Tutorial - 6

Ques 1: Let $X_1 \ldots X_n$ be a random sample from $N(\theta, \sigma^2)$ (σ^2 is known). Show that likelihood ratio test (LRT) for testing $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$ rejects null hypothesis if $\frac{\sqrt{n}(\bar{X}-\theta_0)}{} \geq c_1$. Further find the value of c_1 for $\alpha = 0.05$

Null Hypothesis : $H_0: \theta \leq \theta_0$ Alternate Hypotheses : $H_1: \theta > \theta_0$

$$X_i \sim N(\theta, \sigma^2) \quad (\sigma^2 \text{ known})$$

So,

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2}$$

for LRT, we have $\lambda(x) = \dfrac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)}$ $\dfrac{\text{(Restricted MLE)}}{\text{(UnRestricted MLE)}}$

Unrestricted MLE : $-\infty < \theta < \infty$

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

$$\log L(\theta) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2$$

$$\frac{\partial}{\partial\theta} \log(L(\theta)) = 0 \Rightarrow 0 - \frac{1}{2\sigma^2} \times 2\sum_{i=1}^{n}(x_i-\hat{\theta}) = 0$$

($\because \sigma^2$ known)

$\sum_{i=1}^{} x_i - \hat{\theta} = 0$ $\quad (x_1 + x_2 + \cdots x_n) - n\hat{\theta} = 0$

$$\boxed{\hat{\theta} = \bar{x}}$$

we already know this tho!

⇒ Restricted MLE : $\Theta_0 : \theta \le \theta_0$

$$L(\theta) = \frac{1}{(2\pi\sigma^2)^{n/2}} \times e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta)^2}$$

↳ this would get cancelled in $\lambda(x)$ in Numerator & Denominator. since σ^2 is known.

now, here $\theta \le \theta_0$ and above we saw when θ is unrestricted $\hat{\theta} = \bar{x}$

so if CASE 1 : $\bar{x} \le \theta_0$ in that case

Unrestricted MLE $(\hat{\theta}) = \bar{x}$

CASE 2 : $\bar{x} > \theta_0$. Here we can't say $\bar{x}$ would be MLE for unrestricted parameter space

∴ Max val you can take here is θ_0 ∵ the only critical pt while calculating we found was $\bar{x}$ which is out of range here & ∴

$$\hat{\theta}_{MLE} \text{ for } \theta \in \Theta_0 = \theta_0$$

↳ $\boxed{\theta \le \theta_0}$

you can see it this way

$$\mathcal{L}(\theta) = \frac{1}{(2\pi\sigma^2)^{n/2}} \times e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

Here if you minimise $\sum_{i=1}^{n}(x_i-\theta)^2$, $\mathcal{L}(\theta)$ is maximized.

$\bar{x} > \theta_0$

$$\sum(x_i-\theta)^2 = \sum(x_i-\bar{x})^2 + n(\bar{x}-\theta)^2$$

$\therefore$ we take max + θ_0 so that this is minimized

$$\therefore \lambda(x) = \frac{\max_{\theta \in \Theta_0} \mathcal{L}(\theta) \hookrightarrow \text{MLE}(\theta_0)}{\max_{\theta \in \Theta} \mathcal{L}(\theta) \hookrightarrow \text{MLE}(\bar{X})}$$

$$\lambda(x) = \frac{\cancel{1/(\sqrt{2\pi\sigma^2})^n}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta_0)^2}}{\cancel{1/(\sqrt{2\pi\sigma^2})^n}\, e^{-1/2\sigma^2 \sum(x_i-\bar{X})^2}}$$

$$= e^{1/2\sigma^2\left(\sum_{i=1}^{n}(x_i-\bar{X})^2 - \sum_{i=1}^{n}(x_i-\theta_0)^2\right)}$$

for the rejection space $\{x : \lambda(x) \le c\}$ $0 \le c \le 1$

$$\lambda(x) \le c$$

$$\exp\left(-\frac{1}{2\sigma^2}\left(\sum(x_i-\bar{x})^2 - \sum(x_i-\theta_0)^2\right)\right) \le c$$

$$\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\bar{X})^2 - \underbrace{\sum_{i=1}^{n}(x_i-\theta_0)^2}_{\sum_{i=1}^{n}(x_i-\bar{X})^2 + n(\bar{X}-\theta)^2}\right) \le c_1$$

so,

$$-\frac{n(\bar{X}-\theta)^2}{\sigma^2} \le c_1 \implies \frac{\sqrt{n}(\bar{X}-\theta)}{\sigma} \ge c_1$$

now

to find the value of c_1 here at $\alpha = 0.05$ we'll use power function.

