Quiz 1

Time : 40 minutes Full Marks :15

Problem 1. (7 point) Suppose you have a complete, balanced binary tree data structure with n distinct nodes. You are performing n operations on this tree, each on a distinct node. The cost of an operation on a particular node x is equal to the number of nodes on the path from x to a leaf (including x).

What is the amortized cost per operation? Give the tightest possible asymptotic bound. Show your calculations.

Solution:
(sketch.) From the definition, the cost paid for a node at depth $d = \log_2 n - d + 1$ (assuming n to be a power of 2 and hence ignoring floor ceiling issues). There are 2^d such nodes - follows from the definition of complete balanced binary tree. Hence, the total cost of n opertions is

$$\mathcal{C} = \sum_{d=0}^{\log_2 n - 1} 2^d \cdot (\log_2 n - d + 1)$$

This summation can be computed in three parts. The first part gives us $n \log_2 n$, the third gives is n. The second part is trickier to compute. Here is one way to do it. Define

$$S = \sum_{d=0}^{\log_2 n - 1} d \cdot 2^d = 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 8 + \cdots \frac{n}{2}(\log_2 n - 1)$$

Now multiplying both sides by 2, we have

$$2 \cdot S = 2 \cdot 2 + 3 \cdot 4 + 4 \cdot 8 + \cdots n \cdot (\log_2 n - 1)$$

Subtracting the first equality from the second, we have

$$S = -(2 + 4 + \cdots n/2) + n \log_2 n - n = n \log_2 n - 2n + 2$$

Hence

$$\mathcal{C} = n \log_2 n + n - S = 3n - 2$$

Hence, the amortised cost per operation is $\Theta(1)$. □

Rubric : +1 for correctly writing the summation. +6 for completing the calculation correctly with $Theta(1)$ bound, +3 for proving $O(\log n)$. Just stating $\Theta(1)$ with some intuition will fetch 2 marks , stating $O(\log n)$ with some intuition fetches 1 mark.

Problem 2. In the following problem, in addition to the definition of a 2-universal hash function used in the lectures, we shall use another definition. Let $\mathcal{H}$ be a strongly ℓ-universal family of hash functions from universe U to $\{0, 1, 2, \ldots, s-1\}$ if the following holds. Let $a_1, a_2, \ldots, a_\ell$ be any ℓ distinct elements in U and $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ be elements from $[1, 2, , \cdots s]$, not necessarily distinct. Then the following must hold:

$$\Pr\left[(h(a_1) = \alpha_1) \wedge (h(a_2) = \alpha_2) \wedge \cdots \wedge (h(a_\ell) = \alpha_\ell)\right] = \frac{1}{s^\ell}.$$

For each of the following, say **True** or **False** with at most 2-3 lines of reasons. You get 1 point for just writing True or False without reasons.

Disclaimer. *I realized that I should have defined a 2-Universal Hash function properly. What I had in mind and what really is the correct definition is - a family is 2-Universal if* $\Pr[h(a) = h(b)] \leq 1/s$ *for a randomly drawn function from the Hash family. Instead, I think I made you believe that a family is 2-Universal if pairwise independence property is satisfied. Apologies for that. According to the first definition, the answer to part(b) is True but according to the second, it is False. Hence, I am just giving credit to everyone for this part. The other parts still have valid solutions although (a) is trivial now because of my goof up.*

(a) (2 points) A strongly 2-universal hash family is also 2-universal.

Solution:
True. This is trivially true if you follow the second definition. If you follow first definition. Then you need one line of proof. $\Pr[h(a) = h(b) = \alpha_i] = 1/s^2$ (by definition of 2-strong universality). Hence $\Pr[h(a) = h(b)] = 1/s$ since you just sum the probabilities for $h(a) = h(b) = \alpha_i$ for all $i = 0, 1, 2, \cdots s-1$ and observing that they are all disjoint. □

(b) (3 points) The following hash family from $U = \{a, b\}$ to $\{0, 1\}$ is 2-universal:

	a	b
h_1	0	0
h_2	1	0

Solution:
True. If you follow definition (1). This is because $\Pr[h(a) = h(b)] = 1/2$ since only one out of h_1, h_2 maps both a, b to the same value 0.

False. If you ask for pairwise independence. Clearly $\Pr[h(a) = 0, h(b) = 0] = 1/2$ which is greater than $1/s^2 = 1/4$. □

(c) (3 points) The following hash family from $U = \{a, b, c\}$ to $\{0, 1\}$ is strongly 2-universal:

	a	b	c
h_1	0	0	0
h_2	1	0	1
h_3	0	1	1
h_4	1	1	1

Solution:
False. For instance, $\Pr[h(a) = 1, h(c) = 1] = 1/2$ (since both h_2, h_4 satisfies this) which is larger than $1/s^2 = 1/4$, $\square$