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ME469: Common Discretization Approaches: Control-volume Finite Element Method (CVFEM)

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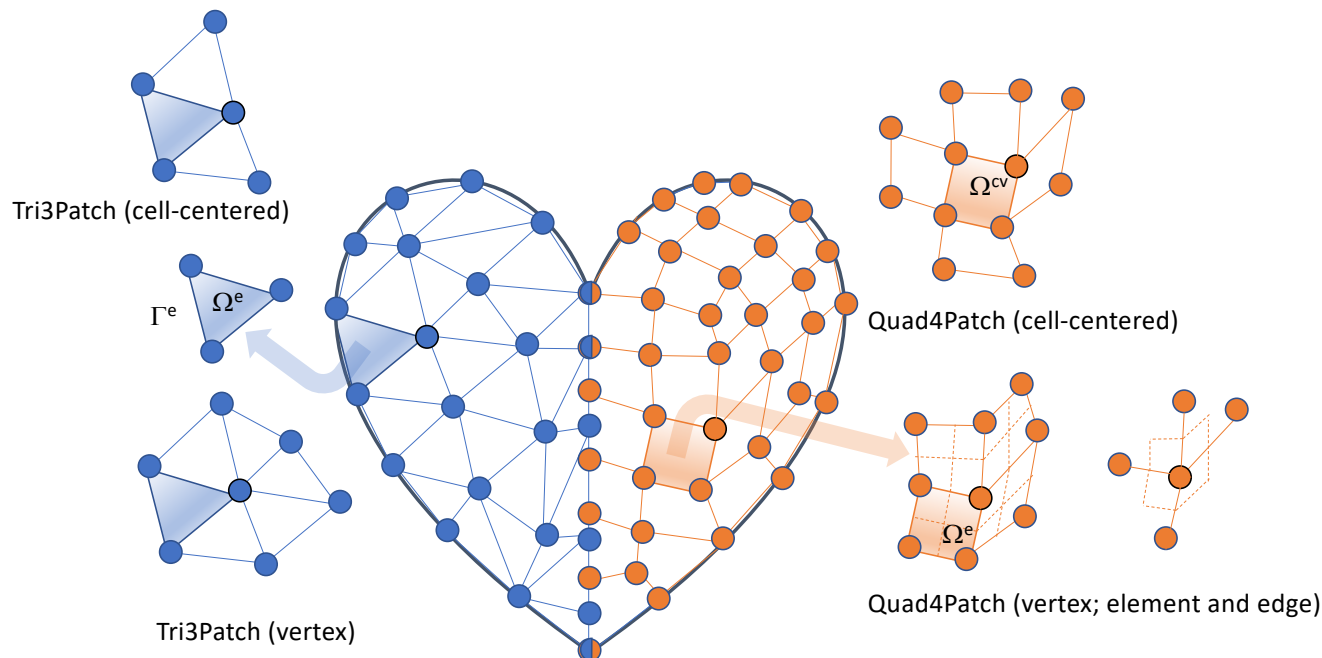
SAND2018-4536 PE





Review of Discretization Options: New, a nodal-basis...

- Degree-of-freedom (DOF) for:
 - Cell-centered: Stencil is based on a element:face:element
 - DOFs at vertices of elements, or “nodes”, element:node (CVFEM, FEM), edge:node (EBVC)



- Definition of an interpolation function:

$$\phi_{ip} = \sum_n N_n^{ip} \phi_n$$

- N_n^{ip} is the Lagrange function associated with node n
- ϕ_n is the value of the DOF at node n
- The nodal basis functions obey equipartition of unity and satisfy, $N_n^{x_j} = \delta_{nj}$



Fundamentals of Discretization: Surface vs Volume Integrations

- Given a partial differential equation (PDE) and associated volumetric form:

$$\int_{\Omega} \frac{\partial F_j}{\partial x_j} d\Omega = \int_{\Omega} S d\Omega$$

- Applying Gauss Divergence provides the standard finite volume form for fluxes in surface integral form (no distinction between internal control volume faces and boundary faces):

$$\sum_k \int_{\Omega_k} \frac{\partial F_j}{\partial x_j} d\Omega_k = \sum_k \int_{\Omega_k} S d\Omega_k \longrightarrow \sum_k \int_{\Gamma_k} F_j n_j d\Gamma_k = \sum_k \int_{\Omega_k} S d\Omega \longrightarrow \int F_j n_j dS = \int S dV$$

- We can also multiple PDE by an arbitrary test function, w , and integrate over a volume,

$$\int w \frac{\partial F_j}{\partial x_j} dV = \int w S dV$$

Next, integrate by parts and apply Gauss-Divergence. Note, that test function must be differentiable – shown here, at least once..

