Question 1:

Consider the Bayesian game given below (Figure 1). First, nature selects player 1's type, either high or low with probabilities p and 1 − p, respectively. Observing his type, player 1 chooses between x or y (when his type is high) and between x′ and y′ (when his type is low). Finally, player 2, neither observing player 1's type nor player 1's chosen action, responds with a or b. Note that this game can be interpreted as players 1 and 2 acting simultaneously, with player 2 being uninformed about player 1's type.

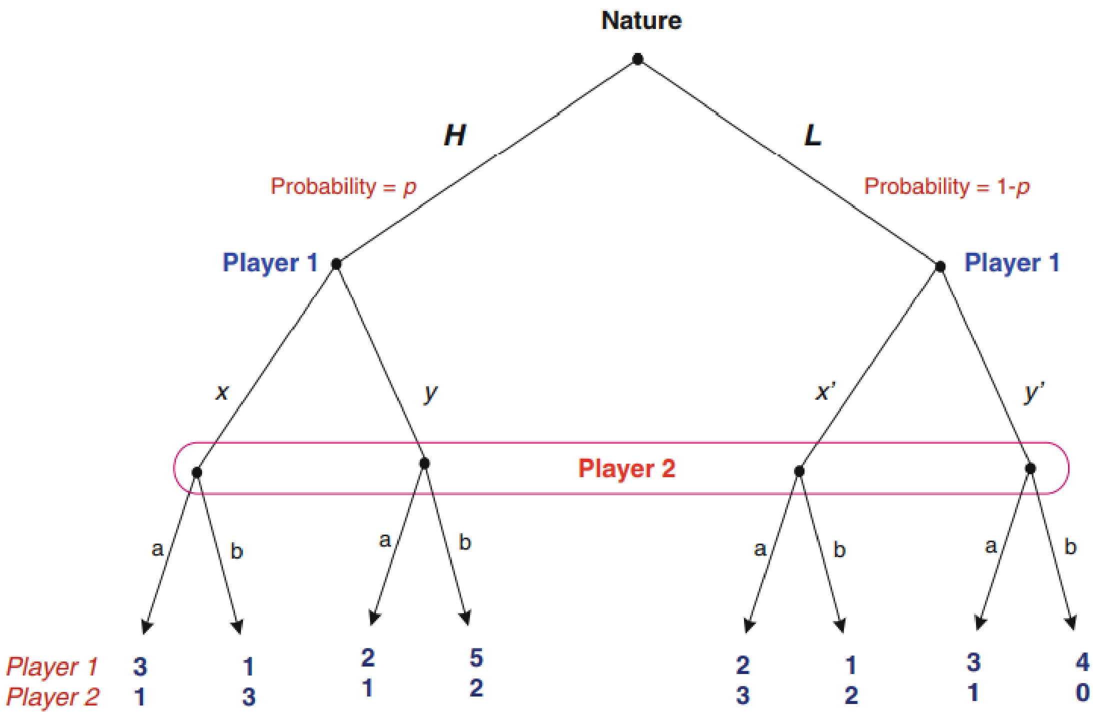

Figure 1

Part (a) Assume that p = 0.75. Find a Bayesian-Nash equilibrium

In order to represent the game tree into its Bayesian normal form representation, we first need to identify the available strategies for each player. In particular, player 2 has only two strategies $S_2 = \{a, b\}$. In contrast, player 1 has four available strategies $S_1 = \{xx', xy', yx', yy'\}$ where the first component of every strategy pair denotes what player 1 chooses when his type is H and the second component reflects what he selects when his type is L. Hence, the Bayesian normal form representation of the game is given by the following 4X2 matrix (Figure 2). Let's find the expected payoffs:

Player 1	Player 2	
	a	b
xx'	$2 + p, 3 - 2p$	$1, 2 + p$
xy'	$3, 1$	$4 - 3p, 3p$
yx'	$2, 3 - 2p$	$1 + 4p, 2$
yy'	$3 - p, 1$	$4 + p, 2p$

Figure 2

Expected payoff for:

$(\boldsymbol{xx'}, \boldsymbol{a})$

$EU_1 = p * 3 + (1 - p) * 2 = 2 + p$

$EU_2 = p * 1 + (1 - p) * 3 = 3 - 2p$

$(\boldsymbol{xy'}, \boldsymbol{a})$

$EU_1 = p * 3 + (1 - p) * 3 = 3$

$EU_2 = p * 1 + (1 - p) * 1 = 1$

And similarly, evaluating for the rest of the strategy profiles, gives you the matrix above.

