline{3}$
b	$\underline{3},\underline{3},\underline{3}$	$2,2,\underline{4}$

a

	a	b
a	$\underline{3},\underline{3},\underline{3}$	$\underline{4},2,2$
b	$2,\underline{4},2$	$1,1,1$

b

Thus, the psNE are $\{b,a,a),(a,b,a),(a,a,b)\}$.

(c) If $n > 3$, show an asymmetric psNE. You may do this by choosing a random value of n. (6)

Answer. When $n > 3$. Thus, $k_b = n - k_a$. For a distribution of k_a and k_b to be a Nash Equilibrium,deviation should be sub-optimal for any player, regardless of whether they have chosen a or b. Thus, this has to be defined by two conditions:

$$4 - (n - k_a) \geq 2(k_a + 1)^2 + 3 \Rightarrow k_a^2 - k_a \leq n$$

and

$$2k_a - k_a^2 + 3 \geq 4 - (n - k_a + 1) \Rightarrow k_a^2 + k_a \geq n$$

which leads to a combined condition of

$$k_a^2 + k_a \geq n \geq k_a^2 - k_a$$

Now, you can pick a value of n to find the NE. For 4, it clearly works for $k_a = 2, k_b = 2$. For 5, it is possible if $k_a = 2, k_b = 3$. For higher values of n, you can try accordingly. **If you did it by brute force, after taking $n = 4$, that is fine too. But it would be a bizarre exercise to check for higher values of n.**

5. **Winner takes all.** There i students, where $i \in \{1,2\}$ competing for a scholarship, but only one can get it in the end. The winner gets a payoff of 36, but the probability of winning is a function of both the effort a student puts, as well as the effort her rival puts. It is given by $\frac{e_i}{e_i+e_j}$, where $j \neq i$. $e_i \in [0,24]$ (think of this as number of average hours per day in preparation. The loss of payoff for player i from exerting a certain amount of effort is given simply by the level of effort exerted, e_i.

   (a) Write the payoff functions. (2)

   Answer. This is actually quite easy, though there was a lot of confusion. Probability of winning is $\frac{e_i}{e_i+e_j}$, so expected revenue is $36\left(\frac{e_i}{e_i+e_j}\right)$. The effort is actually fixed, and there are no probabilities involved. So the payoff function for each player can be given by

$$\pi_i(e_i, e_j) = 36\left(\frac{e_i}{e_i + e_j}\right) - e_i \text{ where } i \in \{1,2\}, i \neq j$$

   You can also write each payoff function as

$$\pi_1(e_1, e_2) = 36\left(\frac{e_1}{e_1 + e_2}\right) - e_1 \quad \& \quad \pi_2(e_1, e_2) = 36\left(\frac{e_1}{e_1 + e_2}\right) - e_2$$

   (b) Find the BRF and draw them (10)

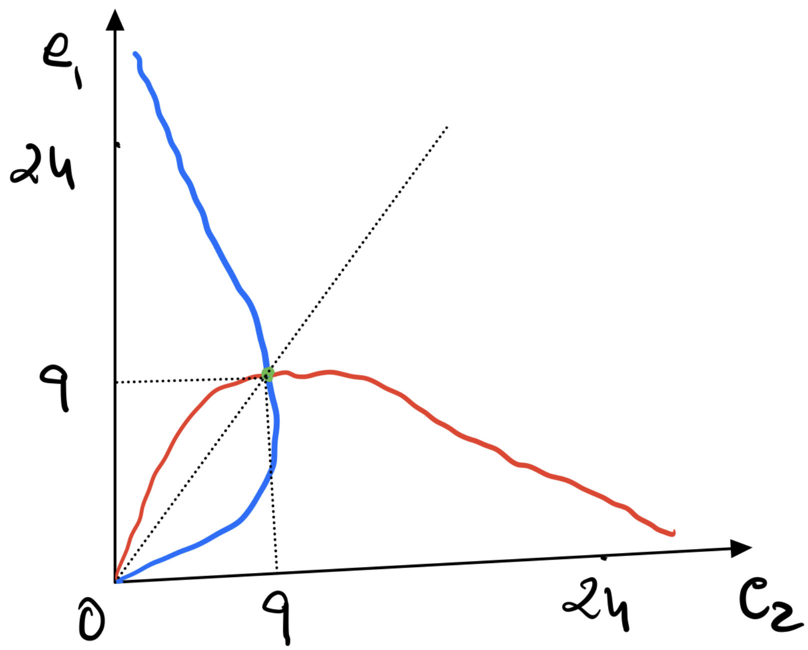
   Answer. The best response function for each player i is given by solving

$$\max_{e_i} \pi_i(e_i, e_j) = 36\left(\frac{e_i}{e_i + e_j}\right) - e_i$$

   Taking FOC, equating to 0 and solving yields

$$e_i(e_j) = 6\sqrt{e_j} - e_j$$

To know the shape of the BRF (to draw the graph), we need to take a couple of derivatives. First, we need to take the derivative that helps us find the maximum. This yields a value of 9 after equation to 0. Then, we take the SOC of the BRF and find that it is negative. The Graph looks like this:



(c) Find the pure strategy NE. (4)

Answer. From the above graph, we can see that the psNE here is each student puts in 9 hours of effort. We cannot really consider $(0, 0)$, as we cannot evaluate the expected payoff at that point. Also, intuitively, anyone putting even a little effort if the rival puts 0 effort wins with probability 1, and is guaranteed a higher payoff. So this cannot be an NE.