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# ME469: Geometric Fidelity

Stefan P. Domino<sup>1,2</sup>

<sup>1</sup> Computational Thermal and Fluid Mechanics, Sandia National Laboratories

<sup>2</sup> Institute for Computational and Mathematical Engineering, Stanford

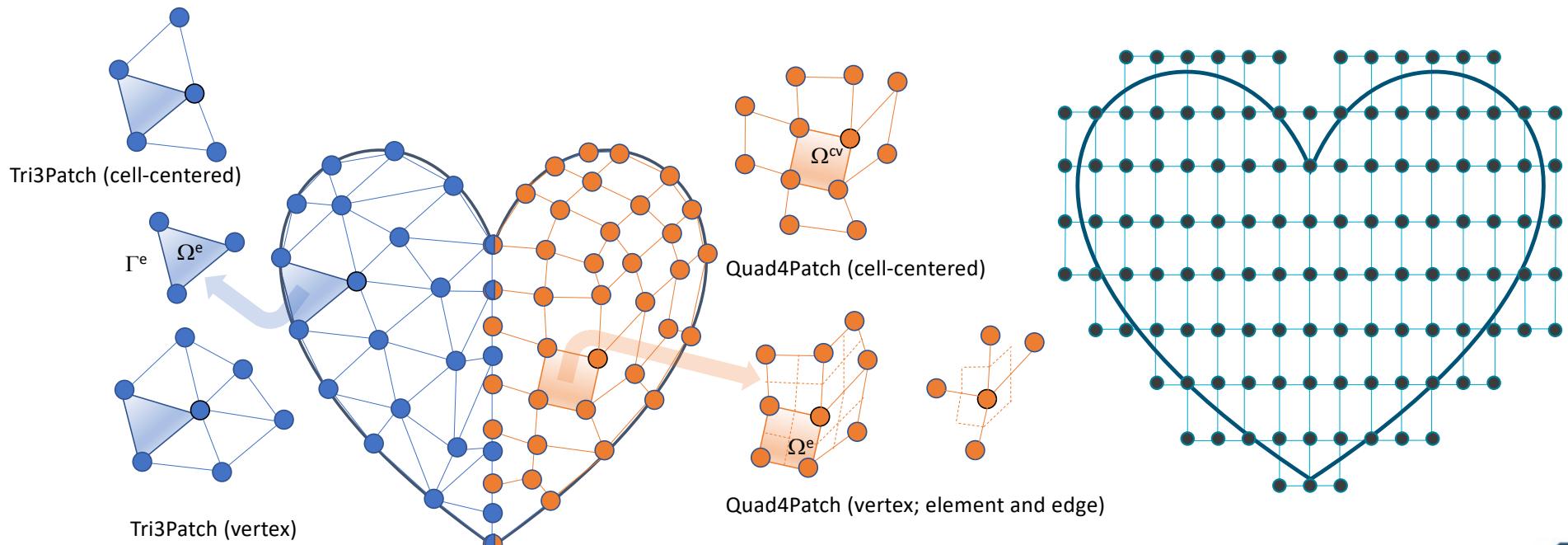
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## Review of Discretization Options

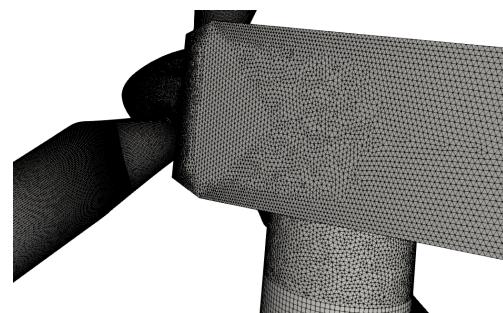
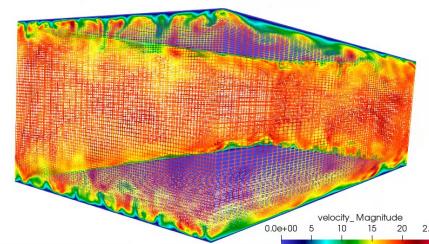
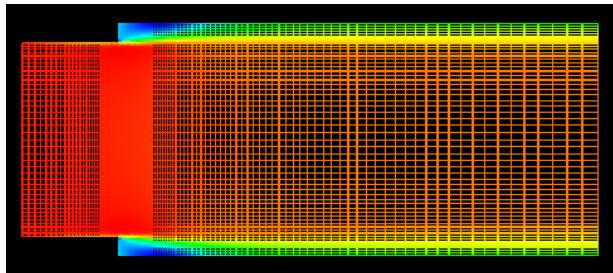
- Degree-of-freedom (DOF) for:
  - Cell-centered: Stencil is based on an element:face:element
  - DOFs at vertices of elements, or “nodes”, element:node (CVFEM, FEM), edge:node (EBVC)





## Simple vs Complex Domains

- Some Geometries are simple, while others are complex





## Simple One-Dimensional Diffusion Example

Let us introduce (briefly) the Method of Manufactured Solutions (MMS)

- Consider a simple diffusion equation:

$$\frac{d^2\phi}{dx^2} = 0$$

with a presumed, or manufactured solution:

$$\phi^{MMS}(x) = x^2$$

$$\frac{d^2\phi^{MMS}}{dx^2} = S^{MMS} = 2$$

Gauss-Divergence

$$\int \frac{d^2\phi^h}{dx^2} dV = \int S^{MMS} dV \quad \int \frac{d\phi^h}{dx} dS = \int S^{MMS} dV$$

Discrete flux-based form at the sub-control volume surface (scs) and sub-control volume IP:

With error:  $\epsilon = \phi^{MMS} - \phi^h$

$$\sum_{scsIP} \frac{d\phi^h}{dx} A_{scsIP} = \sum_{scvIP} S^{MMS} V_{scvIP}$$

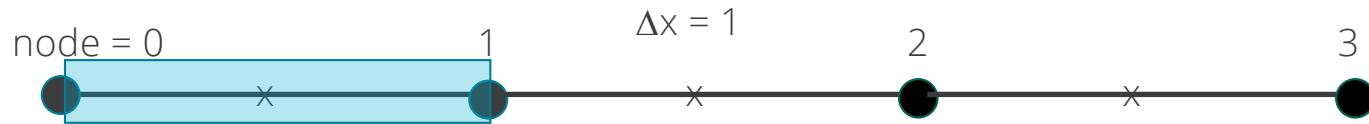


First....

