Q1) (a) The set of all real numbers $\mathbb{R}$ is both open & closed. - True

Proof: A set is open if for every point in the set, there exists an ε-ball around that point which is also contained in the set. Since, $\mathbb{R}$ has no boundary & every point in $\mathbb{R}$ has an open neighbourhood contained in $\mathbb{R}$, it is open.

A set is closed if it contains all its limit points. Since $\mathbb{R}$ contains all its limit points it is closed.

Thus, $\mathbb{R}$ is both open & closed.

(b) A cone is always a convex set. - False

Proof: A set C is a cone if for every $x \in C$ & $\lambda > 0$, we have $\lambda x \in C$. A set is convex if for any $(x, y) \in C$, the line segment between them i.e. $(1-\theta)x + \theta y \ \forall \theta \in [0,1]$ is also in C.

For instance,

$$C = \{(x,0) \mid x \geq 0\} \cup \{(0,y) \mid y \geq 0\}$$

This is a cone since scaling any element with any value of λ keeps it in the set. But, it is not convex since the line segment b/w $(1,0)$ & $(0,1)$ would consist of other points such as $(\frac{1}{2}, \frac{1}{2})$ which are not a part of the cone C.

(c) The function $f(x) = |x-3|$ is convex. - True

A function $f(x)$ is convex if for any $x_1, x_2 \in R$ & $\theta \in [0,1]$ we have,

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$

Now, we may check that the second derivative of function is non-negative. Here, the second derivative would be zero except at $x=3$, implying convexity.

(d) The set $C = A \cup B$ where A, B are convex is always convex. - False

For instance consider,

$$A = \{(x,0) \mid x \geq 0\}$$

$$B = \{(x,1) \mid x \geq 0\}$$

Both A & B are convex individually. However, their union is not convex in all cases. For eg: Let, $x=1$ then, the line connected $(1,0)$ to $(1,1)$ would consist of points which are not a part of $A \cup B$. Hence, this may not necessarily be convex.

Marking Scheme :

Q1) (a) 1 mark for true, 1 mark for correct definition of open set, 1 mark for correct definition of closed set, if student's answer is false then 0.

(b) 1 mark for false, 1 mark for correct explanation, if student's answer is true then 0.

(c) 1 mark for true, 1 mark for correct explanation, if student's answer is false then 0.

(d) 1 mark for false, 2 marks for correct explanation (diagram might or might not be included), if student's answers is true then 0.

Q2) We understand that,

$$\{x_i\}_{i\in\{1,2,3...\}} = i+1$$

$$\therefore \{x_i\} = \{2,3,4,5,6......\}$$

And, $\{y_i\} = -i$

$$\therefore \{y_i\} = \{-1,-2,-3.....\}$$

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Now, $\forall i$

$$\{z_i\} = \{x_i\} + \{y_i\}$$

$$= (i+1) + (-i)$$

$$= 1$$

$$\Rightarrow \{z_i\} = \{1,1,1,1.....\} \quad (\because \forall i)$$

$$\Rightarrow Z = \{1\} \quad (\because Z \text{ is a singleton set})$$

$$\therefore \inf(Z) = 1$$

$$\sup(Z) = 1$$

Yes, Z is trivially bounded with an infimum & supremum of identical values due to it being a singleton set.

Marking Scheme for question 2:

- 1 mark for writing bounded or not
- 1 mark for showing it is singleton set
- 0.5 marks for mentioning supremum and 0.5 for infimum

Q3) To determine whether the set of vectors $\{v_1, v_2, v_3, v_4\}$ is linearly independent, we need to check if the only solution to the equation :

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

is the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$.

Now,

$$A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & 6 & 1 \\ 5 & 1 & 1 & -1 \end{bmatrix}$$

$\therefore$ If Rank $(A) = 4$ then, linearly independent otherwise not.

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 2 & 3 & 1 \\ 1 & 1 & 6 & 1 \\ 5 & 1 & 1 & -1 \end{bmatrix}$$

($\because$ Swap $R_1 \leftrightarrow R_2$ to make the pivot 1).

Then,

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -2 & 5 & -1 \\ 0 & -1 & 7 & 0 \\ 0 & -9 & 6 & -6 \end{bmatrix}$$

($\because$ $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 5R_1$).

Now, divide Row 2 by −2 to make pivot 1 & eliminate the second column,

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -5/2 & 1/2 \\ 0 & 0 & 9/2 & 1/2 \\ 0 & 0 & -39/2 & -3/2 \end{bmatrix}$$

$$\left(\because R_2 \rightarrow R_2/-2 \quad R_3 \rightarrow R_3 + R_2 \quad R_4 \rightarrow R_4 + 9R_2 \right)$$

Divide Row 3 by 9/2 & eliminate the third column,

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -5/2 & 1/2 \\ 0 & 0 & 1 & 1/9 \\ 0 & 0 & 0 & -2/3 \end{bmatrix}$$

As we observe, there are 4 non-zero rows. Therefore, Rank $(A) = 4$ which is equal to the no. of vectors hence, the set of vectors are linearly independent.

Marking Criteria - Those who demonstrated that the linear independence condition holds through their matrix calculations—i.e., by showing full rank or the absence of zero rows using proper row operations received full marks.

One mark was awarded for the matrix calculation, and another mark was given for writing the proper condition of Linear independent along with the correct conclusion.