When p = 0.75 the above matrix becomes (Figure 3):

Player 1	Player 2	
	a	b
xx'	2.75, 1.5	1, $\underline{2.75}$
xy'	$\underline{3}$, 1	1.75, $\underline{2.25}$
yx'	2, 1.5	4, $\underline{2}$
yy'	2.25, 1	$\underline{4.75}$, $\underline{1.5}$

Figure 3

Underlining the payoff associated with the best response of each player, we find that there is a unique BNE in this incomplete information game: (yy', b)

Part (b) For each value of p, find all Bayesian-Nash equilibria.

We can now approach this exercise without assuming a particular value for the probability p. Let us first analyze player 1's best responses. We reproduce the Bayesian normal form representation of the game for any probability p (Figure 4):

Player 1	Player 2	
	a	b
xx'	$2 + p, 3 - 2p$	$1, 2 + p$
xy'	$3, 1$	$4 - 3p, 3p$
yx'	$2, 3 - 2p$	$1 + 4p, 2$
yy'	$3 - p, 1$	$4 + p, 2p$

Figure 4

Player 1's BR:

When player 2 selects the left column (strategy a), player 1 compares the payoff he obtains from xy′, 3, against the payoff arising in his other strategies, i.e., 3 > 2 + p, 3 > 2, and 3 > 3 − p, which hold for all values of p. Hence, player 1's best response to a is xy′. Similarly, when player 2 chooses the right-hand column (strategy b), player 1 compares the payoff from yy′, 4+ p, against that in his other available strategies, i.e., 4 + p > 1, 4p > 4 − 3p, 4 + p >1+4p which hold for all values of p. Hence, player 1's best response to b is yy′. Summarizing, player 1's best responses are $BR_1(a) = xy'$and $BR_1(b) = yy'$.

Player 2's BR:

Let us know examine player 2's to each of the four possible strategies of player 1:

When player 1 selects xx′, he responds with a if 3 − 2p >2+ p , which simplifies to 1 > 3p that is, if 1/3 > p. When player 1 selects xy′, he responds with a if 1 > 3p or 1/3 > p (otherwise he chooses b). When player 1 selects yx′, he responds with a if 3 - 2p> 2; that is, he chooses a if 1/2 > p. When player 1 selects yy′, he responds with a if 1 > 2p, otherwise he chooses b.

Figure 5 summarizes player 2's best responses (either a or b), as a function of the probability p.

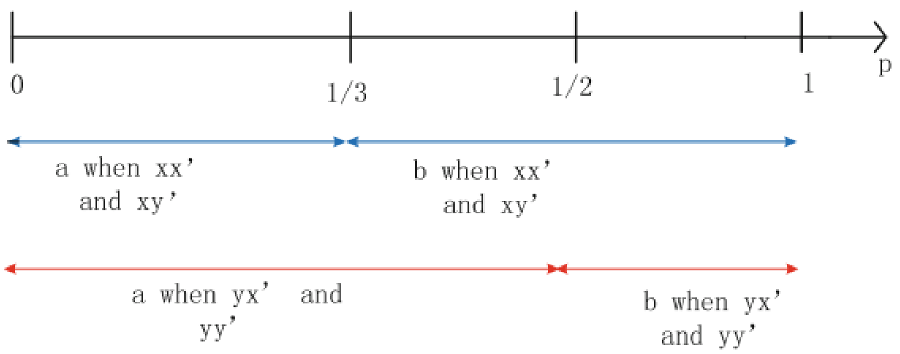

Figure 5

We can then divide our analysis into three different matrices, depending on the specific value of the probability p:

• One matrix for p< 1/3

• A second matrix for $p \in [\frac{1}{3}, \frac{1}{2}]$

• A third matrix for p>1/2

First case: p < 1/3

In this case, player 2 responds with a regardless of the action (row) selected by player 1 (see the range of Fig. 5 where p<1 3). This is indicated by player 2's payoffs in Fig. 6 (below), which are all underlined in the column where player 1 selects a. Hence, strategy a becomes a strictly dominant strategy for player 2 in this context, while player 1's best responses are insensitive to p, namely:

$BR_1(a) = xy'$and $BR_1(b) = yy'$. In this case, the unique BNE when p < 1/3 is (xy′, a)

Player 1	Player 2	
	a	b
xx'	$2 + p, \underline{3 - 2p}$	$1, 2 + p$
xy'	$\underline{3}, \underline{1}$	$4 - 3p, 3p$
yx'	$2, \underline{3 - 2p}$	$1 + 4p, 2$
yy'	$3 - p, \underline{1}$	$\underline{4 + p}, 2p$

Figure 6

Second Case: $\boldsymbol{p \in [\frac{1}{3}, \frac{1}{2}]}$

Player 2 still responds with a when player 1 chooses yx′ and yy′, but with b when player 1 selects xx′ and xy′. This is indicated in player 2's payoffs in the matrix of Figure 7, which are underlined in the column corresponding to a when player 1 chooses yx′ and yy′ (last two rows), but are underlined in the right-hand column (corresponding to strategy b) otherwise. In this case there is no BNE when we restrict players to only use pure strategies.