- Let's review matrix assembly from a previous lecture, with a focus on a verification "patch test"
- Recall, a linear basis was able to reconstruct a quadratic solution perfectly when the norm was computed at the degree of freedom locations

## Simple One-D MMS Example; Edge-based Assemble of Diffusion

We will use a linear basis and solve this system over a small patch of linear bar elements



Recall, simplified equation with RHS from source term already provided

$$\sum_{scsIP} \frac{d\phi^h}{dx}_{scsIP} = \sum_{scvIP} S^{MMS} \frac{\Delta x}{2}$$

Iterate elements with a simple left-hand side rule:  $+= L$  and  $-= R$   
 (note implicit conservation statement)

$$\frac{d\phi^h}{dx}_{scsIP} = \frac{\phi_1^h - \phi_0^h}{\Delta x} + C_0(\Delta x) + \dots$$

$$LHS_0+ = \frac{\phi_1^h - \phi_0^h}{\Delta x}$$

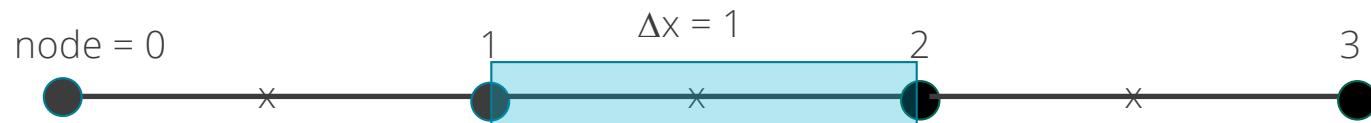
$$LHS_1- = \frac{\phi_1^h - \phi_0^h}{\Delta x}$$

Edge 1



## Simple One-D MMS Example; Edge-based Assemble of Diffusion

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Iterate elements with a simple left-hand side rule:  $+= L$  and  $-= R$

(note implicit conservation statement)

$$\frac{d\phi^h}{dx}_{scsIP} = \frac{\phi_2^h - \phi_1^h}{\Delta x} + C_0(\Delta x) + \dots$$

$$LHS_1+ = \frac{\phi_2^h - \phi_1^h}{\Delta x}$$

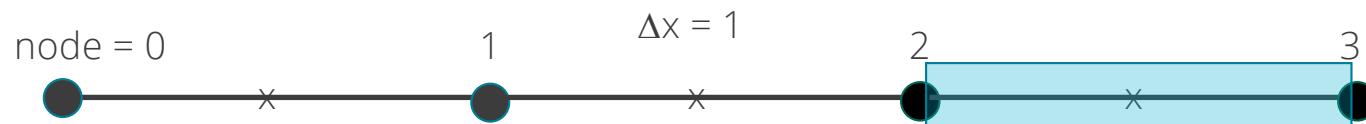
$$LHS_2- = \frac{\phi_2^h - \phi_1^h}{\Delta x}$$

Edge 2



## Simple One-D MMS Example; Edge-based Assemble of Diffusion

We will use a linear basis and solve this system over a small patch of linear bar elements



Recall, simplified equation with RHS from source term already provided

$$\sum_{scsIP} \frac{d\phi^h}{dx}_{scsIP} = \sum_{scvIP} S^{MMS} \frac{\Delta x}{2}$$

Iterate elements with a simple left-hand side rule: += L and -=R

(note implicit conservation statement)

$$\frac{d\phi^h}{dx}_{scsIP} = \frac{\phi_3^h - \phi_2^h}{\Delta x} + C_0(\Delta x) + \dots$$

$$LHS_2+ = \frac{\phi_3^h - \phi_2^h}{\Delta x}$$

$$LHS_3- = \frac{\phi_3^h - \phi_2^h}{\Delta x}$$

Edge 3



## Simple One-D MMS Example; Collect all of the terms

$$LHS_0+ = \frac{\phi_1^h - \phi_0^h}{\Delta x}$$

$$LHS_1- = \frac{\phi_1^h - \phi_0^h}{\Delta x}$$

$$LHS_1+ = \frac{\phi_2^h - \phi_1^h}{\Delta x}$$

$$LHS_2- = \frac{\phi_2^h - \phi_1^h}{\Delta x}$$

$$LHS_2+ = \frac{\phi_3^h - \phi_2^h}{\Delta x}$$

$$LHS_3- = \frac{\phi_3^h - \phi_2^h}{\Delta x}$$

$$A = \frac{1}{\Delta x} \begin{bmatrix} -1 & +1 & 0 & 0 \\ +1 & -2 & +1 & 0 \\ 0 & +1 & -2 & 1 \\ 0 & 0 & +1 & -1 \end{bmatrix}$$

Fully assembled matrix

$$\frac{1}{\Delta x} \begin{bmatrix} -1 & +1 & 0 & 0 \\ +1 & -2 & +1 & 0 \\ 0 & +1 & -2 & 1 \\ 0 & 0 & +1 & -1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

Fully assembled system

$$\frac{1}{\Delta x} \begin{bmatrix} +1 & 0 & 0 & 0 \\ +1 & -2 & +1 & 0 \\ 0 & +1 & -2 & 1 \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 9 \end{bmatrix}$$

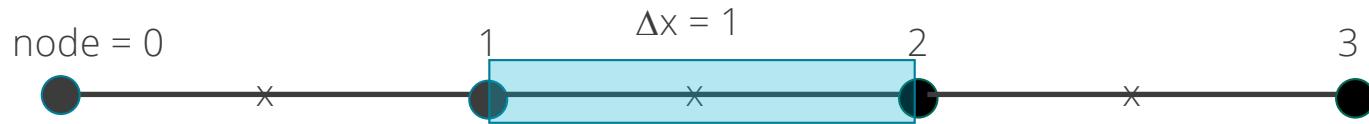
Correct for BCs

$$\phi^h = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix}$$

Solution is exact.... Linear basis exactly captures a quadratic solution

## Simple One-D Advection; Edge-based Assembly

Similar approach: Iteration of edges, definition of Left and Right node, etc.