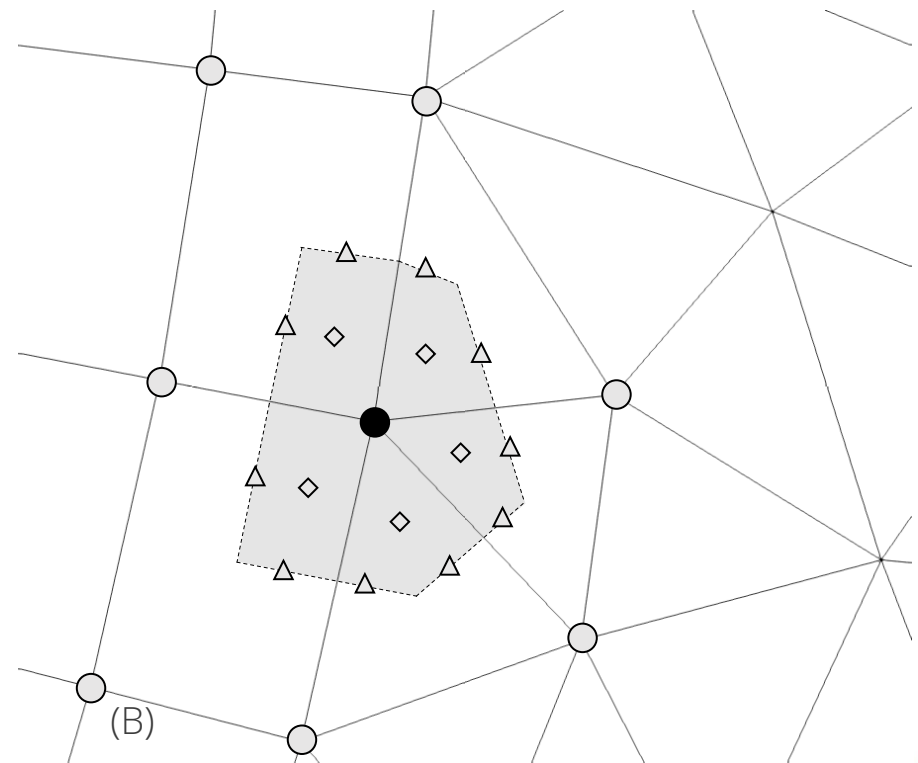
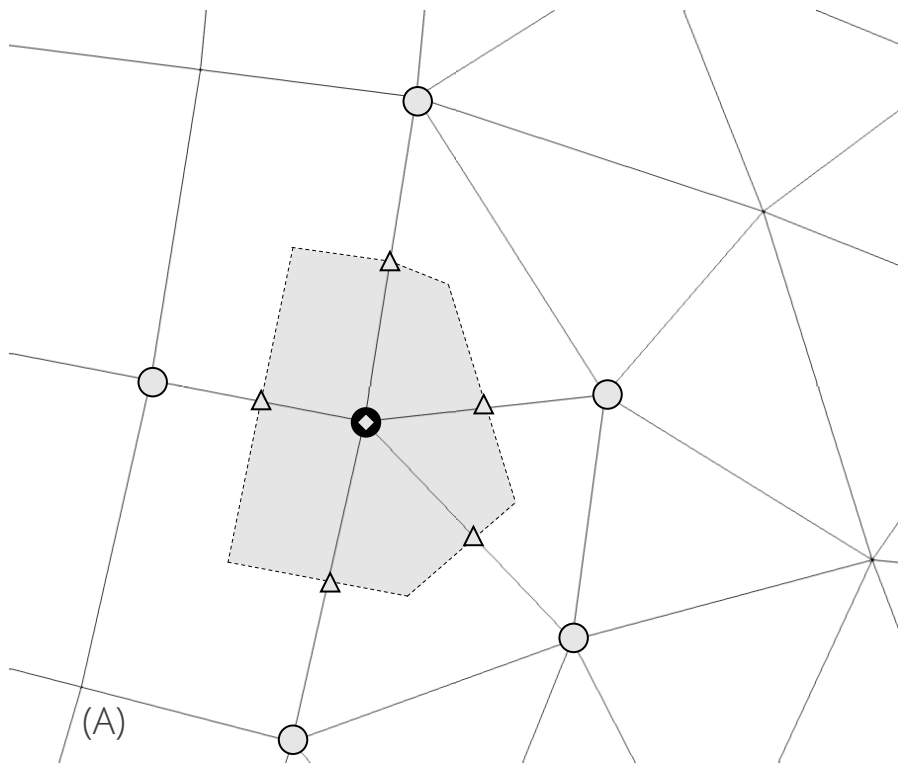
$$-\int F_j \frac{\partial w}{\partial x_j} dV + \int \frac{\partial w F_j}{\partial x_j} dV = -\int \underset{\substack{\uparrow \\ \text{Interior}}}{F_j \frac{\partial w}{\partial x_j}} dV + \int \underset{\substack{\uparrow \\ \text{Boundary}}}{w F_j n_j} dS$$

$$\frac{\partial w F_j}{\partial x_j} = w \frac{\partial F_j}{\partial x_j} + \frac{\partial w}{\partial x_j} F_j$$



Edge-Based Vertex-Centered Leverages the Dual-Volume Element-based Description of a Control Volume Finite Element Method (CVFEM)

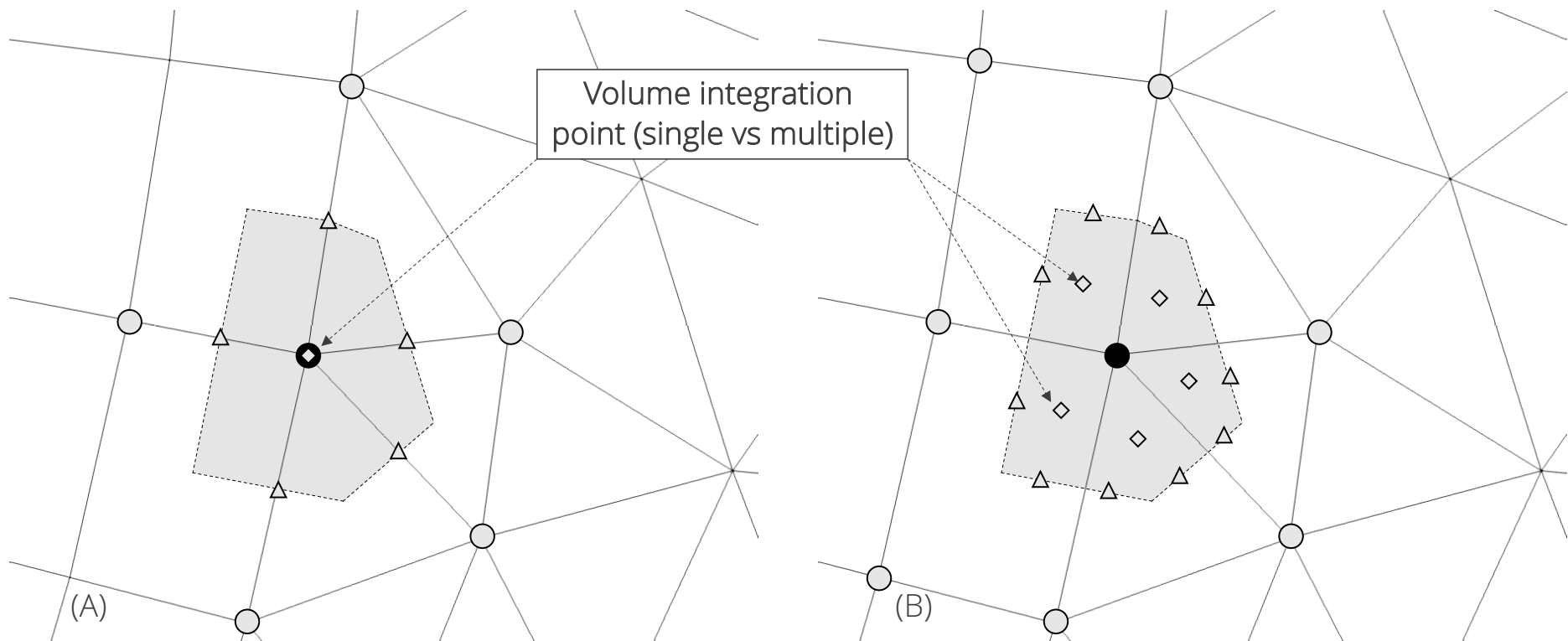
- EBVC (A) and CVFEM (B) – As shown below, the dual-volume and integration point layout is very similar





Edge-Based Vertex-Centered Leverages the Dual-Volume Element-based Description of a Control Volume Finite Element Method (CVFEM)