Player 1	Player 2	
	a	b
xx'	$2 + p, 3 - 2p$	$1, \underline{2 + p}$
xy'	$\underline{3}, 1$	$4 - 3p, \underline{3p}$
yx'	$2, \underline{3 - 2p}$	$1 + 4p, 2$
yy'	$3 - p, \underline{1}$	$\underline{4 + p}, 2p$

Figure 7

Third case: p > 1/2

Player 2 best responds with strategy b regardless of the action (row) selected by player 1, i.e., b becomes a strictly dominant strategy. This result is depicted in the matrix of Figure 8, where player 2's payoffs corresponding to *b* are all underlined (see right-hand column). In this case the unique BNE is (yy′, b). (Note that this BNE is consistent with part (a) of the exercise, where p was assumed to be 0.75, and thus corresponds to the third case analyzed here p > 1/2. Needless to say, we found the same BNE as in the third case.

Player 1	Player 2	
	a	b
xx'	$2+p,\ 3-2p$	$1,\ \underline{2+p}$
xy'	$\underline{3},\ 1$	$4-3p,\ \underline{3p}$
yx'	$2,\ 3-2p$	$1+4p,\ \underline{2}$
yy'	$3-p,\ 1$	$\underline{4+p},\ \underline{2p}$

Figure 8

Question 2:

Let us consider an oligopoly game where two firms compete in quantities. Both firms have the same marginal costs, MC = $1, but they are asymmetrically informed about the actual state of market demand. In particular, Firm 2 does not know what is the actual state of demand, but knows that it is distributed with the following probability distribution:

$p(Q) = (10 - Q)\ with\ prob. 1/2\ \&\ (5 - Q)\ with\ prob. 1/2$

On the other hand, firm 1 knows the actual state of market demand, and firm 2 knows that firm 1 knows this information (i.e., it is common knowledge among the players)

Firm 1.

First, let us focus on Firm 1, the informed player in this game, as we usually do when solving for the BNE of games of incomplete information. When firm 1 observes a high demand market its profits are:

$$Profits_1 = 10q_1^H - {q_1^H}^2 - q_2 q_1^H - 1.q_1^H$$

Differentiating with respect to q_1^H, we can obtain firm 1's best response function when experiencing low costs:

$$10 - 2q_1^H - q_2 - 1 = 0 \Longrightarrow q_1^H(q_2) = 4.5 - q_2/2$$

On the other hand, when firm 1 observes a low demand market, its profits are:

$$Profits_1 = (5 - q_1^L - q_2)q_1^L - q_1^L$$

Differentiating with respect to q_1^L, we can obtain firm 1's best response function when experiencing high costs:

$$5 - 2q_1^L - q_2 - 1 = 0 \Longrightarrow q_1^L(q_2) = 2 - q_2/2$$

Firm 2.

Let us now analyze Firm 2 (the uninformed player in this game). First note that its profits must be expressed in expected terms, since firm 2 does not know whether market demand is high or low.

$$Profits_2 = 1/2[(10 - q_1^H - q_2)q_2 - 1.q_2] + 1/2[(5 - q_1^L - q_2)q_2 - 1.q_2]$$

Rearranging and differentiating with respect to q_2 helps us obtain firm 2's best response function:

$$1/2[(10 - q_1^H - 2q_2) - 1] + 1/2[(5 - q_1^L - 2q_2) - 1] \Longrightarrow q_2(q_1^H, q_1^L) = \frac{13 - q_1^H - q_1^L}{4}$$

After finding the best response functions for both types of Firm 1 and for the unique type of Firm 2 we are ready to plug the first two BRFs into the latter:

$$q_2 = 3.25 - 1.625 + 0.25q_2 \Rightarrow q_2 = 2.167$$

With this information, it is easy to find the particular level of production for firm 1 when experiencing low market demand:

$$q_1^L(q_2) = 2 - \left(\frac{2.167}{2}\right) = 0.916$$

And, high market demand:

$$q_1^H(q_2) = 4.5 - \left(\frac{2.167}{2}\right) = 3.417$$

Therefore, the Bayesian Nash equilibrium (BNE) of this oligopoly game with incomplete information about market demand prescribes the following production levels:

(q_1^H, q_1^L, q_2)=(3.42, 0.92, 2.17)

Question 3:

Two friends want to have a tennis match against each other. The question is whether to play aggressively or defensively (which will depend on their ability). Both players privately observe their ability, and this is explained by their probability of winning if both play aggressively: p_H or p_L, which have corresponding probabilities q and 1 − q, where 0 < p_L < p_H < 1. If a player, attacks and the rival plays defensive, he wins with probability γp_j such that $p_j < \gamma p_j < 1$, j ∈ {H, L}. If a player plays defensive and his rival plays attacking, his probability of winning drops to δp_j where $0 < \delta p_j < p_j$ .If both play aggressive, probability of winning is p_j for a j-type player. Finally, if both players play defensive, there is no winner (payoff is 0, the match takes too long). Reward from winning is R and loss from losing is L, such that R > 0 > L.