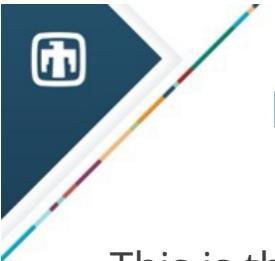
$$\int w \frac{\partial \rho u_j \phi}{\partial x_j} dV \approx \sum_{ip} (\rho u_j)_{ip} \phi_{ip} n_j dS \approx \sum_{ip} \dot{m}_{ip} \phi_{ip} \quad \dot{m}_{ip} = \rho u_j n_j dS|_{ip} = \rho u_j A_j|_{ip}$$

We may also create a local matrix data structure, A[4]: With:

- $A[0] += \text{lhsFac}$  (row node 1, column node 1, "central-coeff")  $LHS_1+ = \frac{\dot{m}}{2} (\phi_1^h + \phi_2^h)$
- $A[1] += \text{lhsFac}$  (row node 1, column node 2, "east-coeff")
- $A[2] -= \text{lhsFac}$  (row node 2, column node 1, "west-coeff")  $LHS_2- = \frac{\dot{m}}{2} (\phi_1^h + \phi_2^h)$
- $A[3] -= \text{lhsFac}$  (row node 2, column node 2, "central-coeff")

With:  $\text{lhsFac} = \frac{\dot{m}}{2}$

Edge 2

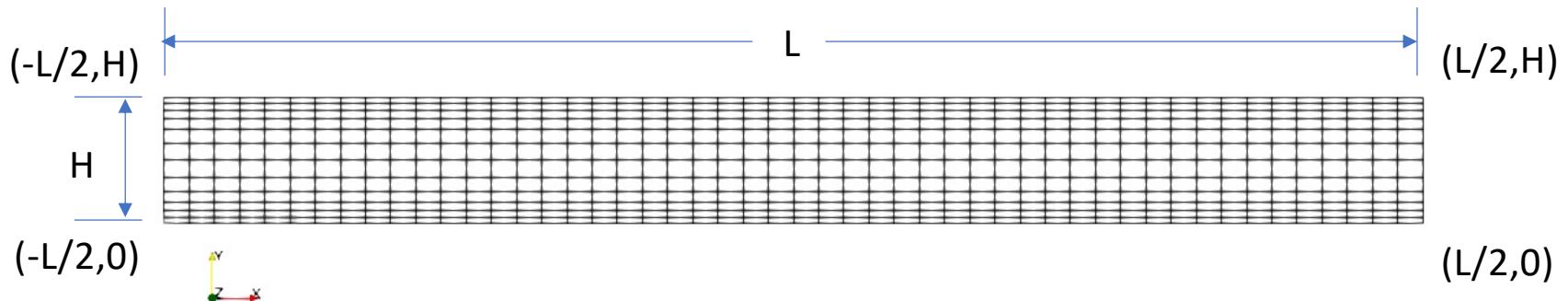


## Let's Revisit a Classic 2-dimensional Channel Flow (laminar)

This is the 2d\_quad4\_channel laboratory:

[https://github.com/NaluCFD/Nalu/blob/master/reg\\_tests/test\\_files/laboratory/2d\\_quad4\\_channel/write\\_up/2d\\_quad4\\_channel\\_laboratory.pdf](https://github.com/NaluCFD/Nalu/blob/master/reg_tests/test_files/laboratory/2d_quad4_channel/write_up/2d_quad4_channel_laboratory.pdf)

- No-slip (walls) top and bottom
- Periodic (left and right) with a constant body force ...or... Open (left and right pressure drop)





## Derivation

### 3.1 Analytical Velocity Profile

Given the assumptions provided in the introduction, the streamwise velocity equation reduces to,

$$\frac{dP}{dx} = \mu \frac{d^2 u_x}{dy^2}. \quad (4)$$

This equation can be integrated twice to obtain,

$$u_x(y) = \frac{1}{\mu} \frac{dP}{dx} \frac{y^2}{2} + k_1 y + k_2, \quad (5)$$

where  $k_1$  and  $k_2$  are constants of integration that are obtained through the application of boundary conditions,  $u_x(y = 0) = 0$  and  $u_x(y = H) = 0$ . Therefore, the final expression for the streamwise velocity is,

$$u_x(y) = \frac{1}{2\mu} \frac{dP}{dx} [y^2 - Hy]. \quad (6)$$

The above equation can be used to determine the vertical location at which the maximum velocity is found via solving  $\frac{du_x}{dy} = 0$ , or  $u_x^{max}$  occurs at  $H/2$ . The functional form for the maximum velocity is,

$$u_x^{max} = \frac{1}{8\mu} \frac{dP}{dx} H^2. \quad (7)$$



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## Sizing

- Let's run at  $Re_\tau = 10$

Reynolds number:

