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ME469: Introduction to the low-Mach Number Approximation

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Consider a Variable Density, non-Isothermal Fluid Flow System

- Consider the variable density (non-isothermal) equations of motion (momentum and continuity) with energy transport:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0,$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} + \frac{\partial P}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i,$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \rho u_j H}{\partial x_j} = -\frac{\partial q_j}{\partial x_j} + \frac{\partial u_i \tau_{ij}}{\partial x_j} + \rho u_i g_i$$

(2+nDim)

Constitutive Relationships

$$E = H - P/\rho,$$

$$H = h + \frac{1}{2} u_k u_k,$$

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij},$$

$$q_i = -k \frac{\partial T}{\partial x_i},$$

$$h = \int_{T_o}^{T} C_p dT$$

$\rho = \frac{PM}{RT}$ Equation of State (EOS) provides the P and ρ relationship

- See Paolucci (1982) or Baum (1978) for the low-Mach pedigree
- Number of Equations = (3+nDim) = Number of unknowns

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Degree of Freedom, Properties, and Constitutive Count

DOF: (3+nDim)

- Density, ρ (Continuity Eq)
- Pressure, P (EOS)
- Velocity, u_i (Momentum Eq)
- Total energy, E (Energy Eq)

Properties:

- Viscosity, μ
- Specific heat, C_p
- Thermal conductivity, λ

Constitutive Relationships:

- Ideal gas law
  - Again, provides density/pressure relationship (a key concept)
- Newtonian stress, τ_{ij}
- Heat flux vector, q_j
- Total enthalpy, H
- Static enthalpy, h
- $dh/dT = C_p$

Dimensionless Form

- Non-dimensionalization is via a characteristic velocity and length scale:

$$Re = \frac{\rho_\infty U_\infty L}{\mu_\infty}, \qquad \text{Reynolds number},$$

$$Pr = \frac{C_{p,\infty}\mu_\infty}{k_\infty}, \qquad \text{Prandtl number},$$

$$Fr_i = \frac{u_\infty^2}{g_i L}, \qquad \text{Froude number}, \quad g_i \neq 0,$$

$$Ma = \sqrt{\frac{u_\infty^2}{\gamma R T_\infty / W}} \qquad \text{Mach Number}$$

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$$\frac{\partial \bar{\rho}\bar{u}_i}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{u}_i}{\partial \bar{x}_j} + \frac{1}{\gamma Ma^2}\frac{\partial \bar{P}}{\partial \bar{x}_i} = \frac{1}{Re}\frac{\partial \bar{\tau}_{ij}}{\partial \bar{x}_j} + \frac{1}{Fr_i}\bar{\rho},$$

$$\frac{\partial \bar{\rho}\bar{h}}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{h}}{\partial \bar{x}_j} = -\frac{1}{Pr}\frac{1}{Re}\frac{\partial \bar{q}_j}{\partial \bar{x}_j} + \frac{\gamma-1}{\gamma}\frac{\partial \bar{P}}{\partial \bar{t}} + \frac{\gamma-1}{\gamma}\frac{Ma^2}{Re}\frac{\partial \bar{u}_i\bar{\tau}_{ij}}{\partial \bar{x}_j} + \bar{\rho}\bar{u}_i\frac{\gamma-1}{\gamma}\frac{Ma^2}{Fr_i} - \frac{\gamma-1}{2}Ma^2\left(\frac{\partial \bar{\rho}\bar{u}_k\bar{u}_k}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{u}_k\bar{u}_k}{\partial \bar{x}_j}\right)$$

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$$\begin{aligned}\frac{\partial \bar\rho \bar h}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j \bar h}{\partial \bar x_j} = &-\frac{1}{Pr}\frac{1}{Re}\frac{\partial \bar q_j}{\partial \bar x_j} + \frac{\gamma-1}{\gamma}\frac{\partial \bar P}{\partial \bar t} \\ &+ \frac{\gamma-1}{\gamma}\frac{Ma^2}{Re}\frac{\partial \bar u_i \bar\tau_{ij}}{\partial \bar x_j} + \bar\rho \bar u_i \frac{\gamma-1}{\gamma}\frac{Ma^2}{Fr_i} \\ &- \frac{\gamma-1}{2}Ma^2\left(\frac{\partial \bar\rho \bar u_k \bar u_k}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j \bar u_k \bar u_k}{\partial \bar x_j}\right)\end{aligned}$$

- Note, as the Mach number approaches zero, the viscous work and kinetic energy terms become negligible

Dimensionless Form

- Non-dimensionalization is via a characteristic velocity and length scale:

$$
\begin{aligned}
\frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j}{\partial \bar{x}_j} &= 0, \\
\frac{\partial \bar{\rho}\bar{u}_i}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{u}_i}{\partial \bar{x}_j} + \frac{1}{\gamma Ma^2}\frac{\partial \bar{P}}{\partial \bar{x}_i} &= \frac{1}{Re}\frac{\partial \bar{\tau}_{ij}}{\partial \bar{x}_j} + \frac{1}{Fr_i}\bar{\rho}, \\
\frac{\partial \bar{\rho}\bar{h}}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{h}}{\partial \bar{x}_j} &= -\frac{1}{Pr}\frac{1}{Re}\frac{\partial \bar{q}_j}{\partial \bar{x}_j} + \frac{\gamma - 1}{\gamma}\frac{\partial \bar{P}}{\partial \bar{t}} \\
&\quad + \frac{\gamma - 1}{\gamma}\frac{Ma^2}{Re}\frac{\partial \bar{u}_i\bar{\tau}_{ij}}{\partial \bar{x}_j} + \bar{\rho}\bar{u}_i\frac{\gamma - 1}{\gamma}\frac{Ma^2}{Fr_i} \\
&\quad - \frac{\gamma - 1}{2}Ma^2\left(\frac{\partial \bar{\rho}\bar{u}_k\bar{u}_k}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{u}_k\bar{u}_k}{\partial \bar{x}_j}\right)
\end{aligned}
$$

