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ME469: Residual-based Advection Stabilization

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Review of Advection Stabilization Options

For the high Peclet number use case, on unstructured meshes, we have identified three strategies:

1. Ad hoc blending between upwind and central using an indicator that is a function of cell Peclet number:

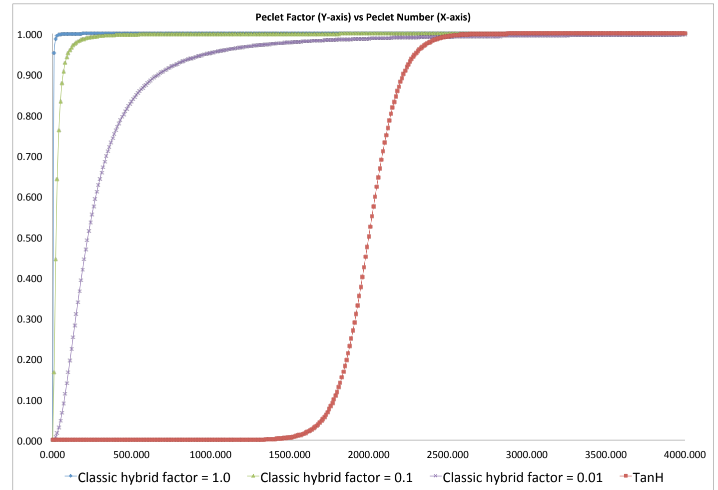
$$\phi_{ip} = \eta \phi^{FOU} + (1 - \eta)\, \phi^{CDS}$$

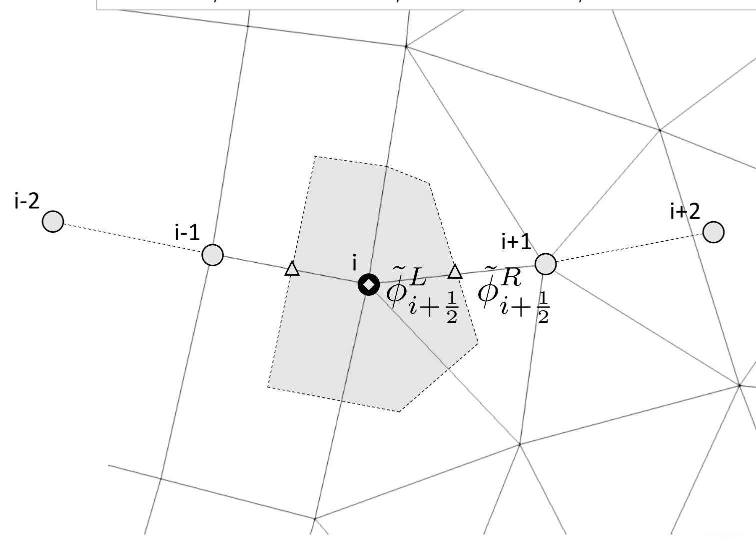
1. Gradient reconstruction (upwind) approaches with limiters

$$\tilde{\phi}^{L}_{i+\frac{1}{2}} = \phi_i + \Phi^L \Delta x^L_j G_j \phi_i,$$
$$\tilde{\phi}^{R}_{i+\frac{1}{2}} = \phi_{i+1} - \Phi^R \Delta x^R_j G_j \phi^{i+1}$$