$$\beta(\theta) = P_\theta(X \in RR) \quad [\theta \in \Theta_0]$$

$P_\theta(X \in RR \mid H_0 \text{ true})$

$\theta \in \Theta_0$

$$= P_\theta\left(\sqrt{n}\left(\frac{\bar{X} - \theta_0}{\sigma}\right) \geq c_1\right)$$

$$= P_\theta\left(\frac{\bar{X} - \theta + \theta - \theta_0}{\sigma/\sqrt{n}} \geq c_1\right)$$

$$\boxed{Z\text{stat} = \frac{\bar{X} - \theta}{\sigma/\sqrt{n}}}$$

$$= P_\theta\left(Z + \frac{\theta - \theta_0}{\sigma/\sqrt{n}} \geq c_1\right)$$

$$= P_\theta\left(Z \geq c_1 - \left(\frac{\theta - \theta_0}{\sigma/\sqrt{n}}\right)\right)$$

we know α = level of significance = P(Type I error)

$$P_\theta\left(Z \geq c_1 - \left(\frac{\theta - \theta_0}{\sigma/\sqrt{n}}\right)\right) = 0.05$$

Ques 28 In general, LRT depends on a MSS. Let $X_1, \dots, X_n$ be random sample from Bernoulli(p). Then, $T = \sum X_i$ is a MSS for p. The 'T' follows Binomial (n, p). Consider n is known & p is unknown, Using the likelihood function of T perform LRT for

$H_0 : p \leq p_0$ vs $H_1 : p > p_0$

Null hypothesis $H_0 : p \leq p_0$ Alt Hypo & $H_1 : p > p_0$

$$L(p) = {}^nC_t \, p^t \, (1-p)^{n-t} \quad (\text{stat } T = \sum X_i \text{ Binomial})$$

Unrestricted MLE &

$$T = \sum X_i \sim \text{Binomial}(n, p) \quad \begin{pmatrix} n \to \text{known} \\ p \to \text{unknown} \end{pmatrix}$$

$$\log L(p) = \log {}^nC_t + t \log p + (n-t) \log (1-p)$$

$$\frac{\partial}{\partial p} \log L(p) = \frac{t}{p} - \frac{(n-t)}{1-p} = 0$$

$$t(1-\hat{p}) = \hat{p}(n-t)$$

$$t - \cancel{\hat{p}t} = \hat{p}n - \cancel{\hat{p}t} \quad \boxed{\hat{p} = t/n}$$

$$\boxed{\hat{p} = \frac{\sum X_i}{n} = \bar{X}}$$

Restricted MLE: H_0 $\Theta_0 : p \le p_0$

similar to previous question, our critical pt is 't/n' or '$\bar{x}$', so,

CASE 1: $\hat{p}_{MLE} = t/n$ or $\bar{x}$ $\left(\frac{t}{n} \le p_0\right)$

CASE 2: $(t/n > p_0)$ $\hat{p}_{MLE} = p_0$

take

$$\log L(p_0) = t\log p_0 + (n-t)\log(1-p_0)$$

$$\log L(t/n) = t\log(t/n) + (n-t)\log(1-t/n)$$

now

$$\log L(p_0) - \log L(t/n) = t\log p_0 + (n-t)\log(1-p_0) - t\log\left(\frac{t}{n}\right) - (n-t)\log\left(1-\frac{t}{n}\right)$$

$$\frac{\partial}{\partial t}\left(\log L(p_0) - \log L(t/n)\right) = \log p_0 + (-1)\log(1-p_0) - \left(\log\frac{t}{n} + \frac{n}{t}\times\frac{t}{n}\right) - \left((-1)\log\left(1-\frac{t}{n}\right) - \frac{(n-t)}{1-t/n}\times\frac{1}{n}\right)$$

$$= \log\left(\frac{p_0 \times (1-t/n)}{(1-p_0)\times t/n}\right)$$

$\because$ $t/n > p_0$ and $1-p_0 > 1-t/n$

$$\frac{p_0(1-t/n)}{(1-p_0)\,t/n} < 1 \qquad \therefore \log(<1) < 0$$

$\therefore$ for $t/n > p_0$, $\log L(p_0) - \log(t/n) < 0$

so, In restricted param space, $p \leq p_0$ when $t/n > p_0$ (i.e critical pt out of space), the fn is increasing since for $t/n > p_0$