- However, the momentum equation notes a singularity in the scaled pressure gradient term

Dimensionless Form

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$$
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\frac{\partial \bar\rho}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j}{\partial \bar x_j} &= 0, \\
\frac{\partial \bar\rho \bar u_i}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j \bar u_i}{\partial \bar x_j} + \frac{1}{\gamma Ma^2}\frac{\partial \bar P}{\partial \bar x_i} &= \frac{1}{Re}\frac{\partial \bar\tau_{ij}}{\partial \bar x_j} + \frac{1}{Fr_i}\bar\rho, \\
\frac{\partial \bar\rho \bar h}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j \bar h}{\partial \bar x_j} &= -\frac{1}{Pr}\frac{1}{Re}\frac{\partial \bar q_j}{\partial \bar x_j} + \frac{\gamma-1}{\gamma}\frac{\partial \bar P}{\partial \bar t} \\
&\quad + \frac{\gamma-1}{\gamma}\frac{Ma^2}{Re}\frac{\partial \bar u_i \bar\tau_{ij}}{\partial \bar x_j} + \bar\rho \bar u_i \frac{\gamma-1}{\gamma}\frac{Ma^2}{Fr_i} \\
&\quad - \frac{\gamma-1}{2} Ma^2 \left(\frac{\partial \bar\rho \bar u_k \bar u_k}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j \bar u_k \bar u_k}{\partial \bar x_j}\right)
\end{aligned}
$$

Conclusions: In the limit of zero Mach number, the energy equation is simplified, while the momentum equation is not well defined and, in fact, singular

Exploration of the Pressure Singularity

To explore the singularity, write each DOF as an asymptotic series:

$$\bar{P} = \bar{p}_0 + \bar{p}_1\epsilon + \bar{p}_2\epsilon^2 \dots$$
$$\bar{u}_i = \bar{u}_{i,0} + \bar{u}_{i,1}\epsilon + \bar{u}_{i,2}\epsilon^2 \dots$$
$$\bar{T} = \bar{T}_0 + \bar{T}_1\epsilon + \bar{T}_2\epsilon^2 \dots$$

The resulting zeroth-order equations are as follows:

$$\frac{\partial \bar{\rho}_0}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j}}{\partial \bar{x}_j} = 0,$$
$$\frac{\partial \bar{\rho}_0 \bar{u}_{0,i}}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j} \bar{u}_{0,i}}{\partial \bar{x}_j} + \frac{1}{\gamma Ma^2}\left(\frac{\partial \bar{p}_0}{\partial \bar{x}_i} + \epsilon \frac{\partial \bar{p}_1}{\partial \bar{x}_i}\right) = \frac{1}{Re}\frac{\partial \bar{\tau}_{0,ij}}{\partial \bar{x}_j},$$
$$\frac{\partial \bar{\rho}_0 \bar{h}_0}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j} \bar{h}_0}{\partial \bar{x}_j} = -\frac{1}{PrRe}\frac{\partial \bar{q}_{0,j}}{\partial \bar{x}_j} + \frac{\gamma - 1}{\gamma}\frac{\partial \bar{p}_0}{\partial \bar{t}}$$

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Exploration of the Pressure Singularity: Ramifications

In order for the zeroth-order momentum equation to be well conditioned in the limit of zero Mach number, $\frac{\partial \bar{p}_o}{\partial \bar{x}_i}$ must be spatially zero with $\epsilon = \gamma Ma^2$

$$\frac{\partial \bar{\rho}_0}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j}}{\partial \bar{x}_j} = 0,$$

$$\frac{\partial \bar{\rho}_0 \bar{u}_{0,i}}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j} \bar{u}_{0,i}}{\partial \bar{x}_j} + \frac{1}{\gamma Ma^2}\left(\frac{\partial \bar{p}_0}{\partial \bar{x}_i} + \epsilon \frac{\partial \bar{p}_1}{\partial \bar{x}_i}\right) = \frac{1}{Re}\frac{\partial \bar{\tau}_{0,ij}}{\partial \bar{x}_j},$$

$$\frac{\partial \bar{\rho}_0 \bar{h}_0}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j} \bar{h}_0}{\partial \bar{x}_j} = -\frac{1}{PrRe}\frac{\partial \bar{q}_{0,j}}{\partial \bar{x}_j} + \frac{\gamma - 1}{\gamma}\frac{\partial \bar{p}_0}{\partial \bar{t}}$$