Q4

Sol^n :

* Weierstrass max^m Th^m: let $S \subset \mathbb{R}^n$ be compact & 'f' a continous real valued function on S. Then

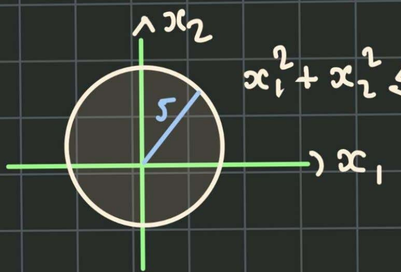
$$\arg\max_{x \in S} f(x) \text{ and } \arg\min_{x \in S} f(x) \text{ both exist}$$

in S $\Rightarrow$ (+1.5)

set $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 25\}$

$f: A \to \mathbb{R}$ $\quad f(x) = d(x, 0) = \|x\|_2 = \sqrt{x_1^2 + x_2^2}$

$\forall x \in A \subseteq \mathbb{R}^2$



By observation set A is:

closed, bounded : compact

contains all boundry points

lies within a finite radius = 5

$\Rightarrow$ (+2.5) for proving set A is closed, bounded and 'f' is continous

also $f(x)$ is continous in $\mathbb{R}^2$

$\Rightarrow$ since $A \subseteq \mathbb{R}^2 \Rightarrow f(x)$ is continous on A

$\therefore$ 'f' is continuous on compact set A

$\Rightarrow$ weierstrass Th^m gurantees the existence of maxima, minima of f in A.

Both maxima and minima exist

maxima : at boundry of set A

$$\Rightarrow \quad x_1^2 + x_2^2 = 25$$

$$\Rightarrow \quad \boxed{f_{max}(x) = \sqrt{25} = 5}$$

minima : at origin

$$f_{min}(x) = 0$$

For Boundedness of set A : Bounded iff $\exists\ r > 0$ s.t. $\forall\ x \in A$

$$\|x\| \leq r$$

$$\Rightarrow \quad x_1^2 + x_2^2 \leq 25 \quad \forall\ x \in A \subseteq \mathbb{R}^2$$

$$\Rightarrow \quad \|x\| = \sqrt{x_1^2 + x_2^2} \leq \sqrt{25}$$

$$\Rightarrow \quad \|x\| \leq 5 \quad \forall\ x \in A \subseteq \mathbb{R}^2$$

$\therefore$ set A is Bounded

For set A to be a closed set : if it contains all its limit points, if a sequence $\{x_n\}$ in A converges to a point $x \in \mathbb{R}^2$ then $x \in A$

let $\{x_n\}$ be a convergent sequence in A

$\Rightarrow \quad x_n = (x_{1n}, x_{2n}) \rightarrow x = (x_1, x_2)$ as $n \rightarrow \infty$

as $\quad x_{1n}^2 + x_{2n}^2 \leq 25$

$$\Rightarrow \quad \lim_{n \to \infty} (x_{1n}^2 + x_{2n}^2) = x_1^2 + x_2^2 \leq 25$$

$\Rightarrow \quad x \in A \quad \therefore$ set A containts all its limit points

$\Rightarrow$ set A is closed

Q5

Sol^n : Convex hull : let set C, convex hull of set C is set of all convex combinations of points in C

⇒ (+1)

⇒ $\text{conv}\, C = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in C,\ \theta_i \geq 0,\ i = 1 \text{ to } k,\ \sum_{i=1}^{k} \theta_i = 1 \right\}$

⇒ (+0.5)

(+1)

set $A = \{ (1,0,0), (0,1,0), (0,0,1) \}$

$$\text{conv}\, A = \left\{ \theta_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \theta_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \theta_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; \theta_1 + \theta_2 + \theta_3 = 1 \right\}$$

(+0.5)

$$\boxed{\text{conv}\, A = \left\{ \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \;\middle|\; \theta_1 + \theta_2 + \theta_3 = 1 \right\}}$$

(+1.5)

" OR "

conv A is the triangle formed by these points as vertices

$$\therefore \boxed{\text{conv}\, A = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0 \right\}}$$

⇒ (+2)

Q6

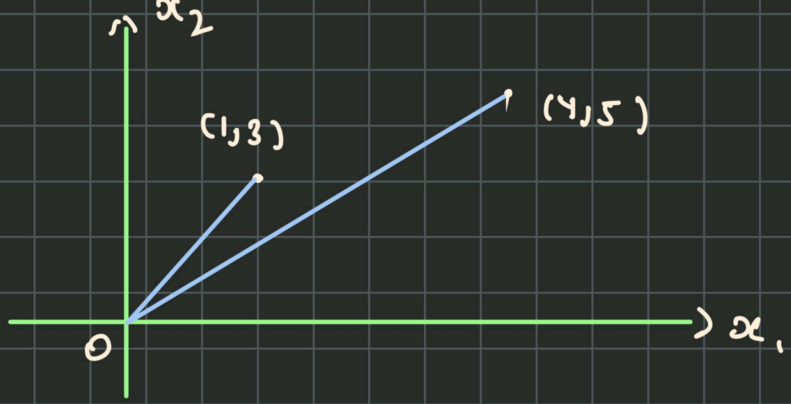
Sol^n: * Cone : a set C is called cone, if for every $x \in C$ & $\theta \geq 0$ we have $\theta x \in C$

* convex cone : for any $x_1, x_2 \in C$ & $\theta_1 \geq 0, \theta_2 \geq 0$

$\Rightarrow \quad \theta_1 x_1 + \theta_2 x_2 \in C$

$\Rightarrow$ (+1)

set $A = \{ \theta_1 x_1 + \theta_2 x_2 \mid x_1 = (4,5),\ x_2 = (1,3);\ \forall\ \theta_1, \theta_2 \geq 0 \}$



(+1)

since for set A, any $x \in A$, $\theta \geq 0$

$\Rightarrow \quad \theta x \in A \Rightarrow$ set A is a cone

$x_1 \in A$, $x_2 \in A$ as $\theta_1 x_1 + \theta_2 x_2 \in A$, $\theta_1, \theta_2 \geq 0$

$\Rightarrow$ set A is a convex cone.

$\therefore$ set A is a convex cone

(+1)