- Ratio of inertial to viscous forces

$$Re_D = \frac{\rho u D}{\mu}$$

- In this channel, define Reynolds number based on the wall friction velocity,  $u_\tau$  (at  $H/2$  length scale)

$$Re_\tau = \frac{\rho u_\tau H/2}{\mu}$$

## 4 Results

Let us test a simulation in which the Reynolds number based on wall friction velocity,  $u^\tau$ , and half-channel height  $H/2$ , is ten:  $Re^\tau = 10$ . By constraining the Reynolds number, wall friction velocity, and density, the consistent viscosity is obtained via,

$$\mu = \frac{\rho u^\tau H}{2Re^\tau}. \quad (8)$$

To obtain the required pressure gradient, we exercise the relationship,

$$u^\tau = \sqrt{\frac{\tau_w}{\rho}}, \quad (9)$$

along with global momentum balance,

$$\int \frac{dP}{dx} dV = \int \tau_w dA, \quad (10)$$

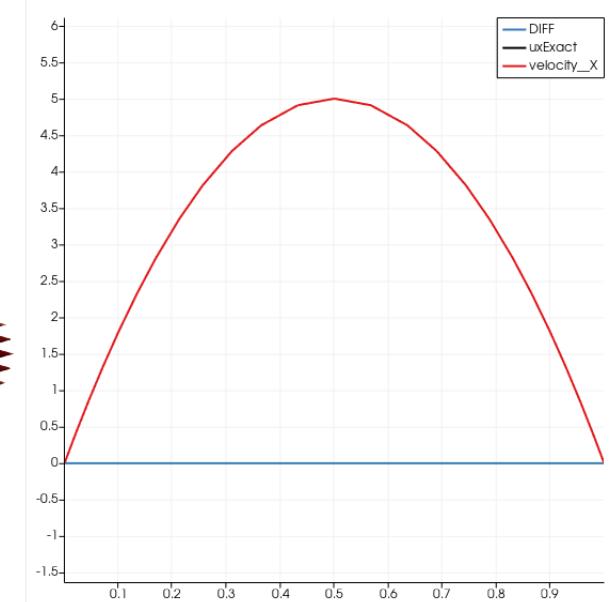
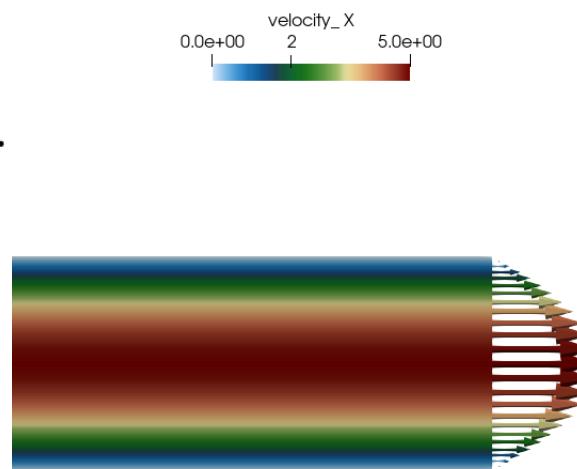
with  $dV = LH$  and  $dA = 2L$ , to obtain the relationship between required pressure gradient and wall shear stress,  $\frac{dP}{dx} = 2\tau_w$ .

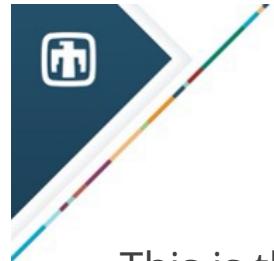
## Simulation Results

Analytical Expression:

$$u_x(y) = \frac{1}{2\mu} \frac{dP}{dx} [y^2 - Hy] .$$

Zero Error when evaluating the norm at the nodes (in general, we may integrate over a domain and choose integration points that are not *nodally-lumped*)



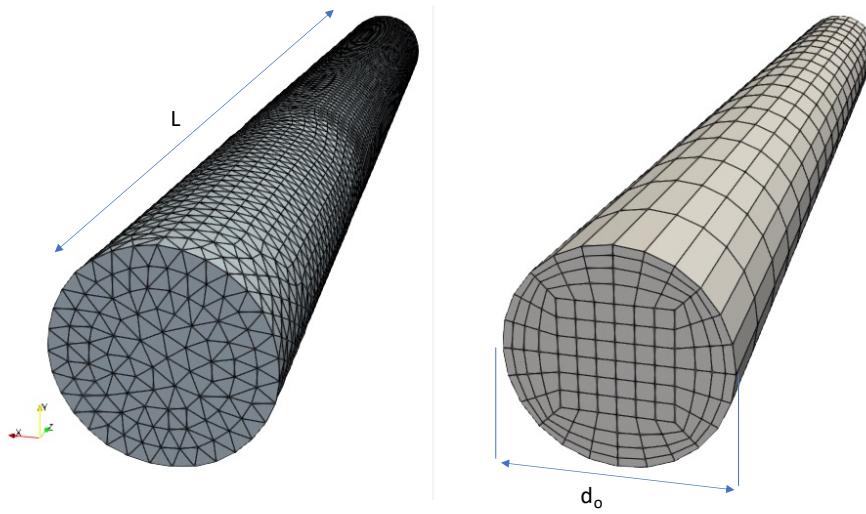


## Let's Revisit a Classic 3-dimensional Pipe Flow (laminar)

This is the 3d\_tet4\_pipe laboratory:

[https://github.com/NaluCFD/Nalu/tree/master/reg\\_tests/test\\_files/laboratory/3d\\_tet4\\_pipe/write\\_up](https://github.com/NaluCFD/Nalu/tree/master/reg_tests/test_files/laboratory/3d_tet4_pipe/write_up)

- No-slip (walls) outer pipe
- Periodic (left and right) with a constant body force ...or... Open (left and right pressure drop)
- We have Tet4 (left) and a Hex8 (right) mesh





