Exceptional service in the national interest

ME469: Boundary Conditions

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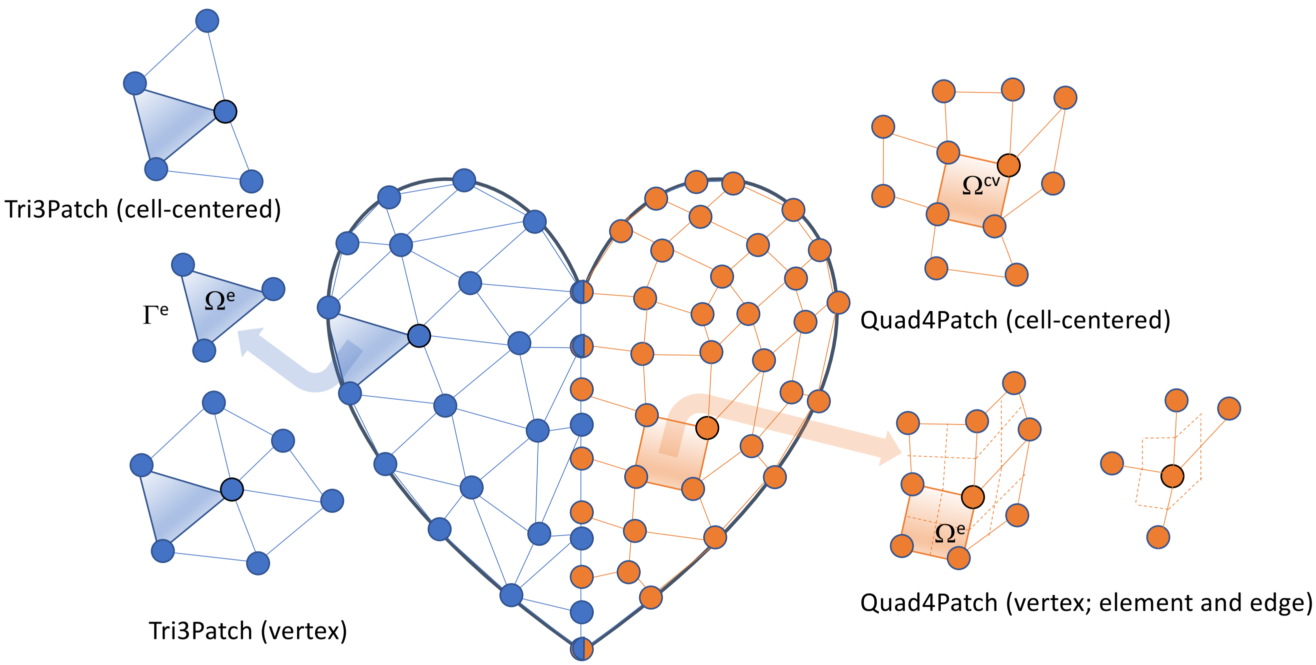
Review of Discretization Options: New, emphasis on boundaries…

- Degree-of-freedom (DOF) for:
  - Cell-centered: Stencil is based on a element:face:element
  - DOFs at vertices of elements, or “nodes”, element:node (CVFEM, FEM), edge:node (EBVC)



- Definition of an interpolation function:

$$\phi_{ip} = \sum_n N_n^{ip} \phi_n$$

- N_n^{ip} is the Lagrange function associated with node n
- ϕ_n is the value of the DOF at node n
- The nodal basis functions obey equipartition of unity and satisfy, $N_n^{x_j} = \delta_{nj}$

