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ME469: Introduction to the low-Mach Number Approximation

Stefan P. Domino[1,2]

[1] Computational Thermal and Fluid Mechanics, Sandia National Laboratories

[2] Institute for Computational and Mathematical Engineering, Stanford

SAND2018-4536 PE

Consider a Variable Density, non-Isothermal Fluid Flow System

- Consider the variable density (non-isothermal) equations of motion (momentum and continuity) with energy transport:

$$\left.\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} &= 0, \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} + \frac{\partial P}{\partial x_i} &= \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i, \\
\frac{\partial \rho E}{\partial t} + \frac{\partial \rho u_j H}{\partial x_j} &= -\frac{\partial q_j}{\partial x_j} + \frac{\partial u_i \tau_{ij}}{\partial x_j} + \rho u_i g_i
\end{aligned}\right\} \text{(2+nDim)}$$

Constitutive Relationships

$$E = H - P/\rho,$$

$$H = h + \frac{1}{2} u_k u_k,$$

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij},$$

$$q_i = -k \frac{\partial T}{\partial x_i},$$

$$h = \int_{T_o}^{T} C_p dT$$

$\rho = \frac{PM}{RT}$ Equation of State (EOS) provides the P and ρ relationship

- See Paolucci (1982) or Baum (1978) for the low-Mach pedigree
- Number of Equations = (3+nDim) = Number of unknowns

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(2+nDim)

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Degree of Freedom, Properties, and Constitutive Count

DOF: (3+nDim)

- Density, ρ (Continuity Eq)
- Pressure, P (EOS)
- Velocity, u_i (Momentum Eq)
- Total energy, E (Energy Eq)

Properties:

- Viscosity, μ
- Specific heat, C_p
- Thermal conductivity, λ

Constitutive Relationships:

- Ideal gas law
  - Again, provides density/pressure relationship (a key concept)
- Newtonian stress, τ_{ij}
- Heat flux vector, q_j
- Total enthalpy, H
- Static enthalpy, h
- $dh/dT = C_p$

Dimensionless Form

- Non-dimensionalization is via a characteristic velocity and length scale:

$$Re = \frac{\rho_\infty U_\infty L}{\mu_\infty}, \qquad \text{Reynolds number,}$$

$$Pr = \frac{C_{p,\infty}\mu_\infty}{k_\infty}, \qquad \text{Prandtl number,}$$

$$Fr_i = \frac{u_\infty^2}{g_i L}, \qquad \text{Froude number,} \quad g_i \neq 0,$$

$$Ma = \sqrt{\frac{u_\infty^2}{\gamma R T_\infty / W}} \qquad \text{Mach Number}$$

Dimensionless Form

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$$\frac{\partial \bar\rho}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j}{\partial \bar x_j} = 0,$$

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$$\frac{\partial \bar\rho \bar h}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j \bar h}{\partial \bar x_j} = -\frac{1}{Pr}\frac{1}{Re}\frac{\partial \bar q_j}{\partial \bar x_j} + \frac{\gamma-1}{\gamma}\frac{\partial \bar P}{\partial \bar t} + \frac{\gamma-1}{\gamma}\frac{Ma^2}{Re}\frac{\partial \bar u_i \bar\tau_{ij}}{\partial \bar x_j} + \bar\rho \bar u_i \frac{\gamma-1}{\gamma}\frac{Ma^2}{Fr_i} - \frac{\gamma-1}{2}Ma^2\left(\frac{\partial \bar\rho \bar u_k \bar u_k}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j \bar u_k \bar u_k}{\partial \bar x_j}\right)$$

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&+ \frac{\gamma - 1}{\gamma}\frac{Ma^2}{Re}\frac{\partial \bar u_i \bar\tau_{ij}}{\partial \bar x_j} + \bar\rho \bar u_i \frac{\gamma - 1}{\gamma}\frac{Ma^2}{Fr_i} \\
&- \frac{\gamma - 1}{2} Ma^2 \left(\frac{\partial \bar\rho \bar u_k \bar u_k}{\partial \bar t} + \frac{\partial \bar\rho \bar u_j \bar u_k \bar u_k}{\partial \bar x_j}\right)
\end{aligned}
$$

- Note, as the Mach number approaches zero, the viscous work and kinetic energy terms become negligible; Fr-based terms also are removed

Dimensionless Form

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\frac{\partial \bar{\rho}\bar{h}}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{h}}{\partial \bar{x}_j} &= -\frac{1}{Pr}\frac{1}{Re}\frac{\partial \bar{q}_j}{\partial \bar{x}_j} + \frac{\gamma-1}{\gamma}\frac{\partial \bar{P}}{\partial \bar{t}} \\
&\quad + \frac{\gamma-1}{\gamma}\frac{Ma^2}{Re}\frac{\partial \bar{u}_i\bar{\tau}_{ij}}{\partial \bar{x}_j} + \bar{\rho}\bar{u}_i\frac{\gamma-1}{\gamma}\frac{Ma^2}{Fr_i} \\
&\quad - \frac{\gamma-1}{2}Ma^2\left(\frac{\partial \bar{\rho}\bar{u}_k\bar{u}_k}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{u}_k\bar{u}_k}{\partial \bar{x}_j}\right)
\end{aligned}
$$

- However, the momentum equation notes a singularity in the scaled pressure gradient term