2. Monotonic upstream-centered scheme for conservation laws (MUSCL) - again, with limiters

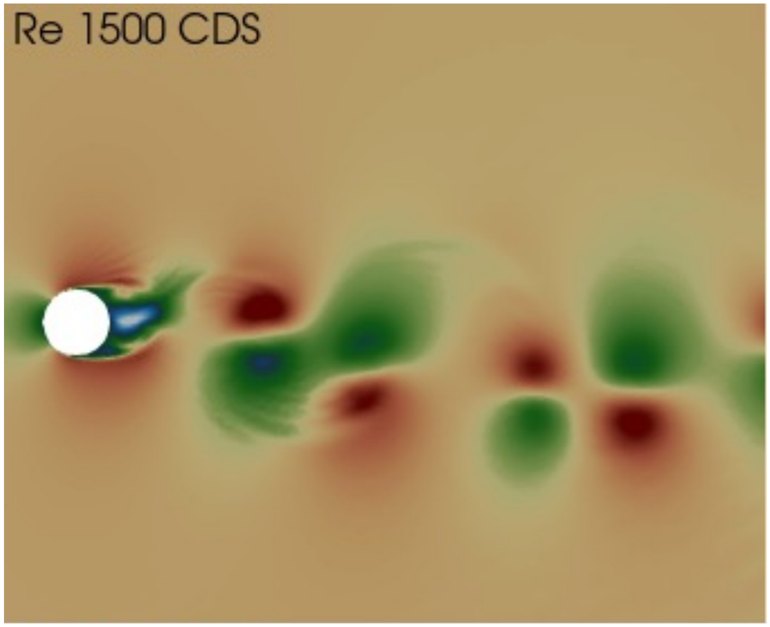
$$\phi_{i+\frac{1}{2}} = \phi^{LOW}_{i+\frac{1}{2}} - \Phi(r_{i+\frac{1}{2}}) \left(\phi^{LOW}_{i+\frac{1}{2}} - \phi^{HIGH}_{i+\frac{1}{2}} \right)$$