- p_0 is a constant-in-space, possibly variable-in-time thermodynamic pressure,
- p_1 is the variable is space pressure, which is also known as the “motion pressure”, p^m
- Recall, this is simply a perturbation about the full thermodynamic pressure:

$$\bar{P} = \bar{p}_0 + \bar{p}_1\epsilon + \bar{p}_2\epsilon^2 \ldots$$

Final low-Mach Equation Set

The resulting Equation set is as follows:

$$\left.\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} &= 0, \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} + \frac{\partial p^m}{\partial x_i} &= \frac{\partial \tau_{ij}}{\partial x_j} + (\rho - \rho_\circ)\, g_i, \\
\frac{\partial \rho h}{\partial t} + \frac{\partial \rho u_j h}{\partial x_j} &= -\frac{\partial q_j}{\partial x_j} + \frac{\partial P_{th}}{\partial t}
\end{aligned}\right\} \underline{\text{2+nDim}}$$

- Equation of state given by the thermodynamic pressure: $\rho = \dfrac{P_{th} M}{RT}$
- Energy transport is only required when the system modeled has a temperature difference

<u>EOS does not provide an equation for closure</u>: alternative approach is required for motion pressure!

The Final low-Mach Number Equation Set: Ramifications

- We have effectively filtered out the acoustics, i.e., the wave speed is infinitely fast
- DOF/Equation system is: ρ, p^m, u_i, h; p^t is a constant for an open domain

In practice, a functional form for the motion pressure is derived and known as a Pressure Poisson Equation (PPE):

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \rho u_i}{\partial t} + ...\right) = -\frac{\partial}{\partial x_i}\left(\frac{\partial p^m}{\partial x_i}\right)$$

With the continuity equation serving as a mass balance constraint

- Note that we have introduced an Elliptic nature of the equation set
- Momentum and other equations can be implicit of explicitly solved, however, the Pressure Poisson Equation requires an implicit solve with dedicated solvers, e.g., multi-grid methods

Why "pressure projection"?

From Domino, 2006 "*Toward verification of formal time accuracy for a family of approximate projection methods using the method of manufactured solutions*"

projection algorithm. In general, any vector can be written as a Hodge decomposition, or in terms of a vector of known divergence and a curl-free part,

$$\mathbf{F} = \mathbf{F}^{\mathbf{kd}} + \nabla\phi, \tag{7.1}$$

with the known divergence given by

$$\nabla \cdot \mathbf{F}^{\mathbf{kd}} = \mathcal{S}. \tag{7.2}$$

The Poisson system is provided by

$$\nabla \cdot \nabla\phi = \nabla \cdot \mathbf{F} - \nabla \cdot \mathbf{F}^{\mathbf{kd}} = \nabla \cdot \mathbf{F} - \mathcal{S} \tag{7.3}$$

with solution,

$$\phi = \Delta^{-1}(\nabla \cdot \mathbf{F} - \mathcal{S}), \tag{7.4}$$

and

$$\begin{aligned}
\mathbf{F}^{\mathbf{kd}} &= \mathbf{F} - \nabla\phi, && (7.5)\\
&= \mathbf{F} - \nabla(\Delta^{-1}(\nabla \cdot \mathbf{F} - \mathcal{S})), && (7.6)\\
&= (\mathbf{I} - \nabla(\Delta^{-1}\nabla\cdot))\mathbf{F} + \nabla\Delta^{-1}\mathcal{S}, && (7.7)\\
&= \mathcal{P}\mathbf{F} + \mathcal{B}, && (7.8)\\
&= \mathcal{P}^{af}\mathbf{F}. && (7.9)
\end{aligned}$$

- For a solenoidal vector, div(u) is zero and P is an "idempotent" projection, i.e., $P=P^2$
- Otherwise, P is an affine-projection operator

Why ”pressure projection”?

From Domino, 2006 “*Toward verification of formal time accuracy for a family of approximate projection methods using the method of manufactured solutions*”

The projection analysis for the equations of motion is completed by the following definitions:

$$\mathbf{F} = -\nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla \cdot \mu (\nabla \mathbf{u} + \nabla^T \mathbf{u}) + \rho \mathbf{g} + \mathbf{s}, \tag{7.10}$$

$$\mathbf{F}^{\mathbf{kd}} = \frac{\partial \rho \mathbf{u}}{\partial t}, \tag{7.11}$$

$$\nabla \cdot \mathbf{F}^{\mathbf{kd}} = -\frac{\partial^2 \rho}{\partial t^2}, \tag{7.12}$$

$$\nabla \phi = \nabla p, \tag{7.13}$$

or

$$\nabla \cdot \nabla p = \nabla \cdot \left(-\nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla \cdot \mu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u}\right) + \rho \mathbf{g} + \mathbf{s}\right) + \frac{\partial^2 \rho}{\partial t^2}. \tag{7.14}$$

Classic Pressure Poisson System

Hodge Decomposition: $$u_i = u_i^{kd} + \frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

Taking the divergence: $$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_i^{kd}}{\partial x_i} + \frac{\partial}{\partial x_i}\frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

While enforcing continuity: $$\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

Finally, the projection step: $$u_i^{n+1} = u_i - \frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

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While enforcing continuity: $$\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

Here is our equation for the motion pressure

Finally, the projection step: $$u_i^{n+1} = u_i - \frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

$$\phi = p^m$$

Thought Experiment…

- Large domain, one door, one “exit”