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\frac{\partial \bar{\rho}\bar{h}}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{h}}{\partial \bar{x}_j} &= -\frac{1}{Pr}\frac{1}{Re}\frac{\partial \bar{q}_j}{\partial \bar{x}_j} + \frac{\gamma-1}{\gamma}\frac{\partial \bar{P}}{\partial \bar{t}} \\
&\quad + \frac{\gamma-1}{\gamma}\frac{Ma^2}{Re}\frac{\partial \bar{u}_i\bar{\tau}_{ij}}{\partial \bar{x}_j} + \bar{\rho}\bar{u}_i\frac{\gamma-1}{\gamma}\frac{Ma^2}{Fr_i} \\
&\quad - \frac{\gamma-1}{2}Ma^2\left(\frac{\partial \bar{\rho}\bar{u}_k\bar{u}_k}{\partial \bar{t}} + \frac{\partial \bar{\rho}\bar{u}_j\bar{u}_k\bar{u}_k}{\partial \bar{x}_j}\right)
\end{aligned}
$$

<u>Conclusions</u>: In the limit of zero Mach number, the energy equation is simplified, while the momentum equation is not well defined and, in fact, **<u>singular</u>**

Exploration of the Pressure Singularity

To explore the singularity, write each DOF as an asymptotic series:

$$\bar{P} = \bar{p}_0 + \bar{p}_1\epsilon + \bar{p}_2\epsilon^2 \dots$$
$$\bar{u}_i = \bar{u}_{i,0} + \bar{u}_{i,1}\epsilon + \bar{u}_{i,2}\epsilon^2 \dots$$
$$\bar{T} = \bar{T}_0 + \bar{T}_1\epsilon + \bar{T}_2\epsilon^2 \dots$$

The resulting zeroth-order equations are as follows:

$$\frac{\partial \bar{\rho}_0}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j}}{\partial \bar{x}_j} = 0,$$
$$\frac{\partial \bar{\rho}_0 \bar{u}_{0,i}}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j} \bar{u}_{0,i}}{\partial \bar{x}_j} + \frac{1}{\gamma M a^2}\left(\frac{\partial \bar{p}_0}{\partial \bar{x}_i} + \epsilon \frac{\partial \bar{p}_1}{\partial \bar{x}_i}\right) = \frac{1}{Re}\frac{\partial \bar{\tau}_{0,ij}}{\partial \bar{x}_j},$$
$$\frac{\partial \bar{\rho}_0 \bar{h}_0}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j} \bar{h}_0}{\partial \bar{x}_j} = -\frac{1}{Pr Re}\frac{\partial \bar{q}_{0,j}}{\partial \bar{x}_j} + \frac{\gamma - 1}{\gamma}\frac{\partial \bar{p}_0}{\partial \bar{t}}$$

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Exploration of the Pressure Singularity: Ramifications

In order for the zeroth-order momentum equation to be well conditioned in the limit of zero Mach number, $\frac{\partial \bar{p}_o}{\partial \bar{x}_i}$ must be spatially zero with $\epsilon = \gamma M a^2$

$$\frac{\partial \bar{\rho}_0}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j}}{\partial \bar{x}_j} = 0,$$

$$\frac{\partial \bar{\rho}_0 \bar{u}_{0,i}}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j} \bar{u}_{0,i}}{\partial \bar{x}_j} + \frac{1}{\gamma M a^2}\left(\frac{\partial \bar{p}_0}{\partial \bar{x}_i} + \epsilon \frac{\partial \bar{p}_1}{\partial \bar{x}_i}\right) = \frac{1}{Re}\frac{\partial \bar{\tau}_{0,ij}}{\partial \bar{x}_j},$$

$$\frac{\partial \bar{\rho}_0 \bar{h}_0}{\partial \bar{t}} + \frac{\partial \bar{\rho}_0 \bar{u}_{0,j} \bar{h}_0}{\partial \bar{x}_j} = -\frac{1}{PrRe}\frac{\partial \bar{q}_{0,j}}{\partial \bar{x}_j} + \frac{\gamma - 1}{\gamma}\frac{\partial \bar{p}_0}{\partial \bar{t}}$$

- p_0 is a constant-in-space, possibly variable-in-time thermodynamic pressure
- p_1 is the variable in space pressure, which is also known as the “motion pressure”, p^m
- Recall, this is simply a perturbation about the full thermodynamic pressure:

$$\bar{P} = \bar{p}_0 + \bar{p}_1 \epsilon + \bar{p}_2 \epsilon^2 \ldots$$

Final low-Mach Equation Set

The resulting Equation set is as follows:

$$\left.\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} &= 0, \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} + \frac{\partial p^m}{\partial x_i} &= \frac{\partial \tau_{ij}}{\partial x_j} + (\rho - \rho_\circ)\, g_i, \\
\frac{\partial \rho h}{\partial t} + \frac{\partial \rho u_j h}{\partial x_j} &= -\frac{\partial q_j}{\partial x_j} + \frac{\partial P_{th}}{\partial t}
\end{aligned}\right\}\ \underline{\text{2+nDim}}$$

- Equation of state given by the thermodynamic pressure: $\rho = \dfrac{P_{th} M}{RT}$
- Energy transport is only required when the system modeled has a temperature difference

<u>EOS does not provide an equation for closure</u>: An alternative approach is required for motion pressure!

