

# Laminar, Two-dimensional Couette Flow

## 1 Introduction

This case provides a description for two-dimensional Couette flow with constant properties, and a zero pressure gradient.

## 2 Theory

The two-dimensional geometry for this tutorial is captured in Figure 1 where the rectangular domain is defined by the height,  $H$ , and length,  $L$ . The streamwise and vertical velocity are defined as  $u_x$  and  $u_y$ , respectively.

The top surface is a no-slip wall boundary specification  $u_x = u_b$ , where  $u_b$  is a bulk velocity of the top moving wall and  $u_y = 0$ . The bottom surface is also a no-slip wall boundary specifications with  $u_x = u_y = 0$ . Finally, the left and right surfaces are periodic. In absence of any external body forces, the flow is aligned to the x-axis and is strictly a function of the vertical-dimension,  $y$ , i.e.,  $u_x = f(y)$ .

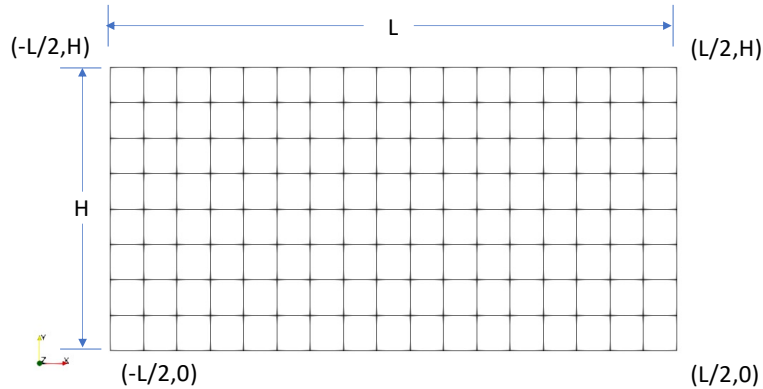


Figure 1: Two-dimensional couette flow in which the height is 2 m and length, 1 m

The variable-density low-Mach equation set is defined by the continuity and momentum equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \quad (1)$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0. \quad (2)$$

In the above equation,  $\rho$  is the fluid density and  $u_j$  is the fluid velocity. The stress tensor is provided by

$$\sigma_{ij} = 2\mu S_{ij}^* - P\delta_{ij}, \quad (3)$$

where the traceless rate-of-strain tensor is defined as

$$S_{ij}^* = S_{ij} - \frac{1}{3}\delta_{ij}S_{kk} = S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}.$$

In a low-Mach flow, the above pressure,  $P$ , is the perturbation about the thermodynamic pressure,  $P^{th}$ .

## 2.1 Analytical Velocity Profile

Given the assumptions provided in the introduction, the streamwise velocity equation reduces to,

$$\mu \frac{d^2 u_x}{dx^2} = 0. \quad (4)$$

We note that a more interesting Couette flow can be derived in the presence of a constant pressure gradient. Activation of a non-zero dynamic viscosity, Equation 4 can be integrated twice to obtain,

$$u_x(y) = k_1 y + k_2, \quad (5)$$

where  $k_1$  and  $k_2$  are constants of integration that are obtained through the application of boundary conditions,  $u_x(y = 0) = 0$  and  $u_x(y = H) = u_b$ . Therefore, the final expression for the streamwise velocity is simply a linear function of vertical distance,

$$u_x(y) = \frac{u_b}{H}y. \quad (6)$$

### 3 Results

Let us test a simulation in which the Reynolds number based on height of the domain, assuming the properties of water ( $\rho = 1000 \text{ kg/m}^3$  and  $\mu = 8.9\text{e-}4 \text{ Pa-s}$ ) at a bulk top velocity of  $u_b = 1\text{e-}3 \text{ m/s}$ , is approximately 1236.

$$Re = \frac{\rho u^b H}{\mu}. \quad (7)$$

#### 3.1 Simulation Specification and Results

The mesh exercised activates a Quad9 topology, thereby exercising a quadratic underlying basis that yields a nominal third-order spatial accurate simulation.

In Figure 2, results are provided for the specifications provided above.

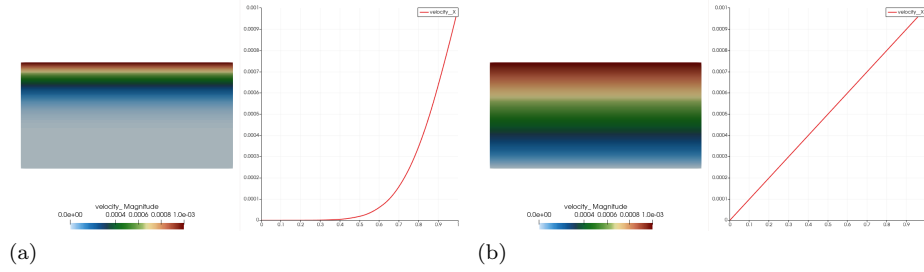


Figure 2: Velocity shadings (left) and velocity profile (right) for the  $Re = 1236$  case.

### 4 Discussion Points

There are several interesting activities associated with this sample case including the following:

- Ensure that derivation of Equation 6 is clear.
- Explore the mesh and input file associated with this case.
- In Figure 2, the flow results demonstrate a linear profile, as expected. Based on past experience with a linear basis, comment on the usage of a quadratic basis.
- Probe all degree-of-freedom results, i.e., velocity and pressure. What is of interest?