Maples Pavilion

Thought Experiment…

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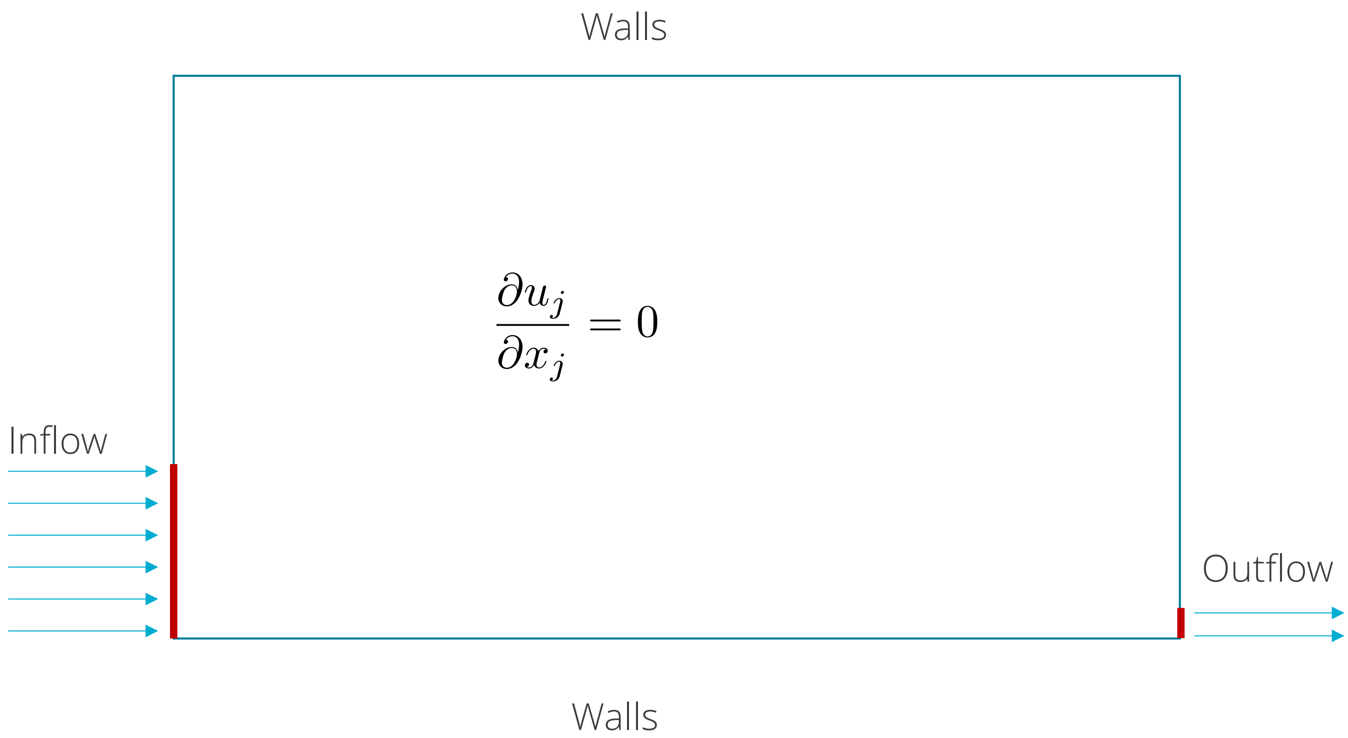

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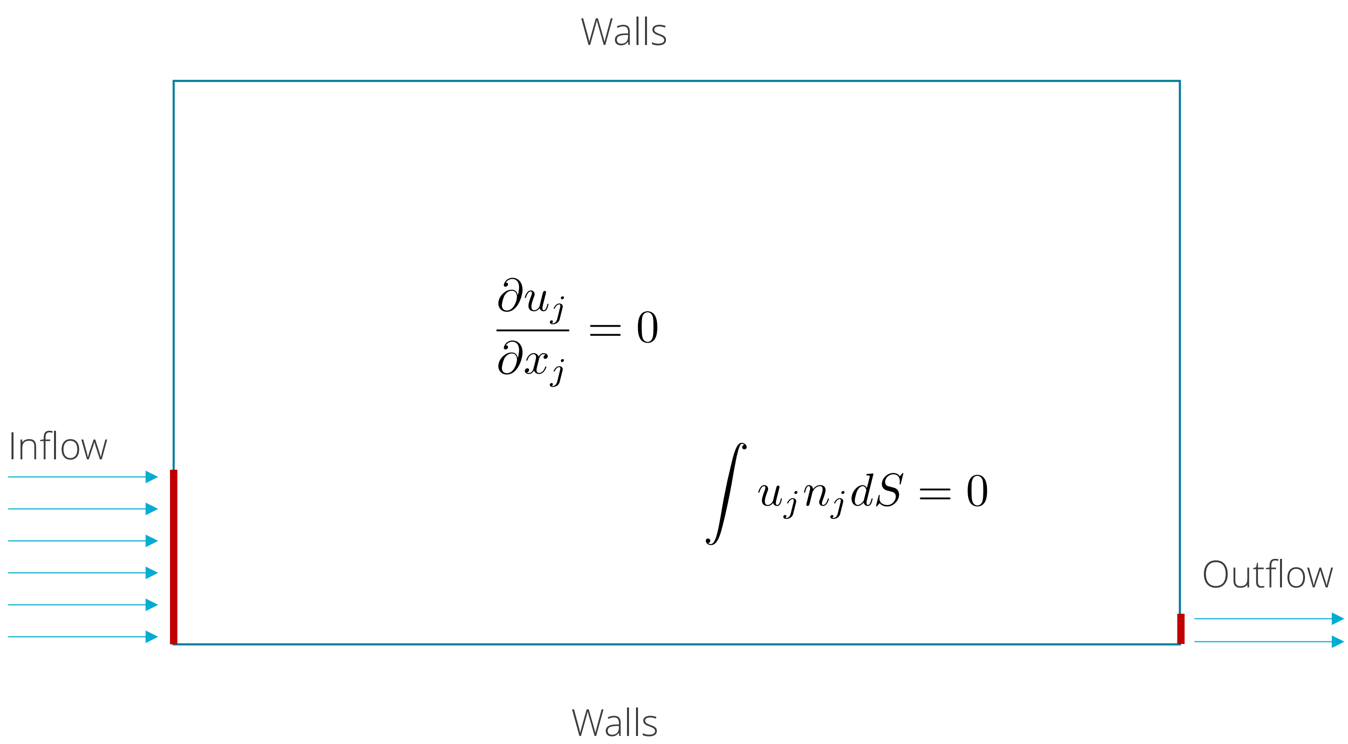