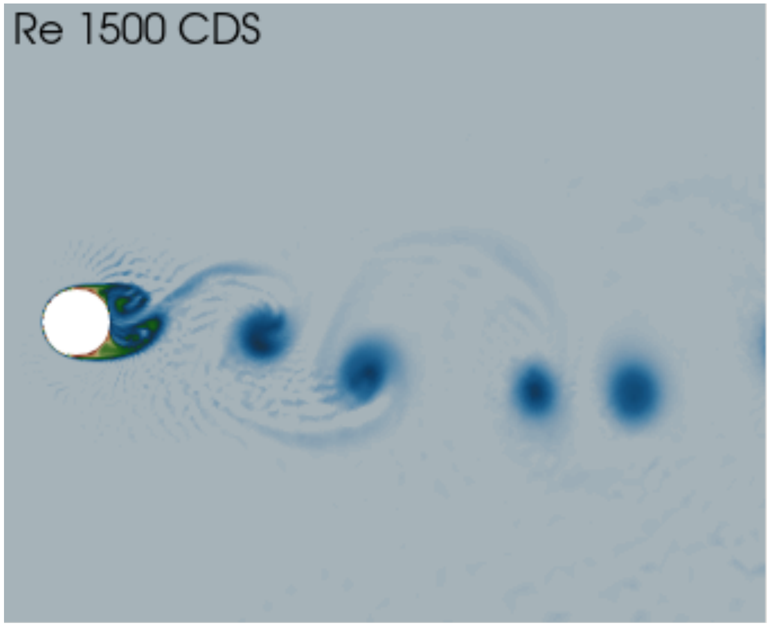




Momentum Field is Tolerant of low-Dissipation Advection Operators

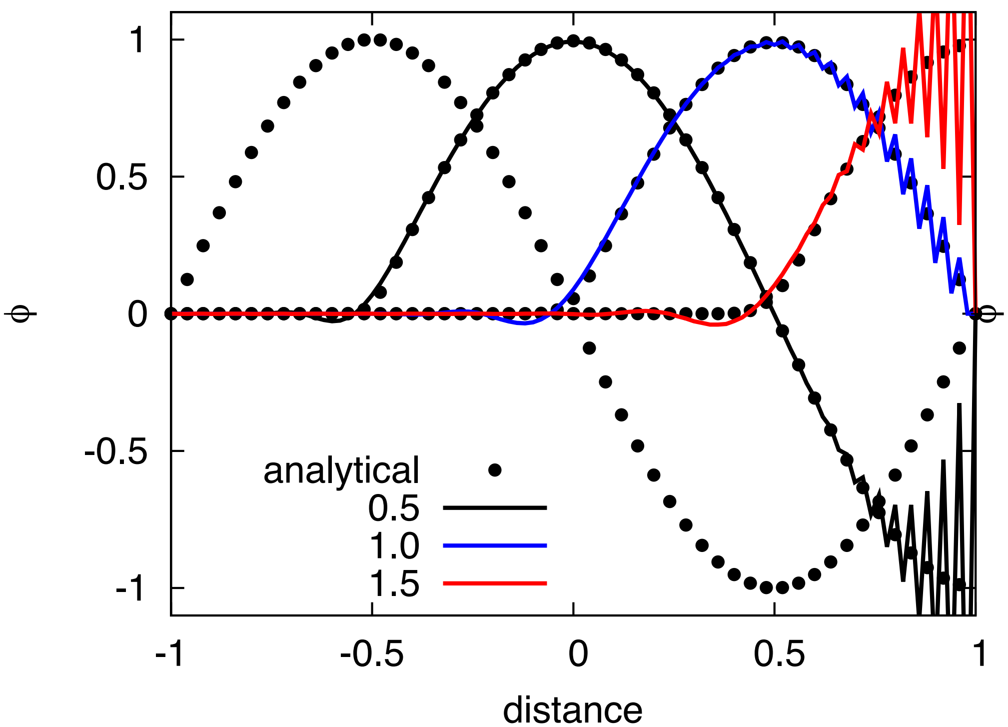
.... While passive scalars are not



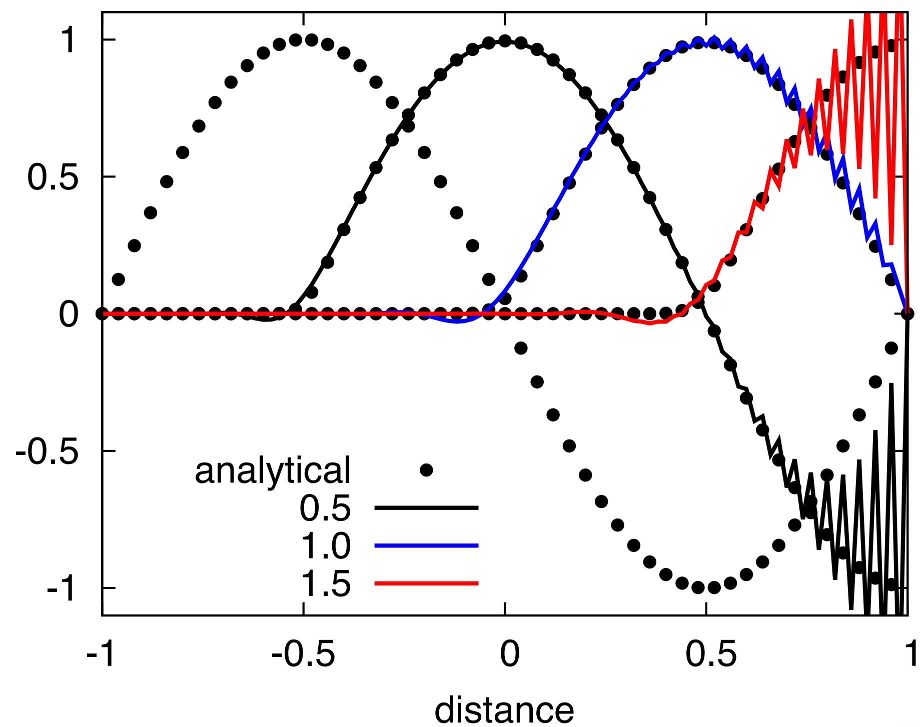


Re = 2000 (local Re 20) – No Stabilization

Analytical solution from Mojtabi and Deville, One-dimensional linear advection–diffusion equation: Analytical and finite element solutions, Comput. Fluids 107 (2015) 189–195.