$$\log(t/n) - \log L(p_0) > 0$$

$\therefore$ Increasing

& hence $\boxed{\hat{p}_{MLE} = p_0}$

CASE 1: $\lambda(X) = 1$ CASE 2: $\dfrac{\cancel{{}^nC_t}\, p_0^t (1-p_0)^{n-t}}{\cancel{{}^nC_t} \left(\frac{t}{n}\right)^t \left(1-\frac{t}{n}\right)^{n-t}}$

Rejection Region: $\{ \lambda(x) \leq c \}$

$$\frac{p_0^t (1-p_0)^{n-t}}{(t/n)^t \left(1-\frac{t}{n}\right)^{n-t}}$$

Ques 36 Let $X_1 \cdots X_n$ be a random sample

$$f_\theta(x) = \frac{1}{\theta} e^{-x/\theta} \quad (0 < x < \infty) \; (\theta > 0)$$

Show likelihood ratio test (LRT) for testing $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$ rejects the null hypothesis if $\overline{X} \leq C_1$ or if $\overline{X} \geq C_2$.

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\Sigma x_i}$$

Unrestricted MLE 6

$$\log L(\theta) = -n \log \theta - \frac{\Sigma x_i}{\theta}$$

$$\frac{\partial}{\partial \theta} \log L(\theta) = -\frac{n}{\theta} - \frac{1}{\theta^2}(\Sigma x_i) = 0$$

$$\boxed{\hat{\theta} = \overline{X}}$$

Restricted MLE 6 $\theta \in \Theta_0 = \{\theta_0\}$

So, straight away this would be $\hat{\theta}_{MLE} = \theta_0$
(Rest.)

So,

$$\lambda(x) = \frac{L(\theta_0)}{L(\bar{x})} = \frac{(1/\theta_0)^n e^{-\Sigma x_i/\theta_0}}{(1/\bar{x})^n e^{-\Sigma x_i/\bar{x}}}$$

$$= \left(\frac{\bar{x}}{\theta_0}\right)^n e^{\frac{\Sigma x_i}{\bar{x}} - \frac{\Sigma x_i}{\theta_0}} = \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n - \frac{\Sigma x_i}{\theta_0}}$$

<u>Rejection Region</u> $\{x : \lambda(x) \leq c\}$ $0 < c \leq 1$

$\lambda(x) \leq c$ $\quad \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n - \frac{\Sigma x_i}{\theta_0}} \leq c$

$$\underbrace{\left(\frac{\bar{x}}{\theta_0}\right)^n}\underbrace{\left(e^{n - \frac{n\bar{x}}{\theta_0}}\right)} \leq c \qquad (\Sigma x_i = n\bar{x})$$

Both entities are +ve

$\therefore$ $\left(\frac{\bar{x}}{\theta_0}\right) \leq c$ or $e^{n - \frac{n\bar{x}}{\theta_0}} \leq c$

$\boxed{\bar{x} \leq c_1}$ or $n - \frac{n\bar{x}}{\theta_0} \leq c$

$$\boxed{\bar{x} > c_2}$$

Ques 48 Let $X_1 \ldots X_n$ be a random sample from $N(\theta, \sigma^2)$, θ, σ^2 unknown. The LRT for testing $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$ rejects Null hypothesis if $T = \frac{\sqrt{n}(\bar{x}-\theta_0)}{S} \ge c_1$ & $T \sim t(n-1)$. Find the value of c_1 at $\alpha = 0.05$.

$H_0: \boxed{\theta \le \theta_0} = \Theta_0$ vs $H_1: \theta > \theta_0$

$$L(\theta) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

Here θ & σ^2 both unknown!!

Unrestricted MLE:

$\theta_{MLE} = \bar{X}$ & $\left\{ \sigma^2_{MLE} = \frac{1}{n}\sum(x_i-\bar{x})^2 \right\}$

this is a nuisance param we just use this to calculate θ_{MLE}!!