The Final low-Mach Number Equation Set: Ramifications

- We have effectively filtered out the acoustics, i.e., the wave speed is infinitely fast
- DOF/Equation system is: ρ, p^m, u_i, h; p^t is a constant for an open domain

In practice, a functional form for the motion pressure is derived and known as a Pressure Poisson Equation (PPE):

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \rho u_i}{\partial t} + ...\right) = -\frac{\partial}{\partial x_i}\left(\frac{\partial p^m}{\partial x_i}\right)$$

With the continuity equation serving as a mass balance constraint

- Note that we have introduced an Elliptic nature to the equation set
- Momentum and other equations can be implicit of explicitly solved, however, the Pressure Poisson Equation requires an implicit solve with dedicated solvers, e.g., multi-grid methods

Later lectures will discuss solution approach in depth, in the meantime, an efficient solver approach for the low-Mach system is a “pressure projection” method

Why "pressure projection"?

From Domino, 2006 "*Toward verification of formal time accuracy for a family of approximate projection methods using the method of manufactured solutions*"

projection algorithm. In general, any vector can be written as a Hodge decomposition, or in terms of a vector of known divergence and a curl-free part,

$$\mathbf{F} = \mathbf{F}^{\mathbf{kd}} + \nabla\phi, \tag{7.1}$$

with the known divergence given by

$$\nabla \cdot \mathbf{F}^{\mathbf{kd}} = \mathcal{S}. \tag{7.2}$$

The Poisson system is provided by

$$\nabla \cdot \nabla\phi = \nabla \cdot \mathbf{F} - \nabla \cdot \mathbf{F}^{\mathbf{kd}} = \nabla \cdot \mathbf{F} - \mathcal{S} \tag{7.3}$$

with solution,

$$\phi = \Delta^{-1}(\nabla \cdot \mathbf{F} - \mathcal{S}), \tag{7.4}$$

and

$$\begin{aligned}
\mathbf{F}^{\mathbf{kd}} &= \mathbf{F} - \nabla\phi, && (7.5)\\
&= \mathbf{F} - \nabla(\Delta^{-1}(\nabla \cdot \mathbf{F} - \mathcal{S})), && (7.6)\\
&= (\mathbf{I} - \nabla(\Delta^{-1}\nabla\cdot))\mathbf{F} + \nabla\Delta^{-1}\mathcal{S}, && (7.7)\\
&= \mathcal{P}\mathbf{F} + \mathcal{B}, && (7.8)\\
&= \mathcal{P}^{af}\mathbf{F}. && (7.9)
\end{aligned}$$

- For a solenoidal vector, div(u) is zero and P is an "idempotent" projection, i.e., $P=P^2$
- Otherwise, P is an affine-projection operator

Why "pressure projection"?

From Domino, 2006 "*Toward verification of formal time accuracy for a family of approximate projection methods using the method of manufactured solutions*"

The projection analysis for the equations of motion is completed by the following definitions:

$$\mathbf{F} = -\nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla \cdot \mu(\nabla \mathbf{u} + \nabla^T \mathbf{u}) + \rho \mathbf{g} + \mathbf{s}, \tag{7.10}$$

$$\mathbf{F}^{\mathbf{kd}} = \frac{\partial \rho \mathbf{u}}{\partial t}, \tag{7.11}$$

$$\nabla \cdot \mathbf{F}^{\mathbf{kd}} = -\frac{\partial^2 \rho}{\partial t^2}, \tag{7.12}$$

$$\nabla \phi = \nabla p, \tag{7.13}$$

or

$$\nabla \cdot \nabla p = \nabla \cdot \left(-\nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla \cdot \mu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u}\right) + \rho \mathbf{g} + \mathbf{s}\right) + \frac{\partial^2 \rho}{\partial t^2}. \tag{7.14}$$

Classic Pressure Poisson System

Hodge Decomposition: $$u_i = u_i^{kd} + \frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

Taking the divergence: $$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_i^{kd}}{\partial x_i} + \frac{\partial}{\partial x_i}\frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

While enforcing continuity: $$\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

Finally, the projection step: $$u_i^{n+1} = u_i - \frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

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Finally, the projection step: $$u_i^{n+1} = u_i - \frac{\Delta t}{\rho}\frac{\partial \phi}{\partial x_i}$$

Here is our equation for the motion pressure:

$$\phi = \Delta p^m = p^{m,n+1} - p^{m,n}$$

This pressure (or change in pressure) acts to enforce the continuity constraint

Sneak Peak: Classic Pressure Projection Algorithm

Solve Momentum Equation (with a provisional motion pressure):

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} + \frac{\partial p^m}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial x_j} + (\rho - \rho_\circ)\, g_i,$$

Solve Continuity, given the computed velocity: $$\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{\Delta t}{\rho} \frac{\partial \phi}{\partial x_i}$$

Update velocity: $$u_i^{n+1} = u_i - \frac{\Delta t}{\rho} \frac{\partial \phi}{\partial x_i}$$

$$\phi = \Delta p^m = p^{m,n+1} - p^{m,n}$$

In future tutorials, we will be more formal in nomenclature, however, for now, the concept of this "*incremental pressure projection scheme*" is to solve momentum with a provisional pressure, and then use the continuity constraint to update (or project) the velocity into the space of known divergence. Monolithic approaches are also viable

Thought Experiment…

- Large domain, one door, one “exit”

Maples Pavilion

Thought Experiment…

- Large domain, one door, one “exit”