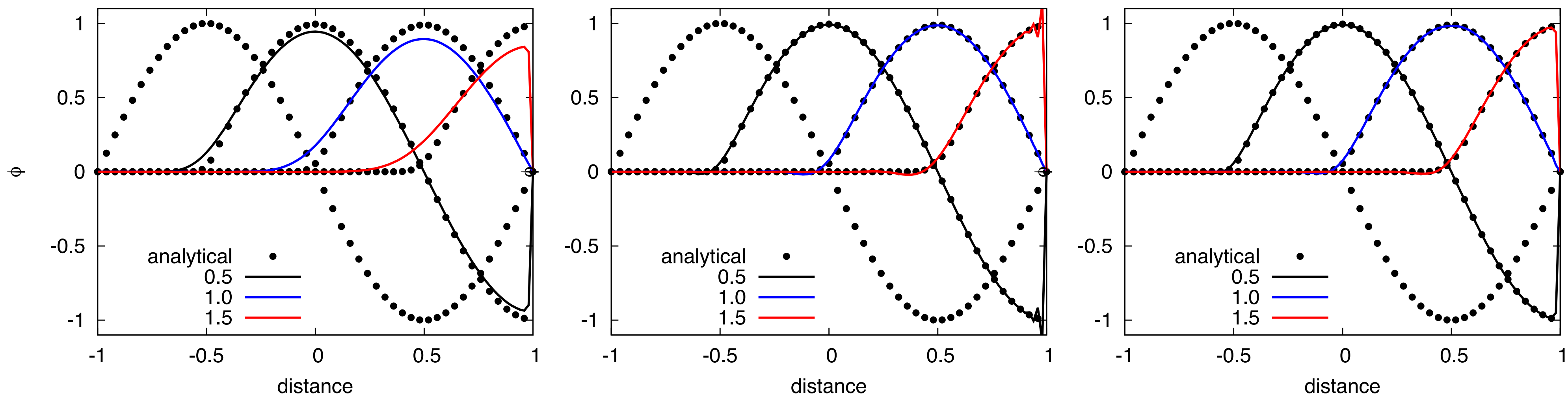
Linear Basis (CVFEM), central/Galerkin



Quadratic Basis (CVFEM), central/Galerkin

Re = 2000 (local Re 20) – Upwind

Analytical solution from Mojtabi and Deville, One-dimensional linear advection–diffusion equation: Analytical and finite element solutions, Comput. Fluids 107 (2015) 189–195.



First-order upwind | Second-order upwind | Second-order upwind + limiter

Recall a Strategy: Central + Diffusion

A general advection system can be written in terms of a central-based scheme and a diffusion term using an effective viscosity

$$\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}+\frac{U}{2\Delta x}\left(\phi_{j+1}^{n+1}-\phi_{j-1}^{n+1}\right)-\nu^{eff}\left(\frac{\phi_{j+1}^{n+1}-2\phi_j^{n+1}+\phi_{j-1}^{n+1}}{\Delta x^2}\right)$$

$$\nu^{eff}=\frac{|U|\Delta x}{2}$$

→ Simply re-casting a first-order upwind scheme

$$\nu^{eff}=\frac{\Delta t U^2}{2}$$

→ Lax-Wendroff, System of Conservation Laws, LA-2285 (1960)
Similar to Taylor-Galerkin

The goals are to be:

1. Consistent, i.e., as you refine the mesh, you revert to the desired PDE
2. Design-order, i.e., converges as Δx^{p+1}

Residual Definition

On a given mesh, h, if we were to evaluate the residual, what would we obtain?

$$R^h(\phi) = \frac{\partial \rho \phi^h}{\partial t} + \frac{\partial \rho u_j \phi^h}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\frac{\mu}{Sc}\frac{\partial \phi^h}{\partial x_j}\right) - S^{\phi^h} \propto O(\Delta x^{p+1})$$

As we refine the mesh, the evaluation of this "fine-scale" residual would approach zero – at least for a consistent discretization approach whose truncation error is well-behaved