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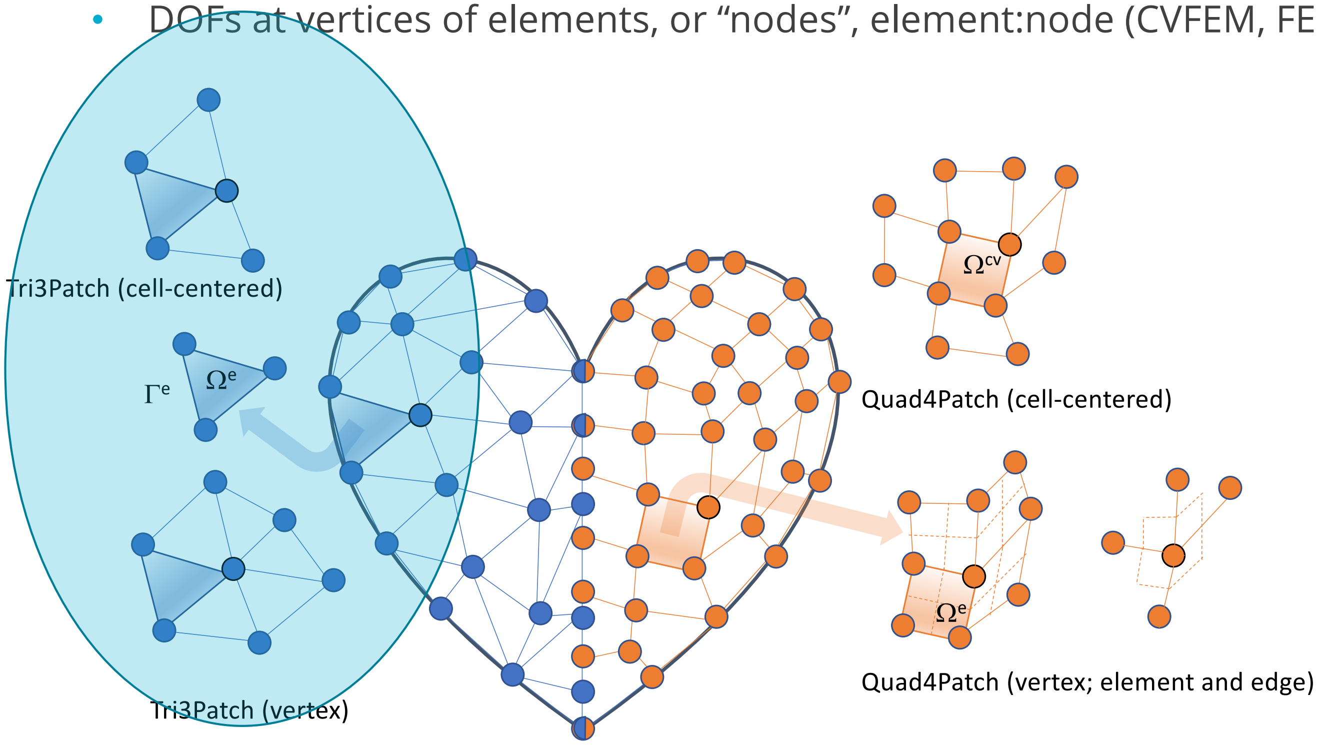

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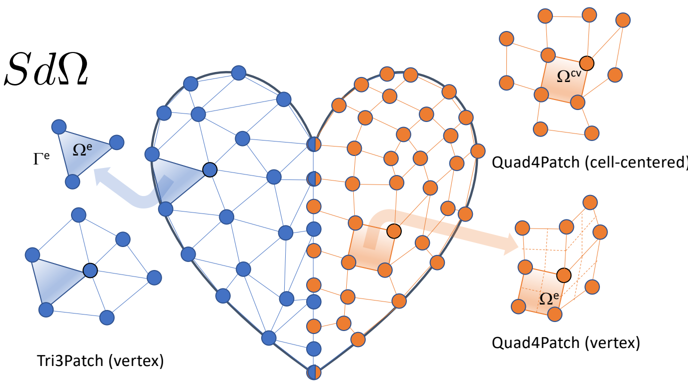
Integration Over the Domain: The "Finite" in Finite-Volume and Finite Element

- Consider a simple model equation with the heart domain in mind:

$$\frac{\partial F_j}{\partial x_j} = S$$

Where F_j is a flux and S is a source term

- Integrating over the entire domain, Ω: $$\int_\Omega \frac{\partial F_j}{\partial x_j} d\Omega = \int_\Omega S d\Omega$$



- Without loss of generality, let us define a set of subdomains, Ω_k:

$$\sum_k \int_{\Omega_k} \frac{\partial F_j}{\partial x_j} d\Omega_k = \sum_k \int_{\Omega_k} S d\Omega_k$$

- As present, only volumetric integrals appear

Note:

- The formality of Σ_k and Ω_k is implied to exist over the full domain and is often times dropped – integral type implied by dV (volume) and dS (domain boundary)

