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$F(x,y)=0 \quad F_x\,dx + F_y\,dy = 0 \quad dy/dx = -\frac{F_x}{F_y}$

Implicit differentiation

$W_t = W_x X_t + W_y Y_t$

$DD = \nabla f \cdot \hat{u}$ $\quad \bar{\nabla} f = (F_x, F_y)$

Most rapid change => $\hat{\nabla} f$
Least rapid change => $-\hat{\nabla} f$
Zero change => $\perp \hat{\nabla} f$

Tangent plane
$\bar{\nabla} F\,(x-x_0, y-y_0, z-z_0) = 0$

Normal plane
$x = x_0 + tF_x$; $z = z_0 + tF_z$
$y = y_0 + tF_y$

Change = DD . ds

Total change
$= f_x|_{(x_0,y_0)}\,dx + f_y|_{(x_0,y_0)}\,dy \ldots$

Linearisation

$L(f(x,y)) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
$+$
HOTs
$+\frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2} f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2$

ERROR
$\le \frac{1}{2} M (|x-x_0| + |y-y_0|)^2 \quad M = \max(|f_{xx}|, |f_{yy}|, |f_{xy}|)_{(x,y)}$

Critical points
$f_x(a,b) = f_y(a,b) = 0$
SDT
$f_{xx}f_{yy} - f_{xy}^2 > 0$
- $f_{xx} > 0$ Local min
- $f_{xx} < 0$ Local max

$f_{xx}f_{yy} - f_{xy}^2 < 0$
Saddle

Langrange $\bar{\nabla} f = \bar{\nabla} g_1 \lambda_1 + \bar{\nabla} g_2 \lambda_2 \ldots$

Taylor Series
$f(a+h, b+k) = f(a,b) + h f_x(a,b) + k f_y(a,b) + \frac{1}{2!}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{a,b}$
+HOTs

$\text{proj}_{\bar{v}} U = |\bar{u}|\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{v}|}\hat{v}$

$\bar{u} = \text{proj}_{\bar{v}} \bar{u} + (\bar{u} - \text{proj}_{\bar{v}} \bar{u})$

Area of ∥gm $= \bar{u} \times \bar{v}$

Area of $\Delta = (\bar{u} \times \bar{v})/2$
$\cos\theta = v \cdot u / |v||u|$

Scalar triple product
$[\bar{u}, \bar{v}, \bar{w}] = (\bar{u} \times \bar{v}) \cdot \bar{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$
$(\bar{v} \times \bar{w}) \cdot \bar{u}$

line passing through P_0 and ∥ to $\bar{v}$
$\bar{x} = \bar{x_0} + t\bar{v} \quad \bar{y} = \bar{y_0} + t\bar{v} \quad \bar{z} = \bar{z_0} + t\bar{v}$

Distance PS ∥ to line V
$PS \sin\theta = |\bar{PS} \times \bar{v}| / |\bar{v}|$

$|u \times v| = |u||v|\sin\theta$

$\bar{T}(\text{tangent}) = \frac{r'}{|r'|} = \frac{d\bar{r}/dt}{ds/dt} = \frac{d\bar{r}}{ds}$

Length
$L = \int_t \left|\frac{dr}{dt}\right| t$

$N = \frac{1}{K}\frac{dT}{ds} = \frac{dT/dt}{|dT/dt|}$

inward of curvature

$K = \left|\frac{d\bar{T}}{ds}\right| = \frac{|d\bar{T}/dt|}{|ds/dt|} = \frac{1}{|\bar{v}|}\left|\frac{d\bar{T}}{dt}\right|$

or 1/radius

ascillatory Circle
- Tangent to P and same curvature as at P
- Lies on concave side

Line integral
$\int_C f\,ds = \int_a^b f(g(t), h(t), k(t))\,|v|\,dt$

$\bar{B} = \bar{T} \times \bar{N} \quad \bar{B}^2 = 1$

$\tau = -\frac{d\bar{B}}{ds}\bar{N}$

$= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$

$M = \int_C \delta(x,y,z)\,ds$

CoM $x = M_{yz}/M$

$I_L = \int_C r^2 \delta\,ds$

$R_G = \sqrt{I_L/M}$

First moment
$M_{yz} = \int_C \delta x\,ds$

$I_x = \int_C (y^2+z^2)\delta\,ds$

$MOI = \int r^2 \delta\,dv$

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Polar form $x^2+y^2=r^2$ $dx\,dy = r\,dr\,d\theta$

$I_L = mh^2 + I_{com}$

Cylindrical Polar Coordinates

$W = \int_a^b F.T\,ds = \int_a^b M\,dx + N\,dy + P\,dz$

Flow = $\int_{t=a}$ circulation (closed loop).

- Evaluate $F(t)$
- Find dr/dt
- $W = \int_{t=a}^{b} F.\frac{dr}{dt}\,dt$

$T\,ds = dr$

$v(t) = (dx, dy, dz)$

$\oint F.dr = 0$ around closed loop

Flux (M, N) $= \int_C F.\hat{n}\,ds = \oint_C M\,dy - N\,dx$

$ds = |v(t)|\,dt$

Test for conservative

$\frac{dP}{dy} = \frac{dN}{dz}$ $\frac{dM}{dz} = \frac{dP}{dx}$

$\frac{dN}{dx} = \frac{dM}{dy}$ $F = (M, N, P)$

Find potential function

$\int_A^B F.dr = f(B) - f(A)$

f is the poten. func.