## Derivation

### 3.1 Analytical Velocity Profile

Given the assumptions provided in the introduction, the axial momentum equation reduces to,

$$\frac{dP}{dz} = \frac{1}{r} \frac{\partial r \tau_{rz}}{\partial r}, \quad (11)$$

where  $\tau_{rz} = \mu \frac{\partial u_z}{\partial r}$ . Integration once results in the following equation,

$$\frac{r^2}{2} \frac{dP}{dz} = r \mu \frac{\partial u_z}{\partial r} + k_1, \quad (12)$$

which, based on  $\frac{\partial u_z}{\partial r} = 0$  at the centerline  $r = 0$ , yields  $k_1 = 0$ . A second integration results in,

$$u_z(r) = \frac{r^2}{4\mu} \frac{dP}{dz} + k_2. \quad (13)$$

Applying  $u_z(r = R) = 0$ , yields the following expression for the axial velocity as a function of radial position,

$$u_z(r) = \frac{1}{4\mu} \frac{dP}{dz} [r^2 - R^2]. \quad (14)$$

The above equation can be used to determine the location at which the maximum velocity is found via solving  $\frac{du_z}{dr} = 0$ , or  $u_z^{max}$  occurs at  $r = 0$ . The functional form for the maximum velocity is,

$$u_x^{max} = -\frac{R^2}{4\mu} \frac{dP}{dZ}. \quad (15)$$



## Sizing

- Let's run at  $Re_\tau = 20$

Reynolds number:

- Ratio of inertial to viscous forces

$$Re_D = \frac{\rho u D}{\mu}$$

- In this channel, define Reynolds number based on the wall friction velocity,  $u_\tau$  (at  $d_o$  length scale)

$$Re_\tau = \frac{\rho u_\tau d_o}{\mu}$$

## 4 Results

Let us test a simulation in which the Reynolds number based on wall friction velocity,  $u^\tau$ , and diameter of the pipe ( $0.02\text{ m}$ ) is  $Re^\tau = 20$ . In this mesh configuration, the length  $L$  is  $0.2\text{ m}$ . By constraining the Reynolds number, wall friction velocity, and density, the consistent viscosity is obtained via,

$$\mu = \frac{\rho u^\tau d_o}{Re^\tau}. \quad (16)$$

To obtain the required pressure gradient, we exercise the relationship,

$$u^\tau = \sqrt{\frac{\tau_w}{\rho}}, \quad (17)$$

along with global momentum balance,

$$\int \frac{dP}{dz} dV = \int \tau_w dA, \quad (18)$$

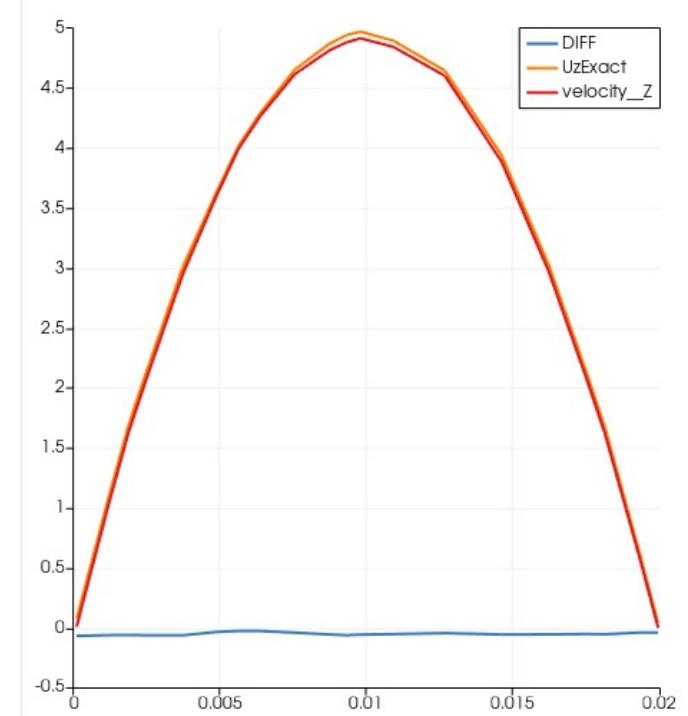
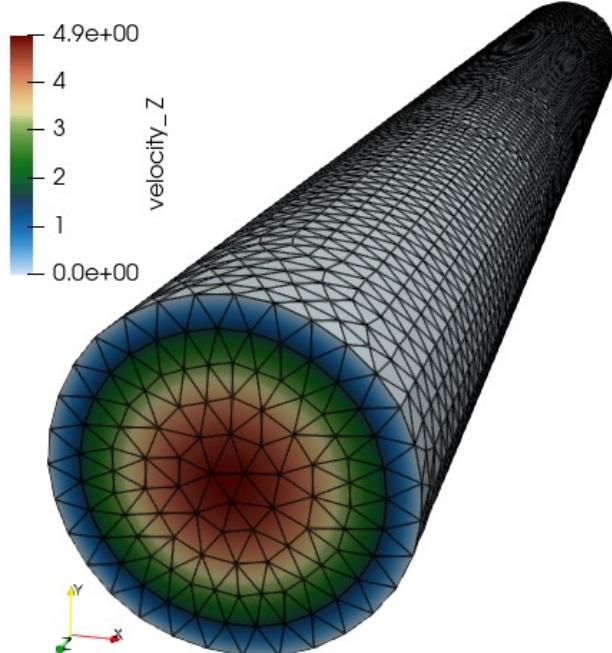
with  $dV = \pi r^2 L$  and  $dA = 2\pi r L$ , to obtain the relationship between required pressure gradient and wall shear stress,  $\frac{dP}{dz} = 2 \frac{\tau_w}{r}$ .

## Simulation Results

Analytical Expression:

$$u_z(r) = \frac{1}{4\mu} \frac{dP}{dz} [r^2 - R^2].$$

Non-zero error when evaluating the norm at the nodes...