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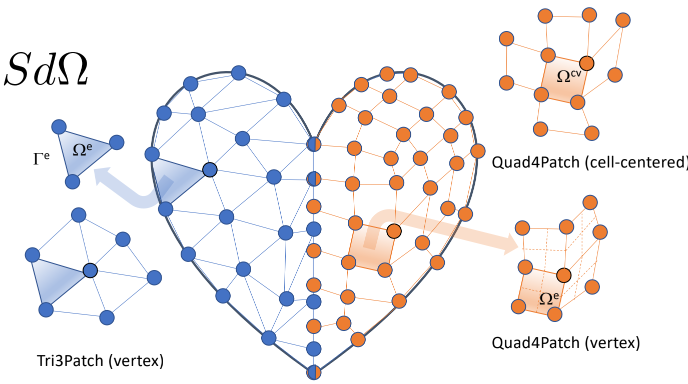

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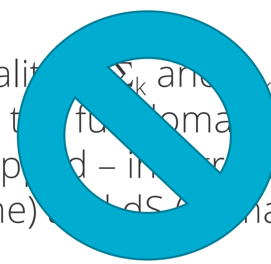

Towards Boundary Contributions

- Given a partial differential equation (PDE) and associated volumetric form:

$$\int_\Omega \frac{\partial F_j}{\partial x_j} d\Omega = \int_\Omega S d\Omega$$

- Applying Gauss Divergence provides the standard finite volume form for fluxes in surface integral form (no distinction between internal control volume faces and boundary faces):

$$\sum_k \int_{\Omega_k} \frac{\partial F_j}{\partial x_j} d\Omega_k = \sum_k \int_{\Omega_k} S d\Omega_k \longrightarrow \sum_k \int_{\Gamma_k} F_j n_j d\Gamma_k = \sum_k \int_{\Omega_k} S d\Omega \longrightarrow \int F_j n_j dS = \int S dV$$

- We can also multiple PDE by an arbitrary test function, w, and integrate over a volume,

$$\int w \frac{\partial F_j}{\partial x_j} dV = \int w S dV$$

Next, integrate by parts and apply Gauss-Divergence. Note, that test function must be differentiable – shown here, at least once..

$$-\int F_j \frac{\partial w}{\partial x_j} dV + \int \frac{\partial w F_j}{\partial x_j} dV = -\underset{\substack{\uparrow \\ \text{Interior}}}{\int} F_j \frac{\partial w}{\partial x_j} dV + \underset{\substack{\uparrow \\ \text{Boundary}}}{\int} w F_j n_j dS$$

$$\frac{\partial w F_j}{\partial x_j} = w \frac{\partial F_j}{\partial x_j} + \frac{\partial w}{\partial x_j} F_j$$

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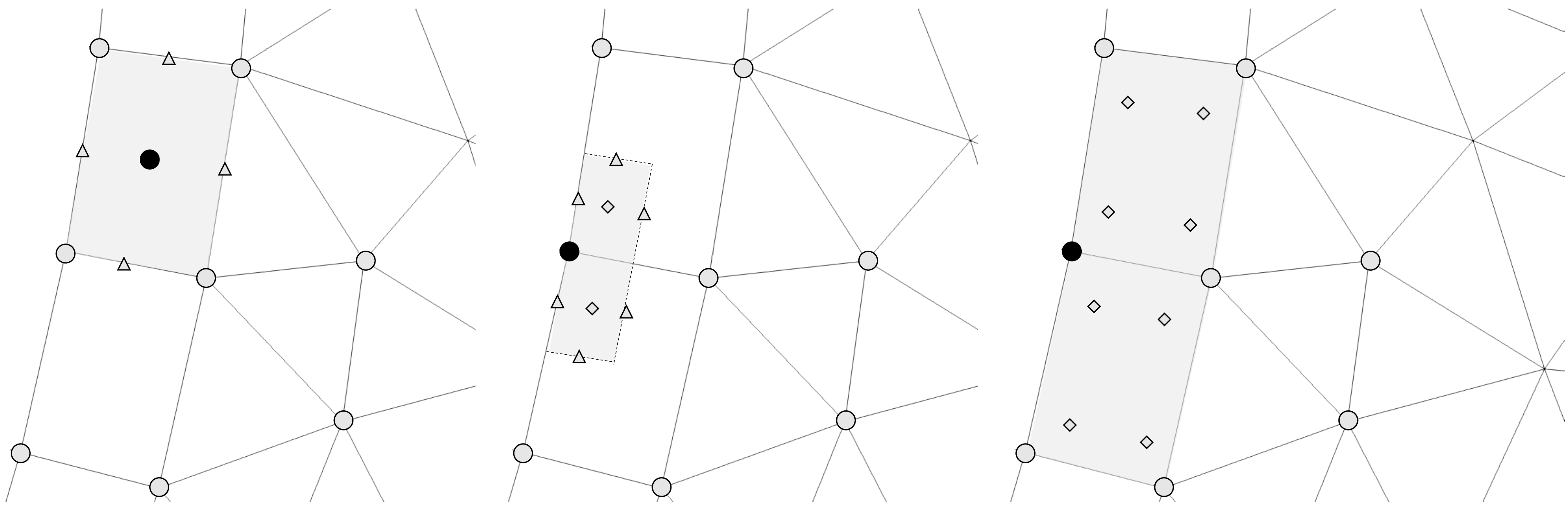
Cell-Centered vs Vertex-Centered Differences