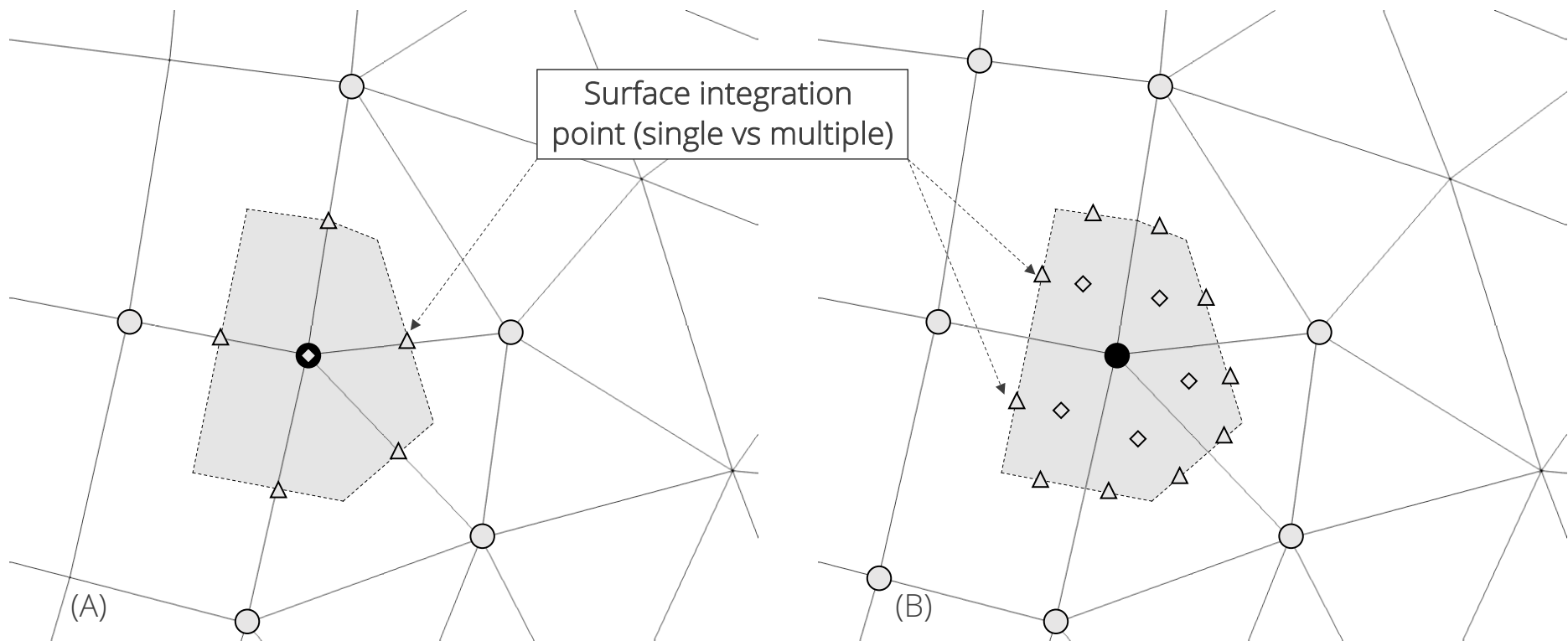
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Edge-Based Vertex-Centered Leverages the Dual-Volume Element-based Description of a Control Volume Finite Element Method (CVFEM)

- EBVC (A) and CVFEM (B) – As shown below, the dual-volume and integration point layout is very similar





Deep Dive on CVFEM

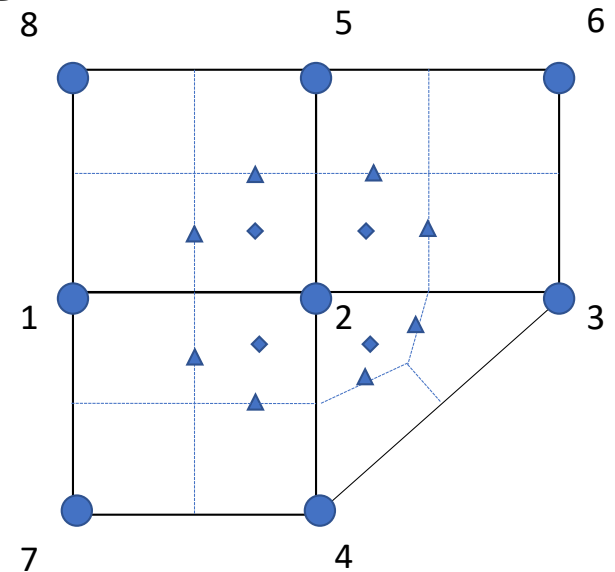
- CVFEM is a discretization scheme that:
 - Iterates over locally-owned elements for Time/Source/etc. (volumetric-based terms)
 - Iterates over locally-owned elements for Advection/Diffusion/etc. (integrated by parts terms)

Below is the patch of elements connected to node 2 (a global matrix row number)

- A *dual-volume* is defined within each element

Any value of the DOF within the element can be computed based on:

$$\phi_{ip} = \sum_n N_n^{ip} \phi_n$$

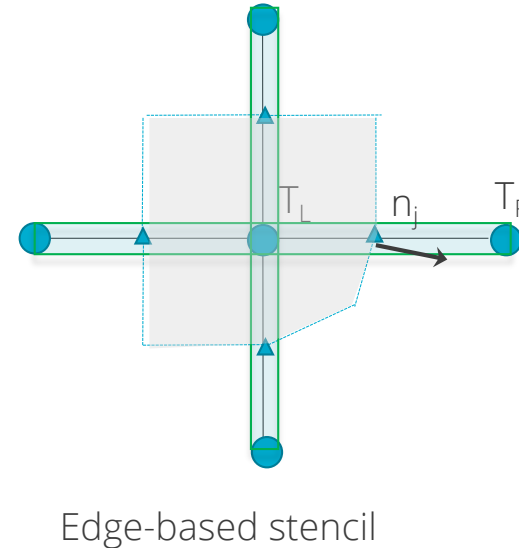
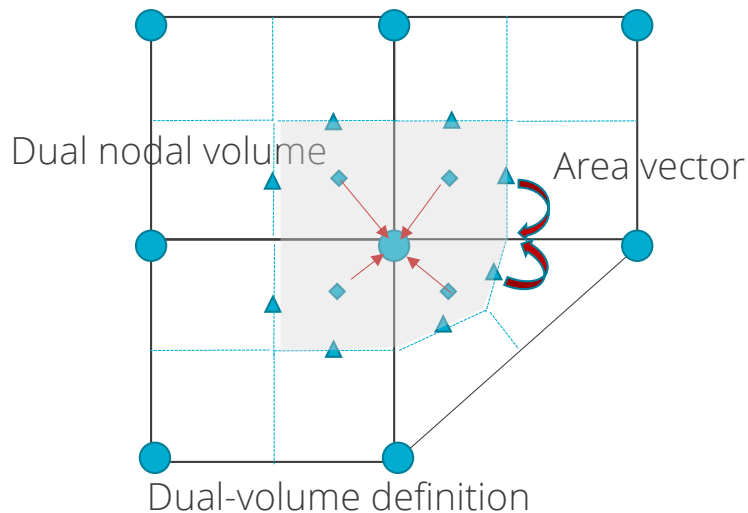