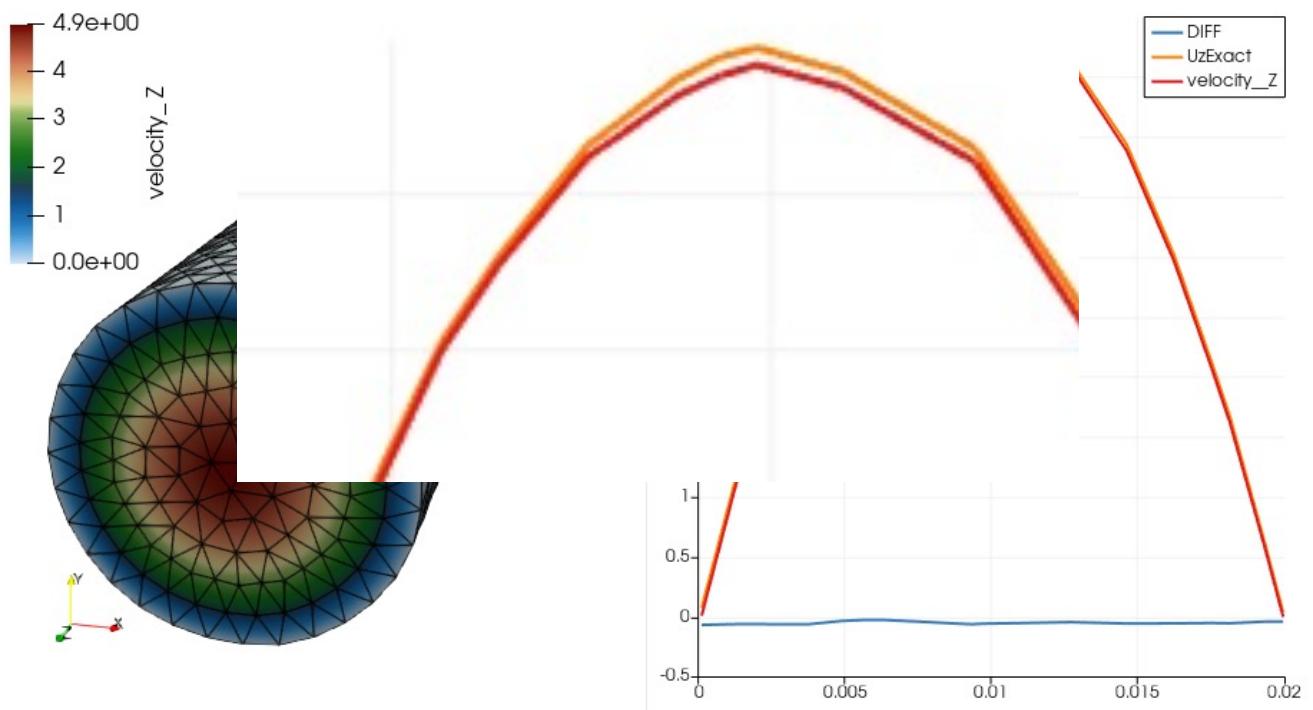
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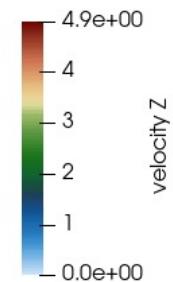
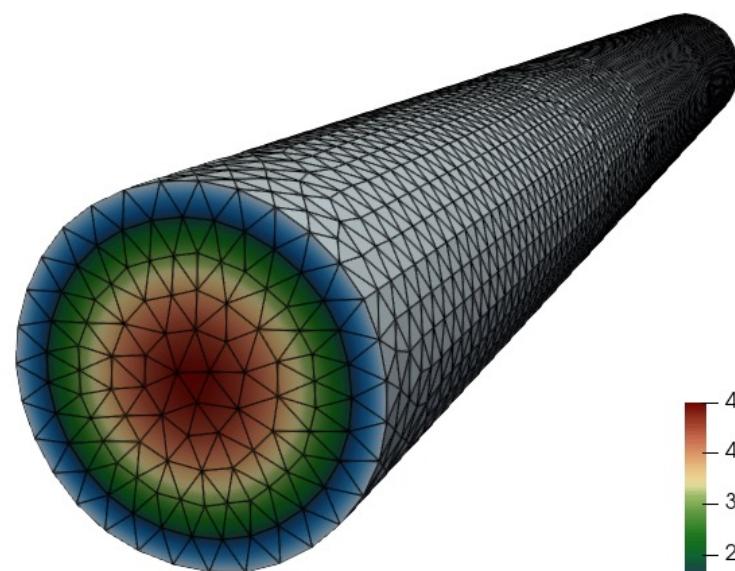
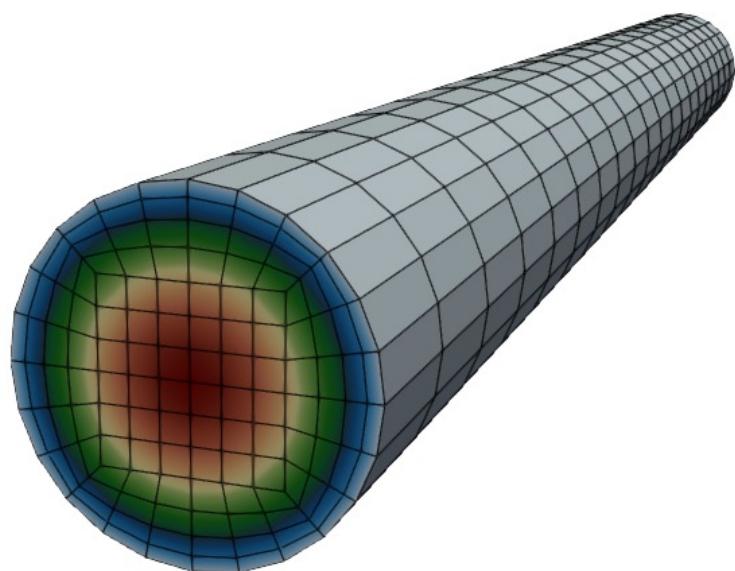
Non-zero error when evaluating the norm at the nodes...