- For a CC scheme, boundary condition is flux-based, while for VC and FEM, there is an option of flux-based, i.e., “weak”, or Dirichlet, i.e., “strong”



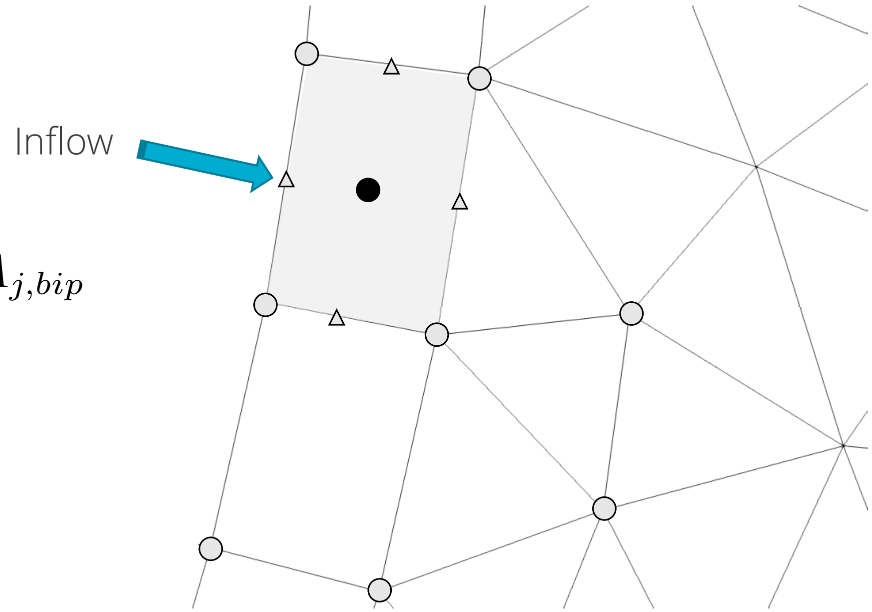
CC VC FEM

For Example, a Cell-Centered Inflow Boundary

Consider an inflow boundary condition where the advection term is of interest

- You may have a pipe flow with a given inflow velocity to match a Reynolds number:
- A well-posed condition provides the inlet density, velocity and scalar values

$$\int \rho u_j \phi n_j dS \approx \sum_{bip} \dot{m}_{bip} \phi_{bip}^{spec} A_{j,bip}$$



For Example, a Vertex-Centered (EBVC/CVFEM) Inflow Boundary Flux-based (Weak treatment)