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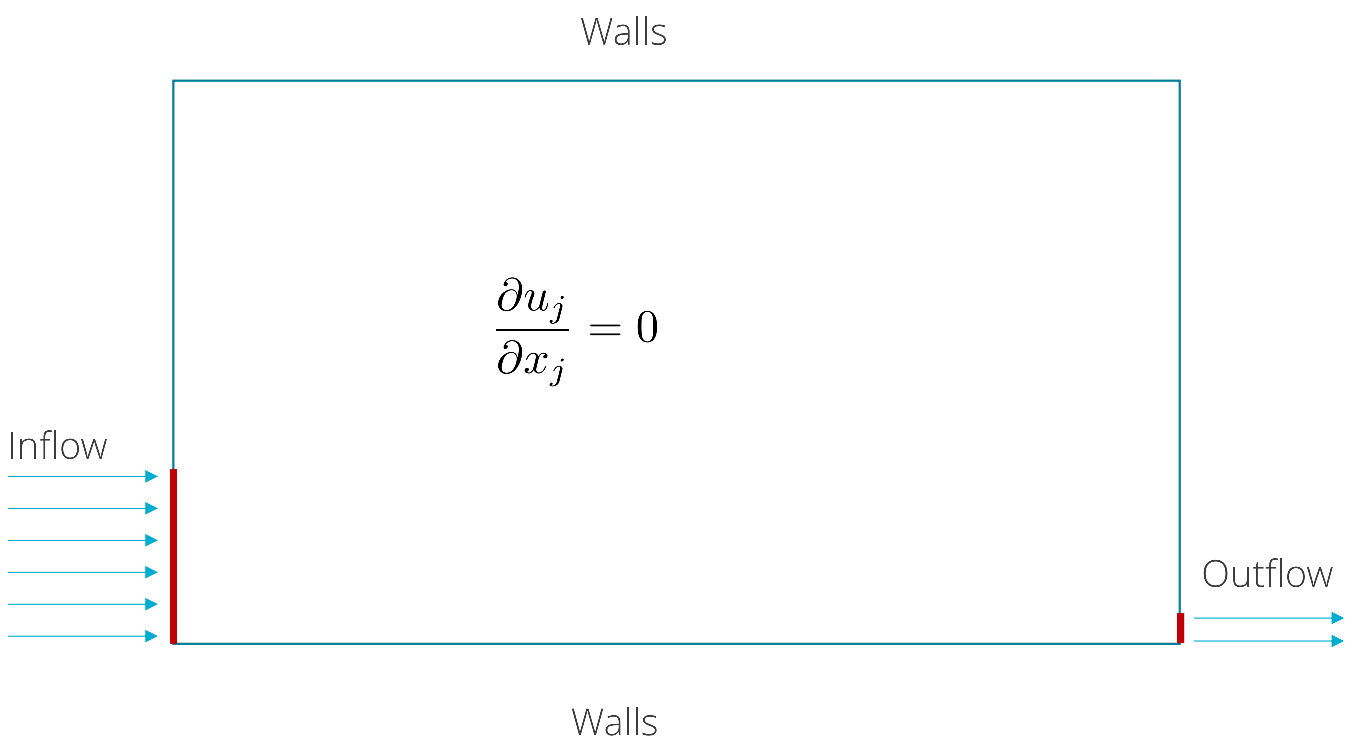
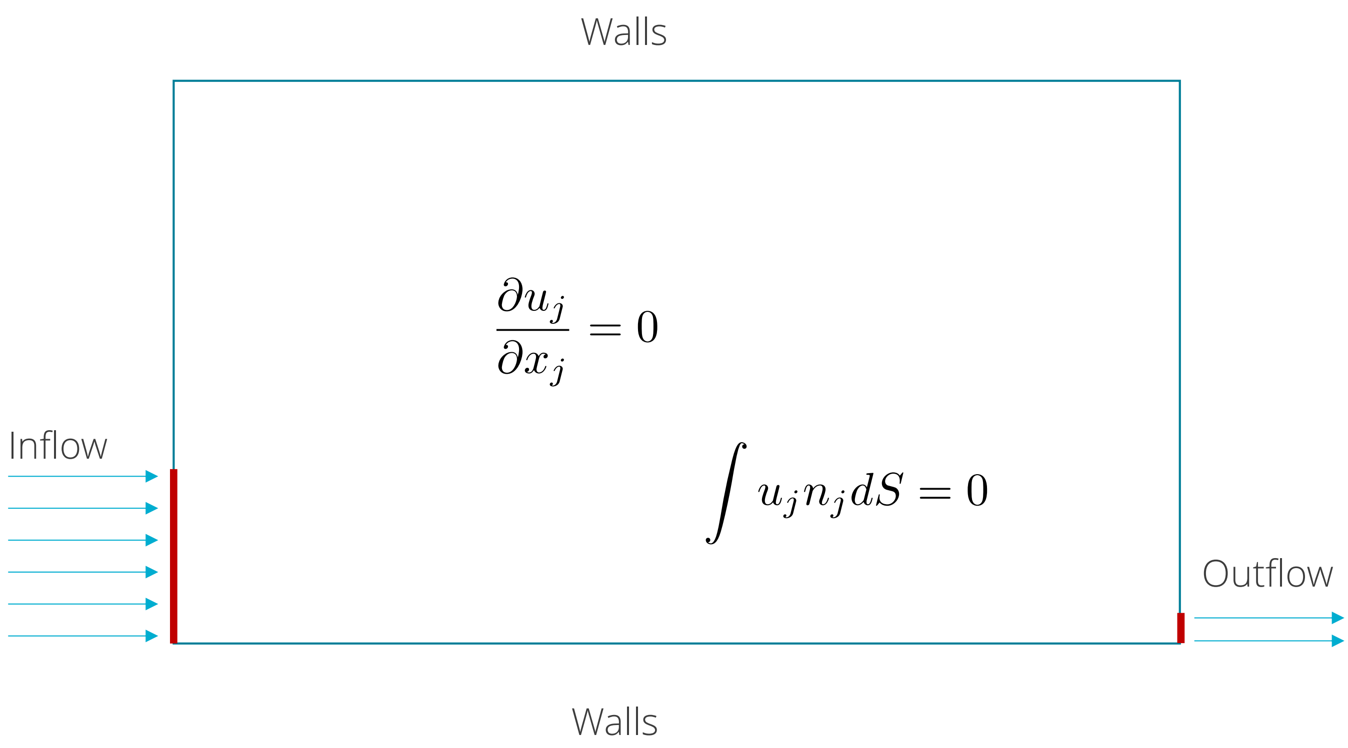