Restricted MLE:

Similar way, we'll have 2 cases here:

$\bar{X} \le \theta_0$ & $\bar{X} > \theta_0$

$\hat{\theta}_{MLE} = \bar{X} \Rightarrow \lambda(X) = 1$ $\quad$ $\hat{\theta}_{MLE} = \theta_0$!

So, $0 \le c \le 1$

$$\beta(\theta) = P_\theta(X \in RR) = P_\theta(\{x : \lambda(x) \le c\})$$

$$= P_\theta\left(\frac{\cancel{(1/2\pi\sigma^2)^n}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta_0)^2}}{\cancel{(1/2\pi\sigma^2)^n}\, e^{-1/2\sigma^2 \sum(x_i-\bar{x})^2}} \le c\right\}$$

$$= P_\theta\left(e^{\frac{1}{2\sigma^2}\left(\sum(x_i-\bar{x})^2 - \sum(x_i-\theta_0)^2\right)} \le c\right\}$$

$$= P_\theta\left(e^{\frac{1}{2\sigma^2}\left(\cancel{\sum(x_i-\bar{x})^2} - \cancel{\sum(x_i-\bar{x})^2} - n(\bar{x}-\theta_0)^2\right)} \le c\right\}$$

$$= P_\theta\left(e^{\frac{1}{2\sigma^2} - n(\bar{x}-\theta_0)^2} \le c\right)$$

<u>Note:</u> Here σ^2 isn't just σ^2. Since it is unknown & $\lambda(x) = \dfrac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)}$, you've to set the value of σ^2 to $\sigma^2_{MLE} = \frac{1}{n}\sum(x_i-\bar{x})^2$ but it'd be same for numerator & denom both.

$\therefore \Theta_0 = \{\theta \le \theta_0,\ \sigma^2 > 0\}$

$\Theta = \{-\infty < \theta < \infty,\ \sigma^2 > 0\}$

$$= P_\theta\left(\frac{1}{\frac{1}{n}\sum(x_i-\bar{x})^2} - n(\bar{x}-\theta_0)^2 \le c_1 \right)$$

$$= P_\theta\left(\frac{\sqrt{n}(\bar{x}-\theta_0)}{S} \geq c_2 \right)$$

t-stat

$$\beta(\theta) = P\left(t_{(n-1)} \geq c_2 \right) = \alpha = 0.05$$

$$c_2 = t_{n-1,\,0.05}$$

Ques58 Suppose we have 2 independent samples e

$X_1 - - - X_n \to$ exponential(θ)

$Y_1 - - - - Y_m \to$ exponential (μ).

a) Find LRT of $H_0: \theta = \mu$ vs $H_1: \theta \neq \mu$

$$\mathcal{L}_x(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x/\theta} = \theta^{-n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}$$

$$\mathcal{L}_y(\mu) = \prod_{i=1}^{m} \frac{1}{\mu} e^{-x/\mu} = \mu^{-m} e^{-1/\mu \sum_{i=1}^{m} x_i}$$

so

$$\mathcal{L}(\theta, \mu) = \mathcal{L}_x(\theta) \mathcal{L}_y(\mu)$$
$$= \theta^{-n} \mu^{-m} e^{-\frac{\sum_{}^{n} x_i}{\theta} - \frac{\sum^{m} y_i}{\mu}}$$

Now

for this likelihood function

$$\log \mathcal{L}(\theta, \mu) = -n\log\theta - m\log\mu - \frac{\sum^{n} x_i}{\theta} - \frac{\sum^{m} y_i}{\mu}$$

so

$$\frac{\partial}{\partial\theta} \log \mathcal{L}(\theta, \mu) = \frac{-n}{\theta} - \frac{\sum x_i}{\theta^2} \Rightarrow \boxed{\theta_{MLE} = \overline{X}}$$

Similarly

$$\frac{\partial}{\partial \mu} \log \mathcal{L}(\theta|\mu) = -\frac{m}{\mu} - \frac{\sum y_i}{\mu^2} \Rightarrow \boxed{\mu_{MLE} = \bar{y}}$$

Above is the case when we have unrestricted MLE. (Refer to the Note at the end of Ques