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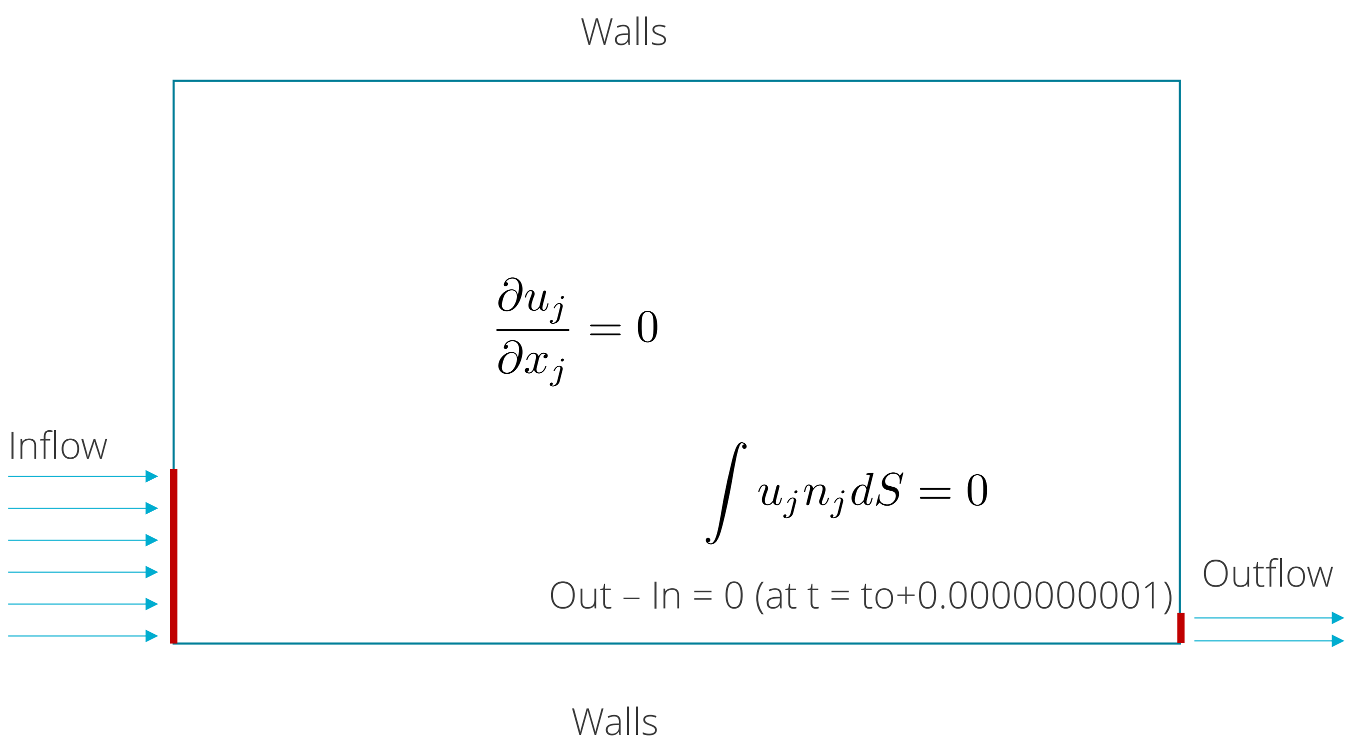
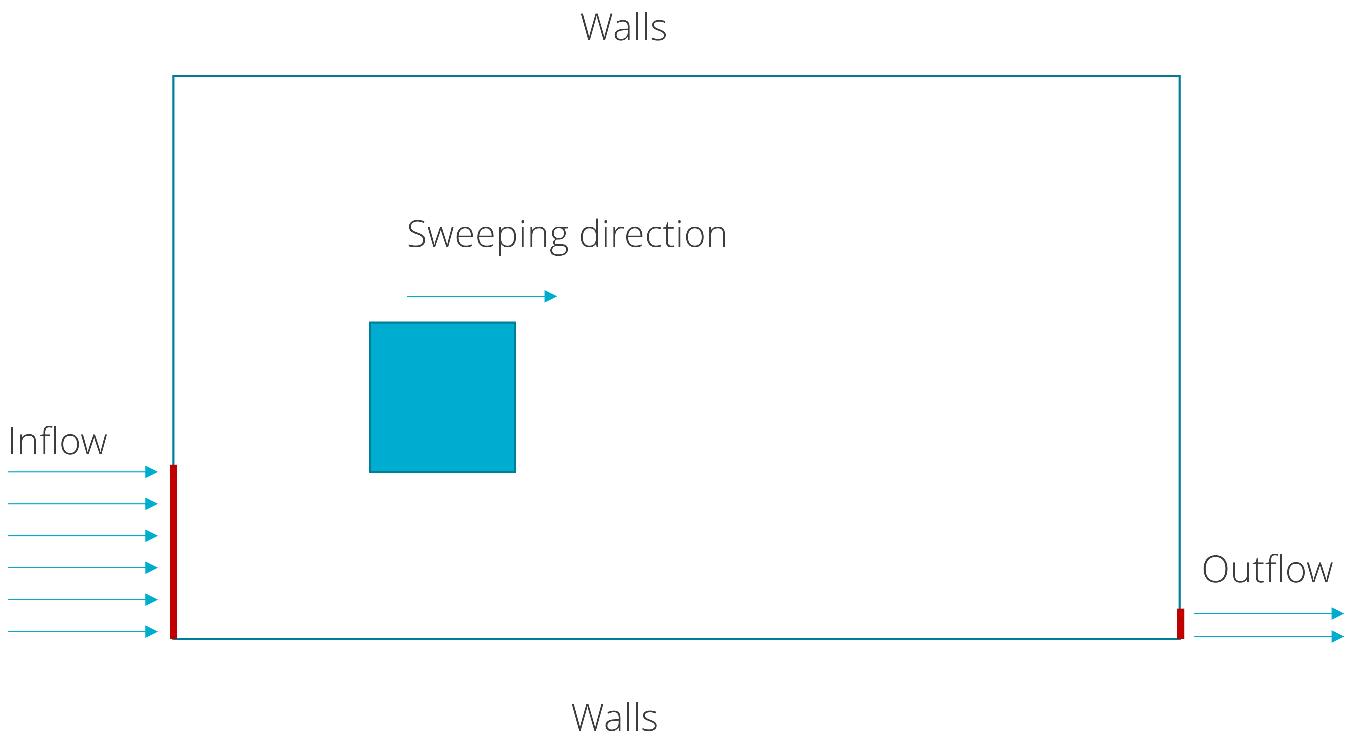