Sample patch of elements



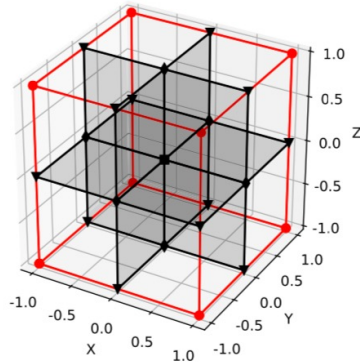
The Control Volume for EBVC is Defined by the **Dual-Volume**

- All primitives are collocated at the vertices of the elements with equal-order interpolation
- A dual mesh is constructed to obtain flux and volume quadrature locations
- Classic two-state, “L” and “R” approach provides spatially second-order accuracy
- Iterate **Nodes** for volume-based contributions
- Iterate **Edges** for surface-based contributions

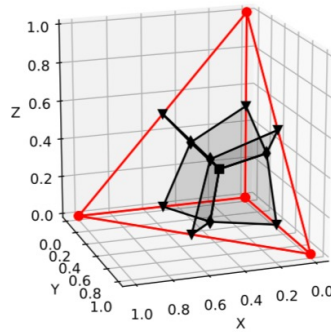




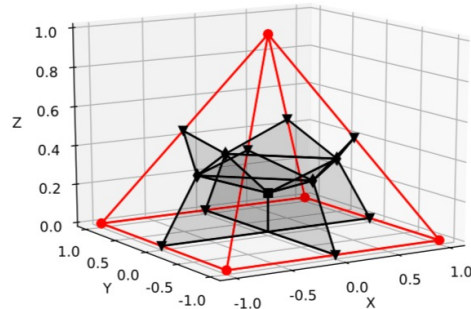
Dual Volume Definitions for Hybrid (Hex/Tet/Pyramid/Wedge) Meshes



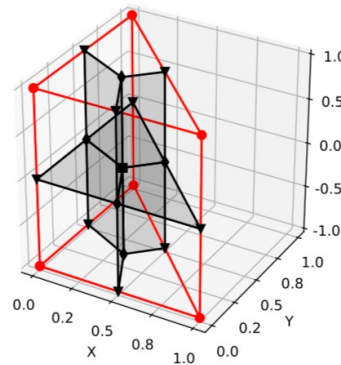
(a) Hexahedral topology (Hex8).



(b) Tetrahedral topology (Tet4).



(c) Pyramid topology (Pyramid5).



(d) Wedge topology (Wedge6).

Fig. 1. CVFEM element and dual-volume definition for the low-order topologies.

- Domino, et. al, "An assessment of atypical mesh topologies for low-Mach large-eddy simulation" 2019
- Generalized unstructured meshes support the ability to capture complex geometries, while minimizing the meshing time
- For example, near a solid wall, you might have a near-structured Hex-based mesh that is transitioned to unstructured
- For the mesh to be conformal, the faces for each adjacent element topology should match
- Drives Hex8 (Quad4 face) : Pyramid5 (Quad4 face); Pyramid5 (Tri3 face) to Tet4 (Tri3 face)
 - The Pyramid is known as a *transition* element

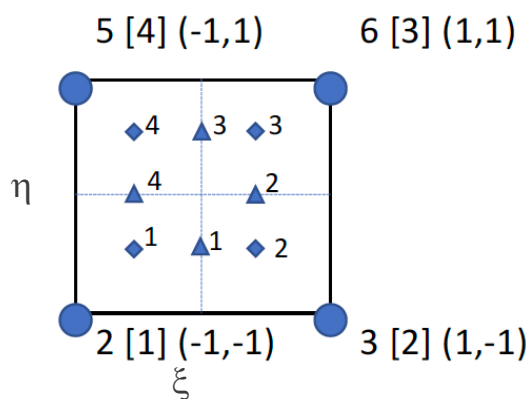


Deep Dive on CVFEM: Element-loops with Rich Basis

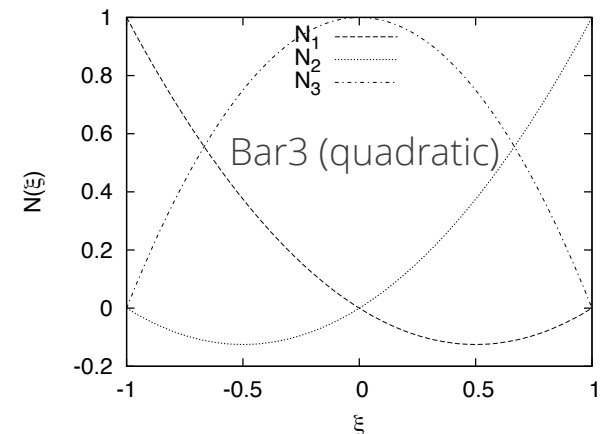
- For each element, recall that a dual volume has been constructed
- Volume-based contributions are evaluated at the sub-control volume integration points (diamonds)
- Surface-based contributions are evaluated at the sub-control surface integration points (triangles)

We define an isoparametric element that ranges from -1:1 in the ξ - (x-direction) and η - (y-direction) direction

Basis Functions for a Quad4



$$\begin{aligned} N_1 &= \frac{1}{4}(1 - \xi)(1 - \eta) \\ N_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\ N_3 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\ N_4 &= \frac{1}{4}(1 - \xi)(1 + \eta) \end{aligned}$$



For example, the isoparametric coordinate for ip 1 is (0.0, -0.5)



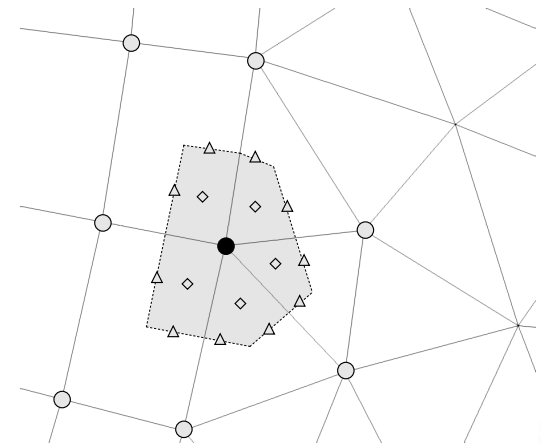
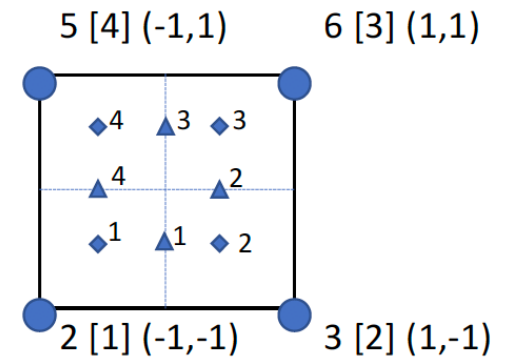
Deep Dive on CVFEM: Implicit Time Discretization