Thought Experiment…

- Large domain, one door, one “exit”

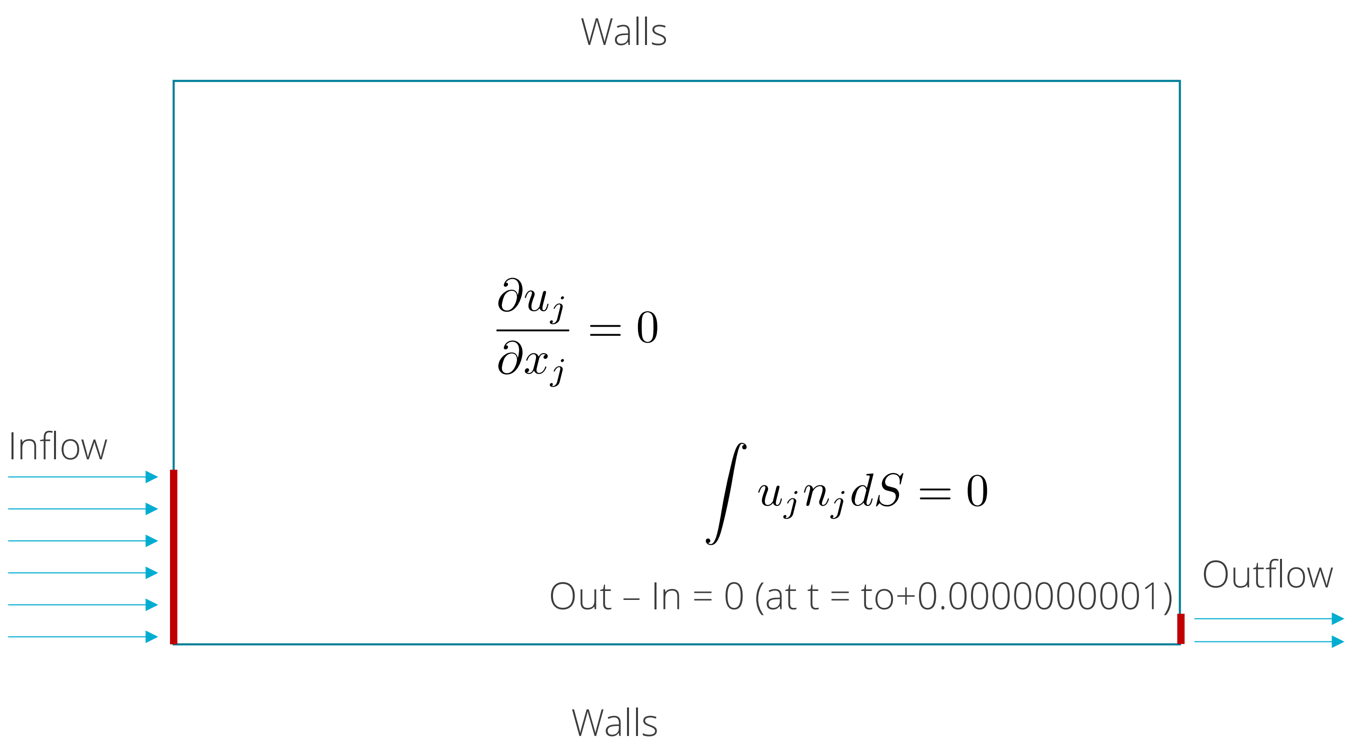


$$\frac{\partial u_j}{\partial x_j} = 0$$

$$\int u_j n_j dS = 0$$