A classic residual-based stabilization, Streamwise Upwind Petrov-Galerkin (Hughes and Brooks, 1982) – more generalized within the Variational Multiscale Method (VMS, Hughes et. al, 1998)

$$\int_\Omega \tilde{w} R^h(\phi) d\Omega = 0 \qquad \tilde{w} = w + \tau^h u_j \frac{\partial w}{\partial x_j} \qquad g^{ij} = \frac{\partial x_i}{\partial \xi_k}\frac{\partial x_j}{\partial \xi_k},$$

$$\tau^h = \beta\left[(\frac{2}{\Delta t})^2 + u_i g_{ij} u_j + c\nu^2 g_{ij} g_{ij}\right]^{-\frac{1}{2}} \qquad g_{ij} = \frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_k}{\partial x_j}$$

Element-metrics

$\beta \sim 1$; $c \sim O(10)$

SUPG Stabilization in a Finite Volume Context

Finite Element Context:

$$\int_\Omega w \frac{\partial \rho\phi}{\partial t} d\Omega - \int_\Omega \frac{\partial w}{\partial x_j} F_j d\Omega + \int_\Gamma w F_j n_j d\Gamma - \int_\Omega w S_\phi d\Omega + \sum_{elem} \int_\Omega \tau^h u_k \frac{\partial w}{\partial x_k} R(\phi^h) d\Omega = 0$$

With:

$$F_j = \rho u_j \phi - \frac{\mu}{Sc} \frac{\partial \phi}{\partial x_j}$$

Finite Volume Context by applying the piecewise constant test function:

$$\int_\Omega \frac{\partial \rho\phi}{\partial t} d\Omega + \int_\Gamma F_j n_j d\Gamma - \int_\Omega S_\phi d\Omega - \sum_{elem} \int_\Gamma \tau^h u_k R(\phi^h) n_k d\Gamma = 0$$

Note: Godunuv's theorem states that a linear stabilization approach is not sufficient to damp out all oscillations

Assuming we have a design-order representation of the residual, the method is consistent

SUPG Stabilization in a Finite Volume Context

Finite Element Context:

$$\int_\Omega w\frac{\partial \rho\phi}{\partial t}d\Omega - \int_\Omega \frac{\partial w}{\partial x_j}F_j d\Omega + \int_\Gamma wF_jn_jd\Gamma - \int_\Omega wS_\phi d\Omega$$
$$+\sum_{elem}\int_\Omega \tau^h u_k \frac{\partial w}{\partial x_k}R(\phi^h)d\Omega = 0$$

With:

$$F_j = \rho u_j\phi - \frac{\mu}{Sc}\frac{\partial \phi}{\partial x_j}$$

Finite Volume Context by applying the piecewise constant test function:

$$\int_\Omega \frac{\partial \rho\phi}{\partial t}d\Omega + \int_\Gamma F_jn_jd\Gamma - \int_\Omega S_\phi d\Omega$$
$$-\sum_{elem}\int_\Gamma \tau^h u_k R(\phi^h)n_k d\Gamma = 0$$

Note: Godunuv's theorem states that a linear stabilization approach is not sufficient to damp out all oscillations

Assuming we have a design-order representation of the residual, the method is consistent

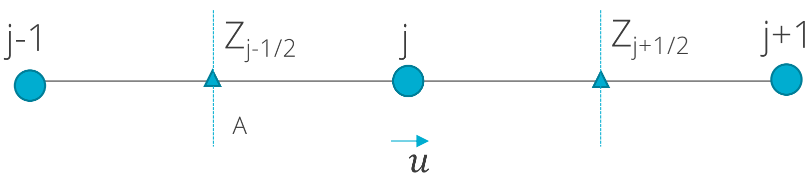
- Note: Diffusion terms, a second-order derivative require special care (projection)

Linear Residual-Based Stabilization: CVFEM closer look

For now, let's remove time, diffusion and source – and assume density and velocity are constant

$$\int_\Gamma \rho u_j \phi n_j d\Gamma - \sum_{elem} \int_\Gamma \tau^h u_k \, n_k \frac{\partial \rho u_j \phi}{\partial x_j} d\Gamma = 0 \longrightarrow \sum_{ip} \dot{m}_{ip}\phi_{ip} - \tau^h u_k A_k^{ip} \rho u_j \left.\frac{\partial \phi}{\partial x_j}\right._{ip} = 0$$