- Backward Euler (two state) and is first-order accurate (A-stable)
- BDF2 (three state) is second-order accurate (A-stable)
- This term is assembled over an element iteration and drives a consistent mass matrix with a full node:element:node connectivity

$$\int w \frac{\partial \rho \phi}{\partial t} dV \approx \sum_{ip} \frac{(\gamma_1 \rho_{ip}^{n+1} \phi_{ip}^{n+1} + \gamma_2 \rho_{ip}^n \phi_{ip}^n + \gamma_3 \rho_{ip}^{n-1} \phi_{ip}^{n-1})}{\Delta t} V_{ip}$$

- For uniform time stepping, the coefficients are give by,
 - $\gamma_1 = 3/2$ $\gamma_2 = -2$ $\gamma_3 = 1/2$
- You can easily see how the underlying basis pulls in the full node:element_node stencil:

$$\phi_{ip} = \sum_n N_n^{ip} \phi_n$$





Deep Dive on CVFEM: Implicit Time Discretization; Code

- <https://github.com/NaluCFD/Nalu/blob/master/src/kernel/ScalarMassElemKernel.C>

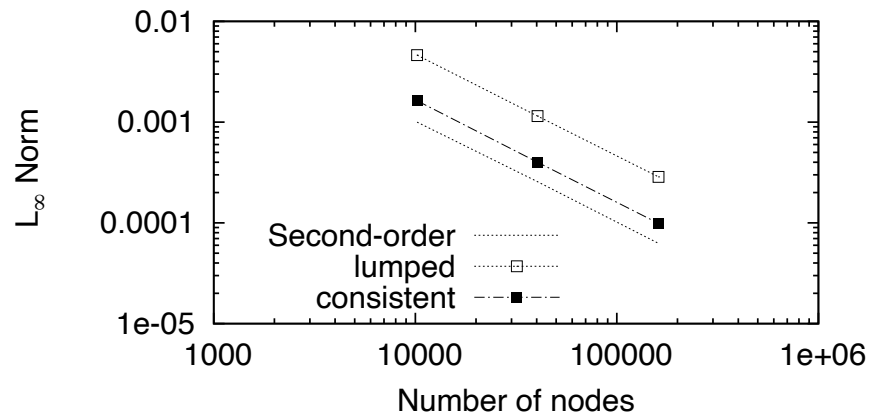


The Consistent Mass Matrix

- When evaluating the volumetric contributions using the full stencil, this approach is known as a *consistent mass matrix* as compared to the previous *lumped mass matrix* that is supported by CC and EBVC

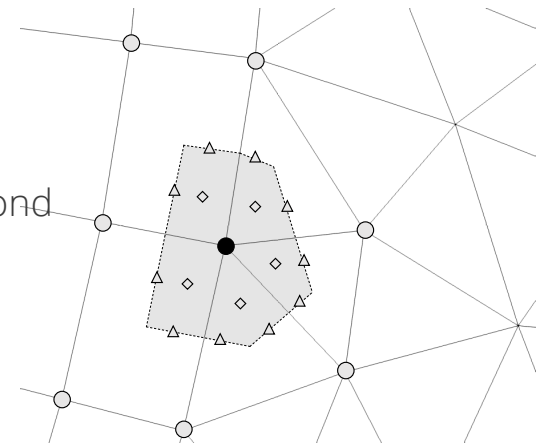
$$\int w \frac{\partial \rho \phi}{\partial t} dV \approx \sum_{ip} \frac{(\gamma_1 \rho_{ip}^{n+1} \phi_{ip}^{n+1} + \gamma_2 \rho_{ip}^n \phi_{ip}^n + \gamma_3 \rho_{ip}^{n-1} \phi_{ip}^{n-1})}{\Delta t} V_{ip} \quad (\text{Consistent}) \quad \phi_{ip} = \sum_n N_n^{ip} \phi_n$$

$$\int w \frac{\partial \rho \phi}{\partial t} dV \approx \sum_{nd} \frac{(\gamma_1 \rho_{nd}^{n+1} \phi_{nd}^{n+1} + \gamma_2 \rho_{nd}^n \phi_{nd}^n + \gamma_3 \rho_{nd}^{n-1} \phi_{nd}^{n-1})}{\Delta t} V_{nd} \quad (\text{Lumped})$$



Convecting Taylor Vortex analytical solution; error computed at 1 second

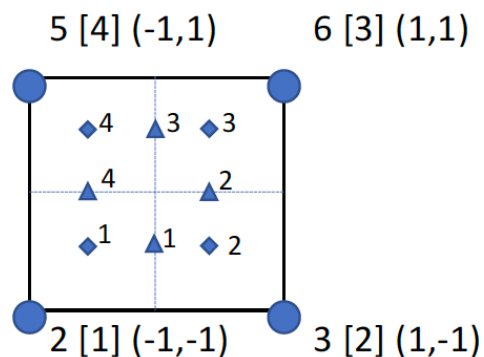
Shown to be the convergence rate, 2nd order





Deep Dive on CVFEM: Source Term Discretization

- Source terms for CVFEM are also assembled over an element or nodal loop
- In some cases, the source term can be complex, i.e., includes gradients, which drives either a nodal assembly of these quantities to the nodes (recall, the projected nodal gradient), or local evaluation using the shape function derivatives (see upcoming diffusion operator)