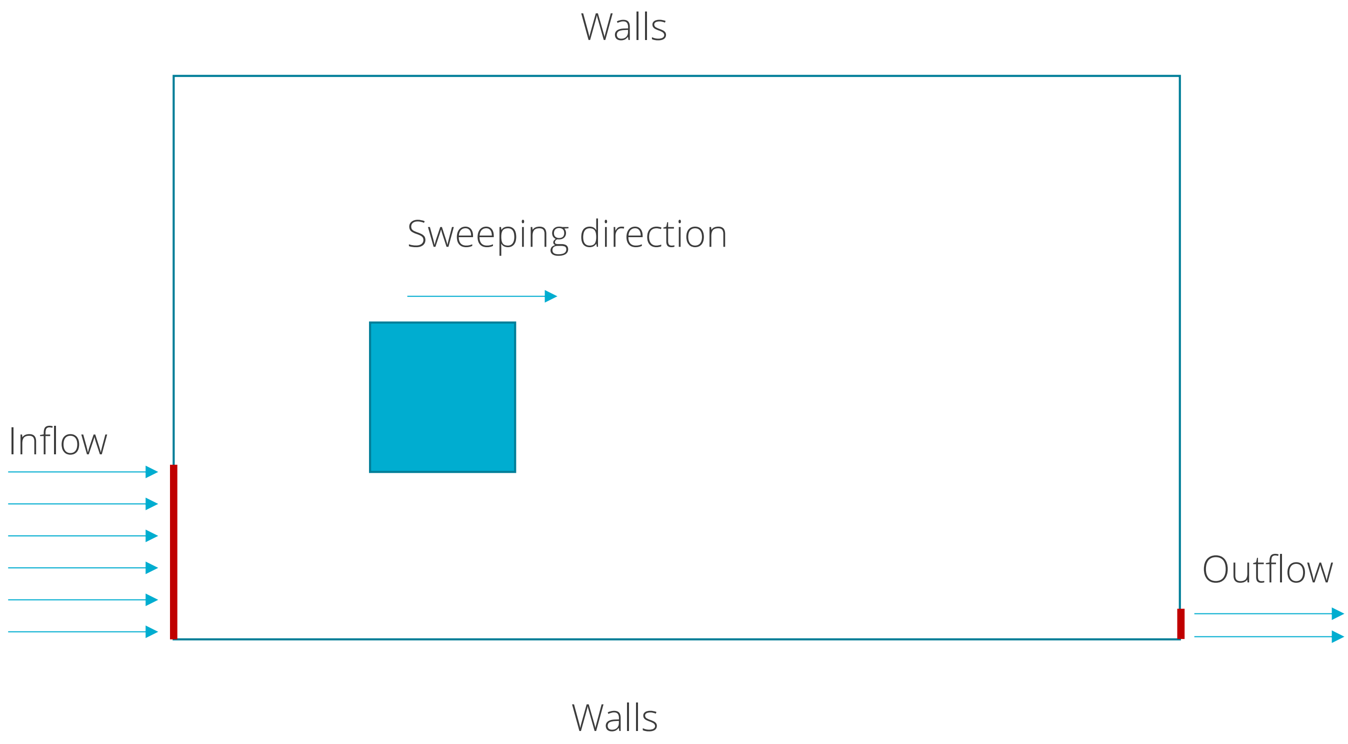
Thought Experiment…

- Large domain, one door, one “exit”



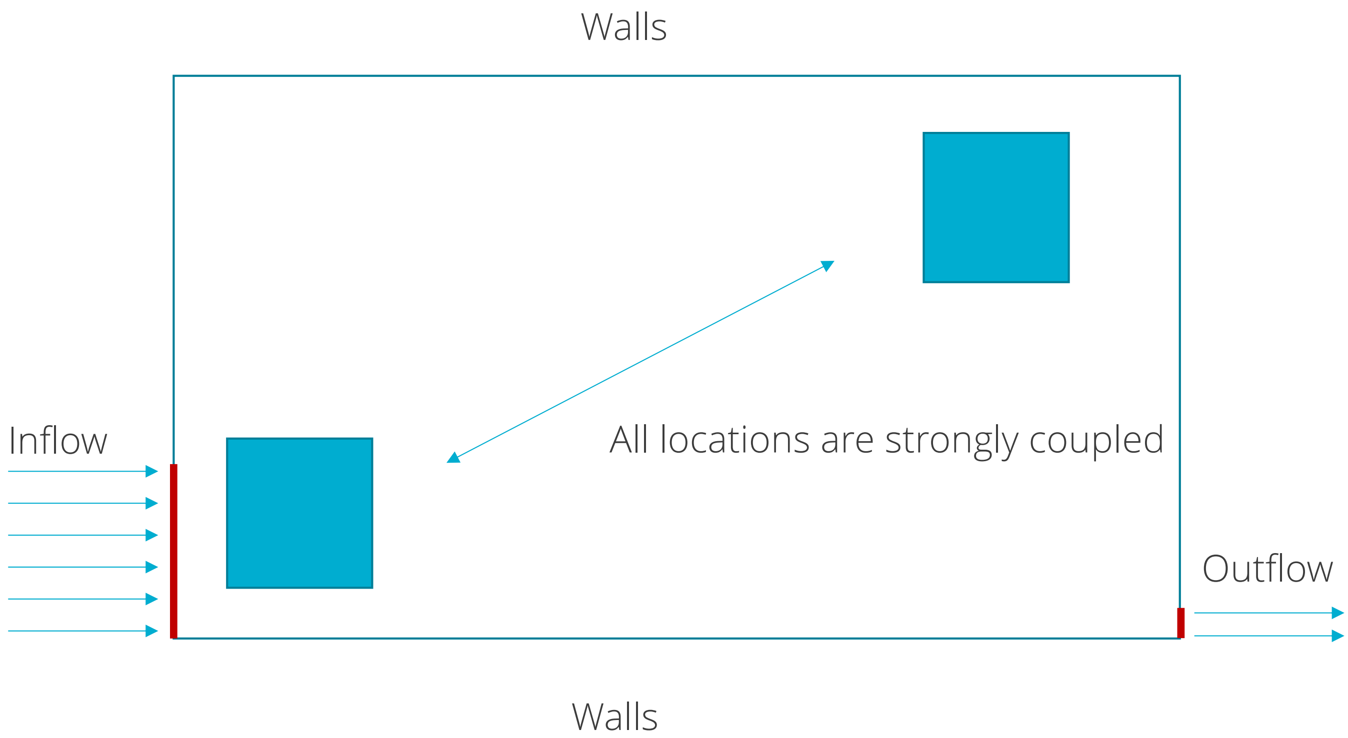
Thought Experiment… Solver Ramifications