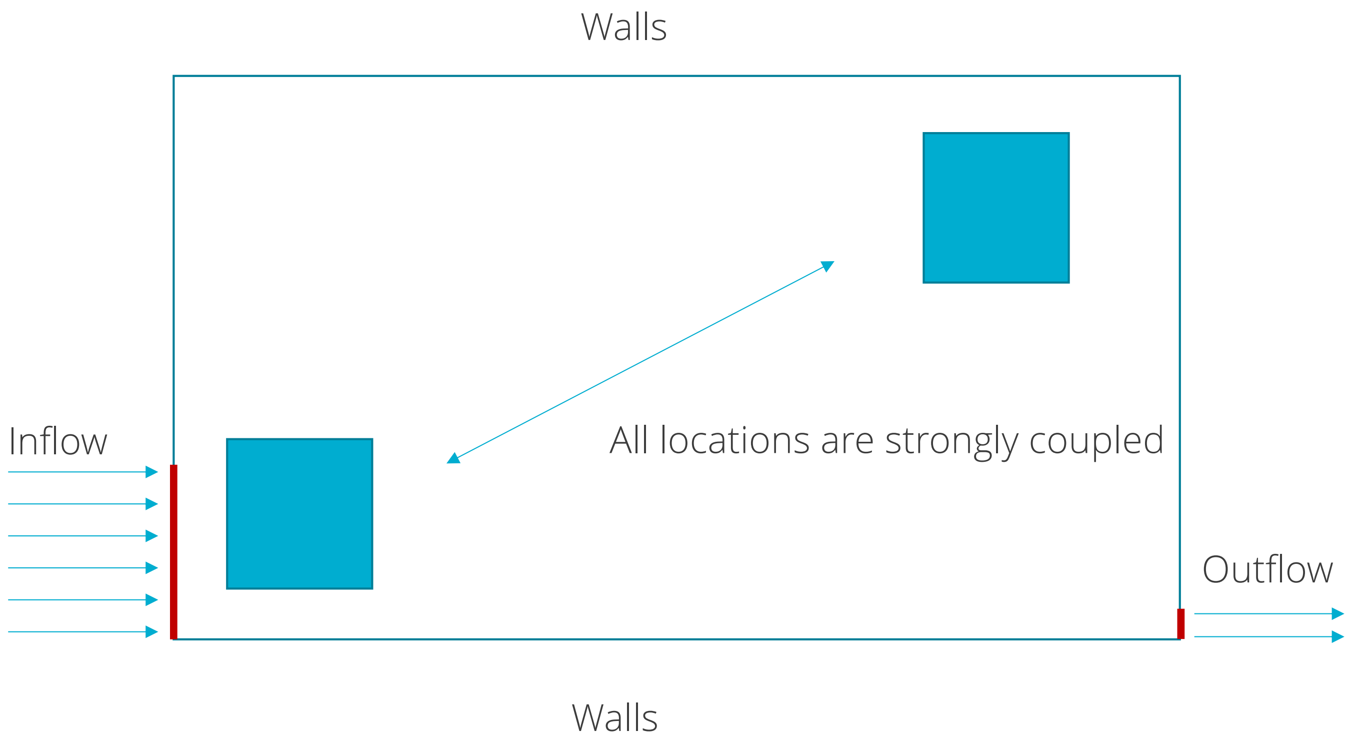
Thought Experiment… Solver Ramifications

