Laminar, Two-dimensional Channel Flow

1 Introduction

This case provides a description for two-dimensional channel flow with constant properties, and a constant pressure gradient.

2 Theory

The two-dimensional geometry for this tutorial is captured in Figure 1 where the rectangular domain is defined by the height, H, and length, L. The streamwise and vertical velocity are defined as u_x and u_y, respectively.

The top and bottom surfaces are no-slip wall boundary specifications with $u_x = u_y = 0$, while the left and right surfaces are open boundaries in which a static pressure is supplied. Zero normal gradients are applied to the open boundary. In absence of any external body forces, the flow is aligned to the x-axis and is strictly a function of the vertical-dimension, y, i.e., $u_x = f(y)$.

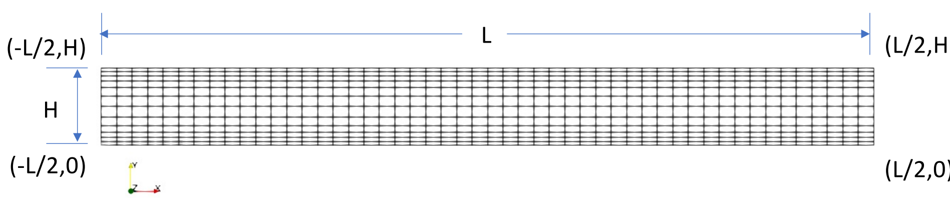

Figure 1: Two-dimensional channel flow in which the height is unity and length, 10.

The variable-density low-Mach equation set is defined by the continuity and momentum equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \tag{1}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0. \tag{2}$$

In the above equation, ρ is the fluid density and u_j is the fluid velocity. The Cauchy stress is provided by

$$\sigma_{ij} = 2\mu S^*_{ij} - P\delta_{ij}, \tag{3}$$

where the traceless rate-of-strain tensor is defined as

$$S^*_{ij} = S_{ij} - \frac{1}{3}\delta_{ij}S_{kk} = S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}.$$

In a low-Mach flow, the above pressure, P, is the perturbation about the thermodynamic pressure, P^{th}.

2.1 Analytical Velocity Profile

Given the assumptions provided in the introduction, the streamwise velocity equation reduces to,

$$\frac{dP}{dx} = \mu\frac{d^2u_x}{dx^2}. \tag{4}$$

This equation can be integrated twice to obtain,

$$u_x(y) = \frac{1}{\mu}\frac{dP}{dx}\frac{y^2}{2} + k_1 y + k_2, \tag{5}$$

where k_1 and k_2 are constants of integration that are obtained through the application of boundary conditions, $u_x(y=0)=0$ and $u_x(y=H)=0$. Therefore, the final expression for the streamwise velocity is,

$$u_x(y) = \frac{1}{2\mu}\frac{dP}{dx}\left[y^2 - Hy\right]. \tag{6}$$

The above equation can be used to determine the vertical location at which the maximum velocity is found via solving $\frac{du_x}{dy} = 0$, or u_x^{max} occurs at $H/2$. The functional form for the maximum velocity is,

$$u_x^{max} = \frac{1}{8\mu}\frac{dP}{dx}H^2. \tag{7}$$

3 Results

Let us test a simulation in which the Reynolds number based on wall friction velocity, u^τ, and half-channel height $H/2$, is ten: $Re^\tau = 10$. By constraining the Reynolds number, wall friction velocity, and density, the consistent viscosity is obtained via,

$$\mu = \frac{\rho u^\tau H}{2Re^\tau}. \tag{8}$$

To obtain the required pressure gradient, we exercise the relationship,

$$u^\tau = \sqrt{\frac{\tau_w}{\rho}}, \tag{9}$$

along with global momentum balance,

$$\int \frac{dP}{dx} dV = \int \tau_w dA, \tag{10}$$

with $dV = LH$ and $dA = 2L$, to obtain the relationship between required pressure gradient and wall shear stress, $\frac{dP}{dx} = 2\tau_w$.

3.1 Simulation Specification and Results

Arbitrarily setting the density and wall friction velocity to unity, along with enforcing the height to be unity, provides a viscosity of 1/20. Moreover, the required pressure gradient given our channel length of 10 is 2. The mesh exercised activates a Quad4 topology, thereby exercising a linear underlying basis that yields a nominal second-order spatial accurate simulation.

In Figure 2, results are provided for the specifications provided above.

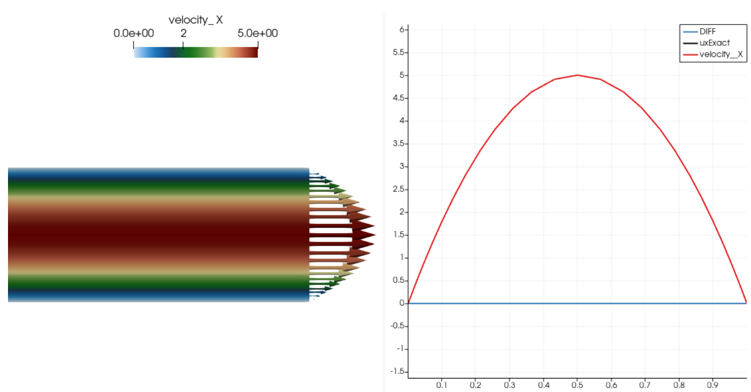

Figure 2: Velocity shadings and profile for the $Re^\tau = 10$ case.

4 Discussion Points

There are several interesting activities associated with this sample case including the following:

- Ensure that derivation of Equation 6 is clear.
- Ensure that derivation of Equation 7 is clear.
- Ensure that derivation of Equation 10 and the expression $\frac{dP}{dx} = 2\tau_w$ is clear.
- Explore the mesh and input file associated with this case.
- In Figure 2, it is noted that the difference between the analytical and simulation result is zero. When you run the two mesh resolutions provided, does this finding hold? Why is this?
- Probe all degree-of-freedom results, i.e., velocity and pressure. What is of interest?
- When the simulation is run and wall shear stress is provided in the output file, what is the value? What are the findings?