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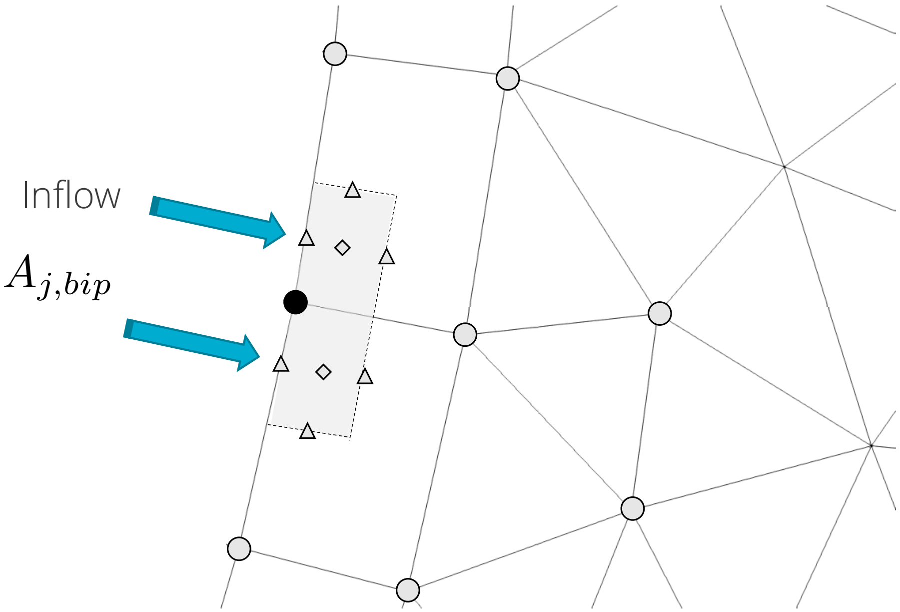


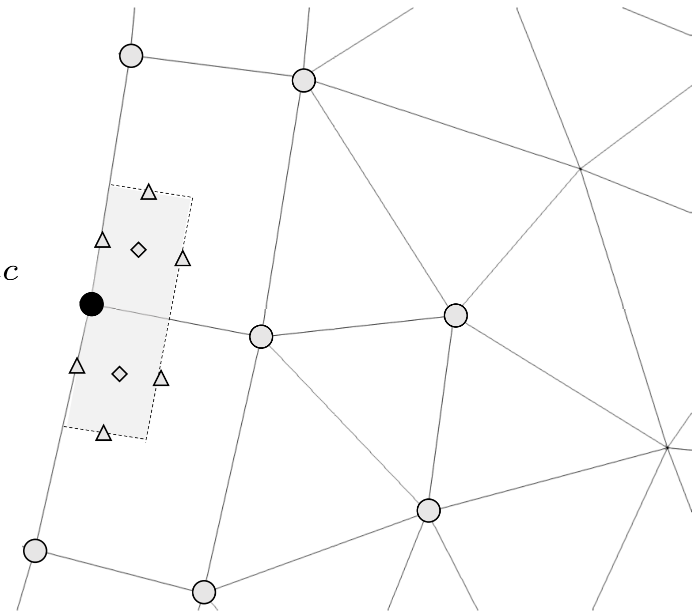
For Example, a Vertex-Centered (EBVC/CVFEM) Inflow Boundary Dirichlet-based (Strong treatment)

Consider an inflow boundary condition where the advection term is of interest

- You may have a pipe flow with a given inflow velocity to match a Reynolds number:
- A well-posed condition provides the inlet density, velocity and scalar values

Inflow

$$\phi = \phi^{spec}$$



Dirichelt conditions are applied when you know the precise value at the boundary, i.e., an inflow, or a wall

Often times, boundary conditions are naturally flux-based, e.g., at an adiabatic wall you know the heat flux, open-bc (zero flux), etc.

Weak vs Strong: A Practical Heat Conduction Example

Consider a simple heat conduction solution in which temperature is known at the boundaries, $T = T^{spec}$

- Governing Equation:

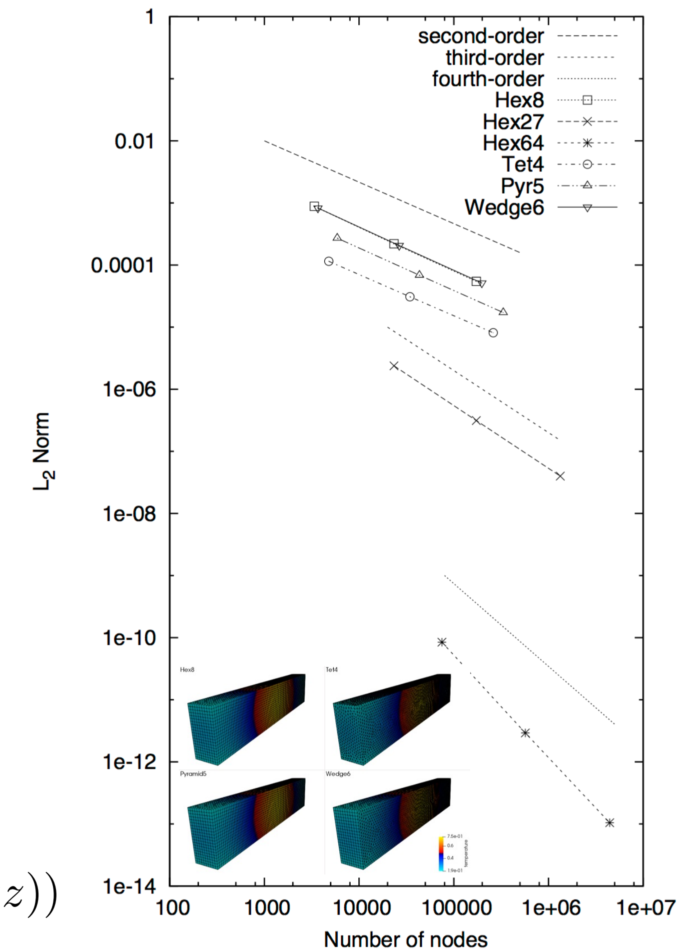
$$\int \rho C_p \frac{\partial T}{\partial t} dV - \int \lambda \frac{\partial T}{\partial x_j} n_j dS = 0$$

- It can be shown (see Svard and Nordstrom, JCP, 2008) that a stable and accurate weak BC implementation is provided by:

$$q_n = -\lambda \frac{\partial T}{\partial x_j} n_j + \gamma \frac{\lambda}{L} \left(T - T^{spec}\right)$$

- Consider the MMS temperature solution (and convergence to the right):

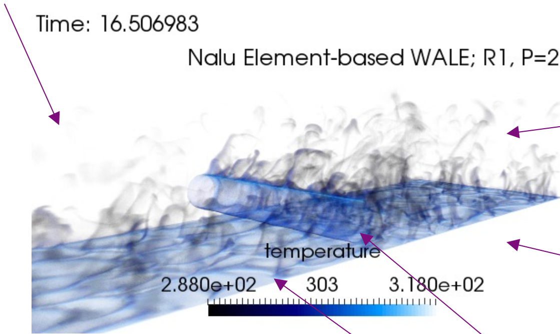
$$T^{mms}(x) = \frac{k}{4}\left(cos(2\pi x) + cos(2\pi y) + cos(2\pi z)\right)$$



Types of Boundary Conditions in Practical CFD

Flow-past a heated cylinder

Symmetry, "flow slips along" (normal stress $pn_i dS$ applied with tangential stress = 0)



Inflow
(known T and u_i)

Open
(known P and far field temperature, zero normal gradient assumption)

Span-wise periodic

Wall (known T and u_i)

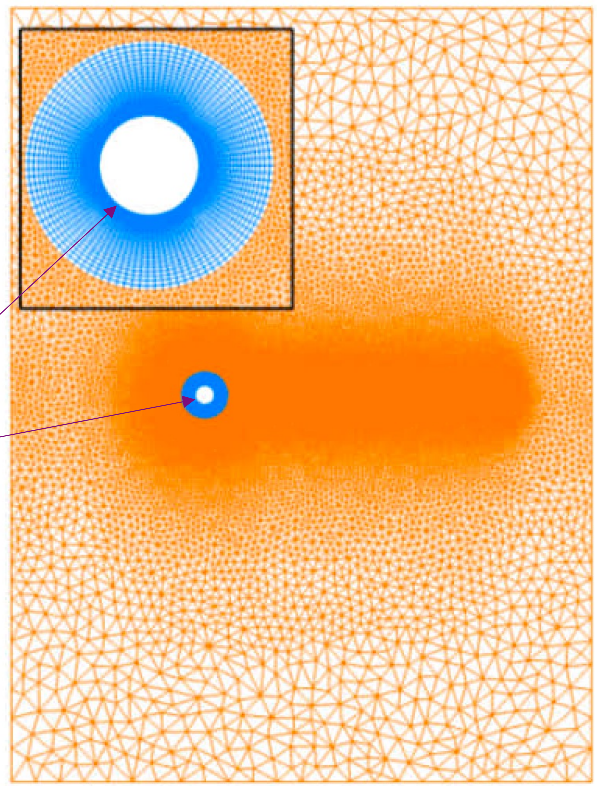
Types of Boundary Conditions in Practical CFD