- Fixed point solvers iterate over the domain sequentially and are effective when this sweeping of the mesh corresponds to a particular physical direction



Thought Experiment... Solver Ramifications

- For Elliptic systems, fixed point iterative solvers fail since the sequential propagation of information is not adequate for a system with infinite wave speeds



Multigrid Methods: The Concept

Briggs, "A Multigrid Tutorial, 2nd Edition" (2000)

- Fixed-point iterations schemes effectively remove high-frequency errors
- Can we take a solution, coarsen, and then solve the new system?

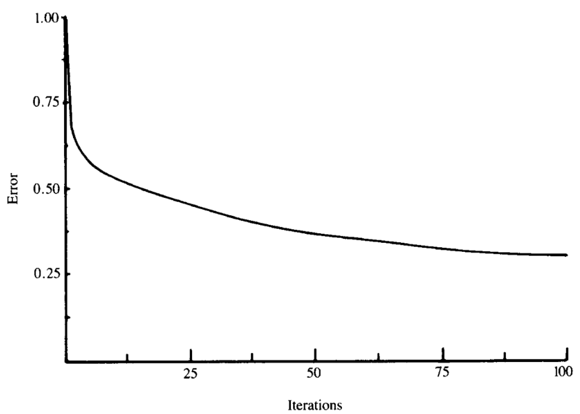


Figure 2.5: *Weighted Jacobi method with* $\omega = \frac{2}{3}$ *applied to the one-dimensional model problem with* $n = 64$ *points and an initial guess* $(\mathbf{v}_1 + \mathbf{v}_6 + \mathbf{v}_{32})/3$. *The maximum norm of the error,* $\|\mathbf{e}\|_\infty$, *is plotted against the iteration number for* 100 *iterations.*

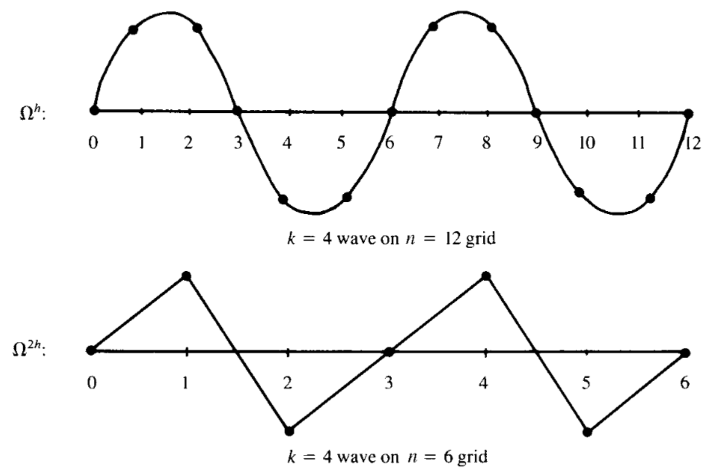


Figure 3.1: *Wave with wavenumber* $k = 4$ *on* Ω^h *(*$n = 12$ *points) projected onto* Ω^{2h} *(*$n = 6$ *points). The coarse grid "sees" a wave that is more oscillatory on the coarse grid than on the fine grid.*

Multigrid Methods: The Approach

- Multigrid methods (MG) are essential for efficient solver performance in fluids-based Elliptic systems
- In simple, structured domains, this can be geometric (GMG), while in unstructured, algebraic (AMG)