- Fixed point solvers iterate over the domain sequentially and are effective when this sweeping of the mesh corresponds to a particular physical direction



Thought Experiment… Solver Ramifications

- For Elliptic systems, fixed point iterative solvers fail since the sequential propagation of information is not adequate for a system with infinite wave speeds



Multigrid Methods: The Concept

Briggs, "A Multigrid Tutorial, 2nd Edition" (2000)

- Fixed-point iterations schemes effectively remove high-frequency errors
- Can we take a solution, coarsen, and then solve the new system?

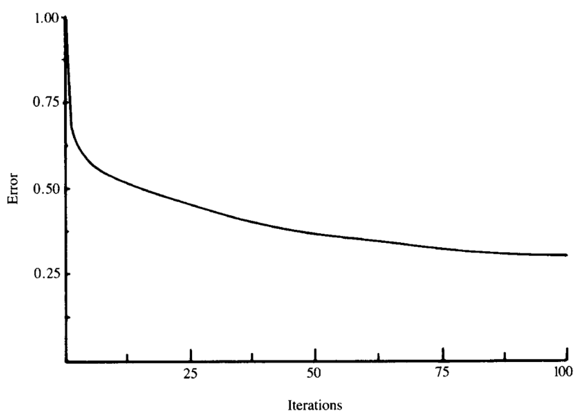


Figure 2.5: *Weighted Jacobi method with $\omega = \frac{2}{3}$ applied to the one-dimensional model problem with $n = 64$ points and an initial guess $(\mathbf{v}_1 + \mathbf{v}_6 + \mathbf{v}_{32})/3$. The maximum norm of the error, $\|\mathbf{e}\|_\infty$, is plotted against the iteration number for 100 iterations.*

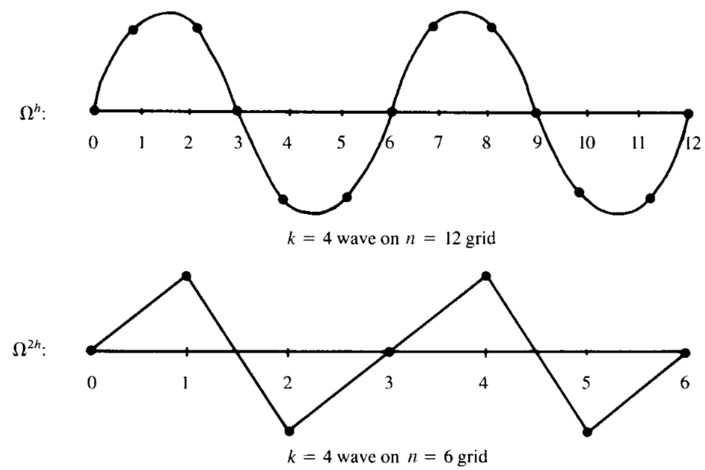


Figure 3.1: *Wave with wavenumber $k = 4$ on Ω^h ($n = 12$ points) projected onto Ω^{2h} ($n = 6$ points). The coarse grid "sees" a wave that is more oscillatory on the coarse grid than on the fine grid.*

Multigrid Methods: The Approach

- Multigrid methods (MG) are essential for efficient solver performance in fluids-based Elliptic systems
- In simple, structured domains, this can be geometric (GMG), while in unstructured, algebraic (AMG)