Flow past a 2-dimensional cylinder (the midterm assignment)

Symmetry, "flow slips along" (normal stress $pn_i dS$ applied with tangential stress = 0)

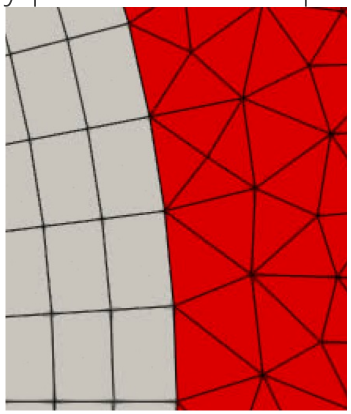
Inflow
(known u_i)

Wall (known Z and u_i)



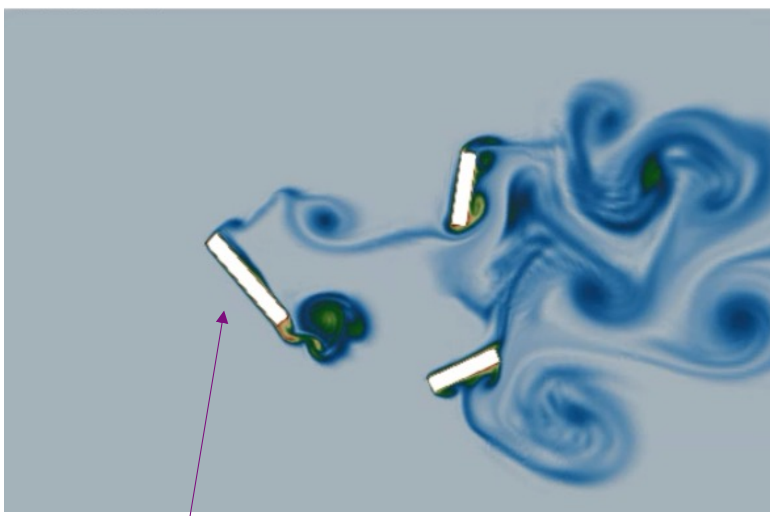
Open
(known P and far field mixture fraction, zero normal gradient assumption)

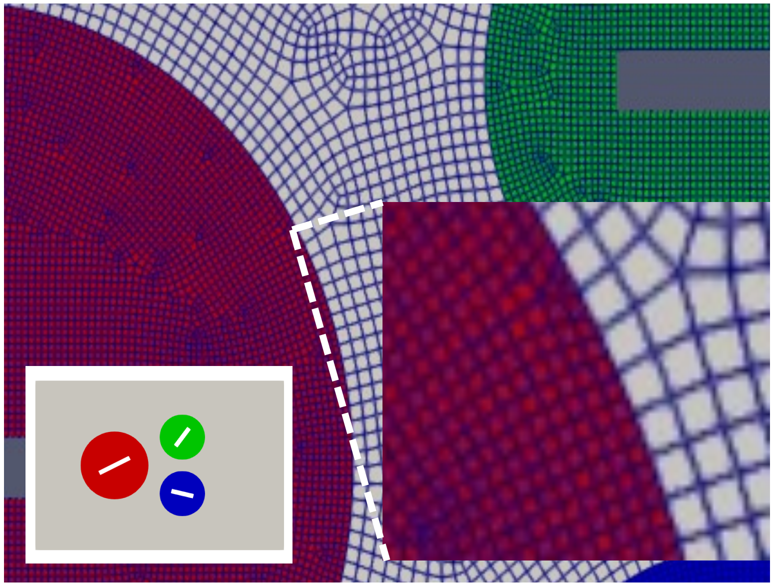
Conformal boundary at the Quad4/Tri3 interface, hence, no boundary procedure is required



Types of Boundary Conditions in Practical CFD

Flow past a series of rotating blades: Zero normal gradient at open bc?





Mathematical Description

See: https://nalu.readthedocs.io/en/latest/source/theory/boundaryConditions.html