<u>In case of restricted MLE i-e when $\theta = \mu$</u>

$$\mathcal{L}(\theta) = \theta^{-n+m} e^{-\frac{\sum^n x_i + \sum^m y_i}{\theta}}$$

$$\log \mathcal{L}(\theta) = -(n+m)\log\theta - \frac{\left(\sum^n x_i + \sum^m y_i\right)}{\theta}$$

$$\frac{\partial}{\partial\theta} \log \mathcal{L}(\theta) = -\frac{(n+m)}{\theta} + \frac{1}{\theta^2}\left(\sum^n x_i + \sum^m y_i\right) = 0$$

$$\Rightarrow \boxed{\theta_{MLE} = \frac{\sum^n x_i + \sum^m y_i}{n+m}}$$

So,

$$\mathcal{L}(\theta)_{\text{unrestricted}} = (\bar{x})^{-n} (\bar{y})^{-m} e^{-n} e^{-m}$$

$$L(\theta)_{\text{Restricted}} = \left(\frac{n\bar{x}+m\bar{y}}{m+n}\right)^{-m+n} e^{-n+m}$$

so,

$$\lambda(x) = \frac{\left(\frac{n\bar{x}+m\bar{y}}{m+n}\right)^{-m+n}}{(\bar{x})^{-n}(\bar{y})^{-m}}$$

$$= \left(\frac{\sum^{n} x_i + \sum^{m} y_i}{m+n}\right)^{-m+n} \times \frac{n^{-n} m^{-m}}{(\sum x_i)^{-n}(\sum y_i)^{-m}}$$

$$= \frac{(m+n)^{m+n}}{m^{n} n^{n}} \times \frac{(\sum^{n} x_i)^{n} (\sum^{m} y_i)^{m}}{(\sum^{n} x_i + \sum^{m} y_i)^{n}}$$

b) Show test can be based on statistic

$$T = \frac{\sum x_i}{\sum x_i + \sum y_i}$$

$$\lambda(x) = \frac{(m+n)^{m+n}}{m^m \; n^n} \times \frac{\left(\sum^{n} X_i\right)^n}{\left(\sum^{n} X_i + \sum^{m} Y_i\right)^n} \times \frac{\left(\sum^{m} Y_i\right)^m}{\left(\sum^{n} X_i + \sum^{m} Y_i\right)^m}$$

we know $T = \dfrac{\sum X_i}{\sum X_i + \sum Y_i}$ and $1-T = \dfrac{\sum Y_i}{\sum X_i + \sum Y_i}$

$$\therefore \lambda(x) = \frac{(m+n)^{m+n}}{m^m \; n^n} \times T^n \times (1-T)^m$$

$\hookrightarrow$ function of T ($m, n \rightarrow$ constants)

c) Find the distribution of T when H_0 is true.

When H_0 is true $\Rightarrow \theta = \mu$

$\therefore X_1 \cdots X_n \sim$ Exponential(θ)

$Y_1 \cdots Y_n \sim$ Exponential $(\mu = \theta)$

$\sum_{i=1}^{n} X_i \sim$ Gamma(n, θ)

$\sum_{i=1}^{m} Y_i \sim$ Gamma(m, θ)

$\therefore \dfrac{\Sigma X_i}{\Sigma X_i + \Sigma Y_i} \sim \text{Beta}(n, m)$

Note :

In (a) part when calculating MLE of θ and μ, we can't just calculate first derivatives & declare it local maximum. we have to see second order derivatives too. In case of 2 diff samples, here we'll make a Hessian matrix denoted by $H = \begin{bmatrix} \dfrac{\partial^2 L}{\partial \mu^2} & \dfrac{\partial^2 L}{\partial \mu \partial \theta} \\ \dfrac{\partial^2 L}{\partial \theta \partial \mu} & \dfrac{\partial^2 L}{\partial \theta^2} \end{bmatrix}$

Here $H = \begin{bmatrix} -n/\bar{x}^2 & 0 \\ 0 & -m/\bar{y}^2 \end{bmatrix}$

Criteria is :

1) If the Hessian is negative definite at a point, it indicates local maximum
2) positive definite $\to$ local minimum
3) indefinite : saddle point.

to check for +ve definite, -ve definite or indefinite, check eigenvalues or leading principal minors. (there are other ways too, these two are simpler ones!).

