

Solution Hint — Robotics Quiz 1

September 16, 2025

Notation

We use the following standard notation for planar manipulators:

- $R(\alpha)$ denotes the 2×2 rotation matrix by angle α (radians or degrees when clearly stated).
- For a 2D homogeneous transform we use 3×3 matrices of the form $\begin{bmatrix} R & p; & 0 & 1 \end{bmatrix}$.
- l_i denotes link lengths, θ_i joint angles. Forward kinematics (position) for a 2-link planar arm (links along x when joint angle zero) with $\phi = \theta_1 + \theta_2$:

$$x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2), \quad y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2).$$

- Jacobian (position part) for 2-link planar arm:

$$J = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}.$$

Q1

Solution. We apply the transforms in the given order to the point P using homogeneous transforms. Let $R(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ and let $T(t_x, t_y)$ denote the translation by (t_x, t_y) .

Using column-vector convention for points,

$$P' = R(45^\circ) P,$$

$$P'' = P' + \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$P_f = R(-45^\circ) P''.$$

Compute numerically (using $\cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2} \approx 0.70710678$):

$$R(45^\circ)P = \begin{bmatrix} 0.70710678 & -0.70710678 \\ 0.70710678 & 0.70710678 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.70710678 \\ 2.12132034 \end{bmatrix},$$

$$P'' = \begin{bmatrix} 0.70710678 \\ 2.12132034 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.70710678 \\ 4.12132034 \end{bmatrix},$$

$$P_f = R(-45^\circ)P'' = \begin{bmatrix} 0.70710678 & 0.70710678 \\ -0.70710678 & 0.70710678 \end{bmatrix} \begin{bmatrix} 1.70710678 \\ 4.12132034 \end{bmatrix} = \begin{bmatrix} 4.12132034 \\ 1.70710678 \end{bmatrix}.$$

Thus the final coordinates are

$$P_f \approx (4.1213, 1.7071).$$

None of the options are correct. (you will get appropriate marks based on the correctness of your derivation. .)

Q2

Solution. Given the orientation constraint $\theta_1 + \theta_2 = 90^\circ$, we have $\cos(\theta_1 + \theta_2) = 0$ and $\sin(\theta_1 + \theta_2) = 1$. The FK simplifies to

$$x = \cos \theta_1 + 0 = \cos \theta_1, \quad y = \sin \theta_1 + 1.$$

For $x = 0$ we need $\cos \theta_1 = 0 \Rightarrow \theta_1 = 90^\circ$ or 270° . If $\theta_1 = 90^\circ$ then $y = \sin 90^\circ + 1 = 2 \neq \sqrt{2}$. If $\theta_1 = 270^\circ$ then $y = -1 + 1 = 0 \neq \sqrt{2}$. Thus there is *no solution* satisfying all three constraints.

No solution.

Q3

Solution. An obstacle occupying the disk of radius 1 centered at $(2, 0)$ removes from the reachable workspace any end-effector positions that lie inside that disk. Thus the region non-reachable is precisely the disk $\{(x, y) \mid (x - 2)^2 + y^2 \leq 1^2\}$.

The region covered by the obstacle disk centered at $(2, 0)$ of radius 1.

This is the correct answer from the given choices, however, the actual unreachable region is slightly bigger than this!! Think, why???

Q4

Solution. We use the standard planar Jacobian (position part):

$$J = \begin{bmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}.$$

Compute trigonometric values: $\sin 90^\circ = 1$, $\cos 90^\circ = 0$. Also $\theta_1 + \theta_2 = 270^\circ$ so $\sin 270^\circ = -1$, $\cos 270^\circ = 0$. Thus

$$J = \begin{bmatrix} -2 - (-1) & -(-1) \\ 0 + 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

This matrix has one nonzero row, so $\text{rank}(J) = 1$.

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Q5

Solution. For the 2-link planar arm the determinant of the position Jacobian equals

$$\det J = l_1 l_2 \sin \theta_2.$$

Hence with $l_1 = l_2 = 1$ and $\theta_2 = 150^\circ$,

$$\det J = \sin 150^\circ = \frac{1}{2}.$$

$\det J = 0.5.$

Q6

Solution. Using the serial product of transforms (rotation about origin then translation along local x by link length, etc.), the net orientation is

$$R_0^2 = R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2) = R(180^\circ) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

End-effector position (column) is

$$p = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 90^\circ + \cos 180^\circ \\ \sin 90^\circ + \sin 180^\circ \end{bmatrix} = \begin{bmatrix} 0 - 1 \\ 1 + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So the homogeneous transform (expressing frame-2 w.r.t frame-0) is

$$T = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$T = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

None of the options are correct. (you will get appropriate marks based on the correctness of your derivation. .)

Q7

Solution. With $\phi = \theta_1 + \theta_2 = 0$, we get $\cos \phi = 1$, $\sin \phi = 0$, and the FK reduces to

$$x = 2 \cos \theta_1 + 1, \quad y = 2 \sin \theta_1.$$

Setting $x = 2$ gives $2 \cos \theta_1 + 1 = 2 \Rightarrow \cos \theta_1 = \frac{1}{2}$. Setting $y = 1$ gives $2 \sin \theta_1 = 1 \Rightarrow \sin \theta_1 = \frac{1}{2}$. There is no single angle with both sine and cosine equal to $1/2$ (that would violate $\sin^2 + \cos^2 = 1$), so the pose is not feasible.

Not feasible.

Q8

Solution.

$$v_0^1 = -R_0^1 v_1^0, \quad v_1^0 = [-l \sin \theta; l \cos \theta] \dot{\theta} \text{ and hence, } v_0^1 = [0; -l\dot{\theta}].$$

Not achievable.

Summary of final answers

1. $P_f \approx (4.1213, 1.7071)$.
2. No solution (pose unattainable).
3. Region occupied by the obstacle disk $(x - 2)^2 + y^2 \leq 1$.
4. $\text{rank}(J) = 1$.

5. $\det J = 0.5$.

6. $T = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$

7. Not feasible.

8. Not achievable.