$$\int w S^\phi dV \approx \sum_{ip} S_{ip}^\phi V_{ip}$$



Deep Dive on CVFEM: Source Term Discretization; Code

- https://github.com/NaluCFD/Nalu/blob/master/src/user_functions/SteadyThermal3dContactSrcElemKernel.C

```
template<typename AlgTraits>
void
SteadyThermal3dContactSrcElemKernel<AlgTraits>::execute(
    SharedMemView<DoubleType*>& /* lhs */,
    SharedMemView<DoubleType*>& rhs,
    ScratchViews<DoubleType>& scratchViews)
{
    // Forcing nDim = 3 instead of using AlgTraits::nDim_ here to avoid compiler
    // warnings when this template is instantiated for 2-D topologies.
    NALU_ALIGNED DoubleType w_scvCoords[3];

    SharedMemView<DoubleType*>& v_coordinates = scratchViews.get_scratch_view_2D(*coordinates_);
    SharedMemView<DoubleType*>& v_scv_volume = scratchViews.get_me_views(CURRENT_COORDINATES).scv_volume;

    // interpolate to ips and evaluate source
    for ( int ip = 0; ip < AlgTraits::numScvIp_; ++ip ) {

        // nearest node to ip
        const int nearestNode = ipNodeMap_[ip];

        // zero out
        for ( int j = 0; j < AlgTraits::nDim_; ++j )
            w_scvCoords[j] = 0.0;

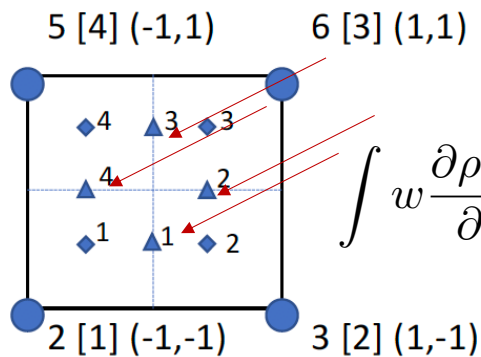
        for ( int ic = 0; ic < AlgTraits::nodesPerElement_; ++ic ) {
            const DoubleType r = v_shape_function_(ip,ic);
            for ( int j = 0; j < AlgTraits::nDim_; ++j )
                w_scvCoords[j] += r*v_coordinates(ic,j);
        }

        rhs(nearestNode) += k_/4.0*(2.0*a_pi_)*(2.0*a_pi_)*
            stk::math::cos(2.0*a_pi_* w_scvCoords[0])
            + stk::math::cos(2.0*a_pi_* w_scvCoords[1])
            + stk::math::cos(2.0*a_pi_* w_scvCoords[2]))*v_scv_volume(ip);
    }
}
```



Deep Dive on CVFEM: Advection Discretization (no stabilization)

- For advection, we have transformed the volume integral to a surface integration
- Therefore, a patch of elements are required for the full assembly at node 2



$$\int w \frac{\partial \rho u_j \phi}{\partial x_j} dV \approx \sum_{ip} (\rho u_j)_{ip} \phi_{ip} n_j dS \approx \sum_{ip} \dot{m}_{ip} \phi_{ip}$$

Notes:

1. Common to integrate-by-parts, however, not required
2. Advection term need not be in divergence form (non-conserved form is suitable)


- Recall, that the mass flow rate at an integration point is prescribed
- Integration points can also be shifted from the sub-control surface to the edge midpoint (while still using the integration point area vector)
- This is a *central-* or *Galerkin-based* advection operator



Deep Dive on CVFEM: Advection Discretization Non-Conserved Form

- The advection term can be integrated by parts, or not; moreover, the PDE can drive a non-conservative equation form:

$$\int w \frac{\partial \rho u_j \phi}{\partial x_j} dV = \int w \rho u_j \phi n_j dS - \int \rho u_j \phi \frac{\partial w}{\partial x_j} dV$$


$$\int w \rho u_j \frac{\partial \phi}{\partial x_j} dV + \int w \phi \frac{\partial \rho u_j}{\partial x_j} dV \quad \int w \frac{\partial \rho \phi}{\partial t} dV = \int w \rho \frac{\partial \phi}{\partial t} dV + \int w \phi \frac{\partial \rho}{\partial t} dV$$

- Non-conserved form, unlike CC and EBVC, provides no added complexity

$$\int w \left(\rho \frac{\partial \phi}{\partial t} + \rho u_j \frac{\partial \phi}{\partial x_j} \right) dV$$



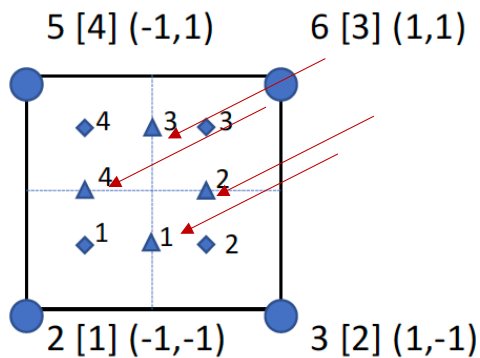
Deep Dive on CVFEM: Advection Discretization; Code

- <https://github.com/NaluCFD/Nalu/blob/master/src/kernel/ScalarAdvDiffElemKernel.C>
- This routine includes advection and diffusion
- Recall that integration point value is provided by nodal loop over the underlying nodal basis for this element
- Also note that this routine is valid for all types of supported elements – both low- and higher-order (polynomial promotion – coming later in the Quarter)



Deep Dive on CVFEM: Diffusion Discretization

- For diffusion, we have transformed the volume integral to a surface integration
- Therefore, a patch of elements are required for the full assembly at node 2
- Note that the CVFEM approach is absent any non-orthogonality corrections;
- However, high aspect ratio elements are now challenging...



$$\int w \frac{\partial q_j}{\partial x_j} dV \approx - \sum_{ip} \frac{\mu}{Sc_{ip}} \frac{\partial \phi}{\partial x_j} n_j dS = - \sum_{ip} \frac{\mu}{Sc_{ip}} \sum_{nd} \frac{\partial N_{nd}^{ip}}{\partial x_j} \phi_{nd} A_j^{ip}$$

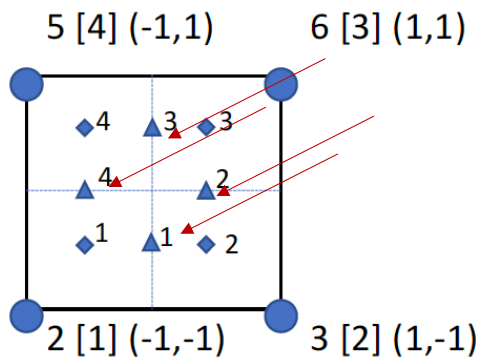
Recall our underlying basis: $\phi_{ip} = \sum_n N_n^{ip} \phi_n$

$$\frac{\partial \phi}{\partial x_j} = \sum_n \frac{\partial N_n^{ip}}{\partial x_j} \phi_n$$



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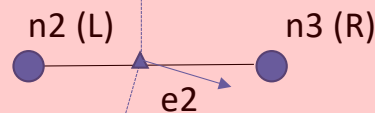


$$\int w \frac{\partial q_j}{\partial x_j} dV \approx - \sum_{ip} \frac{\mu}{Sc_{ip}} \frac{\partial \phi}{\partial x_j} n_j dS = - \sum_{ip} \frac{\mu}{Sc_{ip}} \sum_{nd} \frac{\partial N_{nd}^{ip}}{\partial x_j} \phi_{nd} A_j^{ip}$$

Recall our underlying basis: $\phi_{ip} = \sum_n N_n^{ip} \phi_n$

$$\frac{\partial \phi}{\partial x_j} \Big|_{ip} = \sum_n \frac{\partial N_n^{ip}}{\partial x_j} \phi_n$$

No non-orthogonality!