- Find $\frac{dF}{dx}$
- Integrate $\frac{dF}{dx}$ (get $g(y,z)$)
- differentiate by y (get $h(z)$)
- Compare LHS-RHS

Exact differential

$M\,dx + N\,dy + P\,dz = \frac{df}{dx}dx + \ldots = df$

divergence $F = \frac{dM}{dx} + \frac{dN}{dy}$ (Flux density)

$(\text{curl } F)\,k = \frac{dN}{dx} - \frac{dM}{dy}$ (Circulation density)

Green's theorem

- $\oint_C F.n\,ds = \oint M\,dy - N\,dx = \iint_R \left(\frac{dM}{dx} + \frac{dN}{dy}\right) dx\,dy$
- $\oint F.T\,ds = \oint M\,dx + N\,dy = \iint_R \left(\frac{dN}{dx} - \frac{dM}{dy}\right) dx\,dy$

Area $= \iint_R dy\,dx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\,dx = \oint \frac{1}{2}x\,dy - \frac{1}{2}y\,dx$

Surface Area $= \iint_R \frac{|\nabla f|}{|\nabla f.p|}\,dA$

P unit normal to R

surface integral $= \iint g(x,y,z)\frac{|\nabla f|}{|\nabla f.p|}dA = \iint g\,d\sigma$

Flux $= \iint_S F.\hat{n}\,d\sigma = \iint F.\frac{\pm\nabla g}{|\nabla g.p|}\,dA$

- $r_u = \left(\frac{df}{du}, \frac{dg}{du}, \frac{dh}{du}\right)$
- $r_v = \left(\frac{df}{dv}, \frac{dg}{dv}, \frac{dh}{dv}\right)$
- Area of smooth surface

$r_r \times r_\theta = \sqrt{2}r$

$r_\phi \times r_\theta = r^2 \sin\phi$

$A = \int_c^d \int_a^b |r_u \times r_v|\,du\,dv$

- Parametric surface integral

$\iint_S G(x,y,z)\,d\sigma = \int_c^d\int_a^b G(f(u,v), g(u,v), h(u,v))\,|r_u \times r_v|\,du\,dv$

Flux $= F.n$ $n = \frac{r_u \times r_v}{|r_u \times r_v|}$

curl $F = \nabla \times F = \begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ M & N & P \end{vmatrix}$

Stokes theorem

$\oint_C F.dr = \iint_S \nabla \times F.n\,d\sigma$

curl grad $f = 0$ $\nabla \times \nabla f = 0$

divergence => sum of grads

$\iint_S F.n\,d\sigma = \iiint \nabla.F\,dV$

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Polar form $x^2+y^2=r^2$ $dx\,dy = r\,dr\,d\theta$ | $I_L = mh^2 + I_{com}$

Cylindrical Polar Coordinates

$x = r\cos\theta$ $r^2 = x^2+y^2$
$y = r\sin\theta$ $\tan\theta = y/x$
$z = z$ $0 \le \theta \le 2\pi$

Spherical Polar Coordinates

$x = \rho\sin\varphi\cos\theta$ $0 \le \theta \le 2\pi$ planar
$y = \rho\sin\varphi\sin\theta$ $0 \le \varphi \le \pi$ vertical
$z = \rho\cos\varphi$

Conical Polar coordinates: Same as spherical but θ is fixed instead of r

Jacobian $(J(u,v)) = \left\| \begin{matrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{matrix} \right\|$

$= |\partial_u x\, \partial_v y - \partial_v x\, \partial_u y|$

Polar form $z = r(\cos\theta + i\sin\theta)$
Modulus $|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$
arg (normal) $= \tan^{-1} y/x$
Arg (Principal) $= \tan^{-1} y/x$ but $-\pi < \theta \le \pi$

Triangle inequality
$|z_1+z_2| \le |z_1| + |z_2|$

Multiplication
$z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$
$|z_1 z_2| = |z_1||z_2|$
$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Division
$z_1/z_2 = r_1/r_2\,[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

De Moivre Formula
$z^n = r^n(\cos n\theta + i\sin n\theta) = r^n(\cos\theta + i\sin\theta)^n$

$w = \sqrt[n]{z} = R(\cos\varphi + i\sin\varphi)$
$R = r^{1/n}$
$n\varphi + 2k\pi$, $\varphi = \frac{\theta}{n} + \frac{2k\pi}{n}$
No. of $w = n$
$k = 1, 2, 3 \ldots$

Circle $|z-a| = \rho$ $a \Rightarrow$ centre $\rho \Rightarrow$ radius

For continuity, turn into polar and evaluate

Cauchy Riemann (test for analytic)
$f'(z) = u_x + iv_x = -iu_y + v_y$
$u_r = \frac{1}{r} v_\theta$ $v_r = -\frac{1}{r} u_\theta$

Another Cauchy Riemann $u_x = v_y$
$u_y = -v_x$

Laplace equations (test for analytical and harmonic)
$\nabla^2 u = u_{xx} + u_{yy} = 0$ $\nabla^2 v = v_{xx} + v_{yy} = 0$

Find Conjugate by Cauchy Riemann Equations
- Exactly like potential function

$\exp z = e^z = e^x(\cos y + i\sin y)$
e^z is always analytical $|e^z| = e^x$
$(e^z)' = e^z$

$\cosh z = \frac{1}{2}(e^z + e^{-z})$ $(\cosh z)' = \sinh z$
$\sinh z = \frac{1}{2}(e^z - e^{-z})$ $(\sinh z)' = \cosh z$

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$
$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$
$|\sin z|^2 = \sin^2 x + \sinh^2 y$