What happened?



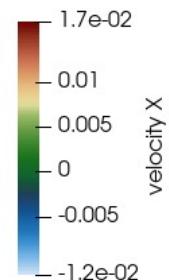
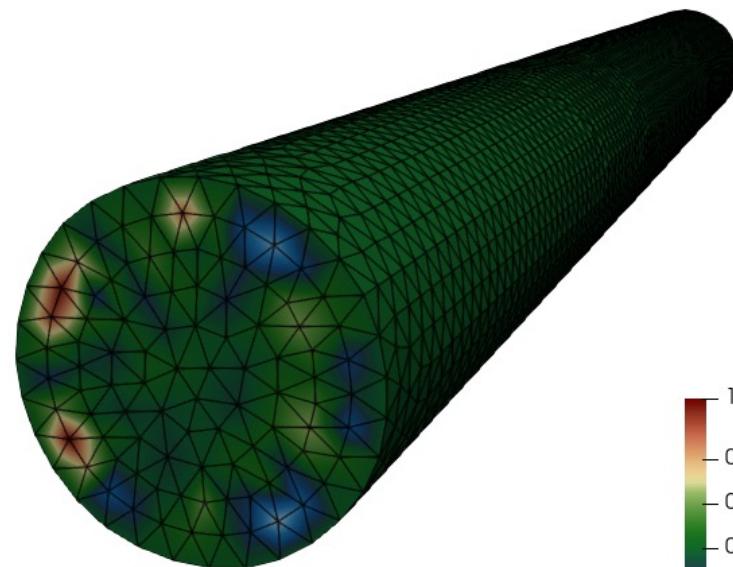
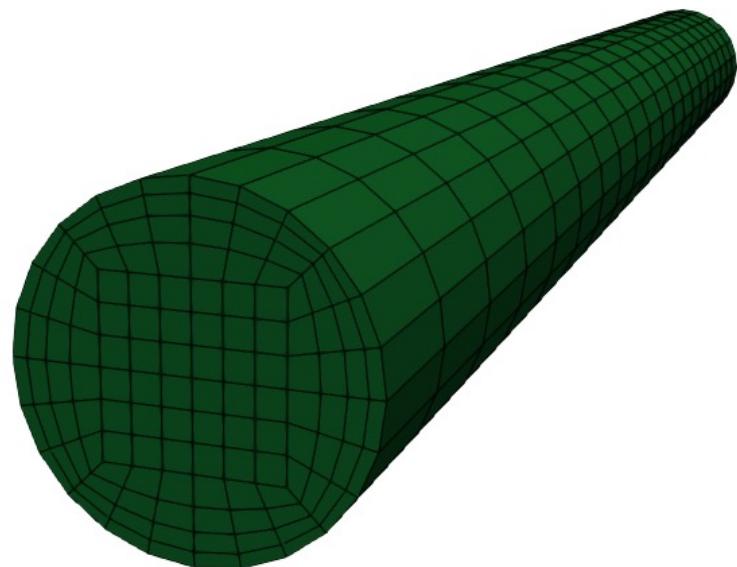


## What About Tet4 vs Hex8?





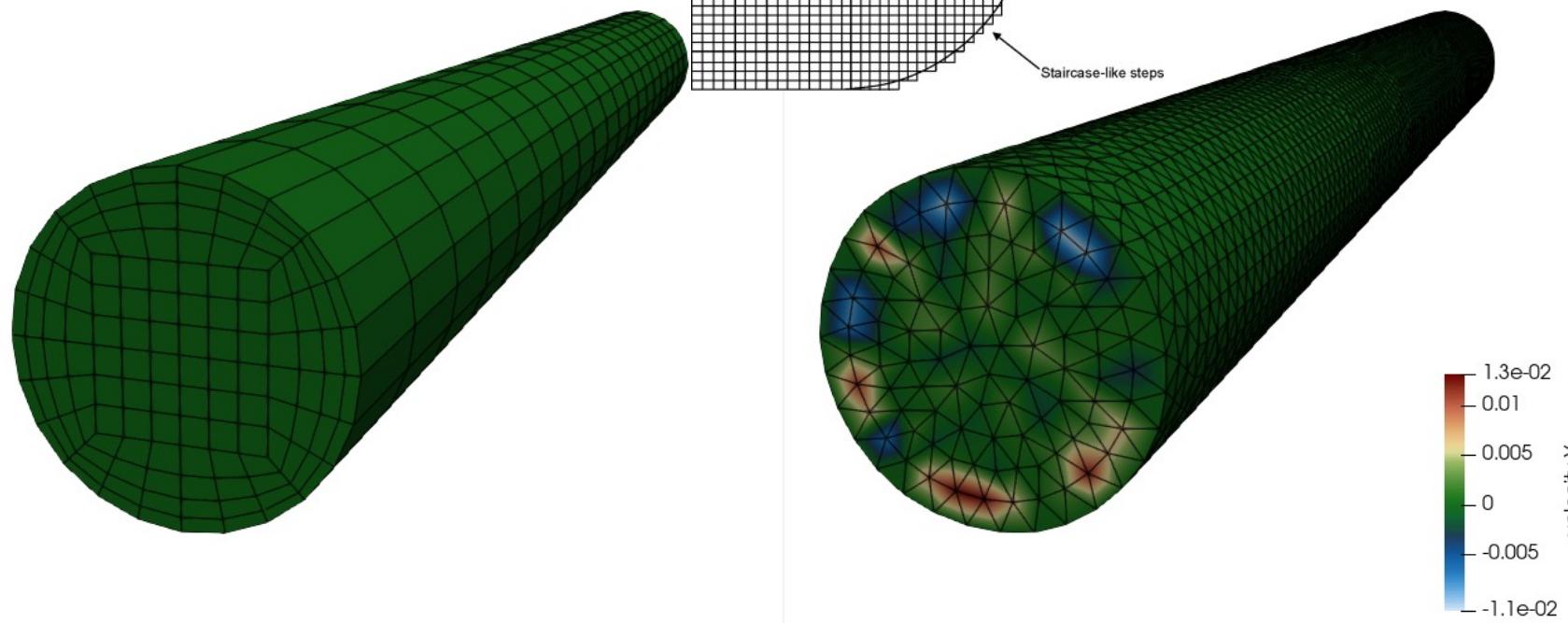
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~0.3% error in non-streamwise velocity components in the tet-based result, ~0% for Hex8!!!