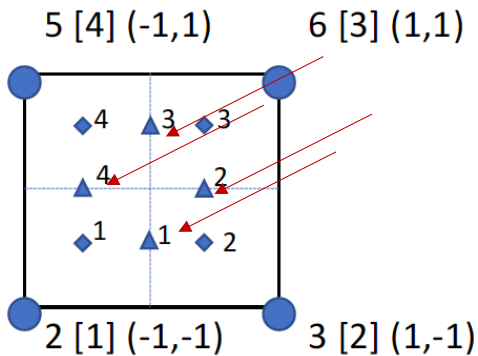


$$\frac{\partial \phi}{\partial x_j} \Big|_{ip} = G_j^{ip} \phi + \left[(\phi_R - \phi_L) - G_l^{ip} \phi \Delta x_l \right] \frac{A_j^{ip}}{A_k \Delta x_k}$$



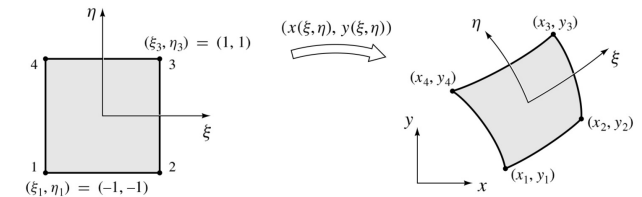
Deep Dive on CVFEM: Diffusion Discretization

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$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{bmatrix}$$

$$\begin{pmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{pmatrix} = J^{-1} \begin{pmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{pmatrix}$$

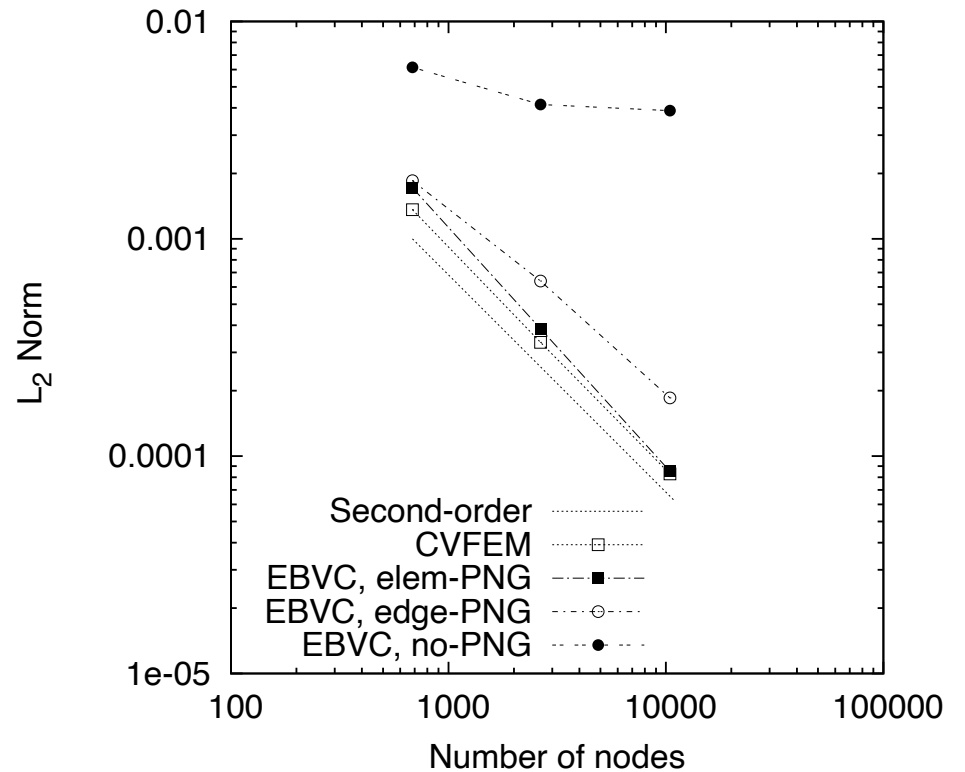
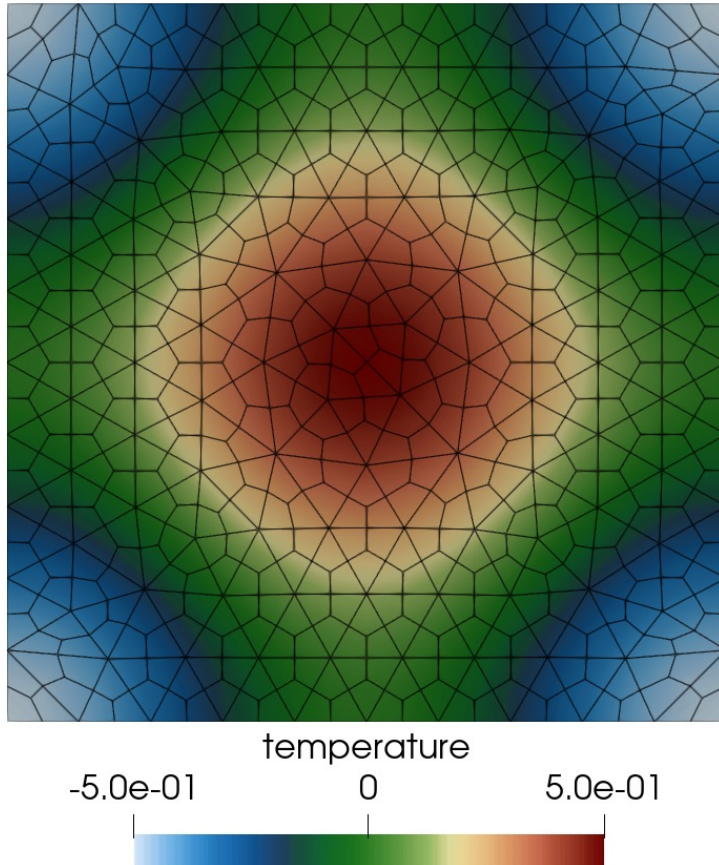


$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\int w \frac{\partial q_j}{\partial x_j} dV \approx - \sum_{ip} \frac{\mu}{S_{c_{ip}}} \frac{\partial \phi}{\partial x_{j_{ip}}} n_j dS = - \sum_{ip} \frac{\mu}{S_{c_{ip}}} \sum_{nd} \frac{\partial N_{nd}^{ip}}{\partial x_j} \phi_{nd} A_j^{ip}$$