Recall central and upwind stencils for our standard 1D configuration (flow left to right)



$$\frac{\dot{m}}{2}(\phi_{j+1} - \phi_{j-1}) - \tau^h u \rho u \frac{(\phi_{j+1} - 2\phi_j + \phi_{j-1})}{\Delta x} A$$

- General interpretation: looks like diffusion with coefficient, $\tau u \rho u$, where τ is a flow-aligned time scale (s)

If we allow Let $\tau = \frac{\Delta x}{2|u|}$, then we recover upwind!

In general, the key is to include the full PDE residual

Linear Residual-Based Stabilization: Consistency

The residual-based stabilization is given by,

$$\int_\Omega \frac{\partial \rho\phi}{\partial t} d\Omega + \int_\Gamma F_j n_j d\Gamma - \int_\Omega S_\phi d\Omega - \sum_{elem} \int_\Gamma \tau^h u_k R(\phi^h) n_k d\Gamma = 0$$

with the fine-scale residual defined as,

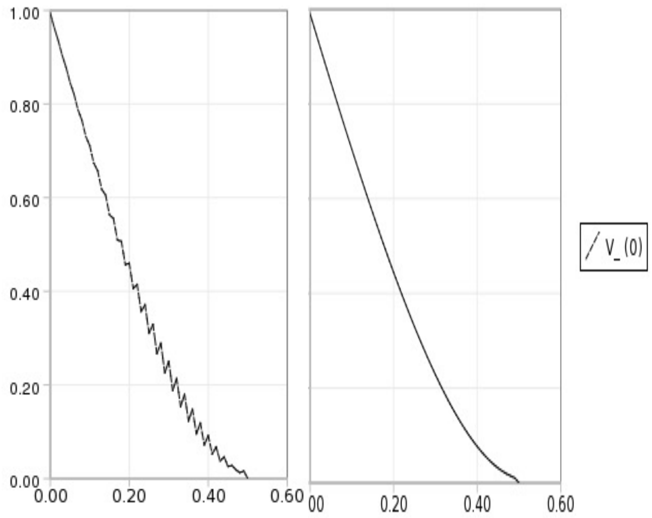
$$R^h(\phi) = \frac{\partial \rho\phi^h}{\partial t} + \frac{\partial \rho u_j \phi^h}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\frac{\mu}{Sc}\frac{\partial \phi^h}{\partial x_j}\right) - S^{\phi^h} \propto O(\Delta x^{p+1})$$

Comments:

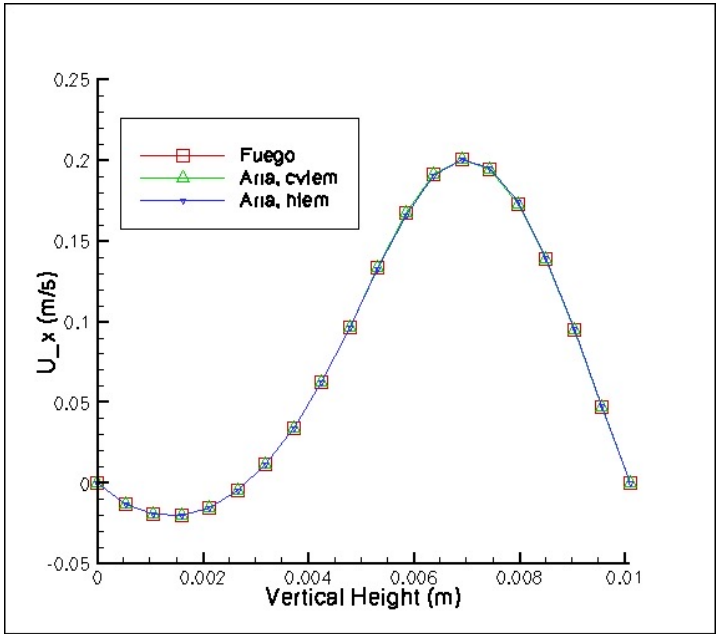
1. If the fine-scale residual is consistently evaluated, then as the mesh is refined the residual-based stabilization is removed
2. Recall that this is a linear stabilization scheme and may still admit oscillations
3. In the pure Finite Element Method description, this form of stabilization is a Petrov-Galerkin approach