$$A\phi = f$$

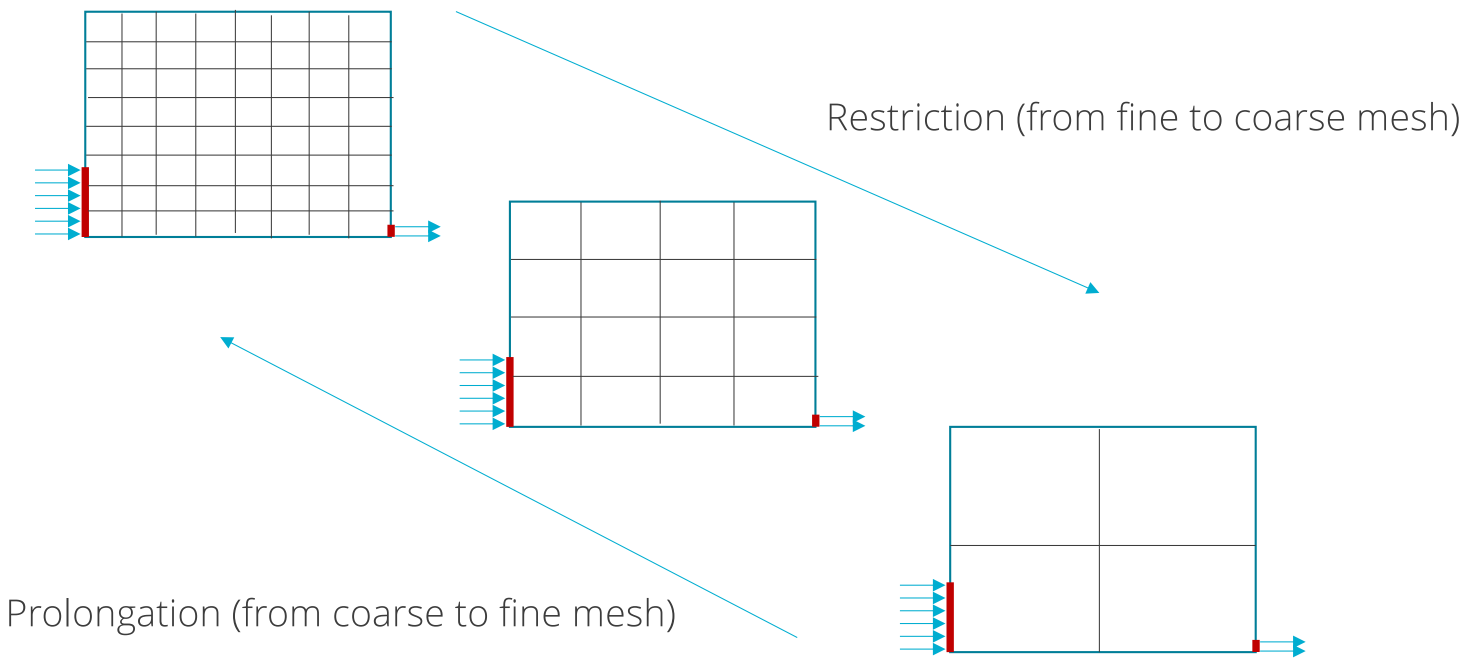
$$\epsilon = \phi - \tilde{\phi}$$

$$r = f - A\tilde{\phi}$$

$$A\left(\tilde{\phi} + \epsilon\right) = f$$

$$A\epsilon = r$$

$$\phi = \tilde{\phi} + \epsilon$$



Multigrid Methods: V- and W-Cycles

Briggs (2000)

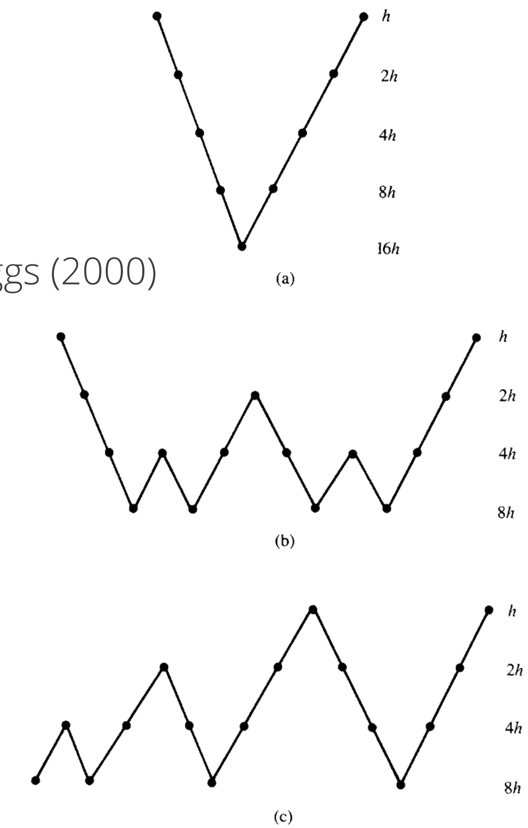
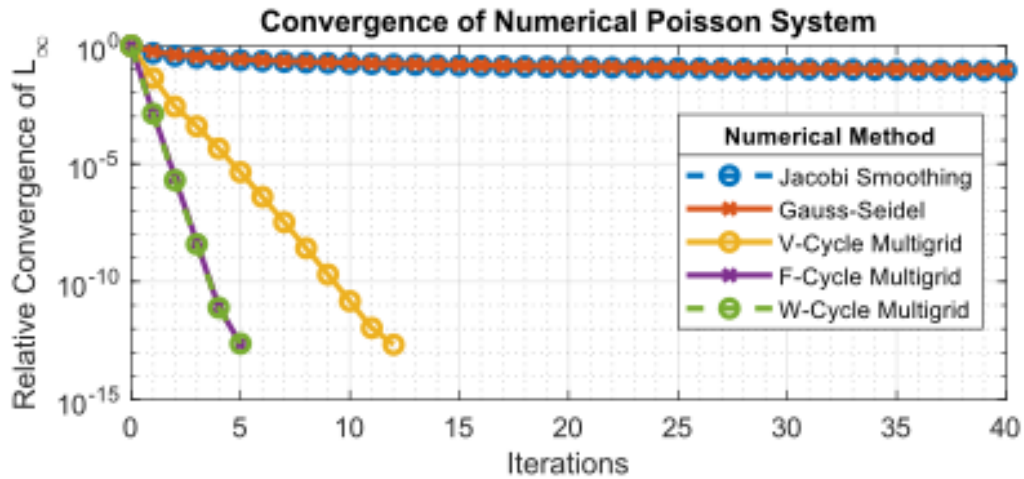


Figure 3.6: *Schedule of grids for* (a) *V-cycle,* (b) *W-cycle, and* (c) *FMG scheme, all on four levels.*



https://en.wikipedia.org/wiki/Multigrid_method