Deep Dive on CVFEM: Diffusion Discretization; Code





A Note on Conservation

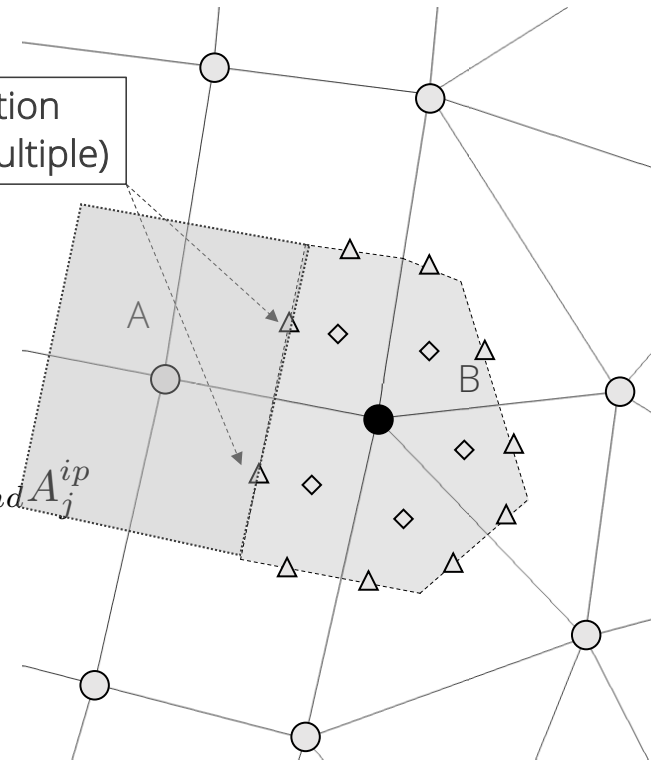
The CC, EBVC, and CVFEM all have one aspect in common.....

- Specifically, a flux contribution is evaluated at the control volume face, and dotted with the area surface normal vector
- This quantity, kg (stuff)/s, “leaves” control volume A, and fully enters control volume B and is *conserved*

Surface integration point (single vs multiple)

$$\int w \frac{\partial q_j}{\partial x_j} dV \approx - \sum_{ip} \frac{\mu}{S_{cip}} \frac{\partial \phi}{\partial x_j} n_j dS = - \sum_{ip} \frac{\mu}{S_{cip}} \sum_{nd} \frac{\partial N_{nd}^{ip}}{\partial x_j} \phi_{nd} A_j^{ip}$$

- For each integration point, the Left and Right nodal state is defined into which flux contributions are assembled
- No loss of mass/momentum/energy/etc.





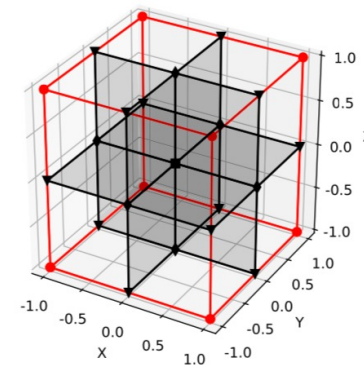
A Note on Polymorphism: Class Shape : Circle/Square/etc.

Recall, that the element type has been characterized by a set of unique attributes, take for example, a Hex8:

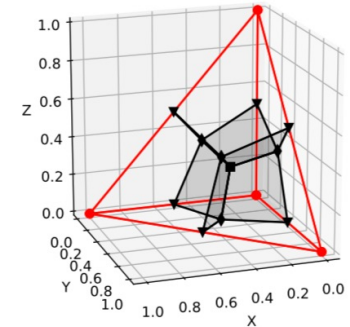
- Nodes Per Element: 8
- Number of surface integration points: 12
- Number of volume integration points: 8
- Number of Faces: 6
- Face Topology: Quad4
- Assembly Algorithms can be templated, or expecting an integration rule defined by the element type

Pseudo C++

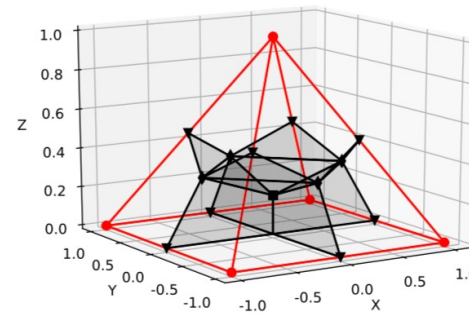
```
>Shape mySquare = new Square();  
>Shape myCircle = new Circle();  
>mySquare->volume();  
>myCircle->volume();
```



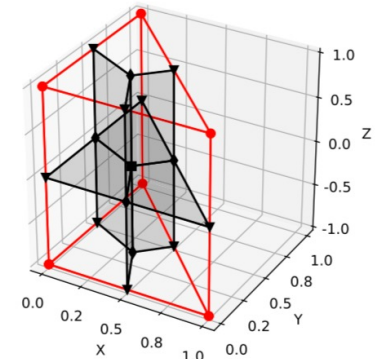
(a) Hexahedral topology (Hex8).



(b) Tetrahedral topology (Tet4).



(c) Pyramid topology (Pyramid5).



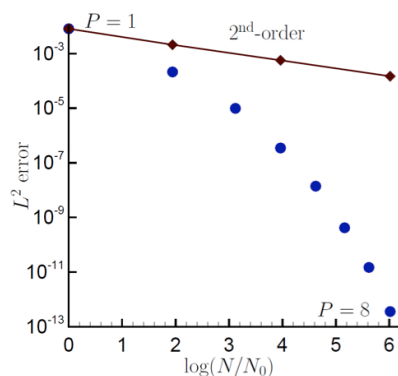
(d) Wedge topology (Wedge6).

Fig. 1. CVFEM element and dual-volume definition for the low-order topologies.

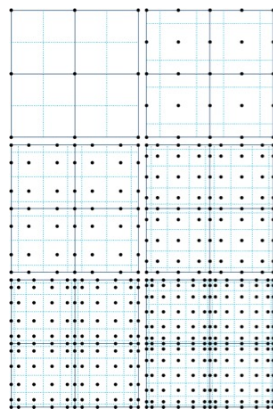


Higher-Order via Polynomial Promotion

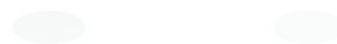
- CVFEM (like EBVC and CC) can be viewed as Petrov-Galerkin method
 - Recall how this is defined?
- Basis can also be promoted (linear to quadratic, etc), i.e., *Polynomial Promotion*, Domino, *CTRSP* (2014) as a first example of low-Mach fluids algorithm – or Domino, *JCP* (2018)
- Research Thrust: Possible higher efficiency on NGP due to increased local work)
- However, suitability of higher-order for LES is an open argument – especially when other errors/uncertainties exist



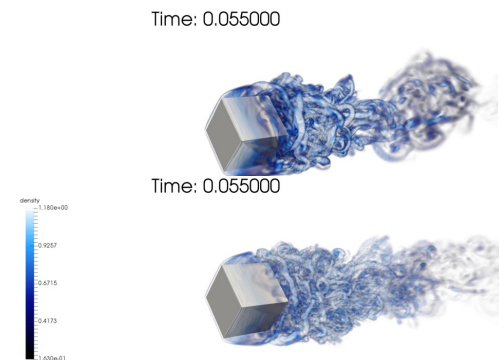
Spectral convergence



Dual-volume for promoted quad4



$P=1$ (left) and $P=4$ (right)
Helium plume (VR-density)



Rotating cube (Re 4000,
RPM 3600) $P=1$ (top) and
 $P=2$ (bottom)