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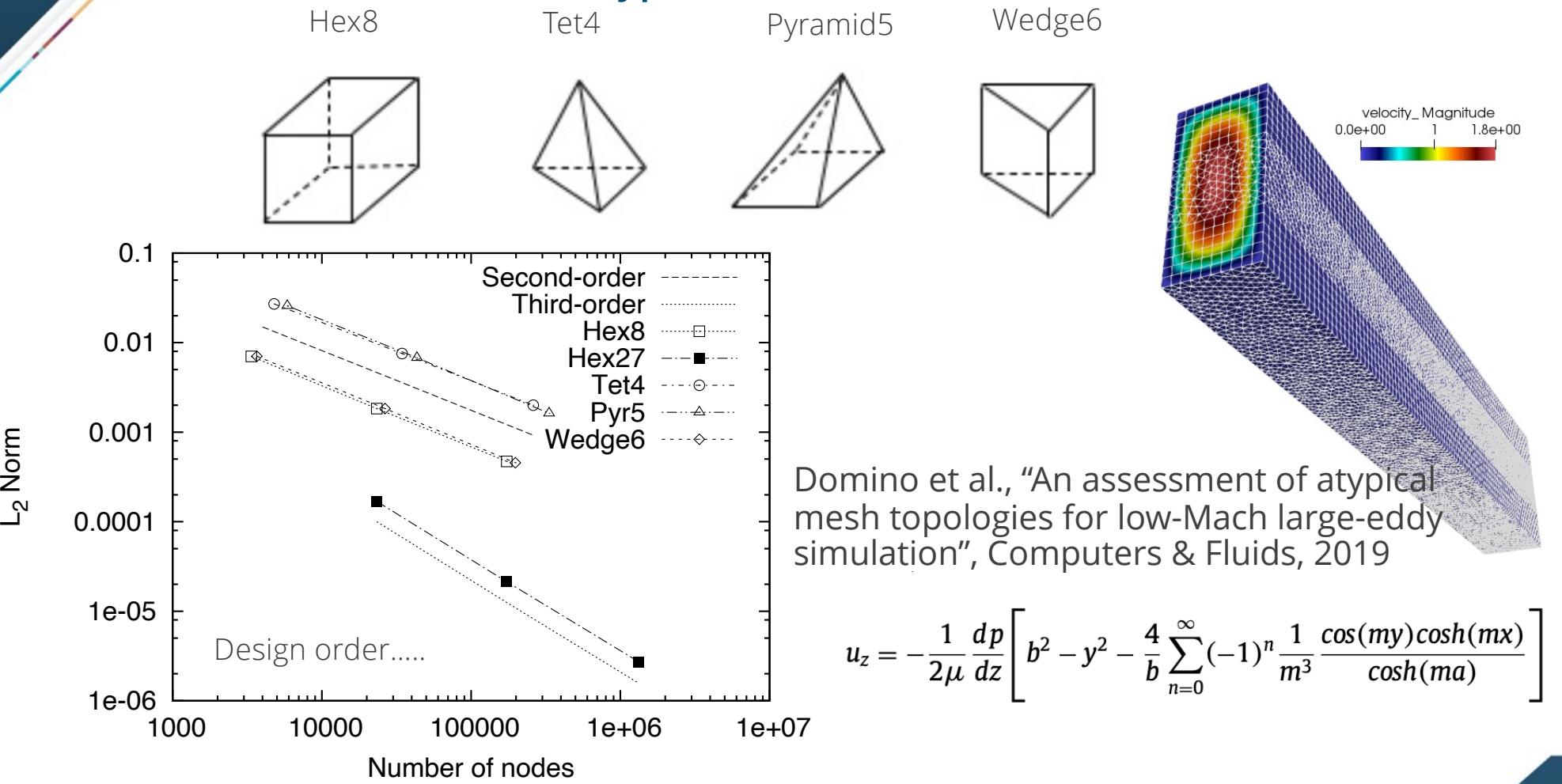


Jiyuan Tu, Guan-Heng Yeoh, Chaoqun Liu,  
Chapter 4 - CFD Mesh Generation: A Practical Guideline,  
Editor(s): Jiyuan Tu, Guan-Heng Yeoh, Chaoqun Liu,  
Computational Fluid Dynamics (Third Edition),  
Butterworth-Heinemann, 2018, Pages 125-154,

~0.3% error in non-streamwise velocity components in the tet-based result, ~0% for Hex8!!!



## Unstructured Mesh Types (Review) and Results





## Conclusions

- Geometric fidelity for curved surfaces can be very important
- Unstructured meshing generally affords easier meshing, with increase throughput (especially when simulating complex geometries)
- Hex-based errors may be lower than Tet-based simulation results, however, balanced with ease of meshing
- The key is that all methods are “design-order”, i.e., demonstrate expected reduction of errors as a function of mesh size