$$A\phi = f$$

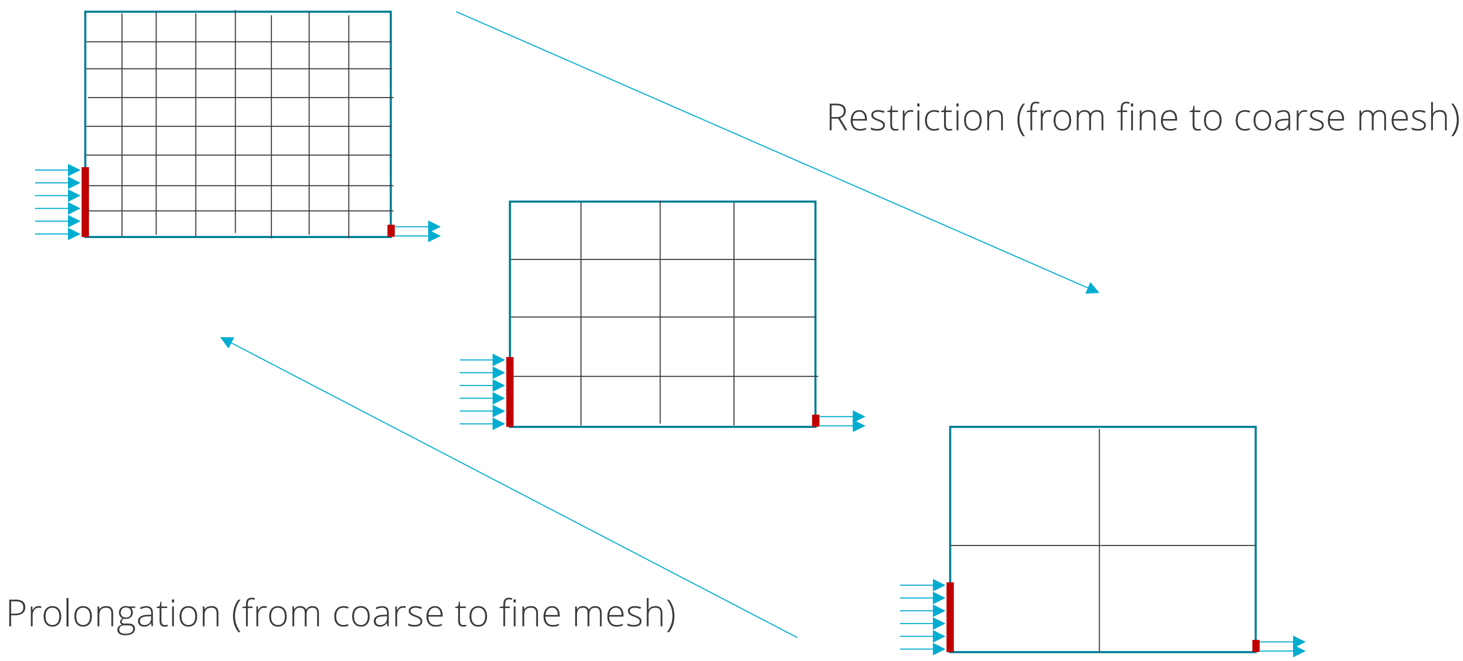
$$\epsilon = \phi - \tilde{\phi}$$

$$r = f - A\tilde{\phi}$$

$$A\left(\tilde{\phi} + \epsilon\right) = f$$

$$A\epsilon = r$$

$$\phi = \tilde{\phi} + \epsilon$$



Multigrid Methods: V- and W-Cycles

Briggs (2000)

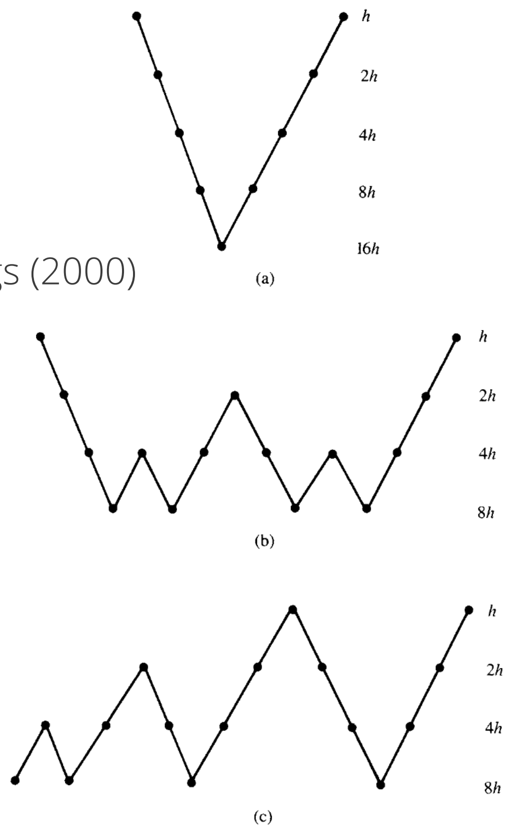
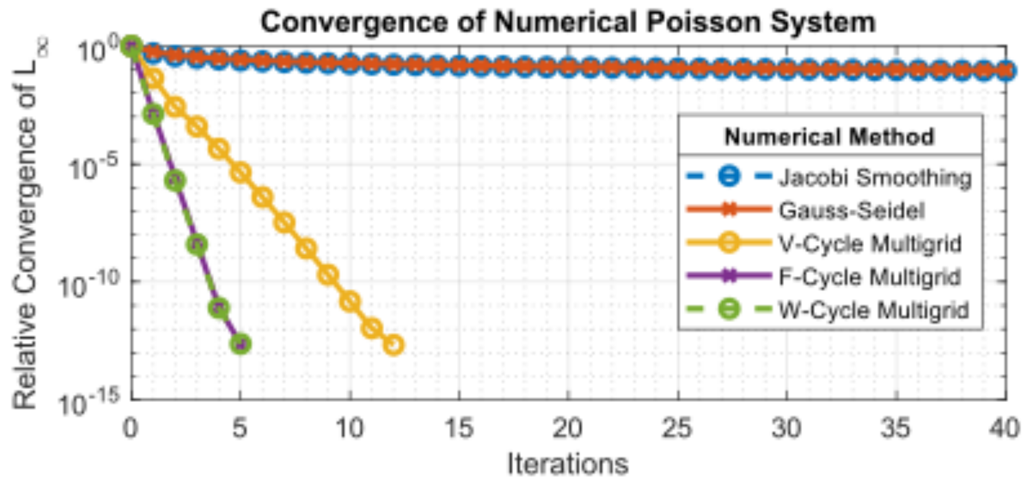


Figure 3.6: *Schedule of grids for* (a) *V-cycle,* (b) *W-cycle, and* (c) *FMG scheme, all on four levels.*



https://en.wikipedia.org/wiki/Multigrid_method