SUPG in Practice (CVFEM and FEM)

SUPG and the analog in CVFEM, Streamwise Upwind Control Volume (SUCV) are effective for low-moderate cell-Peclet numbers



Un-stabilized and SUPG Convecting TV



SUPG/SUCV (Aria, hfem; Aria cvfem) back step velocity prediction compared to upwind (Fuego)

SUCV Generalized to Other Systems

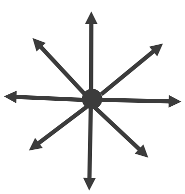
Recall the radiative transport equation – derived in a variational multiscale (VMS) context, Domino et al., Phys. Fluids (2021)

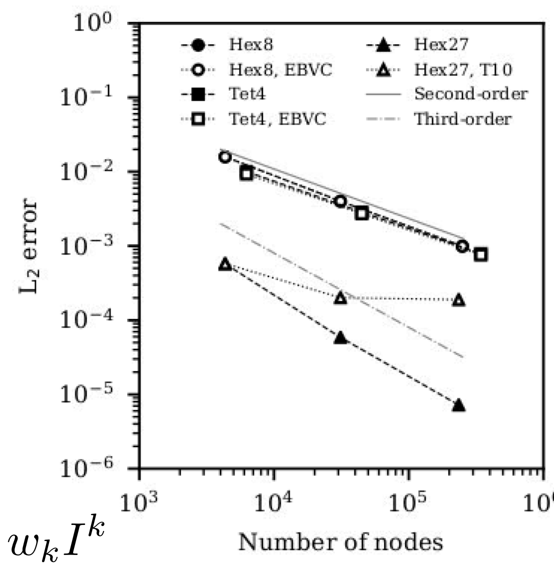
$$\int_{\Omega} I\left(s\right) s_i n_i \mathrm{d}S + \int_{\Gamma} s_i I\left(s\right) n_i \mathrm{d}S$$

$$+ \int_{\Omega} \left((\alpha_a + \alpha_s) I\left(s\right) - \frac{\alpha_a \sigma T^4}{\pi} - \frac{\alpha_s}{4\pi} G \right) \mathrm{d}V$$

$$- \int_{\Omega} \tau^h R^h(s) s_i n_i \mathrm{d}S + \alpha \int_{\Omega} \tau^h (\alpha_a + \alpha_s) R^h(s) \mathrm{d}V$$

$$-\beta \int_{\Gamma} \tau^h R^h(s) s_i n_i \mathrm{d}S = 0.$$





$\alpha = \beta = 0$, SUCV!

$$s_j \frac{\partial I}{\partial x_j} + \left(\mu_a + \mu_s\right) I = \frac{\mu_a \sigma T^4}{\pi} + \frac{\mu_s G}{4\pi}$$

$$G \approx \sum_k w_k I^k$$

Nonlinear Stabilization Operator (NSO)

An artificial viscosity approach is provided with coefficient again related to a fine-scale residual, shown for both finite element and finite volume:

$$\sum_e \int_\Omega \nu^h R^h(\phi) \frac{\partial w}{\partial x_i} g^{ij} \frac{\partial \phi}{\partial x_j} d\Omega \qquad -\sum_e \int_\Gamma \nu^h R^h(\phi) n_i g^{ij} \frac{\partial \phi}{\partial x_j} d\Gamma$$

Here, we link the artificial diffusion coefficient to the fine scale residual:

$$\nu^h = \sqrt{\frac{R^h(\phi)R^h(\phi)}{\frac{\partial \phi}{\partial x_i} g^{ij} \frac{\partial \phi}{\partial x_j}}}$$