Ques66 Find the LRT of $H_0: \theta \le 0$ vs $H_1: \theta > 0$ based on a random sample $X_1 \cdots X_n$ from a population with PDF.

$$f_{\theta,\lambda}(x) = \frac{1}{\lambda} e^{-\frac{(x-\theta)}{\lambda}} I_{(\theta,\infty)}(x)$$

Here the parameters θ and λ are unknown.

$$H_0: \theta \le 0 \quad \text{vs} \quad H_1: \theta > 0$$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\lambda} e^{-\frac{(x-\theta)}{\lambda}} = \frac{1}{\lambda^n} e^{-\frac{1}{\lambda}\sum_{i=1}^{n} x_i - \theta}$$

$$= \lambda^{-n} e^{-\frac{1}{\lambda}\sum_{i=1}^{n}(x_i-\theta)} \prod_{i=1}^{n} I_{(\theta,\infty)}(x_i)$$

now $I_{(\theta,\infty)}(x_i) = \begin{cases} 1 & \text{if } x_i \in (\theta,\infty) \\ 0 & \text{o/w} \end{cases}$

$$= \begin{cases} 1 & x_i > \theta \\ 0 & \text{o/w} \end{cases}$$

for $\prod_{i=1}^{n} I_{(\theta,\infty)}(x_i)$ to be nonzero, we must ha[ve] the smallest x_i: $\min(X_1, X_2 \cdots X_n) = X_{(1)} > \theta$

so,

$$\prod_{i=1}^{n} I_{\theta,\infty}(x_i) = I(X_{(1)} > \theta)$$

so,

$L(\theta) = \lambda^{-n} e^{-\frac{1}{\lambda}\sum_{i=1}^{n}(X_i-\theta)} I(X_{(1)} > \theta)$

so,

<u>Unrestricted MLE</u> & (θ, λ) both unknown.

to maximise $L(\theta)$, $X_{(1)} \geq \theta$ so that $L(\theta)$ is nonzero firstly. $(\theta \leq X_{(1)})$

If we do $\frac{\partial}{\partial\theta} L(\theta)$ we do not get any critical pts. so, here if you see, at $X_{(1)}$ $\Sigma(X_i - \theta)$ value is min. . so, on this parameter space $\theta \leq X_{(1)}$ to be non zero $L(\theta)$, $\hat{\theta}_{MLE} = X_{(1)}$

now taking this $\hat{\theta}_{MLE} = X_{(1)}$ we can calculate λ_{MLE} here

$L(\theta) = \lambda^{-n} e^{-\frac{1}{\lambda}\Sigma X_i - \theta} I(X_{(1)} > \theta)$

$\log L(\theta) = -n\log\lambda - \frac{1}{\lambda}\sum_{i=1}^{n}(X_i - \theta)$

$$\frac{\partial}{\partial \lambda} \log \mathcal{L} = -\frac{n}{\cancel{\lambda}} + \frac{1}{\lambda^{\cancel{2}}} \sum_{i=1}^{n} (X_i - \theta_{MLE}) = 0$$

$$\frac{1}{\lambda_{MLE}} \sum_{i=1}^{n} (X_i - X_{(1)}) = n$$

$$\boxed{\lambda_{MLE} = \frac{1}{n} \sum X_i - X_{(1)}}$$

Restricted MLE of Θ_0 : $\theta \leq 0$

$$\mathcal{L}(\theta) = \lambda^{-n} e^{-\frac{1}{\lambda} \sum_{i=1}^{n} (X_i - \theta)} I(X_{(1)} > \theta)$$

$\theta < X_{(1)}$ and $\theta \leq 0$

$\therefore \theta_{MLE} = 0$ since it already is less than $X_{(1)}$ & if we move away from 0, $\sum (X_i - \theta)$ would increase! $\therefore$ $\mathcal{L}(\theta)$ decrease!

$\therefore$ set $\theta_{MLE} = 0$ to find λ_{MLE}.

$$L(\lambda) = \frac{1}{\lambda^n} \exp\left(-\frac{1}{\lambda}\sum X_i\right)$$

$$\log L(\lambda) = -n\log\lambda - \frac{1}{\lambda}\sum X_i$$

$$\frac{\partial}{\partial\lambda}\log L(\lambda) = -\frac{n}{\lambda} - \frac{\sum X_i}{\lambda^2} \Rightarrow \boxed{\lambda_{MLE} = \bar{X}}$$

So, find $\lambda(x)$

$X_{(1)}$ — $\bar{X} - X_{(1)}$

$$\lambda(x) = \frac{\max_{\theta\in\Theta_0} L(\theta, \lambda)}{\max_{\theta\in\Theta} L(\theta, \lambda)}$$

$\hookrightarrow 0$ $\hookrightarrow \bar{X}$