Part (a): Under which conditions there is a symmetric BNE where both players play aggressive regardless of type.

If one player expects the other to attack for sure (that is, both when his type is high and low), then it is optimal to attack regardless of his type. Doing so results in an expected payoff of $pR + (1-p)L$, where p is the probability of wining the fight when both players attack, and not doing so results in an expected payoff of $\delta pR + (1 - \delta p)L$, where δp is the probability of winning the fight when the player does not attack but his rival does. Comparing these expected payoffs, we obtain that every player prefers to attack if:

$$pR + (1-p)L > \delta pR + (1 - \delta p)L$$

Rearranging yields:

$$R(p - \delta p) > L(p - \delta p)$$

or, further simplifying, R>L, which holds by definition since R>0>L. Therefore, attacking regardless of one's type is a symmetric Bayesian-Nash equilibrium (BNE).

Part (b): Find the conditions to support a symmetric BNE in which both players play aggressive only if their type is high.

Consider a symmetric strategy profile in which a player attacks only if his type is high.

High-type player:

For this strategy profile to be an equilibrium, we need that the expected utility from fighting when being a strong fighter must exceed that from not fighting. In particular,

$$q[p_H R + (1-p_H)L] + (1-q)[\gamma p_H R + (1-\gamma p_H)L] \geq q[\delta p_H R + (1-\delta p_H)L] + (1-q)0$$

Let's start analyzing the *left-hand side* of the inequality:

The first term represents the expected payoff that a type-p_H player obtains when facing a player who is also type p_H (which occurs with probability q). In such a case, this equilibrium prescribes that both players attack (since they are both pH), and the player we analyze wins the fight with probability p_H (and alternatively losses with probability 1 - p_H)

The second term represents the expected payoff that a type p_H player obtains when facing a player who is, instead, type p_L (which occurs with probability 1-q). In this case, his rival doesn't attack in equilibrium, increasing the probability of wining the fight for the player we analyze to γp_H, where $\gamma p_H > p_H$

Let us now examine the *right-hand side* of the inequality, which illustrates the expected payoff that player p_H obtains when he deviates from his equilibrium strategy, i.e., he does not attack despite having a type p_H.

The first term represents the expected payoff that a type p_H player obtains when facing a player who is also p_H (which occurs with probability q). In such a case, the player we analyze doesn't attack while his rival (also of type H) attacks, lowering the chances that the former wins the fight to δp_H, where $\delta p_H < p_H$. If, in contrast, the type of his rival is p_L, then no individual attacks and their payoffs are both zero.

Rearranging the above inequality, we obtain:

$$R[qp_H + \gamma p_H - q\gamma p_H - q\delta p_H] \geq L[qp_H + \gamma p_H - q\gamma p_H - q\delta p_H - (1-q)]$$

Solving for R:

$$R \geq L\left(1 - \frac{(1-q)}{p_H[q(1-\delta) + (1-q)\gamma]}\right)$$

Since both the numerator and denominator are strictly positive, the term in paranthesis satisfies: $\left(1 - \frac{(1-q)}{p_H[q(1-\delta)+(1-q)\gamma]}\right) < 1$, and this condition holds by the initial condition R>L.

Low-type player:

Let us now examine the low type player. Recall that, according to the equilibrium we analyze, this player prefers to not attack. That is, his expected utility must be larger from not attacking than attacking, as follows.

$$q[\delta p_L R + (1-\delta p_L)L] + (1-q)0 \geq q[p_L R + (1-p_L)L] + (1-q)[\gamma p_L R + (1-\gamma p_L)L]$$

An opposite intuition as above can be constructed for this inequality, i.e., a player prefers not to attack when its type is low.

$$R \geq L\left(1 - \frac{(1-q)}{p_L[q\delta - \gamma + q\gamma - q)]}\right)$$

Since both the numerator and denominator is > 0, the term in parenthesis satisfies:

$$\left(1 - \frac{(1-q)}{p_L[q\delta - \gamma + q\gamma - q)]}\right) < 1$$

As a consequence, the above inequality is also implied by the initial condition R> L.