- Or min of:: $\nu^h = C \left(\rho u_i g_{ij} u_j\right)^{\frac{1}{2}}$

- Origin from FEM-based discontinuity capturing operator (DCO), Shakib et al. Comput. Method. Appl. Mech. Engr., (1991)
- Similar concept to Guermond et al. “Entropy-viscosity” approach, J. Comp. Phys. (2011)
  - Variations form an interesting LES subgrid model when the viscosity is a function of the k.e. fine scale residual (Guermond and Larios, 2015)

Re = 2000 (local Re 20) – Residual-Based

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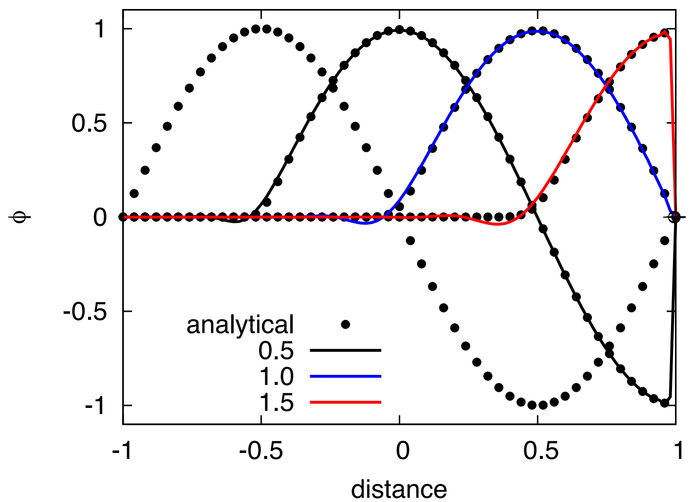
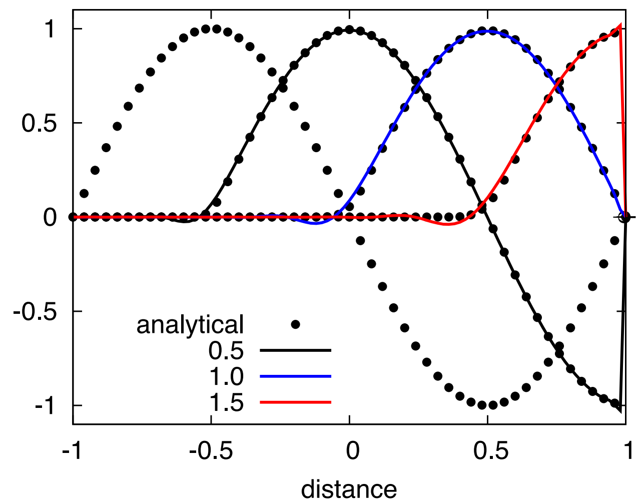
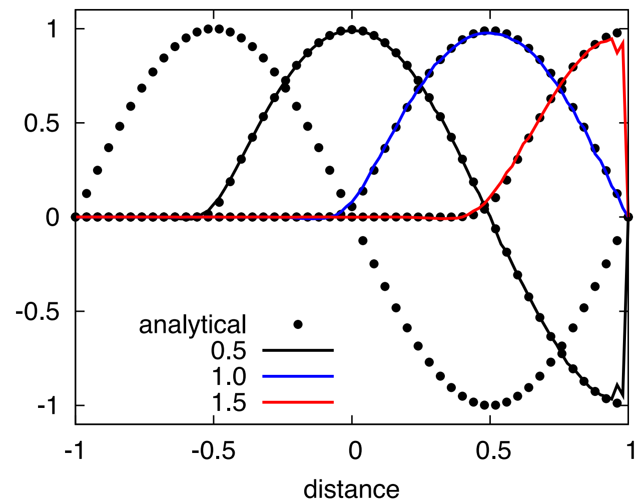


NSO, linear basis (CVFEM)



SUCV, linear basis (CVFEM)



NSO, quadratic basis (CVFEM)

Now, Direct Comparisons, Order of Accuracy

- Convecting Taylor/Vortex simulation at high crossflow velocity
- Results demonstrate the ability to deploy this stabilization approach to high-order